A Further Experiments and Additional Results

In the following, we provide the detailed machine learning models for our experiments:

1) Logistic Regression Model: We use the following min-max regression problem with datasets $\xi_i := \{(\mathbf{a}_{ij}, b_{ij})\}_{j=1}^n$, where $\mathbf{a}_{ij} \in \mathbb{R}^d$ is the feature of the j-th sample of worker i and $b_{ij} \in \{1, -1\}$ is the associated label:

$$\min_{\mathbf{x}\in\mathbb{R}^d} \max_{\mathbf{y}\in\mathbb{R}^n} \frac{1}{m} \sum_{i\in M} f_i(\mathbf{x}, \mathbf{y}),$$

where $f_i(\mathbf{x}, \mathbf{y})$ is defined as:

$$f_i(\mathbf{x}, \mathbf{y}) \triangleq \frac{1}{n} \sum_{j=1}^{n} [y_j l_j(\mathbf{x}) - V(\mathbf{y}) + g(\mathbf{x})], \tag{4}$$

where the loss function $l_i(\mathbf{x}) \triangleq \log\left(1 + \exp\left(-b_{ij}\mathbf{a}_{ij}^\top \mathbf{x}\right)\right)$, $g(\mathbf{x}) \triangleq \lambda_2 \sum_{k=1}^d \frac{\alpha x_k^2}{1+\alpha x_k^2}$, and $V(\mathbf{y}) = \frac{1}{2}\lambda_1 \|n\mathbf{y} - \mathbf{1}\|_2^2$. We choose constants $\lambda_1 = 1/n^2$, $\lambda_2 = 10^{-3}$ and $\alpha = 10$.

2) AUC Maximization: We use a dataset $\{\mathbf{a}_{ij}, b_{ij}\}_{j=1}^n$, where $\mathbf{a}_{ij} \in \mathbb{R}^d$ is the feature of the j-th sample of worker i, $\mathbf{w}_i$ denotes a feature vector and $b_{ij} \in \{-1, +1\}$ denotes the corresponding label. For a scoring function $h_{\mathbf{x}}$ of a classification model parameterized by $\mathbf{x} \in \mathbb{R}^d$, the AUC maximization problem is defined as:

$$\max_{\mathbf{x}} \frac{1}{m^+ m^-} \sum_{b_{ij}=+1, b_{ik}=-1} \mathbb{I}_{\{h_{\mathbf{x}}(\mathbf{a}_{ij}) \geq h_{\mathbf{x}}(\mathbf{a}_{ik})\}}, \tag{5}$$

where m^+ denotes the number of positive samples, m^- denotes the number of negative samples, and $\mathbb{I}_{\{\cdot\}}$ represents the indicator function. The above optimization problem can be reformulated as the following min-max optimization problem [1,2]:

$$\begin{aligned} &\min_{(\mathbf{x},c_1,c_2)\in\mathbb{R}^{d+2}} \max_{\lambda\in\mathbb{R}} f(\mathbf{x}, c_1, c_2, \lambda) \\ :=& \frac{1}{mn} \sum_{i\in M} \sum_{j=1}^{n} \{(1-\tau)(h_{\mathbf{x}}(\mathbf{a}_{ij}) - c_1)^2 \mathbb{I}_{\{b_{ij}=1\}} - \tau(1-\tau)\lambda^2 \\ &+ \tau(h_{\mathbf{x}}(\mathbf{a}_{ij}) - c_2)^2 \mathbb{I}_{\{b_{ij}=-1\}} + 2(1+\lambda)\tau h_{\mathbf{x}}(\mathbf{a}_{ij}) \mathbb{I}_{\{b_{ij}=-1\}} \\ &- 2(1+\lambda)(1-\tau) h_{\mathbf{x}}(\mathbf{a}_{ij}) \mathbb{I}_{\{b_{ij}=1\}}, \}, \end{aligned} \tag{6}$$

where $\tau := m^+ / (m^+ + m^-)$ is the fraction of positive data. Note that $f(\mathbf{x}, c_1, c_2, \cdot)$ is strongly concave for any $(\mathbf{x}, c_1, c_2) \in \mathbb{R}^{d+2}$.

3) Generator Adverserial Networks(GANs): Although our paper is focused on general non-convex-PL min-max problems, we believe that our paper will benefit from comparing further experimental results on the convergence performance of nonconvex-nonconcave problems (e.g., GANs), since the non-convex-PL problem is a special case for nonconvex-nonconcave min-max problems.

In our experiment, generator network is parameterized by $\mathbf{x}$ as $G_{\mathbf{X}}$ and the discriminator network parameterized by $\mathbf{y}$ as $D_{\mathbf{y}}$. We adopt the following loss function:

$$f_i(\mathbf{x}, \mathbf{y}) = \mathbb{E}_{\mathbf{a}_i \sim \mathcal{P}_{true}}[\log D_{\mathbf{y}}(\mathbf{a}_i)] + \mathbb{E}_{\mathbf{z}\sim\mathcal{P}_{\mathbf{z}}}[\log(1 - D_{\mathbf{y}}(G_{\mathbf{x}}(\mathbf{z})))]$$

where $\mathbf{a}_i$ is the data point on client i and $\mathcal{P}_{true}$ is the distribution of the true samples. z denotes the input noise vector and $\mathcal{P}_z$ is the prior distribution of the noise vector for generating samples. We have tested the convergence performance of our algorithms using the MNIST dataset. We chose the learning rates as $\eta_{x,l} = \eta_{y,l} = 10^{-2}$, $\eta_{x,g} = \eta_{y,g} = 2$, local updates

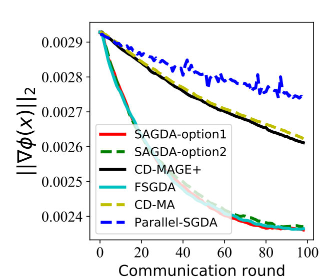
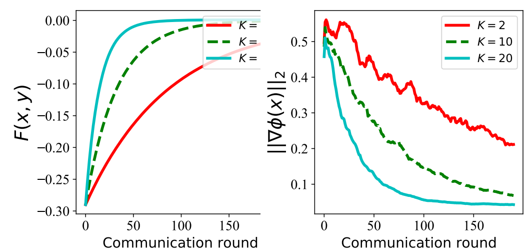


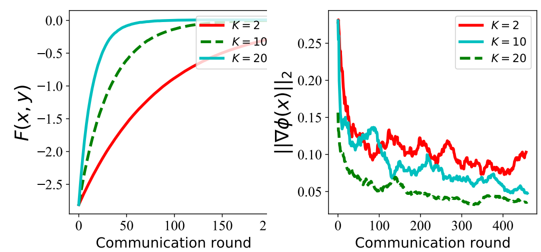
Figure 3: GANs under "MNIST" dataset.

$K = 10$. We have $m = 100$ clients and each client has $n = 100$ samples. Again, from Fig. 3, we can observe that both our proposed algorithms FSGDA and FSGDA have better convergence performance compared with the baselines.

Impact of the Local Steps: In this section, we run additional experiments to investigate the impact of the local steps K on the training performance. We run FSGDA and SAGDA over the hetergenous "a9a" [40] dataset with the regression model mentioned in Section 4. We fix the local step-size at 0.01, worker number at 100, and choose the number of local update rounds K from the discrete set $\{2, 10, 20\}$. In terms of communication round, the gradient norm $\|\nabla \phi(\mathbf{x})\|^2$ decreases as K increases. This is due to the fact that the algorithm needs more communication round while K is small, which matches our Corollary 2 and Corollary 3.



(a) Algorithm SAGDA.

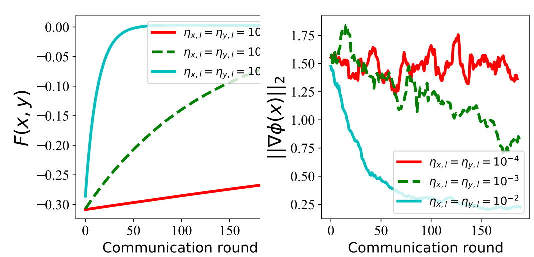


(b) Algorithm FSGDA.

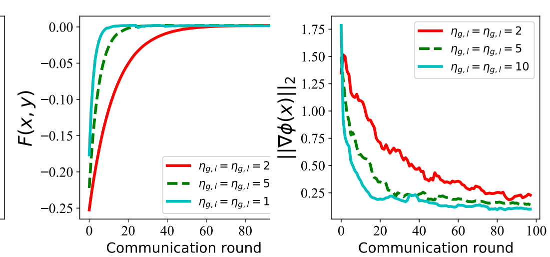
Figure 4: Algorithm performance under different local K steps.

Impact of the Local Step-size: In this experiment, we choose the value of the local step-sizes from the discrete set $\{0.0001, 0.001, 0.01\}$ and fix worker number at 100, local update rounds at 10. As shown in Fig. 5(a) and Fig.6(a) , larger local step-sizes lead to faster convergence rates.

Impact of the Global Step-size: we choose the global step-sizes value from the discrete set $\{2, 5, 10\}$ and fix worker number at 100, local update rounds at 10. As shown in Fig. 5(b) and 6(b) and, larger global step-sizes lead to faster convergence rates.



(a) Different local step-sizes.



(b) Different global step-sizes.

Figure 5: The FSGDA algorithm under different step-sizes.

B Proof

B.1 Proof for FSGDA

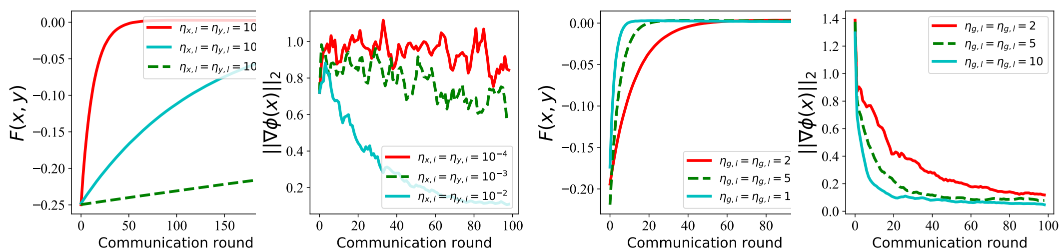
For notational simplicity and clarity, we have the following definitions.

$$\Phi(\mathbf{x}) = \max_{\mathbf{y} \in \mathbb{R}^d} f(\mathbf{x}, \mathbf{y});$$

$$\mathbf{z}_t = (\mathbf{x}_t, \mathbf{y}_t);$$

$$\eta_x = \eta_{x,g}\eta_{x,l}, \eta_y = \eta_{y,g}\eta_{y,l};$$

$$\mathbf{u}_{x,t} = \frac{1}{m} \sum_{i \in S_t} \nabla_x f_i(\mathbf{z}_t), \mathbf{u}_{y,t} = \frac{1}{m} \sum_{i \in S_t} \nabla_y f_i(\mathbf{z}_t).$$



(a) Different local step-sizes. (b) Different global step-sizes.

Figure 6: The SAGDA algorithm under different step-sizes.

