Appendix

Proof of Theorem II.2. The proof follows ideas from [18, Theorem 3.20] or [1, Theorem 3.2] and utilizes the discrete subsampling from [2]. By [2, Lemma 2.2 and Theorem 3.1] we obtain

$$\frac{n}{2} \le \sigma_{\min}^2(\boldsymbol{L}) = \|(\boldsymbol{L}^*\boldsymbol{L})^{-1}\boldsymbol{L}^*\|_{2\to 2}^2 \tag{5}$$

with probability $1-\exp(-t)$. We split the approximation error as follows

$$\begin{aligned}\|f - S_{\boldsymbol{X}}\boldsymbol{f}\|_{L_2}^2 &= \|f-P_I f\|_{L_2}^2 + \|P_I f - S_{\boldsymbol{X}}\boldsymbol{f}\|_{L_2}^2 \\ &\le \|f-P_I f\|_{L_2}^2 + 2\|P_I f - S_{\boldsymbol{X}} P_{I_M}\boldsymbol{f}\|_{L_2}^2 + 2\|S_{\boldsymbol{X}} P_{I_M} f - S_{\boldsymbol{X}}\boldsymbol{f}\|_{L_2}^2 .\end{aligned}$$

By event (5) and the invariance of $S_{\boldsymbol{X}}$ to functions supported on I, we obtain

$$\begin{aligned}\|f - S_{\boldsymbol{X}}\boldsymbol{f}\|_{L_2}^2 &\le \|f-P_I f\|_{L_2}^2 + \frac{4}{n}\sum_{i\in J}|(P_I f - P_{I_M} f)(\boldsymbol{x}^i)|^2 + \frac{4}{n}\sum_{i\in J}|(f - P_{I_M} f)(\boldsymbol{x}^i)|^2 \\ &\le \|f-P_I f\|_{L_2}^2 + \frac{4}{n}\sum_{i\in J}|(P_I f - P_{I_M} f)(\boldsymbol{x}^i)|^2 + 4\|f-P_{I_M} f\|_{\infty}^2 .\end{aligned} \tag{6}$$

We will estimate the middle summand by Bernstein's inequality, cf. [9, Corollary 7.31] or [24, Theorem 6.12]. We define random variables $\xi_i = |(P_I f - P_{I_M} f)(\boldsymbol{x}^i)|^2 - \|P_{I_M} f - P_I f\|_{L_2}^2$. By the reconstructing property, we have $\frac{1}{M}\sum_{j=1}^M |(P_I f - P_{I_M} f)(\boldsymbol{x}^i)|^2 = \|P_{I_M} f - P_I f\|_{L_2}^2$ and, thus, ξ_i are mean zero. Further, we have

$$\|\xi_i\|_\infty \le (\|P_I f - P_{I_M} f\|_\infty + \|P_I f - P_{I_M} f\|_{L_2})^2$$

and

$$\begin{aligned}\mathbb{E}(\xi_i^2) &= \mathbb{E}\Big(\Big(|(P_I f - P_{I_M} f)(\boldsymbol{x}^i)|^2\Big)^2\Big) + \|P_I f - P_{I_M} f\|_{L_2}^4 \\ &\le \|P_I f - P_{I_M} f\|_\infty^2 \sum_{i=1}^M |(P_I f - P_{I_M} f)(\boldsymbol{x}^i)|^2 + \|P_I f - P_{I_M} f\|_{L_2}^4 \\ &\le \|P_I f - P_{I_M} f\|_{L_2}^2 (\|P_I f - P_{I_M} f\|_\infty + \|P_I f - P_{I_M} f\|_{L_2})^2 .\end{aligned}$$

Applying Bernstein's inequality yields

$$\begin{aligned}\frac{1}{n}\sum_{i\in J}|(P_I f - P_{I_M} f)(\boldsymbol{x}^i)|^2 - \|P_I f - P_{I_M} f\|_{L_2}^2 &\le \frac{2t}{3n}(\|P_I f - P_{I_M} f\|_\infty + \|P_I f - P_{I_M} f\|_{L_2})^2 \\ &\quad + \sqrt{\frac{2t}{n}}\|P_I f - P_{I_M} f\|_{L_2}(\|P_I f - P_{I_M} f\|_\infty + \|P_I f - P_{I_M} f\|_{L_2})\end{aligned}$$

with probability $1-\exp(-t)$.

