
A Fast and Provable Algorithm for Sparse Phase Retrieval

Anonymous Author(s)

Affiliation

Address

email

Abstract

We study the sparse phase retrieval problem, which aims to recover a sparse signal from a limited number of phaseless measurements. Existing algorithms for sparse phase retrieval primarily rely on first-order methods with linear convergence rate. In this paper, we propose an efficient second-order algorithm based on Newton projection, which maintains the same per-iteration computational complexity as popular first-order methods. The proposed algorithm is theoretically guaranteed to converge to the ground truth (up to a global sign) at a quadratic convergence rate after at most $\mathcal{O}(\log(\|\mathbf{x}^\natural\|/x_{\min}^\natural))$ iterations, provided a sample complexity of $\mathcal{O}(s^2 \log n)$, where $\mathbf{x}^\natural \in \mathbb{R}^n$ represents an s -sparse ground truth signal. Numerical experiments demonstrate that our algorithm not only outperforms state-of-the-art methods in terms of achieving a significantly faster convergence rate, but also excels in attaining a higher success rate for exact signal recovery from noise-free measurements and providing enhanced signal reconstruction in noisy scenarios.

1 Introduction

We study the phase retrieval problem, which involves reconstructing an n -dimensional signal \mathbf{x}^\natural using its intensity-only measurements:

$$y_i = |\langle \mathbf{a}_i, \mathbf{x}^\natural \rangle|^2, \quad i = 1, 2, \dots, m, \quad (1)$$

where each y_i represents a measurement, \mathbf{a}_i denotes a sensing vector, \mathbf{x}^\natural is the unknown signal to be recovered, and m is the total number of measurements. The phase retrieval problem arises in various applications, including diffraction imaging [1], X-ray crystallography [2, 3], and optics [4], where detectors can only record the squared modulus of Fresnel or Fraunhofer diffraction patterns of radiation scattered from an object. The loss of phase information complicates the understanding of the scattered object, as much of the image’s structural content may be encoded in the phase.

Although the phase retrieval problem is ill-posed and even NP-hard [5], several algorithms have been proven to succeed in recovering target signals under certain assumptions. Algorithms can be broadly categorized into convex and nonconvex approaches. Convex methods, such as PhaseLift [6, 7], PhaseCut [8], and PhaseMax [9, 10], offer optimal sample complexity but are computationally challenging in high-dimensional cases. To improve computational efficiency, nonconvex approaches are explored, including alternating minimization [11], Wirtinger flow [6], truncated amplitude flow [12], Riemannian optimization [13], Gauss-Newton [14, 15], and Kaczmarz [16, 17]. Despite the nonconvex nature of its objective function, the global geometric landscape lacks spurious local minima [18, 19], allowing algorithms with random initialization to work effectively [20, 21].

The nonconvex approaches previously mentioned can guarantee successful recovery of the ground truth (up to a global phase) with a sample complexity $m \sim \mathcal{O}(n \log^a n)$, where $a \geq 0$. This complexity is nearly optimal, as the phase retrieval problem requires $m \geq 2n - 1$ for real signals and $m \geq 4n - 4$ for complex signals [22]. However, in practical situations, especially in high-dimensional cases, the number of available measurements is often less than the signal dimension (*i.e.*, $m < n$), leading to a need for further reduction in sample complexity.

In this paper, we focus on the sparse phase retrieval problem, which aims to recover a sparse signal from a limited number of phaseless measurements. It has been established that the minimal sample complexity required to ensure s -sparse phase retrievability in the real case is only $2s$ for generic sensing vectors [23]. Several algorithms have been proposed to address the sparse phase retrieval problem [24, 25, 26, 27, 28]. These approaches have been demonstrated to effectively reconstruct the ground truth using $\mathcal{O}(s^2 \log n)$ Gaussian measurements. While this complexity is not optimal, it is significantly smaller than that in general phase retrieval.

1.1 Contributions

Existing algorithms for sparse phase retrieval primarily employ first-order methods with linear convergence. Recent work [28] introduced a second-order method, while it fails to obtain a quadratic convergence rate. The main contributions of this paper can be summarized in three key points:

1. We propose a second-order algorithm based on Newton projection for sparse phase retrieval that maintains the same per-iteration computational complexity as popular first-order methods. To ensure fast convergence, we integrate second-order derivative information from intensity-based empirical loss into the search direction; to ensure computational efficiency, we restrict the Newton update to a subset of variables, setting others to zero in each iteration.
2. We establish a non-asymptotic quadratic convergence rate for our proposed algorithm and provide the iteration complexity. Specifically, we prove that the algorithm converges to the ground truth (up to a global sign) at a quadratic rate after at most $\mathcal{O}(\log(|\mathbf{x}^\natural|/x_{\min}^\natural))$ iterations, provided a sample complexity of $\mathcal{O}(s^2 \log n)$. To the best of our knowledge, this is the first algorithm to establish a quadratic convergence rate for sparse phase retrieval.
3. Numerical experiments demonstrate that the proposed algorithm achieves a significantly faster convergence rate in comparison to state-of-the-art methods. Furthermore, the experiments reveal that our algorithm attains a higher success rate in exact signal recovery from noise-free measurements and provides enhanced signal reconstruction performance in noisy scenarios, as evidenced by the improved Peak Signal-to-Noise Ratio (PSNR).

Notation: The p -norm $\|\mathbf{x}\|_p := (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $p \geq 1$. $\|\mathbf{x}\|_0$ denotes the number of nonzero entries of \mathbf{x} , and $\|\mathbf{x}\|$ denotes the 2-norm. For a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\|\mathbf{A}\|$ is the spectral norm of \mathbf{A} . For any $q_1 \geq 1$ and $q_2 \geq 1$, $\|\mathbf{A}\|_{q_2 \rightarrow q_1}$ denotes the induced operator norm from the Banach space $(\mathbb{R}^n, \|\cdot\|_{q_2})$ to $(\mathbb{R}^m, \|\cdot\|_{q_1})$. $\lambda_{\min}(\mathbf{A})$ and $\lambda_{\max}(\mathbf{A})$ denote the smallest and largest eigenvalues of the matrix \mathbf{A} . $|\mathcal{S}|$ denotes the number of elements in \mathcal{S} . $\mathbf{a} \odot \mathbf{b}$ denotes the entrywise product of \mathbf{a} and \mathbf{b} . For functions $f(n)$ and $g(n)$, we write $f(n) \lesssim g(n)$ if $f(n) \leq cg(n)$ for some constant $c \in (0, +\infty)$. For $\mathbf{x}, \mathbf{x}^\natural \in \mathbb{R}^n$, the distance between \mathbf{x} and \mathbf{x}^\natural is defined as $\text{dist}(\mathbf{x}, \mathbf{x}^\natural) := \min \{\|\mathbf{x} - \mathbf{x}^\natural\|, \|\mathbf{x} + \mathbf{x}^\natural\|\}$. x_{\min}^\natural denotes the smallest nonzero entry in magnitude of \mathbf{x}^\natural .

2 Problem Formulation and Related Works

We first present the problem formulation for sparse phase retrieval, and then review related works.

2.1 Problem formulation

The standard sparse phase retrieval problem can be concisely expressed as finding \mathbf{x} that satisfies

$$|\langle \mathbf{a}_i, \mathbf{x} \rangle|^2 = y_i \quad \forall i = 1, \dots, m, \quad \text{and} \quad \|\mathbf{x}\|_0 \leq s, \quad (2)$$

where $\{\mathbf{a}_i\}_{i=1}^m$ are known sensing vectors and $\{y_i\}_{i=1}^m$ represent phaseless measurements with $y_i = |\langle \mathbf{a}_i, \mathbf{x}^\natural \rangle|^2$, where \mathbf{x}^\natural is the ground truth signal ($\|\mathbf{x}^\natural\|_0 \leq s$). While sparsity level s is assumed known a priori for theoretical analysis, our experiments will also explore cases with unknown s .

Table 1: Overview of per-iteration computational cost, numbers of iterations for convergence, loss function, and algorithm types for various methods. \mathbf{x}^\natural represents the ground truth signal with dimension n and sparsity s , and x_{\min}^\natural denotes the smallest nonzero entry in magnitude of \mathbf{x}^\natural .

Methods	Per-iteration cost	Iteration complexity	Loss function	Algorithm types
ThWF [25]	$\mathcal{O}(n^2 \log n)$	$\mathcal{O}(\log(1/\epsilon))$	$f_I(\mathbf{x})$	Grad. Proj.
SPARTA [26]	$\mathcal{O}(ns^2 \log n)$	$\mathcal{O}(\log(1/\epsilon))$	$f_A(\mathbf{x})$	Grad. Proj.
CoPRAM [27]	$\mathcal{O}(ns^2 \log n)$	$\mathcal{O}(\log(1/\epsilon))$	$f_A(\mathbf{x})$	Alt. Min.
HTP [28]	$\mathcal{O}((n + s^2)s^2 \log n)$	$\mathcal{O}(\log(s^2 \log n) + \log(\ \mathbf{x}^\natural\ /x_{\min}^\natural))$	$f_A(\mathbf{x})$	Alt. Min.
Proposed	$\mathcal{O}((n + s^2)s^2 \log n)$	$\mathcal{O}(\log(\log(1/\epsilon)) + \log(\ \mathbf{x}^\natural\ /x_{\min}^\natural))$	$f_I(\mathbf{x})$	Newton Proj.

To address Problem (2), various problem reformulations have been explored. Convex formulations, such as the ℓ_1 -regularized PhaseLift method [24], often use the lifting technique and solve the problem in the $n \times n$ matrix space, resulting in high computational costs. To enhance computational efficiency, nonconvex approaches [25, 26, 28, 29] are explored, which can be formulated as:

$$\underset{\mathbf{x}}{\text{minimize}} f(\mathbf{x}), \quad \text{subject to} \quad \|\mathbf{x}\|_0 \leq s. \quad (3)$$

Both the loss function $f(\mathbf{x})$ and the ℓ_0 -norm constraint in Problem (3) are nonconvex, making it challenging to solve. Two prevalent loss functions are investigated: intensity-based empirical loss

$$f_I(\mathbf{x}) := \frac{1}{4m} \sum_{i=1}^m (|\langle \mathbf{a}_i, \mathbf{x} \rangle|^2 - y_i)^2, \quad (4)$$

and amplitude-based empirical loss

$$f_A(\mathbf{x}) := \frac{1}{2m} \sum_{i=1}^m (|\langle \mathbf{a}_i, \mathbf{x} \rangle| - z_i)^2, \quad (5)$$

where $z_i = \sqrt{y_i}$, $i = 1, \dots, m$. The intensity-based loss $f_I(\mathbf{x})$ is smooth, while the amplitude-based loss $f_A(\mathbf{x})$ is non-smooth because of the modulus.

2.2 Related works

Existing nonconvex sparse phase retrieval algorithms can be broadly classified into two categories: gradient projection methods and alternating minimization methods. Gradient projection methods, such as ThWF [25] and SPARTA [26], employ thresholded gradient descent and iterative hard thresholding, respectively. On the other hand, alternating minimization methods, including CoPRAM [27] and HTP [28], alternate between updating the signal and phase. When updating the signal, formulated as a sparsity-constrained least squares problem, CoPRAM leverages the cosamp method [30], while HTP applies the hard thresholding pursuit algorithm [31]. In this paper, we introduce a Newton projection-based algorithm that incorporates second-order derivative information, resulting in a faster convergence rate compared to gradient projection methods, and, unlike alternating minimization methods, it eliminates the need for separate signal and phase updates. We note that ThWF and our algorithm utilize intensity-based loss as the objective function, while SPARTA, CoPRAM, and HTP employ amplitude-based loss. All these algorithms require a sample complexity of $\mathcal{O}(s^2 \log n)$ under Gaussian measurements for successful recovery.

The majority of sparse phase retrieval algorithms, such as ThWF, SPARTA, and CoPRAM, are first-order methods with linear convergence rates. While HTP is a second-order method that converges in a finite number of iterations, it fails to establish a quadratic convergence rate. We propose a second-order algorithm that attains a non-asymptotic quadratic convergence rate and exhibits lower iteration complexity compared to HTP. Our algorithm maintains the same computational complexity per iteration as popular first-order methods when $s \lesssim \sqrt{n}$. This condition is always assumed to hold true; otherwise, the established sample complexity for sparse phase retrieval algorithms, $\mathcal{O}(s^2 \log n)$, would be reduced to that of general phase retrieval methods. Table 1 presents a comparative overview of the previously discussed methods and our proposed method.

3 Main Results

In this section, we present our proposed algorithm for sparse phase retrieval. Generally, nonconvex methods comprise two stages: initialization and refinement. The first stage generates an initial guess close to the target signal, while the second stage refines the initial guess using various methods, such as ThWF, SPARTA, CoPRAM, and HTP. Our proposed algorithm adheres to this two-stage strategy. In the first stage, we employ an existing effective method to generate an initial point. Our primary focus is on the second stage, wherein we propose an efficient second-order algorithm based on Newton projection to refine the initial guess.

Before delving into the details of our proposed algorithm, we present a unified algorithmic framework for addressing the sparsity-constrained optimization problem in Eq. (3), as summarized in [32]. Given the k -th iterate \mathbf{x}^k , the next iterate \mathbf{x}^{k+1} can be obtained through the following steps:

Step 1 (Hard thresholding):

$$\mathbf{u}^{k+1} = \mathcal{H}_r(\phi(\mathbf{x}^k)), \quad (6)$$

Step 2 (Debiasing):

$$\mathbf{v}^{k+1} = \arg \min_{\mathbf{x}} \psi_k(\mathbf{x}), \quad \text{subject to} \quad \text{supp}(\mathbf{x}) \subseteq \mathcal{S}_{k+1}, \quad (7)$$

Step 3 (Pruning):

$$\mathbf{x}^{k+1} \in \mathcal{H}_s(\mathbf{v}^{k+1}), \quad (8)$$

where $\phi(\mathbf{x}^k)$ is typically chosen as either $\nabla f(\mathbf{x}^k)$ or $\mathbf{x}^k - \eta \nabla f(\mathbf{x}^k)$, $\psi_k(\mathbf{x})$ is designed based on the objective function $f(\mathbf{x})$ and the iterate \mathbf{x}^k , and \mathcal{S}_{k+1} is usually defined as the support of \mathbf{u}^{k+1} . The hard-thresholding operator, denoted by \mathcal{H}_s , is defined with a sparsity level of s as follows:

$$\mathcal{H}_s(\mathbf{w}) := \arg \min_{\mathbf{x}} \|\mathbf{x} - \mathbf{w}\|^2, \quad \text{subject to} \quad \|\mathbf{x}\|_0 \leq s. \quad (9)$$

A variety of well-known algorithms for solving sparsity-constrained optimization problems adhere to the three-step algorithmic framework mentioned earlier. For instance, the Iterative Hard Thresholding (IHT) algorithm solely performs Step 1 using $\mathcal{H}_s(\mathbf{x}^k - \eta \nabla f(\mathbf{x}^k))$ with η the stepsize; the Hard Thresholding Pursuit (HTP) implements the first two steps by computing \mathbf{u}^{k+1} via one-step IHT in Step 1, and then solving the support-constrained problem in Step 2 with $\mathcal{S}_{k+1} = \text{supp}(\mathbf{u}^{k+1})$; the Compressive Sampling Matching Pursuit (CoSaMP) executes all three steps, calculating $\mathbf{u}^{k+1} = \mathcal{H}_{2s}(\nabla f(\mathbf{x}^k))$ in Step 1, performing Step 2 with $\mathcal{S}_{k+1} = \text{supp}(\mathbf{u}^{k+1}) \cup \text{supp}(\mathbf{x}^k)$, and pruning the result in Step 3 to ensure an s -sparse level.

Several state-of-the-art methods for sparse phase retrieval share strong connections with the previously described popular algorithms for sparsity-constrained optimization, and thus relate closely to the algorithmic framework. SPARTA combines IHT with gradient truncation to eliminate erroneously estimated signs. HTP merges hard thresholding pursuit with alternating minimization, updating the signal and phase alternately. CoPRAM integrates CoSaMP with alternating minimization. Our proposed algorithm will also be presented using this algorithmic framework.

3.1 Proposed algorithm

In this subsection, we introduce our proposed algorithm for sparse phase retrieval, which utilizes the intensity-based loss f_I defined in Eq. (4) as the objective function. The algorithm incorporates the first two steps of the previously discussed algorithmic framework.

Our algorithm is developed based on the Newton projection method. It is worth mentioning that Newton-type methods typically require solving a linear system at each iteration to determine the Newton direction. This generally results in a computational cost of $\mathcal{O}(n^3)$ for our problem, rendering it impractical in high-dimensional situations. To address this challenge, we categorize variables into two groups at each iteration: *free* and *fixed*, updating them separately. The *free* variables, consisting of at most s variables, are updated according to the (approximate) Newton direction, while the *fixed* variables are set to zero. This strategy requires solving a linear system of size $s \times s$, substantially decreasing the computational expense from $\mathcal{O}(n^3)$ to $\mathcal{O}(s^3)$.

154 In the first step, we identify the set of *free* variables using one-step IHT of the loss $f_A(\mathbf{x})$ in (5):

$$\mathcal{S}_{k+1} = \text{supp}(\mathcal{H}_s(\mathbf{x}^k - \eta \nabla f_A(\mathbf{x}^k))),$$

155 where η is the stepsize. Since f_A is non-smooth, we adopt the generalized gradient [33] as ∇f_A . The
156 s -sparse hard thresholding limits $|\mathcal{S}_{k+1}|$ to s , implying that there are at most s *free* variables. We
157 only update *free* variables along the approximate Newton direction and set others to zero.

