Consistent Non-Parametric Methods for Maximizing Robustness

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Abstract

Learning classifiers that are robust to adversarial examples has received a great deal of recent attention. A major drawback of the standard robust learning framework is there is an artificial robustness radius r that applies to all inputs. This ignores the fact that data may be highly heterogeneous, in which case it is plausible that robustness regions should be larger in some regions of data, and smaller in others. In this paper, we address this limitation by proposing a new limit classifier, called the neighborhood optimal classifier, that extends the Bayes optimal classifier outside its support by using the label of the closest in-support point. We then argue that this classifier maximizes the size of its robustness regions subject to the constraint of having accuracy equal to the Bayes optimal. We then present sufficient conditions under which general non-parametric methods that can be represented as weight functions converge towards this limit, and show that both nearest neighbors and kernel classifiers satisfy them under certain conditions.

1 Introduction

Adversarially robust classification, that has been of much recent interest, is typically formulated as follows. We are given data drawn from an underlying distribution D, a metric d, as well as a pre-specified robustness radius r. We say that a classifier c is r-robust at an input x if it predicts the same label on a ball of radius r around x. Our goal in robust classification is to find a classifier c that maximizes astuteness, which is defined as accuracy on those examples where c is also r-robust.

While this formulation has inspired a great deal of recent work, both theoretical and empirical [5, 17, 19, 20, 26, 15, 18, 21, 22, 23, 30], a major limitation is that enforcing a pre-specified robustness radius r may lead to sub-optimal accuracy *and* robustness. To see this, consider what would be an ideally robust classifier the example in Figure 1. For simplicity, suppose that we know the data distribution. In this case, a classifier that has an uniformly large robustness radius r will misclassify some points from the blue cluster on the left, leading to lower accuracy. This is illustrated in panel (a), in which large robustness radius leads to intersecting robustness regions. On the other hand, in panel (b), the blue cluster on the right is highly separated from the red cluster, and could be accurately classified with a high margin. But this will not happen if the robustness radius is set small enough to avoid the problems posed in panel (a). Thus, enforcing a fixed robustness radius that applies to the entire dataset may lead to lower accuracy and lower robustness.

In this work, we propose an alternative formulation of robust classification that ensures that in the large sample limit, there is no robustness-accuracy trade off, and that regions of space with higher separation are classified more robustly. An extra advantage is that our formulation is achievable by existing methods. In particular, we show that two very common non-parametric algorithms – nearest neighbors and kernel classifiers – achieve these properties in the large sample limit.

Our formulation is built on the notion of a new large-sample limit. In the standard statistical learning framework, the large-sample ideal is the Bayes optimal classifier that maximizes accuracy on the

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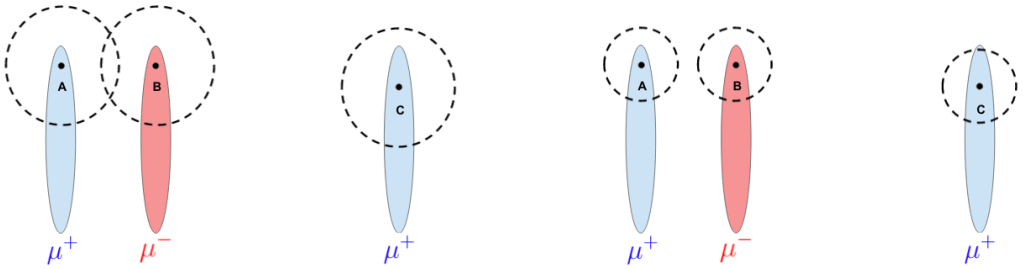


(a) Large robustness radii

(b) Small robustness radii

Figure 1: A data distribution demonstrating the difficulties with fixed radius balls for robustness regions. The red represents negatively labeled points, and the blue positive. If the robustness radius is set too large (panel (a)), then the regions of A and B intersect leading to a loss of accuracy. If the radius is set too small (panel (b)), this leads to a loss of robustness at point C where in principle it should be possible to defend against a larger amount of adversarial attacks.

data distribution, and is undefined outside. Since this is not always robust with radius r, prior work introduces the notion of an r-optimal classifier [31] that maximizes accuracy on points where it is also r-robust. However, this classifier also suffers from the same challenges as the example in Figure 1.

We depart from both by introducing a new limit that we call the neighborhood preserving Bayes optimal classifier, described as follows. Given an input x that lies in the support of the data distribution D, it predicts the same label as the Bayes optimal. On an x outside the support, it outputs the prediction of the Bayes Optimal on the nearest neighbor of x *within* the support of D. The first property ensures that there is no loss of accuracy – since it always agrees with the Bayes Optimal within the data distribution. The second ensures higher robustness in regions that are better separated. Our goal is now to design classifiers that converge to the neighborhood preserving Bayes optimal in the large sample limit; this ensures that with enough data, the classifier will have accuracy approaching that of the Bayes optimal, as well as higher robustness where possible without sacrificing accuracy.

We next investigate how to design classifiers with this convergence property. Our starting point is classical statistical theory [25] that shows that a class of methods known as weight functions will converge to a Bayes optimal in the large sample limit provided certain conditions hold; these include k-nearest neighbors under certain conditions on k and n, certain kinds of decision trees as well as kernel classifiers. Through an analysis of weight functions, we next establish precise conditions under which they converge to the neighborhood preserving Bayes optimal in the large sample limit. As expected, these are stronger than standard convergence to the Bayes optimal. In the large sample limit, we show that k_n-nearest neighbors converge to the neighborhood preserving Bayes optimal provided $k_n = \omega(\log n)$, and kernel classifiers converge to the neighborhood preserving Bayes optimal provided certain technical conditions (such as the bandwidth shrinking sufficiently slowly). By contrast, certain types of histograms do not converge to the neighborhood preserving Bayes optimal, even if they do converge to the Bayes optimal. We round these off with a lower bound that shows that for nearest neighbor, the condition that $k_n = \omega(\log n)$ is tight. In particular, for $k_n = O(\log n)$, there exist distributions for which k_n-nearest neighbors provably fails to converge towards the neighborhood preserving Bayes optimal (despite converging towards the standard Bayes optimal).

In summary, the contributions of the paper are as follows. First, we propose a new large sample limit the neighborhood preserving Bayes optimal and a new formulation for robust classification. We then establish conditions under which weight functions, a class of non-parametric methods, converge to the neighborhood preserving Bayes optimal in the large sample limit. Using these conditions, we show that k_n-nearest neighbors satisfy these conditions when $k_n = \omega(\log n)$, and kernel classifiers satisfy these conditions provided the kernel function K has faster than polynomial decay, and the bandwidth parameter h_n decreases sufficiently slowly.

To complement these results, we also include negative examples of non-parametric classifiers that do not converge. We provide an example where histograms do not converge to the neighborhood preserving Bayes optimal with increasing n. We also show a lower bound for nearest neighbors, indi-

cating that $k_n = \omega(\log n)$ is both necessary and sufficient for convergence towards the neighborhood preserving Bayes optimal.

Our results indicate that the neighborhood preserving Bayes optimal formulation shows promise and has some interesting theoretical properties. We leave open the question of coming up with other alternative formulations that can better balance both robustness and accuracy for all kinds of data distributions, as well as are achievable algorithmically. We believe that addressing this would greatly help address the challenges in adversarial robustness.

2 Preliminaries

We consider binary classification over $\mathbb{R}^d \times \{\pm 1\}$, and let ρ denote any distance metric on $\mathbb{R}^d$. We let μ denote the measure over $\mathbb{R}^d$ corresponding to the probability distribution over which instances $x \in \mathbb{R}^d$ are drawn. Each instance x is then labeled as $+1$ with probability $\eta(x)$ and -1 with probability $1 - \eta(x)$. Together, μ and η comprise our data distribution $\mathcal{D} = (\mu, \eta)$ over $\mathbb{R}^d \times \{\pm 1\}$.

For comparison to the robust case, for a classifier $f : \mathbb{R}^d \to \{\pm 1\}$ and a distribution $\mathcal{D}$ over $\mathbb{R}^d \times \{\pm 1\}$, it will be instructive to consider its **accuracy**, denoted $A(f, \mathcal{D})$, which is defined as the fraction of examples from $\mathcal{D}$ that f labels correctly. Accuracy is maximized by the **Bayes Optimal classifier**: which we denote by g. It can be shown that for any $x \in supp(\mu)$, $g(x) = 1$ if $\eta(x) \geq \frac{1}{2}$, and $g(x) = -1$ otherwise.

Our goal is to build classifiers $\mathbb{R}^d \to \{\pm 1\}$ that are both accurate and robust to small perturbations. For any example x, perturbations to it are constrained to taking place in the **robustness region** of x, denoted U_x. We will let $\mathcal{U} = \{U_x : x \in \mathbb{R}^d\}$ denote the collections of all robustness regions.

We say that a classifier $f : \mathbb{R}^d \to \{\pm 1\}$ is **robust** at x if for all $x' \in U_x$, $f(x') = f(x)$. Combining robustness and accuracy, we say that classifier is **astute** at a point x if it is both accurate and robust. Formally, we have the following definition.

Definition 1. *A classifier $f : \mathbb{R}^d \to \{\pm 1\}$ is said to be **astute** at (x, y) with respect to robustness collection $\mathcal{U}$ if $f(x) = y$ and f is robust at x with respect to $\mathcal{U}$. If $\mathcal{D}$ is a data distribution over $\mathbb{R}^d \times \{\pm 1\}$, the **astuteness** of f over $\mathcal{D}$ with respect to $\mathcal{U}$, denoted $A_{\mathcal{U}}(f, \mathcal{D})$, is the fraction of examples $(x, y) \sim \mathcal{D}$ for which f is astute at (x, y) with respect to $\mathcal{U}$. Thus*

$$A_{\mathcal{U}}(f, \mathcal{D}) = P_{(x,y)\sim\mathcal{D}}[f(x') = y, \forall x' \in \mathcal{U}_x].$$

Non-parametric Classifiers We now briefly review several kinds of non-parametric classifiers that we will consider throughout this paper. We begin with *weight functions*, which are a general class of non-parametric algorithms that encompass many classic algorithms, including nearest neighbors and kernel classifiers.

Weight functions are built from training sets, $S = \{(x_1, y_1), (x_2, y_2,), \ldots, (x_n, y_n)\}$ by assigning a function $w_i^S : \mathbb{R}^d \to [0, 1]$ that essentially scores how relevant the training point (x_i, y_i) is to the example being classified. The functions w_i^S are allowed to depend on $x_1, \ldots, x_n$ but must be independent of the labels $y_1, \ldots, y_n$. Given these functions, a point x is classified by just checking whether $\sum y_i w_i^S(x) \geq 0$ or not. If it is nonnegative, we output $+1$ and otherwise -1. A complete description of weight functions is included in the appendix.

Next, we enumerate several common Non-parametric classifiers that can be construed as weight functions. Details can be found in the appendix.

Histogram classifiers partition the domain $\mathbb{R}^d$ into cells recursively by splitting cells that contain a sufficiently large number of points x_i. This corresponds to a weight function in which $w_i^S(x) = \frac{1}{k_x}$ if x_i is in the same cell as x, where k_x denotes the number of points in the cell containing x.

k_n**-nearest neighbors** corresponds to a weight function in which $w_i^S(x) = \frac{1}{k_n}$ if x_i is one of the k_n nearest neighbors of x, and $w_i^S(x) = 0$ otherwise.

Kernel-Similarity classifiers are weight functions built from a kernel function $K : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and a window size $(h_n)_1^\infty$ such that $w_i^S(x) \propto K(\rho(x, x_i)/h_n)$ (we normalize by dividing by $\sum_1^n K((\rho(x, x_i)/h_n))$).

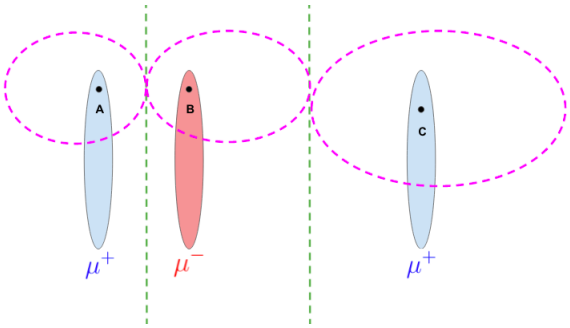


Figure 2: The decision boundary of the neighborhood preserving Bayes optimal classifier is shown in green, and the neighborhood preserving robust region of x is shown in pink. The former consists of points equidistant from μ^+, μ^-, and the latter consists of points equidistant from x, μ^+.

3 The Neighborhood preserving Bayes optimal classifier

Robust classification is typically studied by setting the robustness regions, $\mathcal{U} = \{U_x\}_{x \in \mathbb{R}^d}$, to be balls of radius r centered at x, $U_x = \{x' : \rho(x, x') \leq r\}$. The quantity r is the robustness radius, and is typically set by the practitioner (before any training has occurred).

This method has a limitation with regards to trade-offs between accuracy and robustness. To increase the margin or robustness, we must have a large robustness radius (thus allowing us to defend from larger adversarial attacks). However, with large robustness radii, this can come at a cost of accuracy, as it is not possible to robustly give different labels to points with intersecting robustness regions.

For an illustration, consider Figure 1. Here we consider a data distribution $D = (\mu, \eta)$ in which the blue regions denote all points with $\eta(x) > 0.5$ (and thus should be labeled $+$), and the red regions denote all points with $\eta(x) < 0.5$ (and thus should be labeled $-$). Observe that it is not possible to be simultaneously accurate and robust at points A, B while enforcing a large robustness radius, as demonstrated by the intersecting balls. While this can be resolved by using a smaller radius, this results in losing out on potential robustness at point C. In principal, we should be able to afford a large margin of robustness about C due to its relatively far distance from the red regions.

Motivated by this issue, we seek to find a formalism for robustness that allows us to simultaneously avoid paying for any accuracy-robustness trade-offs and *adaptively* size robustness regions (thus allowing us to defend against a larger range of adversarial attacks at points that are located in more homogenous zones of the distribution support). To approach this, we will first provide an ideal limit object: a classifier that has the same accuracy as the Bayes optimal (thus meeting our first criteria) that has good robustness properties. We call this the the neighborhood preserving Bayes optimal classifier, defined as follows.

Definition 2. *Let $\mathcal{D} = (\mu, \eta)$ be a distribution over $\mathbb{R}^d \times \{\pm 1\}$. Then the **neighborhood preserving Bayes optimal classifier of** $\mathcal{D}$, denoted $g_{neighbor}$, is the classifier defined as follows. Let $\mu^+ = \{x : \eta(x) \geq \frac{1}{2}\}$ and $\mu^- = \{x : \eta(x) < \frac{1}{2}\}$. Then for any $x \in \mathbb{R}^d$, $g_{neighbor}(x) = +1$ if $\rho(x, \mu^+) \leq \rho(x, \mu^-)$, and $g_{neighbor}(x) = -1$ otherwise.*

This classifier can be thought of as the most robust classifier that matches the accuracy of the Bayes optimal. We call it *neighborhood preserving* because it extends the Bayes optimal classifier into a local neighborhood about every point in the support. For an illustration, refer to Figure 2, which plots the decision boundary of the neighborhood preserving Bayes optimal for an example distribution.

