A Comments on EPIC

In this appendix, we will show that EPIC has a number of undesirable properties. For the sake of readability, the proofs of the theorems in this section are given in Appendix F.3, rather than here.

First of all, while EPIC does induce an upper bound on a form of regret, this is not the type of regret that is typically relevant in practice. To demonstrate this, we will first provide a generalisation of the regret bound given in Gleave et al. (2020):

Theorem 3. *There exists a positive constant U, such that for any reward functions R_1 and R_2, and any τ and μ_0, if two policies π_1 and π_2 satisfy that $J_2(\pi_2) \geqslant J_2(\pi_1)$, then we have that*

$$J_1(\pi_1) - J_1(\pi_2) \leqslant U \cdot L_2(R_1) \cdot D^{\mathrm{EPIC}}(R_1, R_2).$$

Theorem 3 covers the special case when π_1 is optimal under R_1, and π_2 is optimal under R_2. This means that this theorem is a generalisation of the bound given in Gleave et al. (2020). However, we will next show that EPIC does not induce a regret bound of the form given in Definition 5:

Theorem 4. *EPIC is not sound, for any choice of τ or μ_0.*

These results may at first seem paradoxical, since the definition of soundness is quite similar to the statement of Theorem 3. The reason why these statements can both be true simultaneously is that $L_2(R_1)$ can be arbitrarily large even as $\max_\pi J_1(\pi) - \min_\pi J_1(\pi)$ becomes arbitrarily small. Moreover, we argue that it is more relevant to compare $J_1(\pi_1) - J_1(\pi_2)$ to $\max_\pi J_1(\pi) - \min_\pi J_1(\pi)$, rather than $L_2(R_1)$. First of all, note that it is not informative to just know the absolute value of $J_1(\pi_1) - J_1(\pi_2)$, since this depends on the scale of R_1. Rather, what we want to know how much reward we might lose, relative to the total amount of reward that could be had. This quantity is measured by $\max_\pi J_1(\pi) - \min_\pi J_1(\pi)$, not $L_2(R_1)$. For example, if $J_1(\pi_1) - J_1(\pi_2) = \epsilon$, but $\max_\pi J_1(\pi) - \min_\pi J_1(\pi)$ is just barely larger than ϵ, then this should be considered to be a large loss of reward, even if $L_2(R_1) \gg \epsilon$. For this reason, we consider the regret bound given in Definition 5 to be more informative than the regret bound given in Gleave et al. (2020) and in Theorem 3.

Another relevant question is whether EPIC induces a *lower* bound on worst-case regret. We next show that this is not the case, regardless of whether we consider the type of regret used in Gleave et al. (2020), or the type of regret used in Theorem 3, or the type of regret used in Definition 5:

Theorem 5. *There exist rewards R_1, R_2 such that $D^{\mathrm{EPIC}}(R_1, R_2) > 0$, but where R_1 and R_2 induce the same ordering of policies for any choice of τ or μ_0.*

This means that EPIC cannot induce a lower bound on worst-case regret, for almost any way of defining regret. Together, we think these results show that EPIC lacks the theoretical guarantees that we desire in a reward function pseudometric.

B Comments on DARD

In this appendix, we briefly discuss some of the properties of DARD. In so doing, we will criticise some of the choices made in the design of DARD, and argue that STARC metrics offer a better way to incorporate information about the environment dynamics.

To start with, recall that DARD uses the canonicalisation function C^{DARD}, which is the function where $C^{\mathrm{DARD}}(R)(s, a, s')$ is given by

$$R(s, a, s') + \mathbb{E}[\gamma R(s', A, S'') - R(s, A, S') - \gamma R(S', A, S'')],$$

where $A \sim \mathcal{D}_\mathcal{A}$, $S' \sim \tau(s, A)$, and $S'' \sim \tau(s', A)$. Moreover, also recall that C^{DARD} only is designed to remove potential shaping, whereas the canonicalisation functions we specify in Definition 1 are designed to remove both potential shaping and S'-redistribution.

Now, first and foremost, note that while C^{DARD} is designed to remove potential shaping, it does this in a somewhat strange way. In particular, while it is shown in Wulfe et al. (2022) that $C^{\mathrm{DARD}}(R_1) = C^{\mathrm{DARD}}(R_2)$ if R_1 and R_2 differ by potential shaping, it is in general *not* the case that R and $C^{\mathrm{DARD}}(R)$ differ by potential shaping. To see this, note that the term $\gamma R(S', A, S'')$ depends on both s and s', which a potential shaping function cannot do. This has a few important consequences.

In particular, it is unclear if $C^{\text{DARD}}(R_1) = C^{\text{DARD}}(R_2)$ *only if* R_1 and R_2 differ by potential shaping. It is also unclear if R and $C^{\text{DARD}}(R)$ in general even have the same policy ordering. Note also that $C^{\text{DARD}}(R)$ is not monotonic, in the sense that $C^{\text{DARD}}(C^{\text{DARD}}(R))$ and $C^{\text{DARD}}(R)$ may be different. This seems undesirable.

Another thing to note is that $\mathbb{E}[C^{\text{DARD}}(R)(S, A, S')]$ may not be 0. This means that it is unclear whether or not DARD can be expressed in terms of norms, like EPIC can (c.f. Proposition 12). It is also unclear if DARD induces an upper bound on regret, since Wulfe et al. (2022) do not provide a regret bound. The fact that $C^{\text{DARD}}(R)$ is not a potential shaping function does not necessarily imply that DARD does not induce an upper bound on regret. However, without a proof, there is a worry that there might be reward pairs with a bounded DARD distance but unbounded regret.

Yet another thing to note is that, while DARD is designed to be used in cases where the environment dynamics are known, it can still be influenced by the reward of transitions that are impossible according to the environment dynamics. For example, the final term of $C^{\text{DARD}}(R)(s, a, s')$ can be influenced by impossible transitions. This gives us the following result:

Proposition 8. *There exists transition functions τ and initial state distributions μ_0 for which DARD is not complete.*

Proof. Consider an environment $(\mathcal{S}, \mathcal{A}, \tau, \mu_0, \_, \gamma)$ where $\mathcal{S} = \{s_1, s_2, s_3\}$, $\mathcal{A} = \{a_1, a_2\}$, and where the transition function is given by $\tau(s_1, a) = s_2$, $\tau(s_2, a) = s_3$, and $\tau(s_3, a) = s_1$, for any $a \in \mathcal{A}$. We may also suppose $\mu_0 = s_1$, and $\gamma = 0.9$.

Next, let $R_1 = R_2 = 0$ for all transitions which are possible under τ, but let $R_1(s, a_1, s') = 1$, $R_1(s, a_2, s') = 0$, $R_2(s, a_1, s') = 0$, and $R_2(s, a_2, s') = 1$ for all transitions which are impossible under τ. Now $D^{\text{DARD}}(R_1, R_2) > 0$, even though $(\mathcal{S}, \mathcal{A}, \tau, \mu_0, R_1, \gamma)$ and $(\mathcal{S}, \mathcal{A}, \tau, \mu_0, R_2, \gamma)$ have exactly the same policy ordering. $\square$

Together, the above leads us to worry that DARD might lead to misleading measurements. Therefore, we believe that STARC metrics offer a better way to incorporate knowledge about the transition dynamics into the reward metric, especially in the light of our results from Section 3.

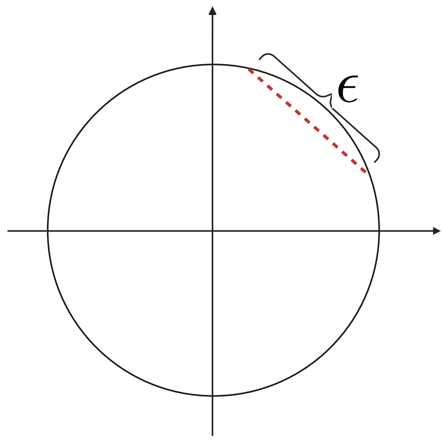
C A Geometric Intuition for STARC Metrics

In this section, we will provide a geometric intuition for how STARC metrics work. This will help to explain why STARC metrics are designed in the way that they are, and how they work. It may also make it easier to understand some of our proofs.

First of all, note that the space of all reward functions $\mathcal{R}$ forms an $|\mathcal{S}||\mathcal{A}||\mathcal{S}|$-dimensional vector space. Next, recall that if two reward functions R_1 and R_2 differ by (some combination of) potential shaping and S'-redistribution, then R_1 and R_2 induce the same ordering of policies. Moreover, both of these transformations are *additive*. In other words, they correspond to a set of reward functions $\{R_0\}$, such that R_1 and R_2 differ by a combination of potential shaping and S'-redistribution if and only if $R_1 - R_2 \in \{R_0\}$. This means that $\{R_0\}$ is a linear subspace of $\mathcal{R}$, and that for any reward function R, the set of all reward functions that differ from R by a combination of potential shaping and S'-redistribution together form an affine subspace of $\mathcal{R}$.

A canonicalisation function is a linear map that removes the dimensions that are associated with $\{R_0\}$. In other words, they map $\mathcal{R}$ to an $|\mathcal{S}|(|\mathcal{A}| - 1)$-dimensional subspace of $\mathcal{R}$ in which no reward functions differ by potential shaping or S'-redistribution. The null space of a canonicalisation function is always $\{R_0\}$. The canonicalisation function that is minimal for the L_2-norm is the orthogonal map that satisfies these properties, whereas other canonicalisation functions are non-orthogonal.

When we normalise the resulting reward functions by dividing by a norm n, we project the entire vector space onto the unit ball of n (except the zero reward, which remains at the origin). The metric m then measures the distance between the resulting reward functions on the surface of this sphere:



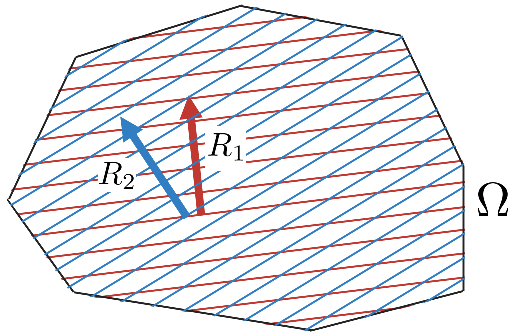
To make this more clear, it may be worth considering the case of non-sequential decision making. Suppose we have a finite set of *choices* C, and a utility function $U : C \to \mathbb{R}$. Given two distributions $\mathcal{D}_1$, $\mathcal{D}_2$ over C, we say that we prefer $\mathcal{D}_1$ over $\mathcal{D}_2$ if $\mathbb{E}_{c\sim\mathcal{D}_1}[U(c)] > \mathbb{E}_{c\sim\mathcal{D}_2}[U(c)]$. The set of all utility functions over C forms a $|C|$-dimensional vector space. Moreover, in this setting, it is well-known that two utility functions U_1, U_2 induce the same preferences between all possible distributions over C if and only if they differ by an affine transformation. Therefore, if we wanted to represent the set of all *non-equivalent* utility functions over C, we may consider requiring that $U(c_0) = 0$ for some $c_0 \in C$, and that $L_2(U) = 1$ unless $U(c) = 0$ for all $c \in C$. Any utility function over C is equivalent to some utility function in this set, and this set can in turn be represented as the surface of a $(|C| - 1)$-dimensional sphere, together with the origin.

This is essentially analogous to the normalisation that the canonicalisation function c and the normalisation function n perform for STARC metrics. Here C is analogous to the set of all trajectories, the trajectory return function G is analogous to U, and a policy π induces a distribution over trajectories. It is worth knowing that affine transformations of the trajectory return function, G, correspond exactly to potential shaping and positive linear scaling of R (see Skalse et al., 2022a, their Theorem 3.12). However, while the cases are analogous, it is not a direct correspondence, because not all distributions over trajectories can be realised as a policy in a given MDP.

Another perspective that may help with understanding STARC metrics comes from considering *occupancy measures*. Specifically, for a given policy π, let its occupancy measure η^π be the $|\mathcal{S}||\mathcal{A}||\mathcal{S}|$-dimensional vector in which the value of the (s, a, s')'th dimension is

$$\sum_{t=0}^{\infty} \gamma^t \mathbb{P}_{\xi\sim\pi}(S_t = s, A_t = a, S_{t+1} = s').$$

Now note that $J(\pi) = \eta^\pi \cdot R$. Therefore, by computing occupancy measures, we can divide the computation of J into two parts, the first of which is independent of R, and the second of which is a linear function. Moreover, let $\Omega = \{\eta^\pi : \pi \in \Pi\}$ be the set of all occupancy measures. We now have that the policy value function J of a reward function R can be visualised as a linear function on this set. Moreover, if we have two reward functions R_1, R_2, then they can be visualised as two different linear functions on this set:



Ω is located in a $|\mathcal{S}|(|\mathcal{A}|-1)$-dimensional affine subspace of $\mathbb{R}^{|\mathcal{S}||\mathcal{A}||\mathcal{S}|}$, and contains a set which is open in this space (Skalse & Abate, 2023). Moreover, it can be represented as the convex hull of a finite set of points (Feinberg & Rothblum, 2012). It is thus a polytope.

From this image, it is visually clear that the worst-case regret of maximising R_1 instead of R_2, should be proportional to the angle between the projections of R_1 and R_2 onto Ω. Moreover, this is what STARC metrics measure; any STARC metric is bilipschitz equivalent to the angle between reward functions projected onto Ω. This in should in turn give an intuition for why the STARC distance between two rewards provide both an upper and lower bound on their worst-case regret.

D Approximating STARC Metrics in Large Environments

In small MDPs, STARC metrics can be computed exactly (in time that is polynomial in $|\mathcal{S}|$ and $|\mathcal{A}|$). However, most realistic MDPs are too large for this to be feasible. As such, we will here discuss how to approximate STARC metrics in large environments, including continuous environments.

First, recall the VAL canonicalisation function (Proposition 2), given by

$$c(R)(s,a,s') = \mathbb{E}_{S' \sim \tau(s,a)}\left[R(s,a,S') - V^\pi(s) + \gamma V^\pi(S')\right],$$

where π can be any (fixed) policy. This canonicalisation function is straightforward to approximate in any environment where reinforcement learning can be used, including large-scale environments. To do this, first pick an arbitrary policy π, such as e.g. the uniformly random policy. Then compute an approximation of V^π – this can be done using a neural network updated with *on-policy* Bellman updates (Sutton & Barto, 2018). Note that π should not be updated, and must remain fixed. Then simply estimate the expected value of $R(s,a,S') - V^\pi(s) + \gamma V^\pi(S')$ by sampling from τ. This approximation can be computed in any environment where it is possible to sample from τ and approximate V^π (which is to say, any environment where reinforcement learning is applicable). Note that we do not require direct access to τ, we only need to be able to sample from it.