158 In the second step, we update the *free* variables in \mathcal{S}_{k+1} by solving a support-constrained problem
159 in Eq. (7). Note that we adopt the intensity-based loss f_I as the objective function. To accelerate
160 convergence, we choose function $\psi_k(\mathbf{x})$ in (7) as the second-order Taylor expansion of f_I at \mathbf{x}^k :

$$\psi_k(\mathbf{x}) := f_I(\mathbf{x}^k) + \langle \nabla f_I(\mathbf{x}^k), \mathbf{x} - \mathbf{x}^k \rangle + \frac{1}{2} \langle \mathbf{x} - \mathbf{x}^k, \nabla^2 f_I(\mathbf{x}^k)(\mathbf{x} - \mathbf{x}^k) \rangle.$$

161 Let \mathbf{x}^* denote the minimizer of Problem (7). For notational simplicity, define $\mathbf{g}_{\mathcal{S}_{k+1}}^k =$
162 $[\nabla f_I(\mathbf{x}^k)]_{\mathcal{S}_{k+1}}$, which denotes the sub-vector of $\nabla f_I(\mathbf{x}^k)$ indexed by \mathcal{S}_{k+1} , $\mathbf{H}_{\mathcal{S}_{k+1}}^k =$
163 $[\nabla^2 f_I(\mathbf{x}^k)]_{\mathcal{S}_{k+1}}$, which represents the principle sub-matrix of the Hessian indexed by \mathcal{S}_{k+1} , and
164 $\mathbf{H}_{\mathcal{S}_{k+1}, \mathcal{S}_{k+1}^c}^k = [\nabla^2 f_I(\mathbf{x}^k)]_{\mathcal{S}_{k+1}, \mathcal{S}_{k+1}^c}$, denoting the sub-matrix of the Hessian whose rows and
165 columns are indexed by \mathcal{S}_{k+1} and \mathcal{S}_{k+1}^c , respectively. Following from the first-order optimality
166 condition of Problem (7), we obtain that $\mathbf{x}_{\mathcal{S}_{k+1}^c}^* = \mathbf{0}$ and $\mathbf{x}_{\mathcal{S}_{k+1}}^*$ satisfies

$$\mathbf{H}_{\mathcal{S}_{k+1}}^k (\mathbf{x}_{\mathcal{S}_{k+1}}^* - \mathbf{x}_{\mathcal{S}_{k+1}}^k) = \mathbf{H}_{\mathcal{S}_{k+1}, \mathcal{S}_{k+1}^c}^k \mathbf{x}_{\mathcal{S}_{k+1}^c}^k - \mathbf{g}_{\mathcal{S}_{k+1}}^k. \quad (10)$$

167 As a result, we obtain the next iterate \mathbf{x}^{k+1} by

$$\mathbf{x}_{\mathcal{S}_{k+1}}^{k+1} = \mathbf{x}_{\mathcal{S}_{k+1}}^k - \mathbf{p}_{\mathcal{S}_{k+1}}^k, \quad \text{and} \quad \mathbf{x}_{\mathcal{S}_{k+1}^c}^{k+1} = \mathbf{0}, \quad (11)$$

168 where $\mathbf{p}_{\mathcal{S}_{k+1}}^k$ represents the approximate Newton direction over \mathcal{S}_{k+1} , which can be calculated by

$$\mathbf{H}_{\mathcal{S}_{k+1}}^k \mathbf{p}_{\mathcal{S}_{k+1}}^k = -\mathbf{H}_{\mathcal{S}_{k+1}, J_{k+1}}^k \mathbf{x}_{J_{k+1}}^k + \mathbf{g}_{\mathcal{S}_{k+1}}^k. \quad (12)$$

169 where $J_{k+1} := \mathcal{S}_k \setminus \mathcal{S}_{k+1}$ with $|J_{k+1}| \leq s$. In contrast to Eq. (10), we replace $\mathbf{x}_{\mathcal{S}_{k+1}^c}^k$ with $\mathbf{x}_{J_{k+1}}^k$ in
170 (12), as J_{k+1} captures all nonzero elements in $\mathbf{x}_{\mathcal{S}_{k+1}^c}^k$ as follows:

$$\mathcal{G}(\mathbf{x}_{\mathcal{S}_{k+1}^c}^k) = \begin{bmatrix} \mathbf{x}_{\mathcal{S}_{k+1}^c \cap \mathcal{S}_k}^k \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{\mathcal{S}_k \setminus \mathcal{S}_{k+1}}^k \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{J_{k+1}}^k \\ \mathbf{0} \end{bmatrix}, \quad (13)$$

171 where operator \mathcal{G} arranges all nonzero elements of a vector to appear first, followed by zero elements.
172 The first equality in (13) follows from the fact that $\text{supp}(\mathbf{x}^k) \subseteq \mathcal{S}_k$. By calculating $\mathbf{H}_{\mathcal{S}_{k+1}, J_{k+1}}^k$
173 rather than $\mathbf{H}_{\mathcal{S}_{k+1}, \mathcal{S}_{k+1}^c}^k$ as in (12), the computational cost is substantially reduced from $\mathcal{O}(smn)$ to
174 $\mathcal{O}(s^2m)$. The costs for computing $\mathbf{H}_{\mathcal{S}_{k+1}}^k$ and solving the linear system in (12) are $\mathcal{O}(s^2m)$ and
175 $\mathcal{O}(s^3)$, respectively. Therefore, the overall computational cost for Step 2 is $\mathcal{O}(s^2m)$, while the cost
176 for Step 1 amounts to $\mathcal{O}(mn)$, which involves calculating $\nabla f_A(\mathbf{x}^k)$.

177 In summary, the computational costs for Steps 1 and 2 are $\mathcal{O}(mn)$ and $\mathcal{O}(s^2m)$, respectively, making
178 the total cost per iteration $\mathcal{O}(n + s^2)m$, with $m \sim \mathcal{O}(s^2 \log n)$ that is required for successful
179 recovery. Since $s \lesssim \sqrt{n}$ is always assumed to hold true as discussed in Section 2.2, the per-iteration
180 computational complexity of our algorithm is equivalent to that of popular first-order methods, which
181 is $\mathcal{O}(ns^2 \log n)$. The pruning step is omitted as \mathbf{x}^{k+1} in (11) is already s -sparse.

Algorithm 1 Proposed algorithm

Input: Data $\{\mathbf{a}_i, y_i\}_{i=1}^m$, sparsity s , initial estimate \mathbf{x}^0 , and stepsize η .

1: **for** $k = 0, 1, 2, \dots$ **do**

2: Identify the set of *free* variables $\mathcal{S}_{k+1} = \text{supp}(\mathcal{H}_s(\mathbf{x}^k - \eta \nabla f_A(\mathbf{x}^k)))$;

3: Compute the approximate Newton direction $\mathbf{p}_{\mathcal{S}_{k+1}}^k$ over \mathcal{S}_{k+1} by solving (12).

4: Update \mathbf{x}^{k+1} :

$$\mathbf{x}_{\mathcal{S}_{k+1}}^{k+1} = \mathbf{x}_{\mathcal{S}_{k+1}}^k - \mathbf{p}_{\mathcal{S}_{k+1}}^k, \quad \text{and} \quad \mathbf{x}_{\mathcal{S}_{k+1}^c}^{k+1} = \mathbf{0}.$$

5: **end for**

Output: \mathbf{x}^{k+1} .

3.2 Initialization

The nonconvex nature of phase retrieval problems often requires a well-designed initial guess to find a global minimizer. Spectral initialization is a common approach [6]. In this paper, we adopt a sparse variant of the spectral initialization method to obtain a favorable initial guess for Algorithm 1.

Assuming $\{\mathbf{a}_i\}_{i=1}^m$ are independently drawn from a Gaussian distribution $\mathcal{N}(\mathbf{0}, \mathbf{I}_n)$, the expectation of the matrix $\frac{1}{m} \sum_{i=1}^m y_i \mathbf{a}_i \mathbf{a}_i^T$ is $\mathbf{M} := \|\mathbf{x}^\natural\|^2 \mathbf{I}_n + 2\mathbf{x}^\natural (\mathbf{x}^\natural)^T$. The leading eigenvector of \mathbf{M} is precisely $\pm \mathbf{x}^\natural$. Hence, the leading eigenvector of $\frac{1}{m} \sum_{i=1}^m y_i \mathbf{a}_i \mathbf{a}_i^T$ can be close to $\pm \mathbf{x}^\natural$ [6]. However, this method requires the sample complexity of at least $\mathcal{O}(n)$, which is excessively high for sparse phase retrieval. Leveraging the sparsity of \mathbf{x}^\natural is crucial to lower this complexity.

We adopt the sparse spectral initialization method proposed in [27]. Specifically, we first collect the indices of the largest s values from $\{\frac{1}{m} \sum_{i=1}^m y_i [\mathbf{a}_i]_j^2\}_{j=1}^n$ and obtain the set \hat{S} , which serves as an estimate of the support of the true signal \mathbf{x}^\natural . Next, we construct the initial guess \mathbf{x}^0 as follows: $\mathbf{x}_{\hat{S}}^0$ is the leading eigenvector of $\frac{1}{m} \sum_{i=1}^m y_i [\mathbf{a}_i]_{\hat{S}} [\mathbf{a}_i]_{\hat{S}}^T$, and $\mathbf{x}_{\hat{S}^c}^0 = \mathbf{0}$. Finally, we scale \mathbf{x}^0 such that $\|\mathbf{x}^0\|^2 = \frac{1}{m} \sum_{i=1}^m y_i$, ensuring the power of \mathbf{x}^0 closely aligns with the power of \mathbf{x}^\natural .

The study in [27] demonstrates that, given a sample complexity $m \sim \mathcal{O}(s^2 \log n)$, the aforementioned sparse spectral initialization method can produce an initial estimate \mathbf{x}^0 that is sufficiently close to the ground truth. Specifically, it holds $\text{dist}(\mathbf{x}^0, \mathbf{x}^\natural) \leq \gamma \|\mathbf{x}^\natural\|$ for any $\gamma \in (0, 1)$, with a probability of at least $1 - 8m^{-1}$.

3.3 Theoretical results

Given the nonconvex nature of both the objective function and the constraint set in the sparse phase retrieval problem, a thorough theoretical analysis is essential for ensuring the convergence of our algorithm to the ground truth. In this subsection, we provide a comprehensive analysis of the convergence of our algorithm for both noise-free and noisy scenarios.

3.3.1 Noise-free case

We begin by the noise-free case, in which each measurement $y_i = |\langle \mathbf{a}_i, \mathbf{x}^\natural \rangle|^2$. Starting with an initial guess obtained via the sparse spectral initialization method, the following theorem shows that our algorithm exhibits a quadratic convergence rate after at most $\mathcal{O}(\log(\|\mathbf{x}^\natural\|/x_{\min}^\natural))$ iterations.

Theorem 3.1. *Let $\{\mathbf{a}_i\}_{i=1}^m$ be i.i.d. random vectors distributed as $\mathcal{N}(\mathbf{0}, \mathbf{I}_n)$, and $\mathbf{x}^\natural \in \mathbb{R}^n$ be any signal with $\|\mathbf{x}^\natural\|_0 \leq s$. Let $\{\mathbf{x}^k\}_{k \geq 1}$ be the sequence generated by Algorithm 1 with the input measurements $y_i = |\langle \mathbf{a}_i, \mathbf{x}^\natural \rangle|^2$, $i = 1, \dots, m$, and the initial guess \mathbf{x}^0 generated by the sparse spectral initialization method mentioned earlier. There exists positive constants $\rho, \eta_1, \eta_2, C_1, C_2, C_3, C_4, C_5$ such that if the stepsize $\eta \in [\eta_1, \eta_2]$ and $m \geq C_1 s^2 \log n$, then with probability at least $1 - (C_2 K + C_3)m^{-1}$, the sequence $\{\mathbf{x}^k\}_{k \geq 1}$ converges to the ground truth \mathbf{x}^\natural at a quadratic rate after at most $\mathcal{O}(\log(\|\mathbf{x}^\natural\|/x_{\min}^\natural))$ iterations, i.e.,*

$$\text{dist}(\mathbf{x}^{k+1}, \mathbf{x}^\natural) \leq \rho \cdot \text{dist}^2(\mathbf{x}^k, \mathbf{x}^\natural), \quad \forall k \geq K,$$

where $K \leq C_4 \log(\|\mathbf{x}^\natural\|/x_{\min}^\natural) + C_5$, and x_{\min}^\natural is the smallest nonzero entry in magnitude of \mathbf{x}^\natural .

The proof of Theorem 3.1 is available in Appendix B.2.

Remark 3.2. Theorem 3.1 establishes the non-asymptotic quadratic convergence rate of our algorithm as it converges to the ground truth, leading to an iteration complexity of $\mathcal{O}(\log(\log(1/\epsilon)) + \log(\|\mathbf{x}^\natural\|/x_{\min}^\natural))$ for achieving an ϵ -accurate solution. This convergence rate is significantly faster than those of state-of-the-art methods such as ThWF [25], SPARTA [26], and CoPRAM [27], which, as first-order methods, exhibit only linear convergence. Although HTP [28] is a second-order approach, it fails to establish a quadratic convergence rate, and its iteration complexity, $\mathcal{O}(\log(\log(n^{s^2})) + \log(\|\mathbf{x}^\natural\|/x_{\min}^\natural))$, is higher than that of our algorithm.

Remark 3.3. It is worth emphasizing that while the superlinear convergence is extensively established for Newton-type methods in existing literature, it often holds only asymptotically: the ratio of the distance to the optimal solution at $(k+1)$ -th and k -th iterations tends to zero as k goes to infinity. Consequently, the overall iteration complexity cannot be explicitly characterized. This fact highlights the significance of establishing a non-asymptotic superlinear convergence rate.

3.3.2 Noisy case

In real-world scenarios, observations are frequently affected by noise. In what follows, we demonstrate the robustness of our proposed algorithm in the presence of noise within phaseless measurements. Building upon [25, 34], we assume that the noisy measurements are given by:

$$y_i = |\langle \mathbf{a}_i, \mathbf{x}^\natural \rangle|^2 + \epsilon_i, \quad \text{for } i = 1, \dots, m,$$

where ϵ represents a vector of stochastic noise that is independent of $\{\mathbf{a}_i\}_{i=1}^m$. Throughout this paper, we assume, without loss of generality, that the expected value of ϵ is 0.

Theorem 3.4. *Let $\{\mathbf{a}_i\}_{i=1}^m$ be i.i.d. random vectors distributed as $\mathcal{N}(\mathbf{0}, \mathbf{I}_n)$, and $\mathbf{x}^\natural \in \mathbb{R}^n$ be any signal with $\|\mathbf{x}^\natural\|_0 \leq s$. Let $\{\mathbf{x}^k\}_{k \geq 1}$ be the sequence generated by Algorithm 1 with noisy input $y_i = |\langle \mathbf{a}_i, \mathbf{x}^\natural \rangle|^2 + \epsilon_i$, $i = 1, \dots, m$. There exists positive constants $\eta_1, \eta_2, C_6, C_7, C_8$, and $\gamma \in (0, 1/8]$, such that if the stepsize $\eta \in [\eta_1, \eta_2]$, $m \geq C_6 s^2 \log n$ and the initial guess \mathbf{x}^0 obeys $\text{dist}(\mathbf{x}^0, \mathbf{x}^\natural) \leq \gamma \|\mathbf{x}^\natural\|$ with $\|\mathbf{x}^0\|_0 \leq s$, then with probability at least $1 - (C_7 K' + C_8)m^{-1}$,*

$$\text{dist}(\mathbf{x}^{k+1}, \mathbf{x}^\natural) \leq \rho' \cdot \text{dist}(\mathbf{x}^k, \mathbf{x}^\natural) + v \|\epsilon\|, \quad \forall 0 \leq k \leq K',$$

where $\rho' \in (0, 1)$, $v \in (0, 1)$, and K' is a positive integer.

The proof of Theorem 3.4 is provided in Appendix B.3. Theorem 3.4 validates the robustness of our algorithm, demonstrating its ability to effectively recover the signal from noisy measurements.

4 Experimental Results

In this section, we present a series of numerical experiments designed to validate the efficiency and accuracy of our proposed algorithm. All experiments were conducted on a 2 GHz Intel Core i5 processor with 16 GB of RAM, and all compared methods were implemented using MATLAB.

Unless explicitly specified, the sensing vectors $\{\mathbf{a}_i\}_{i=1}^m$ were generated by the standard Gaussian distribution. The true signal \mathbf{x}^\natural has s nonzero entries, where the support is selected uniformly from all subsets of $[n]$ with cardinality s , and their values are independently generated from the standard Gaussian distribution $\mathcal{N}(0, 1)$. In the case of noisy measurements, we have:

$$y_i = |\langle \mathbf{a}_i, \mathbf{x}^\natural \rangle|^2 + \sigma \epsilon_i, \quad \text{for } i = 1, \dots, m, \quad (14)$$

where $\{\epsilon_i\}_{i=1}^m$ follow i.i.d standard Gaussian distribution, and $\sigma > 0$ determines the noise level.

