SUPPLEMENTARY MATERIAL: LAU: A NOVEL TWO-PARAMETER LEARNABLE LOGMOID ACTIVATION UNIT

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A PROOF OF THEOREM 1

This proof is inspired by the results in Refs.(Chen & Cao, 2009; 2015). From the property (3) in Proposition 1 and Eq.(19) in the main text, we obtain that

$$\begin{aligned}
& |f(x) - G(f,x)| \\
= \quad & \left| f(x) \sum_{k=-\infty}^{\infty} \Phi(nx-k) - \sum_{k=-n}^{n} f(\frac{k}{n})\Phi(nx-k) \right| \\
\leq \quad & \left| \sum_{k=-n}^{n} \left(f(x) - f(\frac{k}{n}) \right) \Phi(nx-k) \right| \\
& + \left| \sum_{|k|\geq n+1} f(x)\Phi(nx-k) \right| \qquad (1) \\
\leq \quad & \sum_{k=-n}^{n} \left| f(x) - f(\frac{k}{n}) \right| \Phi(nx-k) \\
& + \|f\|_\infty \sum_{|k|\geq n+1} \Phi(nx-k) \qquad (2) \\
=: \quad & \triangle_1 + \|f\|_\infty \triangle_2 \,. \qquad (3)
\end{aligned}$$

Here, we have used the triangle inequality of $|x+y| \leq |x| + |y|$ to get the inequality (1). The inequality (2) is followed from the inequality of $\Phi(x) > 0$ in Proposition 1 and the triangle inequality, where $\|f\|_\infty$ is the uniform norm of f on $\mathbb{R}$.

In what follows, we estimate $\triangle_1$ and $\triangle_2$. For any $\alpha \in (0,1]$, from Eq.(3) we have

$$\begin{aligned}
\triangle_1 \quad = \quad & \sum_{k:\left|x-\frac{k}{n}\right|\leq\frac{1}{n^\alpha}} \left| f(x) - f(\frac{k}{n}) \right| \Phi(nx-k) \\
& + \sum_{k:\left|x-\frac{k}{n}\right|>\frac{1}{n^\alpha}} \left| f(x) - f(\frac{k}{n}) \right| \Phi(nx-k) \\
\leq \quad & w(f, \frac{1}{n^\alpha}) \sum_{k=-\infty}^{\infty} \Phi(nx-k) \\
& + 2\|f\|_\infty \sum_{k:\left|x-\frac{k}{n}\right|>\frac{1}{n^\alpha}} \Phi(nx-k) \qquad (4) \\
\leq \quad & w(f, \frac{1}{n^\alpha}) + 2\|f\|_\infty \sum_{k:|nx-k|>n^{1-\alpha}} \Phi(nx-k). \\
& \qquad (5)
\end{aligned}$$

Here, we have used Eq.(17) and the triangle inequality to get the inequality (4). We have used property (3) in the last step.

Consider the property (5) in Proposition 1 and the fact that $\Phi(x)$ is strictly decreasing for any $x > T_0$ given by the property (4) Proposition 1. It follows that

$$\begin{aligned}\sum_{k:|nx-k|>n^{1-\alpha}} \Phi(nx-k) &\leq 2\int_{n^{1-\alpha}-1}^{\infty} \Phi(x)dx \\ &\leq 2\int_{n^{1-\alpha}-1}^{\infty} e^{-\frac{1}{2}x-\frac{1}{2}}dx \\ &= 4e^{-\frac{1}{2}n^{1-\alpha}} \end{aligned} \tag{6}$$

Since $-n \leq nx \leq n$, $|k| \geq n+1$ and $|nx-k| \geq 1$, we get that

$$\begin{aligned}\triangle_2 = \sum_{|k|\geq n+1} \Phi(nx-k) &\leq 2\int_{1}^{+\infty} \Phi(x)dx \\ &\leq 2\int_{0}^{+\infty} e^{-\frac{1}{2}x-\frac{1}{2}}dx \qquad (7) \\ &= \frac{4}{e} \qquad (8)\end{aligned}$$

where we have used the properties (4) and (5) in Proposition 1 for the inequality (7). From Eqs.(6),(8) and (3), we get

$$|f(x) - G(f,x)| \leq w(f, \frac{1}{n^{\alpha}}) + (8e^{-\frac{1}{2}n^{1-\alpha}} + \frac{4}{e})\|f\|_{\infty}$$

This has completed the proof.

B THE PROOF OF THEOREM 2

The proof is similar to its of Theorem 1. From the property (3) in Proposition 1 and Eq.(20) we have

$$\begin{aligned}& \left|f(x) - \overline{G}(f,x)\right| \\ =& \left|f(x)\sum_{k=-\infty}^{\infty}\Phi(nx-k) - \sum_{k=-\infty}^{\infty} f(\frac{k}{n})\Phi(nx-k)\right| \\ \leq& \sum_{k=-\infty}^{\infty}\left|f(x) - f(\frac{k}{n})\right|\Phi(nx-k) \qquad (9) \\ =& \sum_{k:\left|x-\frac{k}{n}\right|\leq\frac{1}{n^{\alpha}}}\left|f(x) - f(\frac{k}{n})\right|\Phi(nx-k) \\ &+ \sum_{k:\left|x-\frac{k}{n}\right|>\frac{1}{n^{\alpha}}}\left|f(x) - f(\frac{k}{n})\right|\Phi(nx-k) \\ :=& \triangle_1 \\ \leq& w(f,\frac{1}{n^{\alpha}}) + 8e^{-\frac{1}{2}n^{1-\alpha}}\|f\|_{\infty} \qquad (10)\end{aligned}$$

where we have used the triangle inequality for the inequality (9). The proof is completed.

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