Next, we turn our attention towards measuring its robustness, which must be done with respect to some set of robustness regions $\mathcal{U} = \{U_x\}$. While these regions U_x can be nearly arbitrary, we seek regions U_x such that $A_{\mathcal{U}}(g_{max}, \mathcal{D}) = A(g_{bayes}, \mathcal{D})$ (our astuteness equals the maximum possible accuracy) and U_x are "as large as possible" (representing large robustness). To this end, we propose the following regions.

Definition 3. *Let $\mathcal{D} = (\mu, \eta)$ be a data distribution over $\mathbb{R}^d \times \{\pm 1\}$. Let $\mu^+ = \{x : \eta(x) > \frac{1}{2}\}$, $\mu^- = \{x : \eta(x) < \frac{1}{2}\}$, and $\mu^{1/2} = \{x : \eta(x) = \frac{1}{2}\}$. For $x \in \mu^+$, we define the **neighborhood***

***preserving robustness region**, denoted V_x, as*

$$V_x = \{x' : \rho(x, x') < \rho(\mu^- \cup \mu^{\frac{1}{2}}, x')\}.$$

It consists of all points that are closer to x than they are to $\mu^- \cup \mu^{1/2}$ (points oppositely labeled from x). We can use a similar definition for $x \in \mu^-$. Finally, if $x \in \mu^{1/2}$, we simply set $V_x = \{x\}$.

These robustness regions take advantage of the structure of the neighborhood preserving Bayes optimal. They can essentially be thought of as regions that maximally extend from any point x in the support of $\mathcal{D}$ to the decision boundary of the neighborhood preserving Bayes optimal. We include an illustration of the regions V_x for an example distribution in Figure 2.

As a technical note, for $x \in supp(\mathcal{D})$ with $\eta(x) = 0.5$, we give them a trivial robustness region. The rational for doing this is that $\eta(x) = 0.5$ is an edge case that is arbitrary to classify, and consequently enforcing a robustness region at that point is arbitrary and difficult to enforce.

We now formalize the robustness and accuracy guarantees of the max-margin Bayes optimal classifier with the following two results.

Theorem 4. *(Accuracy) Let $\mathcal{D}$ be a data distribution. Let $\mathcal{V}$ denote the collection of neighborhood preserving robustness regions, and let g denote the Bayes optimal classifier. Then the neighborhood preserving Bayes optimal classifier, $g_{neighbor}$, satisfies $A_{\mathcal{V}}(g_{neighbor}, \mathcal{D}) = A(g, \mathcal{D})$, where $A(g, \mathcal{D})$ denotes the accuracy of the Bayes optimal. Thus, $g_{neighbor}$ maximizes accuracy.*

Theorem 5. *(Robustness) Let $\mathcal{D}$ be a data distribution, let f be a classifier, and let $\mathcal{U}$ be a set of robustness regions. Suppose that $A_{\mathcal{U}}(f, \mathcal{D}) = A(g, \mathcal{D})$, where g denotes the Bayes optimal classifier. Then there exists $x \in supp(\mathcal{D})$ such that $V_x \not\subset U_x$, where V_x denotes the neighborhood preserving robustness region about x. In particular, we cannot have V_x be a strict subset of U_x for all x.*

Theorem 4 shows that the neighborhood preserving Bayes classifier achieves maximal accuracy, while Theorem 5 shows that achieving a strictly higher robustness (while maintaining accuracy) is not possible; while it is possible to make accurate classifiers which have higher robustness than $g_{neighbor}$ in some regions of space, it is not possible for this to hold across all regions. Thus, the neighborhood preserving Bayes optimal classifier can be thought of as a local maximum to the constrained optimization problem of maximizing robustness subject to having maximum (equal to the Bayes optimal) accuracy.

3.1 Neighborhood Consistency

Having defined the neighborhood preserving Bayes optimal classifier, we now turn our attention towards building classifiers that converge towards it. Before doing this, we must precisely define what it means to converge. Intuitively, this consists of building classifiers whose robustness regions "approach" the robustness regions of the neighborhood preserving Bayes optimal classifier. This motivates the definition of *partial neighborhood preserving robustness regions*.

Definition 6. *Let $0 < \kappa < 1$ be a real number, and let $\mathcal{D} = (\mu, \eta)$ be a data distribution over $\mathbb{R}^d \times \{\pm 1\}$. Let $\mu^+ = \{x : \eta(x) > \frac{1}{2}\}$, $\mu^- = \{x : \eta(x) < \frac{1}{2}\}$, and $\mu^{1/2} = \{x : \eta(x) = \frac{1}{2}\}$. For $x \in \mu^+$, we define the **neighborhood preserving robustness region**, denoted V_x, as*

$$V_x = \{x' : \rho(x, x') < \kappa\rho(\mu^- \cup \mu^{\frac{1}{2}}, x')\}.$$

It consists of all points that are closer to x than they are to $\mu^- \cup \mu^{1/2}$ (points oppositely labeled from x) by a factor of κ. We can use a similar definition for $x \in \mu^-$. Finally, if $\eta(x) = \frac{1}{2}$, we simply set $V_x^\kappa = \{x\}$.

Observe that $V_x^\kappa \subset V_x$ for all $0 < \kappa < 1$, and thus being robust with respect to V_x^κ is a milder condition than V_x. Using this notion, we can now define margin consistency.

Definition 7. *A learning algorithm A is said to be **neighborhood consistent** if the following holds for any data distribution $\mathcal{D}$. For any $0 < \epsilon, \delta, \kappa < 1$, there exists N such that for all $n \geq N$, with probability at least $1 - \delta$ over $S \sim \mathcal{D}^n$,*

$$A_{\mathcal{V}^\kappa}(A_S, D) \geq A(g, \mathcal{D}) - \epsilon,$$

where g denotes the Bayes optimal classifier and A_S denotes the classifier learned by algorithm A from dataset S.

This condition essentially says that the astuteness of the classifier learned by the algorithm converges towards the accuracy of the Bayes optimal classifier. Furthermore, we stipulate that this holds as long as the astuteness is measured with respect to some $\mathcal{V}^\kappa$. Observe that as $\kappa \to 1$, these regions converge towards the neighborhood preserving robustness regions, thus giving us a classifier with robustness effectively equal to that of the neighborhood preserving Bayes optimal classifier.

4 Neighborhood Consistent Non-Parametric Classifiers

Having defined neighborhood consistency, we turn to the following question: which non-parametric algorithms are neighborhood consistent? Our starting point will be the standard literature for the convergence of non-parametric classifiers with regard to accuracy. We begin by considering the standard conditions for k_n-nearest neighbors to converge (in accuracy) towards the Bayes optimal.

k_n-nearest neighbors is *consistent* if and only if the following two conditions are met: $\lim_{n\to\infty} k_n = \infty$, and $\lim_{n\to\infty} \frac{k_n}{n} = 0$. The first condition guarantees that each point is classified by using an increasing number of nearest neighbors (thus making the probability of a misclassification small), and the second condition guarantees that each point is classified using only points very close to it. We will refer to the first condition as *precision*, and the second condition as *locality.* A natural question is whether the same principles suffice for neighborhood consistency as well. We began by showing that without any additional constraints, the answer is no.

Theorem 8. *Let $\mathcal{D} = (\mu, \eta)$ be the data distribution where μ denotes the uniform distribution over $[0, 1]$ and η is defined as: $\eta(x) = x$. Over this space, let ρ be the euclidean distance metric. Suppose $k_n = O(\log n)$ for $1 \leq n < \infty$. Then k_n-nearest neighbors is not neighborhood consistent with respect to $\mathcal{D}$.*

The issue in the example above is that for smaller k_n, k_n-nearest neighbors lacks sufficient precision. For neighborhood consistnecy, points must be labeled using even more training points than are needed accuracy. This is because the classifier must be uniformly correct across the entirety of V_x^κ. Thus, to build neighborhood consistent classifiers, we must bolster the precision from the standard amount used for standard consistency. To do this, we begin by introducing *splitting numbers*, a useful tool for bolstering the precision of weight functions.

4.1 Splitting Numbers

We will now generalize beyond nearest neighbors to consider weight functions. Doing so will allow us to simultaneously analyze nearest neighbors and kernel classifiers. To do so, we must first rigorously substantiate our intuitions about increasing precision into concrete requirements. This will require several technical definitions.

Definition 9. *Let μ be a probability measure over $\mathbb{R}^d$. For any $x \in \mathbb{R}^d$, the **probability radius** $r_p(x)$ is the smallest radius for which $B(x, r_p(x))$ has probability mass at least p. More precisely, $r_p(x) = \inf\{r : \mu(B(x, r)) \geq p\}$.*

Definition 10. *Let W be a weight function and let $S = \{x_1, x_2, \ldots, x_n\}$ be any finite subset of $\mathbb{R}^d$. For any $x \in \mathbb{R}^d$, $\alpha \geq 0$, and $0 \leq \beta \leq 1$, let $W_{x,\alpha,\beta} = \{i : \rho(x, x_i) \leq \alpha, w_i^S(x) \geq \beta\}$. Then the **splitting number** of W with respect to S, denoted as $T(W, S)$ is the number of distinct subsets generated by $W_{x,\alpha\beta}$ as x ranges over $\mathbb{R}^d$, α ranges over $[0, \infty)$, and β ranges over $[0, 1]$. Thus $T(W, S) = |\{W_{x,\alpha,\beta} : x \in \mathbb{R}^d, 0 \leq \alpha, 0 \leq \beta \leq 1\}|$.*

Splitting numbers allow us to ensure high amounts of precision over a weight function. To prove neighborhood consistency, it is necessary for a classifier to be correct at *all* points in a given region. Consequently, techniques that consider a single point will be insufficient. The splitting number provides a mechanism for studying entire regions simultaneously. For more details on splitting numbers, we include several examples in the appendix.

4.2 Sufficient Conditions for Neighborhood Consistency

We now state our main result.

Theorem 11. *Let W be a weight function, $\mathcal{D}$ a distribution over $\mathbb{R}^d \times \{\pm 1\}$, $\mathcal{U}$ a neighborhood preserving collection, and $(t_n)_1^\infty$ be a sequence of positive integers such that the following four conditions hold.*

1. W is consistent (with resp. to accuracy) with resp. to $\mathcal{D}$.

2. For any $0 < p < 1$, $\lim_{n\to\infty} E_{S\sim\mathcal{D}^n}[\sup_{x\in\mathbb{R}^d} \sum_1^n w_i^S(x) 1_{\rho(x,x_i)>r_p(x)}] = 0$.

3. $\lim_{n\to\infty} E_{S\sim D^n}[t_n \sup_{x\in\mathbb{R}^d} w_i^S(x)] = 0$.

4. $\lim_{n\to\infty} E_{S\sim D^n} \frac{\log T(W,S)}{t_n} = 0$.

Then W is neighborhood consistent with respect to $\mathcal{D}$.

Remarks: Condition 1 is necessary because neighborhood consistency implies standard consistency – or, convergence in accuracy to the Bayes Optimal. Standard consistency has been well studied for non-parametric classifiers, and there are a variety of results that can be used to ensure it – for example, Stone's Theorem (included in the appendix).

Conditions 2. and 3. are stronger version of conditions 2. and 3. of Stone's theorem. In particular, both include a supremum taken over all $x \in \mathbb{R}^d$ as opposed to simply considering a random point $x \sim \mathcal{D}$. This is necessary for ensuring correct labels on entire regions of points simultaneously. We also note that the dependence on $r_p(x)$ (as opposed to some fixed r) is a key property used for adaptive robustness. This allows the algorithm to adjust to potential differing distance scales over different regions in $\mathbb{R}^d$. This idea is reminiscent of the analysis given in [6], which also considers probability radii.

Condition 4. is an entirely new condition which allows us to simultaneously consider all $T(W, S)$ subsets of S. This is needed for analyzing weighted sums with arbitrary weights.

Next, we apply Theorem 11 to get specific examples of margin consistent non-parametric algorithms.

4.3 Nearest Neighbors and Kernel Classifiers

We now provide sufficient conditions for k_n-nearest neighbors to be neighborhood consistent.

Corollary 12. *Suppose $(k_n)_1^\infty$ satisfies (1) $\lim_{n\to\infty} \frac{k_n}{n} = 0$, and (2) $\lim_{n\to\infty} \frac{\log n}{k_n} = 0$. Then k_n-nearest neighbors is neighborhood consistent.*

As a result of Theorem 8, corollary 12 is tight for nearest neighbors. Thus k_n nearest neighbors is neighborhood consistent if and only if $k_n = \omega(\log n)$.

Next, we give sufficient conditions for a kernel-similarity classifier.

Corollary 13. *Let W be a kernel classifier over $\mathbb{R}^d \times \{\pm 1\}$ constructed from $K : \mathbb{R}^+ \to \mathbb{R}^+$ and h_n. Suppose the following properties hold.*

1. K is decreasing, and satisfies $\int_{\mathbb{R}^d} K(||x||)dx < \infty$.

2. $\lim_{n\to\infty} h_n = 0$ and $\lim_{n\to\infty} nh_n^d = \infty$.

3. For any $c > 1$, $\lim_{x\to\infty} \frac{K(cx)}{K(x)} = 0$.

4. For any $x \geq 0$, $\lim_{n\to\infty} \frac{n}{\log n} K(\frac{x}{h_n}) = \infty$.

Then W is neighborhood consistent.

Observe that conditions 1. 2. and 3. are satisfied by many common Kernel functions such as the Gaussian or Exponential kernel ($K(x) = \exp(-x^2)$/ $K(x) = \exp(-x)$). Condition 4. can be similarly satisfied by just increasing h_n to be sufficiently large. Overall, this theorem states that Kernel classification is neighborhood consistent as long as the bandwidth shrinks slowly enough.