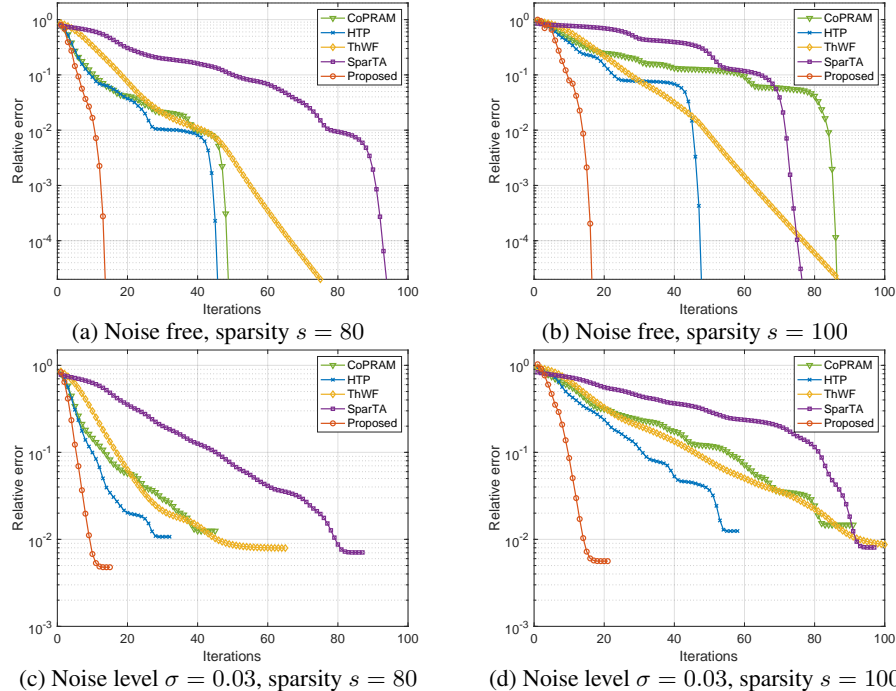


Figure 1: Relative error versus iterations for various algorithms, with fixed signal dimension $n = 5000$ and sample size $m = 3000$. The results represent the average of 100 independent trial runs.

We compare our proposed algorithm with state-of-the-art methods, including ThWF [25], SPARTA [26], CoPRAM [27], and HTP [28]. For ThWF, we set parameters as recommended in [25]. For SPARTA, we set parameters as follows: $\mu = 1$ and $|Z| = \lceil m/6 \rceil$. Both HTP and our algorithm use a step size η of 0.95. The maximum number of iterations for each algorithm is 100. The Relative Error (RE) between the estimated signal \hat{x} and the ground truth x^h is defined as $RE := \frac{\text{dist}(\hat{x}, x^h)}{\|x^h\|}$. A recovery is deemed successful if $RE < 10^{-3}$. We provide additional experimental results on robustness to noise levels and phase transition with varying sparsity levels in Appendix A.

Figure 1 compares the number of iterations required for convergence across various algorithms. In these experiments, we set the number of measurements to $m = 3000$, the dimension of the true signal to $n = 5000$, and the sparsity levels to $s = 80$ and 100. We consider both noise-free measurements and noisy measurements with a noise level of $\sigma = 0.03$. As depicted in Figure 1, all five algorithms perform well under both noise-free and noisy conditions; however, our algorithm converges with significantly fewer iterations compared to state-of-the-art methods.

Table 2: Comparison of running times (in seconds) for different algorithms in the recovery of signals with sparsity levels of 80 and 100 for both noise-free and noisy scenarios.

Methods		ThWF	SPARTA	CoPRAM	HTP	Proposed
Noise free ($\sigma = 0$)	Sparsity 80	0.3630	1.0059	0.9762	0.0813	0.0530
	Sparsity 100	0.6262	1.2966	3.3326	0.2212	0.1024
Noisy ($\sigma = 0.03$)	Sparsity 80	0.2820	1.1082	1.3426	0.1134	0.0803
	Sparsity 100	0.4039	1.6368	4.1006	0.2213	0.1187

Table 2 presents a comparison of the convergence running times for various algorithms, corresponding to the experiments depicted in Figure 1. For noise-free measurements, all algorithms are set to terminate when the iterate satisfies the following condition: $\frac{\text{dist}(x^k, x^h)}{\|x^h\|} < 10^{-3}$, which indicates a successful recovery. In the case of noisy measurements, the termination criterion is set as $\frac{\text{dist}(x^{k+1}, x^k)}{\|x^k\|} < 10^{-3}$. As evidenced by the results in Table 2, our algorithm consistently outperforms state-of-the-art methods in terms of running time, for both noise-free and noisy cases, highlighting its superior efficiency for sparse phase retrieval applications.

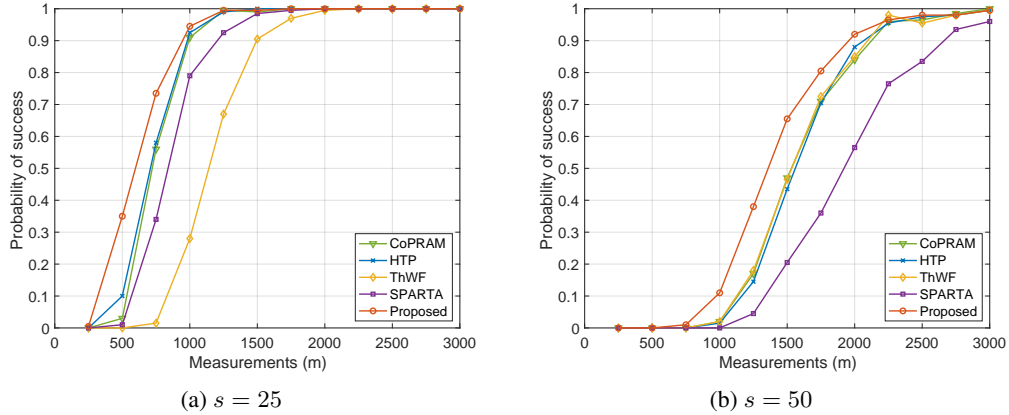


Figure 2: Phase transition performance of various algorithms for signals of dimension $n = 3000$ with sparsity levels $s = 25$ and 50. The results represent the average of 100 independent trial runs.

Figure 2 depicts the phase transitions of different algorithms, with the true signal dimension fixed at $n = 3000$ and sparsity levels set to $s = 25$ and 50. The phase transition graph is generated by evaluating the successful recovery rate of each algorithm over 100 independent trial runs. Figure 2 shows that the probability of successful recovery for each algorithm transitions from zero to one as the sample size m increases. Furthermore, our algorithm consistently outperforms state-of-the-art methods, achieving a higher successful recovery rate across various measurement counts.

279 In practical applications, natural signals may not be inherently sparse; however, their wavelet
 280 coefficients often exhibit sparsity. Figure 3 illustrates the reconstruction performance of a signal from
 281 noisy phaseless measurements, where the true signal, with a dimension of 30,000, exhibits sparsity
 282 and contains 208 nonzero entries under the wavelet transform, using 20,000 samples. The sampling
 283 matrix $\mathbf{A} \in \mathbb{R}^{20,000 \times 30,000}$ is constructed from a random Gaussian matrix and an inverse wavelet
 284 transform generated using four levels of Daubechies 1 wavelet. The noise level is set to $\sigma = 0.03$. To
 285 evaluate recovery accuracy, we use the Peak Signal-to-Noise Ratio (PSNR), defined as:

$$\text{PSNR} = 10 \cdot \log \frac{V^2}{\text{MSE}},$$

286 where V represents the maximum fluctuation in the ground truth signal, and MSE denotes the mean
 287 squared error of the reconstruction. A higher PSNR value generally indicates better reconstruction
 288 quality. As depicted in Figure 3, our proposed algorithm outperforms state-of-the-art methods in
 289 terms of both reconstruction time and PSNR. It achieves a higher PSNR while requiring considerably
 290 less time for reconstruction. In the experiments, the sparsity level is assumed to be unknown, and the
 291 hard thresholding sparsity level is set to 300 for various algorithms.

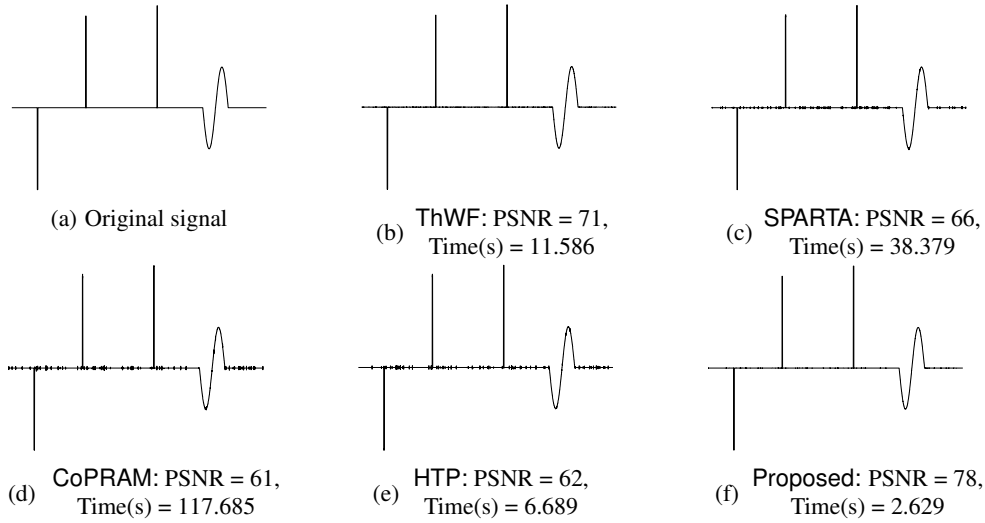


Figure 3: Reconstruction of the signal with a dimension of 30,000 from noisy phaseless measurements by various algorithms. The proposed algorithm requires significantly less time for reconstruction than state-of-the-art methods while preserving the highest PSNR. Time(s) is the running time in seconds.

292 5 Conclusions and Discussions

293 In this paper, we have introduced an efficient Newton projection-based algorithm for sparse phase
 294 retrieval. Our algorithm attains a non-asymptotic quadratic convergence rate while maintaining the
 295 same per-iteration computational complexity as popular first-order methods, which exhibit linear
 296 convergence limitations. Empirical results have demonstrated a significant improvement in the
 297 convergence rate of our algorithm. Furthermore, experiments have revealed that our algorithm excels
 298 in attaining a higher success rate for exact signal recovery with noise-free measurements and provides
 299 superior signal reconstruction performance when dealing with noisy data.

300 Finally, we discuss the limitations of our paper, which also serve as potential avenues for future
 301 research. Both our algorithm and state-of-the-art methods share the same sample complexity
 302 of $\mathcal{O}(s^2 \log n)$ for successful recovery; however, our algorithm requires this complexity in both
 303 the initialization and refinement stages, while state-of-the-art methods require $\mathcal{O}(s^2 \log n)$ for
 304 initialization and $\mathcal{O}(s \log n/s)$ for refinement. Investigating ways to achieve tighter complexity
 305 in our algorithm's refinement stage is a worthwhile pursuit for future studies.

306 Currently, the initialization stage exhibits a sub-optimal sample complexity of $\mathcal{O}(s^2 \log n)$. A key
 307 challenge involves reducing its quadratic dependence on s . Recent work [27] attained a complexity of
 308 $\mathcal{O}(s \log n)$, closer to the information-theoretic limit, but relied on the strong assumption of power law
 309 decay for sparse signals. Developing an initialization method that offers optimal sample complexity
 310 for a broader range of sparse signals is an engaging direction for future research.

References

- [1] Andrew M Maiden and John M Rodenburg. An improved ptychographical phase retrieval algorithm for diffractive imaging. *Ultramicroscopy*, 109(10):1256–1262, 2009.
- [2] Rick P Millane. Phase retrieval in crystallography and optics. *Journal of the Optical Society of America A*, 7(3):394–411, 1990.
- [3] Jianwei Miao, Tetsuya Ishikawa, Qun Shen, and Thomas Earnest. Extending x-ray crystallography to allow the imaging of noncrystalline materials, cells, and single protein complexes. *Annual Review of Physical Chemistry*, 59(1):387–410, 2008.
- [4] Yoav Shechtman, Yonina C Eldar, Oren Cohen, Henry Nicholas Chapman, Jianwei Miao, and Mordechai Segev. Phase retrieval with application to optical imaging: a contemporary overview. *IEEE Signal Processing Magazine*, 32(3):87–109, 2015.
- [5] Matthew Fickus, Dustin G Mixon, Aaron A Nelson, and Yang Wang. Phase retrieval from very few measurements. *Linear Algebra and its Applications*, 449:475–499, 2014.
- [6] Emmanuel J Candes, Xiaodong Li, and Mahdi Soltanolkotabi. Phase retrieval via wirtinger flow: Theory and algorithms. *IEEE Transactions on Information Theory*, 61(4):1985–2007, 2015.
- [7] Emmanuel J Candes, Thomas Strohmer, and Vladislav Voroninski. Phaselift: Exact and stable signal recovery from magnitude measurements via convex programming. *Communications on Pure and Applied Mathematics*, 66(8):1241–1274, 2013.
- [8] Irene Waldspurger, Alexandre d’Aspremont, and Stéphane Mallat. Phase recovery, maxcut and complex semidefinite programming. *Mathematical Programming*, 149(1):47–81, 2015.
- [9] Tom Goldstein and Christoph Studer. Phasemax: Convex phase retrieval via basis pursuit. *IEEE Transactions on Information Theory*, 64(4):2675–2689, 2018.
- [10] Paul Hand and Vladislav Voroninski. An elementary proof of convex phase retrieval in the natural parameter space via the linear program phasemax. *arXiv preprint arXiv:1611.03935*, 2016.
- [11] Praneeth Netrapalli, Prateek Jain, and Sujay Sanghavi. Phase retrieval using alternating minimization. *Advances in Neural Information Processing Systems*, 26, 2013.
- [12] Gang Wang, Georgios B Giannakis, and Yonina C Eldar. Solving systems of random quadratic equations via truncated amplitude flow. *IEEE Transactions on Information Theory*, 64(2):773–794, 2017.
- [13] Jian-Feng Cai and Ke Wei. Solving systems of phaseless equations via riemannian optimization with optimal sampling complexity. *Journal of Computational Mathematics*, 2018.
- [14] Bing Gao and Zhiqiang Xu. Phaseless recovery using the gauss–newton method. *IEEE Transactions on Signal Processing*, 65(22):5885–5896, 2017.
- [15] Chao Ma, Xin Liu, and Zaiwen Wen. Globally convergent levenberg-marquardt method for phase retrieval. *IEEE Transactions on Information Theory*, 65(4):2343–2359, 2018.
- [16] Ke Wei. Solving systems of phaseless equations via Kaczmarz methods: A proof of concept study. *Inverse Problems*, 31(12):125008, 2015.
- [17] Yuejie Chi and Yue M Lu. Kaczmarz method for solving quadratic equations. *IEEE Signal Processing Letters*, 23(9):1183–1187, 2016.
- [18] Ju Sun, Qing Qu, and John Wright. A geometric analysis of phase retrieval. *Foundations of Computational Mathematics*, 18(5):1131–1198, 2018.

- [19] Zhenzhen Li, Jian-Feng Cai, and Ke Wei. Toward the optimal construction of a loss function without spurious local minima for solving quadratic equations. *IEEE Transactions on Information Theory*, 66(5):3242–3260, 2019.
- [20] Yuxin Chen, Yuejie Chi, Jianqing Fan, and Cong Ma. Gradient descent with random initialization: Fast global convergence for nonconvex phase retrieval. *Mathematical Programming*, 176(1):5–37, 2019.
- [21] Irene Waldspurger. Phase retrieval with random gaussian sensing vectors by alternating projections. *IEEE Transactions on Information Theory*, 64(5):3301–3312, 2018.
- [22] Aldo Conca, Dan Edidin, Milena Hering, and Cynthia Vinzant. An algebraic characterization of injectivity in phase retrieval. *Applied and Computational Harmonic Analysis*, 38(2):346–356, 2015.
- [23] Yang Wang and Zhiqiang Xu. Phase retrieval for sparse signals. *Applied and Computational Harmonic Analysis*, 37(3):531–544, 2014.
- [24] Henrik Ohlsson, Allen Yang, Roy Dong, and Shankar Sastry. Cpri—an extension of compressive sensing to the phase retrieval problem. *Advances in Neural Information Processing Systems*, 25, 2012.
- [25] T Tony Cai, Xiaodong Li, and Zongming Ma. Optimal rates of convergence for noisy sparse phase retrieval via thresholded wirtinger flow. *The Annals of Statistics*, 44(5):2221–2251, 2016.
- [26] Gang Wang, Liang Zhang, Georgios B Giannakis, Mehmet Akçakaya, and Jie Chen. Sparse phase retrieval via truncated amplitude flow. *IEEE Transactions on Signal Processing*, 66(2):479–491, 2017.
- [27] Gauri Jagatap and Chinmay Hegde. Sample-efficient algorithms for recovering structured signals from magnitude-only measurements. *IEEE Transactions on Information Theory*, 65(7):4434–4456, 2019.
- [28] Jian-Feng Cai, Jingzhi Li, Xiliang Lu, and Juntao You. Sparse signal recovery from phaseless measurements via hard thresholding pursuit. *Applied and Computational Harmonic Analysis*, 56:367–390, 2022.
- [29] Mahdi Soltanolkotabi. Structured signal recovery from quadratic measurements: Breaking sample complexity barriers via nonconvex optimization. *IEEE Transactions on Information Theory*, 65(4):2374–2400, 2019.
- [30] Deanna Needell and Joel A Tropp. Cosamp: Iterative signal recovery from incomplete and inaccurate samples. *Applied and computational harmonic analysis*, 26(3):301–321, 2009.
- [31] Simon Foucart. Hard thresholding pursuit: an algorithm for compressive sensing. *SIAM Journal on Numerical Analysis*, 49(6):2543–2563, 2011.
- [32] Shenglong Zhou. Gradient projection newton pursuit for sparsity constrained optimization. *Applied and Computational Harmonic Analysis*, 61:75–100, 2022.
- [33] Huishuai Zhang, Yi Zhou, Yingbin Liang, and Yuejie Chi. A nonconvex approach for phase retrieval: Reshaped wirtinger flow and incremental algorithms. *Journal of Machine Learning Research*, 18, 2017.
- [34] Yuxin Chen and Emmanuel J Candès. Solving random quadratic systems of equations is nearly as easy as solving linear systems. *Communications on pure and applied mathematics*, 70(5):822–883, 2017.
- [35] Emmanuel J Candes and Terence Tao. Decoding by linear programming. *IEEE Transactions on Information Theory*, 51(12):4203–4215, 2005.