For simplicity, we write the update step uniformly:

$$\mathbf{x}_{t+1} = \mathbf{x}_t - \eta_x K(\mathbf{u}_{x,t} - \mathbf{e}_{x,t}),$$
$$\mathbf{y}_{t+1} = \mathbf{y}_t + \eta_y K(\mathbf{u}_{y,t} - \mathbf{e}_{y,t}).$$

For FSGDA , the update rule is:

$$\mathbf{x}_{t+1} = \mathbf{x}_t - \eta_{x,g}\eta_{x,l}\left(\frac{1}{m}\sum_{i\in S_t}\sum_{j\in[K]}\nabla_x f_i(\mathbf{z}_{t,i}^j,\xi_{t,i}^j)\right),$$
$$\mathbf{y}_{t+1} = \mathbf{y}_t + \eta_{y,g}\eta_{y,l}\left(\frac{1}{m}\sum_{i\in S_t}\sum_{j\in[K]}\nabla_y f_i(\mathbf{z}_{t,i}^j,\xi_{t,i}^j)\right),$$
$$\mathbf{e}_{x,t} = \frac{1}{mK}\sum_{i\in S_t}\sum_{j\in[K]}\left(\nabla_x f_i(\mathbf{z}_t) - \nabla_x f_i(\mathbf{z}_{t,i}^j,\xi_{t,i}^j)\right),$$
$$\bar{\mathbf{e}}_{x,t} = \mathbb{E}[\mathbf{e}_{x,t}] = \frac{1}{mK}\sum_{i\in S_t}\sum_{j\in[K]}\left(\nabla_x f_i(\mathbf{z}_t) - \nabla_x f_i(\mathbf{z}_{t,i}^j)\right),$$
$$\mathbf{e}_{y,t} = \frac{1}{mK}\sum_{i\in S_t}\sum_{j\in[K]}\left(\nabla_y f_i(\mathbf{z}_t) - \nabla_y f_i(\mathbf{z}_{t,i}^j,\xi_{t,i}^j)\right),$$
$$\bar{\mathbf{e}}_{y,t} = \mathbb{E}[\mathbf{e}_{y,t}] = \frac{1}{mK}\sum_{i\in S_t}\sum_{j\in[K]}\left(\nabla_y f_i(\mathbf{z}_t) - \nabla_y f_i(\mathbf{z}_{t,i}^j)\right).$$

Note the above expectation is only on the stochastic noise.

Lemma 1.

$$\mathbb{E}\left\|\Delta\mathbf{x}_t\right\|^2 = \mathbb{E}\|\left(\mathbf{u}_{x,t} - \mathbf{e}_{x,t}\right)\|^2 \le 4\mathbb{E}\left\|\bar{\mathbf{e}}_{x,t}\right\|^2 + 4\mathbb{E}\left\|\mathbf{u}_{x,t}\right\|^2 + \frac{2}{mK}\sigma_x^2,$$
$$\mathbb{E}\left\|\Delta\mathbf{x}_t\right\|^2 = \mathbb{E}\|\left(\mathbf{u}_{y,t} - \mathbf{e}_{y,t}\right)\|^2 \le 4\mathbb{E}\left\|\bar{\mathbf{e}}_{y,t}\right\|^2 + 4\mathbb{E}\left\|\mathbf{u}_{y,t}\right\|^2 + \frac{2}{mK}\sigma_y^2.$$

Proof.

$$\begin{aligned}\mathbb{E}\|\left(\mathbf{u}_{x,t} - \mathbf{e}_{x,t}\right)\|^2 &= \mathbb{E}\|\left(\mathbf{u}_{x,t} - \bar{\mathbf{e}}_{x,t}\right) + \left(\bar{\mathbf{e}}_{x,t} - \mathbf{e}_{x,t}\right)\|^2\\ &\le 2\mathbb{E}\|\left(\mathbf{u}_{x,t} - \bar{\mathbf{e}}_{x,t}\right)\|^2 + 2\mathbb{E}\|\left(\bar{\mathbf{e}}_{x,t} - \mathbf{e}_{x,t}\right)\|^2\\ &\le 4\mathbb{E}\|\bar{\mathbf{e}}_{x,t}\|^2 + 4\mathbb{E}\|\mathbf{u}_{x,t}\|^2 + \frac{2}{mK}\sigma_x^2,\end{aligned}$$

where the second inequality follows from the fact that $\{\nabla_x f_i(\mathbf{z}_{t,i}^j,\xi_{t,i}^j) - \nabla_x f_i(\mathbf{z}_{t,i}^j)\}$ the martingale difference sequence (see Lemma 4 in [28]).

The bound of $\|\left(\mathbf{u}_{y,t} - \mathbf{e}_{y,t}\right)\|^2$ follows from the similar proof. □

Lemma 2 (One Round Progress for Φ)**.**

$$\mathbb{E}\Phi(\mathbf{x}_{t+1}) - \Phi(\mathbf{x}_t) \le -\frac{1}{2}\eta_x K\|\nabla\Phi(\mathbf{x}_t)\|^2 - \frac{1}{4}\eta_x K\|\nabla_x f(\mathbf{z}_t)\|^2 + 2L\eta_x^2 K^2 \mathbb{E}\left\|\mathbf{u}_{x,t}\right\|^2 + \eta_x K\left(\frac{3}{2} + 2L\eta_x K\right)\mathbb{E}\left\|\bar{\mathbf{e}}_{x,t}\right\|^2 + \eta_x K\frac{L_f^2}{\mu^2}\left\|\nabla_y f(\mathbf{z}_t)\right\|^2 + \frac{L\eta_x^2 K}{m}\sigma_x^2.$$

Proof. Due to the L-smoothness of $\Phi(\mathbf{x})$, we have one step update in expectation conditioned on step t:

$$\begin{aligned}\mathbb{E}\Phi(\mathbf{x}_{t+1}) - \Phi(\mathbf{x}_t) &\le \langle\nabla\Phi(\mathbf{x}_t), \mathbb{E}[\mathbf{x}_{t+1} - \mathbf{x}_t]\rangle + \frac{L}{2}\mathbb{E}\|\mathbf{x}_{t+1} - \mathbf{x}_t\|^2 \\ &= \underbrace{\langle\nabla\Phi(\mathbf{x}_t), -\eta_x K\mathbb{E}\left[\mathbf{u}_{x,t} - \mathbf{e}_{x,t}\right]\rangle}_{A_1} + \underbrace{\frac{L}{2}\mathbb{E}\|\eta_x K\left(\mathbf{u}_{x,t} - \mathbf{e}_{x,t}\right)\|^2}_{A_2}.\end{aligned}$$

$$\begin{aligned}&A_1 = \langle\nabla\Phi(\mathbf{x}_t), -\eta_x K\mathbb{E}\left(\nabla_x f(\mathbf{z}_t) - \bar{\mathbf{e}}_{x,t}\right)\rangle \\ &= -\frac{1}{2}\eta_x K\|\nabla\Phi(\mathbf{x}_t)\|^2 - \frac{1}{2}\eta_x K\mathbb{E}\|\nabla_x f(\mathbf{z}_t) - \bar{\mathbf{e}}_{x,t}\|^2 + \frac{1}{2}\eta_x K\mathbb{E}\|\nabla\Phi(\mathbf{x}_t) - \nabla_x f(\mathbf{z}_t) + \bar{\mathbf{e}}_{x,t}\|^2 \\ &\le -\frac{1}{2}\eta_x K\|\nabla\Phi(\mathbf{x}_t)\|^2 - \frac{1}{4}\eta_x K\|\nabla_x f(\mathbf{z}_t)\|^2 + \frac{3}{2}\eta_x K\mathbb{E}\|\bar{\mathbf{e}}_{x,t}\|^2 + \eta_x K\|\nabla\Phi(\mathbf{x}_t) - \nabla_x f(\mathbf{z}_t)\|^2,\end{aligned}$$

where the last inequality follows from $\|\mathbf{a}+\mathbf{b}\|^2 \ge \frac{1}{2}\|\mathbf{a}\|^2 - \|\mathbf{b}\|^2$ and $\|\mathbf{a}+\mathbf{b}\|^2 \le 2\|\mathbf{a}\|^2 + 2\|\mathbf{b}\|^2$.

$$A_2 \le 2L\eta_x^2 K^2\mathbb{E}\left\|\bar{\mathbf{e}}_{x,t}\right\|^2 + 2L\eta_x^2 K^2\mathbb{E}\left\|\mathbf{u}_{x,t}\right\|^2 + \frac{L\eta_x^2 K}{m}\sigma_x^2,$$

where the inequality is due to Lemma 1.

$$\begin{aligned}\|\nabla\Phi(\mathbf{x}_t) - \nabla_x f(\mathbf{z}_t)\|^2 &= L_f^2\|\mathbf{y}(\mathbf{x}_t) - \mathbf{y}^*\|^2 \\ &\le \frac{L_f^2}{\mu^2}\left\|\nabla_y f(\mathbf{z}_t)\right\|^2,\end{aligned}$$

where the last inequality is due to the PL condition (Theorem 2 in [42]).

Combining pieces together, we have:

$$\begin{aligned}\mathbb{E}\Phi(\mathbf{x}_{t+1}) - \Phi(\mathbf{x}_t) &= \underbrace{\langle\nabla\Phi(\mathbf{x}_t), -\eta_x K\mathbb{E}\left[\mathbf{u}_{x,t} - \mathbf{e}_{x,t}\right]\rangle}_{A_1} + \underbrace{\frac{L}{2}\mathbb{E}\|\eta_x K\left(\mathbf{u}_{x,t} - \mathbf{e}_{x,t}\right)\|^2}_{A_2} \\ &\le -\frac{1}{2}\eta_x K\|\nabla\Phi(\mathbf{x}_t)\|^2 - \frac{1}{4}\eta_x K\|\nabla_x f(\mathbf{z}_t)\|^2 + 2L\eta_x^2 K^2\mathbb{E}\left\|\mathbf{u}_{x,t}\right\|^2 \\ &\quad + \eta_x K\left(\frac{3}{2} + 2L\eta_x K\right)\mathbb{E}\left\|\bar{\mathbf{e}}_{x,t}\right\|^2 + \eta_x K\frac{L_f^2}{\mu^2}\left\|\nabla_y f(\mathbf{z}_t)\right\|^2 + \frac{L\eta_x^2 K}{m}\sigma_x^2.\end{aligned}$$

□

Lemma 3 (One Round Progress for f)**.**

$$\begin{aligned}&f(\mathbf{z}_t) - \mathbb{E}f(\mathbf{z}_{t+1}) \\ &\le \frac{3}{2}\eta_x K\left\|\nabla_x f(\mathbf{z}_t)\right\|^2 + 2L_f\eta_x^2 K^2\mathbb{E}\left\|\mathbf{u}_{x,t}\right\|^2 + \eta_x K\left(\frac{1}{2} + 2L_f\eta_x K\right)\mathbb{E}\left\|\bar{\mathbf{e}}_{x,t}\right\|^2 + \frac{L_f\eta_x^2 K}{m}\sigma_x^2 \\ &\quad - \frac{1}{2}\eta_y K\|\nabla_y f(\mathbf{z}_t)\|^2 + 2L_f\eta_y^2 K^2\mathbb{E}\left\|\mathbf{u}_{y,t}\right\|^2 + \eta_y K\left(\frac{1}{2} + 2L_f\eta_y K\right)\mathbb{E}\left\|\bar{\mathbf{e}}_{y,t}\right\|^2 + \frac{L_f\eta_y^2 K}{m}\sigma_y^2.\end{aligned}$$