Next, note that if v is an n-dimensional vector, and $\mathcal{U}$ is the uniform distribution over $\{1 \dots n\}$, then

$$L_p(v) = (n \cdot \mathbb{E}_{i \sim \mathcal{U}}[|v_i|^p])^{1/p} = n^{1/p} \cdot \mathbb{E}_{i \sim \mathcal{U}}[|v_i|^p]^{1/p}.$$

This in turn means that

$$L_p\left(\frac{v}{L_p(v)}, \frac{w}{L_p(w)}\right) = \mathbb{E}_{i \sim \mathcal{U}}\left[\left|\frac{v_i}{\mathbb{E}_{j \sim \mathcal{U}}[|v_j|^p]^{1/p}} - \frac{w_i}{\mathbb{E}_{j \sim \mathcal{U}}[|w_j|^p]^{1/p}}\right|^p\right]^{1/p}$$

since the $n^{1/p}$-terms cancel out. Therefore, the normalisation step and distance step can also be estimated through sampling; simply sample enough random transitions to approximate each of the expectations in the expression above. Indeed, this can even be done if the state space and action space are continuous. Recall that the L_p-norm of an infinite-dimensional vector v is defined as

$$L_{p,X}(v) = \left(\int_X |v_x|^p \mathrm{d}x\right)^{1/p}$$

where $X \subseteq \mathbb{R}^n$. This value can also be approximated through sampling.

It is also possible to approximate division by n and taking the distance with m using the Pearson distance, which is what EPIC does. In particular, let $\mathcal{D}$ be a distribution over $\mathcal{S} \times \mathcal{A} \times \mathcal{S}$ that assigns positive probability to all transitions, and let R_1, R_2 be reward functions such that $\mathbb{E}_{S,A,S' \sim \mathcal{D}}[R_1(S,A,S')] = \mathbb{E}_{S,A,S' \sim \mathcal{D}}[R_1(S,A,S')] = 0$. We then have that the Pearson distance $\sqrt{(1 - \rho(R_1(S,A,S'), R_2(S,A,S')))/2}$ between $R_1(S,A,S')$ and $R_2(S,A,S')$, where $S, A, S' \sim \mathcal{D}$, is equal to

$$\frac{1}{2} \cdot L_{2,W}\left(\frac{R_1}{L_{2,W}(R_1)}, \frac{R_2}{L_{2,W}(R_2)}\right),$$

where W is a weight matrix depending on $\mathcal{D}$, and ρ denotes the Pearson correlation. For details, see the proof of Proposition 12. For this identity to hold, it is crucial that $\mathbb{E}_{S,A,S' \sim \mathcal{D}}[R_1(S,A,S')] = \mathbb{E}_{S,A,S' \sim \mathcal{D}}[R_1(S,A,S')] = 0$. However, this can easily be ensured; for an arbitrary canonicalisation function c_1, let $c_2(R) = c_1(R) - \mathbb{E}_{S,A,S' \sim \mathcal{D}}[c_1(R)(S,A,S')]$. If c_1 is a valid canonicalisation function, then so is c_2. The Pearson correlation can of course be estimated through sampling.

E PROOFS OF MISCELLANEOUS CLAIMS

In this Appendix, we provide proofs for several miscellaneous claims and minor propositions made throughout the paper, especially in Section 2.2.

Proposition 9. *For any policy π, the function $c : \mathcal{R} \to \mathcal{R}$ given by*

$$c(R)(s,a,s') = \mathbb{E}_{S' \sim \tau(s,a)}\left[R(s,a,S') - V^\pi(s) + \gamma V^\pi(S')\right]$$

is a canonicalisation function.

Proof. To prove that c is a canonicalisation function, we must show

1. that c is linear,
2. that $c(R)$ and R only differ by potential shaping and S'-redistribution, and
3. that $c(R_1) = c(R_2)$ if and only if R_1 and R_2 only differ by potential shaping and S'-redistribution.

We first show that c is linear. Given a state s, let v_s be the $|\mathcal{S}||\mathcal{A}||\mathcal{S}|$-dimensional vector where

$$v_s[s', a, s''] = \sum_{i=0}^{\infty} \gamma^i \cdot \mathbb{P}(S_i = s', A_i = a, S_{i+1} = s''),$$

where the probability is given for a trajectory that is generated from π and τ, starting in s. Now note that $V^\pi(s) = v_s \cdot R$, where R is represented as a vector. Using these vectors $\{v_s\}$, it is possible to express c as a linear transformation.

To see that $c(R)$ and R differ by potential shaping and S'-redistribution, it is sufficient to note that V^π acts as a potential function, and that setting $R_2(s,a,s') = \mathbb{E}_{S' \sim \tau(s,a)}[R_1(s,a,S')]$ is a form of S'-redistribution.

To see that $c(R_1) = c(R_2)$ if R_1 and R_2 differ by potential shaping and S'-redistribution, first note that if R_1 and R_2 differ by potential shaping, so that $R_2(s,a,s') = R_1(s,a,s') + \gamma\Phi(s') - \Phi(s)$ for some Φ, then $V_2^\pi(s) = V_1^\pi(s) - \Phi(s)$ (see e.g. Lemma B.1 in Skalse et al., 2022a). This means that

$$\begin{aligned}
c(R_2)(s,a,s') &= \mathbb{E}[R_2(s,a,S') + \gamma \cdot V_2^\pi(S') - V_2^\pi(s)] \\
&= \mathbb{E}[R_1(s,a,S') + \gamma \cdot \Phi(S') - \Phi(s) + \gamma \cdot (V_1^\pi(S') - \Phi(S')) - (V_1^\pi(s) - \Phi(s))] \\
&= \mathbb{E}[R_1(s,a,S') + \gamma \cdot V_1^\pi(S') - V_1^\pi(s)] \\
&= c(R_1)(s,a,s').
\end{aligned}$$

To see that $c(R_1) = c(R_2)$ only if R_1 and R_2 differ by potential shaping and S'-redistribution, first note that we have already shown that R and $c(R)$ differ by potential shaping and S'-redistribution for all R. This implies that R_1 and $c(R_1)$ differ by potential shaping and S'-redistribution, and likewise for R_2 and $c(R_2)$. Then if $c(R_1) = c(R_2)$, we can combine these transformations, and obtain that R_1 and R_2 also differ by potential shaping and S'-redistribution. This completes the proof. ☐

Note that this holds regardless of which environment V^π is calculated in. Therefore, V^π could be computed in an environment that is entirely distinct from the training environment. For example, V^π could be computed in an environment (and using a policy) designed specifically to make this computation easy.

Proposition 10. *For any weighted L_2-norm, a minimal canonicalisation function exists and is unique.*

Proof. Let R_0 be the reward function that is 0 for all transitions. First note that the set of all reward functions that differ from R_0 by potential shaping and S'-redistribution form a linear subspace of $\mathcal{R}$. Let this space be denoted by $\mathcal{Y}$, and let $\mathcal{X}$ denote the orthogonal complement of $\mathcal{Y}$ in $\mathcal{R}$. Now any reward function $R \in \mathcal{R}$ can be uniquely expressed in the form $R_\mathcal{X} + R_\mathcal{Y}$, where $R_\mathcal{X} \in \mathcal{X}$ and $R_\mathcal{Y} \in \mathcal{Y}$. Consider the function $c : \mathcal{R} \to \mathcal{R}$ where $c(R) = R_\mathcal{X}$. Now this function is a canonicalisation function such that $n(c(R)) \leqslant R'$ for all R' such that $c(R) = c(R')$, assuming that n is a weighted L_2-norm.

To see this, we must show that

1. c is linear,
2. $c(R)$ and R differ by potential shaping and S'-redistribution,
3. $c(R_1) = c(R_2)$ if R_1 and R_2 differ by potential shaping and S'-redistribution, and
4. $n(c(R)) \leqslant n(R')$ for all R' such that $c(R) = c(R')$.

It follows directly from the construction that c is linear. To see that $c(R)$ and R differ by potential shaping and S'-redistribution, simply note that $c(R) = R - R_\mathcal{Y}$, where $R_\mathcal{Y}$ is given by a combination of potential shaping and S'-redistribution of R_0. To see that $c(R_1) = c(R_2)$ if R_1 and R_2 differ by potential shaping and S'-redistribution, let $R_2 = R_1 + R'$, where R' is given by potential shaping and S'-redistribution of R_0, and let $R_1 = R_\mathcal{X} + R_\mathcal{Y}$, where $R_\mathcal{X} \in \mathcal{X}$ and $R_\mathcal{Y} \in \mathcal{Y}$. Now $c(R_1) = R_\mathcal{X}$. Moreover, $R_2 = R_\mathcal{X} + R_\mathcal{Y} + R'$. We also have that $R' \in \mathcal{Y}$. We can thus express R_2 as $R_\mathcal{X} + (R_\mathcal{Y} + R')$, where $R_\mathcal{X} \in \mathcal{X}$ and $(R_\mathcal{Y} + R') \in \mathcal{Y}$, which implies that $c(R_2) = R_\mathcal{X}$. Therefore, if R_1 and R_2 differ by potential shaping and S'-redistribution, then $c(R_1) = c(R_2)$. To see that $c(R_1) = c(R_2)$ only if R_1 and R_2 differ by potential shaping and S'-redistribution, first note that we have already shown that R and $c(R)$ differ by potential shaping and S'-redistribution for all R. This implies that R_1 and $c(R_1)$ differ by potential shaping and S'-redistribution, and likewise for R_2 and $c(R_2)$. Then if $c(R_1) = c(R_2)$, we can combine these transformations, and obtain that R_1 and R_2 also differ by potential shaping and S'-redistribution.

To see that $n(c(R)) \leqslant n(R')$ for all R' such that $c(R) = c(R')$, first note that if $c(R) = c(R')$, then $R = R_\mathcal{X} + R_\mathcal{Y}$ and $R' = R_\mathcal{X} + R'_\mathcal{Y}$, where $R_\mathcal{X} \in \mathcal{X}$ and $R_\mathcal{Y}, R'_\mathcal{Y} \in \mathcal{Y}$. This means that $n(c(R)) = n(R_\mathcal{X})$, and $n(R') = n(R_\mathcal{X} + R'_\mathcal{Y})$. Moreover, since n is a weighted L_2-norm, and since $R_\mathcal{X}$ and $R'_\mathcal{Y}$ are orthogonal, we have that $n(R_\mathcal{X} + R_\mathcal{Y}) = \sqrt{n(R_\mathcal{X})^2 + n(R_\mathcal{Y})^2} \geqslant n(R_\mathcal{X})$. This means that $n(c(R)) \leqslant n(R')$.

To see that this canonicalisation function is the unique minimal canonicalisation function for any weighted L_2-norm n, consider an arbitrary reward function R. Now, the set of all reward functions that differ from R by potential shaping and S'-redistribution forms an affine space of $\mathcal{R}$, and a minimal canonicalisation function must map R to a point R' in this space such that $n(R') \leqslant n(R'')$ for all other points R'' in that space. If n is a weighted L_2-norm, then this specifies a convex optimisation problem with a unique solution. □

Note that this proof only shows that a minimal canonicalisation function exists and is unique when n is a (weighted) L_2-norm. It does not show that such a canonicalisation function *only* exists for these norms, nor does it show that it is unique for all norms.

Proposition 11. *If c is a canonicalisation function, then the function $n : \mathcal{R} \to \mathcal{R}$ given by $n(R) = \max_\pi J(\pi) - \min_\pi J(\pi)$ is a norm on* $\mathrm{Im}(c)$.

Proof. To show that a function n is a norm on $\mathrm{Im}(c)$, we must show that it satisfies:

1. $n(R) \geqslant 0$ for all $R \in \text{Im}(c)$.
2. $n(R) = 0$ if and only if $R = R_0$ for all $R \in \text{Im}(c)$.
3. $n(\alpha \cdot R) = \alpha \cdot n(R)$ for all $R \in \text{Im}(c)$ and all scalars α.
4. $n(R_1 + R_2) \leqslant n(R_1) + n(R_2)$ for all $R_1, R_2 \in \text{Im}(c)$.

Here R_0 is the reward function that is 0 everywhere. It is trivial to show that Axioms 1 and 3 are satisfied by n. For Axiom 2, note that $n(R) = 0$ exactly when $\max_\pi J(\pi) = \min_\pi J(\pi)$. If R is R_0, then $J(\pi) = 0$ for all π, and so the "if" part holds straightforwardly. For the "only if" part, let R be a reward function such that $\max_\pi J(\pi) = \min_\pi J(\pi)$. Then R and R_0 induce the same policy ordering under τ and μ_0, which means that they differ by potential shaping, S'-redistribution, and positive linear scaling (see Proposition 1). Moreover, since R_0 is 0 everywhere, this means that R and R_0 in fact differ by potential shaping and S'-redistribution. However, from the definition of canonicalisation functions, if $R_1, R_2 \in \text{Im}(c)$ differ by potential shaping and S'-redistribution, then it must be that $R_1 = R_2$. Hence Axiom 2 holds as well.

We can show that Axiom 4 holds algebraically:

$$\begin{aligned} n(R_1 + R_2) &= \max_\pi (J_1(\pi) + J_2(\pi)) - \min_\pi (J_1(\pi) + J_2(\pi)) \\ &\leqslant \max_\pi J_1(\pi) + \max_\pi J_2(\pi) - \min_\pi J_1(\pi) - \min_\pi J_2(\pi) \\ &= (\max_\pi J_1(\pi) - \min_\pi J_1(\pi)) + (\max_\pi J_2(\pi) - \min_\pi J_2(\pi)) \\ &= n(R_1) + n(R_2) \end{aligned}$$

This means that $n(R) = \max_\pi J(\pi) - \min_\pi J(\pi)$ is a norm on $\text{Im}(c)$. $\square$

This means that we can normalise the reward functions so that $\max_\pi J(\pi) - \min_\pi J(\pi) = 1$, which is nice. This proposition will also be useful in our later proofs.

Proposition 12. *EPIC can be expressed as*

$$D^{\text{EPIC}}(R_1, R_2) = \frac{1}{2} \cdot L_{2,\mathcal{D}} \left(\frac{C^{\text{EPIC}}(R_1)}{L_{2,\mathcal{D}}(C^{\text{EPIC}}(R_1))}, \frac{C^{\text{EPIC}}(R_2)}{L_{2,\mathcal{D}}(C^{\text{EPIC}}(R_2))} \right),$$

where $L_{2,\mathcal{D}}$ is a weighted L_2-norm.