- 398 [36] Simon Foucart and Holger Rauhut. An invitation to compressive sensing. In *A Mathematical*
399 *Introduction to Compressive Sensing*, pages 1–39. Springer, 2013.
- 400 [37] Thomas Blumensath and Mike E Davies. Iterative hard thresholding for compressed sensing.
401 *Applied and Computational Harmonic Analysis*, 27(3):265–274, 2009.

402 In Appendix A, we present two additional experimental results, illustrating the robustness of the
 403 proposed algorithm against noise in measurements and offering phase transition comparisons. The
 404 proofs for Theorem 3.1 and Theorem 3.4 are provided in Appendix B. Auxiliary lemmas can be
 405 found in Appendix C.

406 A Additional Experiments

407 In this section, we carry out experiments to assess the robustness of the proposed algorithm against
 408 noise in measurements and to evaluate its performance in phase transition for signal recovery.

409 We first investigate the impact of the noise level of measurements on the recovery error of our
 410 proposed algorithm. Noise levels are represented by signal-to-noise ratios (SNR), *i.e.*, $\|\mathbf{x}^{\natural}\|/\sigma$, where
 411 \mathbf{x}^{\natural} is the ground truth signal and σ is a parameter determining the standard deviation of Gaussian
 412 noise, as defined in (14). We set the dimension of the ground truth signal to $n = 2000$, the sparsity
 413 level to $s = 20$, and the number of measurements to $m = 1500$.

414 Figure 4 depicts the relative error of the proposed algorithm as a function of signal-to-noise ratios
 415 (SNR) in dB. We observe a nearly linear decrease in the relative error as the SNR increases, implying
 416 that the recovery error of our algorithm can be controlled by a multiple of the measurement noise level.
 417 This result demonstrates the robustness of our proposed algorithm against noise in measurements.
 418 Additionally, the small error bars shown in Figure 4 emphasize the low variability of recovery errors
 419 of our proposed algorithm.

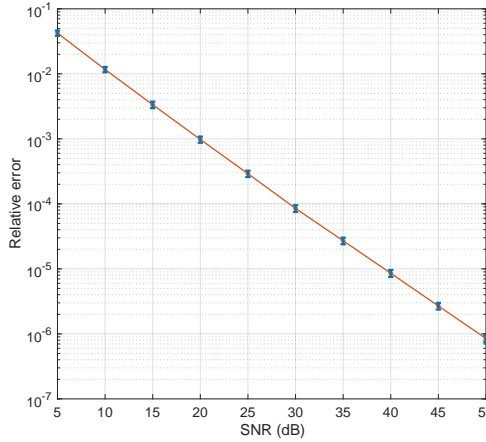


Figure 4: Robustness of the proposed algorithm against additive Gaussian noise. The y -axis represents the relative error of the proposed algorithm, while the x -axis corresponds to the signal-to-noise ratios (SNR) of the measurements. The results are averaged over 100 independent trial runs, with error bars indicating the standard deviation of the recovery error. We set $n = 2000$, $m = 1500$, and $s = 20$.

420 Figure 5 illustrates the success rate of various algorithms as a function of varying sparsity levels s
 421 and the number of measurements m . With a fixed signal dimension of $n = 3000$, we vary the signal
 422 sparsity s from 6 to 120 and the number of measurements m from 150 to 3000. A signal recovery is
 423 considered successful if the relative error $\frac{\text{dist}(\hat{\mathbf{x}}, \mathbf{x}^{\natural})}{\|\mathbf{x}^{\natural}\|} < 10^{-3}$. The gray level of a block represents the
 424 success rate: black corresponds to a 0% successful recovery, white to a 100% successful recovery,
 425 and gray to a rate between 0% and 100%.

426 As shown in Figure 5, our proposed algorithm performs better than the state-of-the-art methods when
 427 the sparsity s is large. In contrast, when s is small, our algorithm, along with SPARTA, CoPRAM,
 428 and HTP, exhibits a slightly better performance than ThWF. These successful recovery rates are
 429 computed by averaging the results over 100 independent trial runs.

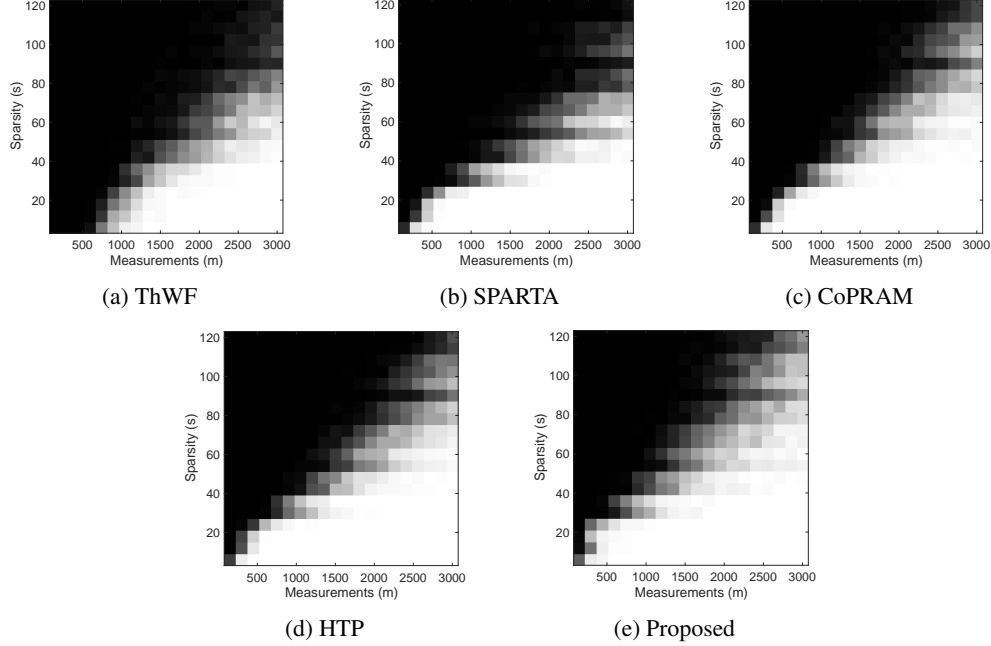


Figure 5: Phase transition comparison for various algorithms with a signal dimension of $n = 3000$. The successful recovery rates are represented by different grey levels in the corresponding block. Black signifies a 0% successful recovery rate, white indicates 100%, and grey represents values between 0% and 100%. All the results are averaged over 100 independent trial runs.

B Proofs

We provide proofs for Theorem 3.1, the recovery guarantee for noise-free measurements, and Theorem 3.4, the recovery guarantee for noisy measurements. First, we present several essential lemmas in Appendix B.1. Following that, we provide the proofs for Theorem 3.1 and Theorem 3.4 separately in Appendix B.2 and Appendix B.3, respectively.

For a more concise representation, we arrange the sampling vectors and observations as follows:

$$\mathbf{A} := [\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_m]^T, \quad \mathbf{y} := [y_1 \ y_2 \ \cdots \ y_m]^T, \quad \mathbf{z} := [z_1 \ z_2 \ \cdots \ z_m]^T, \quad (15)$$

where $z_i = \sqrt{y_i}$, $i = 1, \dots, m$.

B.1 Key Lemmas

In this section, we introduce and prove several essential lemmas that serve as the foundation for proving Theorem 3.1 and Theorem 3.4.

The first lemma asserts that, given a sufficiently large m , the normalized random Gaussian matrix \mathbf{A} obeys the restricted isometry property (RIP) with overwhelming probability. This conclusion is well-established in the compressive sensing literature [35, 36].

Lemma B.1. [36, Theorem 9.27] *Let \mathbf{A} be defined as in (15) with each vector \mathbf{a}_i distributed as the random Gaussian vector $\mathbf{a} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_n)$ independently for $i = 1, 2, \dots, m$. There exists some universal positive constants C'_1, C'_2 such that for any positive integer $r \leq n$ and any $\delta_r \in (0, 1)$, if $m \geq C'_1 \delta_r^{-2} r \log(n/r)$, then $\frac{1}{\sqrt{m}} \mathbf{A}$ satisfies r -RIP with constant δ_r , namely*

$$(1 - \delta_r) \|\mathbf{x}\|^2 \leq \left\| \frac{1}{\sqrt{m}} \mathbf{A} \mathbf{x} \right\|^2 \leq (1 + \delta_r) \|\mathbf{x}\|^2, \quad \forall \mathbf{x} \in \mathbb{R}^n \text{ with } \|\mathbf{x}\|_0 \leq r, \quad (16)$$

with probability at least $1 - e^{-C'_2 m}$.

448 The next lemma, derived from [25], asserts that the induced norm of the submatrix of random
 449 Gaussian matrix \mathbf{A} is bounded by its size.

450 **Lemma B.2.** [25, Lemma A.5] Let $\{\mathbf{a}_i\}_{i=1}^m \subseteq \mathbb{R}^n$ be identically and independently distributed as
 451 $\mathcal{N}(\mathbf{0}, \mathbf{I}_n)$ and define the matrix \mathbf{A} as in (15). Given a support $\mathcal{S} \subseteq [n]$ with cardinality s , with
 452 probability at least $1 - 4m^{-2}$, there holds

$$\|\mathbf{A}_{\mathcal{S}}\|_{2 \rightarrow 4} \leq (3m)^{1/4} + s^{1/2} + 2\sqrt{\log m}. \quad (17)$$

453

454 The following lemma serves as an extension of Lemma 7.4 found in [6].

455 **Lemma B.3.** For any s -sparse vector $\mathbf{x}^{\natural} \in \mathbb{R}^n$ with support $\mathcal{S}^{\natural} = \text{supp}(\mathbf{x}^{\natural})$, let $\{\mathbf{a}_i\}_{i=1}^m \subseteq$
 456 \mathbb{R}^n be identically and independently distributed as $\mathcal{N}(\mathbf{0}, \mathbf{I}_n)$. For any subset $\mathcal{S} \subseteq [n]$ such that
 457 $\text{supp}(\mathbf{x}^{\natural}) \subseteq \mathcal{S}$ and $|\mathcal{S}| \leq r$ for some integer $r \leq n$. With probability at least $1 - m^{-4} - c_a \delta^{-2} m^{-1} -$
 458 $c_b \exp(-c_c \delta^2 m / \log m)$, we have

$$\left\| \frac{1}{m} \sum_{i=1}^m |\mathbf{a}_{i,\mathcal{S}}^T \mathbf{x}^{\natural}|^2 \mathbf{a}_{i,\mathcal{S}} \mathbf{a}_{i,\mathcal{S}}^T - \left(\|\mathbf{x}^{\natural}\|^2 (\mathbf{I}_n)_{\mathcal{S},\mathcal{S}} + 2\mathbf{x}_{\mathcal{S}}^{\natural} (\mathbf{x}_{\mathcal{S}}^{\natural})^T \right) \right\| \leq \delta \|\mathbf{x}^{\natural}\|^2 \quad (18)$$

459 provided $m \geq C(\delta)r \log(n/r)$. Here $C(\delta)$ is a constant depending on δ and c_a, c_b and c_c are positive
 460 absolute constants.

461 *Proof.* If $|\mathcal{S}| = |\mathcal{S}^{\natural}|$, then $\mathcal{S} = \mathcal{S}^{\natural}$ and the result is established by Lemma C.6. We now focus on the
 462 case where $\mathcal{T} := \mathcal{S} \setminus \mathcal{S}^{\natural} \neq \emptyset$. We first define two matrices as follows:

$$\mathbf{G} = \begin{bmatrix} \mathbf{a}_{i,\mathcal{S}^{\natural}}^T \mathbf{a}_{i,\mathcal{S}^{\natural}}^T & \mathbf{a}_{i,\mathcal{S}^{\natural}}^T \mathbf{a}_{i,\mathcal{T}}^T \\ \mathbf{a}_{i,\mathcal{T}}^T \mathbf{a}_{i,\mathcal{S}^{\natural}}^T & \mathbf{a}_{i,\mathcal{T}}^T \mathbf{a}_{i,\mathcal{T}}^T \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} \|\mathbf{x}^{\natural}\|^2 (\mathbf{I}_n)_{\mathcal{S}^{\natural},\mathcal{S}^{\natural}} + 2\mathbf{x}_{\mathcal{S}^{\natural}}^{\natural} (\mathbf{x}_{\mathcal{S}^{\natural}}^{\natural})^T & \mathbf{0} \\ \mathbf{0} & \|\mathbf{x}^{\natural}\|^2 (\mathbf{I}_n)_{\mathcal{T},\mathcal{T}} \end{bmatrix}.$$

463 Then we rephrase the term on the left-hand side of (18) as follows:

$$\begin{aligned} \left\| \frac{1}{m} \sum_{i=1}^m |\mathbf{a}_{i,\mathcal{S}^{\natural}}^T \mathbf{x}^{\natural}|^2 \mathbf{G} - \mathbf{H} \right\| &\leq \left\| \frac{1}{m} \sum_{i=1}^m |\mathbf{a}_{i,\mathcal{S}^{\natural}}^T \mathbf{x}^{\natural}|^2 \mathbf{a}_{i,\mathcal{S}^{\natural}} \mathbf{a}_{i,\mathcal{S}^{\natural}}^T - \left(\|\mathbf{x}^{\natural}\|^2 (\mathbf{I}_n)_{\mathcal{S}^{\natural},\mathcal{S}^{\natural}} + 2\mathbf{x}_{\mathcal{S}^{\natural}}^{\natural} (\mathbf{x}_{\mathcal{S}^{\natural}}^{\natural})^T \right) \right\| \\ &+ \left\| \frac{1}{m} \sum_{i=1}^m |\mathbf{a}_{i,\mathcal{S}^{\natural}}^T \mathbf{x}^{\natural}|^2 \mathbf{a}_{i,\mathcal{S}^{\natural}} \mathbf{a}_{i,\mathcal{T}}^T \right\| + \left\| \frac{1}{m} \sum_{i=1}^m |\mathbf{a}_{i,\mathcal{S}^{\natural}}^T \mathbf{x}^{\natural}|^2 \mathbf{a}_{i,\mathcal{T}} \mathbf{a}_{i,\mathcal{T}}^T - \|\mathbf{x}^{\natural}\|^2 (\mathbf{I}_n)_{\mathcal{T},\mathcal{T}} \right\|. \end{aligned}$$

464 The first term can be bounded with overwhelming probability via a direct application of Lemma C.6
 465 as below whenever $m \geq c_1 \delta^{-2} s \log(n/s)$:

$$\left\| \frac{1}{m} \sum_{i=1}^m |\mathbf{a}_{i,\mathcal{S}^{\natural}}^T \mathbf{x}^{\natural}|^2 \mathbf{a}_{i,\mathcal{S}^{\natural}} \mathbf{a}_{i,\mathcal{S}^{\natural}}^T - \left(\|\mathbf{x}^{\natural}\|^2 (\mathbf{I}_n)_{\mathcal{S}^{\natural},\mathcal{S}^{\natural}} + 2\mathbf{x}_{\mathcal{S}^{\natural}}^{\natural} (\mathbf{x}_{\mathcal{S}^{\natural}}^{\natural})^T \right) \right\| \leq \delta/4.$$

466 For the other two terms, it is enough to consider $\mathbf{x}^{\natural} = \mathbf{e}_1$ because the unitary invariance of the
 467 Gaussian measure and rescaling. Then for a prefixed subset \mathcal{S} (thus \mathcal{T} is also fixed), there exists a
 468 unit vector $\mathbf{u} \in \mathbb{R}^n$ with $\text{supp}(\mathbf{u}) \subseteq \mathcal{T}$ such that the second term is equal to

$$\left\| \frac{1}{m} \sum_{i=1}^m (a_i(1))^3 \mathbf{a}_{i,\mathcal{T}}^T \right\| = \frac{1}{m} \sum_{i=1}^m (a_i(1))^3 (\mathbf{a}_i^T \mathbf{u}).$$

469 We emphasize that $\mathbf{a}_i^T \mathbf{u} = \mathbf{a}_{i,\mathcal{T}}^T \mathbf{u}_{\mathcal{T}}$ is a random Gaussian variable distributed as $\mathcal{N}(0, 1)$ for all
 470 $\mathbf{u} \in \mathbb{R}^n$, $\|\mathbf{u}\| = 1$, $\text{supp}(\mathbf{u}) \subseteq \mathcal{T}$ and independent of $a_i(1)$ for all $i \in [m]$. So for one realization of
 471 $\{a_i(1)\}$, Lemma C.1 (Hoeffding-type inequality) implies

$$\mathbb{P} \left\{ \left| \frac{1}{m} \sum_{i=1}^m (a_i(1))^3 \mathbf{a}_i^T \mathbf{u} \right| > t \right\} \leq e \exp \left(- \frac{c_2 m^2 t^2}{\sum_{i=1}^m |a_i(1)|^6} \right),$$

472 for any $t > 0$. Define the set \mathbb{W}_r^n to be a collection of all the index set in $[n]$ with cardinality no
 473 larger than r , i.e. $\mathbb{W}_r^n := \{\mathcal{S} \subseteq [n] : |\mathcal{S}| \leq r\}$. Note that the cardinality of \mathbb{W}_r^n can be bounded by

474 $\sum_{j=1}^r \binom{n}{j} \leq \left(\frac{en}{r}\right)^r$. Taking $t = \delta/8$, together with a union bound on the set \mathbb{W}_r^n , we obtain for any
 475 subset $\mathcal{T} = \mathcal{S} \setminus \mathcal{S}^\natural$,

$$\mathbb{P}\left\{\left\|\frac{1}{m} \sum_{i=1}^m (a_i(1))^3 \mathbf{a}_{i,\mathcal{T}}^T\right\| > \delta/4\right\} \leq e \exp\left(-\frac{c_2 m^2 \delta^2}{64 \sum_{i=1}^m |a_i(1)|^6} + 6r \log(n/r)\right).$$

476 Now with an application of Chebyshev's inequality, we have $\sum_{i=1}^m |a_i(1)|^6 \leq 20m$ with probability
 477 at least $1 - c_3 m^{-1}$. Substituting this into the above, we conclude that for any subset $\mathcal{S} \subseteq \mathbb{W}_r^n$,
 478 whenever $m \geq c_4 \delta^{-2} r \log(n/r)$ for some sufficiently large c_4 ,

$$\left\|\frac{1}{m} \sum_{i=1}^m (a_i(1))^3 \mathbf{a}_{i,\mathcal{T}}^T\right\| \leq \delta/4,$$

479 with probability at least $1 - c_3 m^{-1} - e \exp(-c_5 \delta^2 m)$.