Plugging this in (6) and using $\|P_{I_M} f - P_I f\|_{L_2} \le \|f - P_I f\|_{L_2}$, we obtain

$$\begin{aligned}\|f - S_{\boldsymbol{X}}\boldsymbol{f}\|_{L_2}^2 &\le 5\|f-P_I f\|_{L_2}^2 + 2\sqrt{\frac{8t}{n}}\|f-P_I f\|_{L_2}\Big(\|P_{I_M} f - P_I f\|_\infty + \|f-P_I f\|_{L_2}\Big) \\ &\quad + \frac{8t}{3n}\Big(\|P_{I_M} f - P_I f\|_\infty + \|f-P_I f\|_{L_2}\Big)^2 + 4\|f-P_{I_M} f\|_\infty^2 \\ &\le \Big(\sqrt{5}\|f-P_I f\|_{L_2} + \sqrt{\frac{8t}{3n}}\Big(\|P_{I_M} f - P_I f\|_\infty + \|f-P_I f\|_{L_2}\Big)\Big)^2 \\ &\quad + 4\|f-P_{I_M} f\|_\infty^2 .\end{aligned}$$

Using the assumption on n, we have $8t/3n \le 2/9|I| \le 2/9$ and further estimate

$$\|f - S_{\boldsymbol{X}}\boldsymbol{f}\|_{L_2}^2 \le \Big(3\|f-P_I f\|_{L_2} + \sqrt{\frac{2}{9|I|}}\|P_{I_M} f - P_I f\|_\infty\Big)^2 + 4\|f-P_{I_M} f\|_\infty^2 .$$

The second part of the bound is obtained by applying Hölder's inequality:

$$\begin{aligned}\|P_{I_M}f - P_I f\|_\infty^2 &= \Big\| \sum_{\boldsymbol{k}\in I_M\setminus I} \hat{f}_{\boldsymbol{k}} \exp(2\pi\mathrm{i}\langle \boldsymbol{k},\cdot\rangle)\Big\|_\infty^2 \\ &\le \Big\| \sum_{\boldsymbol{k}\in I_M\setminus I} |\exp(2\pi\mathrm{i}\langle \boldsymbol{k},\cdot\rangle)|^2 \Big\|_\infty \sum_{\boldsymbol{k}\in I_M\setminus I} |\hat{f}_{\boldsymbol{k}}|^2 \\ &= |I_M\setminus I| \|P_{I_M}f - P_I f\|_{L_2}^2 \le |I_M\setminus I| \|f - P_I f\|_{L_2}^2 \,.\end{aligned}$$

By union bound we obtain the overall probability. ∎

Lemma A.1. *For X uniformly distributed on $\mathbb{T}^d$, we have*

$$\mathbb{P}(|g(X)| \ge \delta') \ge \frac{\|g\|_{L_2}^2 - \delta'^2}{\|g\|_\infty^2}\,.$$

Proof. Consequence of [19, Par. 9.3.A] for $h(t) = t^2$. ∎

Proof of Theorem III.1. From Lemma A.1 follows

$$\mathbb{P}(|\hat{f}_{\{1,\dots,t\},\boldsymbol{k}}(\boldsymbol{\xi})| \ge \delta') \ge \frac{\|\hat{f}_{\{1,\dots,t\},\boldsymbol{k}}\|_{L_2}^2 - \delta'^2}{\|\hat{f}_{\{1,\dots,t\},\boldsymbol{k}}\|_\infty^2} =: p_{\boldsymbol{k}}\,.$$

Let $\overline{p} = \max_{\boldsymbol{k}\in\mathcal{P}_{\{1,\dots,t\}}(I_\delta)} p_{\boldsymbol{k}}$. By union bound, we have the successful detection with probability exceeding $1 - |I_\delta|(1-\overline{p})^r$. This is smaller or equal $1-\varepsilon$ whenever

$$r \le -\frac{\log|I_\delta| + \log(1/\varepsilon)}{\log(1-\overline{p})}\,. \tag{7}$$

Since $p_{\boldsymbol{k}} \in (0,1)$, we obtain

$$-\frac{1}{\log(1-p_{\boldsymbol{k}})} \le \frac{1}{p_{\boldsymbol{k}}} = \frac{\|\hat{f}_{\{1,\dots,t\},\boldsymbol{k}}\|_\infty^2}{\|\hat{f}_{\{1,\dots,t\},\boldsymbol{k}}\|_{L_2}^2 - \delta'^2} \le \frac{(\sum_{(\boldsymbol{k},\boldsymbol{l})\in\mathbb{Z}^{d-t}} |\hat{f}_{(\boldsymbol{k},\boldsymbol{l})}|)^2}{\sum_{(\boldsymbol{k},\boldsymbol{l})\in\mathbb{Z}^{d-t}} |\hat{f}_{(\boldsymbol{k},\boldsymbol{l})}|^2 - \delta'^2}\,.$$

With $\boldsymbol{k}\in\mathcal{P}_{\{1,\dots,t\}}(I_\delta)$ we have

$$\delta'^2 \le \delta^2/2 \le \frac{1}{2}\sum_{(\boldsymbol{k},\boldsymbol{l})\in\mathbb{Z}^{d-t}} |\hat{f}_{(\boldsymbol{k},\boldsymbol{l})}|^2\,.$$