Proof. Recall that by default, $D^{\text{EPIC}}(R_1, R_2)$ is defined to be the Pearson distance between $C^{\text{EPIC}}(R_1)(S, A, S')$ and $C^{\text{EPIC}}(R_2)(S, A, S')$, where $S, S' \sim \mathcal{D}_\mathcal{S}$ and $A \sim \mathcal{D}_\mathcal{A}$, and where the "Pearson distance" between two random variables X and Y be defined as $\sqrt{(1 - \rho(X, Y))/2}$, where ρ denotes the Pearson correlation. Recall also that $C^{\text{EPIC}}(R)(s, a, s')$ is equal to

$$R(s, a, s') + \mathbb{E}[\gamma R(s', A, S') - R(s, A, S') - \gamma R(S, A, S')].$$

For the sake of brevity, let $R_1^C = C^{\text{EPIC}}(R_1)$ and $R_2^C = C^{\text{EPIC}}(R_2)$, and let $\mathcal{D}$ be the distribution over $\mathcal{S} \times \mathcal{A} \times \mathcal{S}$ given by

$$\mathbb{P}_{(S,A,S') \sim \mathcal{D}}(S = s, A = a, S' = s) = \mathbb{P}_{S \sim \mathcal{D}_\mathcal{S}}(S = s) \cdot \mathbb{P}_{A \sim \mathcal{D}_\mathcal{A}}(A = a) \cdot \mathbb{P}_{S' \sim \mathcal{D}_\mathcal{S}}(S' = s').$$

Moreover, let $X = R_1^C(T)$ and $Y = R_2^C(T')$, where T and T' are random transitions distributed according to $\mathcal{D}$. We now have that $D^{\text{EPIC}}(R_1, R_2) = \sqrt{(1 - \rho(X, Y))/2}$, where ρ is the Pearson correlation.

Next, recall that the Pearson correlation between two random variables X and Y is defined as

$$\frac{\mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]}{\sigma_X \sigma_Y},$$

where σ_X and σ_Y are the standard deviations of X and Y. Recall also that the standard deviation σ_X of a random variable X is equal to $\sqrt{\mathbb{E}[X^2] - \mathbb{E}[X]^2}$.

Next, note that we in this case have that $\mathbb{E}[X]$ and $\mathbb{E}[Y]$ both are equal to 0. This follows from the way that these variables were defined, together with the linearity of expectation. Therefore, we can rewrite $\rho(X,Y)$ as

$$\frac{\mathbb{E}[XY]}{\sqrt{\mathbb{E}[X^2]}\sqrt{\mathbb{E}[Y^2]}}.$$

Let W be the $(|\mathcal{S}||\mathcal{A}||\mathcal{S}|) \times (|\mathcal{S}||\mathcal{A}||\mathcal{S}|)$-dimensional diagonal matrix in which the diagonal value that corresponds to transition t is equal to $\sqrt{\mathcal{P}_{T\sim\mathcal{D}}(T=t)}$. We now have that $\sqrt{\mathbb{E}[X^2]} = L_2(WR_1^C)$ and $\sqrt{\mathbb{E}[Y^2]} = L_2(WR_2^C)$. Moreover, we also have that $\mathbb{E}[XY] = (WR_1^C) \cdot (WR_2^C)$. Next, recall that the dot product $v \cdot w$ between two vectors v and w can be written as $L_2(v) \cdot L_2(w) \cdot \cos(\theta)$, where θ is the angle between v and w. This means that we can rewrite $\rho(X,Y)$ as

$$\frac{L_2(WR_1^C) \cdot L_2(WR_2^C) \cdot \cos(\theta)}{L_2(WR_1^C) \cdot L_2(WR_2^C)} = \cos(\theta),$$

where θ is the angle between WR_1^C and WR_2^C.

Since the angle between two vectors is unaffected by the scale of those vectors, we have that θ is also the angle between $WR_1^C/L_2(WR_1^C)$ and $WR_2^C/L_2(WR_2^C)$. We can now apply the Law of Cosines, and conclude that the L_2-distance between $WR_1^C/L_2(WR_1^C)$ and $WR_2^C/L_2(WR_2^C)$ is equal to $\sqrt{2-2\cos(\theta)} = \sqrt{2-2\rho(X,Y)} = 2 \cdot D^{\text{EPIC}}(R_1,R_2)$. This means that

$$D^{\text{EPIC}}(R_1,R_2) = \frac{1}{2} \cdot L_2\left(\frac{WR_1^C}{L_2(WR_1^C)}, \frac{WR_2^C}{L_2(WR_2^C)}\right).$$

Rewriting this completes the proof. □

The fact that EPIC can be expressed in this form is also asserted in Gleave et al. (2020), but a proof is not given. We have provided the proof here, to make the equivalence more accessible and intuitive. It is worth noting that this equivalence only holds when the "coverage distribution" is the same as the distribution used to compute C^{EPIC}, which is left somewhat ambiguous in Gleave et al. (2020).

F Main Proofs

In this section, we give the proofs of all our results. We have divided it into four parts. In the first part, we prove that STARC metrics are sound. In the second part, we prove that STARC metrics are complete. In the third part, we prove that EPIC (and a class of similar metrics) all are subject to the results discussed in Appendix A. In the final part, we prove a few remaining theorems.

F.1 Soundness

Before we can give the proof of Theorem 1, we will first state and prove several supporting lemmas.

Lemma 1. *For any reward functions R_1 and R_2, and any policy π, we have that*

$$|J_1(\pi) - J_2(\pi)| \leqslant \left(\frac{1}{1-\gamma}\right) L_\infty(R_1,R_2).$$

Proof. This follows from straightforward algebra:

$$
\begin{aligned}
|J_1(\pi) - J_2(\pi)| =& \Bigg|\mathbb{E}_{\xi\sim\pi}\left[\sum_{t=0}^{\infty}\gamma^t R_1(S_t, A_t, S_{t+1})\right] \\
&- \mathbb{E}_{\xi\sim\pi}\left[\sum_{t=0}^{\infty}\gamma^t R_2(S_t, A_t, S_{t+1})\right]\Bigg| \\
&= \sum_{t=0}^{\infty}\gamma^t |\mathbb{E}_{\xi\sim\pi}[R_1(S_t, A_t, S_{t+1}) - R_2(S_t, A_t, S_{t+1})]| \\
&\leqslant \sum_{t=0}^{\infty}\gamma^t \mathbb{E}_{\xi\sim\pi}[|R_1(S_t, A_t, S_{t+1}) - R_2(S_t, A_t, S_{t+1})|] \\
&\leqslant \sum_{t=0}^{\infty}\gamma^t L_\infty(R_1, R_2) = \left(\frac{1}{1-\gamma}\right) L_\infty(R_1, R_2).
\end{aligned}
$$

Here the second line follows from the linearity of expectation, and the third line follows from Jensen's inequality. □

Thus, the L_∞-distance between two rewards bounds the difference between their policy evaluation functions. Since all norms are bilipschitz equivalent on any finite-dimensional vector space, this extends to all norms:

Lemma 2. *If p is a* norm*, then there is a positive constant K_p such that, for any reward functions R_1 and R_2, and any policy π, $|J_1(\pi) - J_2(\pi)| \leqslant K_p \cdot p(R_1, R_2)$.*

Proof. If p and q are norms on a finite-dimensional vector space, then there are constants k and K such that $k \cdot p(x) \leqslant q(x) \leqslant K \cdot p(x)$. Since $\mathcal{S}$ and $\mathcal{A}$ are finite, $\mathcal{R}$ is a finite-dimensional vector space. This means that there is a constant K such that $L_\infty(R_1, R_2) \leqslant K \cdot p(R_1, R_2)$. Together with Lemma 1, this implies that

$$
|J_1(\pi) - J_2(\pi)| \leqslant \left(\frac{1}{1-\gamma}\right) \cdot K \cdot m(R_1, R_2).
$$

Letting $K_p = \left(\frac{K}{1-\gamma}\right)$ completes the proof. □

Note that the constant K_p given by Lemma 2 may not be the smallest value of K for which this statement holds of a given norm p. This fact can be used to compute tighter bounds for particular STARC-metrics.

Lemma 3. *Let R_1 and R_2 be reward functions, and π_1, π_2 be two policies. If $|J_1(\pi) - J_2(\pi)| \leqslant U$ for $\pi \in \{\pi_1, \pi_2\}$, and if $J_2(\pi_2) \geqslant J_2(\pi_1)$, then*

$$
J_1(\pi_1) - J_1(\pi_2) \leqslant 2 \cdot U.
$$

Proof. First note that U must be non-negative. Next, note that if $J_1(\pi_1) < J_1(\pi_2)$ then $J_1(\pi_1) - J_1(\pi_2) < 0$, and so the lemma holds. Now consider the case when $J_1(\pi_1) \geqslant J_1(\pi_2)$:

$$
\begin{aligned}
J_1(\pi_1) - J_1(\pi_2) &= J_1(\pi_1) - J_2(\pi_2) + J_2(\pi_2) - J_1(\pi_2) \\
&\leqslant |J_1(\pi_1) - J_2(\pi_2)| + |J_2(\pi_2) - J_1(\pi_2)|
\end{aligned}
$$

Our assumptions imply that $|J_2(\pi_2) - J_1(\pi_2)| \leqslant U$. We will next show that $|J_1(\pi_1) - J_2(\pi_2)| \leqslant U$ as well. Our assumptions imply that

$$
\begin{aligned}
&|J_1(\pi_1) - J_2(\pi_1)| \leqslant U \\
\implies &J_2(\pi_1) \geqslant J_1(\pi_1) - U \\
\implies &J_2(\pi_2) \geqslant J_1(\pi_1) - U
\end{aligned}
$$

Here the last implication uses the fact that $J_2(\pi_2) \geqslant J_2(\pi_1)$. A symmetric argument also shows that $J_1(\pi_1) \geqslant J_2(\pi_2) - U$ (recall that we assume that $J_1(\pi_1) \geqslant J_1(\pi_2)$). Together, this implies that $|J_1(\pi_1) - J_2(\pi_2)| \leqslant U$. We have thus shown that if $J_1(\pi_1) \geqslant J_1(\pi_2)$ then

$$|J_1(\pi_1) - J_2(\pi_2)| + |J_2(\pi_2) - J_1(\pi_2)| \leqslant 2 \cdot U,$$

and so the lemma holds. This completes the proof. □

Lemma 4. *For any linear function $c : \mathbb{R}^n \to \mathbb{R}^n$ and any norm n, there is a positive constant K_n such that $n(c(v)) \leqslant K_n \cdot n(v)$ for all $v \in \mathbb{R}^n$.*

Proof. First consider the case when $n(v) > 0$. In this case, we can find an upper bound for $n(c(v))$ in terms of $n(v)$ by finding an upper bound for $\frac{n(c(R))}{n(R)}$. Since c is linear, and since n is absolutely homogeneous, we have that for any $v \in \mathbb{R}^n$ and any non-zero $\alpha \in \mathbb{R}$,

$$\frac{n(c(\alpha \cdot v))}{n(\alpha \cdot v)} = \left(\frac{\alpha}{\alpha}\right) \frac{n(c(v))}{n(v)} = \frac{n(c(v))}{n(v)}.$$

In other words, $\frac{n(c(v))}{n(v)}$ is unaffected by scaling of v. We may thus restrict our attention to the unit ball of n. Next, since the surface of the unit ball of n is a compact set, and since $\frac{n(c(v))}{n(v)}$ is continuous on this surface, the extreme value theorem implies that $\frac{n(c(v))}{n(v)}$ must take on some maximal value K_n on this domain. Together, the above implies that $n(c(v)) \leqslant K_n \cdot n(v)$ for all R such that $n(v) > 0$.

Next, suppose $n(v) = 0$. In this case, v is the zero vector. Since c is linear, this implies that $c(v) = v$, which means that $n(c(v)) = 0$ as well. Therefore, if $n(v) = 0$, then the statement holds for any K_n. In particular, it holds for the value K_n selected above. □

Lemma 5. *Let c be a linear function $c : \mathcal{R} \to \mathcal{R}$, and let n be a norm on $\mathrm{Im}(c)$. Let R be any reward function, let $R_C = c(R)$, and let $R_S = \left(\frac{R_C}{n(R_C)}\right)$ if $n(R_C) > 0$, and R_C otherwise. Assume there is a constant B such that $J_C(\pi) = J(\pi) + B$ for all π. Then $J(\pi_1) - J(\pi_2) = n(c(R)) \cdot J_S(\pi_1) - J_S(\pi_2)$.*

Proof. Let us first consider the case where $n(R) = 0$. Since n is a norm, R must be the reward function that is 0 everywhere. Since c is linear, this also implies that $n(c(R)) = 0$. In that case, both $J(\pi_1) - J(\pi_2) = 0$ for all π_1 and π_2, and $n(c(R)) \cdot x = 0$ for all x. Therefore, the statement holds.