480 Assuming $\mathbf{x}^\natural = \mathbf{e}_1$ and fixing a subset $\mathcal{S} \subseteq [n]$ (thus \mathcal{T} is also fixed), we can simplify the last term
 481 as follows:

$$\left\|\frac{1}{m} \sum_{i=1}^m |a_i(1)|^2 \mathbf{a}_{i,\mathcal{T}} \mathbf{a}_{i,\mathcal{T}}^T - (\mathbf{I}_n)_{\mathcal{T},\mathcal{T}}\right\| \leq \left\|\frac{1}{m} \sum_{i=1}^m |a_i(1)|^2 (\mathbf{a}_{i,\mathcal{T}} \mathbf{a}_{i,\mathcal{T}}^T - \mathbf{I}_{|\mathcal{T}|})\right\| + \left|\frac{1}{m} \sum_{i=1}^m |a_i(1)|^2 - 1\right|,$$

482 for which there exists a unit vector $\mathbf{u} \in \mathbb{R}^n$ with $\text{supp}(\mathbf{u}) \subseteq \mathcal{T}$ such that

$$\left\|\frac{1}{m} \sum_{i=1}^m |a_i(1)|^2 (\mathbf{a}_{i,\mathcal{T}} \mathbf{a}_{i,\mathcal{T}}^T - \mathbf{I}_{|\mathcal{T}|})\right\| = \frac{1}{m} \sum_{i=1}^m |a_i(1)|^2 \left|(\mathbf{a}_{i,\mathcal{T}}^T \mathbf{u}_{\mathcal{T}})^2 - 1\right|.$$

483 Note that $\{a_i(1)\}$ is independent of $\{\mathbf{a}_{i,\mathcal{T}}\}$, an application of Bernstein's inequality implies

$$\mathbb{P}\left\{\frac{1}{m} \sum_{i=1}^m |a_i(1)|^2 \left|(\mathbf{a}_{i,\mathcal{T}}^T \mathbf{u}_{\mathcal{T}})^2 - 1\right| > t\right\} \leq 2 \exp(-c_6 \min(d_1, d_2)),$$

484 where $d_1 = \frac{t^2}{c_7^2 \sum_{i=1}^m |a_i(1)|^4}$, and $d_2 = \frac{t}{c_7 \max_{i \in [m]} |a_i(1)|^2}$. Taking $t = \delta/8$, together with a union
 485 bound on the subset \mathbb{W}_r^n , we obtain for any subset $\mathcal{S} \in \mathbb{W}_r^n$,

$$\mathbb{P}\left\{\left\|\frac{1}{m} \sum_{i=1}^m |a_i(1)|^2 (\mathbf{a}_{i,\mathcal{T}} \mathbf{a}_{i,\mathcal{T}}^T - \mathbf{I}_{|\mathcal{T}|})\right\| > \delta/4\right\} \leq 2 \exp(-c_6 \min(d_3, d_4) + 6r \log(n/r)),$$

486 where $d_3 = \frac{m^2 \delta^2}{64 c_7^2 \sum_{i=1}^m |a_i(1)|^4}$ and $d_4 = \frac{m \delta}{8 c_7 \max_{i \in [m]} |a_i(1)|^2}$. Applying Chebyshev's inequality and
 487 the union bound, we obtain:

$$\sum_{i=1}^m |a_i(1)|^4 \leq 10m \quad \text{and} \quad \max_{i \in [m]} |a_i(1)|^2 \leq 10 \log m$$

488 hold with probability at least $1 - c_8 m^{-1} - m^{-4}$. To conclude, for any subset $\mathcal{S} \in \mathbb{W}_r^n$, when
 489 $m \geq c_9 \delta^{-2} r \log(n/r)$ for some sufficiently large constant c_9 ,

$$\left\|\frac{1}{m} \sum_{i=1}^m |a_i(1)|^2 (\mathbf{a}_{i,\mathcal{T}} \mathbf{a}_{i,\mathcal{T}}^T - \mathbf{I}_{|\mathcal{T}|})\right\| \leq \delta/4$$

490 with probability at least $1 - c_8 m^{-1} - m^{-4} - 2 \exp(-c_{10} \delta^2 m / \log m)$. Finally, an application of
 491 Chebyshev's inequality implies

$$\left|\frac{1}{m} \sum_{i=1}^m |a_i(1)|^2 - 1\right| \leq \delta/4$$

492 with probability at least $1 - c_{11} \delta^{-2} m^{-1}$. The proof is finished by combining the above bounds and
 493 probabilities. \square

494 The subsequent result concerning the spectral norm of submatrices of $\frac{1}{\sqrt{m}}\mathbf{A}$ can be deduced by
 495 employing the restricted isometry property of matrix $\frac{1}{\sqrt{m}}\mathbf{A}$.

496 **Lemma B.4.** [30, Proposition 3.1] Assume the matrix \mathbf{A} satisfies RIP inequality (16) with $r = s$
 497 and $r = s'$. Then for any disjoint subsets \mathcal{S} and \mathcal{T} of $\{1, 2, \dots, m\}$ satisfying $|\mathcal{S}| \leq s$ and $|\mathcal{T}| \leq s'$,
 498 the following inequalities hold:

$$\left\| \frac{1}{\sqrt{m}} \mathbf{A}_{\mathcal{S}}^T \right\| \leq \sqrt{1 + \delta_s}, \quad (19)$$

$$\left\| \frac{1}{m} \mathbf{A}_{\mathcal{S}}^T \mathbf{A}_{\mathcal{T}} \right\| \leq \delta_{s+s'}, \quad (20)$$

$$\forall \mathbf{u} \in \mathbb{R}^{|\mathcal{S}|}, \quad (1 - \delta_s) \|\mathbf{u}\| \leq \left\| \frac{1}{m} \mathbf{A}_{\mathcal{S}}^T \mathbf{A}_{\mathcal{S}} \mathbf{u} \right\| \leq (1 + \delta_s) \|\mathbf{u}\|, \quad (21)$$

499

500 The following lemma is derived through the application of Hölder's inequality and Lemma B.2:

501 **Lemma B.5.** Given two vectors $\mathbf{x}, \mathbf{z} \in \mathbb{R}^n$ each with sparsity no larger than s and two subsets
 502 $\mathcal{S}, \mathcal{T} \subseteq [n]$. Define the subset $\mathcal{R} := \mathcal{S} \cup \mathcal{T} \cup \text{supp}(\mathbf{x}) \cup \text{supp}(\mathbf{z})$. Under the event (17) with support
 503 \mathcal{R} , there holds

$$\|\nabla_{\mathcal{S}, \mathcal{T}}^2 f_I(\mathbf{x}) - \nabla_{\mathcal{S}, \mathcal{T}}^2 f_I(\mathbf{z})\| \leq \frac{3}{m} \left((3m)^{1/4} + (|\mathcal{R}|)^{1/2} + 2\sqrt{\log m} \right)^4 \|\mathbf{x} + \mathbf{z}\| \|\mathbf{x} - \mathbf{z}\|. \quad (22)$$

504 *Proof.* A simple calculation yields

$$\nabla_{\mathcal{S}, \mathcal{T}}^2 f_I(\mathbf{x}) - \nabla_{\mathcal{S}, \mathcal{T}}^2 f_I(\mathbf{z}) = \frac{3}{m} \mathbf{A}_{\mathcal{S}}^T D(|\mathbf{A}\mathbf{x}|^2 - |\mathbf{A}\mathbf{z}|^2) \mathbf{A}_{\mathcal{T}}.$$

505 There exist unit vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ with support $\text{supp}(\mathbf{u}) \subseteq \mathcal{S}$ and $\text{supp}(\mathbf{v}) \subseteq \mathcal{T}$ such that

$$\begin{aligned} & \|\nabla_{\mathcal{S}, \mathcal{T}}^2 f_I(\mathbf{x}) - \nabla_{\mathcal{S}, \mathcal{T}}^2 f_I(\mathbf{z})\| \\ &= \frac{3}{m} \left\| \mathbf{A}_{\mathcal{S}}^T D(|\mathbf{A}\mathbf{x}|^2 - |\mathbf{A}\mathbf{z}|^2) \mathbf{A}_{\mathcal{T}} \right\| \\ &= \frac{3}{m} \left| \sum_{i=1}^m (|\mathbf{a}_i^T \mathbf{x}|^2 - |\mathbf{a}_i^T \mathbf{z}|^2) (\mathbf{a}_{i, \mathcal{S}}^T \mathbf{u}_{\mathcal{S}}) (\mathbf{a}_{i, \mathcal{T}}^T \mathbf{v}_{\mathcal{T}}) \right| \\ &= \frac{3}{m} \left| \sum_{i=1}^m (|\mathbf{a}_{i, \mathcal{R}}^T \mathbf{x}_{\mathcal{R}}|^2 - |\mathbf{a}_{i, \mathcal{R}}^T \mathbf{z}_{\mathcal{R}}|^2) (\mathbf{a}_{i, \mathcal{R}}^T \mathbf{u}_{\mathcal{R}}) (\mathbf{a}_{i, \mathcal{R}}^T \mathbf{v}_{\mathcal{R}}) \right| \\ &\leq \frac{3}{m} \sum_{i=1}^m |(\mathbf{a}_{i, \mathcal{R}}^T (\mathbf{x}_{\mathcal{R}} + \mathbf{z}_{\mathcal{R}})) (\mathbf{a}_{i, \mathcal{R}}^T (\mathbf{x}_{\mathcal{R}} - \mathbf{z}_{\mathcal{R}})) (\mathbf{a}_{i, \mathcal{R}}^T \mathbf{u}_{\mathcal{R}}) (\mathbf{a}_{i, \mathcal{R}}^T \mathbf{v}_{\mathcal{R}})| \\ &\leq \frac{3}{m} \left(\sum_{i=1}^m (\mathbf{a}_{i, \mathcal{R}}^T (\mathbf{x}_{\mathcal{R}} + \mathbf{z}_{\mathcal{R}}))^4 \right)^{\frac{1}{4}} \left(\sum_{i=1}^m (\mathbf{a}_{i, \mathcal{R}}^T (\mathbf{x}_{\mathcal{R}} - \mathbf{z}_{\mathcal{R}}))^4 \right)^{\frac{1}{4}} \left(\sum_{i=1}^m (\mathbf{a}_{i, \mathcal{R}}^T \mathbf{u}_{\mathcal{R}})^4 \right)^{\frac{1}{4}} \left(\sum_{i=1}^m (\mathbf{a}_{i, \mathcal{R}}^T \mathbf{v}_{\mathcal{R}})^4 \right)^{\frac{1}{4}} \\ &\leq \frac{3}{m} \|\mathbf{A}_{\mathcal{R}}\|_{2 \rightarrow 4}^4 \|\mathbf{x} + \mathbf{z}\| \|\mathbf{x} - \mathbf{z}\| \\ &\leq \frac{3}{m} \left((3m)^{\frac{1}{4}} + (|\mathcal{R}|)^{1/2} + 2\sqrt{\log m} \right)^4 \|\mathbf{x} + \mathbf{z}\| \|\mathbf{x} - \mathbf{z}\|, \end{aligned}$$

506 where the second inequality is implied by Hölder's inequality and the fourth inequality is implied by
 507 Lemma B.2. \square

508 The following inequalities in Lemma B.6 can be derived using Lemmas B.5, B.3, C.5, and C.4.

509 **Lemma B.6.** Let $\mathbf{x}^{\natural} \in \mathbb{R}^n$ be any signal with sparsity $\|\mathbf{x}^{\natural}\|_0 \leq s$ and support \mathcal{S}^{\natural} . Let $\{\mathbf{a}_i\}_{i=1}^m$
 510 be random Gaussian vectors identically and independently distributed as $\mathcal{N}(\mathbf{0}, \mathbf{I}_n)$. Define \mathbf{A}, \mathbf{y}
 511 and $f_I(\mathbf{x})$ as in (15) and (4) respectively. Given two subsets $\mathcal{S}, \mathcal{T} \subseteq [n]$ satisfying $|\mathcal{S}| \leq s$ and
 512 $|\mathcal{T}| \leq s$. Then if $m \geq 30s^2$, for any s -sparse vector $\mathbf{x} \in \mathbb{R}^n$ with $\text{supp}(\mathbf{x}) \subseteq \mathcal{T}$ and obeying
 513 $\text{dist}(\mathbf{x}, \mathbf{x}^{\natural}) \leq \gamma \|\mathbf{x}^{\natural}\|$ with $0 < \gamma \leq 0.1$, under events (17) and (18), the following inequalities hold:

514 (i)
$$l_1 \|\mathbf{u}\| \leq \|\nabla_{\mathcal{S}}^2 f_I(\mathbf{x}) \mathbf{u}\| \leq l_2 \|\mathbf{u}\|, \quad \forall \mathbf{u} \in \mathbb{R}^{|\mathcal{S}|}, \quad (23)$$

515 where $l_1 := (2 - 2\delta - 10\gamma(2 + \gamma)) \|\mathbf{x}^\natural\|^2$ and $l_2 := (6 + 2\delta + 10\gamma(2 + \gamma)) \|\mathbf{x}^\natural\|^2$.