4.4 Histogram Classifiers

Having discussed neighborhood consistent nearest-neighbors and kernel classifier, we now turn our attention towards another popular weight function, histogram classifiers. Recall that histogram

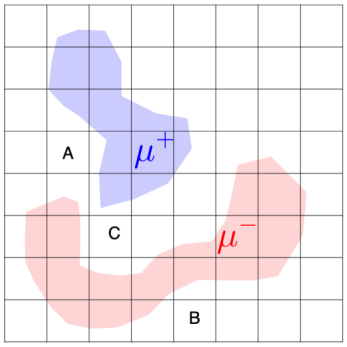
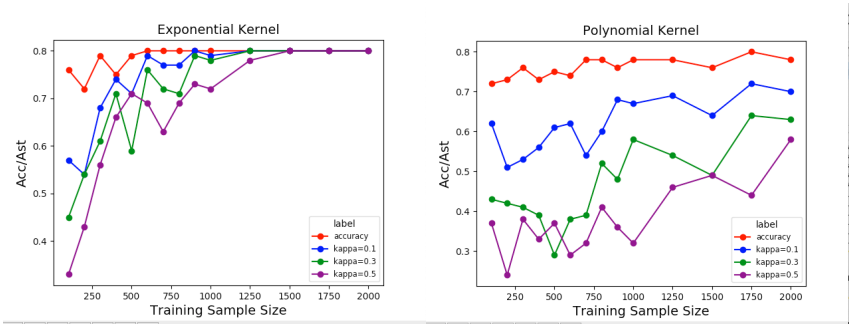


Figure 3: we have a histogram classifier being applied to the blue and red regions. The classifier will be unable to construct good labels in the cells labeled A, B, C, and consequently will not be robust with respect to V_x^κ for sufficiently large κ.

classifiers operate by partitioning their input space into increasingly small cells, and then classifying each cell by using a majority vote from the training examples within that cell (a detailed description can be found in the appendix). We seek to answer the following question: is increasing precision sufficient for making histogram classifiers neighborhood consistent? Unfortunately, the answer this turns out not to be no. The main issue is that histogram classifiers have no mechanism for performing classification outside the support of the data distribution.

For an example of this, refer to Figure 3. Here we see a distribution being classified by a histogram classifier. Observe that the cell labeled A contains points that are strictly closer to μ^+ than μ^-, and consequently, for sufficiently large κ, V_x^κ will intersect A for some point $x \in \mu^+$. A similar argument holds for the cells labeled B and C.. However, since A, B, C are all in cells that will never contain any data, they will never be labeled in a meaningful way. Because of this, histogram classifiers are not neighborhood consistent.

5 Validation



(a) exponential kernel (b) polynomial kernel

Figure 4: Plots of astuteness against the training sample size. In both panels, accuracy is plotted in red, and the varying levels of robustness regions ($\kappa = 0.1, 0.3, 0.5$) are givne in blue, green and purple. In panel (a), observe that as sample size increases, every measure of astuteness converges towards 0.8 which is as predicted by Corollary 13. In panel (b), although the accuracy appears to converge, none of the robustness measure. In fact, they get progressively worse the larger κ gets.

To complement our theoretical large sample results for non-parametric classifiers, we now include several experiments to understand their behavior for finite samples. We seek to understand how quickly non-parametic classifiers converge towards the neighborhood preserving Bayes optimal.

We focus our attention on kernel classifiers and use two different kernel similarity functions: the first, an exponential kernel, and the second, a polynomial kernel. These classifiers were chosen so that the

former meets the conditions of Corollary 13, and the latter does not. Full details on these classifiers can be found in the appendix.

To be able to measure performance with increasing data size, we look at a simple synthetic dataset over overlayed circles (see Figure 5 for an illustration) with support designed so that the data is intrinsically multiscaled. In particular, this calls for different levels of robustness in different regions. For simplicity, we use a global label noise parameter of 0.2, meaning that any sample drawn from this distribution is labeled differently than its support with probability 0.2. Further details about our dataset are given in section D.

Performance Measure. For a given classifier, we evaluate its astuteness at a test point x with respect to the robustness region V_x^κ (Definition 6). While these regions are not computable in practice due to their dependency on the support of the data distribution, we are able to approximate them for this synthetic example due to our explicit knowledge of the data distribution. Details for doing this can be found in the appendix. To compute the empirical astuteness of a kernel classifier W_K about test point x, we perform a grid search over all points in V_x^κ to ensure that all points in the robustness region are labeled correctly.

For each classifier, we measure the empirical astuteness by using three trials of 20 test points and taking the average. While this is a relatively small amount of test data, it suffices as our purpose is to just verify that the algorithm roughly converges towards the optimal possible astuteness. Recall that for any neighborhood consistent algorithm, as $n \to \infty$, $A_{\mathcal{V}^\kappa}$ should converge towards A^*, the accuracy of the Bayes optimal classifier, for *any* $0 < \kappa < 1$. Thus, to verify this holds, we use $\kappa = 0.1, 0.3, 0.5$. For each of these values, we plot the empirical astuteness as the training sample size n gets larger and larger. As a baseline, we also plot their standard accuracy on the test set.

Results and Discussion: The results are presented in Figure 4; the left panel is for the exponential kernel, while the right one is for the polynomial kernel. As predicted by our theory, we see that in all cases, the exponential kernel converges towards the maximum astuteness regardless of the value of κ: the only difference is that the rate of convergence is slower for larger values of κ. This is, of course, expected because larger values of κ entail larger robustness regions.

By contrast, the polynomial kernel performs progressively worse for larger values of κ. This kernel was selected specifically to violate the conditions of Corollary 13, and in particular fails criteria 3. However, note that the polynomial kernel nevertheless performs will with respect to accuracy thus giving another example demonstrating the added difficulty of neighborhood consistency.

Our results bridge the gap between our asymptotic theoretical results and finite sample regimes. In particular, we see that kernel classifiers that meet the conditions of Corollary 13 are able to converge in astuteness towards the neighborhood preserving Bayes optimal classifier, while classifiers that do not meet these conditions fail.

6 Related Work

There is a wealth of literature on robust classification, most of which impose the same robustness radius r on the entire data. [5, 17, 19, 20, 26, 15, 16, 18, 21, 22, 23], among others, focus primarily on neural networks, and robustness regions that are ℓ_1, ℓ_2, or ℓ_∞ norm balls of a given radius r.

[7] and [12] show how to train neural networks with different robustness radii at different points by trading off robustness and accuracy; their work differ from ours in that they focus on neural networks, their robustness regions are still norm balls, and that their work is largely empirical.

Our framework is also related to large margin classification – in the sense that the robustness regions $\mathcal{U}$ induce a *margin constraint* on the decision boundary. The most popular large margin classifier is the Support Vector Machine[9, 3, 14] – a large margin linear classifier that minimizes the worst-case margin over the training data. Similar ideas have also been used to design classifiers that are more flexible than linear; for example, [27] shows how to build large margin Lipschitz classifiers by rounding globally Lipschitz functions. Finally, there has also been purely empirical work on achieving large margins for more complex classifiers – such as [13] for deep neural networks that minimizes the worst case margin, and [29] for metric learning to find large margin nearest neighbors. Our work differs from these in that our goal is to ensure a high enough local margin at each x, (by considering the neighborhood preserving regions V_x) as opposed to optimizing a global margin.

Finally, our analysis builds on prior work on robust classification for non-parametric methods in the standard framework. [1, 24, 28, 31] provide adversarial attacks on non-parametric methods. Wang et. al. [28] develops a defense for 1-NN that removes a subset of the training set to ensure higher robustness. Yang et. al [31] proposes the r-optimal classifier – which is the maximally astute classifier in the standard robustness framework – and proposes a defense called Adversarial Pruning.

Theoretically, [4] provide conditions under which weight functions converge towards the r-optimal classifier in the large sample limit. They show that for r-separated distributions, where points from different classes are at least distance $2r$ or more apart, nearest neighbors and kernel classifiers satisfy these conditions. In the more general case, they use Adversarial Pruning as a preprocessing step to ensure that the training data is r-separated, and show that this preprocessing step followed by nearest neighbors or kernel classifiers leads to solutions that are robust and accurate in the large sample limit. Our result fundamentally differs from theirs in that we analyze a different algorithm, and our proof techniques are quite different. In particular, the fundamental differences between the r-optimal classifier and the neighborhood preserving Bayes optimal classifier call for different algorithms and different analysis techniques.

In concurrent work, [8] proposes a similar limit to the neighborhood preserving Bayes optimal which they refer to as the margin canonical Bayes. However, their work then focuses on a data augmentation technique that leads to convergence whereas we focus on proving the neighborhood consistency of classical non-parametric classifiers.

References

[1] Laurent Amsaleg, James Bailey, Dominique Barbe, Sarah M. Erfani, Michael E. Houle, Vinh Nguyen, and Milos Radovanovic. The vulnerability of learning to adversarial perturbation increases with intrinsic dimensionality. In *2017 IEEE Workshop on Information Forensics and Security, WIFS 2017, Rennes, France, December 4-7, 2017*, pages 1–6, 2017.

[2] Robert B. Ash. *Information theory*. Dover Publications, 1990.

[3] Kristin P. Bennett and Erin J. Bredensteiner. Duality and geometry in SVM classifiers. In Pat Langley, editor, *Proceedings of the Seventeenth International Conference on Machine Learning (ICML 2000), Stanford University, Stanford, CA, USA, June 29 - July 2, 2000*, pages 57–64. Morgan Kaufmann, 2000.

[4] Robi Bhattacharjee and Kamalika Chaudhuri. When are non-parametric methods robust? In *Proceedings of the 37th International Conference on Machine Learning, ICML 2020, 13-18 July 2020, Virtual Event*, volume 119 of *Proceedings of Machine Learning Research*, pages 832–841. PMLR, 2020.

[5] Nicholas Carlini and David A. Wagner. Towards evaluating the robustness of neural networks. In *2017 IEEE Symposium on Security and Privacy, SP 2017, San Jose, CA, USA, May 22-26, 2017*, pages 39–57, 2017.

[6] Kamalika Chaudhuri and Sanjoy Dasgupta. Rates of convergence for nearest neighbor classification. In Z. Ghahramani, M. Welling, C. Cortes, N. D. Lawrence, and K. Q. Weinberger, editors, *Advances in Neural Information Processing Systems 27*, pages 3437–3445. Curran Associates, Inc., 2014.

[7] Minhao Cheng, Qi Lei, Pin-Yu Chen, Inderjit S. Dhillon, and Cho-Jui Hsieh. CAT: customized adversarial training for improved robustness. *CoRR*, abs/2002.06789, 2020.

[8] Sadia Chowdhury and Ruth Urner. On the (un-)avoidability of adversarial examples. *CoRR*, abs/2106.13326, 2021.

[9] Corinna Cortes and Vladimir Vapnik. Support-vector networks. *Mach. Learn.*, 20(3):273–297, 1995.

[10] Sanjoy Dasgupta, Daniel J. Hsu, and Claire Monteleoni. A general agnostic active learning algorithm. In John C. Platt, Daphne Koller, Yoram Singer, and Sam T. Roweis, editors, *Advances in Neural Information Processing Systems 20, Proceedings of the Twenty-First Annual Conference on Neural Information Processing Systems, Vancouver, British Columbia, Canada, December 3-6, 2007*, pages 353–360. Curran Associates, Inc., 2007.

[11] Luc Devroye, László Györfi, and Gábor Lugosi. *A Probabilistic Theory of Pattern Recognition*, volume 31 of *Stochastic Modelling and Applied Probability*. Springer, 1996.

[12] Gavin Weiguang Ding, Yash Sharma, Kry Yik Chau Lui, and Ruitong Huang. MMA training: Direct input space margin maximization through adversarial training. In *8th International Conference on Learning Representations, ICLR 2020, Addis Ababa, Ethiopia, April 26-30, 2020*. OpenReview.net, 2020.

[13] Gamaleldin F. Elsayed, Dilip Krishnan, Hossein Mobahi, Kevin Regan, and Samy Bengio. Large margin deep networks for classification. In Samy Bengio, Hanna M. Wallach, Hugo Larochelle, Kristen Grauman, Nicolò Cesa-Bianchi, and Roman Garnett, editors, *Advances in Neural Information Processing Systems 31: Annual Conference on Neural Information Processing Systems 2018, NeurIPS 2018, 3-8 December 2018, Montréal, Canada*, pages 850–860, 2018.

[14] Yoav Freund and Robert E. Schapire. Large margin classification using the perceptron algorithm. *Mach. Learn.*, 37(3):277–296, 1999.

[15] Matthias Hein and Maksym Andriushchenko. Formal guarantees on the robustness of a classifier against adversarial manipulation. In I. Guyon, U. V. Luxburg, S. Bengio, H. Wallach, R. Fergus, S. Vishwanathan, and R. Garnett, editors, *Advances in Neural Information Processing Systems 30*, pages 2266–2276. Curran Associates, Inc., 2017.

[16] Guy Katz, Clark W. Barrett, David L. Dill, Kyle Julian, and Mykel J. Kochenderfer. Towards proving the adversarial robustness of deep neural networks. In *Proceedings First Workshop on Formal Verification of Autonomous Vehicles, FVAV@iFM 2017, Turin, Italy, 19th September 2017.*, pages 19–26, 2017.

[17] Yanpei Liu, Xinyun Chen, Chang Liu, and Dawn Song. Delving into transferable adversarial examples and black-box attacks. In *5th International Conference on Learning Representations, ICLR 2017, Toulon, France, April 24-26, 2017, Conference Track Proceedings*, 2017.

[18] Aleksander Madry, Aleksandar Makelov, Ludwig Schmidt, Dimitris Tsipras, and Adrian Vladu. Towards deep learning models resistant to adversarial attacks. In *6th International Conference on Learning Representations, ICLR 2018, Vancouver, BC, Canada, April 30 - May 3, 2018, Conference Track Proceedings*, 2018.

[19] Nicolas Papernot, Patrick D. McDaniel, Ian J. Goodfellow, Somesh Jha, Z. Berkay Celik, and Ananthram Swami. Practical black-box attacks against deep learning systems using adversarial examples. *ASIACCS*, 2017.

[20] Nicolas Papernot, Patrick D. McDaniel, Somesh Jha, Matt Fredrikson, Z. Berkay Celik, and Ananthram Swami. The limitations of deep learning in adversarial settings. In *IEEE European Symposium on Security and Privacy, EuroS&P 2016, Saarbrücken, Germany, March 21-24, 2016*, pages 372–387, 2016.

[21] Nicolas Papernot, Patrick D. McDaniel, Xi Wu, Somesh Jha, and Ananthram Swami. Distillation as a defense to adversarial perturbations against deep neural networks. In *IEEE Symposium on Security and Privacy, SP 2016, San Jose, CA, USA, May 22-26, 2016*, pages 582–597, 2016.

[22] Aditi Raghunathan, Jacob Steinhardt, and Percy Liang. Certified defenses against adversarial examples. In *6th International Conference on Learning Representations, ICLR 2018, Vancouver, BC, Canada, April 30 - May 3, 2018, Conference Track Proceedings*, 2018.

[23] Aman Sinha, Hongseok Namkoong, and John C. Duchi. Certifying some distributional robustness with principled adversarial training. In *6th International Conference on Learning Representations, ICLR 2018, Vancouver, BC, Canada, April 30 - May 3, 2018, Conference Track Proceedings*, 2018.