Proof. Similarly, due to L-smoothness of $f(\mathbf{z})$, we have:

$$
\begin{aligned}
& f(\mathbf{z}_t)-\mathbb{E}f(\mathbf{z}_{t+1}) \leq \eta_x K \mathbb{E}\left\langle\nabla_x f(\mathbf{z}_t), \mathbf{u}_{x,t}-\mathbf{e}_{x,t}\right\rangle-\eta_y K \mathbb{E}\left\langle\nabla_y f(\mathbf{z}_t), \mathbf{u}_{y,t}-\mathbf{e}_{y,t}\right\rangle \\
& \quad+\frac{L_f \eta_x^2 K^2}{2} \mathbb{E}\left\|\mathbf{u}_{x,t}-\mathbf{e}_{x,t}\right\|^2+\frac{L_f \eta_y^2 K^2}{2} \mathbb{E}\left\|\mathbf{u}_{y,t}-\mathbf{e}_{y,t}\right\|^2 \\
& =\eta_x K \mathbb{E}\left\langle\nabla_x f(\mathbf{z}_t), \nabla_x f(\mathbf{z}_t)-\bar{\mathbf{e}}_{x,t}\right\rangle-\eta_y K \mathbb{E}\left\langle\nabla_y f(\mathbf{z}_t), \nabla_y f(\mathbf{z}_t)-\bar{\mathbf{e}}_{y,t}\right\rangle \\
& \quad+\frac{L_f \eta_x^2 K^2}{2} \mathbb{E}\left\|\mathbf{u}_{x,t}-\mathbf{e}_{x,t}\right\|^2+\frac{L_f \eta_y^2 K^2}{2} \mathbb{E}\left\|\mathbf{u}_{y,t}-\mathbf{e}_{y,t}\right\|^2 \\
& \leq \frac{3}{2} \eta_x K\left\|\nabla_x f(\mathbf{z}_t)\right\|^2+\frac{1}{2} \eta_x K \mathbb{E}\left\|\bar{\mathbf{e}}_{x,t}\right\|^2-\frac{1}{2} \eta_y K\left\|\nabla_y f(\mathbf{z}_t)\right\|^2+\frac{1}{2} \eta_y K \mathbb{E}\left\|\bar{\mathbf{e}}_{y,t}\right\|^2 \\
& \quad+\frac{L_f \eta_x^2 K^2}{2} \mathbb{E}\left\|\mathbf{u}_{x,t}-\mathbf{e}_{x,t}\right\|^2+\frac{L_f \eta_y^2 K^2}{2} \mathbb{E}\left\|\mathbf{u}_{y,t}-\mathbf{e}_{y,t}\right\|^2 \\
& \leq \frac{3}{2} \eta_x K\left\|\nabla_x f(\mathbf{z}_t)\right\|^2+2 L_f \eta_x^2 K^2 \mathbb{E}\left\|\mathbf{u}_{x,t}\right\|^2+\eta_x K\left(\frac{1}{2}+2 L_f \eta_x K\right) \mathbb{E}\left\|\bar{\mathbf{e}}_{x,t}\right\|^2+\frac{L_f \eta_x^2 K}{m} \sigma_x^2 \\
& \quad-\frac{1}{2} \eta_y K\|\nabla_y f(\mathbf{z}_t)\|^2+2 L_f \eta_y^2 K^2 \mathbb{E}\left\|\mathbf{u}_{y,t}\right\|^2+\eta_y K\left(\frac{1}{2}+2 L_f \eta_y K\right) \mathbb{E}\left\|\bar{\mathbf{e}}_{y,t}\right\|^2+\frac{L_f \eta_y^2 K}{m} \sigma_y^2 .
\end{aligned}
$$

□

Lemma 4 (Bounded Error for FSGDA**).**

$$
\begin{aligned}
\mathbb{E}\|\bar{\mathbf{e}}_{x,t}\|^2 \leq L_f^2\Big[& 40K^2\eta_{x,l}^2\left\|\nabla_x f(\mathbf{z}_t)\right\|^2+40K^2\eta_{y,l}^2\left\|\nabla_y f(\mathbf{z}_t)\right\|^2+40K^2\eta_{x,l}^2\sigma_{x,G}^2+40K^2\eta_{y,l}^2\sigma_{y,G}^2 \\
& +5K\eta_{x,l}^2\sigma_x^2+5K\eta_{y,l}^2\sigma_y^2\Big], \\
\mathbb{E}\|\bar{\mathbf{e}}_{y,t}\|^2 \leq L_f^2\Big[& 40K^2\eta_{x,l}^2\left\|\nabla_x f(\mathbf{z}_t)\right\|^2+40K^2\eta_{y,l}^2\left\|\nabla_y f(\mathbf{z}_t)\right\|^2+40K^2\eta_{x,l}^2\sigma_{x,G}^2+40K^2\eta_{y,l}^2\sigma_{y,G}^2 \\
& +5K\eta_{x,l}^2\sigma_x^2+5K\eta_{y,l}^2\sigma_y^2\Big].
\end{aligned}
$$

Proof.

$$
\begin{aligned}
\mathbb{E}\|\bar{\mathbf{e}}_{x,t}\|^2 &= \mathbb{E}\left\|\frac{1}{mK}\sum_{i\in S_t}\sum_{j\in[K]}\left(\nabla_x f_i(\mathbf{z}_t)-\nabla_x f_i(\mathbf{z}_{t,i}^j)\right)\right\|^2 \\
&\leq \mathbb{E}\left[\frac{1}{K}\sum_{i\in S_t}\sum_{j\in[K]}\left\|\left(\nabla_x f_i(\mathbf{z}_t)-\nabla_x f_i(\mathbf{z}_{t,i}^j)\right)\right\|^2\right] \\
&\leq \frac{L_f^2}{MK}\sum_{i\in[M]}\sum_{j\in[K]}\mathbb{E}\left\|\left(\mathbf{z}_t-\mathbf{z}_{t,i}^j\right)\right\|^2
\end{aligned}
$$

$$
\begin{aligned}
& \mathbb{E}\left\|\left(\mathbf{z}_t-\mathbf{z}_{t,i}^{j+1}\right)\right\|^2=\mathbb{E}\left[\left\|\mathbf{x}_{t,i}^j-\mathbf{x}_t-\eta_{x,l}\nabla_x f_i(\mathbf{z}_{t,i}^j,\xi_{t,i}^j)\right\|^2\right]+\mathbb{E}\left[\left\|\mathbf{y}_{t,i}^j-\mathbf{y}_t-\eta_{y,l}\nabla_y f_i(\mathbf{z}_{t,i}^j,\xi_{t,i}^j)\right\|^2\right] \\
& \leq \mathbb{E}\left[\left\|\mathbf{x}_{t,i}^j-\mathbf{x}_t-\eta_{x,l}\nabla_x f_i(\mathbf{z}_{t,i}^j)\right\|^2\right]+\mathbb{E}\left[\left\|\mathbf{y}_{t,i}^j-\mathbf{y}_t-\eta_{y,l}\nabla_y f_i(\mathbf{z}_{t,i}^j)\right\|^2\right]+\eta_{x,l}^2\sigma_x^2+\eta_{y,l}^2\sigma_y^2 \\
& \leq\left(1+\frac{1}{2K-1}\right)\left\|\mathbf{z}_{t,i}^j-\mathbf{z}_t\right\|^2+2K\eta_{x,l}^2\left\|\nabla_x f_i\left(\mathbf{z}_{t,i}^j\right)\right\|^2+2K\eta_{y,l}^2\left\|\nabla_y f_i\left(\mathbf{z}_{t,i}^j\right)\right\|^2+\eta_{x,l}^2\sigma_x^2+\eta_{y,l}^2\sigma_y^2 \\
& \leq\left(1+\frac{1}{2K-1}\right)\left\|\mathbf{z}_{t,i}^j-\mathbf{z}_t\right\|^2+4K\eta_{x,l}^2\left\|\nabla_x f_i\left(\mathbf{z}_{t,i}^j\right)-\nabla_x f_i\left(\mathbf{z}_t\right)\right\|^2+4K\left\|\nabla_x f_i\left(\mathbf{z}_t\right)\right\|^2 \\
& \quad+4K\eta_{y,l}^2\left\|\nabla_y f_i\left(\mathbf{z}_{t,i}^j\right)-\nabla_y f_i\left(\mathbf{z}_t\right)\right\|^2+4K\eta_{y,l}^2\left\|\nabla_y f_i\left(\mathbf{z}_t\right)\right\|^2+\eta_{x,l}^2\sigma_x^2+\eta_{y,l}^2\sigma_y^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left(1+\frac{1}{2K-1}+4K\max\{L_f^2\eta_{x,l}^2, L_f^2\eta_{y,l}^2\}\right)\left\|\mathbf{z}_{t,i}^j-\mathbf{z}_t\right\|^2 \\
&\quad + 4K\eta_{x,l}^2\left\|\nabla_x f_i\left(\mathbf{z}_t\right)\right\|^2 + 4K\eta_{y,l}^2\left\|\nabla_y f_i\left(\mathbf{z}_t\right)\right\|^2 + \eta_{x,l}^2\sigma_x^2 + \eta_{y,l}^2\sigma_y^2 \\
&\leq \left(1+\frac{1}{K-1}\right)\left\|\mathbf{z}_{t,i}^j-\mathbf{z}_t\right\|^2 + 4K\left\|\nabla_x f_i\left(\mathbf{z}_t\right)\right\|^2 + 4K\eta_{y,l}^2\left\|\nabla_y f_i\left(\mathbf{z}_t\right)\right\|^2 + \eta_{x,l}^2\sigma_x^2 + \eta_{y,l}^2\sigma_y^2 \\
&\leq \sum_{\tau=0}^{j-1}\left(1+\frac{1}{K-1}\right)^{\tau}\left[4K\eta_{x,l}^2\left\|\nabla_x f_i\left(\mathbf{z}_t\right)\right\|^2 + 4K\eta_{y,l}^2\left\|\nabla_y f_i\left(\mathbf{z}_t\right)\right\|^2 + \eta_{x,l}^2\sigma_x^2 + \eta_{y,l}^2\sigma_y^2\right] \\
&\leq 20K^2\eta_{x,l}^2\left\|\nabla_x f_i\left(\mathbf{z}_t\right)\right\|^2 + 20K^2\eta_{y,l}^2\left\|\nabla_y f_i\left(\mathbf{z}_t\right)\right\|^2 + 5K\eta_{x,l}^2\sigma_x^2 + 5K\eta_{y,l}^2\sigma_y^2 \\
&\leq 40K^2\eta_{x,l}^2\left\|\nabla_x f(\mathbf{z}_t)\right\|^2 + 40K^2\eta_{y,l}^2\left\|\nabla_y f(\mathbf{z}_t)\right\|^2 + 40K^2\eta_{x,l}^2\sigma_{x,G}^2 + 40K^2\eta_{y,l}^2\sigma_{y,G}^2 \\
&\quad + 5K\eta_{x,l}^2\sigma_x^2 + 5K\eta_{y,l}^2\sigma_y^2,
\end{aligned}
$$

where the first inequality is due to bounded variance of stochastic gradient, the second and third inequalities follow from the fact $\|\mathbf{a}+\mathbf{b}\|^2 \leq \left(1+\frac{1}{\epsilon}\right)\|\mathbf{a}\|^2 + (1+\epsilon)\|\mathbf{b}\|^2$, the forth inequality is due to smoothness of f in x and y, fifth inequality holds if

$$
4K\max\{L_f^2\eta_{x,l}^2, L_f^2\eta_{y,l}^2\} \leq \frac{1}{2(K-1)(2K-1)}, \tag{7}
$$

the second last inequality follows from the $\sum_{\tau=0}^{j-1}\left(1+\frac{1}{K-1}\right)^{\tau} \leq (K-1)\left[\left(1+\frac{1}{K-1}\right)^{K}-1\right] \leq 5K$, and the last inequality is due to the Assumption 4.

Plugging into the bound of $\|\bar{\mathbf{e}}_{x,t}\|^2$, we have:

$$
\begin{aligned}
\|\bar{\mathbf{e}}_{x,t}\|^2 \leq L_f^2\Big[&40K^2\eta_{x,l}^2\left\|\nabla_x f(\mathbf{z}_t)\right\|^2 + 40K^2\eta_{y,l}^2\left\|\nabla_y f(\mathbf{z}_t)\right\|^2 + 40K^2\eta_{x,l}^2\sigma_{x,G}^2 + 40K^2\eta_{y,l}^2\sigma_{y,G}^2 \\
&+ 5K\eta_{x,l}^2\sigma_x^2 + 5K\eta_{y,l}^2\sigma_y^2\Big].
\end{aligned}
$$

The bound of $\|\bar{\mathbf{e}}_{y,t}\|^2$ follows from the similar proof.