516 (ii)
$$\left\| \nabla_{\mathcal{S}, \mathcal{S}^\natural \setminus \mathcal{S}}^2 f_I(\mathbf{x}) \right\| \leq l_3 := (2 + 2\delta + 10\gamma(2 + \gamma)) \|\mathbf{x}^\natural\|^2. \quad (24)$$

517 *Proof.* (i) Define $\mathcal{R} = \mathcal{T} \cup \mathcal{S} \cup \mathcal{S}^\natural$. Then $|\mathcal{R}| \leq 3s$. Applying Lemma B.5 yields that

$$\begin{aligned} \left\| \nabla_{\mathcal{R}}^2 f_I(\mathbf{x}) - \nabla_{\mathcal{R}}^2 f_I(\mathbf{x}^\natural) \right\| &\leq \frac{3}{m} \left((3m)^{1/4} + (3s)^{1/2} + 2\sqrt{\log m} \right)^4 \|\mathbf{x} + \mathbf{x}^\natural\| \|\mathbf{x} - \mathbf{x}^\natural\| \\ &\leq 10 \|\mathbf{x} + \mathbf{x}^\natural\| \|\mathbf{x} - \mathbf{x}^\natural\|, \end{aligned}$$

518 provided $m \geq 30s^2$. Furthermore, it follows from Weyl theorem in Lemma C.4, the inequalities in
519 Lemma B.5 and Lemma B.3 and the interlacing inequality that

$$\begin{aligned} \lambda_{\min}(\nabla_{\mathcal{S}}^2 f_I(\mathbf{x})) &\geq \lambda_{\min}(\nabla_{\mathcal{R}}^2 f_I(\mathbf{x})) \\ &\geq \lambda_{\min}(\nabla_{\mathcal{R}}^2 f_I(\mathbf{x}^\natural)) - \left\| \nabla_{\mathcal{R}}^2 f_I(\mathbf{x}) - \nabla_{\mathcal{R}}^2 f_I(\mathbf{x}^\natural) \right\| \\ &\geq \lambda_{\min}(\mathbb{E}[\nabla_{\mathcal{R}}^2 f_I(\mathbf{x}^\natural)]) - \left\| \nabla_{\mathcal{R}}^2 f_I(\mathbf{x}^\natural) - \mathbb{E}[\nabla_{\mathcal{R}}^2 f_I(\mathbf{x}^\natural)] \right\| - 10 \|\mathbf{x} + \mathbf{x}^\natural\| \|\mathbf{x} - \mathbf{x}^\natural\| \\ &\geq 2\|\mathbf{x}^\natural\|^2 - 2\delta\|\mathbf{x}^\natural\|^2 - 10\|\mathbf{x} + \mathbf{x}^\natural\| \|\mathbf{x} - \mathbf{x}^\natural\| \\ &\geq (2 - 2\delta - 10\gamma(2 + \gamma)) \|\mathbf{x}^\natural\|^2, \end{aligned}$$

520 where the last inequality is implied by $\text{dist}(\mathbf{x}, \mathbf{x}^\natural) \leq \gamma\|\mathbf{x}^\natural\|$. Similarly, the upper bound for the
521 largest eigenvalue of $\nabla_{\mathcal{S}}^2 f_I(\mathbf{x})$ can be bounded as follows:

$$\begin{aligned} \lambda_{\max}(\nabla_{\mathcal{S}}^2 f_I(\mathbf{x})) &\leq \lambda_{\max}(\nabla_{\mathcal{R}}^2 f_I(\mathbf{x})) \\ &\leq \lambda_{\max}(\nabla_{\mathcal{R}}^2 f_I(\mathbf{x}^\natural)) + \left\| \nabla_{\mathcal{R}}^2 f_I(\mathbf{x}) - \nabla_{\mathcal{R}}^2 f_I(\mathbf{x}^\natural) \right\| \\ &\leq \lambda_{\max}(\mathbb{E}[\nabla_{\mathcal{R}}^2 f_I(\mathbf{x}^\natural)]) + \left\| \nabla_{\mathcal{R}}^2 f_I(\mathbf{x}^\natural) - \mathbb{E}[\nabla_{\mathcal{R}}^2 f_I(\mathbf{x}^\natural)] \right\| + 10 \|\mathbf{x} + \mathbf{x}^\natural\| \|\mathbf{x} - \mathbf{x}^\natural\| \\ &\leq 6\|\mathbf{x}^\natural\|^2 + 2\delta\|\mathbf{x}^\natural\|^2 + 10\|\mathbf{x} + \mathbf{x}^\natural\| \|\mathbf{x} - \mathbf{x}^\natural\| \\ &\leq (6 + 2\delta + 10\gamma(2 + \gamma)) \|\mathbf{x}^\natural\|^2. \end{aligned}$$

522 (ii) Define $\mathcal{R} = \mathcal{T} \cup \mathcal{S} \cup \mathcal{S}^\natural$. Thus $|\mathcal{R}| \leq 3s$. For the disjoint subsets \mathcal{S} and $\mathcal{S}^\natural \setminus \mathcal{S}$, we consider
523 $\nabla_{\mathcal{S}, \mathcal{S}^\natural \setminus \mathcal{S}}^2 f_I(\mathbf{x})$, which is a submatrix of $\nabla_{\mathcal{R}}^2 f_I(\mathbf{x}) - 4\|\mathbf{x}^\natural\|^2 \mathbf{I}$. We note that the spectral norm of a
524 submatrix never exceeds the norm of the entire matrix. By employing the result from part (i), we can
525 deduce that

$$\begin{aligned} \left\| \nabla_{\mathcal{S}, \mathcal{S}^\natural \setminus \mathcal{S}}^2 f_I(\mathbf{x}) \right\| &\leq \left\| \nabla_{\mathcal{R}}^2 f_I(\mathbf{x}) - 4\|\mathbf{x}^\natural\|^2 \mathbf{I} \right\| \\ &\leq \max\{6 + 2\delta + 10\gamma(2 + \gamma) - 4, 4 - (2 - 2\delta - 10\gamma(2 + \gamma))\} \cdot \|\mathbf{x}^\natural\|^2 \\ &= (2 + 2\delta + 10\gamma(2 + \gamma)) \|\mathbf{x}^\natural\|^2, \end{aligned}$$

526 completing the proof. □

527 The next lemma is adapted from Lemma 3 in [28].

528 **Lemma B.7.** Let $\{\mathbf{x}^k, \mathcal{T}_k\}_{k \geq 1}$ be generated by the Algorithm 1. Let $\mathbf{z}^k := \mathbf{z} \odot \text{sgn}(\mathbf{A}\mathbf{x}^k)$. Assume
529 $\|\mathbf{x}^k - \mathbf{x}^\natural\| \leq \gamma\|\mathbf{x}^\natural\|$. Then under the event (16) with $r = s, 2s$ and the event (41), it holds that

$$\frac{1}{m} \left\| \mathbf{A}_{\mathcal{T}_{k+1}}^T (\mathbf{z}^k - \mathbf{A}\mathbf{x}^\natural) \right\| \leq \sqrt{C_\gamma(1 + \delta_s)} \|\mathbf{x}^k - \mathbf{x}^\natural\|,$$

530 where $C_\gamma = \frac{4}{(1-\gamma^2)}(\epsilon_0 + \gamma\sqrt{\frac{21}{20}})^2$ with $\epsilon_0 = 10^{-3}$.

531 *Proof.* Define the sets $\{\mathcal{G}_k\}_{k \geq 1}$ as follows:

$$\mathcal{G}_k = \{i \mid \text{sgn}(\mathbf{a}_i^T \mathbf{x}^k) = \text{sgn}(\mathbf{a}_i^T \mathbf{x}^\natural), 1 \leq i \leq m\}.$$

532 With $\mathbf{z}^k := \mathbf{z} \odot \text{sgn}(\mathbf{A}\mathbf{x}^k)$, where \mathbf{z} is defined in (15), we deduce that

$$\begin{aligned}
\left(\frac{1}{\sqrt{m}} \|\mathbf{z}^k - \mathbf{A}\mathbf{x}^\natural\| \right)^2 &= \frac{1}{m} \sum_{i=1}^m (\text{sgn}(\mathbf{a}_i^T \mathbf{x}^k) - \text{sgn}(\mathbf{a}_i^T \mathbf{x}^\natural))^2 |\mathbf{a}_i^T \mathbf{x}^\natural|^2 \\
&\leq \frac{4}{m} \sum_{i \in \mathcal{G}_k^c} |\mathbf{a}_i^T \mathbf{x}^\natural|^2 \cdot \mathbf{1}_{(\mathbf{a}_i^T \mathbf{x}^k)(\mathbf{a}_i^T \mathbf{x}^\natural) \leq 0} \\
&\leq \underbrace{\frac{4}{(1-\gamma)^2} (\epsilon_0 + \gamma \sqrt{\frac{21}{20}})^2}_{C_\gamma} \|\mathbf{x}^k - \mathbf{x}^\natural\|^2,
\end{aligned} \tag{25}$$

533 where the first inequality follows from $|\text{sgn}(\mathbf{a}_i^T \mathbf{x}^k) - \text{sgn}(\mathbf{a}_i^T \mathbf{x}^\natural)| \leq 2$ and $\text{sgn}(\mathbf{a}_i^T \mathbf{x}^k) -$
534 $\text{sgn}(\mathbf{a}_i^T \mathbf{x}^\natural) = 0$ on \mathcal{G}_k , and the second inequality follows from Lemma C.7. Together with (19) in
535 Lemma B.4, (25) leads to

$$\frac{1}{m} \|\mathbf{A}_{\mathcal{T}_{k+1}}^T (\mathbf{z}^k - \mathbf{A}\mathbf{x}^\natural)\| \leq \sqrt{C_\gamma(1 + \delta_s)} \|\mathbf{x}^k - \mathbf{x}^\natural\|,$$

536 completing the proof. \square

537 Given a point \mathbf{x}^k that is close to the ground truth signal, the following lemma provides an upper
538 bound on the estimation error for the vector obtained after one iteration of IHT, as described in [37].
539 To make this paper self-contained, we include the details of the proof for the reader's convenience.

540 **Lemma B.8.** *Given an s -sparse estimate \mathbf{x}^k satisfying $\|\mathbf{x}^k - \mathbf{x}^\natural\| \leq \gamma \|\mathbf{x}^\natural\|$. Define the vector*
541 *obtained by one iteration of IHT with stepsize η to be*

$$\mathbf{u}^k := \mathcal{H}_s(\mathbf{x}^k - \eta \nabla f_A(\mathbf{x}^k)).$$

542 Under the RIP event (16), it holds that

$$\|\mathbf{u}^k - \mathbf{x}^\natural\| \leq \zeta \|\mathbf{x}^k - \mathbf{x}^\natural\|,$$

543 where $\zeta = 2 \left(\sqrt{2} \max\{\eta \delta_{3s}, 1 - \eta(1 - \delta_{2s})\} + \eta \sqrt{C_\gamma(1 + \delta_{2s})} \right)$.

544 *Proof.* Define $\mathcal{S}^\natural := \text{supp}(\mathbf{x}^\natural)$, $\mathcal{T}_{k+1} := \mathcal{S}_{k+1} \cup \mathcal{S}^\natural$, and

$$\mathbf{v}^k := \mathbf{x}^k - \eta \nabla f_A(\mathbf{x}^k).$$

545 Since \mathbf{u}^k is the best s -term approximation of \mathbf{v}^k , we have

$$\|\mathbf{u}^k - \mathbf{v}^k\| \leq \|\mathbf{x}^\natural - \mathbf{v}^k\|,$$

546 which implies

$$\|\mathbf{u}_{\mathcal{T}_{k+1}}^k - \mathbf{v}_{\mathcal{T}_{k+1}}^k\| \leq \|\mathbf{x}_{\mathcal{T}_{k+1}}^\natural - \mathbf{v}_{\mathcal{T}_{k+1}}^k\|,$$

547 because of the relation $\text{supp}(\mathbf{u}^k) \subseteq \mathcal{T}_{k+1}$ and $\text{supp}(\mathbf{x}^\natural) \subseteq \mathcal{T}_{k+1}$. Then, it follows from the triangle
548 inequality and the inequality above that

$$\begin{aligned}
\|\mathbf{u}^k - \mathbf{x}^\natural\| &= \|\mathbf{u}_{\mathcal{T}_{k+1}}^k - \mathbf{x}_{\mathcal{T}_{k+1}}^\natural\| = \left\| \mathbf{u}_{\mathcal{T}_{k+1}}^k - \mathbf{v}_{\mathcal{T}_{k+1}}^k + \mathbf{v}_{\mathcal{T}_{k+1}}^k - \mathbf{x}_{\mathcal{T}_{k+1}}^\natural \right\| \\
&\leq \|\mathbf{u}_{\mathcal{T}_{k+1}}^k - \mathbf{v}_{\mathcal{T}_{k+1}}^k\| + \|\mathbf{v}_{\mathcal{T}_{k+1}}^k - \mathbf{x}_{\mathcal{T}_{k+1}}^\natural\| \\
&\leq 2\|\mathbf{v}_{\mathcal{T}_{k+1}}^k - \mathbf{x}_{\mathcal{T}_{k+1}}^\natural\|.
\end{aligned} \tag{26}$$

549 Define $\mathbf{z}^k := \mathbf{z} \odot \text{sgn}(\mathbf{A}\mathbf{x}^k)$ with \mathbf{z} as in (15). Using the definition of \mathbf{v}^k , a direct calculation yields

$$\begin{aligned}
\|\mathbf{v}_{\mathcal{T}_{k+1}}^k - \mathbf{x}_{\mathcal{T}_{k+1}}^\natural\| &= \left\| \mathbf{x}_{\mathcal{T}_{k+1}}^k - \mathbf{x}_{\mathcal{T}_{k+1}}^\natural - \frac{\eta}{m} \mathbf{A}_{\mathcal{T}_{k+1}}^T \mathbf{A}(\mathbf{x}^k - \mathbf{x}^\natural) + \frac{\eta}{m} \mathbf{A}_{\mathcal{T}_{k+1}}^T (\mathbf{z}^k - \mathbf{A}\mathbf{x}^\natural) \right\| \\
&\leq \underbrace{\left\| \frac{\eta}{m} \mathbf{A}_{\mathcal{T}_{k+1}}^T (\mathbf{z}^k - \mathbf{A}\mathbf{x}^\natural) \right\|}_{I_1} + \underbrace{\left\| \left(\mathbf{I} - \frac{\eta}{m} \mathbf{A}_{\mathcal{T}_{k+1}}^T \mathbf{A}_{\mathcal{T}_{k+1}} \right) (\mathbf{x}_{\mathcal{T}_{k+1}}^k - \mathbf{x}_{\mathcal{T}_{k+1}}^\natural) \right\|}_{I_2} \\
&\quad + \underbrace{\left\| \frac{\eta}{m} \mathbf{A}_{\mathcal{T}_{k+1}}^T \mathbf{A}_{\mathcal{T}_k \setminus \mathcal{T}_{k+1}} [\mathbf{x}^k - \mathbf{x}^\natural]_{\mathcal{T}_k \setminus \mathcal{T}_{k+1}} \right\|}_{I_3}.
\end{aligned}$$

550 We will now proceed to estimate I_1 , I_2 , and I_3 sequentially.

551 For I_1 : An application of Lemma B.7 yields that

$$\left\| \frac{\eta}{m} \mathbf{A}_{\mathcal{T}_{k+1}}^T (\mathbf{z}^k - \mathbf{A}\mathbf{x}^\natural) \right\| \leq \eta \sqrt{C_\gamma(1 + \delta_{2s})} \|\mathbf{x}^k - \mathbf{x}^\natural\|. \quad (27)$$

552 For I_2 : Let $\eta \in (0, \frac{1}{1 + \delta_{2s}})$. It follows from (21) in Lemma B.4 as well as Weyl's inequality that

$$(1 - \eta(1 + \delta_{2s})) \|\mathbf{u}\| \leq \left\| \left(\mathbf{I} - \frac{\eta}{m} \mathbf{A}_{\mathcal{T}_{k+1}}^T \mathbf{A}_{\mathcal{T}_{k+1}} \right) \mathbf{u} \right\| \leq (1 - \eta(1 - \delta_{2s})) \|\mathbf{u}\|,$$

553 for any $\mathbf{u} \in \mathbb{R}^{|\mathcal{T}_{k+1}|}$, which deducts

$$\left\| \left(\mathbf{I} - \frac{\eta}{m} \mathbf{A}_{\mathcal{T}_{k+1}}^T \mathbf{A}_{\mathcal{T}_{k+1}} \right) (\mathbf{x}_{\mathcal{T}_{k+1}}^k - \mathbf{x}_{\mathcal{T}_{k+1}}^\natural) \right\| \leq (1 - \eta(1 - \delta_{2s})) \left\| \mathbf{x}_{\mathcal{T}_{k+1}}^k - \mathbf{x}_{\mathcal{T}_{k+1}}^\natural \right\|.$$

554 For I_3 : Eq.(20) in Lemma B.4 implies that

$$\left\| \frac{\eta}{m} \mathbf{A}_{\mathcal{T}_{k+1}}^T \mathbf{A}_{\mathcal{T}_k \setminus \mathcal{T}_{k+1}} [\mathbf{x}^k - \mathbf{x}^\natural]_{\mathcal{T}_k \setminus \mathcal{T}_{k+1}} \right\| \leq \eta \delta_{3s} \|\mathbf{x}^k - \mathbf{x}^\natural\|_{\mathcal{T}_k \setminus \mathcal{T}_{k+1}}.$$

555 Combining all terms together, we obtain

$$\begin{aligned} \|\mathbf{v}_{\mathcal{T}_{k+1}}^{k+1} - \mathbf{x}_{\mathcal{T}_{k+1}}^\natural\| &\leq I_1 + I_2 + I_3 \leq \sqrt{2(I_1^2 + I_2^2)} + I_3 \\ &\leq \sqrt{2} \max\{\eta \delta_{3s}, 1 - \eta(1 - \delta_{2s})\} \|\mathbf{x}^k - \mathbf{x}^\natural\| + \eta \sqrt{C_\gamma(1 + \delta_{2s})} \|\mathbf{x}^k - \mathbf{x}^\natural\| \\ &= \left(\sqrt{2} \max\{\eta \delta_{3s}, 1 - \eta(1 - \delta_{2s})\} + \eta \sqrt{C_\gamma(1 + \delta_{2s})} \right) \|\mathbf{x}^k - \mathbf{x}^\natural\|. \end{aligned}$$

556 We complete the proof by using (26). □

557 B.2 Proof of Theorem 3.1

558 *Proof.* In this proof, we consider the case where $\|\mathbf{x}^0 - \mathbf{x}^\natural\| \leq \|\mathbf{x}^0 + \mathbf{x}^\natural\|$. Consequently, the distance
559 between the initial estimate and the true signal is given by $\text{dist}(\mathbf{x}^0, \mathbf{x}^\natural) = \|\mathbf{x}^0 - \mathbf{x}^\natural\|$. Note that the
560 case where $\|\mathbf{x}^0 + \mathbf{x}^\natural\| \leq \|\mathbf{x}^0 - \mathbf{x}^\natural\|$ can be addressed in a similar manner. For the purpose of this
561 proof, we assume, without loss of generality, that the true signal has a unit norm, i.e., $\|\mathbf{x}^\natural\| = 1$.