Let us next consider the case when $n(R) > 0$. Since c is linear, this means that $n(c(R)) > 0$. Moreover, since $R_S = R_C / n(R_C)$, and since $J_C(\pi) = J(\pi) + B$, we have that

$$J_S(\pi) = \left(\frac{1}{n(c(R))}\right)(J(\pi) + B)$$

This further implies that

$$J_S(\pi_1) - J_S(\pi_2) = \left(\frac{1}{n(c(R))}\right)(J(\pi_1) - J(\pi_2))$$

since the B-terms cancel out. By rearranging, we get that

$$J(\pi_1) - J(\pi_2) = n(c(R))(J_S(\pi_1) - J_S(\pi_2)).$$

This completes the proof. □

Lemma 6. *If R_1 and R_2 differ by potential shaping with Φ, then for any τ and μ_0, we have that $J_2(\pi) = J_1(\pi) - \mathbb{E}_{S_0 \sim \mu_0}[\Phi(S_0)]$. Moreover, if R_1 and R_2 differ by potential shaping with Φ and S'-redistribution for τ, then for any μ_0, we have that $J_2(\pi) = J_1(\pi) - \mathbb{E}_{S_0 \sim \mu_0}[\Phi(S_0)]$.*

Proof. The first part follows from Lemma B.1 in Skalse et al. (2022a). The second part then follows straightforwardly from the properties of S'-redistribution. □

Theorem 1. *Any STARC metric is sound.*

Proof. Consider any transition function τ and any initial state distribution μ_0, and let d be a STARC metric. We wish to show that there exists a positive constant U, such that for any R_1 and R_2, and any pair of policies π_1 and π_2 such that $J_2(\pi_2) \geqslant J_2(\pi_1)$, we have that

$$J_1(\pi_1) - J_1(\pi_2) \leqslant (\max_\pi J_1(\pi) - \min_\pi J_1(\pi)) \cdot K_d \cdot d(R_1, R_2).$$

Recall that $d(R_1, R_2) = m(s(R_1), s(R_2))$, where m is an admissible metric. Since m is admissible, we have that $p(s(R_1), s(R_2)) \leqslant K_m \cdot m(s(R_1), s(R_2))$ for some norm p and constant K_m. Moreover, since p is a norm, we can apply Lemma 2 to conclude that there is a constant K_p such that for any policy π, we have that

$$|J_1^S(\pi) - J_2^S(\pi)| \leqslant K_p \cdot p(s(R_1), s(R_2)),$$

where J_1^S is the policy evaluation function of $s(R_1)$, and J_2^S is the policy evaluation function of $s(R_2)$. Combining this with the fact that $p(s(R_1), s(R_2)) \leqslant K_m \cdot m(s(R_1), s(R_2))$, we get

$$\begin{aligned} |J_1^S(\pi) - J_2^S(\pi)| &\leqslant K_p \cdot p(s(R_1), s(R_2)) \\ &\leqslant K_p \cdot K_m \cdot m(s(R_1), s(R_2)) \\ &= K_{mp} \cdot d(R_1, R_2) \end{aligned}$$

where $K_{mp} = K_p \cdot K_m$. We have thus established that, for any π, we have

$$|J_1^S(\pi) - J_2^S(\pi)| \leqslant K_{mp} \cdot d(R_1, R_2).$$

Let π_1 and π_2 be any two policies such that $J_2(\pi_2) \geqslant J_2(\pi_1)$. Note that $J_2(\pi_2) \geqslant J_2(\pi_1)$ if and only if $J_2^S(\pi_2) \geqslant J_2^S(\pi_1)$. We can therefore apply Lemma 3 and conclude that

$$J_1^S(\pi_1) - J_1^S(\pi_2) \leqslant 2 \cdot K_{mp} \cdot d(R_1, R_2).$$

By Lemma 6, there is a constant B such that $J_1^C = J_1 + B$. We can therefore apply Lemma 5:

$$J_1(\pi_1) - J_1(\pi_2) \leqslant n(c(R_1)) \cdot 2 \cdot K_{mp} \cdot d(R_1, R_2).$$

We have that n is a norm on $\mathrm{Im}(c)$. Moreover, $\max_\pi J_1(\pi) - \min_\pi J_1(\pi)$ is also a norm on $\mathrm{Im}(c)$ (Proposition 4). Since $\mathrm{Im}(c)$ is a finite-dimensional vector space, this means that there is a constant K_s such that $n(c(R_1)) \leqslant K_s \cdot (\max_\pi J_1(\pi) - \min_\pi J_1(\pi))$ for all $R_1 \in \mathcal{R}$. Let $U = 2 \cdot K_{mp} \cdot K_s$. We have now established that, for any π_1 and π_2 such that $J_2(\pi_2) \geqslant J_2(\pi_1)$, we have

$$J_1(\pi_1) - J_1(\pi_2) \leqslant (\max_\pi J_1(\pi) - \min_\pi J_1(\pi)) \cdot U \cdot d(R_1, R_2).$$

This completes the proof. □

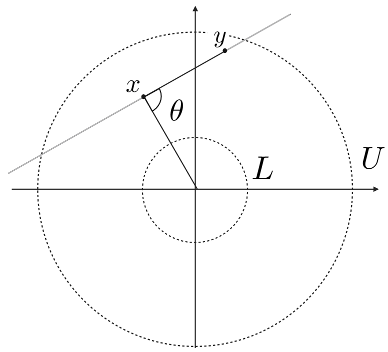
F.2 COMPLETENESS

In this section we give the proofs that concern completeness. We will need the following lemma:

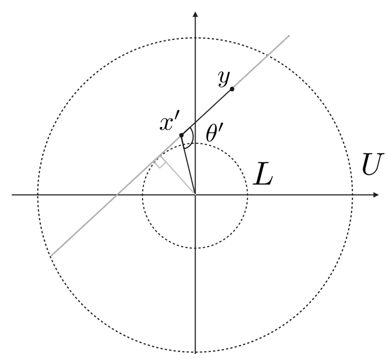
Lemma 7. *Let $S \subset \mathbb{R}^n$ be the boundary of a bounded convex set whose interior includes the origin. Then there is an $\alpha > 0$ such that for any $x, y \in S$, the angle between x and $y - x$ is at least α.*

Proof. Let L be the largest sphere which is centred around the origin, and whose interior does not intersect S. Note that since the interior of S contains the origin, the radius of L is positive. Similarly, let U be the smallest sphere which is centred around the origin, and whose exterior does not intersect S. (In other words, L and U are two spheres such that S lies "between" L and U. Note that if S is a sphere centred around the origin, then $L = U$.)

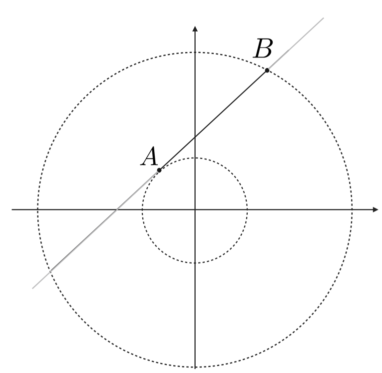
Let x and y be two arbitrary points in S, and let θ be the angle between $-x$ and $y - x$:



Consider the line that passes through x and y. This line cannot intersect the interior of L, since S is the boundary of a *convex* set. Note also that θ gets bigger if we reduce the magnitude of x. Thus, let x' be the vector that results from reducing the magnitude of x until the line between x' and y is a tangent of L:



Now the angle θ between $-x$ and $y - x$ is at most as big as the angle θ' between $-x'$ and $y - x'$. Next, let the point where the line between x' and y intersects L be called A, and the point where it intersects U be called B. Consider the line segment between A and B:



This line segment is a compact set, which means that there is a point x'' along this line which maximises the angle between $-x''$ and $y - x''$ (note that this point in fact is equal to B, but we will not need this fact in our proof). Let this angle be θ''. We now have that $\theta > \theta' > \theta''$. Moreover, the value of θ'' does not depend on x or y, which means that the angle θ between $-x$ and $y - x$ is at most θ for all points $x, y \in S$. This in turn means that the angle between x and $y - x$ is at least $\alpha = \pi - \theta$, which completes the proof. $\square$

Using this, we can now show that we can get a lower bound on the *angle* between two standardised reward functions in terms of their STARC-distance:

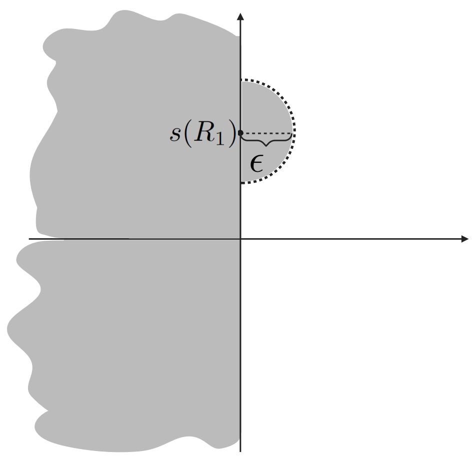
Lemma 8. *For any STARC metric d, there exist an $\ell_1 \in \mathbb{R}^+$ such that the angle θ between $s(R_1)$ and $s(R_2)$ satisfies $\ell_1 \cdot d(R_1, R_2) \leqslant \theta$ for all R_1, R_2 for which neither $s(R_1)$ or $s(R_2)$ is 0.*

Proof. Let d be an arbitrary STARC-metric, and let R_1 and R_2 be two arbitrary reward functions for which neither $s(R_1)$ or $s(R_2)$ is 0. Recall that $d(R_1, R_2) = m(s(R_1), s(R_2))$, where m is a metric that is bilipschitz equivalent to some norm. Since all norms are bilipschitz equivalent on any finite-dimensional vector space, this means that m is bilipschitz equivalent to the L_2-norm. Thus, there are positive constants p, q such that

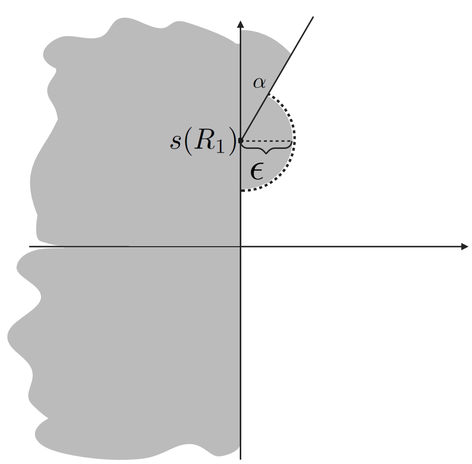
$$p \cdot m(s(R_1), s(R_2)) \leqslant L_2(s(R_1), s(R_2)) \leqslant q \cdot m(s(R_1), s(R_2)).$$

In particular, the L_2-distance between $s(R_1)$ and $s(R_2)$ is at least $\epsilon = p \cdot d(R_1, R_2)$. For the rest of our proof, it will be convenient to assume that $\epsilon < L_2(s(R_1))$; this can be ensured by picking a p that is sufficiently small.

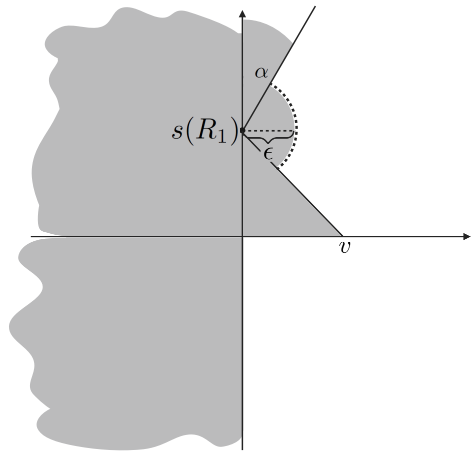
Let us plot the plane which contains $s(R_1)$, $s(R_2)$, and the origin, and orient it so that $s(R_1)$ points straight up, and so that $s(R_2)$ is not on the left-hand side:



Since the distance between $s(R_1)$ and $s(R_2)$ is at least ϵ, and since $s(R_2)$ is not on the left-hand side, we know that $s(R_2)$ cannot be inside of the region shaded grey in the figure above (though it may be on the boundary). Moreover, as per Lemma 7, we know that the angle between $s(R_1)$ and $s(R_2) - s(R_1)$ is at least α, where $\alpha > 0$. This means that we also can rule out the following region:

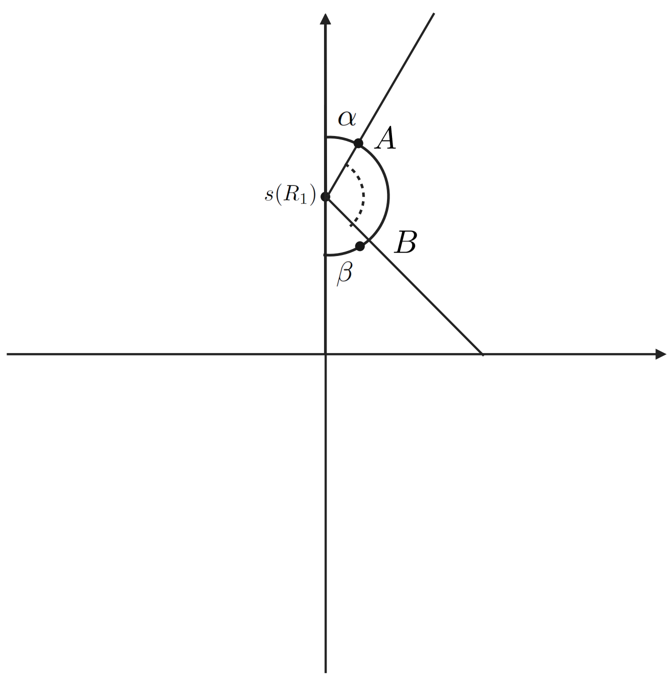


Moreover, let v be the element of $\mathrm{Im}(s)$ that is perpendicular to $s(R_1)$, lies on a plane with $s(R_1)$, $s(R_2)$, and the origin, and points in the same direction as $s(R_2)$ within this plane. Since $\mathrm{Im}(s)$ is convex, we know that $s(R_2)$ cannot lie within the triangle formed by the x-axis, the y-axis, and the line between $s(R_1)$ and v:

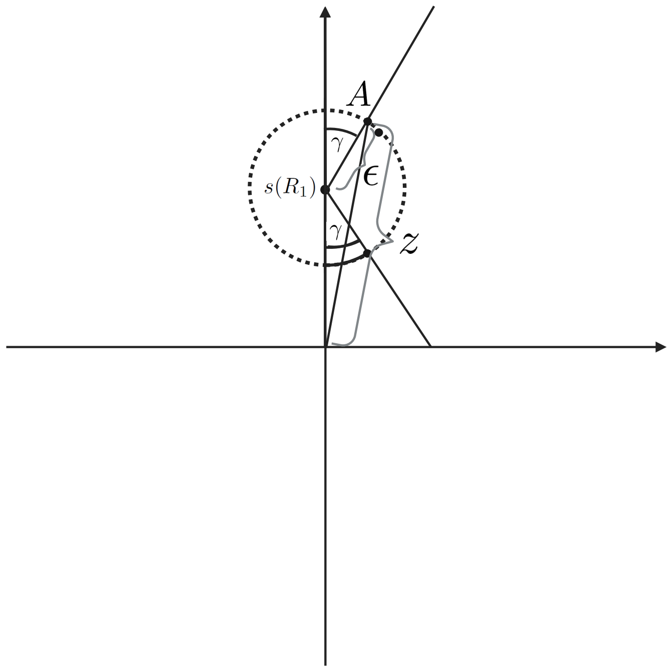


Since $\mathrm{Im}(s)$ is closed and convex, we know that there is a vector a in $\mathrm{Im}(s)$ whose L_2-norm is bigger than all other vectors in $\mathrm{Im}(s)$, and a (non-zero) vector b in $\mathrm{Im}(s)$ whose L_2-norm is smaller than all other (non-zero) vectors in $\mathrm{Im}(s)$. From this, we can infer that the angle between $s(R_1)$ and $v - s(R_1)$ is at least $\beta = \arctan(b/a)$. Also note that $\beta > 0$.