[24] Chawin Sitawarin and David A. Wagner. On the robustness of deep k-nearest neighbors. In *2019 IEEE Security and Privacy Workshops, SP Workshops 2019, San Francisco, CA, USA, May 19-23, 2019*, pages 1–7, 2019.

[25] Charles Stone. Consistent nonparametric regression. *The Annals of Statistics*, 5(4):595–645, 1977.

[26] Christian Szegedy, Wojciech Zaremba, Ilya Sutskever, Joan Bruna, Dumitru Erhan, Ian J. Goodfellow, and Rob Fergus. Intriguing properties of neural networks. In *2nd International Conference on Learning Representations, ICLR 2014, Banff, AB, Canada, April 14-16, 2014, Conference Track Proceedings*, 2014.

[27] Ulrike von Luxburg and Olivier Bousquet. Distance-based classification with lipschitz functions. In Bernhard Schölkopf and Manfred K. Warmuth, editors, *Computational Learning Theory and Kernel Machines, 16th Annual Conference on Computational Learning Theory and 7th Kernel Workshop, COLT/Kernel 2003, Washington, DC, USA, August 24-27, 2003, Proceedings*, volume 2777 of *Lecture Notes in Computer Science*, pages 314–328. Springer, 2003.

[28] Yizhen Wang, Somesh Jha, and Kamalika Chaudhuri. Analyzing the robustness of nearest neighbors to adversarial examples. In *Proceedings of the 35th International Conference on Machine Learning, ICML 2018, Stockholmsmässan, Stockholm, Sweden, July 10-15, 2018*, pages 5120–5129, 2018.

[29] Kilian Q. Weinberger, John Blitzer, and Lawrence K. Saul. Distance metric learning for large margin nearest neighbor classification. In *Advances in Neural Information Processing Systems 18 [Neural Information Processing Systems, NIPS 2005, December 5-8, 2005, Vancouver, British Columbia, Canada]*, pages 1473–1480, 2005.

[30] Yao-Yuan Yang, Cyrus Rashtchian, Ruslan Salakhutdinov, and Kamalika Chaudhuri. A closer look at robustness vs. accuracy. In *Neural Information Processing Systems (NeuRIPS)*, 2020.

[31] Yao-Yuan Yang, Cyrus Rashtchian, Yizhen Wang, and Kamalika Chaudhuri. Robustness for non-parametric methods: A generic attack and a defense. In *Artificial Intelligence and Statistics (AISTATS)*, 2020.

Checklist

The checklist follows the references. Please read the checklist guidelines carefully for information on how to answer these questions. For each question, change the default **[TODO]** to [Yes] , [No] , or [N/A] . You are strongly encouraged to include a **justification to your answer**, either by referencing the appropriate section of your paper or providing a brief inline description. For example:

- Did you include the license to the code and datasets? [Yes] See Section **??**.
- Did you include the license to the code and datasets? [No] The code and the data are proprietary.
- Did you include the license to the code and datasets? [N/A]

Please do not modify the questions and only use the provided macros for your answers. Note that the Checklist section does not count towards the page limit. In your paper, please delete this instructions block and only keep the Checklist section heading above along with the questions/answers below.

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes] we express our claims through theorems
   (b) Did you describe the limitations of your work? [Yes]
   (c) Did you discuss any potential negative societal impacts of your work? [Yes]
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]
2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes] In the theorem statements
   (b) Did you include complete proofs of all theoretical results? [Yes] in the appendix
3. If you ran experiments...

(a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [Yes] in the appendix

(b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [Yes] Many details are given in the main body, but a full explanation with all details is in the appendix.

(c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [Yes] In the appendix: this was not particularly needed for our very light experiments.

(d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [Yes] Just a simple personal computer.

4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...

(a) If your work uses existing assets, did you cite the creators? [N/A]

(b) Did you mention the license of the assets? [N/A]

(c) Did you include any new assets either in the supplemental material or as a URL? [N/A]

(d) Did you discuss whether and how consent was obtained from people whose data you're using/curating? [N/A]

(e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [N/A]

5. If you used crowdsourcing or conducted research with human subjects...

(a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]

(b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]

(c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]

A Further Details of Definitions and Theorems

A.1 Non-Parametric Classifiers

In this section, we precisely define weight functions, histogram classifiers and kernel classifiers.

Definition 14. *[11] A **weight function** W is a non-parametric classifier with the following properties.*

1. *Given input $S = \{(x_1, y_1), (x_2, y_2,), \ldots, (x_n, y_n)\} \sim \mathcal{D}^n$, W constructs functions $w_1^S, w_2^S, \ldots, w_n^S : \mathbb{R}^d \to [0, 1]$ such that for all $x \in \mathbb{R}^d$, $\sum_1^n w_i^S(x) = 1$. The functions w_i^S are allowed to depend on $x_1, x_2, \ldots x_n$ but must be independent of $y_1, y_2, \ldots, y_n$.*
2. *W has output W_S defined as*

$$W_S(x) = \begin{cases} +1 & \sum_1^n w_i^S(x) y_i > 0 \\ -1 & \sum_1^n w_i^S(x) y_i \leq 0 \end{cases}$$

As a result, $w_i^S(x)$ can be thought of as the weight that (x_i, y_i) has in classifying x.

Definition 15. *A **histogram classifier**, H, is a non-parametric classification algorithm over $\mathbb{R}^d \times \{\pm 1\}$ that works as follows. For a distribution $\mathcal{D}$ over $\mathbb{R} \times \{\pm 1\}$, H takes $S = \{(x_i, y_i) : 1 \leq i \leq n\} \sim \mathcal{D}^n$ as input. Let k_i be a sequence with $\lim_{i\to\infty} k_i = \infty$ and $\lim_{i\to\infty} \frac{k_i}{i} = 0$. H constructs a set of hypercubes $C = \{c_1, c_2, \ldots, c_m\}$ as follows:*

1. *Initially $C = \{c\}$, where $S \subset c$.*
2. *For $c \in C$, if c contains more than k_n points of S, then partition c into 2^d equally sized hypercubes, and insert them into C.*
3. *Repeat step 2 until all cubes in C have at most k_n points.*

For $x \in \mathbb{R}$ let $c(x)$ denote the unique cell in C containing x. If $c(x)$ doesn't exist, then $H_S(x) = -1$ by default. Otherwise,

$$H_S(x) = \begin{cases} +1 & \sum_{x_i \in c(x)} y_i > 0 \\ -1 & \sum_{x_i \in c(x)} y_i \leq 0 \end{cases}.$$

Definition 16. *A **partitioning rule** is a weight function W over $\mathcal{X} \times \{\pm 1\}$ constructed in the following manner. Given $S = \{(x_i, y_i)\} \sim \mathcal{D}^n$, as a function of $\{x_1, \ldots, x_n\}$, we partition $\mathbb{R}^d$ into regions with $A(x)$ denoting the region containing x. Then, for any $x \in \mathbb{R}^d$ we have*

$$w_i^S(x) = \begin{cases} 1 & x_i \in A(x) \\ 0 & \textit{otherwise} \end{cases}.$$

To achieve $\sum w_i^S(x) = 1$, we can simply normalize weights for any x by $\sum_1^n w_i^S(X)$.

Definition 17. *A **kernel classifier** is a weight function W over $\mathbb{R}^d \times \{\pm 1\}$ constructed from function $K : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+$ and some sequence $\{h_n\} \subset \mathbb{R}^+$ in the following manner. Given $S = \{(x_i, y_i)\} \sim \mathcal{D}^n$, we have*

$$w_i^S(x) = \frac{K(\frac{\rho(x,x_i)}{h_n})}{\sum_{j=1}^n K(\frac{\rho(x,x_j)}{h_n})}.$$

Then, as above, W has output

$$W_S(x) = \begin{cases} +1 & \sum_1^n w_i^S(x) y_i > 0 \\ -1 & \sum_1^n w_i^S(x) y_i \leq 0 \end{cases}$$

A.2 Splitting Numbers

We begin by restating definitions 9 and 10.

Definition 9. *Let μ be a probability measure over $\mathbb{R}^d$. For any $x \in \mathbb{R}^d$, the **probability radius** $r_p(x)$ is the smallest radius for which $B(x, r_p(x))$ has probability mass at least p. More precisely, $r_p(x) = \inf\{r : \mu(B(x, r)) \geq p\}$.*

Definition 10. *Let W be a weight function and let $S = \{x_1, x_2, \ldots, x_n\}$ be any finite subset of $\mathbb{R}^d$. For any $x \in \mathbb{R}^d$, $\alpha \geq 0$, and $0 \leq \beta \leq 1$, let $W_{x,\alpha,\beta} = \{i : \rho(x, x_i) \leq \alpha, w_i^S(x) \geq \beta\}$. Then the* ***splitting number*** *of W with respect to S, denoted as $T(W, S)$ is the number of distinct subsets generated by $W_{x,\alpha\beta}$ as x ranges over $\mathbb{R}^d$, α ranges over $[0, \infty)$, and β ranges over $[0, 1]$. Thus $T(W, S) = |\{W_{x,\alpha,\beta} : x \in \mathbb{R}^d, 0 \leq \alpha, 0 \leq \beta \leq 1\}|$.*

The main idea behind splitting numbers is that they allow us to ensure uniform convergence properties over a weight function. To prove neighborhood consistency, it is necessary for a classifier to be correct at *all* points in a given region. Consequently, techniques that consider a single point will be insufficient. The splitting number provides a mechanism for studying entire regions simultaneously. For clarity, we include a quick example in which we bound the splitting number for a given weight function.

Example: Let W denote any kernel classifier corresponding such that $K : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a decreasing function. For any $S \sim \mathcal{D}^n$, observe that the condition $w_i^S(x) \geq \beta$ precisely corresponds to $\rho(x, x_i) \leq \gamma$ for some value of γ. This is because $w_i^S(x) > w_j^S(x)$ if and only if $\rho(x, x_i) < \rho(x, x_j)$. Thus, the regions $W_{x,\alpha,\beta}$ correspond to $\{i : \rho(x, x_i) \leq \gamma\}$, where γ is a positive real number that depends on x, α, β. These sets precisely correspond to subsets of S that are contained within $B(x, \gamma)$. Since balls have VC dimension at most $d+2$, by Sauer's lemma, the number of subsets of S that can be obtained in this manner is $O(n^{d+2})$. Therefore, we have that $T(W, S) = O(n^{d+2})$ for all $S \sim \mathcal{D}^n$.

A.3 Stone's Theorem

Theorem 18. *[25] Let W be weight function over $\mathbb{R}^d \times \{\pm 1\}$. Suppose the following conditions hold for any distribution $\mathcal{D}$ over $\mathbb{R}^d \times \{\pm 1\}$. Let X be a random variable with distribution $\mathcal{D}_{\mathbb{R}^d}$, and $S = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\} \sim \mathcal{D}^n$. All expectations are taken over X and S.*

1. There is a constant c such that, for every nonnegative measurable function f satisfying $\mathbb{E}[f(X)] < \infty$, and $\mathbb{E}[\sum_1^n w_i^S(X) f(x_i)] \leq c\mathbb{E}[f(x)]$.

2. $\forall a > 0$, $\lim_{n\to\infty} \mathbb{E}[\sum_1^n w_i^S(x) I_{||x_i - X|| > a||}] = 0$.

3. $\lim_{n\to\infty} \mathbb{E}[\max_{1\leq i\leq n} w_i^S(X)] = 0$.

Then W is consistent.

B Proofs

Notation:

- We let ρ denote our distance metric over $\mathbb{R}^d$. For sets $X_1, X_2 \subset \mathbb{R}^d$, we let $\rho(X_1, X_2) = \inf_{x_1 \in X_1, x_2 \in X_2} \rho(x_1, x_2)$.
- For any $x \in \mathbb{R}^d$, $B(x, a) = \{x : \rho(x, x') \leq a\}$.
- For any measure over $\mathbb{R}^d$, μ, we let $supp(\mu) = \{x : \mu(B(x, a)) > 0 \text{ for all } a > 0\}$.
- Given some measure μ over $\mathbb{R}^d$ and some $x \in \mathbb{R}^d$, we let $r_p(x)$ denote the probability radius (Definition 9) of x with probability p. that is, $r_p(x) = \inf\{r : \mu(B(x, r)) \geq p\}$.
- For weight function W and training sample S, we let W_S denote the weight function learned by W from S.

B.1 Proofs of Theorems 4 and 5

Proof. (Theorem 4) Let $\mathcal{D} = (\mu, \eta)$ be a data distribution, and let μ^+, μ^- be as described in section **??**. Observe that for any $x \in \mu^+$, the Bayes optimal classifier and the neighborhood preserving Bayes optimal both have the same output, and furthermore the neighborhood preserving Bayes gives this output (by definition) throughout the entirety of V_x, the neighborhood preserving robustness region of x. It follows that the neighborhood preserving Bayes optimal has optimal astuteness, as desired. □

Proof. (Theorem 5) Let $\mathcal{D} = (\mu, \eta)$ be a data distribution, and assume towards a contradiction that there exists classifier f which has maximal astuteness with respect towards some set of robustness regions $\mathcal{U} = \{U_x\}$ such that $V_x \subseteq U_x$ for all x. The key observation is that because f has maximal astuteness, we must have $f(x) = g(x)$ for almost all points $x \sim \mu$ (where g is the Bayes optimal classifier). Furthermore, for those values of x, we must have g be robust at x (meaning it uniformly outputs the same output through U_x).

In order for U_x to be strictly larger than V_x for some x, it *necessarily* must intersect with $U_{x'}$ for some x' with $g(x') \neq g(x)$, and this is what causes the contradiction: f cannot be astute at both x and x' if they are differently labeled and their robustness regions intersect. □

B.2 Proof of Theorem 8

Let $\mathcal{D} = (\mu, \eta)$ be the distribution with μ being the uniform distribution over $[0, 1]$ and $\eta : [0, 1] \to [0, 1]$ be $\eta(x) = x$. For example, if $(x, y) \sim \mathcal{D}$, then $\Pr[y = 1|x = 0.3] = 0.3$.

We desire to show that k_n-nearest neighbors is not neighborhood consistent with respect to $\mathcal{D}$. We begin with the following key lemma.