□

Theorem 2 (Convergence Rate for FSGDA). *Under Assumptions 1- 4, define $\mathcal{L}_t = \Phi(\mathbf{x}_t) - \frac{1}{10}f(\mathbf{x}_t,\mathbf{y}_t)$, if the learning rates $\eta_{x,g}$, $\eta_{x,l}$, $\eta_{y,g}$, and $\eta_{y,l}$ satisfy:*

$$
\begin{aligned}
&8K(K-1)(2K-1)L_f^2\max\{\eta_{x,l}^2, \eta_{y,l}^2\} \leq 1, \\
&a_1 - a_3 40L_f^2K^2\eta_{x,l}^2 - \frac{\eta_y}{\eta_x}a_4 40L_f^2K^2\eta_{x,l}^2 \geq 0, \\
&a_2 - a_3\frac{\eta_x}{\eta_y}40L_f^2K^2\eta_{y,l}^2 - a_4 40L_f^2K^2\eta_{y,l}^2 \geq 0,
\end{aligned}
$$

where $a_1 = \left(\frac{1}{10} - 2(2L+\frac{1}{5}L_f)\eta_x K\right)$, $a_2 = \left(\frac{1}{20} - \frac{2}{5}L_f\eta_y K - \frac{\eta_x}{\eta_y}\frac{L_f^2}{\mu^2}\right)$, $a_3 = \left(\frac{31}{20} + (2L+\frac{1}{5}L_f)\eta_x K\right)$ and $a_4 = \left(\frac{1}{20} + \frac{1}{5}L_f\eta_y K\right)$, then the output sequence $\{\mathbf{x}_t\}$ generated by FSGDA satisfies:

$$
\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}\|\nabla\Phi(\mathbf{x}_t)\|^2 \leq \underbrace{\frac{2\left(\mathcal{L}_0-\mathcal{L}_T\right)}{\eta_x KT}}_{\text{optimization error}} + \underbrace{\frac{2\eta_x}{m}\left(L+\frac{L_f}{100}\right)\sigma_x^2 + \frac{L_f\eta_y^2}{5m\eta_x}\sigma_y^2}_{\text{statistical error}} + \underbrace{\psi_3}_{\substack{\text{local}\\\text{update error}}} + \underbrace{\psi_4}_{\substack{\text{sampling}\\\text{variance}}}.
$$

Here, ψ_3 and ψ_4 are defined as follows:

$$
\begin{aligned}
\psi_3 &= 2\left(a_3L_f^2 + a_4\frac{\eta_y}{\eta_x}L_f^2\right)\left[40K^2\eta_{x,l}^2\sigma_{x,G}^2 + 40K^2\eta_{y,l}^2\sigma_{y,G}^2 + 5K\eta_{x,l}^2\sigma_x^2 + 5K\eta_{y,l}^2\sigma_y^2\right], \\
\psi_4 &= \left((2L+\frac{1}{5}L_f)\eta_x K\right)\left(1-\frac{m}{M}\right)\frac{2}{m}\sigma_{x,G}^2 + \frac{2}{5m}L_f\eta_y K\frac{\eta_y}{\eta_x}\left(1-\frac{m}{M}\right)\sigma_{y,G}^2,
\end{aligned}
$$

Proof. Define potential function $\mathcal{L}_t = \Phi(\mathbf{x}_t) - \frac{1}{10} f(\mathbf{z}_t)$,

$$
\begin{aligned}
&\mathbb{E}L_{t+1} - \mathcal{L}_t = \mathbb{E}\Phi(\mathbf{x}_{t+1}) - \Phi(\mathbf{x}_t) + \frac{1}{10}\left(f(\mathbf{z}_t) - \mathbb{E}f(\mathbf{z}_{t+1})\right) \\
&\leq -\frac{1}{2}\eta_x K\|\nabla\Phi(\mathbf{x}_t)\|^2 - \frac{1}{10}\eta_x K\|\nabla_x f(\mathbf{z}_t)\|^2 + \left((2L + \frac{1}{5}L_f)\eta_x^2 K^2\right)\mathbb{E}\left\|\mathbf{u}_{x,t}\right\|^2 \\
&\quad - \eta_y K\left(\frac{1}{20} - \frac{\eta_x}{\eta_y}\frac{L_f^2}{\mu^2}\right)\|\nabla_y f(\mathbf{z}_t)\|^2 + \frac{1}{5}L_f\eta_y^2 K^2\mathbb{E}\left\|\mathbf{u}_{y,t}\right\|^2 \\
&\quad + \eta_x K\left(\frac{31}{20} + (2L + \frac{1}{5}L_f)\eta_x K\right)\mathbb{E}\left\|\bar{\mathbf{e}}_{x,t}\right\|^2 + \eta_y K\left(\frac{1}{20} + \frac{1}{5}L_f\eta_y K\right)\mathbb{E}\left\|\bar{\mathbf{e}}_{y,t}\right\|^2 \\
&\quad + \frac{\eta_x^2 K}{m}\left(L + \frac{L_f}{10}\right)\sigma_x^2 + \frac{L_f\eta_y^2 K}{10m}\sigma_y^2 \\
&\leq -\frac{1}{2}\eta_x K\|\nabla\Phi(\mathbf{x}_t)\|^2 - \eta_x K\underbrace{\left(\frac{1}{10} - 2(2L + \frac{1}{5}L_f)\eta_x K\right)}_{a_1}\|\nabla_x f(\mathbf{z}_t)\|^2 \\
&\quad - \eta_y K\underbrace{\left(\frac{1}{20} - \frac{2}{5}L_f\eta_y K - \frac{\eta_x}{\eta_y}\frac{L_f^2}{\mu^2}\right)}_{a_2}\|\nabla_y f(\mathbf{z}_t)\|^2 \\
&\quad + \left((2L + \frac{1}{5}L_f)\eta_x^2 K^2\right)\left(1 - \frac{m}{M}\right)\frac{2}{m}\sigma_{x,G}^2 + \frac{1}{5}L_f\eta_y^2 K^2\left(1 - \frac{m}{M}\right)\frac{2}{m}\sigma_{y,G}^2 \\
&\quad + \eta_x K\underbrace{\left(\frac{31}{20} + (2L + \frac{1}{5}L_f)\eta_x K\right)}_{a_3}\mathbb{E}\left\|\bar{\mathbf{e}}_{x,t}\right\|^2 + \eta_y K\underbrace{\left(\frac{1}{20} + \frac{1}{5}L_f\eta_y K\right)}_{a_4}\mathbb{E}\left\|\bar{\mathbf{e}}_{y,t}\right\|^2 \\
&\quad + \frac{\eta_x^2 K}{m}\left(L + \frac{L_f}{10}\right)\sigma_x^2 + \frac{L_f\eta_y^2 K}{10m}\sigma_y^2 \\
&\leq -\frac{1}{2}\eta_x K\|\nabla\Phi(\mathbf{x}_t)\|^2 + \frac{\eta_x^2 K}{m}\left(L + \frac{L_f}{10}\right)\sigma_x^2 + \frac{L_f\eta_y^2 K}{10m}\sigma_y^2 \\
&\quad + \left((2L + \frac{1}{5}L_f)\eta_x^2 K^2\right)\left(1 - \frac{m}{M}\right)\frac{2}{m}\sigma_{x,G}^2 + \frac{1}{5}L_f\eta_y^2 K^2\left(1 - \frac{m}{M}\right)\frac{2}{m}\sigma_{y,G}^2 \\
&\quad + K\left(a_3 L_f^2\eta_x + a_4\eta_y L_f^2\right)\left[40K^2\eta_{x,l}^2\sigma_{x,G}^2 + 40K^2\eta_{y,l}^2\sigma_{y,G}^2 + 5K\eta_{x,l}^2\sigma_x^2 + 5K\eta_{y,l}^2\sigma_y^2\right],
\end{aligned}
$$

where the second inequality is due to $\mathbb{E}\|\mathbf{u}_{x,t}\|^2 \leq 2\|\nabla_x f(\mathbf{z}_t)\|^2 + 2\left(1 - \frac{m}{M}\right)\frac{\sigma_{x,G}^2}{m}$ and $\mathbb{E}\|\mathbf{u}_{y,t}\|^2 \leq 2\|\nabla_y f(\mathbf{z}_t)\|^2 + 2\left(1 - \frac{m}{M}\right)\frac{\sigma_{y,G}^2}{m}$, the last inequality follows from the conditions:

$$
a_1 - a_3 40L_f^2 K^2\eta_{x,l}^2 - \frac{\eta_y}{\eta_x}a_4 40L_f^2 K^2\eta_{x,l}^2 \geq 0, \tag{8}
$$

$$
a_2 - a_3\frac{\eta_x}{\eta_y}40L_f^2 K^2\eta_{y,l}^2 - a_4 40L_f^2 K^2\eta_{y,l}^2 \geq 0. \tag{9}
$$

Telescoping and rearranging, we have:

$$
\begin{aligned}
\frac{1}{T}\sum_{t=0}^{T-1}\|\nabla\Phi(\mathbf{x}_t)\|^2 &\leq \frac{2\left(\mathcal{L}_0 - \mathcal{L}_*\right)}{\eta_x KT} + \frac{2\eta_x}{m}\left(L + \frac{L_f}{100}\right)\sigma_x^2 + \frac{L_f\eta_y^2}{5m\eta_x}\sigma_y^2 \\
&\quad + \left((2L + \frac{1}{5}L_f)\eta_x K\right)\left(1 - \frac{m}{M}\right)\frac{2}{m}\sigma_{x,G}^2 + \frac{1}{5}L_f\eta_y K\frac{\eta_y}{\eta_x}\left(1 - \frac{m}{M}\right)\frac{2}{m}\sigma_{y,G}^2 \\
&\quad + 2\left(a_3 L_f^2 + a_4\frac{\eta_y}{\eta_x}L_f^2\right)\left[40K^2\eta_{x,l}^2\sigma_{x,G}^2 + 40K^2\eta_{y,l}^2\sigma_{y,G}^2 + 5K\eta_{x,l}^2\sigma_x^2 + 5K\eta_{y,l}^2\sigma_y^2\right].
\end{aligned}
$$

□

B.2 Proof for SAGDA Option I

For SAGDA Option I, the update rule is:

$$
\begin{aligned}
\mathbf{x}_{t+1} &= \mathbf{x}_t - \eta_{x,g}\eta_{x,l}\left[\frac{1}{m}\sum_{i\in S_t}\sum_{j\in[K]}\left(\nabla_x f_i(\mathbf{z}_{t,i}^j,\xi_{t,i}^j) - \mathbf{v}_x^i + \bar{\mathbf{v}}_{x,t}\right)\right],\\
\mathbf{y}_{t+1} &= \mathbf{y}_t + \eta_{y,g}\eta_{y,l}\left[\frac{1}{m}\sum_{i\in S_t}\sum_{j\in[K]}\left(\nabla_y f_i(\mathbf{z}_{t,i}^j,\xi_{t,i}^j) - \mathbf{v}_y^i + \bar{\mathbf{v}}_{y,t}\right)\right],\\
\mathbf{e}_{x,t} &= \frac{1}{mK}\sum_{i\in S_t}\sum_{j\in[K]}\left[\nabla_x f_i(\mathbf{z}_t) - \left(\nabla_x f_i(\mathbf{z}_{t,i}^j,\xi_{t,i}^j) - \mathbf{v}_x^i + \bar{\mathbf{v}}_{x,t}\right)\right]\\
\bar{\mathbf{e}}_{x,t} &= \mathbb{E}[\mathbf{e}_{x,t}] = \frac{1}{mK}\sum_{i\in S_t}\sum_{j\in[K]}\left(\nabla_x f_i(\mathbf{z}_t) - \nabla_x f_i(\mathbf{z}_{t,i}^j)\right),\\
\mathbf{e}_{y,t} &= \frac{1}{mK}\sum_{i\in S_t}\sum_{j\in[K]}\left[\nabla_y f_i(\mathbf{z}_t) - \left(\nabla_y f_i(\mathbf{z}_{t,i}^j,\xi_{t,i}^j) - \mathbf{v}_y^i + \bar{\mathbf{v}}_{y,t}\right)\right]\\
\bar{\mathbf{e}}_{y,t} &= \mathbb{E}[\mathbf{e}_{y,t}] = \frac{1}{mK}\sum_{i\in S_t}\sum_{j\in[K]}\left(\nabla_y f_i(\mathbf{z}_t) - \nabla_y f_i(\mathbf{z}_{t,i}^j)\right),
\end{aligned}
$$

where we define $\mathbf{v}_x^i = \nabla_x f_i(\mathbf{w}_{t,i},\xi)$ and $\bar{\mathbf{v}}_{x,t} = \frac{1}{M}\sum_{i\in[M]}\mathbf{v}_x^i$ with a sequence of parameters $\mathbf{w}_{t,i}$ such that

$$
\mathbf{w}_{t,i} := \begin{cases}\mathbf{z}_{t-1}, \ \textit{if } i\in S_{t-1},\\ \mathbf{w}_{t-1,i}, \ \textit{otherwise}.\end{cases}
$$