We now have everything we need to derive a lower bound on the angle θ between $s(R_1)$ and $s(R_2)$. First note that this angle can be no greater than the angle between $s(R_1)$ and the points marked A and B in the figure below (whichever is smaller):



To make things easier, replace both α and β with $\gamma = \min(\alpha, \beta)$. Since this makes the shaded region smaller, we still have that $s(R_2)$ cannot be in the interior of the new shaded region. Moreover, in this case, we know that the angle between $s(R_1)$ and $s(R_2)$ is no smaller than the angle θ' between $s(R_1)$ and the point marked A:



Deriving this angle is now just a matter of trigonometry. Letting z denote $L_2(A)$, we have that:

$$\frac{\epsilon}{\sin(x)} = \frac{z}{\sin(\pi - \gamma)} = \frac{z}{\sin(\gamma)}$$

From this, we get that

$$\begin{aligned}\theta' &= \arcsin\left(\left(\frac{\epsilon}{z}\right)\sin(\gamma)\right)\\ &\geqslant \left(\frac{\epsilon}{z}\right)\sin(\gamma)\end{aligned}$$

Moreover, it is also straightforward to find an upper bound z' for z. Specifically, we have that $z^2 = L_2(s(R_1))^2 + \epsilon^2 - 2L_2(s(R_1))\epsilon\cos(\pi - \gamma)$. Since $\epsilon < L_2(s(R_1))$, this means that

$$z < \sqrt{2L_2(s(R_1))^2 - 2L_2(s(R_1))^2\cos(\pi - \gamma)}.$$

Moreover, since $\mathrm{Im}(s)$ is compact, there is a vector a in $\mathrm{Im}(s)$ whose L_2-norm is bigger than all other vectors in $\mathrm{Im}(s)$. We thus know that

$$z < z' = \sqrt{2L_2(a)^2 - 2L_2(a)^2\cos(\pi - \gamma)}.$$

Putting this together, we have that

$$\theta \geqslant \theta' \geqslant \left(\frac{\epsilon}{z'}\right)\sin(\gamma) = m(s(R_1), s(R_2)) \cdot p \cdot \left(\frac{\sin(\gamma)}{z'}\right).$$

Setting $\ell_1 = p \cdot \left(\frac{\sin(\gamma)}{z'}\right)$ thus completes the proof. $\square$

Finally, before we can give the full proof, we will also need the following:

Lemma 9. *For any invertible matrix $M : \mathbb{R}^n \to \mathbb{R}^n$ there is an $\ell_2 \in (0, 1]$ such that for any $v, w \in \mathbb{R}^n$, the angle θ' between Mv and Mw satisfies $\theta' \geqslant \ell_2 \cdot \theta$, where θ is the angle between v and w.*

Proof. We will first prove that this holds in the 2-dimensional case, and then extend this proof to the general n-dimensional case.

Let M be an arbitrary invertible matrix $\mathbb{R}^2 \to \mathbb{R}^2$. First note that we can factor M via Singular Value Decomposition into three matrices U, Σ, V, such that $M = U\Sigma V^\top$, where U and V are orthogonal matrices, and Σ is a diagonal matrix with non-negative real numbers on the diagonal. Since M is invertible, we also have that Σ cannot have any zeroes along its diagonal. Next, recall that orthogonal matrices preserve angles. This means that we can restrict our focus to just Σ.[6]

Let α and β be the singular values of M. We may assume, without loss of generality, that

$$\Sigma = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

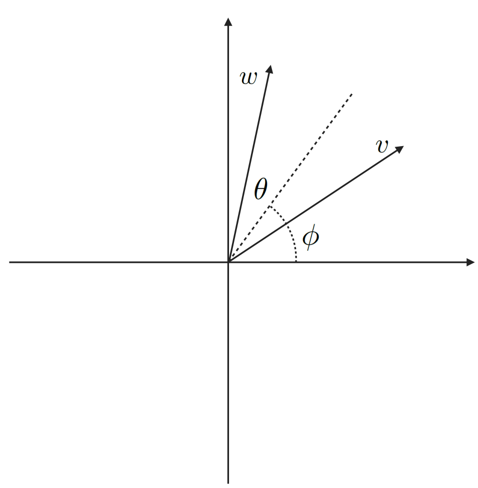
Moreover, since scaling the x and y-axes uniformly will not affect the angle between any vectors after multiplication, we can instead equivalently consider the matrix

$$\Sigma = \begin{pmatrix} \alpha/\beta & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $v, w \in \mathbb{R}^2$ be two arbitrary vectors with angle θ, and let θ' be the angle between Σv and Σw. We will derive a lower bound on θ' expressed in terms of θ. Moreover, since the angle between v and w is not affected by their magnitude, we will assume (without loss of generality) that both v and w have length 1 (under the L_2-norm).

First, note that if $\theta = \pi$ then $v = -w$. This means that $\Sigma v = -\Sigma w$, since Σ is a linear transformation, which in turn means that $\theta' = \pi$. Thus $\theta' \geqslant \ell_2 \cdot \theta$ as long as $\ell_2 \leqslant 1$. Next, assume that $\theta < \pi$.

We may assume (without loss of generality) that the angle between v and the x-axis is no bigger than the angle between w and the x-axis. Let ϕ be the angle between the x-axis and the vector that is in the middle between v and w. This means that we can express v as $(\cos(\phi - \theta/2), \sin(\phi - \theta/2))$ and w as $(\cos(\phi + \theta/2), \sin(\phi + \theta/2))$. Moreover, since reflection along either of the axes will not change the angle between either v and w or Σv and Σw, we may assume (without loss of generality) that $\phi \in [0, \pi/2]$. For convenience, let $\sigma = \alpha/\beta$.



[6] If there are vectors x, y such that the angle between x and y is θ and the angle between Mx and My is θ', then there are vectors v, w such that the angle between x and y is θ and the angle between Σv and Σw is θ', and vice versa.

(Note that we can visualise the action of Σ as scaling the x-axis in the figure above by σ.)

We now have that $\Sigma v = (\sigma\cos(\phi - \theta/2), \sin(\phi - \theta/2))$ and $\Sigma w = (\sigma\cos(\phi + \theta/2), \sin(\phi + \theta/2))$. Using the dot product, we get that

$$\cos(\theta') = \frac{\sigma^2\cos(\phi - \theta/2)\cos(\phi + \theta/2) + \sin(\phi - \theta/2)\sin(\phi + \theta/2)}{\sqrt{\sigma^2\cos^2(\phi - \theta/2) + \sin^2(\phi - \theta/2)}\sqrt{\sigma^2\cos^2(\phi + \theta/2) + \sin^2(\phi + \theta/2)}}.$$

We next note that if $\theta \in [0, \pi)$ and $\phi \in [0, \pi/2]$, then the derivative of this expression with respect to ϕ can only be 0 when $\phi \in \{0, \pi/2\}$.[7] This means that $\cos(\theta')$ must be maximised or minimised when ϕ is either 0 or $\pi/2$, which in turn means that the angle θ' must be minimised or maximised when ϕ is either 0 or $\pi/2$.

It is now easy to see that if $\sigma > 1$ then θ' is minimised when $\phi = 0$, and that if $\sigma < 1$ then θ' is minimised when $\phi = \pi/2$. Moreover, if $\phi = \pi/2$, then

$$\theta' = 2\arctan\left(\frac{\sigma\cos(\pi/2 - \theta/2)}{\sin(\pi/2 - \theta/2)}\right) = 2\arctan\left(\sigma\tan(\theta/2)\right),$$

which in turn is greater than $\theta \cdot \sigma$ when $\sigma < 1$. Similarly, if $\phi = 0$, then

$$\theta' = 2\arctan\left(\frac{\sin(\theta/2)}{\sigma\cos(\theta/2)}\right) = 2\arctan\left(\sigma^{-1}\tan(\theta/2)\right),$$

which is in turn greater than $\sigma^{-1} \cdot \theta$ when $\sigma > 1$. In either case, we thus have that

$$\theta' \geqslant \theta \cdot \min(\sigma, \sigma^{-1}) = \theta \cdot \min(\beta/\alpha, \alpha/\beta).$$

We have therefore show that, for any invertible matrix $M : \mathbb{R}^2 \to \mathbb{R}^2$, there exists a positive constant $\min(\beta/\alpha, \alpha/\beta)$, where α and β are the singular values of M, such that if $v, w \in \mathbb{R}^2$ have angle θ, then the angle between Mv and Mw is at least $\theta \cdot \min(\beta/\alpha, \alpha/\beta)$.

To generalise this to the general n-dimensional case, let $v, w \in \mathbb{R}^n$ be two arbitrary vectors. Consider the 2-dimensional linear subspace given by $S = \text{span}(v, w)$, and note that $M(S)$ also is a 2-dimensional linear subspace of $\mathbb{R}^n$ (since M is linear and invertible). The linear transformation which M induces between S and $M(S)$ is isomorphic to a linear transformation $M' : \mathbb{R}^2 \to \mathbb{R}^2$.[8] We can thus apply our previous result for the two-dimensional case, and conclude that if the angle between v and w is θ, then the angle between Mv and Mw is at least $\theta \cdot \min(\beta/\alpha, \alpha/\beta)$, where α and β are the singular values of M'. Next, note that the singular values of M' cannot be smaller than the smallest singular values of M or bigger than the biggest singular values of M. We can therefore let $\ell_2 = \alpha/\beta$, where α is the smallest singular value of M and β is the greatest singular value of M, and conclude that the angle between Mv and Mw must be at least $\ell_2 \cdot \theta$. Since the value of ℓ_2 does not depend on v or w, this completes the proof. □

With these lemmas, we can now finally prove that all STARC metrics are complete:

Theorem 2. *Any STARC metric is complete.*

Proof. Let d be an arbitrary STARC metric. We need to show that there exists a positive constant L such that, for any reward functions R_1 and R_2, there are two policies π_1, π_2 with $J_2(\pi_2) \geqslant J_2(\pi_1)$ and

$$J_1(\pi_1) - J_1(\pi_2) \geqslant L \cdot (\max_\pi J_1(\pi) - \min_\pi J_1(\pi)) \cdot d(R_1, R_2),$$

and moreover, if both $\max_\pi J_1(\pi) - \min_\pi J_1(\pi) = 0$ and $\max_\pi J_2(\pi) - \min_\pi J_2(\pi) = 0$, then we have that $d(R_1, R_2) = 0$.

[7]For example, this may be verified using tools such as Wolfram Alpha.

[8]To see this, let A be an orthonormal matrix that rotates $\mathbb{R}^2$ to align with S, and let B be an orthonormal matrix that rotates $M(S)$ to align with $\mathbb{R}^2$. Now $M' = BMA$ is an invertible linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$. Moreover, since orthonormal matrices preserve the angles between vectors, we have that $v, w \in S$ have angle θ and $Mv, Mw \in M(S)$ have angle θ', if and only if $A^{-1}v, A^{-1}w \in \mathbb{R}^2$ have angle θ and $BMv, BMw \in \mathbb{R}^2$ have angle θ'. Note that $M'A^{-1}v = BMv$ and $M'A^{-1}w = BMw$. This means that there are $v, w \in S$ such that v, w have angle θ and Mv, Mw have angle θ', if and only if there are $v', w' \in \mathbb{R}^2$ such that v', w' have angle θ and $M'v'$ and $M'w'$ have angle θ' (with $v' = A^{-1}v$ and $w' = A^{-1}w$).

We first note that the last condition holds straightforwardly. If $\max_\pi J_1(\pi) - \min_\pi J_1(\pi) = 0$ and $\max_\pi J_2(\pi) - \min_\pi J_2(\pi) = 0$ then R_1 and R_2 have the same policy order, which means that Proposition 6 implies that $d(R_1, R_2) = 0$. This condition is therefore satisfied.

For the first condition, first note that if $\max_\pi J_1(\pi) - \min_\pi J_1(\pi) = 0$, then the statement holds trivially for any non-negative L (since the other two terms on the right-hand side of the inequality are strictly non-negative).

Let us next consider the case where both $\max_\pi J_1(\pi) - \min_\pi J_1(\pi) > 0$ and $\max_\pi J_1(\pi) - \min_\pi J_1(\pi) > 0$. We need to introduce a new definition. Let $m : \Pi \to \mathbb{R}^{|\mathcal{S}||\mathcal{A}||\mathcal{S}|}$ be the function that takes a policy π, and returns the vector where $m(\pi)[s, a, s'] = \sum_{t=0}^{\infty} \gamma^t \mathbb{P}(S_t, A_t, S_{t+1} = s, a, s')$, where the probability is for a trajectory sampled from π under τ and μ_0. In other words, m returns the long-run discounted cumulative probability with which π visits each transition. Next, note that $J(\pi) = m(\pi) \cdot R$. This means that m can be used to decompose J into two steps, the first of which is independent of the reward function, and the second of which is a linear function.

We will use d to derive a lower bound on the angle θ between the level sets of J_1 and J_2 in $\text{Im}(m)$. We will then show that $\text{Im}(m)$ contains an open set with a certain diameter. From this, we can find two policies that incur a certain amount of regret.

First, by Lemma 8, there exists an ℓ_1 such that for any non-trivial R_1 and R_2, the angle between $s(R_1)$ and $s(R_2)$ is at least $\ell_1 \cdot d(R_1, R_2)$. To make our proof easier, we will assume that we pick an ℓ_1 that is small enough to ensure that $\ell_1 \cdot d(R_1, R_2) \leqslant \pi/2$ for all R_1, R_2.

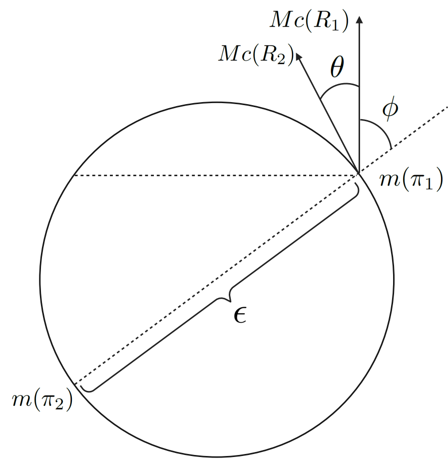
Note that $s(R_1)$ and $s(R_2)$ may not be parallel with $\text{Im}(m)$, which means that the angle between $s(R_1)$ and $s(R_2)$ may not be the same as the angle between the level sets of J_1 and J_2 in $\text{Im}(m)$. Therefore, consider the matrix M that projects $\text{Im}(c)$ onto the linear subspace of $\mathcal{R}$ that is parallel to $\text{Im}(m)$, where c is the canonicalisation function of d. Now the angle between $Ms(R_1)$ and $Ms(R_2)$ is the same as the angle between the level sets of the linear functions which J_1 and J_2 induce on $\text{Im}(m)$. Moreover, note that M is invertible, since any two reward functions in $\text{Im}(c)$ induce different policy orderings except when they differ by positive linear scaling (Proposition 1). We can therefore apply Lemma 8, and conclude that there exists an $\ell_2 \in (0, 1]$, such that the angle θ between the level sets of J_1 and J_2 in $\text{Im}(m)$ is at least $\ell_2 \cdot \ell_1 \cdot d(R_1, R_2)$. Moreover, since $\ell_1 \cdot d(R_1, R_2)$ is at most $\pi/2$, and since $\ell_2 \leqslant 1$, we have that $\ell_2 \cdot \ell_1 \cdot d(R_1, R_2)$ is at most $\pi/2$.