Lemma 19. *For any $n > 0$, let f_n denote the k_n-nearest neighbor classifier learned from $S \sim \mathcal{D}^n$. There exists some constant $\Delta > 0$ such that for all sufficiently large n, with probability at least $\frac{1}{2}$ over $S \sim \mathcal{D}^n$, there exists $x \in [0, 1]$ with $\frac{1}{2} - \Delta \leq x \leq \frac{1}{2} - \frac{3\Delta}{4}$ and $f_n(x) = +1$.*

Proof. Let C be a constant such that $k_n \leq C \log n$ for all $2 \leq n < \infty$. Set Δ as

$$\frac{1}{2} \log_2 \frac{1}{1 - 2\Delta} + \frac{1}{2} \log_2 \frac{1}{1 + 2\Delta} < \frac{1}{C}. \tag{1}$$

Let $A \subset [0, 1]$ denote the interval $[\frac{1}{2} - \Delta, \frac{1}{2} - \frac{3\Delta}{4}]$. For $S \sim \mathcal{D}^n$, with high probability, there exist at least $\frac{\Delta n}{8}$ instances x_i that are in A. Let us relabel these x_i as $x_1, x_2, \ldots, x_m$ as

$$\frac{1}{2} - \Delta \leq x_1 < x_2 < \cdots < x_m \leq \frac{1}{2} - \frac{3\Delta}{4}.$$

Next, suppose that for some i, at least half of $y_i, y_{i+1}, \ldots, y_{i+k_n-1}$ are $+1$. Then it follows that $f_n(x) = +1$ for $x = \frac{x_{i+k_n} + x_i}{2}$ because the k_n nearest neighbors of x are precisely $x_i, x_{i+1}, \ldots x_{i+k_n-1}$ (as a technical note we make x just slightly smaller to break the tie between x_i and x_{i+k_n}). To lower bound the probability that this occurs for some i, we partition $y_1, y_2, \ldots y_m$ into at least $\frac{m}{2k_n}$ disjoint groups each containing k_n consecutive values of y_i. We then bound the probability that each group will have at least $k_n/2$ $+1$s.

Consider any group of k_n y_is. We have that $\Pr[y_i] = +1 = \eta(x_i) = x_i \geq \frac{1}{2} - \Delta$. Since the variables y_i are independent (even conditioning on x_i), it follows that the probability that at least half of them are $+1$ is at least $\Pr[\text{Bin}(k_n, \frac{1}{2} - \Delta) \geq \frac{k_n}{2}]$. For simplicity, assume that k_n is even. Then using a standard lower bound for the tail of a binomial distribution (see, for example, Lemma 4.7.2 of [2]), we have that

$$\Pr[\text{Bin}(k_n, \frac{1}{2} - \Delta) \geq \frac{k_n}{2}] \geq \frac{1}{\sqrt{2k_n}} \exp(-k_n D(\frac{1}{2}||(\frac{1}{2} - \Delta)),$$

where $D(\frac{1}{2}||(\frac{1}{2} - \Delta)) = \frac{1}{2} \log_2 \frac{1}{1-2\Delta} + \frac{1}{2} \log_2 \frac{1}{1+2\Delta}$.

To simplify notation, let $D_\Delta = D(\frac{1}{2}||(\frac{1}{2} - \Delta))$. Then because we have $\frac{m}{2k_n}$ independent groups of y_is, we have that

$$\begin{aligned}
\Pr_{S \sim \mathcal{D}^n}[\exists x \in [\frac{1}{2} - \Delta, \frac{1}{2} - \frac{3\Delta}{4}] \text{ s.t. } f_n(x) = +1] &\geq 1 - (1 - \frac{1}{\sqrt{2k_n}} \exp(-k_n D_\Delta))^{\frac{m}{2k_n}} \\
&\geq 1 - \exp(-\frac{m}{2k_n\sqrt{2k_n}} e^{-k_n D_\Delta}) \\
&\geq 1 - \exp(-\frac{n\Delta}{(16C \log n)^{3/2}} e^{-CD_\Delta \log n}),
\end{aligned}$$

with the inequalities holding because $m \geq \frac{n\Delta}{8}$ and $k_n \leq C \log n$. By equation 1, $CD_\Delta < 1$. Therefore, $\lim_{n\to\infty} \frac{n}{(2C\log n)^{3/2}} e^{-CD_\Delta \log n} = \infty$, which implies that for n sufficiently large,

$$\Pr_{S\sim\mathcal{D}^n}[\exists x \in [\frac{1}{2} - \Delta, \frac{1}{2} - \frac{3\Delta}{4}] \text{ s.t. } f_n(x) = +1] \geq \frac{1}{2},$$

as desired. □

We now complete the proof of Theorem 8.

Proof. (Theorem 8) Let Δ be as described in Lemma 19, and let $\kappa = \frac{1}{2}$. For all $x < \frac{1}{2}$, we have that $[x, \frac{2x}{3} + \frac{1}{6}] \subseteq V_x^\kappa$. This is because we can easily verify that all points inside that interval are closer to x than they are to $\frac{1}{2}$ (and consequently all points in $\mu^+ \cup \mu^{1/2}$) by factor of 2. It follows that for all $x \in [\frac{1}{2} - \frac{7\Delta}{8}, \frac{1}{2} - \Delta]$,

$$[\frac{1}{2} - \Delta, \frac{1}{2} - \frac{3\Delta}{4}] \subseteq V_x^\kappa.$$

However, applying Lemma 19, we know that with probability at least $\frac{1}{2}$, there exists some point $x' \in [\frac{1}{2} - \Delta, \frac{1}{2} - \frac{3\Delta}{4}]$ such that $f_n(x') = +1$. It follows that with probability at least $\frac{1}{2}$, f_n lacks astuteness at *all* $x \in [\frac{1}{2} - \frac{7\Delta}{8}, \frac{1}{2} - \Delta]$. Since this set of points has total probability mass $\Delta/8$, it follows that with probability at least $\frac{1}{2}$, there is a fixed gap between $A_{\mathcal{V}^\kappa}(f_n, \mathcal{D})$ and $A(g, \mathcal{D})$ (as they differ in a region of probability mass at least $\Delta/8$). This implies that k_n-nearest neighbors is not neighborhood consistent. □

B.3 Proof of Theorem 11

Let $\mathcal{D} = (\mu, \eta)$ is a distribution over $\mathbb{R}^d \times \{\pm 1\}$. We will use the following notation: let $\mathcal{D}^+ = \{x : \eta(x) > \frac{1}{2}\}$, $\mathcal{D}^- = \{x : \eta(x) < \frac{1}{2}$ and $\mathcal{D}_{1/2} = \{x : \eta(x) = \frac{1}{2}\}$. In particular, we have that $\mathcal{D}^+ = \mu^+, \mathcal{D}^- = \mu^-$ and $\mathcal{D}_{1/2} = \mu^{1/2}$. This notation serve will be convenient throughout this section since it allows us to avoid overloading the symbol μ.

To show that an algorithm is neighborhood consistent with respect to $\mathcal{D}$, we must show that for any $0 < \kappa < 1$, the astuteness with respect to $\mathcal{V}^\kappa$ converges towards the accuracy of the Bayes optimal. To this end, we fix any $0 < \kappa < 1$ and consider $\mathcal{V}^\kappa$.

For our proofs, it will be useful to have the additional assumption that the robustness regions, V_x^κ are *closed*. To obtain this, we let $\mathcal{U} = \{U_x\}$ where $U_x = \overline{V_x^\kappa}$. Each U_x is the closure of the corresponding V_x^κ, and in particular we have $V_x^\kappa \subset U_x$. Because of this, it will suffice for us to consider $A_\mathcal{U}$ as opposed to $A_{\mathcal{V}^\kappa}$ since $A_\mathcal{U}(f, \mathcal{D}) \leq A_{\mathcal{V}^\kappa}(f, \mathcal{D})$ for all classifiers f.

We now begin by first proving several useful properties of $\mathcal{U}$ that we will use throughout this entire section.

Lemma 20. *The collection of sets $\mathcal{U} = \{U_x\}$ defined as $U_x = \overline{V_x^\kappa}$ satisfies the following properties.*

1. *U_x is closed for all x.*
2. *if $x \in \mathcal{D}^+$, for all $x' \in U_x$, $\rho(x, x') < \rho(\mathcal{D}^+ \cup \mathcal{D}_{1/2}, x')$.*
3. *if $x \in \mathcal{D}^-$, for all $x' \in U_x$, $\rho(x, x') < \rho(\mathcal{D}^- \cup \mathcal{D}_{1/2}, x')$.*
4. *$U_x = \{x\}$ for all $x \in \mathcal{D}_{1/2}$.*
5. *U_x is bounded for all x.*

Here $\mu^+, \mu^-, \mu^{1/2}$ are as described in section **??**.

Proof. Property (1) is given the by definition, and properties (2), (3) follow from the fact that κ is strictly less than 1. In particular, the distance function ρ is continuous and consequently all limit points of a set have distances that are limits of distances within the set. Property (4) is since $V_x^\kappa = \{x\}$ for all $x \in \mathcal{D}_{1/2}$.

Finally, property (5) follows from the fact that $\kappa < 1$. As x gets arbitrarily far away from x the ratio of its distance to x with its distance to μ^- gets arbitrarily close to 1, and consequently there is some maximum radius R so that $V_x^\kappa \subset B(x, R)$. Since $B(x, R)$ is closed, it follows that $U_x \subset B(x, R)$ as well. □

Next, fix W as a weight function and t_n is a sequence of positive integers such that the conditions of Theorem 11 hold, that is:

1. W is consistent (with resp. to accuracy) with resp. to $\mathcal{D}$.
2. For any $0 < p < 1$, $\lim_{n\to\infty} E_{S\sim\mathcal{D}^n}[\sup_{x\in\mathbb{R}^d} \sum_1^n w_i^S(x) 1_{\rho(x,x_i)>r_p(x)}] = 0$.
3. $\lim_{n\to\infty} E_{S\sim D^n}[t_n \sup_{x\in\mathbb{R}^d} w_i^S(x)] = 0$.
4. $\lim_{n\to\infty} E_{S\sim D^n} \frac{\log T(W,S)}{t_n} = 0$.

Finally, we will also make the additional assumption that $\mathcal{D}$ has infinite support. Cases where $\mathcal{D}$ has finite support can be somewhat trivially handled: when the sample size goes to infinity, we will have perfect labels for every point in the support, and consequently condition 2. will ensure that any $x' \in V_x^\kappa$ is labeled according to the label of x.

We also use the following notation. For any classifier $f : \mathbb{R}^d \to \{\pm 1\}$, we let

$$\mathcal{D}_f^+ = \{x : f(x' = +1 \text{ for all } x' \in U_x\}, \text{ and } \mathcal{D}_f^- = \{x : f(x' = -1 \text{ for all } x' \in U_x\}. \quad (2)$$

These sets represent the examples that f robustly labels as $+1$ and -1 respectively. These sets are useful since they allows us to characterize the astuteness of f, which we do with the following lemma.

Lemma 21. *For any classifier $f : \mathbb{R}^d \to \{\pm 1\}$, we have*

$$A_{\mathcal{U}}(f, \mathcal{D}) \geq A(g, \mathcal{D}) - \mu(\mathcal{D}^+ \setminus \mathcal{D}_f^+) - \mu(D^- \setminus \mathcal{D}_f^-),$$

where g denotes the Bayes optimal classifier.

Proof. By property 4 of Lemma 20, $U_x = \{x\}$ for all $x \in \mathcal{D}_{1/2}$. Consequently, if $x \in \mathcal{D}_{1/2}$, there is a $\frac{1}{2}$ chance that any classifier is astute at (x, y). Using this along with the definition of astuteness, we see that

$$\begin{aligned} A_{\mathcal{U}}(f, \mathcal{D}) &= \Pr_{(x,y)\sim\mathcal{D}}[f(x') = y \text{ for all } x' \in U_x] \\ &= \Pr_{(x,y)\sim\mathcal{D}}[y = +1 \text{ and } x \in (D^+ \cap D_f^+)] + \Pr_{(x,y)\sim\mathcal{D}}[y = -1 \text{ and } x \in (D^- \cap D_f^-)] + \frac{1}{2} \Pr_{(x,y)\sim\mathcal{D}}[x \in \mathcal{D}_{1/2}] \end{aligned}$$

However, observe by the definitions of $\mathcal{D}^+, \mathcal{D}^-$ and $\mathcal{D}_{1/2}$ that

$$A(g, \mathcal{D}) = \Pr_{(x,y)\sim\mathcal{D}}[y = +1 \text{ and } x \in D^+] + \Pr_{(x,y)\sim\mathcal{D}}[y = -1 \text{ and } x \in D^-] + \frac{1}{2} \Pr_{(x,y)\sim\mathcal{D}}[x \in \mathcal{D}_{1/2}].$$

Substituting this, we find that

$$\begin{aligned} A_{\mathcal{U}}(f, \mathcal{D}) &\geq A(g, \mathcal{D}) - \Pr_{(x,y)\sim\mathcal{D}}[x \in (D^+ \setminus D_f^+)] - \Pr_{(x,y)\sim\mathcal{D}}[x \in (D^- \setminus D_f^-)] \\ &= A(g, \mathcal{D}) - \mu(\mathcal{D}^+ \setminus \mathcal{D}_f^+) - \mu(D^- \setminus \mathcal{D}_f^-), \end{aligned}$$

as desired. □

Lemma 21 shows that to understand how W_S converges in astuteness, it suffices to understand how the regions $\mathcal{D}_{W_S}^+$ and $\mathcal{D}_{W_S}^-$ converge towards D^+ and D^- respectively. This will be our main approach for proving Theorem 11. Due to the inherent symmetry between $+$ and $-$, we will focus on showing how the region $\mathcal{D}_{W_S}^+$ converges towards D^+. The case for $-$ will be analogous. To that end, we have the following key definition.

Definition 22. *Let $p, \Delta > 0$. We say $x \in \mathcal{D}^+$ is (p, Δ)-**covered** if for all $x' \in U_x$ and for all $x'' \in B(x', r_p(x')) \cap \text{supp}(\mu)$, $\eta(x'') > \frac{1}{2} + \Delta$. Here r_p denotes the probability radius (Definition 9). We also let $\mathcal{D}_{p,\Delta}^+$ denote the set of all $x \in \mathcal{D}^+$ that are (p, Δ)-covered.*

If x is (p, Δ)-covered, it means that for all $x' \in U_x$, there is a set of points with measure p around x' that are both close to x', and likely (with at least probability $\frac{1}{2} + \Delta$) to be labeled as $+1$. Our main idea will be to show that if x is (p, Δ) covered and n is sufficiently large, x is likely to be in $\mathcal{D}^+_{W_S}$.

We begin this process by first showing that all x are (p, Δ)-covered for some p, Δ. To do so, it will be useful to have one more piece of notation which we will also use throughout the rest of the section. We let

$$\mathcal{D}^-_{1/2} = \mathcal{D}^- \cup \mathcal{D}_{1/2} = supp(\mu) \setminus \mathcal{D}^+.$$

This set will be useful, since Lemma 20 implies that for all $x \in \mathcal{D}^+$ and for all $x' \in U_x$, $\rho(x, x') < \rho(\mathcal{D}^-_{1/2}, x')$. We now return to showing that all x are $(p, \Delta$-covered for some p, Δ.