We further have the following definition for notational clarity:

$$
\begin{aligned}
\Delta\mathbf{x}_t &= \frac{1}{mK}\sum_{i\in S_t}\sum_{j\in[K]}\left[\nabla_x f_i(\mathbf{z}_{t,i}^j,\xi_{t,i}^j) - \mathbf{v}_x^i + \bar{\mathbf{v}}_{x,t}\right],\\
\Delta\mathbf{y}_t &= \frac{1}{mK}\sum_{i\in S_t}\sum_{j\in[K]}\left[\nabla_y f_i(\mathbf{z}_{t,i}^j,\xi_{t,i}^j) - \mathbf{v}_y^i + \bar{\mathbf{v}}_{y,t}\right],\\
\Psi_t &= \frac{1}{MK}\sum_{i\in[M]}\sum_{j\in[K]}\mathbb{E}\left\|\mathbf{z}_{t,i}^j - \mathbf{z}_t\right\|^2,\\
\Gamma_t &= \frac{1}{M}\sum_{i\in[M]}\mathbb{E}\left\|\mathbf{w}_{t,i} - \mathbf{z}_t\right\|^2.
\end{aligned}
$$

Lemma 5 (Iterative Control Variate)**.**

$$
\Gamma_t = \left(1 - \frac{m}{2M}\right)\Gamma_{t-1} + \left(\frac{m}{M} + \frac{M}{m} - 1\right)\mathbb{E}\left\|\mathbf{z}_t - \mathbf{z}_{t-1}\right\|^2.
$$

Proof.

$$
\begin{aligned}
\Gamma_t &= \frac{1}{M}\sum_{i\in[M]}\mathbb{E}\left\|\mathbf{w}_{t,i} - \mathbf{z}_t\right\|^2\\
&= \left(1 - \frac{m}{M}\right)\frac{1}{M}\sum_{i\in[M]}\mathbb{E}\left\|\mathbf{w}_{t-1,i} - \mathbf{z}_t\right\|^2 + \frac{m}{M}\mathbb{E}\left\|\mathbf{z}_{t-1} - \mathbf{z}_t\right\|^2\\
&\le \left(1 - \frac{m}{M}\right)\left(1 + \frac{1}{b}\right)\Gamma_{t-1} + \left[\left(1 - \frac{m}{M}\right)(1+b) + \frac{m}{M}\right]\mathbb{E}\left\|\mathbf{z}_t - \mathbf{z}_{t-1}\right\|^2\\
&= \left(1 - \frac{m}{2M}\right)\Gamma_{t-1} + \left(\frac{m}{M} + \frac{M}{m} - 1\right)\mathbb{E}\left\|\mathbf{z}_t - \mathbf{z}_{t-1}\right\|^2,
\end{aligned}
$$

where we set $b = \frac{2M}{m} - 1$. □

Lemma 6 (Local Step Distance for SAGDA Option I). *$\forall i \in [M], j \in [K]$, we can bound the local step distance as follows:*

$$\frac{1}{M}\sum_{i\in[M]}\mathbb{E}\left\|\left(\mathbf{z}_{t,i}^{j}-\mathbf{z}_{t}\right)\right\|^{2} \leq 160K^{2}\left(\eta_{x,l}^{2}+\eta_{y,l}^{2}\right)L_{f}^{2}\Gamma_{t}+10K^{2}\left(\eta_{x,l}^{2}\sigma_{x}^{2}+\eta_{y,l}^{2}\sigma_{y}^{2}\right)$$
$$+40K^{2}\left(\eta_{x,l}^{2}\mathbb{E}\left\|\nabla_{x}f(\mathbf{z}_{t})\right\|^{2}+\eta_{y,l}^{2}\mathbb{E}\left\|\nabla_{y}f(\mathbf{z}_{t})\right\|^{2}\right).$$

Proof. First, we bound the local update as follows:

$$\begin{aligned}
&\mathbb{E}\left\|\left(\nabla_{x}f_{i}(\mathbf{z}_{t,i}^{j},\xi_{t,i}^{j})-\mathbf{v}_{x}^{i}+\bar{\mathbf{v}}_{x,t}\right)\right\|^{2}\\
&\leq 4\Bigg[\mathbb{E}\left\|\nabla_{x}f_{i}(\mathbf{z}_{t,i}^{j})-\nabla_{x}f_{i}(\mathbf{z}_{t})\right\|^{2}+\mathbb{E}\left\|\mathbb{E}[\mathbf{v}_{x}^{i}]-\nabla_{x}f_{i}(\mathbf{z}_{t})\right\|^{2}+\mathbb{E}\left\|\mathbb{E}[\bar{\mathbf{v}}_{x,t}]-\nabla_{x}f(\mathbf{z}_{t})\right\|^{2}\\
&\quad+\left\|\nabla_{x}f(\mathbf{z}_{t})\right\|^{2}\Bigg]+\sigma_{x}^{2}\\
&\leq 4L_{f}^{2}\mathbb{E}\left\|\mathbf{z}_{t,i}^{j}-\mathbf{z}_{t}\right\|^{2}+4L_{f}^{2}\mathbb{E}\left\|\mathbf{w}_{t,i}-\mathbf{z}_{t}\right\|^{2}+4L_{f}^{2}\mathbb{E}\left\|\mathbb{E}[\bar{\mathbf{v}}_{x,t}]-\nabla_{x}f(\mathbf{z}_{t})\right\|^{2}\\
&\quad+4\mathbb{E}\left\|\nabla_{x}f(\mathbf{z}_{t})\right\|^{2}+\sigma_{x}^{2}.
\end{aligned}$$

That is,

$$\begin{aligned}
&\frac{1}{M}\sum_{i\in[M]}\mathbb{E}\left\|\left(\nabla_{x}f_{i}(\mathbf{z}_{t,i}^{j},\xi_{t,i}^{j})-\mathbf{v}_{x}^{i}+\bar{\mathbf{v}}_{x,t}\right)\right\|^{2}\\
&\leq 4L_{f}^{2}\frac{1}{M}\sum_{i\in[M]}\mathbb{E}\left\|\mathbf{z}_{t,i}^{j}-\mathbf{z}_{t}\right\|^{2}+8L_{f}^{2}\Gamma_{t}+\sigma_{x}^{2}+4\mathbb{E}\left\|\nabla_{x}f(\mathbf{z}_{t})\right\|^{2}.
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{M}\sum_{i\in[M]}\mathbb{E}\left[\left\|\mathbf{x}_{t,i}^{j+1}-\mathbf{x}_{t}\right\|^{2}\right]=\frac{1}{M}\sum_{i\in[M]}\mathbb{E}\left[\left\|\mathbf{x}_{t,i}^{j}-\mathbf{x}_{t}-\eta_{x,l}\left(\nabla_{x}f_{i}(\mathbf{z}_{t,i}^{j},\xi_{t,i}^{j})-\mathbf{v}_{x,t}^{i}+\bar{\mathbf{v}}_{x,t}\right)\right\|^{2}\right]\\
&\leq\left(1+\frac{1}{2K-1}\right)\frac{1}{M}\sum_{i\in[M]}\mathbb{E}\left\|\mathbf{x}_{t,i}^{j}-\mathbf{x}_{t}\right\|^{2}+2K\eta_{x,l}^{2}\frac{1}{M}\sum_{i\in[M]}\mathbb{E}\left\|\nabla_{x}f_{i}(\mathbf{z}_{t,i}^{j},\xi_{t,i}^{j})-\mathbf{v}_{x,t}^{i}+\bar{\mathbf{v}}_{x,t}\right\|^{2}\\
&\leq\left(1+\frac{1}{2K-1}+8KL_{f}^{2}\eta_{x,l}^{2}\right)\frac{1}{M}\sum_{i\in[M]}\mathbb{E}\left\|\mathbf{x}_{t,i}^{j}-\mathbf{x}_{t}\right\|^{2}+32K\eta_{x,l}^{2}L_{f}^{2}\Gamma_{t}\\
&\quad+2K\eta_{x,l}^{2}\sigma_{x}^{2}+8K\eta_{x,l}^{2}\mathbb{E}\left\|\nabla_{x}f(\mathbf{z}_{t})\right\|^{2}.
\end{aligned}$$

We can bound $\left\|\mathbf{y}_{t,i}^{j+1}-\mathbf{y}_{t}\right\|^{2}$ in the same way, and then we have

$$\begin{aligned}
&\frac{1}{M}\sum_{i\in[M]}\mathbb{E}\left\|\left(\mathbf{z}_{t,i}^{j+1}-\mathbf{z}_{t}\right)\right\|^{2}\\
&\leq\left(1+\frac{1}{2K-1}+8KL_{f}^{2}\max\{\eta_{x,l}^{2},\eta_{y,l}^{2}\}\right)\frac{1}{M}\sum_{i\in[M]}\mathbb{E}\left\|\mathbf{z}_{t,i}^{j}-\mathbf{z}_{t}\right\|^{2}+\Bigg[32K\left(\eta_{x,l}^{2}+\eta_{y,l}^{2}\right)L_{f}^{2}\Gamma_{t}\\
&\quad+2K\left(\eta_{x,l}^{2}\sigma_{x}^{2}+\eta_{y,l}^{2}\sigma_{y}^{2}\right)+8K\left(\eta_{x,l}^{2}\mathbb{E}\left\|\nabla_{x}f(\mathbf{z}_{t})\right\|^{2}+\eta_{y,l}^{2}\mathbb{E}\left\|\nabla_{y}f(\mathbf{z}_{t})\right\|^{2}\right)\Bigg]\\
&\leq\left(1+\frac{1}{K-1}\right)\frac{1}{M}\sum_{i\in[M]}\mathbb{E}\left\|\mathbf{z}_{t,i}^{j}-\mathbf{z}_{t}\right\|^{2}\Bigg[32K\left(\eta_{x,l}^{2}+\eta_{y,l}^{2}\right)L_{f}^{2}\Gamma_{t}\\
&\quad+2K\left(\eta_{x,l}^{2}\sigma_{x}^{2}+\eta_{y,l}^{2}\sigma_{y}^{2}\right)+8K\left(\eta_{x,l}^{2}\mathbb{E}\left\|\nabla_{x}f(\mathbf{z}_{t})\right\|^{2}+\eta_{y,l}^{2}\mathbb{E}\left\|\nabla_{y}f(\mathbf{z}_{t})\right\|^{2}\right)\Bigg]
\end{aligned}$$

$$\leq \sum_{\tau=0}^{j-1}\left(1+\frac{1}{K-1}\right)^{\tau}\Big[32K\left(\eta_{x,l}^2+\eta_{y,l}^2\right)L_f^2\Gamma_t$$
$$+2K\left(\eta_{x,l}^2\sigma_x^2+\eta_{y,l}^2\sigma_y^2\right)+8K\left(\eta_{x,l}^2\mathbb{E}\left\|\nabla_x f(\mathbf{z}_t)\right\|^2+\eta_{y,l}^2\mathbb{E}\left\|\nabla_y f(\mathbf{z}_t)\right\|^2\right)\Big]$$
$$\leq 160K^2\left(\eta_{x,l}^2+\eta_{y,l}^2\right)L_f^2\Gamma_t+10K^2\left(\eta_{x,l}^2\sigma_x^2+\eta_{y,l}^2\sigma_y^2\right)$$
$$+40K^2\left(\eta_{x,l}^2\mathbb{E}\left\|\nabla_x f(\mathbf{z}_t)\right\|^2+\eta_{y,l}^2\mathbb{E}\left\|\nabla_y f(\mathbf{z}_t)\right\|^2\right).$$

The learning rates should satisfy

$$4K\max\{L_f^2\eta_{x,l}^2, L_f^2\eta_{y,l}^2\}\leq\frac{1}{2(K-1)(2K-1)}, \tag{10}$$