This gives us that, for any two policies π_1, π_2, we have:

$$
\begin{aligned}
J_1(\pi_1) - J_1(\pi_2) &= J_1^C(\pi_1) - J_1^C(\pi_2) \\
&= c(R_1)m(\pi_1) - c(R_1)m(\pi_2) \\
&= c(R_1)(m(\pi_1) - m(\pi_2)) \\
&= M(c(R_1))(m(\pi_1) - m(\pi_2)) \\
&= L_2(M(c(R_1))) \cdot L_2(m(\pi_1) - m(\pi_2)) \cdot \cos(\phi)
\end{aligned}
$$

where ϕ is the angle between $M(c(R_1))$ and $m(\pi_1) - m(\pi_2)$, and J_1^C is the evaluation function of $c(R_1)$. Note that the first line follows from Lemma 6. We can thus derive a lower bound on worst-case regret by deriving a lower bound for the greatest value of this expression.

We have that $\text{Im}(m)$ contains a set that is open in the smallest affine space which contains $\text{Im}(m)$ (see Skalse et al., 2022b). This means that there is an ϵ such that $\text{Im}(m)$ contains a sphere of diameter ϵ. We will show that we always can find two policies within this sphere that incur a certain amount of regret. Consider the 2-dimensional cut which goes through the middle of this sphere and is parallel with the normal vectors of the level sets of J_1 and J_2. The intersection between this cut and the ϵ-sphere forms a 2-dimensional circle with diameter ϵ. Let π_1, π_2 be the two policies for which $m(\pi_1)$ and $m(\pi_2)$ lie opposite to each other on this circle, and satisfy that $J_2(\pi_1) = J_2(\pi_2)$ (or, equivalently, that $Mc(R_1) \cdot m(\pi_1) = Mc(R_1) \cdot m(\pi_2)$). Without loss of generality, we may assume that $J_1(\pi_1) \geqslant J_1(\pi_2)$.



Now note that $L_2(m(\pi_1) - m(\pi_2)) = \epsilon$. Moreover, recall that the angle θ between $Mc(R_1)$ and $Mc(R_2)$ is at least $\theta' = \ell_1 \cdot \ell_2 \cdot d(R_1, R_2)$, and that this quantity is at most $\pi/2$. This means that the angle ϕ is at most $\pi/2 - \theta'$, and so $\cos(\phi)$ is at least $\cos(\pi/2 - \theta') = \cos(\pi/2 - \ell_1 \cdot \ell_2 \cdot d(R_1, R_2))$. This means that we have two policies π_1, π_2 where $J_2(\pi_2) = J_2(\pi_1)$ and such that

$$\begin{aligned} J_1(\pi_1) - J_1(\pi_2) &= L_2(M(c(R_1))) \cdot L_2(m(\pi_1) - m(\pi_2)) \cos(\phi) \\ &= L_2(M(c(R_1))) \cdot \epsilon \cdot cos(\pi/2 - \ell_2 \cdot \ell_1 \cdot d(R_1, R_2)). \end{aligned}$$

Note that $\cos(\pi/2 - x) \geqslant x \cdot 2/\pi$ when $x \leqslant \pi/2$, and that $\ell_2 \cdot \ell_1 \cdot d(R_1, R_2) \leqslant \pi/2$. Putting this together, we have that there must exist two policies π_1, π_2 with $J_2(\pi_2) = J_2(\pi_1)$ such that

$$J_1(\pi_1) - J_1(\pi_2) \geqslant L_2(M(c(R_1))) \cdot \left(\frac{\epsilon \cdot \ell_1 \cdot \ell_2 \cdot 2}{\pi} \right) \cdot d(R_1, R_2).$$

Next, note that, if p is a norm and M is an invertible matrix, then $p \circ M$ is also a norm. Furthermore, recall that $\max_\pi J_1(\pi) - \min_\pi J_1(\pi)$ is a norm on $\mathrm{Im}(c)$, when c is a canonicalisation function (Proposition 4). Since all norms are equivalent on a finite-dimensional vector space, this means that there must exist a positive constant ℓ_3 such that $L_2(M(c(R_1))) \geqslant \ell_3 \cdot (\max_\pi J_1(\pi) - \min_\pi J_1(\pi))$. We can therefore set $L = (\epsilon \cdot \ell_1 \cdot \ell_2 \cdot \ell_3 \cdot 2/\pi)$, and obtain the result that we want:

$$J_1(\pi_1) - J_1(\pi_2) \geqslant L \cdot (\max_\pi J_1(\pi) - \min_\pi J_1(\pi)) \cdot d(R_1, R_2).$$

Finally, we must consider the case where R_2 is trivial under τ and μ_0, but where R_1 is not. In this case, $J_2(\pi_2) \geqslant J_2(\pi_1)$ for all π_1 and π_2, which means that $\max_{\pi_1, \pi_2 : J_2(\pi_2) \geqslant J_2(\pi_1)} J_1(\pi_1) - J_1(\pi_2) = \max_\pi J_1(\pi) - \min_\pi J_1(\pi)$. Therefore, the statement holds for any L as long as we ensure that $L \cdot d(R_1, R_0) \leqslant 1$ for all R_1 and R_2. This completes the proof. $\square$

F.3 Issues with EPIC, and Similar Metrics

In this appendix, we prove the results from in Appendix A. Moreover, we state and prove versions of these theorems that are more general than the versions given in the main text. First, we need a few new definitions:

Definition 7. A function $c : \mathcal{R} \to \mathcal{R}$ is an *EPIC-like canonicalisation function* if c is linear, $c(R)$ and R differ by potential shaping, and $c(R_1) = c(R_2)$ if and only if R_1 and R_2 only differ by potential shaping.

Definition 8. A function $d : \mathcal{R} \times \mathcal{R} \to \mathbb{R}$ is an *EPIC-like metric* if there is an EPIC-like canonicalisation function c, a function n that is a norm on $\mathrm{Im}(c)$, and a metric m that is admissible on $\mathrm{Im}(c)$, such that $d(R_1, R_2) = m(s(R_1), s(R_2))$, where $s(R) = c(R)/n(c(R))$ when $n(c(R)) \neq 0$, and $c(R)$ otherwise.

Note that C^{EPIC} is an EPIC-like canonicalisation function, and that EPIC is an EPIC-like metric.

Theorem 3. *For any EPIC-like metric d there exists a positive constant U, such that for any reward functions R_1 and R_2, if two policies π_1 and π_2 satisfy that $J_2(\pi_2) \geqslant J_2(\pi_1)$, then we have that*

$$J_1(\pi_1) - J_1(\pi_2) \leqslant U \cdot L_2(R_1) \cdot d(R_1, R_2).$$

Proof. We wish to show that there is a positive constant U, such that for any R_1 and R_2, and any pair of policies π_1 and π_2 such that $J_2(\pi_2) \geqslant J_2(\pi_1)$, we have

$$J_1(\pi_1) - J_1(\pi_2) \leqslant U \cdot L_2(R_1) \cdot d(R_1, R_2).$$

Moreover, this must hold for any choice of τ and μ_0.

Recall that $d(R_1, R_2) = m(s(R_1), s(R_2))$, where m is an admissible metric. Since m is admissible, we have that $p(s(R_1), s(R_2)) \leqslant K_m \cdot m(s(R_1), s(R_2))$ for some norm p and constant K_m. Moreover, since p is a norm, we can apply Lemma 2 to conclude that there is a constant K_p such that for any policy π, any transition function τ, and any initial state distribution μ_0, we have that

$$|J_1^S(\pi) - J_2^S(\pi)| \leqslant K_p \cdot p(s(R_1), s(R_2)).$$

Combining this with the fact that $p(s(R_1), s(R_2)) \leqslant K_m \cdot m(s(R_1), s(R_2))$, we get

$$\begin{aligned} K_p \cdot p(s(R_1), s(R_2)) &\leqslant K_p \cdot K_m \cdot m(s(R_1), s(R_2)) \\ &= K_{mp} \cdot d(R_1, R_2) \end{aligned}$$

where $K_{mp} = K_p \cdot K_m$. We have thus established that, for any π, τ, and μ_0, we have

$$|J_1^S(\pi) - J_2^S(\pi)| \leqslant K_{mp} \cdot d(R_1, R_2).$$

Consider an arbitrary transition function τ and initial state distribution μ_0, and let π_1 and π_2 be any two policies such that $J_2(\pi_2) \geqslant J_2(\pi_1)$ under τ and μ_0. Note that $J_2(\pi_2) \geqslant J_2(\pi_1)$ if and only if $J_2^S(\pi_2) \geqslant J_2^S(\pi_1)$. We can therefore apply Lemma 3 and conclude that

$$J_1^S(\pi_1) - J_1^S(\pi_2) \leqslant 2 \cdot K_{mp} \cdot d(R_1, R_2).$$

By Lemma 6, there is a constant B such that $J_1^C = J_1 + B$. We can therefore apply Lemma 5:

$$J_1(\pi_1) - J_1(\pi_2) \leqslant n(c(R_1)) \cdot 2 \cdot K_{mp} \cdot d(R_1, R_2).$$

By Lemma 4, there is a positive constant K_n such that $n(c(R)) \leqslant K_n \cdot n(R)$ for all $R \in \mathcal{R}$.

$$J_1(\pi_1) - J_1(\pi_2) \leqslant K_n \cdot n(R_1) \cdot 2 \cdot K_{mp} \cdot d(R_1, R_2).$$

Moreover, since n is a norm, and since $\mathcal{R}$ is a finite-dimensional vector space, we have that there is a constant K_2 such that $n(R) \leqslant K_2 \cdot L_2(R)$ for all $R \in \mathcal{R}$. Let $U = 2 \cdot K_n \cdot K_{mp} \cdot K_2$. We have now established that, for any π_1 and π_2 such that $J_2(\pi_2) \geqslant J_2(\pi_1)$, we have that

$$J_1(\pi_1) - J_1(\pi_2) \leqslant U \cdot L_2(R_1) \cdot d(R_1, R_2).$$

Note that U does not depend on τ or μ_0. This completes the proof. □

Theorem 4. *No EPIC-like metric is sound.*

Proof. Consider an arbitrary transition function τ and an arbitrary initial state distribution μ_0, and let d be an EPIC-like metric with canonicalisation function c, normalisation function n, and admissible metric m.

Let $\mathcal{X}$ be a linear subspace of $\mathrm{Im}(c)$, such that there, for any reward function $R \in \mathrm{Im}(c)$, is exactly one reward function $R' \in \mathcal{X}$ such that R and R' differ by S'-redistribution under τ.

Let R_1 be an arbitrary reward function in $\mathcal{X}$ such that $n(R_1) = 1$, and let $R_2 = -R_1$. Note that R_1 and the reward function that is 0 everywhere do not differ by potential shaping and S'-redistribution – this is ensured by the fact that they are distinct, and both included in $\mathcal{X}$. As per Proposition 1, this implies that R_1 does not have the same policy order as the reward function that is 0 everywhere. This, in turn, means that $\max_\pi J_1(\pi) - \min_\pi J_1(\pi) > 0$. Moreover, since $R_2 = -R_1$, this implies that, if π_1 is a policy that is optimal under R_1, and π_2 is a policy that is optimal under R_2, then π_2

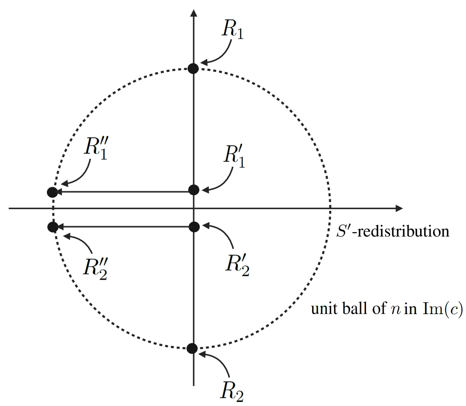
is maximally bad under R_1, and π_1 is maximally bad under R_2. In other words, there are policies π_1, π_2 such that $J_2(\pi_2) \geqslant J_2(\pi_1)$, and

$$\frac{J_1(\pi_1) - J_1(\pi_2)}{\max_\pi J_1(\pi) - \min_\pi J_1(\pi)} = 1.$$

This is the greatest value for this expression, and so the regret for R_1 and R_2 is maximally high.

Next, let $R_1' = \epsilon \cdot R_1$, and $R_2' = \epsilon \cdot R_2$, for some small positive value ϵ. Since positive linear scaling does not affect the regret, we have that the regret for R_1' and R_2' also is 1.

Let x be a vector in $\mathrm{Im}(c)$ that is orthogonal to $\mathcal{X}$. Note that movement along x corresponds to S'-redistribution under τ. Next, let $R_1'' = R_1' + \alpha \cdot x$ and $R_2'' = R_2' + \beta \cdot x$, where α and β are two positive constants such that $n(R_1'') = 1$ and $n(R_2'') = 1$. Since movement along x corresponds to S'-redistribution under τ, and since S'-redistribution under τ does not affect regret, we have that the regret for R_1'' and R_2'' is 1.



Now, since R_1'' and R_2'' are in $\mathrm{Im}(c)$, and since $n(R_1'') = 1$ and $n(R_2'') = 1$, we have that $d(R_1'', R_2'') = m(R_1'', R_2'')$. By making ϵ sufficiently small, we can ensure that this value is arbitrarily close to 0. Therefore, for any simple STARC metric d and any environment, there are reward functions such that R_1'' and R_2'' have maximally high regret, but $d(R_1'', R_2'')$ is arbitrarily close to 0. $\square$

Theorem 5. *There exist reward functions R_1, R_2 such that $d(R_1, R_2) > 0$ for any EPIC-like metric d, but where R_1 and R_2 induce the same ordering of policies for any choice of transition function and any choice of initial state distribution.*

Proof. Recall that $\mathcal{S}$ must contain at least two states s_1, s_2, and $\mathcal{A}$ must contain at least two actions a_1, a_2. Let $R_1(s_1, a_1, s_1) = 1$, $R_1(s_1, a_1, s_2) = \epsilon$, $R_2(s_1, a_1, s_1) = \epsilon$, and $R_2(s_1, a_1, s_2) = 1$, and let R_1 and R_2 be 0 for all other transitions. R_1 and R_2 do not differ by potential shaping or positive linear scaling; this means that $d(R_1, R_2) > 0$ for any EPIC-like metric d. However, R_1 and R_2 have the same policy ordering for all τ and μ_0. $\square$

F.4 Other Proofs

In this section, we provide a the remaining proofs of the results mentioned in the main text.