Lemma 23. *For any $x \in \mathcal{D}^+$, there exists $p, \Delta > 0$ such that x is (p, Δ)-covered.*

Proof. Fix any x. Let $f : U_x \to \mathbb{R}$ be the function defined as $f(x') = \rho(x', \mathcal{D}^-_{1/2}) - \rho(x', x)$. Observe that f is continuous. By assumption, U_x is closed and bounded, and consequently must attain its minimum. However, by Lemma 20, we have that $f(x') > 0$ for all $x' \in U_x$. it follows that $\min_{x' \in U_x} f(x') = \gamma$ where $\gamma > 0$.

Next, let $p = \mu(B(x, \gamma/2))$. $p > 0$ since $x \in supp(\mu)$. Observe that for any $x' \in U_x$, $r_p(x') \leq \rho(x, x') + \gamma/2$, where, $r_p(x')$ denotes the probability radius of x'. This is because $B(x', (\rho(x, x') + \gamma/2))$ contains $B(x, \gamma/2)$ which has probability mass p. It follows that for any $x' \in U_x$, $\rho(x', \mathcal{D}^-_{1/2}) \geq r_p(x') + \gamma/2$. Motivated by this observation, let A be the region defined as

$$A = \bigcup_{x' \in U_x} B(x', r_p(x')).$$

Then by our earlier observation, we have that $\rho(A, \mathcal{D}^-_{1/2}) \geq \frac{\gamma}{2}$. Since distance is continuous, it follows that $\rho(\overline{A}, \mathcal{D}^-_{1/2}) \geq \frac{\gamma}{2}$ as well, where $\overline{A}$ denotes the closure of A.

This means that for any $x'' \in \overline{A} \cap supp(\mu)$, $\eta(x'') > \frac{1}{2}$, since otherwise $\rho(\overline{A}, \mathcal{D}^-_{1/2})$ would equal 0 (as the two sets would literally intersect). Finally, $supp(\mu)$ is a closed set (see Appendix C.1), and thus $\overline{A} \cap supp(\mu)$ is closed as well. Since η is continuous (by assumption from Definition **??**), it follows that η must maintain its minimum value over $\overline{A} \cap supp(\mu)$. It follows that there exists $2\Delta > 0$ such that $\eta(x'') \geq \frac{1}{2} + 2\Delta > \frac{1}{2} + \Delta$ for all $x'' \in \overline{A} \cap supp(\mu)$.

Finally, by the definition of A, for all $x' \in U_x$, $B(x', r_p(x')) \subset A$. It consequently follows from the definition that x is (p, Δ)-covered, as desired. □

While the previous lemma show that some p, Δ cover any $x \in \mathcal{D}^+$, this does not necessarily mean that there are some fixed p, Δ that cover *all* $x \in \mathcal{D}^+$. Nevertheless, we can show that this is almost true, meaning that there are some p, Δ that cover *most* $x \in \mathcal{D}^+$. Formally, we have the following lemma.

Lemma 24. *For any $\epsilon > 0$, there exists p, Δ such that $\mu(\mathcal{D}^+ \setminus \mathcal{D}^+_{p,\Delta}) < \epsilon$, where $\mathcal{D}^+_{p,\Delta}$ is as defined in Definition 22.*

Proof. Observe that if x is (p, Δ)-covered, then it is also (p', Δ')-covered for any $p' < p$ and $\Delta' < \Delta$. This is because $B(x', r_{p'}(x')) \subset B(x', r_p(x))$ and $\frac{1}{2} + \Delta > \frac{1}{2} + \Delta'$. Keeping this in mind, define

$$\mathcal{A} = \{\mathcal{D}^+_{1/i,1/j} : i, j \in \mathbb{N}\}.$$

For any $x \in \mathcal{D}^+$, by Lemma 23 and our earlier observation, there exists $A \in \mathcal{A}$ such that $x \in A$. It follows that $\cup_{A \in \mathcal{A}} A = \mathcal{D}^+$. By applying Lemma 41, we see that there exists a finite subset of $\mathcal{A}$, $\{A_1, \ldots, A_m\}$ such that

$$\mu(A_1 \cup \cdots \cup A_m\}) > \mu(\mathcal{D}^+) - \epsilon.$$

Let $A_k = \mathcal{D}^+_{1/i_k,1/j_k}$ for $1 \leq k \leq m$. From our previous observation once again, we see that $\cup A_i \subset \mathcal{D}^+_{1/I,1/J}$ where $I = \max(i_k)$ and $J = \max(j_k)$. It follows that setting $p = 1/I$ and $\Delta = 1/J$ suffices. □

Recall that our overall goal is to show that if x is (p, Δ)-covered, n is sufficiently large, then x is very likely to be in $\mathcal{D}^+_{W_S}$ (defined in equation 2). To do this, we will need to find sufficient conditions on S for x to be in W_S. This requires the following definitions, that are related to *splitting numbers* (Definition 10).

Definition 25. *Let $x \in \mathbb{R}^d$ be a point, and let $S = \{(x_1, y_1), \ldots, (x_n, y_n)\}$ be a training set sampled from $\mathcal{D}^n$. For $0 \leq \alpha$, $0 \leq \beta \leq 1$, and $0 < \Delta < \frac{1}{2}$, we define*

$$W^{\Delta,S}_{x,\alpha,\beta} = \{i : \rho(x, x_i) \leq \alpha, w^S_i(x) \geq \beta, \eta(x_i) > \frac{1}{2} + \Delta\}.$$

Definition 26. *Let $0 < \Delta < \frac{1}{2}$, and let $S = \{(x_1, y_1), \ldots, (x_n, y_n)\}$ be a training set sampled from $\mathcal{D}^n$. Then we let*

$$W^{\Delta,S} = \{W^{\Delta,S}_{x,\alpha,\beta} : x \in \mathbb{R}^d, 0 \leq \alpha, 0 \leq \beta \leq 1\}.$$

These convoluted looking sets will be useful for determining the behavior of W_s at some $x \in \mathcal{D}^+_{p,\Delta}$. Broadly speaking, the idea is that if every set of indices $R \subset W^{\Delta,S}$ is relatively well behaved (i.e. the number of y_is that are $+1$ is close to $(|R|(\frac{1}{2} + \Delta)$, the expected amount), then $W_s(x') = +1$ for all $x' \in U_x$. Before showing this, we will need a few more lemmas.

Lemma 27. *Fix any $\delta > 0$ and let $0 < \Delta < \frac{1}{2}$. There exists N such that for all $n > N$ the following holds. With probability $1 - \delta$ over $S \sim \mathcal{D}^n$, for all $R \in W^{\Delta,S}$ with $|R| > t_n$, $\frac{1}{|R|}\sum_{i \in R} y_i \geq \Delta$*

Proof. The key idea is to observe that the set $W^{\Delta,S}$ and the value $T(W, S)$ are completely determined by $\{x_1, \ldots, x_n\}$. This is because weight functions choose their weights only through dependence on $x_1, \ldots, x_n$. Consequently, we can take the equivalent formulation of first drawing $x_1, \ldots, x_n \sim \mu^n$, and then drawing y_i independently according to $y_i = 1$ with probability $\eta(x_1)$ and 0 with probability $1-\eta(x_i)$. In particular, we can treat $y_1, \ldots, y_n$ as independent from $W^{\Delta,S}$ and $T(W, S)$ conditioning on $x_1, \ldots, x_n$.

Fix any $x_1, \ldots, x_n$. First, we see that $|W^{\Delta,S}| \leq T(W, S)$. This is because $W^{\Delta,S}_{x,\alpha,\beta}$ is a subset that is uniquely defined by $W_{x,\alpha,\beta}$ (see Definitions 25 and 10). Second, for any $R \in W^{\Delta,S}$, observe that for all $i \in R$, y_i is a binary variable in $[-1, 1]$ with expected value at least $(\frac{1}{2} + \Delta) - (\frac{1}{2} - \Delta) = 2\Delta$ (again by the definition). It follows that if $|R| \geq t_n$, by Hoeffding's inequality

$$\Pr_{y_1 \ldots y_n}[\sum_{i \in R} y_i < \Delta] \leq \exp\left(-\frac{2|R|^2\Delta^2}{4|R|}\right) \leq \exp\left(-\frac{t_n\Delta^2}{2}\right).$$

Since there at most $T(W, S)$ sets R, it follows that

$$\Pr_{y_1 \ldots y_n}[\sum_{i \in R} y_i < \Delta \text{ for some } R \in W^{\Delta,S} \text{ with } |R| > t_n] \leq T(W, S) \exp\left(-\frac{t_n\Delta^2}{2}\right).$$

However, by condition 4. of Theorem 11, it is not difficult to see that this quantity has expectation that tends to 0 as $n \to \infty$ (unless $T(W, S)$ uniformly equals 1, but this degenerate case can easily be handled on its own). Thus, for any $\delta > 0$, it follows that there exists N such that for all $n > N$, with probability at least $1 - \frac{\delta}{2}$, $T(W, S) \exp\left(-\frac{t_n\Delta^2}{2}\right) \leq \frac{\delta}{2}$. This value of N consequently suffices for our lemma. □

We now relate $\mathcal{D}^+_{W_S}$ (Equation 2) to $W^{\Delta,S}$ as well as the conditions of Theorem 11.

Lemma 28. *Let $S = \{(x_1, y_1), \ldots, (x_n, y_n)\}$ and let $0 < \Delta \leq \frac{1}{2}$ and $0 < p < 1$ such that the following conditions hold.*

1. *For all $R \in W^{\Delta,S}$ with $|R| > t_n$, $\frac{1}{|R|}\sum_{i \in R} y_i \geq \Delta$.*
2. *$\sup_{x \in \mathbb{R}^d} \sum_1^n w^S_i(x) 1_{\rho(x,x_i) > r_p(x)} < \frac{\Delta}{5}$.*
3. *$t_n \sup_{x \in \mathbb{R}^d} w^S_i(x) < \frac{\Delta}{5}$.*

Then $\mathcal{D}^+_{p,\Delta} \subseteq \mathcal{D}^+_{W_S}$.

Proof. Let $x \in \mathcal{D}^+_{p,\Delta}$, and let $x' \in U_x$ be arbitrary. It suffices to show that $W_S(x') = +1$ (as x, x' were arbitrarily chosen). From the definition of W_S, this is equivalent to showing that $\sum_1^n w_i^S(x')y_i > 0$. Thus, our strategy will be to lower bound this sum using the conditions given in the lemma statement.

We first begin by simplifying notation. Since S and x' are both fixed, we use w_i to denote $w_i^S(x')$. Since n is fixed, we will also use t to denote t_n. Next, suppose that $|\{x_1, \ldots, x_n\} \cap B(x', r_p(x'))| = k$. Without loss of generality, we can rename indices such that $\{x_1, \ldots, x_n\} \cap B(x', r_p(x')) \cap B(x', r_p(x')) = \{x_1, \ldots, x_k\}$, and $w_1 \geq w_2 \geq \cdots \geq w_k$.

Let $Y_j = \sum_{i=1}^j y_i$. Our main idea will be to express the sum in terms of these Y_js as follows.

$$\begin{aligned}
\sum_1^n w_i y_i &= \sum_1^k w_i y_i + \sum_{k+1}^n w_i y_i \\
&= w_k Y_k + (w_{k-1} - w_k)Y_{k-1} + \cdots + (w_{t+1} - w_{t+2})Y_{t+1} + \sum_{i=1}^t (w_i - w_{t+1})y_i + \sum_{k+1}^n w_i y_i \\
&= \underbrace{w_k Y_k + \sum_{i=t+1}^{k-1} (w_i - w_{i+1})Y_i}_{\alpha} + \underbrace{\sum_{i=1}^t (w_i - w_{t+1})y_i}_{\beta} + \underbrace{\sum_{k+1}^n w_i y_i}_{\tau}.
\end{aligned}$$

We now bound α, β and τ in terms of Δ by using the conditions given in the lemma. We begin with β and τ, which are considerably easier to handle.

For β, we have that

$$\beta = \sum_{i=1}^t (w_i - w_{t+1})y_i \geq \sum_{i=1}^t (w_i - w_{t+1})(-1) \geq -tw_1.$$

By condition 2 of the lemma, we see that $tw_1 < \frac{\Delta}{5}$, which implies that $\beta \geq -\frac{\Delta}{5}$.

For γ, we have that $\gamma = \sum_{k+1}^n w_i y_i \geq -\sum_{k+1}^n w_i$. However, for all $k+1 \leq i \leq n$, by definition of k, $\rho(x', x_i) > r_p(x')$. It follows from condition 3 of the lemma that $\gamma \geq -\frac{\Delta}{5}$.

Finally, we handle α. Recall that x is (p, Δ)-covered. It follows that for all $x'' \in supp(\mu) \cap B(x', r_p(x'))$, $\eta(x'') > \frac{1}{2} + \Delta$. Thus, by the definition of k, $\eta(x_i) > \frac{1}{2} + \Delta$ for $1 \leq i \leq k$. It follows that if $w_i > w_{i+1}$ or $i = k$, then

$$\begin{aligned}
W^{\Delta,S}_{x', r_p(x'), w_i} &= \{j : \rho(x', x_j) \leq r_p(x'), w_j \geq w_i, \eta(x_j) > \frac{1}{2} + \Delta\} \\
&= \{1, \ldots, i\}.
\end{aligned}$$

This implies that $\{1, \ldots, i\} \in W^{\Delta,S}$, and consequently that $Y_i \geq i\Delta$, from condition 1 of the lemma. It follows that for all $t < i \leq k$, $(w_i - w_{i+1})Y_i \geq i(w_i - w_{i+1})\Delta$, and that $w_k Y_k \geq k w_k \Delta$. Substituting these, we find that

$$\begin{aligned}
\alpha &= w_k Y_k + \sum_{i=t+1}^{k-1} (w_i - w_{i+1})Y_i \\
&\geq k w_k \Delta + \sum_{i=t+1}^{k-1} i(w_i - w_{i+1})\Delta \\
&= w_k \Delta + w_{k-1}\Delta + \cdots + w_{t+1}\Delta + (t+1)w_{t+1}\Delta. \\
&\geq (1 - \sum_{1^t} w_i - \sum_{k+1}^n w_i)\Delta \\
&\geq (1 - \frac{2\Delta}{5})\Delta \\
&\geq (\frac{4\Delta}{5}),
\end{aligned}$$

with the last inequalities holding from the arguments given for β and γ along with the fact that $0 < \Delta \leq \frac{1}{2}$. Finally, substituting these, we find that $\alpha + \beta + \gamma \geq \frac{4\Delta}{5} - \frac{2\Delta}{5} = \frac{2\Delta}{5} > 0$, as desired. □

We are now ready to prove the key lemma that forms one half of the main theorem (the other half corresponding to $\mathcal{D}^-_{W_S}$).