Consequently,

$$\begin{aligned}-\frac{1}{\log(1-p_{\boldsymbol{k}})} &\le 2\frac{(\sum_{(\boldsymbol{k},\boldsymbol{l})\in\mathbb{Z}^{d-t}} |\hat{f}_{(\boldsymbol{k},\boldsymbol{l})}|)^2}{\sum_{(\boldsymbol{k},\boldsymbol{l})\in\mathbb{Z}^{d-t}} |\hat{f}_{(\boldsymbol{k},\boldsymbol{l})}|^2} \\ &\le 4\frac{(\sum_{(\boldsymbol{k},\boldsymbol{l})\in\mathcal{P}_{\{1,\dots,t\},\boldsymbol{k}}(I_\delta)} |\hat{f}_{(\boldsymbol{k},\boldsymbol{l})}|)^2}{\sum_{(\boldsymbol{k},\boldsymbol{l})\in\mathbb{Z}^{d-t}} |\hat{f}_{(\boldsymbol{k},\boldsymbol{l})}|^2} + 4\frac{(\sum_{(\boldsymbol{k},\boldsymbol{l})\notin\mathcal{P}_{\{1,\dots,t\},\boldsymbol{k}}(I_\delta)} |\hat{f}_{(\boldsymbol{k},\boldsymbol{l})}|)^2}{\sum_{(\boldsymbol{k},\boldsymbol{l})\in\mathbb{Z}^{d-t}} |\hat{f}_{(\boldsymbol{k},\boldsymbol{l})}|^2} \\ &\le 4|\mathcal{P}_{\{1,\dots,t\},\boldsymbol{k}}(I_\delta)| + \frac{4}{\delta^2}\Big(\sum_{(\boldsymbol{k},\boldsymbol{l})\notin\mathcal{P}_{\{1,\dots,t\},\boldsymbol{k}}(I_\delta)} |\hat{f}_{(\boldsymbol{k},\boldsymbol{l})}|\Big)^2 \\ &\le 4|I_\delta| + \frac{4}{\delta^2}\Big(\sum_{\boldsymbol{l}\notin I_\delta} |\hat{f}_{\boldsymbol{l}}|\Big)^2\,.\end{aligned}$$

Plugging this into (7) we obtain the assertion. ∎

Proof of Theorem III.2. By Theorem III.1 we have

$$\begin{aligned}\frac{\delta}{\sqrt{2}} &\le \max_{i=1,\dots,r} |\hat{f}_{\{1,\dots,t\},\boldsymbol{k}}(\boldsymbol{\xi}^i)| \\ &\le \max_{i=1,\dots,r} |\hat{f}_{\{1,\dots,t\},\boldsymbol{k}}(\boldsymbol{\xi}^i) - \hat{g}_{\{1,\dots,t\},\boldsymbol{k}}(\boldsymbol{\xi}^i)| + |\hat{g}_{\{1,\dots,t\},\boldsymbol{k}}(\boldsymbol{\xi}^i)|\,.\end{aligned}$$

Next, we show a bound on the above difference in order to obtain the desired threshold. We have

$$\begin{aligned}|\hat{f}_{\{1,\dots,t\},\boldsymbol{k}}(\boldsymbol{\xi}^i) - \hat{g}_{\{1,\dots,t\},\boldsymbol{k}}(\boldsymbol{\xi}^i)|^2 &\le \sum_{k\in I_{\{1,\dots,t\}}} |\hat{f}_{\{1,\dots,t\},\boldsymbol{k}}(\boldsymbol{\xi}^i) - \hat{g}_{\{1,\dots,t\},\boldsymbol{k}}(\boldsymbol{\xi}^i)|^2 \\ &\le \|f(\cdot,\boldsymbol{\xi}^i) - S_{\boldsymbol{X}}f(\cdot,\boldsymbol{\xi}^i\|_{L_2}^2\,.\end{aligned}$$

Using Theorem 2.2 and the assumption on the carnality of the frequency index sets, we obtain

$$\begin{aligned}
&|\hat{f}_{\{1,\dots,t\},\boldsymbol{k}}(\boldsymbol{\xi}^i) - \hat{g}_{\{1,\dots,t\},\boldsymbol{k}}(\boldsymbol{\xi}^i)|^2 \\
&\quad \leq \Big(3 + \sqrt{\frac{2|I^M_{\{1,\dots,t\}} \setminus I_{\{1,\dots,t\}}|}{9|I_{\{1,\dots,t\}}|}}\Big)^2 \|f(\cdot,\boldsymbol{\xi}^i) - P_{I_{\{1,\dots,t\}}} f(\cdot,\boldsymbol{\xi}^i)\|_{L_2}^2 \\
&\qquad + 4\|f(\cdot,\boldsymbol{\xi}^i) - P_{I^M_{\{1,\dots,t\}}} f(\cdot,\boldsymbol{\xi}^i)\|_\infty^2 \\
&\quad \leq 16\|f(\cdot,\boldsymbol{\xi}^i) - P_{I_{\{1,\dots,t\}}} f(\cdot,\boldsymbol{\xi}^i)\|_{L_2}^2 + 4\|f(\cdot,\boldsymbol{\xi}^i) - P_{I^M_{\{1,\dots,t\}}} f(\cdot,\boldsymbol{\xi}^i)\|_\infty^2
\end{aligned}$$

with probability

$$1 - 2r\exp(-t) \quad \text{and} \quad |\boldsymbol{X}| \geq 12|I_{\{1,\dots,t\}}|(\log|I_{\{1,\dots,t\}}| + t)\,,$$

where union bound was used over the detection iterations. Setting $t = \log(2r/\varepsilon)$ we obtain the assertion. ■