Proposition 1. *Any STARC metric is a pseudometric on $\mathcal{R}$.*

Proof. To show that d is a pseudometric, we must show that

1. $d(R, R) = 0$
2. $d(R_1, R_2) = d(R_2, R_1)$

3. $d(R_1, R_3) \leqslant d(R_1, R_2) + d(R_2, R_3)$

1 follows from the fact that m is a metric, and 2 follows directly from the fact that the definition of STARC metrics is symmetric in R_1 and R_2. For 3, the fact that m is a metric again implies that $d(R_1, R_3) = m(s(R_1), s(R_3)) \leqslant m(s(R_1), s(R_2)) + m(s(R_2), s(R_3)) = d(R_1, R_2) + d(R_2, R_3)$. This completes the proof. $\square$

Proposition 2. *All STARC metrics have the property that $d(R_1, R_2) = 0$ if and only if R_1 and R_2 induce the same ordering of policies.*

Proof. This is immediate from Proposition 1, together with the fact that if R_1 and R_2 differ by potential shaping, S'-redistribution, and positive linear scaling, applied in any order, then $R_2 = \alpha \cdot R_3$ for some scalar α and some R_3 that differs from R_1 via potential shaping and S'-redistribution. $\square$

Proposition 3. *If two pseudometrics d_1, d_2 on $\mathcal{R}$ are both sound and complete, then d_1 and d_2 are bilipschitz equivalent.*

Proof. Since d_1 is complete, we have that

$$L_1 \cdot d_1(R_1, R_2) \cdot (\max_\pi J_1(\pi) - \min_\pi J_1(\pi)) \leqslant \max_{\pi_1, \pi_2 : J_2(\pi_2) \geqslant J_2(\pi_1)} J_1(\pi_1) - J_1(\pi_2).$$

Similarly, since d_2 is sound, we also have that

$$\max_{\pi_1, \pi_2 : J_2(\pi_2) \geqslant J_2(\pi_1)} J_1(\pi_1) - J_1(\pi_2) \leqslant U_2 \cdot d_2(R_1, R_2) \cdot (\max_\pi J_1(\pi) - \min_\pi J_1(\pi)).$$

This implies that

$$L_1 \cdot d_1(R_1, R_2) \cdot (\max_\pi J_1(\pi) - \min_\pi J_1(\pi)) \leqslant U_2 \cdot d_2(R_1, R_2) \cdot (\max_\pi J_1(\pi) - \min_\pi J_1(\pi)).$$

First suppose that $(\max_\pi J_1(\pi) - \min_\pi J_1(\pi)) > 0$. We can then divide both sides, and obtain that

$$d_1(R_1, R_2) \leqslant \left(\frac{U_2}{L_1}\right) d_2(R_1, R_2).$$

Similarly, we also have that

$$\left(\frac{L_2}{U_1}\right) d_2(R_1, R_2) \leqslant d_1(R_1, R_2).$$

This means that we have constants $\left(\frac{U_2}{L_1}\right)$ and $\left(\frac{L_2}{U_1}\right)$ not depending on R_1 or R_2, such that

$$\left(\frac{L_2}{U_1}\right) d_2(R_1, R_2) \leqslant d_1(R_1, R_2) \leqslant \left(\frac{U_2}{L_1}\right) d_2(R_1, R_2)$$

for all R_1 and R_2 such that $(\max_\pi J_1(\pi) - \min_\pi J_1(\pi)) > 0$.

Next, assume $(\max_\pi J_1(\pi) - \min_\pi J_1(\pi)) = 0$ but $(\max_\pi J_2(\pi) - \min_\pi J_2(\pi)) > 0$. Since d_1 and d_2 are pseudometrics, we have that $d_1(R_1, R_2) = d_1(R_2, R_1)$ and $d_2(R_1, R_2) = d_2(R_2, R_1)$. Therefore, $\left(\frac{L_2}{U_1}\right) d_2(R_1, R_2) \leqslant d_1(R_1, R_2) \leqslant \left(\frac{U_2}{L_1}\right) d_2(R_1, R_2)$ in this case as well.

Finally, assume that $(\max_\pi J_1(\pi) - \min_\pi J_1(\pi)) = 0$ and $(\max_\pi J_2(\pi) - \min_\pi J_2(\pi)) = 0$. In this case, R_1 and R_2 induce the same policy order (namely, the order where $\pi_1 \equiv \pi_2$ for all π_1, π_2). This in turn means that $d_1(R_1, R_2) = d_2(R_1, R_2) = 0$, and so $\left(\frac{L_2}{U_1}\right) d_2(R_1, R_2) \leqslant d_1(R_1, R_2) \leqslant \left(\frac{U_2}{L_1}\right) d_2(R_1, R_2)$ in this case as well. This completes the proof. $\square$

G EXPERIMENTAL SETUP OF SMALL MDPS

In this appendix, we give the precise details required to reproduce our experimental results, as well as more of the raw data than what is provided in the main text.

G.1 Environments and Rewards

As mentioned in the main text, we used Markov Decision Processes with 32 states and 4 actions. The discount factor was set to 0.95 and the initial state distribution was uniform.

The transition distribution $\tau(s, a, s')$ was generated as follows:

1. Sample i.i.d. Gaussians ($\mu = 0, \sigma = 1$) to generate a matrix of shape $[32, 4, 32]$.
2. For each item in the matrix: if the item is below 1, set its value to -20. This is done to ensure the transition distribution is sparse and therefore more similar to real-world environments. Without this step, when an agent is in state S and takes action A, the distribution $\tau(S, A, s')$ would be close to uniform, meaning that the choice of the action would not make much of a difference.
3. Softmax along the last dimension (which corresponds to s') to get a valid probability distribution.

We then generated pairs of rewards. This worked in two stages: random generation and interpolation.

In the random generation stage, we choose two random rewards R_1, R_2 using the following procedure:

1. Sample i.i.d. Gaussians ($\mu = 0, \sigma = 1$) to generate a matrix of shape $[32, 4, 32]$ corresponding to $R(s, a, s')$.
2. With a 20% probability, make the function sparse in the following way – for each item in the matrix: if the item is below 3, set its value to 0.
3. With a 70% probability, scale the reward function in the following way – sample a uniform distribution between 0 and 10, multiply the matrix by this number.
4. With a 30% probability, translate the reward function in the following way – sample a uniform distribution between 0 and 10, add this number to the matrix.
5. With a 50% probability, apply random potential shaping in the following way – sample 32 i.i.d. Gaussians ($\mu = 0, \sigma = 1$) to get a potential vector Φ. Then sample a uniform distribution between 0 and 10 and multiply the vector by this number. Then sample a uniform distribution between 0 and 1 and add this number to the vector. Then apply potential shaping to the reward function: $R_{\text{new}}(s, a, s') = R(s, a, s') + \gamma\Phi(s') - \Phi(s)$.

When we say "With an X% probability", we are sampling a random number from a uniform distribution between 0 and 1 and if the number is above (100-X)%, we perform the action described, otherwise we skip the step.

Then in the interpolation stage, we take the pair of reward functions generated above and do a linear interpolation between them, finding 16 functions which lie between R_1 and R_2. More precisely, we set $R_{(i)} = R_1 + id$ where $d = (R_2 - R_1)/16$ and with i ranging from 1 to 16.

The interpolation step exists to give us pairs of rewards which are relatively close to each other – for instance, R_1 and $R_{(1)}$ are very similar. This is important because nearly all reward functions generated with the random generation process described above will be orthogonal to each other, and thus their distances to each other would always be quite large. By including this interpolation step, we ensure a greater variety in the range of distance values and regret values we expect to see.

For each environment, we generated 16 pairs of reward functions, and then for each pair we performed 16 interpolation steps. This means that for each transition distribution, we compared 256 different reward functions.

We then compute all distance metrics as well as rollout regret between R_1 and $R_{(i)}$ for all i.

G.2 Rollout Regret

We calculate rollout regret in 3 stages: 1, find optimal and anti-optimal policies, 2, compute returns under various policies and reward functions, 3, calculate regret.

In stage 1, we use value iteration to find policies π_1 which maximises reward R_1, $\pi_{(i)}$ which maximises reward $R_{(i)}$, π_x which minimises reward R_1 (in other words maximising reward $-R_1$, meaning π_x is the worst possible policy under R_1), and π_y which minimises reward R_2. Note that all of these policies are deterministic, i.e. $\pi : \mathcal{S} \to \mathcal{A}$.

In stage 2, we simulate a number of episodes to determine the average return of the policy. Specifically, we simulate 32 episodes such that no two episodes start in the same initial state (this helps reduce noise in the return estimates). The episode terminates when the discount factor being applied (i.e. γ^t) is below 10^{-5}.

In stage 3, we calculate regret as follows. The regret is the average of two regrets: the regret of using $\pi_{(i)}$ instead of π_1 when evaluating using R_1, and the regret of using π_1 instead of $\pi_{(i)}$ when evaluating using $R_{(i)}$ (with both of these being normalised by the range of possible returns):

$$\begin{aligned}
\text{Reg} &= \frac{\text{Reg}_1 + \text{Reg}_{(i)}}{2} \\
\text{Reg}_1 &= \frac{J_1(\pi_1) - J_1(\pi_{(i)})}{J_1(\pi_1) - J_1(\pi_x)} \\
\text{Reg}_{(i)} &= \frac{J_{(i)}(\pi_{(i)}) - J_{(i)}(\pi_1)}{J_{(i)}(\pi_{(i)}) - J_{(i)}(\pi_y)}
\end{aligned}$$

In cases where the denominator is zero, we simply replace it with 1 (since the numerator in these cases is also necessarily 0).

G.3 List of Metrics

Our experiment covers hundreds of metrics, derived by creating different combinations of canonicalisation functions, normalisations, and distance metrics. Specifically, we used 6 "pseudo-canonicalisations" (some of which, like C^{EPIC} and C^{DARD}, do not meet the conditions of Definition 1), 7 normalisation functions, and 6 distance norms.

For canonicalisations, we used `None` (which simply skips the canonicalisation step), C^{EPIC}, C^{DARD}, `MinimalPotential` (which is the minimal "pseudo-canonicalisation" that removes potential shaping but not S'-redistribution, and therefore is easier to compute), `VALPotential` (which is given by $R(s, a, s') - V^\pi(s) + \gamma V^\pi(s')$), and `VAL` (defined in Proposition 2 as $\mathbb{E}_{S' \sim \tau(s,a)}[R(s, a, S') - V^\pi(s) + \gamma V^\pi(S')]$). For both C^{EPIC} and C^{DARD}, both $\mathcal{D}_\mathcal{S}$ and $\mathcal{D}_\mathcal{A}$ were chosen to be uniform over $\mathcal{S}$ and $\mathcal{A}$. For both `VALPotential` and `VAL`, π was chosen to be the uniformly random policy. [9] Note that `VAL` is the only canonicalisation which removes both potential shaping and S'-redistribution, and thus the only one that meets the STARC definition of a canonicalisation function (Definition 1). The other pseudo-canonicalisations were used for comparison. It is worth noting that our experiment does *not* include the minimal canonicalisation functions, given in Definition 4, because these functions are prohibitively expensive to compute. They are therefore better suited for theoretical analysis, rather than practical evaluations.

The normalisation step and the distance step used L_1, L_2, L_∞, `weighted_`L_1, `weighted_`L_2, and `weighted_`L_∞. The weighted norms are weighted by the transition function τ, i.e. $L_p^\tau(R)(s, a, s') = (\sum_{s,a,s'} \tau(s, a, s')|R(s, a, s')|^p)^{1/p}$. We also considered metrics that skip the normalisation step.

We used almost all combinations of these – the only exception was that we did not combine `MinimalPotential` with normalisation norms L_∞ or `weighted_`L_∞ because the optimisation algorithm for `MinimalPotential` does not converge for these norms.

G.4 Number of Reward Pairs

We used 49,152 reward pairs. This number was chosen in advance as the stopping point – it corresponds to using 96 CPU cores, generating 2 environments on each core, choosing 16 reward pairs within each environment and then performing 16 interpolation steps between them.

[9] Since `VAL` is a valid canonicalisation function for any choice of policy π, we simply picked a policy for which V^π would be easy to estimate. The reason for choosing a uniformly random policy, rather than some deterministic policy, is that this policy has exploration build in.

H Full Results Of Small MDP Experiments

We used the Balrog GPU cluster at UC Berkeley, which consists of 8 A100 GPUs, each with 40 GB memory, along with 96 CPU cores.

The notation in this table is in the format `Canonicalisation-normalisation-distance`. For instance, `VAL-2-weighted_1` means using the `Val` canonicalisation function, L_2 normalisation function, and then taking the distance with the weighted L_1 norm. `0` means normalisation is skipped.

Table 2: Full experimental results

Distance function	Correlation to regret
VALPotential-1-weighted_1	0.876
VAL-1-weighted_1	0.873
VAL-1-1	0.873
VAL-weighted_1-weighted_1	0.873
VAL-weighted_1-1	0.873
VAL-1-2	0.870
VAL-weighted_1-2	0.870
VAL-1-weighted_2	0.870
VAL-weighted_1-weighted_2	0.870
VALPotential-weighted_1-weighted_1	0.867
DARD-1-weighted_1	0.861
VAL-weighted_2-inf	0.858
VAL-2-inf	0.858
VAL-weighted_2-weighted_2	0.856
VAL-2-2	0.856
VAL-weighted_2-2	0.856
VAL-2-weighted_2	0.856
VALPotential-weighted_2-weighted_2	0.845
DARD-weighted_1-weighted_1	0.835
DARD-weighted_2-weighted_2	0.831
EPIC-weighted_1-weighted_1	0.830
VALPotential-2-weighted_2	0.828
DARD-2-weighted_2	0.826
DARD-1-1	0.824
EPIC-weighted_2-weighted_2	0.823
EPIC-1-weighted_1	0.819
VAL-weighted_inf-inf	0.816
MinimalPotential-1-1	0.815
MinimalPotential-2-weighted_2	0.814
EPIC-2-weighted_2	0.814
EPIC-1-1	0.814
VALPotential-weighted_2-weighted_inf	0.807
EPIC-weighted_2-weighted_inf	0.806
VAL-2-1	0.804
VAL-2-weighted_1	0.804
VAL-weighted_2-1	0.804
VAL-weighted_2-weighted_1	0.804
VALPotential-1-1	0.800
VALPotential-weighted_2-weighted_1	0.784
VAL-weighted_2-weighted_inf	0.783
VAL-2-weighted_inf	0.783
VALPotential-2-2	0.782
DARD-2-2	0.782
MinimalPotential-2-2	0.778
EPIC-2-2	0.778
VALPotential-1-weighted_2	0.776