□

Lemma 7.

$$\mathbb{E}\|\Delta\mathbf{x}_t\|^2\leq 4L_f^2\Psi_t+4L_f^2\Gamma_t+4\left\|\nabla_x f(\mathbf{z}_t)\right\|^2+\frac{9}{mK}\sigma_x^2,$$
$$\mathbb{E}\|\Delta\mathbf{y}_t\|^2\leq 4L_f^2\Psi_t+4L_f^2\Gamma_t+4\left\|\nabla_y f(\mathbf{z}_t)\right\|^2+\frac{9}{mK}\sigma_y^2,$$
$$\mathbb{E}\|\mathbf{z}_{t+1}-\mathbf{z}_t\|^2\leq 4L_f^2K^2\left(\eta_x^2+\eta_y^2\right)\Psi_t+4L_f^2K^2\left(\eta_x^2+\eta_y^2\right)\Gamma_t$$
$$+4K^2\left(\eta_x^2\left\|\nabla_x f(\mathbf{z}_t)\right\|^2+\eta_y^2\left\|\nabla_y f(\mathbf{z}_t)\right\|^2\right)+\frac{9K}{m}\left(\eta_x^2\sigma_x^2+\eta_y^2\sigma_y^2\right).$$

Proof.

$$\mathbb{E}\|\Delta\mathbf{x}_t\|^2\leq\mathbb{E}\left\|\frac{1}{mK}\sum_{i\in S_t}\sum_{j\in[K]}\left[\nabla_x f_i(\mathbf{z}_{t,i}^j)-\mathbb{E}[\mathbf{v}_x^i]+\mathbb{E}[\bar{\mathbf{v}}_{x,t}]\right]\right\|^2+\frac{9}{mK}\sigma_x^2$$
$$\leq\frac{4}{MK}\sum_{i\in[M]}\sum_{j\in[K]}\Big[\mathbb{E}\left\|\nabla_x f_i(\mathbf{z}_{t,i}^j)-\nabla_x f_i(\mathbf{z}_t)\right\|^2+\mathbb{E}\left\|\mathbb{E}[\mathbf{v}_x^i]-\nabla_x f_i(\mathbf{z}_t)\right\|^2$$
$$+\mathbb{E}\left\|\mathbb{E}[\bar{\mathbf{v}}_{x,t}]-\nabla_x f(\mathbf{z}_t)\right\|^2+\left\|\nabla_x f(\mathbf{z}_t)\right\|^2\Big]+\frac{9}{mK}\sigma_x^2$$
$$\leq\frac{4}{MK}\sum_{i\in[M]}\sum_{j\in[K]}\left[L_f^2\mathbb{E}\left\|\mathbf{z}_{t,i}^j-\mathbf{z}_t\right\|^2+L_f^2\mathbb{E}\left\|\mathbf{w}_{t,i}-\mathbf{z}_t\right\|^2+\left\|\nabla_x f(\mathbf{z}_t)\right\|^2\right]+\frac{9}{mK}\sigma_x^2$$
$$=4L_f^2\Psi_t+4L_f^2\Gamma_t+4\left\|\nabla_x f(\mathbf{z}_t)\right\|^2+\frac{9}{mK}\sigma_x^2,$$

$\mathbb{E}[\mathbf{v}_{x,t}^i]=\nabla_x f_i(\mathbf{z}_t)$ and $\mathbb{E}[\bar{\mathbf{v}}_{x,t}]=\nabla_x f(\mathbf{z}_t)$ where the second inequality is due to Lemma 4 in [28]). The bound of $\|\left(\mathbf{u}_{y,t}-\mathbf{e}_{y,t}\right)\|^2$ follows from the similar proof. □

Lemma 8 (Bounded Error for SAGDA Option I)**.**

$$\mathbb{E}\|\bar{\mathbf{e}}_{x,t}\|^2\quad\leq L_f^2\Psi_t,$$
$$\mathbb{E}\|\bar{\mathbf{e}}_{y,t}\|^2\quad\leq L_f^2\Psi_t.$$

Proof.

$$\mathbb{E}\|\bar{\mathbf{e}}_{x,t}\|^2=\mathbb{E}\left\|\frac{1}{mK}\sum_{i\in S_t}\sum_{j\in[K]}\left(\nabla_x f_i(\mathbf{z}_t)-\nabla_x f_i(\mathbf{z}_{t,i}^j)\right)\right\|^2$$
$$\leq\frac{1}{mK}\mathbb{E}\left[\sum_{i\in S_t}\sum_{j\in[K]}\left\|\left(\nabla_x f_i(\mathbf{z}_t)-\nabla_x f_i(\mathbf{z}_{t,i}^j)\right)\right\|^2\right]$$

$$\leq \frac{L_f^2}{MK} \sum_{i\in[M],j\in[K]} \mathbb{E}\left\| \left(\mathbf{z}_t - \mathbf{z}_{t,i}^j\right)\right\|^2$$
$$= L_f^2 \Psi_t.$$

$\mathbb{E}\|\bar{\mathbf{e}}_{y,t}\|^2$ has the same bounds. □

Theorem 1 (Convergence Rate of SAGDA)**.** *Under Assumptions 1- 3, define $\mathcal{L}_t = \Phi(\mathbf{x}_t) - \frac{1}{10}f(\mathbf{x}_t, \mathbf{y}_t)$, the output sequence $\{\mathbf{x}_t\}$ generated by SAGDA satisfies:*

- *For Option I with learning rates $\eta_{x,g}$, $\eta_{x,l}$, $\eta_{y,g}$, and $\eta_{y,l}$ satisfying*

$$8K(K-1)(2K-1)L_f^2 \max\{\eta_{x,l}^2, \eta_{y,l}^2\} \leq 1,$$
$$\frac{1}{2} - 4a_2 L_f^2 K^2\left(\eta_x^2 + \eta_y^2\right) - \left(a_1 + a_2 4L_f^2 K^2\left(\eta_x^2 + \eta_y^2\right)\right) 160K^2\left(\eta_{x,l}^2 + \eta_{y,l}^2\right) L_f^2 \geq 0,$$
$$\left[\frac{1}{10}\eta_x K - 4a_2 K^2 \eta_x^2\right] - \left[a_1 + a_2 4L_f^2 K^2\left(\eta_x^2 + \eta_y^2\right)\right] 40K^2 \eta_{x,l}^2 \geq 0,$$
$$\left[\eta_y K\left(\frac{1}{20} - \frac{\eta_x}{\eta_y}\frac{L_f^2}{\mu^2}\right) - 4a_2 K^2 \eta_y^2\right] - \left[a_1 + a_2 4L_f^2 K^2\left(\eta_x^2 + \eta_y^2\right)\right] 40K^2 \eta_{y,l}^2 \geq 0,$$

where $a_1 = KL_f^2\left(\frac{31}{20}\eta_x + \frac{1}{20}\eta_y\right)$ and $a_2 = \frac{1}{2}\left(L + \frac{L_f}{10}\right) + 1 + \frac{M^2}{m^2} - \frac{M}{m}$, it holds that

$$\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}\|\nabla\Phi(\mathbf{x}_t)\|^2 \leq \underbrace{\frac{2\left(\mathcal{L}_0 - \mathcal{L}_*\right)}{\eta_x KT}}_{\text{optimization error}} + \underbrace{\left[\left(L + \frac{L_f}{10}\right) + 4\right]\frac{9}{m\eta_x}\left(\eta_x^2\sigma_x^2 + \eta_y^2\sigma_y^2\right)}_{\text{statistical error}} + \underbrace{\psi_1}_{\text{local update error}}$$

where ψ_1 is defined as follows:

$$\psi_1 = \left[L_f^2\left(\frac{31}{20} + \frac{1}{20}\frac{\eta_y}{\eta_x}\right) + \left[\frac{1}{2}\left(L + \frac{L_f}{10}\right) + 2\right] 4L_f^2 K\left(\eta_x + \frac{\eta_y^2}{\eta_x}\right)\right]\left[20K^2\left(\eta_{x,l}^2\sigma_x^2 + \eta_{y,l}^2\sigma_y^2\right)\right].$$

- *For Option II with learning rates $\eta_{x,g}$, $\eta_{x,l}$, $\eta_{y,g}$, and $\eta_{y,l}$ satisfying*

$$8K(K-1)(2K-1)L_f^2 \max\{\eta_{x,l}^2, \eta_{y,l}^2\} \leq 1,$$
$$\frac{1}{10}\eta_x K - \left(2\left(L + \frac{L_f}{10}\right)\eta_x^2 K^2 + 40K^2\eta_{x,l}^2 b_1\right) \geq 0,$$
$$\eta_y K\left(\frac{1}{20} - \frac{\eta_x}{\eta_y}\frac{L_f^2}{\mu^2}\right) - \left(\frac{1}{5}L_f \eta_y^2 K^2 + 40K^2\eta_{y,l}^2 b_1\right) \geq 0,$$

where $b_1 = L_f^2\left[\frac{31}{20}\eta_x K + \frac{1}{20}\eta_y K + 2\left(L + \frac{L_f}{10}\right)\eta_x^2 K^2 + \frac{1}{5}L_f\eta_y^2 K^2\right]$, it holds that

$$\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}\|\nabla\Phi(\mathbf{x}_t)\|^2 \leq \frac{2\left(\mathcal{L}_0 - \mathcal{L}_*\right)}{\eta_x KT} + \left[\left(L + \frac{L_f}{10}\right)\frac{9\eta_x}{m}\sigma_x^2 + + \frac{9}{10}L_f\frac{\eta_y^2}{m\eta_x}\sigma_y^2\right] + \psi_2.$$

where ψ_2 is defined as follows:

$$\psi_2 = L_f^2\left[\frac{31}{20}K + \frac{1}{20}\frac{\eta_y}{\eta_x}K + 2\left(L + \frac{L_f}{10}\right)\eta_x K^2 + \frac{1}{5}L_f\frac{\eta_y^2}{\eta_x}K^2\right]\left[10\left(16K+1\right)\right]\left(\eta_{x,l}^2\sigma_x^2 + \eta_{y,l}^2\sigma_y^2\right).$$

Proof. Similar to the bound of Φ and f in (2) and (3), we have the following results:

$$\mathbb{E}\Phi(\mathbf{x}_{t+1}) - \Phi(\mathbf{x}_t) \leq -\frac{1}{2}\eta_x K\|\nabla\Phi(\mathbf{x}_t)\|^2 - \frac{1}{4}\eta_x K\|\nabla_x f(\mathbf{z}_t)\|^2 + \frac{3}{2}\eta_x K\mathbb{E}\left\|\bar{\mathbf{e}}_{x,t}\right\|^2$$

$$+\eta_x K\frac{L_f^2}{\mu^2}\left\|\nabla_y f(\mathbf{z}_t)\right\|^2+\frac{1}{2}L\eta_x^2K^2\mathbb{E}\left\|\mathbf{u}_{x,t}-\mathbf{e}_{x,t}\right\|^2.$$

$$\begin{aligned}f(\mathbf{z}_t)-\mathbb{E}f(\mathbf{z}_{t+1})\le&\frac{3}{2}\eta_x K\left\|\nabla_x f(\mathbf{z}_t)\right\|^2+\frac{1}{2}\eta_x K\mathbb{E}\left\|\bar{\mathbf{e}}_{x,t}\right\|^2+\frac{1}{2}\eta_y K\mathbb{E}\left\|\bar{\mathbf{e}}_{y,t}\right\|^2-\frac{1}{2}\eta_y K\|\nabla_y f(\mathbf{z}_t)\|^2\\&+\frac{1}{2}L_f\eta_x^2K^2\left\|\mathbf{u}_{x,t}-\mathbf{e}_{x,t}\right\|^2+\frac{1}{2}L_f\eta_y^2K^2\left\|\mathbf{u}_{y,t}-\mathbf{e}_{y,t}\right\|^2.\end{aligned}$$