Lemma 29. *Let $\delta, \epsilon > 0$. There exists N such that for all $n > N$, with probability $1 - \delta$ over $S \sim \mathcal{D}^n$, $\mu(\mathcal{D}^+ \setminus \mathcal{D}^+_{W_S}) < \epsilon$.*

Proof. First, by Lemma 24, let $0 < p$ and $0 < \Delta$ be such that $\mu(\mathcal{D}^+ \setminus \mathcal{D}^+_{p,\Delta}) < \epsilon$. By combining Lemma 27, condition 3 of Theorem 11, and condition 2 of Theorem 11 respectively, we see that there exists N such that for all $n > N$, the following hold:

1. With probability at least $1 - \frac{\delta}{3}$ over $S \sim \mathcal{D}^n$, for all $R \in W^{\Delta,S}$ with $|R| > t_n$, $\frac{1}{|R|}\sum_{i \in R} y_i \geq \Delta$.
2. With probability at least $1 - \frac{\delta}{3}$ over $S \sim \mathcal{D}^n$, $\sup_{x \in \mathbb{R}^d} \sum_1^n w_i^S(x) 1_{\rho(x,x_i) > r_p(x)} < \frac{\Delta}{5}$.
3. With probability at least $1 - \frac{\delta}{3}$ over $S \sim \mathcal{D}^n$, $t_n \sup_{x \in \mathbb{R}^d} w_i^S(x) < \frac{\Delta}{5}$.

By a union bound, this implies that p, Δ, S satisfy the conditions of Lemma 28 with probability at least $1 - \delta$. Thus, applying the Lemma, we see that with probability $1 - \delta$, $\mathcal{D}^+_{p,\Delta} \subset \mathcal{D}^+_{W_S}$. This immediately implies our claim. □

By replicating all of the work in this section for $\mathcal{D}^-$ and $\mathcal{D}^-_{p,\Delta}$, we can similarly show the following:

Lemma 30. *Let $\delta, \epsilon > 0$. There exists N such that for all $n > N$, with probability $1 - \delta$ over $S \sim \mathcal{D}^n$, $\mu(\mathcal{D}^- \setminus \mathcal{D}^-_{W_S}) < \epsilon$.*

Combining these two lemmas with Lemma 21 immediately implies that for all $\delta, \epsilon > 0$, there exists N such that for all $n > N$, with probability $1 - \delta$ over $S \sim \mathcal{D}^n$,

$$A_{\mathcal{U}}(W_S, \mathcal{D}) \geq A(g, \mathcal{D}) - \epsilon.$$

Since $V_x^\kappa \subset U_x$ and since κ was arbitrary, this implies Theorem 11, which completes our proof.

B.4 Proof of Corollary 12

Recall that k_n-nearest neighbors can be interpreted as a weight function, in which $w_i^S(x) = \frac{1}{k_n}$ if x_i is one of the k_n closest points to x, and 0 otherwise. Therefore, it suffices to show that the conditions of Theorem 11 are met.

We let W denote the weight function associated with k_n-nearest neighbors.

Lemma 31. *W is consistent.*

Proof. It is well known (for example [6]) that k_n-nearest neighbors is consistent for $\lim_{n\to\infty} k_n = \infty$ and $\lim_{n\to\infty} \frac{k_n}{n} = 0$. These can easily be verified for our case. □

Lemma 32. *For any $0 < p < 1$, $\lim_{n\to\infty} \mathbb{E}_{S\sim\mathcal{D}^n}[\sup_{x\in\mathbb{R}^d} \sum_1^n w_i^S(x) 1_{\rho(x,x_i)>r_p(x)}] = 0$.*

Proof. It suffices to show that for n sufficiently large, all k_n-nearest neighbors of x are located inside $B(x, r_p(x))$ for all $x \in \mathbb{R}^d$. We do this by using a VC-dimension type argument to show that all balls $B(x, r)$ contain a number of points from $S \sim \mathcal{D}^n$ that is close to their expectation.

For $x \in \mathbb{R}^d$ and $r \geq 0$, let $f_{x,r}$ denote the $0-1$ function defined as $f_{x,r}(x') = 1_{x' \in B(x,r)}$. Let $F = \{f_{x,r} : x \in \mathbb{R}^d, r \geq 0\}$ denote the class of all such functions. It is well known that the VC dimension of F is at most $d + 2$.

For $f \in F$, let $\mathbb{E}f$ denote $\mathbb{E}_{(x',y)\sim\mathcal{D}} f(x')$ and $\mathbb{E}_n f$ denote $\frac{1}{n}\sum_1^n f(x_i)$, where $\mathbb{E}_n f$ is defined with respect to some sample $S \sim \mathcal{D}^n$. By the standard generalization result of Vapnik and Chervonenkis (see [10] for a proof), we have that with probability $1-\delta$ over $S \sim \mathcal{D}^n$,

$$-\beta_n\sqrt{\mathbb{E}f} \leq \mathbb{E}f - \mathbb{E}_n f \leq \beta_n\sqrt{\mathbb{E}f} \tag{3}$$

holds for all $f \in F$, where $\beta_n = \sqrt{(4/n)((d+2)\ln 2n + \ln(8/\delta)}$.

Suppose n is sufficiently large so that $\beta_n \leq \frac{p}{2}$ and $\frac{k_n}{n} < \frac{p}{2}$, and suppose that equation 3 holds. Pick any $x \in \mathbb{R}^d$ and consider $f_{x,r}$ where $r > r_p(x)$. This implies $\mathbb{E}f_{x,r} \geq p$. Then by equation 3, we see that $\mathbb{E}_n f \geq \frac{p}{2}$. This implies that all k_n nearest neighbors of x are in the ball $B(x,r)$, and that consequently $\sum_1^n w_i^S(x)1_{\rho(x,x_i)>r} = 0$. Because this holds for all x, r with $x \in \mathbb{R}^d$ and $r > r_p(x)$, it follows that equation 2 implies that

$$\sup_{x\in X}\sum_1^n w_i^S(x)1_{\rho(x,x_i)>r_p(x)} = 0.$$

Because equation 3 holds with probability at least $1-\delta$, and δ can be made arbitrarily small, the desired claim follows. □

Let $t_n = \sqrt{dk_n \log n}$.

Lemma 33. $\lim_{n\to\infty} E_{S\sim D^n}[t_n \sup_{x\in\mathbb{R}^d} w_i^S(x)] = 0.$

Proof. Let $S \sim \mathcal{D}^n$. By the definition of k_n nearest neighbors, $\sup_{x\in\mathbb{R}^d} w_i^S(x) = \frac{1}{k_n}$. Therefore, $t_n \sup_{x\in\mathbb{R}^d} w_i^S(x) = \sqrt{\frac{d\log n}{k_n}}$. By assumption 2. of corollary 12, $\lim_{n\to\infty}\frac{d\log n}{k_n} = 0$, which implies that

$$\lim_{n\to\infty}\mathbb{E}_{S\sim D^n}[t_n \sup_{x\in\mathbb{R}^d} w_i^S(x)] = \lim_{n\to\infty}\sqrt{\frac{d\log n}{k_n}} = \lim_{n\to\infty}\frac{d\log n}{k_n} = 0,$$

as desired. □

Lemma 34. $\lim_{n\to\infty} E_{S\sim D^n}\frac{\log T(W,S)}{t_n} = 0.$

Proof. For $S \sim \mathcal{D}^n$, recall that $T(W,S)$ was defined as

$$T(W,S)|\{W_{x,\alpha,\beta} : x \in \mathbb{R}^d, 0 \leq \alpha, 0 \leq \beta \leq 1\}|,$$

where $W_{x,\alpha,\beta}$ denotes

$$W_{x,\alpha,\beta} = \{i : \rho(x,x_i) \leq \alpha, w_i^S(x) \geq \beta\}.$$

Our goal will to be upper bound $\log T(W,S)$.

To do so, we first need a tie-breaking mechanism for k_n-nearest neighbors. For each $x_i \in S$, we independently sample $z_i \in [0,1]$ from the uniform distribution. We then tie break based upon the value of z_i, i.e. if $\rho(x,x_i) = \rho(x,x_j)$, we say that x_i is closer to x than x_j if $z_i < z_j$. With probability 1, no two values z_i, z_j will be equal, so this ensures that this method always works.

Let $A_{x,\alpha} = \{i : \rho(x,x_i) \leq \alpha\}$ and let $B_{x,c} = \{i : z_i \leq c\}$. The key observation is that for any α, β, $W_{x,\alpha,\beta} = A_{x,\alpha} \cap B_{x,c}$ for some value of c. This can be seen by noting that the nearest neighbors of x are uniquely determined by $\rho(x,x_i)$ and z_i. Therefore, it suffices to bound $|A = A_{x,\alpha} : x \in \mathbb{R}^d, \alpha \geq 0\}|$ and $|B = \{B_{x,c} : x \in \mathbb{R}^d, c \geq 0\}|$.

To bound $|A|$, observe that the set of closed balls in $\mathbb{R}^d$ has VC-dimension at most $d+2$. Thus by Sauer's lemma, there are at most $O(n^{d+2}$ subsets of $\{x_1, x_2, \ldots, x_n\}$ that can be obtained from closed balls. Thus $|A| \leq O(n^{d+2}$.

To bound $|B|$, we simply note that $B_{x,c}$ consists of all i for which $z_i \leq c$. Since the z_i can be sorted, there are at most $n+1$ such sets. Thus $|B| \leq n+1$.

Combining this, we see that $T(W,S) \leq |A||B| \leq O(n^{d+3})$. Finally, we see that

$$\lim_{n\to\infty} \frac{\log T(W,S)}{t_n} = \lim_{n\to\infty} \frac{O(d\log n)}{\sqrt{k_n d \log n}} = \lim_{n\to\infty} \sqrt{\frac{O(d\log n)}{k_n}} = 0,$$

with the last inequality holding by condition 2. of Corollary 12.

□

Finally, we note that Corollary 12 is an immediate consequence of the previous 4 lemmas as we can simply apply Theorem 11.

B.5 Proof of Corollary 13

Let W be a kernel classifier constructed from K and h_n such that the conditions of Corollary 13 hold: that is,

1. $K : [0,\infty) \to [0,\infty)$ is decreasing and satisfies $\int_{\mathbb{R}^d} K(x)dx < \infty$.
2. $\lim_{n\to\infty} h_n = 0$ and $\lim_{n\to\infty} nh_n^d = \infty$.
3. For any $c > 1$, $\lim_{x\to\infty} \frac{K(cx)}{K(x)} = 0$.
4. For any $x \geq 0$, $\lim_{n\to\infty} \frac{n}{\log n} K(\frac{x}{h_n}) = \infty$.

It suffices to show that the conditions of Theorem 11 are met for W. Before doing this, we will describe one additional assumption we make for this case.

Additional Assumption: We assume that $\mathcal{D}, \mathcal{U}$ are such that there exists some compact set $\mathcal{X} \subset \mathbb{R}^d$ such that for all $x \in supp(\mu)$, $U_x \subset \mathcal{X}$. This is primarily for convenience: observe that any distribution can be approximated arbitrarily closely by distributions satisfying these properties (as each U_x is bounded by assumption). Importantly, because of this, we will note that it is possible for conditions 2. and 3. of Theorem 11 to be relaxed to taking supremums over $\mathcal{X}$ rather than $\mathbb{R}^d$. This is because in our proof, we only ever used these conditions in their restriction to $\bigcup_{x\in supp(\mu)} \bigcup x' \in U_x B(x', r_p(x'))$.

Using this assumption, we return to proving the corollary.

Lemma 35. *W is consistent with respect to $\mathcal{D}$.*

Proof. Condition 1. of Corollary 13 imply that K is a regular kernel. This together with Condition 2. implies that W is consistent: a proof can be found in [11]. □

To verify the second condition, it will be useful to have the following definition.

Definition 36. *For any $p, \epsilon > 0$ and $x \in \mathcal{X}$, define r_p^ϵ as*

$$r_p^\epsilon(x) = \sup\{r : \mu(B(x,r)) - \mu(B(x, r_p(x)) \leq \epsilon\}.$$

Lemma 37. *For any $p, \epsilon > 0$, there exists a constant $c_p^\epsilon > 1$ such that $\frac{r_p^\epsilon(x)}{r_p(x)} \geq c_p^\epsilon$ for all $x \in \mathcal{X}$, where we set $\frac{r_p^\epsilon(x)}{r_p(x)} = \infty$ if $r_p(x) = 0$.*

Proof. The basic idea is to use the fact that $\mathcal{X}$ is compact. Our strategy will be to analyze the behavior of $\frac{r_p^\epsilon(x)}{r_p(x)}$ over small balls $B(x_0, r)$ centered around some fixed x_0, and then use compactness to pick some finite set of balls $B(x_0, r)$. This must be done carefully because the function $x \to \frac{r_p^\epsilon(x)}{r_p(x)}$ is not necessarily continuous.

Fix any $x_0 \in \mathcal{X}$. First, observe that $r_p^\epsilon(x_0) > r_p(x_0)$. This is because $B(x_0, r_p(x_0)) = \cap_{r>r_p(x_0)} B(x_0, r)$, and consequently $\lim_{r\downarrow r_p(x_0)} \mu(B(x_0, r)) = \mu(B(x_0, r_p(x_)))$.

Next, define

$$s_p^\epsilon(x) = \inf\{r : \mu(B(x, r_p(x)) - \mu(B(x,r)) \leq \epsilon\}.$$

We can similarly show that $r_p(x_0) > s_p^\epsilon(x_0)$.

Finally, define

$$r_0 = \frac{1}{3}\min(r_p^\epsilon(x_0) - r_p(x_0), r_p(x_0) - s_p^\epsilon(x_0)).$$

Consider any $x \in B^o(x_0, r_0)$ where B^o denotes the open ball, and let $\alpha = \rho(x_0, x)$. Then we have the following.

1. $r_p(x) \le r_p(x_0) + \alpha$. This holds because $B(x, r_p(x_0) + \alpha)$ contains $B(x_0, r_p(x_0))$, which has probability mass at least p.
2. $r_p(x) \ge r_p(x_0) - \alpha$. This holds because if $r_p(x) < r_p(x_0) - \alpha$, then there would exists $r < r_p(x_0)$ such that $\mu(B(x_0, r)) \ge p$ which is a contradiction.
3. $B(x_0, s_p^\epsilon(x_0)) \subset B(x, r_p(x))$. This is just a consequence of the definition of r_0 and the previous observation.