DARD-weighted_2-weighted_1	0.774
DARD-2-weighted_1	0.767
VALPotential-2-weighted_1	0.767
VAL-weighted_inf-weighted_2	0.766
VAL-weighted_inf-2	0.766
DARD-1-weighted_2	0.761
VAL-inf-inf	0.756
DARD-weighted_1-weighted_2	0.754
VALPotential-2-1	0.752
DARD-2-1	0.751
EPIC-weighted_1-1	0.749
VAL-1-inf	0.749
VAL-weighted_1-inf	0.749
MinimalPotential-2-weighted_1	0.746
EPIC-2-weighted_1	0.746
VAL-1-weighted_inf	0.741
VAL-weighted_1-weighted_inf	0.741
EPIC-weighted_2-weighted_1	0.738
VALPotential-weighted_inf-weighted_inf	0.735
EPIC-2-1	0.734
VAL-weighted_inf-weighted_inf	0.734
MinimalPotential-2-1	0.733
EPIC-weighted_1-weighted_2	0.730
VAL-inf-weighted_2	0.723
VAL-inf-2	0.723
VALPotential-weighted_inf-weighted_1	0.722
MinimalPotential-1-weighted_1	0.718
DARD-weighted_inf-weighted_2	0.718
DARD-weighted_inf-weighted_1	0.713
DARD-weighted_2-1	0.711
DARD-weighted_2-weighted_inf	0.708
VAL-inf-weighted_1	0.708
VAL-inf-1	0.708
VAL-inf-weighted_inf	0.707
VAL-weighted_inf-1	0.707
VAL-weighted_inf-weighted_1	0.707
DARD-weighted_inf-weighted_inf	0.698
VALPotential-weighted_1-weighted_inf	0.692
EPIC-weighted_inf-weighted_2	0.692
EPIC-weighted_inf-weighted_1	0.686
VALPotential-1-weighted_inf	0.685
DARD-weighted_inf-1	0.685
EPIC-weighted_2-1	0.680
DARD-weighted_1-weighted_inf	0.679
VALPotential-2-weighted_inf	0.677
DARD-2-weighted_inf	0.675
EPIC-weighted_inf-weighted_inf	0.661
DARD-inf-weighted_1	0.657
DARD-1-weighted_inf	0.654
VALPotential-inf-weighted_1	0.653
DARD-inf-weighted_2	0.652
VALPotential-inf-weighted_2	0.648
EPIC-1-2	0.647
EPIC-weighted_inf-1	0.642
DARD-inf-1	0.639
DARD-inf-2	0.637
MinimalPotential-2-weighted_inf	0.637
EPIC-2-weighted_inf	0.637
VALPotential-inf-1	0.636

VALPotential-inf-2	0.634
MinimalPotential-2-inf	0.634
EPIC-2-inf	0.634
None-2-weighted_2	0.633
DARD-inf-weighted_inf	0.632
DARD-2-inf	0.630
EPIC-inf-weighted_2	0.630
EPIC-inf-weighted_1	0.629
VALPotential-2-inf	0.625
VALPotential-inf-weighted_inf	0.624
EPIC-1-weighted_2	0.622
None-weighted_2-weighted_inf	0.622
DARD-weighted_1-1	0.621
EPIC-inf-1	0.620
None-2-2	0.618
EPIC-inf-2	0.617
None-weighted_2-weighted_2	0.615
EPIC-weighted_1-weighted_inf	0.607
None-weighted_1-weighted_1	0.598
None-1-1	0.597
None-2-weighted_1	0.579
MinimalPotential-1-2	0.576
None-weighted_2-weighted_1	0.573
None-2-1	0.571
EPIC-inf-weighted_inf	0.571
MinimalPotential-1-weighted_2	0.568
VALPotential-inf-inf	0.557
None-inf-inf	0.555
EPIC-inf-inf	0.554
DARD-inf-inf	0.552
EPIC-1-inf	0.542
None-inf-weighted_2	0.539
None-weighted_inf-weighted_1	0.539
None-inf-2	0.538
None-inf-weighted_1	0.537
None-inf-1	0.537
None-weighted_inf-weighted_inf	0.530
EPIC-weighted_1-2	0.529
EPIC-1-weighted_inf	0.517
None-1-weighted_1	0.515
None-2-weighted_inf	0.513
MinimalPotential-1-inf	0.493
MinimalPotential-1-weighted_inf	0.489
None-inf-weighted_inf	0.487
DARD-1-2	0.453
None-2-inf	0.444
EPIC-weighted_1-inf	0.436
None-weighted_1-weighted_inf	0.429
None-1-weighted_2	0.390
DARD-1-inf	0.376
None-1-weighted_inf	0.353
None-1-2	0.352
None-0-inf	0.333
DARD-weighted_inf-2	0.319
None-0-weighted_inf	0.312
None-0-2	0.304
None-0-weighted_2	0.303
None-0-weighted_1	0.296
None-0-1	0.296

None-1-inf	0.278
DARD-weighted_1-2	0.241
VALPotential-1-2	0.240
EPIC-weighted_2-2	0.224
DARD-weighted_1-inf	0.220
EPIC-weighted_inf-2	0.197
EPIC-0-inf	0.184
DARD-weighted_2-2	0.178
VALPotential-1-inf	0.171
VAL-0-weighted_inf	0.171
VALPotential-0-inf	0.171
DARD-0-inf	0.170
VAL-0-weighted_1	0.149
VAL-0-1	0.149
VAL-0-weighted_2	0.146
VAL-0-2	0.146
VALPotential-0-weighted_inf	0.141
DARD-0-weighted_inf	0.141
VAL-0-inf	0.140
EPIC-0-weighted_inf	0.140
DARD-0-weighted_1	0.137
VALPotential-0-weighted_1	0.136
DARD-0-weighted_2	0.136
VALPotential-0-weighted_2	0.135
EPIC-0-weighted_2	0.131
EPIC-0-weighted_1	0.130
EPIC-weighted_2-inf	0.129
DARD-0-2	0.125
DARD-0-1	0.124
VALPotential-0-2	0.123
DARD-weighted_2-inf	0.122
VALPotential-0-1	0.122
EPIC-0-1	0.122
EPIC-0-2	0.122
VALPotential-weighted_2-1	0.112
DARD-weighted_inf-inf	0.095
VALPotential-weighted_2-2	0.093
VALPotential-weighted_inf-1	0.077
VALPotential-weighted_inf-weighted_2	0.073
VALPotential-weighted_2-inf	0.065
EPIC-weighted_inf-inf	0.052
VALPotential-weighted_inf-2	0.051
VALPotential-weighted_inf-inf	0.024
None-weighted_2-1	-0.034
VALPotential-weighted_1-1	-0.035
None-weighted_inf-1	-0.035
VALPotential-weighted_1-weighted_2	-0.037
None-weighted_inf-weighted_2	-0.040
VALPotential-weighted_1-2	-0.040
None-weighted_inf-2	-0.043
None-weighted_2-2	-0.043
VALPotential-weighted_1-inf	-0.044
None-weighted_1-1	-0.045
None-weighted_inf-inf	-0.046
None-weighted_2-inf	-0.046
None-weighted_1-weighted_2	-0.047
None-weighted_1-2	-0.047
None-weighted_1-inf	-0.048

H.1 COMPARISON OF EXPERIMENTAL PERFORMANCE BASED ON CHOICE OF NORMS

As discussed in the main text, the choice of normalisation and metric functions can make a noticeable difference to the performance of a reward metric. To make it easier to see the impact that this choice has, this appendix contains the same data as Appendix H, but organised together by canonicalisation function, and then arranged by normalisation and metric.

	1	2	inf	weighted_1	weighted_2	weighted_inf
0	0.296	0.304	0.333	0.296	0.303	0.312
1	0.597	0.352	0.278	0.515	0.39	0.353
2	0.571	0.618	0.444	0.579	**0.633**	0.513
inf	0.537	0.538	0.555	0.537	0.539	0.487
weighted_1	-0.045	-0.047	-0.048	0.598	-0.047	0.429
weighted_2	-0.034	-0.043	-0.046	0.573	0.615	0.622
weighted_inf	-0.035	-0.043	-0.046	0.539	-0.04	0.53

Table 3: Correlation to regret for the `None` canonicalisation for each normalization and distance metric. Each row corresponds to a normalisation function, and each column corresponds to a metric function.

	1	2	inf	weighted_1	weighted_2	weighted_inf
0	0.122	0.122	0.184	0.13	0.131	0.14
1	0.814	0.647	0.542	0.819	0.622	0.517
2	0.734	0.778	0.634	0.746	0.814	0.637
inf	0.62	0.617	0.554	0.629	0.63	0.571
weighted_1	0.749	0.529	0.436	**0.83**	0.73	0.607
weighted_2	0.68	0.224	0.129	0.738	0.823	0.806
weighted_inf	0.642	0.197	0.052	0.686	0.692	0.661

Table 4: Correlation to regret for the `EPIC` canonicalisation for each normalization and distance metric. Each row corresponds to a normalisation function, and each column corresponds to a metric function.

	1	2	inf	weighted_1	weighted_2	weighted_inf
0	0.124	0.125	0.17	0.137	0.136	0.141
1	0.824	0.453	0.376	**0.861**	0.761	0.654
2	0.751	0.782	0.63	0.767	0.826	0.675
inf	0.639	0.637	0.552	0.657	0.652	0.632
weighted_1	0.621	0.241	0.22	0.835	0.754	0.679
weighted_2	0.711	0.178	0.122	0.774	0.831	0.708
weighted_inf	0.685	0.319	0.095	0.713	0.718	0.698

Table 5: Correlation to regret for the `DARD` canonicalisation for each normalization and distance metric. Each row corresponds to a normalisation function, and each column corresponds to a metric function.

	1	2	inf	weighted_1	weighted_2	weighted_inf
1	**0.815**	0.576	0.493	0.718	0.568	0.489
2	0.733	0.778	0.634	0.746	0.814	0.637

Table 6: Correlation to regret for the `MinimalPotential` canonicalisation for each normalization and distance metric. Each row corresponds to a normalisation function, and each column corresponds to a metric function.

	1	2	inf	weighted_1	weighted_2	weighted_inf
0	0.122	0.123	0.171	0.136	0.135	0.141
1	0.8	0.24	0.171	**0.876**	0.776	0.685
2	0.752	0.782	0.625	0.767	0.828	0.677
inf	0.636	0.634	0.557	0.653	0.648	0.624
weighted_1	-0.035	-0.04	-0.044	0.867	-0.037	0.692
weighted_2	0.112	0.093	0.065	0.784	0.845	0.807
weighted_inf	0.077	0.051	0.024	0.722	0.073	0.735

Table 7: Correlation to regret for the `VALPotential` canonicalisation for each normalization and distance metric. Each row corresponds to a normalisation function, and each column corresponds to a metric function.

	1	2	inf	weighted_1	weighted_2	weighted_inf
0	0.149	0.146	0.14	0.149	0.146	0.171
1	**0.873**	0.87	0.749	**0.873**	0.87	0.741
2	0.804	0.856	0.858	0.804	0.856	0.783
inf	0.708	0.723	0.756	0.708	0.723	0.707
weighted_1	**0.873**	0.87	0.749	**0.873**	0.87	0.741
weighted_2	0.804	0.856	0.858	0.804	0.856	0.783
weighted_inf	0.707	0.766	0.816	0.707	0.766	0.734

Table 8: Correlation to regret for the `VAL` canonicalisation for each normalization and distance metric. Each row corresponds to a normalisation function, and each column corresponds to a metric function.

I Experimental Setup Of Reacher Environment

In this appendix, we elaborate on the details of our Reacher experiments. The discount rate for this experiment was $\gamma = 0.99$, following the original MuJoCo environment.

I.1 Reward functions

As mentioned in the main text, we used 7 different reward functions.

The `GroundTruth` is simply copied from the original Reacher environment. It computes the Euclidean distance between the fingertip and target, denoted d. It also computes a penalty for taking large actions, which is computed by squaring the values of the action and summing them, $p = a_0^2 + a_1^2$. It then returns $-(d + p)$.

The `PotentialShaped` reward applies randomly generated (but deterministic) potential shaping on top of `GroundTruth`. When the experiment starts, 11 weights (one for each dimension in observation space) and 1 bias are randomly sampled from a normal Gaussian. The potential function is then simply $\Phi(s) = w \cdot x + b$, so the full reward is $\texttt{PotentialShaped}(s, a, s') = \texttt{GroundTruth}(s, a, s') + \gamma\Phi(s') - \Phi(s)$.

`SPrime` returns the same value as `GroundTruth` if $s' = \tau(s, a)$, and the same value as `Random` otherwise. At the start of the experiment, a new instance of `Random` is initialised (see below for details). We consider s' to follow from $\tau(s, a)$ if the two quantities are either less than 1% apart, or less than 0.01 apart (even if they aren't perfectly equal).

`SecondPeak` creates a second, smaller "peak" (corresponding to a second, less important target) in the environment, alongside the original "peak" from `GroundTruth`. When initialised at the start of the experiment, it picks a random position on the same 2D plane where the fingertip and target are, such that the Euclidean distance between the second peak and the original peak is at least 0.5 (note that the size of the whole plane is 1×1). The reward is then determined by first computing the Euclidean distance between the fingertip and the second peak, denoted d, and then simply adding $-0.2d$ on top of `GroundTruth`, ie. $\texttt{SecondPeak} = \texttt{GroundTruth} - 0.2d$.

`SemanticallyIdentical` creates a reward peak around the target, similarly to `GroundTruth`, but this peak has a different shape. It is a 2D Gaussian with a standard deviation of 0.1 along both axes. The values of the Gaussian are then rounded to the nearest 0.01.

`NegativeGroundReward` simply returns $-1 * \texttt{GroundTruth}$.

`Random` returns random (but deterministic) values. When the experiment starts, 11 s-weights, 2 a-weights, 11 s'-weights, and 1 bias are generated by sampling a normal Gaussian. The reward function then simply returns $s \cdot w_s + a \cdot w_a + s' \cdot w_{s'} + b$.

I.2 Canonicalising and normalising in continuous settings

The state value function in VAL is based on a uniformly random policy π. We implemented V^π using SARSA (Rummery & Niranjan, 1994) updates with AdamW (Loshchilov & Hutter, 2019) and a reply buffer on a 4-layer MLP (which maps observations onto real values).

For the norm, we effectively need to compute the norm of a function, which means taking the norm of an infinite-dimensional vector. This can be written precisely as $(\int |f(x)|^p \, dx)^{1/p}$, and approximated using Monte Carlo sampling as $(\frac{1}{N} \sum |f(x)|^p \, dx)^{1/p}$. When taking a sample of the reward function, we sample s, a uniformly, and then set $s' = \tau(s, a)$ – this removes impossible transitions from the sample space, while also reducing the dimensionality of the space we need to cover from 22 dimensions down to 12. As a special case, when taking the L_∞ norm, we simply look for the maximum value of $|f(x)|$. This could be approximated using optimisation algorithms by assuming $|f(x)|$ is convex, but we chose not to make this assumption and instead simply choose the maximum among the samples.