Define potential function $\mathcal{L}_t=\Phi(\mathbf{x}_t)-\frac{1}{10}f(\mathbf{z}_t)$,

$$\begin{aligned}&\mathbb{E}\mathcal{L}_{t+1}-\mathcal{L}_t=\mathbb{E}\Phi(\mathbf{x}_{t+1})-\Phi(\mathbf{x}_t)+\frac{1}{10}\left(f(\mathbf{z}_t)-\mathbb{E}f(\mathbf{z}_{t+1})\right)\\
&\le-\frac{1}{2}\eta_x K\|\nabla\Phi(\mathbf{x}_t)\|^2-\frac{1}{10}\eta_x K\left\|\nabla_x f(\mathbf{z}_t)\right\|^2-\eta_y K\left(\frac{1}{20}-\frac{\eta_x}{\eta_y}\frac{L_f^2}{\mu^2}\right)\left\|\nabla_y f(\mathbf{z}_t)\right\|^2\\
&\quad+\frac{31}{20}\eta_x K\left\|\bar{\mathbf{e}}_{x,t}\right\|^2+\frac{1}{20}\eta_y K\left\|\bar{\mathbf{e}}_{y,t}\right\|^2+\frac{1}{2}\left(L+\frac{L_f}{10}\right)\eta_x^2K^2\mathbb{E}\left\|\Delta\mathbf{x}_t\right\|^2+\frac{1}{20}L_f\eta_y^2K^2\mathbb{E}\left\|\Delta\mathbf{y}_t\right\|^2\\
&\le-\frac{1}{2}\eta_x K\|\nabla\Phi(\mathbf{x}_t)\|^2-\frac{1}{10}\eta_x K\left\|\nabla_x f(\mathbf{z}_t)\right\|^2-\eta_y K\left(\frac{1}{20}-\frac{\eta_x}{\eta_y}\frac{L_f^2}{\mu^2}\right)\left\|\nabla_y f(\mathbf{z}_t)\right\|^2\\
&\quad+KL_f^2\left(\frac{31}{20}\eta_x+\frac{1}{20}\eta_y\right)\Psi_t+\frac{1}{2}\left(L+\frac{L_f}{10}\right)\mathbb{E}\left\|\mathbf{z}_{t+1}-\mathbf{z}_t\right\|^2\end{aligned}$$

$$\begin{aligned}&(\mathbb{E}\mathcal{L}_{t+1}+\alpha\Gamma_{t+1})-(\mathcal{L}_t+\alpha\Gamma_t)\\
&\le-\frac{1}{2}\eta_x K\|\nabla\Phi(\mathbf{x}_t)\|^2-\frac{1}{10}\eta_x K\left\|\nabla_x f(\mathbf{z}_t)\right\|^2-\eta_y K\left(\frac{1}{20}-\frac{\eta_x}{\eta_y}\frac{L_f^2}{\mu^2}\right)\left\|\nabla_y f(\mathbf{z}_t)\right\|^2\\
&\quad+KL_f^2\left(\frac{31}{20}\eta_x+\frac{1}{20}\eta_y\right)\Psi_t+\frac{1}{2}\left(L+\frac{L_f}{10}\right)\mathbb{E}\left\|\mathbf{z}_{t+1}-\mathbf{z}_t\right\|^2+\alpha\Gamma_{t+1}-\alpha\Gamma_t\\
&\le-\frac{1}{2}\eta_x K\|\nabla\Phi(\mathbf{x}_t)\|^2-\frac{1}{10}\eta_x K\left\|\nabla_x f(\mathbf{z}_t)\right\|^2-\eta_y K\left(\frac{1}{20}-\frac{\eta_x}{\eta_y}\frac{L_f^2}{\mu^2}\right)\left\|\nabla_y f(\mathbf{z}_t)\right\|^2\\
&\quad+\underbrace{KL_f^2\left(\frac{31}{20}\eta_x+\frac{1}{20}\eta_y\right)}_{a_1}\Psi_t+\underbrace{\left[\frac{1}{2}\left(L+\frac{L_f}{10}\right)+\alpha\left(\frac{m}{M}+\frac{M}{m}-1\right)\right]}_{a_2}\mathbb{E}\left\|\mathbf{z}_{t+1}-\mathbf{z}_t\right\|^2-\alpha\frac{m}{2M}\Gamma_t\\
&\le-\frac{1}{2}\eta_x K\|\nabla\Phi(\mathbf{x}_t)\|^2-\frac{1}{10}\eta_x K\left\|\nabla_x f(\mathbf{z}_t)\right\|^2-\eta_y K\left(\frac{1}{20}-\frac{\eta_x}{\eta_y}\frac{L_f^2}{\mu^2}\right)\left\|\nabla_y f(\mathbf{z}_t)\right\|^2\\
&\quad+\left[a_1+a_24L_f^2K^2\left(\eta_x^2+\eta_y^2\right)\right]\Psi_t+\left[4a_2L_f^2K^2\left(\eta_x^2+\eta_y^2\right)-\alpha\frac{m}{2M}\right]\Gamma_t\\
&\quad+a_2\left[4K^2\left(\eta_x^2\left\|\nabla_x f(\mathbf{z}_t)\right\|^2+\eta_y^2\left\|\nabla_y f(\mathbf{z}_t)\right\|^2\right)+\frac{9K}{m}\left(\eta_x^2\sigma_x^2+\eta_y^2\sigma_y^2\right)\right]\\
&\le-\frac{1}{2}\eta_x K\|\nabla\Phi(\mathbf{x}_t)\|^2-\left[\frac{1}{10}\eta_x K-4a_2K^2\eta_x^2\right]\left\|\nabla_x f(\mathbf{z}_t)\right\|^2\\
&\quad-\left[\eta_y K\left(\frac{1}{20}-\frac{\eta_x}{\eta_y}\frac{L_f^2}{\mu^2}\right)-4a_2K^2\eta_y^2\right]\left\|\nabla_y f(\mathbf{z}_t)\right\|^2+\left[a_1+a_24L_f^2K^2\left(\eta_x^2+\eta_y^2\right)\right]\times\\
&\quad\left[10K^2\left(\eta_{x,l}^2\sigma_x^2+\eta_{y,l}^2\sigma_y^2\right)+40K^2\left(\eta_{x,l}^2\mathbb{E}\left\|\nabla_x f(\mathbf{z}_t)\right\|^2+\eta_{y,l}^2\mathbb{E}\left\|\nabla_y f(\mathbf{z}_t)\right\|^2\right)\right]\\
&\quad-\left[\alpha\frac{m}{2M}-4a_2L_f^2K^2\left(\eta_x^2+\eta_y^2\right)-\left(a_1+a_24L_f^2K^2\left(\eta_x^2+\eta_y^2\right)\right)160K^2\left(\eta_{x,l}^2+\eta_{y,l}^2\right)L_f^2\right]\Gamma_t\\
&\quad+a_2\frac{9K}{m}\left(\eta_x^2\sigma_x^2+\eta_y^2\sigma_y^2\right),\end{aligned}$$

where we can set $\alpha = \frac{M}{m}$ and requires the learning rates η_x, η_y and $\eta_{x,l}, \eta_{y,l}$ satisfy

$$\left[\alpha \frac{m}{2M} - 4a_2 L_f^2 K^2 \left(\eta_x^2 + \eta_y^2\right) - \left(a_1 + a_2 4 L_f^2 K^2 \left(\eta_x^2 + \eta_y^2\right)\right) 160 K^2 \left(\eta_{x,l}^2 + \eta_{y,l}^2\right) L_f^2\right] \geq 0, \tag{11}$$

$$\left[\frac{1}{10}\eta_x K - 4a_2 K^2 \eta_x^2\right] - \left[a_1 + a_2 4 L_f^2 K^2 \left(\eta_x^2 + \eta_y^2\right)\right] 40 K^2 \eta_{x,l}^2 \geq 0, \tag{12}$$

$$\left[\eta_y K \left(\frac{1}{20} - \frac{\eta_x}{\eta_y}\frac{L_f^2}{\mu^2}\right) - 4a_2 K^2 \eta_y^2\right] - \left[a_1 + a_2 4 L_f^2 K^2 \left(\eta_x^2 + \eta_y^2\right)\right] 40 K^2 \eta_{y,l}^2 \geq 0. \tag{13}$$

$$\begin{aligned}
&(\mathbb{E}\mathcal{L}_{t+1} + \alpha\Gamma_{t+1}) - (\mathcal{L}_t + \alpha\Gamma_t) \\
&\leq -\frac{1}{2}\eta_x K \|\nabla\Phi(\mathbf{x}_t)\|^2 + \left[a_1 + a_2 4 L_f^2 K^2 \left(\eta_x^2 + \eta_y^2\right)\right]\left[10K^2\left(\eta_{x,l}^2\sigma_x^2 + \eta_{y,l}^2\sigma_y^2\right)\right] + a_2 \frac{9K}{m}\left(\eta_x^2\sigma_x^2 + \eta_y^2\sigma_y^2\right) \\
&\leq -\frac{1}{2}\eta_x K \|\nabla\Phi(\mathbf{x}_t)\|^2 + \left[\frac{1}{2}\left(L + \frac{L_f}{10}\right) + 2\right]\frac{9K}{m}\left(\eta_x^2\sigma_x^2 + \eta_y^2\sigma_y^2\right) \\
&\quad + \left[K L_f^2 \left(\frac{31}{20}\eta_x + \frac{1}{20}\eta_y\right) + \left[\frac{1}{2}\left(L + \frac{L_f}{10}\right) + 2\right] 4 L_f^2 K^2 \left(\eta_x^2 + \eta_y^2\right)\right]\left[10K^2\left(\eta_{x,l}^2\sigma_x^2 + \eta_{y,l}^2\sigma_y^2\right)\right]
\end{aligned}$$

Note that $\Gamma_0 = 0$.

Telescoping and rearranging, we have:

$$\begin{aligned}
\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}\|\nabla\Phi(\mathbf{x}_t)\|^2 &\leq \frac{2\left(\mathcal{L}_0 - \mathcal{L}_*\right)}{\eta_x K T} + \left[\left(L + \frac{L_f}{10}\right) + 4\right]\frac{9}{m\eta_x}\left(\eta_x^2\sigma_x^2 + \eta_y^2\sigma_y^2\right) \\
&\quad + \left[L_f^2\left(\frac{31}{20} + \frac{1}{20}\frac{\eta_y}{\eta_x}\right) + \left[\frac{1}{2}\left(L + \frac{L_f}{10}\right) + 2\right] 4 L_f^2 K \left(\eta_x + \frac{\eta_y^2}{\eta_x}\right)\right]\left[20K^2\left(\eta_{x,l}^2\sigma_x^2 + \eta_{y,l}^2\sigma_y^2\right)\right].
\end{aligned}$$