562 Let \mathbf{x}^k represent the k -th iterate generated by Algorithm 1. Given an s -sparse estimate \mathbf{x}^k with
563 support \mathcal{S}_k , which is close to the target signal, i.e., $\text{dist}(\mathbf{x}^k, \mathbf{x}^\natural) \leq \gamma \|\mathbf{x}^\natural\|$. For any $0 \leq t \leq 1$, denote
564 $\mathbf{x}(t) := \mathbf{x}^\natural + t(\mathbf{x}^k - \mathbf{x}^\natural)$. It is evident that $\text{supp}(\mathbf{x}(t)) \subseteq \text{supp}(\mathbf{x}^\natural) \cup \text{supp}(\mathbf{x}^k)$, and the size of the
565 support of $\mathbf{x}(t)$ is at most $2s$, i.e., $|\text{supp}(\mathbf{x}(t))| \leq 2s$. Furthermore, we obtain

$$\|\mathbf{x}^k - \mathbf{x}(t)\| \leq (1 - t) \|\mathbf{x}^k - \mathbf{x}^\natural\| \leq \|\mathbf{x}^k - \mathbf{x}^\natural\|,$$

566 and

$$\begin{aligned} \|\mathbf{x}^k + \mathbf{x}(t)\| &= \|(1 + t)\mathbf{x}^k + (1 - t)\mathbf{x}^\natural\| \leq (1 + t) \|\mathbf{x}^k\| + (1 - t) \|\mathbf{x}^\natural\| \\ &\leq (1 + t)(1 + \gamma) \|\mathbf{x}^\natural\| + (1 - t) \|\mathbf{x}^\natural\| \leq 2(\gamma + 1) \|\mathbf{x}^\natural\|, \end{aligned}$$

567 where the last inequality holds because $0 \leq t \leq 1$.

568 Assume events (17) and (18) hold, then by (22) in Lemma B.5 with $\mathbf{x} = \mathbf{x}^k, \mathbf{z} = \mathbf{x}(t), \mathcal{S} =$
569 $\mathcal{S}_{k+1}, \mathcal{T} = \mathcal{S}_k \cup \mathcal{S}^\natural$ there holds

$$\begin{aligned} &\left\| \nabla_{\mathcal{S}_{k+1}, \mathcal{S}_k \cup \mathcal{S}^\natural}^2 f_I(\mathbf{x}^k) - \nabla_{\mathcal{S}_{k+1}, \mathcal{S}_k \cup \mathcal{S}^\natural}^2 f_I(\mathbf{x}(t)) \right\| \\ &\leq \frac{3}{m} \left((3m)^{1/4} + (3s)^{1/2} + 2\sqrt{\log m} \right)^4 \|\mathbf{x}^k + \mathbf{x}(t)\| \|\mathbf{x}^k - \mathbf{x}(t)\| \\ &\leq 10 \|\mathbf{x}^k + \mathbf{x}(t)\| \|\mathbf{x}^k - \mathbf{x}(t)\| \\ &\leq L_h \|\mathbf{x}^k - \mathbf{x}^\natural\|, \end{aligned} \quad (28)$$

where $L_h := 20(\gamma + 1)\|\mathbf{x}^\natural\|$. Also, Eq. (23) in Lemma B.6 with $\mathbf{x} = \mathbf{x}^k$, $\mathcal{S} = \mathcal{S}_{k+1}$ indicates that

$$\left\| \left(\nabla_{\mathcal{S}_{k+1}, \mathcal{S}_{k+1}}^2 f_I(\mathbf{x}^k) \right)^{-1} \right\| = \left(\lambda_{\min} \left(\nabla_{\mathcal{S}_{k+1}, \mathcal{S}_{k+1}}^2 f_I(\mathbf{x}^k) \right) \right)^{-1} \leq \frac{1}{l_1}. \quad (29)$$

Moreover, by the mean value theorem, one has

$$\nabla f_I(\mathbf{x}^k) - \nabla f_I(\mathbf{x}^\natural) = \int_0^1 \nabla^2 f_I(\mathbf{x}(t))(\mathbf{x}^k - \mathbf{x}^\natural) dt. \quad (30)$$

We also have the following chain of equalities

$$\begin{aligned} \|\mathbf{x}^{k+1} - \mathbf{x}^\natural\| &= \left[\|\mathbf{x}_{\mathcal{S}_{k+1}}^{k+1} - \mathbf{x}_{\mathcal{S}_{k+1}}^\natural\|^2 + \|\mathbf{x}_{\mathcal{S}_{k+1}^c}^{k+1} - \mathbf{x}_{\mathcal{S}_{k+1}^c}^\natural\|^2 \right]^{1/2} \\ &= \left[\underbrace{\|\mathbf{x}_{\mathcal{S}_{k+1}^c}^\natural\|^2}_{I_1} + \underbrace{\|\mathbf{x}_{\mathcal{S}_{k+1}}^k - \mathbf{x}_{\mathcal{S}_{k+1}}^\natural + \mathbf{p}_{\mathcal{S}_{k+1}}^k\|^2}_{I_2} \right]^{1/2}. \end{aligned} \quad (31)$$

We first bound the term I_1 in (31). Note that $\mathbf{x}_{\mathcal{S}_{k+1}^c}^\natural$ is a subvector of $\mathbf{x}^\natural - \mathbf{u}^k$. Based on this observation and applying Lemma B.8, we deduce that

$$I_1 = \|\mathbf{x}_{\mathcal{S}_{k+1}^c}^\natural\| \leq \|\mathbf{x}^\natural - \mathbf{u}^k\| \leq \zeta \|\mathbf{x}^k - \mathbf{x}^\natural\|, \quad (32)$$

where $\zeta = 2 \left(\sqrt{2} \max\{\eta\delta_{3s}, 1 - \eta(1 - \delta_{2s})\} + \eta\sqrt{C_\gamma(1 + \delta_{2s})} \right)$ with $\eta < \frac{1}{1 + \delta_{2s}}$.

We now proceed to bound the term I_2 in (31). By plugging the expression of $\mathbf{p}_{\mathcal{S}_{k+1}}^k$, we obtain that

$$\begin{aligned} I_2 &= \left\| \mathbf{x}_{\mathcal{S}_{k+1}}^k - \mathbf{x}_{\mathcal{S}_{k+1}}^\natural + \mathbf{p}_{\mathcal{S}_{k+1}}^k \right\| \\ &= \left\| \left(\nabla_{\mathcal{S}_{k+1}}^2 f_I(\mathbf{x}^k) \right)^{-1} \left(\nabla_{\mathcal{S}_{k+1}, \mathcal{S}_{k+1}^c}^2 f_I(\mathbf{x}^k) \mathbf{x}_{\mathcal{S}_{k+1}^c}^k - \nabla_{\mathcal{S}_{k+1}} f_I(\mathbf{x}^k) \right) + \mathbf{x}_{\mathcal{S}_{k+1}}^k - \mathbf{x}_{\mathcal{S}_{k+1}}^\natural \right\|. \end{aligned} \quad (33)$$

We further can deduce that:

$$\begin{aligned} I_2 &\leq \frac{1}{l_1} \left\| \nabla_{\mathcal{S}_{k+1}, \mathcal{S}_{k+1}^c}^2 f_I(\mathbf{x}^k) \mathbf{x}_{\mathcal{S}_{k+1}^c}^k - \nabla_{\mathcal{S}_{k+1}} f_I(\mathbf{x}^k) + \nabla_{\mathcal{S}_{k+1}}^2 f_I(\mathbf{x}^k) (\mathbf{x}_{\mathcal{S}_{k+1}}^k - \mathbf{x}_{\mathcal{S}_{k+1}}^\natural) \right\| \\ &\leq \frac{1}{l_1} \left\| \int_0^1 \left[\nabla_{\mathcal{S}_{k+1}, \mathcal{S}_{k+1}^c}^2 f_I(\mathbf{x}^k) - \nabla_{\mathcal{S}_{k+1}, \mathcal{S}_{k+1}^c}^2 f_I(\mathbf{x}(t)) \right] (\mathbf{x}^k - \mathbf{x}^\natural) dt \right\| + \frac{l_3}{l_1} \|\mathbf{x}_{\mathcal{S}_{k+1}^c}^\natural\| \\ &\leq \frac{1}{l_1} \int_0^1 \left\| \nabla_{\mathcal{S}_{k+1}, \mathcal{S}_{k+1}^c}^2 f_I(\mathbf{x}^k) - \nabla_{\mathcal{S}_{k+1}, \mathcal{S}_{k+1}^c}^2 f_I(\mathbf{x}(t)) \right\| \|\mathbf{x}^k - \mathbf{x}^\natural\| dt + \frac{\zeta l_3}{l_1} \|\mathbf{x}^k - \mathbf{x}^\natural\| \\ &\leq \frac{L_h}{l_1} \|\mathbf{x}^k - \mathbf{x}^\natural\|^2 \int_0^1 (1-t) dt + \frac{\zeta l_3}{l_1} \|\mathbf{x}^k - \mathbf{x}^\natural\| \\ &= \rho_1 \|\mathbf{x}^k - \mathbf{x}^\natural\|, \end{aligned}$$

where the first inequality follows from (29), the second inequality is based on Lemma B.6 and (30) together with the fact that $\nabla f_I(\mathbf{x}^\natural) = \mathbf{0}$, and the third inequality is derived from (32), and the last inequality is obtained from (28). The equality includes ρ_1 , defined as follows:

$$\begin{aligned} \rho_1 &:= \frac{L_h}{2l_1} \|\mathbf{x}^k - \mathbf{x}^\natural\| + \frac{\zeta l_3}{l_1} \leq \frac{20(1 + \gamma)\gamma}{2(2 - 2\delta - 10\gamma(2 + \gamma))} + \frac{\zeta(2 + 2\delta + 10\gamma(2 + \gamma))}{2 - 2\delta - 10\gamma(2 + \gamma)} \\ &= \frac{2\zeta(1 + \delta) + 10\gamma(1 + \gamma + \zeta(2 + \gamma))}{2 - 2\delta - 10\gamma(2 + \gamma)}. \end{aligned}$$

By substituting the upper bounds of terms I_1 and I_2 into (31), we obtain:

$$\|\mathbf{x}^{k+1} - \mathbf{x}^\natural\| \leq \sqrt{\rho_1^2 + \zeta^2} \|\mathbf{x}^k - \mathbf{x}^\natural\|. \quad (34)$$

Let $\rho := \sqrt{\rho_1^2 + \zeta^2}$, then $\rho < 1$ can be ensured by properly choosing parameters. For example, when $\delta_{3s} \leq 0.05$ and $\delta = 0.001$, and set $\eta = 0.95$, then $\rho \leq 0.6 < 1$ provided that $\gamma \leq 0.01$. Therefore, $\|\mathbf{x}^{k+1} - \mathbf{x}^\natural\| \leq \rho \|\mathbf{x}^k - \mathbf{x}^\natural\| \leq \rho\gamma \|\mathbf{x}^\natural\|$ for some $\rho \in (0, 1)$.

Let $\mathcal{R}_k = \mathcal{S}_k \cup \mathcal{S}_{k+1} \cup \mathcal{S}^\natural$. Assume that the event (17) with $\mathcal{R} = \mathcal{R}_k$ holds for k_1 iterations. As stated in Theorem IV.1 of [27], the initial guess \mathbf{x}^0 is guaranteed to satisfy $\text{dist}(\mathbf{x}^0, \mathbf{x}^\natural) \leq \gamma \|\mathbf{x}^\natural\|$. By applying mathematical induction, we can show that for any integer $0 \leq k \leq k_1$, there exists a constant $\rho \in (0, 1)$ such that:

$$\text{dist}(\mathbf{x}^{k+1}, \mathbf{x}^\natural) \leq \rho \cdot \text{dist}(\mathbf{x}^k, \mathbf{x}^\natural).$$

Let K be the minimum integer such that it holds

$$\gamma \|\mathbf{x}^\natural\| \rho^K < x_{\min}^\natural. \quad (35)$$

We then assert that $\mathcal{S}^\natural \subseteq \mathcal{S}_k$ for all $k \geq K$. This is because if it were not the case, there would exist an index $i \in \mathcal{S}^\natural \setminus \mathcal{S}_k \neq \emptyset$, such that $\|\mathbf{x}^k - \mathbf{x}^\natural\| \geq |x_i^\natural| \geq x_{\min}^\natural$. However, this contradicts our assumption $\|\mathbf{x}^k - \mathbf{x}^\natural\| \leq \gamma \|\mathbf{x}^\natural\| \rho^k \leq \gamma \|\mathbf{x}^\natural\| \rho^K < x_{\min}^\natural$. Consequently, based on (35), we derive:

$$K = \left\lceil \frac{\log(\gamma \|\mathbf{x}^\natural\| / x_{\min}^\natural)}{\log(\rho^{-1})} \right\rceil + 1 \leq C_a \log \left(\|\mathbf{x}^\natural\| / x_{\min}^\natural \right) + C_b.$$

Note that $\mathcal{S}_k = \mathcal{S}^\natural$ for all $k \geq K$ implies $\mathcal{R}_k = \mathcal{R}_{k+1}$ for all $k \geq K$. As a result, the probability of event (17) occurring for all $k \geq 0$ can be bounded by $1 - 4Km^{-2}$. To conclude, with probability at least $1 - (4K + C_c)m^{-1} - C_de^{-C_em/\log m}$, there holds

$$\text{dist}(\mathbf{x}^{k+1}, \mathbf{x}^\natural) \leq \rho \cdot \text{dist}(\mathbf{x}^k, \mathbf{x}^\natural).$$

with some $\rho \in (0, 1)$ provided $m \geq C_f s^2 \log(n/s)$. Moreover, for $k \geq K$, utilizing the result the result $\mathcal{S}_{k+1} = \mathcal{S}^\natural$ and consequently $\mathbf{x}_{\mathcal{S}_{k+1}}^c = \mathbf{0}$, we obtain:

$$\|\mathbf{x}^{k+1} - \mathbf{x}^\natural\| = \left[\left\| \mathbf{x}_{\mathcal{S}_{k+1}}^{k+1} - \mathbf{x}_{\mathcal{S}_{k+1}}^\natural \right\|^2 + \left\| \mathbf{x}_{\mathcal{S}_{k+1}}^{k+1} - \mathbf{x}_{\mathcal{S}_{k+1}}^c \right\|^2 \right]^{1/2} = \left\| \mathbf{x}_{\mathcal{S}_{k+1}}^k - \mathbf{x}_{\mathcal{S}_{k+1}}^\natural + \mathbf{p}_{\mathcal{S}_{k+1}}^k \right\|.$$

We can further obtain

$$\begin{aligned} \|\mathbf{x}^{k+1} - \mathbf{x}^\natural\| &\leq \frac{1}{l_1} \left\| \nabla_{\mathcal{S}_{k+1},:}^2 f_I(\mathbf{x}^k) \mathbf{x}^k - \nabla_{\mathcal{S}_{k+1}} f_I(\mathbf{x}^k) - \nabla_{\mathcal{S}_{k+1}}^2 f_I(\mathbf{x}^k) \mathbf{x}_{\mathcal{S}_{k+1}}^\natural \right\| \\ &\leq \frac{1}{l_1} \left\| \nabla_{\mathcal{S}_{k+1},:}^2 f_I(\mathbf{x}^k) (\mathbf{x}^k - \mathbf{x}^\natural) - \int_0^1 \nabla_{\mathcal{S}_{k+1},:}^2 f_I(\mathbf{x}(t)) (\mathbf{x}^k - \mathbf{x}^\natural) dt \right\| \\ &\leq \frac{1}{l_1} \left\| \int_0^1 \left[\nabla_{\mathcal{S}_{k+1},:}^2 f_I(\mathbf{x}^k) - \nabla_{\mathcal{S}_{k+1},:}^2 f_I(\mathbf{x}(t)) \right] (\mathbf{x}^k - \mathbf{x}^\natural) dt \right\| \\ &\leq \frac{1}{l_1} \int_0^1 \left\| \nabla_{\mathcal{S}_{k+1}, \mathcal{S}_k \cup \mathcal{S}^\natural}^2 f_I(\mathbf{x}^k) - \nabla_{\mathcal{S}_{k+1}, \mathcal{S}_k \cup \mathcal{S}^\natural}^2 f_I(\mathbf{x}(t)) \right\| \|\mathbf{x}^k - \mathbf{x}^\natural\| dt \\ &\leq \frac{L_h}{l_1} \|\mathbf{x}^k - \mathbf{x}^\natural\|^2 \int_0^1 (1-t) dt \\ &= \frac{L_h}{2l_1} \|\mathbf{x}^k - \mathbf{x}^\natural\|^2, \end{aligned}$$

where the first inequality follows from (29) and (33), and the last inequality follows from (28). Consequently, the sequence $\{\mathbf{x}^k\}$ converges quadratically. \square

B.3 Proof of Theorem 3.4

Proof. Owing to the independence between the sensing matrix and noise, along with the assumption $\mathbb{E}(\epsilon) = \mathbf{0}$, Lemma C.5 remains valid in the noisy setting. Therefore, by applying Lemma B.3, if $m \geq \mathcal{O}(s \log s)$, then with overwhelming probability, it holds that

$$\left\| \nabla_{\mathcal{S}}^2 f_I(\mathbf{x}^\natural) - \mathbb{E} [\nabla_{\mathcal{S}}^2 f_I(\mathbf{x}^\natural)] \right\| \leq (\delta + \epsilon) \|\mathbf{x}^\natural\|^2.$$

Under the assumptions in Lemma B.6, it follows that

$$l'_1 \|\mathbf{u}\| \leq \left\| \nabla_{\mathcal{S}}^2 f_I(\mathbf{x}) \mathbf{u} \right\| \leq l'_2 \|\mathbf{u}\|, \quad \forall \mathbf{u} \in \mathbb{R}^{|\mathcal{S}|}, \quad (36)$$