By the definitions of r_p^ϵ and s_p^ϵ, we see that $\mu(B(x_0, r_p^\epsilon(x_0)) - \mu(B(x_0, s_p^\epsilon(x_0)) \le 2\epsilon$. By the triangle inequality, $B(x, r_p^\epsilon(x_0) - \alpha) \subset B(x_0, r_p^\epsilon(x_0))$ and $B(x_0, s_p^\epsilon(x_0)) \subset B(x, r_p(x))$. it follows that

$$\mu(B(x, r_p^\epsilon(x_0) - \alpha)) - \mu(B(x, r_p(x))) \le 2\epsilon,$$

which implies that $r_p^{2\epsilon}(x) \ge r_p^\epsilon(x_0) - \alpha$. Therefore we have the for all $x \in B(x_0, r_0)$,

$$\frac{r_p^{2\epsilon}(x)}{r_p(x)} \ge \frac{r_p^\epsilon(x_0) - \alpha}{r_p(x_0) + \alpha} \ge \frac{2r_p^\epsilon(x_0) + r_p(x_0)}{r_p^\epsilon(x_0) + 2r_p(x_0)}.$$

Notice that the last expression is a constant that depends only on x_0, and moreover, since $r_p^\epsilon(x_0) > r_p(x_0)$, this constant is strictly larger than 1. Let us denote this as $c(x_0)$. Then we see that $\frac{r_p^{2\epsilon}(x)}{r_p(x)} \ge c(x_0)$ for all $x \in B^o(x_0, r_0)$.

Finally, observe that $\{B^o(x_0, r_0) : x_0 \in \mathcal{X}\}$ forms an open cover of $\mathcal{X}$ and therefore has a finite sub-cover C. Therefore, taking $c = \min_{B^o(x_0,r_0)\in C} c(x_0)$, we see that $\frac{r_p^{2\epsilon}(x)}{r_p(x)} \ge c > 1$ for all $x \in \mathcal{X}$. Because ϵ was arbitrary, the claim holds. □

Lemma 38. *For any* $0 < p < 1$, $\lim_{n\to\infty} \mathbb{E}_{S\sim\mathcal{D}^n}[\sup_{x\in\mathcal{X}} \sum_1^n w_i^S(x) 1_{\rho(x,x_i) > r_p(x)}] = 0$.

Proof. Fix $p > 0$, and fix any $\epsilon, \delta > 0$. Pick n sufficiently large so that the following hold.

1. Let c_p^ϵ be as defined from Lemma 37.

$$\sup_{x\in\mathcal{X}} \frac{K(c_p^\epsilon r_p(x)/h_n)}{K(r_p(x)/h_n)} < \delta. \tag{4}$$

   This is possible because of conditions 2. and 3. of Corollary 13, and because the function $x \to r_p(x)$ is continuous.

2. With probability at least $1 - \delta$ over $S \sim \mathcal{D}^n$, for all $r > 0$, and $x \in \mathcal{X}$,

$$|\mu(B(x, r)) - \frac{1}{n}\sum_1^n 1_{x_i \in B(x,r)}| \le \epsilon. \tag{5}$$

   This is possible because the set of balls $B(x, r)$ has VC dimension at most $d + 2$.

We now bound $\mathbb{E}_{S\sim\mathcal{D}^n}[\sup_{x\in\mathcal{X}} \sum_1^n w_i^S(x) 1_{\rho(x,x_i) > r_p(x)}]$ by dividing into cases where S satisfies and doesn't satisfy equation 5.

Suppose S satisfies equation 5. By condition 1. of Corollary 13, K is decreasing, and by Lemma 37, $r_p^\epsilon(x) \ge c_p^\epsilon r_p(x)$. Therefore, we have that for any $x \in \mathcal{X}$,

$$\begin{aligned}\sum_1^n K(\rho(x, x_i)/h_n) 1_{\rho(x,x_i)\ge r_p^\epsilon(x)} &\le \sum_1^n K(c_p^\epsilon r_p(x)/h_n) \\ &\le n\delta K(r_p(x)/h_n)),\end{aligned}$$

where the second inequality comes from equation 4.

Next, by the definition of $r_p^\epsilon(x)$, we have that $\mu(B(x, r_p^\epsilon(x)) - \mu(B(x, r_p(x))) \leq \epsilon$. Therefore, by applying equation 5 two times, we see that for any $x \in \mathcal{X}$

$$\sum_1^n K(\rho(x, x_i)/h_n) 1_{r_p(x) < \rho(x,x_i) \leq r_p^\epsilon(x)} \leq 3n\epsilon K(r_p(x)/h_n).$$

Finally, we have that

$$\sum_1^n w_i^S(x) \geq \sum_1^n K(r_p(x)/h_n) 1_{\rho(x,x_i) \leq r_p(x)} \geq n(p-\epsilon) K(r_p(x)/h_n).$$

Therefore, using all three of our inequalities, we have that for any $x \in \mathcal{X}$

$$\begin{aligned}
\sum_1^n w_i^S(x) 1_{\rho(x,x_i) > r_p(x)} &= \sum_1^n w_i^S(x) 1_{\rho(x,x_i) > r_p^\epsilon(x)} + \sum_1^n w_i^S(x) 1_{r_p^\epsilon \geq \rho(x,x_i) > r_p(x)} \\
&= \frac{\sum_1^n K(\rho(x, x_i)/h_n) 1_{\rho(x,x_i) > r_p^\epsilon(x)} + \sum_1^n K(\rho(x, x_i)/h_n) 1_{r_p^\epsilon \geq \rho(x,x_i) > r_p(x)}}{\sum_1^n K(\rho(x, x_i)/h_n)} \\
&\leq \frac{n\delta K(r_p(x)/h_n)) + 3n\epsilon K(r_p(x)/h_n)}{n(p-\epsilon) K(r_p(x)/h_n).} \\
&= \frac{\delta + 3\epsilon}{p - \epsilon}.
\end{aligned}$$

If S does *not* satisfy equation 5, then we simply have $\sup_{x \in \mathcal{X}} \sum_1^n w_i^S(x) 1_{\rho(x,x_i) > r_p(x)} \leq 1$. Combining all of this, we have that

$$E_{S \sim \mathcal{D}^n} \sum_1^n w_i^S(x) 1_{\rho(x,x_i) > r_p(x)} \leq \delta(1) + (1-\delta)\frac{\delta + 3\epsilon}{p - \epsilon}.$$

Since δ, ϵ can be made arbitrarily small, the result follows. □

By assumption, $\mathcal{X}$ is compact and therefore has diameter $D < \infty$. Define

$$t_n = \sqrt{n \log n K(\frac{D}{h_n})} \text{ for } 1 \leq n < \infty.$$

Lemma 39. $\lim_{n \to \infty} E_{S \sim D^n}[t_n \sup_{x \in \mathcal{X}} w_i^S(x)] = 0.$

Proof. Because K is a decreasing function, we have that $K(D/h_n) \leq K(\rho(x, x_i)/h_n) \leq K(0)$. As a result, we have that for any $x \in \mathcal{X}$,

$$\begin{aligned}
t_n \sup_{1 \leq i \leq n} w_i^S(x) &= \frac{t_n \sup_{1 \leq i \leq n} K(\rho(x, x_i)/h_n)}{\sum_1^n K(\rho(x, x_i)/h_n)} \\
&\leq \frac{t_n K(0)}{nK(D/h_n)} \\
&= K(0)\sqrt{\frac{n \log n K(D/h_n)}{n^2 K(D/h_n)^2}} \\
&= K(0)\sqrt{\frac{\log n}{nK(D/h_n)}}.
\end{aligned}$$

However, by condition 4. of Corollary 13, $\lim_{n \to \infty} \frac{n}{\log n} K(D/h_n) = \infty$. Therefore, since the above inequality holds for all $x \in \mathcal{X}$, we have that

$$\lim_{n \to \infty} E_{S \sim D^n}[t_n \sup_{x \in \mathcal{X}} w_i^S(x)] \leq \lim_{n \to \infty} K(0)\sqrt{\frac{\log n}{nK(D/h_n)}} = 0.$$

□

Lemma 40. $\lim_{n\to\infty} E_{S\sim D^n} \frac{\log T(W,S)}{t_n} = 0.$

Proof. For $S \sim \mathcal{D}^n$, recall that $T(W,S)$ was defined as

$$T(W,S)|\{W_{x,\alpha,\beta} : x \in \mathcal{X}, 0 \le \alpha, 0 \le \beta \le 1\}|,$$

where $W_{x,\alpha,\beta}$ denotes

$$W_{x,\alpha,\beta} = \{i : \rho(x,x_i) \le \alpha, w_i^S(x) \ge \beta\}.$$

Our goal will to be upper bound $\log T(W,S)$.

The key observation is that $W_{x,\alpha,\beta}$ is precisely the set of x_i for which $\rho(x,x_i) \le r$ where r is some threshold. This is because the restriction that $w_i^S(x) \ge \beta$ can be directly translated into $\rho(x,x_i) \le r$ for some value of r, as K is a monotonically decreasing function. Thus, $T(W,S)$ is the number of subsets of S that can be obtained by considering the interior of some ball $B(x,r)$ centered at x with radius r.

We now observe that the set of closed balls in $\mathbb{R}^d$ has VC-dimension at most $d+2$. Thus by Sauer's lemma, there are at most $O(n^{d+2}$ subsets of $\{x_1, x_2, \ldots, x_n\}$ that can be obtained from closed balls. Thus $T(W,S) \le O(n^{d+2}$.

Finally, we see that

$$\lim_{n\to\infty} \frac{\log T(W,S)}{t_n} = \lim_{n\to\infty} \frac{O(d\log n)}{\sqrt{n\log nK(\frac{D}{h_n})}} \le \lim_{n\to\infty} \sqrt{\frac{O(d\log n)}{nK(\frac{D}{h_n})}} = 0,$$

with the last equality holding by condition 4. of Corollary 13. □

Finally, we note that Corollary 13 is an immediate consequences of Lemmas 35, 38, 39, and 40, as we can simply apply Theorem 11.

C Useful Technical Definitions and Lemmas

Lemma 41. *Let μ be a measure over $\mathbb{R}^d$, and let $\mathcal{A}$ denote a countable collections of measurable sets A_i such that $\mu(\bigcup_{A\in\mathcal{A}} A) < \infty$. Then for all $\epsilon > 0$, there exists a finite subset of $\mathcal{A}$, $\{A_1, \ldots, A_m\}$ such that*

$$\mu(A_1 \cup A_2 \cup \cdots \cup A_m) > \mu(\bigcup_{A\in\mathcal{A}} A) - \epsilon.$$

Proof. Follows directly from the definition of a measure. □

C.1 The support of a distribution

Let μ be a probability measure over $\mathbb{R}^d$.

Definition 42. *The **support** of μ, $supp(\mu)$, is defined as all $x \in \mathbb{R}^d$ such that for all $r > 0$, $\mu(B(x,r)) > 0$.*

From this definition, we can show that $supp(\mu)$ is closed.

Lemma 43. *$supp(\mu)$ is closed.*

Proof. Let x be a point such that $B(x,r) \cap supp(\mu) \neq \emptyset$ for all $r > 0$. It suffices to show that $x \in supp(\mu)$, as this will imply closure.

Let x be such a point, and fix $r > 0$. Then there exists $x' \in B(x, r/2)$ such that $x' \in supp(\mu)$. By definition, we see that $\mu(B(x', r/3)) > 0$. However, $B(x', r/3) \subset B(x,r)$ by the triangle inequality. it follows that $\mu(B(x,r)) > 0$. Since r was arbitrary, it follows that $x \in supp(\mu)$. □

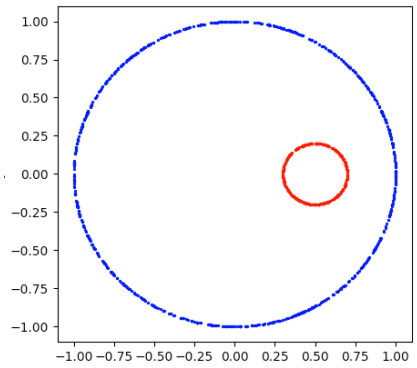


Figure 5: Our data distribution $\mathcal{D} = (\mu, \eta)$ with μ^+ shown in blue and μ^- shown in red. Observe that this simple distribution captures varying distances between the red and blue regions, which necessitates having varying sizes for robustness regions.

D Experiment Details

Data Distribution Our data distribution $\mathcal{D} = (\mu, \eta)$ is over $\mathbb{R}^2 \times \{\pm 1\}$, and is defined as follows. We let μ^+ consist of a uniform distribution over the circle $x^2 + y^2 = 1$, and μ^- consist of the uniform distribution over the circle $(x - 0.5)^2 + y^2 = 0.04$. The two distributions are weighted so that we draw a point from μ^+ with probability 0.7, and μ^- with probability 0.3. Finally, we utilize label noise 0.2 meaning that the label y matches that given by the Bayes optimal with probability 0.2. In summary, $\mathcal{D}$ can be described with the following 4 cases:

1. With probability 0.7×0.8, we select (x, y) with $x \in \mu^+$ and $y = +1$.
2. With probability 0.7×0.2, we select (x, y) with $x \in \mu^+$ and $y = -1$.
3. With probability 0.3×0.8, we select (x, y) with $x \in \mu^-$ and $y = -1$.
4. With probability 0.3×0.2, we select (x, y) with $x \in \mu^-$ and $y = +1$.

We also include a drawing (Figure 5) of the support of $\mathcal{D}$, with the positive portion μ^+ shown in blue and the negative portion, μ^- shown in red.

Computing Robustness Regions Recall that in order to measure robustness, we utilize the so-called partial neighborhood preserving regions V_x^κ (Definition 6) for varying values of κ. In the case of our data distribution $\mathcal{D}$, V_x^κ consists of points closer to x by a factor of κ than they are to μ^- (resp. μ^+) when $x \in \mu^+$ (resp. μ^-). To represent a region V_x^κ, we simply use a function f that verifies whether a given point $x' \in V_x^\kappa$. While this methodology is not sufficient for training general classifiers (for a whole litany of reasons: to begin with it assumes full knowledge of the distribution), it will suffice for our toy synthetic experiments.

Trained Classifiers We train two classifiers, both of which are kernel classifiers.

The first classifier is an exponential kernel classifier with bandwidth function $h_n = \frac{1}{10\sqrt{\log n}}$ and kernel function $K(x) = e^{-x}$.

The second classifier is a polynomial kernel classifier with bandwidth function $h_n = \frac{1}{10n^{1/3}}$ and kernel function $K(x) = \frac{1}{1+x^2}$.

Both of these kernels are regular kernels, and both bandwidths satisfy sufficient conditions for consistency with respect to accuracy. In other words, both of these classifiers will converge towards the accuracy of the Bayes optimal.

However, the first classifier is selected to satisfy the criterion of Corollary 13, whereas the second is not. This distinction is reflected in our experiments.

Verifying Robustness To verify the robustness of classifier f at point x (with respect to V_x^κ), we simply do a grid search with grid parameter 0.01. We grid the entire regions into points with distance at most 0.01 between them, and then verify that f has the desired value at all of those points. To ensure proper robustness, we also simply verify that f cannot change enough within a distance of 0.01 by constructing an upper bound on how much f can possibly change. For kernel classifiers, this is simple to do as there is a relatively straightforward upper bound on the gradient of a Kernel classifier.