606 and

$$\|\nabla_{\mathcal{T},S}^2 f_I(\mathbf{x})\| \leq l'_3, \quad (37)$$

607 where $l'_1 = (2 - 2\delta - 2\epsilon - 10\gamma(2 + \gamma))\|\mathbf{x}^\natural\|$, $l'_2 = (6 + 2\delta + 2\epsilon + 10\gamma(2 + \gamma))\|\mathbf{x}^\natural\|$, and
 608 $l'_3 = (2 + 2\delta + 2\epsilon + 10\gamma(2 + \gamma))\|\mathbf{x}^\natural\|$. In the noisy case, $\{\mathbf{z}^k\}_{k \geq 0}$ is given by

$$\mathbf{z}^k = (|\mathbf{A}\mathbf{x}^\natural|^2 + \epsilon)^{\frac{1}{2}} \odot \text{sgn}(\mathbf{A}\mathbf{x}^k).$$

609 Then using the same argument as the proof of the inequality in Lemma B.7, we have

$$\begin{aligned} \left\| \frac{\eta}{m} \mathbf{A}_{\mathcal{T}_{k+1}}^T (\mathbf{z}^k - \mathbf{A}\mathbf{x}^\natural) \right\| &\leq \frac{\eta}{m} \left\| \mathbf{A}_{\mathcal{T}_{k+1}}^T (|\mathbf{A}\mathbf{x}^\natural| \odot \text{sgn}(\mathbf{A}\mathbf{x}^k) - \mathbf{A}\mathbf{x}^\natural) \right\| \\ &\quad + \frac{C'\eta}{m} \left\| \mathbf{A}_{\mathcal{T}_{k+1}}^T (\epsilon \odot \text{sgn}(\mathbf{A}\mathbf{x}^k)) \right\| \\ &\leq \eta \sqrt{C_\gamma(1 + \delta_{2s})} \|\mathbf{x}^k - \mathbf{x}^\natural\| + \frac{C'\eta}{\sqrt{m}} \sqrt{1 + \delta_{2s}} \|\epsilon\|, \end{aligned} \quad (38)$$

610 where the last inequality follows from Lemma B.7 and (19) in Lemma B.4. Then, we modify
 611 Lemma B.8 to estimate $\|\mathbf{u}^k - \mathbf{x}^\natural\|$ in the noisy case. All arguments in Lemma B.8 go except that the
 612 estimation of I_1 in (27) should be replaced by (38). Thus we obtain

$$\|\mathbf{u}^k - \mathbf{x}^\natural\| \leq \zeta \|\mathbf{x}^k - \mathbf{x}^\natural\| + \frac{C'\eta}{\sqrt{m}} \sqrt{1 + \delta_{2s}} \|\epsilon\|, \quad (39)$$

613 where \mathbf{u}^k, η are the same as those in Lemma B.8.

614 We now proceed to prove the convergence property for the noisy case, employing a similar argument
 615 as in the proof of Theorem 3.1. Notably, equality (31) remains valid in the presence of noise. As for
 616 the second term in (31), since $\mathbf{x}_{\mathcal{S}_{k+1}^c}^\natural$ is a subvector of $\mathbf{x}^\natural - \mathbf{u}^k$, it follows from (39) that

$$\|\mathbf{x}_{\mathcal{S}_{k+1}^c}^\natural\| \leq \|\mathbf{x}^\natural - \mathbf{u}^k\| \leq \zeta \|\mathbf{x}^k - \mathbf{x}^\natural\| + \frac{C'\eta}{\sqrt{m}} \sqrt{1 + \delta_{2s}} \|\epsilon\|,$$

617 where $\zeta = 2 \left(\sqrt{2} \max\{\eta\delta_{3s}, 1 - \eta(1 - \delta_{2s})\} + \eta \sqrt{C_\gamma(1 + \delta_{2s})} \right)$ with $\eta < 1/(1 + \delta_{2s})$.

618 Furthermore, the first term in (31) can now be estimated as

$$\begin{aligned} &\|\mathbf{x}_{\mathcal{S}_{k+1}}^k - \mathbf{x}_{\mathcal{S}_{k+1}}^\natural + \mathbf{p}_{\mathcal{S}_{k+1}}^k\| \\ &\leq \frac{1}{l'_1} \left\| \nabla_{\mathcal{S}_{k+1},:}^2 f_I(\mathbf{x}^k) \mathbf{x}^k - \nabla_{\mathcal{S}_{k+1},:}^2 f_I(\mathbf{x}^k) \mathbf{x}^\natural + \nabla_{\mathcal{S}_{k+1},\mathcal{S}_{k+1}^c}^2 f_I(\mathbf{x}^k) \mathbf{x}_{\mathcal{S}_{k+1}^c}^\natural \right. \\ &\quad \left. - \nabla_{\mathcal{S}_{k+1}} f_I(\mathbf{x}^k) + \nabla_{\mathcal{S}_{k+1}} f_I(\mathbf{x}^\natural) + \frac{1}{m} \mathbf{A}_{\mathcal{S}_{k+1}}^T (\epsilon \odot (\mathbf{A}\mathbf{x}^\natural)) \right\| \\ &\leq \frac{1}{l'_1} \left\| \nabla_{\mathcal{S}_{k+1},:}^2 f_I(\mathbf{x}^k) (\mathbf{x}^k - \mathbf{x}^\natural) - \int_0^1 \nabla_{\mathcal{S}_{k+1},:}^2 f_I(\mathbf{x}(t)) (\mathbf{x}^k - \mathbf{x}^\natural) dt \right\| \\ &\quad + \frac{1}{l'_1} \left\| \nabla_{\mathcal{S}_{k+1},\mathcal{S}_{k+1}^c}^2 f_I(\mathbf{x}^k) \mathbf{x}_{\mathcal{S}_{k+1}^c}^\natural \right\| + \frac{1}{l'_1 m} \left\| \mathbf{A}_{\mathcal{S}_{k+1}}^T (\epsilon \odot (\mathbf{A}\mathbf{x}^\natural)) \right\| \\ &\leq \frac{1}{l'_1} \left\| \int_0^1 \left[\nabla_{\mathcal{S}_{k+1},:}^2 f_I(\mathbf{x}^k) - \nabla_{\mathcal{S}_{k+1},:}^2 f_I(\mathbf{x}(t)) \right] (\mathbf{x}^k - \mathbf{x}^\natural) dt \right\| + \frac{l'_3}{l'_1} \|\mathbf{x}_{\mathcal{S}_{k+1}^c}^\natural\| + \frac{\sqrt{1 + \delta_s} \|\mathbf{A}\mathbf{x}^\natural\|_\infty}{l'_1 \sqrt{m}} \|\epsilon\| \\ &\leq \frac{1}{l'_1} \int_0^1 \left\| \nabla_{\mathcal{S}_{k+1},\mathcal{S}_{k+1} \cup \mathcal{S}^\natural}^2 f_I(\mathbf{x}^k) - \nabla_{\mathcal{S}_{k+1},\mathcal{S}_{k+1} \cup \mathcal{S}^\natural}^2 f_I(\mathbf{x}(t)) \right\| \|\mathbf{x}^k - \mathbf{x}^\natural\| dt \\ &\quad + \frac{\zeta l'_3}{l'_1} \|\mathbf{x}^k - \mathbf{x}^\natural\| + \frac{C' l'_3 \eta \sqrt{1 + \delta_{2s}} + \sqrt{1 + \delta_s} \|\mathbf{A}\mathbf{x}^\natural\|_\infty}{l'_1 \sqrt{m}} \|\epsilon\| \\ &\leq \frac{L_h}{l'_1} \|\mathbf{x}^k - \mathbf{x}^\natural\|^2 \int_0^1 (1 - t) dt + \frac{\zeta l'_3}{l'_1} \|\mathbf{x}^k - \mathbf{x}^\natural\| + \frac{C' l'_3 \eta \sqrt{1 + \delta_{2s}} + \sqrt{1 + \delta_s} \|\mathbf{A}\mathbf{x}^\natural\|_\infty}{l'_1 \sqrt{m}} \|\epsilon\| \\ &= \rho_2 \|\mathbf{x}^k - \mathbf{x}^\natural\| + \frac{C' l'_3 \eta \sqrt{1 + \delta_{2s}} + \sqrt{1 + \delta_s} \|\mathbf{A}\mathbf{x}^\natural\|_\infty}{l'_1 \sqrt{m}} \|\epsilon\|, \end{aligned} \quad (40)$$

where the first inequality follows from (36), together with the equality $\nabla f_I(\mathbf{x}^\natural) + \mathbf{A}^T (\boldsymbol{\epsilon} \odot (\mathbf{A}\mathbf{x}^\natural)) = \mathbf{0}$, and the third inequality is based on (37) and the inequality $\|\mathbf{a} \odot \mathbf{b}\| \leq \|\mathbf{a}\|_\infty \|\mathbf{b}\|$ for any two vectors \mathbf{a}, \mathbf{b} . The last equality includes ρ_2 , defined as follows:

$$\begin{aligned} \rho_2 &:= \frac{L_h}{2m'_s} \|\mathbf{x}^k - \mathbf{x}^\natural\| + \frac{\zeta h'_s}{m'_s} \leq \frac{20(1+\gamma)\gamma}{2(2-2\delta-2\epsilon-10\gamma(2+\gamma))} + \frac{\zeta(2+2\delta+2\epsilon+10\gamma(2+\gamma))}{2-2\delta-2\epsilon-10\gamma(2+\gamma)} \\ &= \frac{2\zeta(1+\delta+\epsilon)+10\gamma(1+\gamma+\zeta(2+\gamma))}{2-2\delta-2\epsilon-10\gamma(2+\gamma)}. \end{aligned}$$

Following from $(a^2 + b^2)^{1/2} \leq a + b$ for $ab \geq 0$ and putting the two terms together yields

$$\|\mathbf{x}^{k+1} - \mathbf{x}^\natural\| \leq \rho' \|\mathbf{x}^k - \mathbf{x}^\natural\| + v \|\boldsymbol{\epsilon}\|,$$

where $\rho' = \rho_2 + \zeta$ and $v = \frac{C'(l'_1+l'_3)\eta\sqrt{1+\delta_{2s}}+\sqrt{1+\delta_s}\|\mathbf{A}\mathbf{x}^\natural\|_\infty}{l'_1\sqrt{m}}$. Noticing that

$$\frac{1}{\sqrt{m}} \|\mathbf{A}\mathbf{x}^\natural\|_\infty = \frac{1}{\sqrt{m}} \max_{i \in [m]} |\mathbf{a}_i^T \mathbf{x}^\natural| \leq \frac{1}{\sqrt{m}} \max_{i \in [m], j \in \mathcal{S}^\natural} |a_{ij}| \cdot \|\mathbf{x}^\natural\| \leq 3\sqrt{\frac{\log(ms)}{m}} \|\mathbf{x}^\natural\|$$

with probability at least $1 - (ms)^{-2}$. Consequently, $\frac{1}{\sqrt{m}} \|\mathbf{A}\mathbf{x}^\natural\|_\infty$ can be quite small. As a result, properly setting parameters can lead to $v \in (0, 1)$ and $\rho' \in (0, 1)$. \square

C Auxiliary Lemmas

In this section, we present some auxiliary lemmas.

Lemma C.1. (Hoeffding-type inequality) Let X_1, \dots, X_N be independent centered sub-Gaussian random variables, and let $K = \max_{i \in [N]} \|X_i\|_{\psi_2}$, where the sub-Gaussian norm

$$\|X_i\|_{\psi_2} := \sup_{p \geq 1} p^{-1/2} (\mathbb{E} [|X|^p])^{1/p}.$$

Then for every $\mathbf{b} = [b_1; \dots; b_N] \in \mathbb{R}^N$ and every $t \geq 0$, we have

$$\mathbb{P} \left\{ \left| \sum_{k=1}^N b_k X_k \right| \geq t \right\} \leq e \exp \left(- \frac{ct^2}{K^2 \|\mathbf{b}\|^2} \right).$$

Here c is a universal constant.

Lemma C.2. (Bernstein-type inequality) Let X_1, \dots, X_N be independent centered sub-exponential random variables, and let $K = \max_i \|X_i\|_{\psi_1}$, where the sub-exponential norm

$$\|X_i\|_{\psi_1} := \sup_{p \geq 1} p^{-1} (\mathbb{E} [|X|^p])^{1/p}.$$

Then for every $\mathbf{b} = [b_1; \dots; b_N] \in \mathbb{R}^N$ and every $t \geq 0$, we have

$$\mathbb{P} \left\{ \left| \sum_{k=1}^N b_k X_k \right| \geq t \right\} \leq 2 \exp \left(-c \min \left(\frac{t^2}{K^2 \|\mathbf{b}\|^2}, \frac{t}{K \|\mathbf{b}\|_\infty} \right) \right).$$

Here c is a universal constant.

Lemma C.3. (Bernstein's inequality for bounded distributions) Let X_1, \dots, X_N be independent mean zero random variables, such that $|X_i| \leq K$ for all i . Then, for every $t \geq 0$, we have

$$\mathbb{P} \left\{ \left| \sum_{i=1}^N X_i \right| \geq t \right\} \leq 2 \exp \left(- \frac{t^2/2}{\sigma^2 + Kt/3} \right).$$

Here $\sigma^2 = \sum_{i=1}^N \mathbb{E} X_i^2$ is the variance of the sum.

The next lemma is the so-called Weyl Theorem, which is a classical linear algebra result.

640 **Lemma C.4.** Suppose $\mathbf{M}, \mathbf{N} \in \mathbb{R}^{n \times n}$ are two symmetric matrices. The eigenvalues of \mathbf{M} are
641 denoted as $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ and the eigenvalues of \mathbf{N} are denoted as $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n$. Then we
642 have

$$|\mu_i - \lambda_i| \leq \|\mathbf{M} - \mathbf{N}\|_2, \quad \forall i = 1, 2, \dots, n.$$

643 The following lemma provides the expectation of the sub-Hessian of the intensity-based loss f_I at \mathbf{x}^\natural .
644 As this result can be easily derived through basic calculations, we will not delve into the details here.

645 **Lemma C.5.** For any subset $\mathcal{S} \subseteq [n]$ such that $\text{supp}(\mathbf{x}^\natural) \subseteq \mathcal{S}$, the expectation of $\nabla_{\mathcal{S}}^2 f_I(\mathbf{x}^\natural)$ is

$$\mathbb{E} [\nabla_{\mathcal{S}}^2 f_I(\mathbf{x}^\natural)] = 2 \left(\|\mathbf{x}^\natural\|^2 (\mathbf{I}_n)_{\mathcal{S}} + 2\mathbf{x}_{\mathcal{S}}^\natural (\mathbf{x}_{\mathcal{S}}^\natural)^T \right),$$

646 and it has one eigenvalue of $6\|\mathbf{x}^\natural\|^2$ and all other eigenvalues are $2\|\mathbf{x}^\natural\|^2$.

647 The results presented below have been previously established in [25].

648 **Lemma C.6.** [25, Lemma A.6] On an event with probability at least $1 - m^{-1}$, we have

$$\left\| \frac{1}{m} \sum_{i=1}^m |\mathbf{a}_{i,\mathcal{S}^\natural}^T \mathbf{x}^\natural|^2 \mathbf{a}_{i,\mathcal{S}^\natural} \mathbf{a}_{i,\mathcal{S}^\natural}^T - \left(\|\mathbf{x}^\natural\|^2 (\mathbf{I}_n)_{\mathcal{S}^\natural} + 2\mathbf{x}_{\mathcal{S}^\natural}^\natural (\mathbf{x}_{\mathcal{S}^\natural}^\natural)^T \right) \right\| \leq \delta \|\mathbf{x}^\natural\|^2,$$

649 provided $m \geq C(\delta)s \log s$. Here $C(\delta)$ is a constant depending on δ .

650 The subsequent lemma, a direct outcome from [29], plays a crucial role in bounding the term
651 $\|\mathbf{A}_{\mathcal{T}_{k+1}}^T (\mathbf{z}^k - \mathbf{A}\mathbf{x}^\natural)\|$.

652 **Lemma C.7.** [29, Lemma 25] Let $\{\mathbf{a}_i\}_{i=1}^m$ be i.i.d. Gaussian random vectors with mean $\mathbf{0}$ and
653 variance matrix \mathbf{I} . Let γ be any constant in $(0, \frac{1}{8}]$. Fixing any $\epsilon_0 > 0$, then for any s -sparse vector \mathbf{x}
654 satisfying $\text{dist}(\mathbf{x}, \mathbf{x}^\natural) \leq \lambda_0 \|\mathbf{x}^\natural\|$, with probability at least $1 - e^{-C'_6 m}$ there holds that

$$\frac{1}{m} \sum_{i=1}^m |\mathbf{a}_i^T \mathbf{x}^\natural|^2 \cdot \mathbf{1}_{\{(\mathbf{a}_i^T \mathbf{x})(\mathbf{a}_i^T \mathbf{x}^\natural) \leq 0\}} \leq \frac{1}{(1 - \lambda_0)^2} \left(\epsilon_0 + \lambda_0 \sqrt{\frac{21}{20}} \right)^2 \|\mathbf{x} - \mathbf{x}^\natural\|^2, \quad (41)$$

655 provided $m \geq C'_5 s \log(n/s)$. Here C'_5 and C'_6 are some universal positive constants.