□

B.3 Proof for SAGDA Option II

For SAGDA Option II, the update rule is:

$$\begin{aligned}
\mathbf{x}_{t+1} &= \mathbf{x}_t - \eta_{x,g}\eta_{x,l}\left[\frac{1}{m}\sum_{i\in S_t}\sum_{j\in[K]}\left(\nabla_x f_i(\mathbf{z}_{t,i}^j, \xi_{t,i}^j) - \nabla_x f_i(\mathbf{z}_t, \xi_{t,i}) + \frac{1}{m}\sum_{i\in S_t}\nabla_x f_i(\mathbf{z}_t, \xi_{t,i})\right)\right], \\
\mathbf{y}_{t+1} &= \mathbf{y}_t + \eta_{y,g}\eta_{y,l}\left[\frac{1}{m}\sum_{i\in S_t}\sum_{j\in[K]}\left(\nabla_y f_i(\mathbf{z}_{t,i}^j, \xi_{t,i}^j) - \nabla_y f_i(\mathbf{z}_t, \xi_{t,i}) + \frac{1}{m}\sum_{i\in S_t}\nabla_y f_i(\mathbf{z}_t, \xi_{t,i})\right)\right], \\
\mathbf{e}_{x,t} &= \frac{1}{mK}\sum_{i\in S_t}\sum_{j\in[K]}\left[\nabla_x f_i(\mathbf{z}_t) - \left(\nabla_x f_i(\mathbf{z}_{t,i}^j, \xi_{t,i}^j) - \nabla_x f_i(\mathbf{z}_t, \xi_{t,i}) + \frac{1}{m}\sum_{i\in S_t}\nabla_x f_i(\mathbf{z}_t, \xi_{t,i})\right)\right] \\
&= \frac{1}{mK}\sum_{i\in S_t}\sum_{j\in[K]}\left[\nabla_x f_i(\mathbf{z}_t) - \nabla_x f_i(\mathbf{z}_{t,i}^j, \xi_{t,i}^j)\right], \\
\bar{\mathbf{e}}_{x,t} &= \mathbb{E}[\mathbf{e}_{x,t}] = \frac{1}{mK}\sum_{i\in S_t}\sum_{j\in[K]}\left(\nabla_x f_i(\mathbf{z}_t) - \nabla_x f_i(\mathbf{z}_{t,i}^j)\right), \\
\mathbf{e}_{y,t} &= \frac{1}{mK}\sum_{i\in S_t}\sum_{j\in[K]}\left[\nabla_y f_i(\mathbf{z}_t) - \left(\nabla_y f_i(\mathbf{z}_{t,i}^j, \xi_{t,i}^j) - \nabla_y f_i(\mathbf{z}_t, \xi_{t,i}) + \frac{1}{m}\sum_{i\in S_t}\nabla_y f_i(\mathbf{z}_t, \xi_{t,i})\right)\right] \\
&= \frac{1}{mK}\sum_{i\in S_t}\sum_{j\in[K]}\left(\nabla_y f_i(\mathbf{z}_t) - \nabla_y f_i(\mathbf{z}_{t,i}^j, \xi_{t,i}^j)\right),
\end{aligned}$$

$$\bar{\mathbf{e}}_{y,t} = \mathbb{E}[\mathbf{e}_{y,t}] = \frac{1}{mK} \sum_{i \in S_t} \sum_{j \in [K]} \left(\nabla_y f_i(\mathbf{z}_t) - \nabla_y f_i(\mathbf{z}_{t,i}^j) \right).$$

Lemma 9.

$$\mathbb{E}\| \left(\mathbf{u}_{x,t} - \mathbf{e}_{x,t} \right) \|^2 \leq \frac{4}{MK} \sum_{i \in [M]} \sum_{j \in [K]} \left[L_f^2 \mathbb{E} \left\| \mathbf{z}_{t,i}^j - \mathbf{z}_t \right\|^2 + \|\nabla_x f(\mathbf{z}_t)\|^2 \right] + \frac{9}{mK} \sigma_x^2,$$
$$\mathbb{E}\| \left(\mathbf{u}_{y,t} - \mathbf{e}_{y,t} \right) \|^2 \leq \frac{4}{MK} \sum_{i \in [M]} \sum_{j \in [K]} \left[L_f^2 \mathbb{E} \left\| \mathbf{z}_{t,i}^j - \mathbf{z}_t \right\|^2 + \|\nabla_y f(\mathbf{z}_t)\|^2 \right] + \frac{9}{mK} \sigma_y^2.$$

Proof.

$$\mathbb{E}\| \left(\mathbf{u}_{x,t} - \mathbf{e}_{x,t} \right) \|^2 \leq \mathbb{E} \left\| \frac{1}{mK} \sum_{i \in S_t} \sum_{j \in [K]} \left[\nabla_x f_i(\mathbf{z}_{t,i}^j) - \mathbb{E}[\mathbf{v}_{x,t}^i] + \mathbb{E}[\bar{\mathbf{v}}_{x,t}] \right] \right\|^2 + \frac{9}{mK} \sigma_x^2$$
$$\leq \frac{4}{MK} \sum_{i \in [M]} \sum_{j \in [K]} \left[\mathbb{E} \left\| \nabla_x f_i(\mathbf{z}_{t,i}^j) - \nabla_x f_i(\mathbf{z}_t) \right\|^2 + \mathbb{E} \left\| \mathbb{E}[\mathbf{v}_{x,t}^i] - \nabla_x f_i(\mathbf{z}_t) \right\|^2 \right.$$
$$\left. + \mathbb{E} \left\| \mathbb{E}[\bar{\mathbf{v}}_{x,t}] - \nabla_x f(\mathbf{z}_t) \right\|^2 + \|\nabla_x f(\mathbf{z}_t)\|^2 \right] + \frac{9}{mK} \sigma_x^2$$
$$\leq \frac{4}{MK} \sum_{i \in [M]} \sum_{j \in [K]} \left[L_f^2 \mathbb{E} \left\| \mathbf{z}_{t,i}^j - \mathbf{z}_t \right\|^2 + \|\nabla_x f(\mathbf{z}_t)\|^2 \right] + \frac{9}{mK} \sigma_x^2,$$

where the last inequality is due to $\mathbb{E}[\mathbf{v}_{x,t}^i] = \nabla_x f_i(\mathbf{z}_t)$ and $\mathbb{E}[\bar{\mathbf{v}}_{x,t}] = \nabla_x f(\mathbf{z}_t)$, and the second inequality is due to Lemma 4 in [28]).

The bound of $\| \left(\mathbf{u}_{y,t} - \mathbf{e}_{y,t} \right) \|^2$ follows from the similar proof. □

Lemma 10 (Bounded Error for SAGDA Option II)**.**

$$\mathbb{E}\|\bar{\mathbf{e}}_{x,t}\|^2 \leq \tfrac{L_f^2}{MK} \textstyle\sum_{i \in [M], j \in [K]} \mathbb{E} \left\| \left(\mathbf{z}_t - \mathbf{z}_{t,i}^j \right) \right\|^2,$$
$$\mathbb{E}\|\bar{\mathbf{e}}_{y,t}\|^2 \leq \tfrac{L_f^2}{MK} \textstyle\sum_{i \in [M], j \in [K]} \mathbb{E} \left\| \left(\mathbf{z}_t - \mathbf{z}_{t,i}^j \right) \right\|^2.$$

Proof.

$$\mathbb{E}\|\bar{\mathbf{e}}_{x,t}\|^2 = \mathbb{E} \left\| \frac{1}{mK} \sum_{i \in S_t} \sum_{j \in [K]} \left(\nabla_x f_i(\mathbf{z}_t) - \nabla_x f_i(\mathbf{z}_{t,i}^j) \right) \right\|^2$$
$$\leq \frac{1}{mK} \mathbb{E} \left[\sum_{i \in S_t} \sum_{j \in [K]} \left\| \left(\nabla_x f_i(\mathbf{z}_t) - \nabla_x f_i(\mathbf{z}_{t,i}^j) \right) \right\|^2 \right]$$
$$\leq \frac{L_f^2}{MK} \sum_{i \in [M], j \in [K]} \mathbb{E} \left\| \left(\mathbf{z}_t - \mathbf{z}_{t,i}^j \right) \right\|^2.$$

$\mathbb{E}\|\bar{\mathbf{e}}_{y,t}\|^2$ has the same bounds. □

Lemma 11 (Local Step Distance for SAGDA Option II)**.** *$\forall i \in [M], j \in [K]$, we can bound the local step distance as follows:*

$$\mathbb{E} \left\| \left(\mathbf{z}_t - \mathbf{z}_{t,i}^j \right) \right\|^2$$
$$\leq 5K \left(16K + 1 \right) \eta_{x,l}^2 \sigma_x^2 + 5K \left(16K + 1 \right) \eta_{y,l}^2 \sigma_y^2 + 40K^2 \left(\eta_{x,l}^2 \mathbb{E} \left\| \nabla_x f(\mathbf{z}_t) \right\|^2 + \eta_{y,l}^2 \mathbb{E} \left\| \nabla_y f(\mathbf{z}_t) \right\|^2 \right).$$

Proof.

$$
\begin{aligned}
&\mathbb{E}\left\|\left(\mathbf{z}_t-\mathbf{z}_{t,i}^{j+1}\right)\right\|^2=\mathbb{E}\left[\left\|\mathbf{x}_{t,i}^j-\mathbf{x}_t-\eta_{x,l}\left(\nabla_x f_i(\mathbf{z}_{t,i}^j,\xi_{t,i}^j)-\mathbf{v}_{x,t}^i+\bar{\mathbf{v}}_{x,t}\right)\right\|^2\right]\\
&\quad+\mathbb{E}\left[\left\|\mathbf{y}_{t,i}^j-\mathbf{y}_t+\eta_{y,l}\left(\nabla_y f_i(\mathbf{z}_{t,i}^j,\xi_{t,i}^j)-\mathbf{v}_{x,t}^i+\bar{\mathbf{v}}_{y,t}\right)\right\|^2\right]\\
&=\mathbb{E}\left[\left\|\mathbf{x}_{t,i}^j-\mathbf{x}_t-\eta_{x,l}\left(\nabla_x f_i(\mathbf{z}_{t,i}^j)-\mathbf{v}_{x,t}^i+\bar{\mathbf{v}}_{x,t}\right)\right\|^2\right]+\eta_{x,l}^2\sigma_x^2\\
&\quad+\mathbb{E}\left[\left\|\mathbf{y}_{t,i}^j-\mathbf{y}_t+\eta_{y,l}\left(\nabla_y f_i(\mathbf{z}_{t,i}^j)-\mathbf{v}_{x,t}^i+\bar{\mathbf{v}}_{y,t}\right)\right\|^2\right]+\eta_{y,l}^2\sigma_y^2\\
&=\left(1+\frac{1}{2K-1}\right)\mathbb{E}\left\|\mathbf{x}_{t,i}^j-\mathbf{x}_t\right\|^2+2K\mathbb{E}\left\|\eta_{x,l}\left(\nabla_x f_i(\mathbf{z}_{t,i}^j)-\mathbf{v}_{x,t}^i+\bar{\mathbf{v}}_{x,t}\right)\right\|^2+\eta_{x,l}^2\sigma_x^2\\
&\quad+\left(1+\frac{1}{2K-1}\right)\mathbb{E}\left\|\mathbf{y}_{t,i}^j-\mathbf{y}_t\right\|^2+2K\mathbb{E}\left\|\eta_{y,l}\left(\nabla_y f_i(\mathbf{z}_{t,i}^j)-\mathbf{v}_{x,t}^i+\bar{\mathbf{v}}_{y,t}\right)\right\|^2+\eta_{y,l}^2\sigma_y^2\\
&=\left(1+\frac{1}{2K-1}\right)\mathbb{E}\left\|\mathbf{z}_{t,i}^j-\mathbf{z}_t\right\|^2+2K\mathbb{E}\left\|\eta_{x,l}\left(\nabla_x f_i(\mathbf{z}_{t,i}^j)-\mathbf{v}_{x,t}^i+\bar{\mathbf{v}}_{x,t}\right)\right\|^2+\eta_{x,l}^2\sigma_x^2\\
&\quad+2K\mathbb{E}\left\|\eta_{y,l}\left(\nabla_y f_i(\mathbf{z}_{t,i}^j)-\mathbf{v}_{x,t}^i+\bar{\mathbf{v}}_{y,t}\right)\right\|^2+\eta_{y,l}^2\sigma_y^2\\
&\le\left(1+\frac{1}{2K-1}\right)\mathbb{E}\left\|\mathbf{z}_{t,i}^j-\mathbf{z}_t\right\|^2+2K\eta_{x,l}^2\left[4L_f^2\mathbb{E}\left\|\mathbf{z}_{t,i}^j-\mathbf{z}_t\right\|^2+8\sigma_x^2+4\mathbb{E}\left\|\nabla_x f(\mathbf{z}_t)\right\|^2\right]\\
&\quad+2K\eta_{y,l}^2\left[4L_f^2\mathbb{E}\left\|\mathbf{z}_{t,i}^j-\mathbf{z}_t\right\|^2+8\sigma_y^2+4\mathbb{E}\left\|\nabla_y f(\mathbf{z}_t)\right\|^2\right]+\eta_{x,l}^2\sigma_x^2+\eta_{y,l}^2\sigma_y^2\\
&\le\left(1+\frac{1}{2K-1}+8K\max\{L_f^2\eta_{x,l}^2,L_f^2\eta_{y,l}^2\}\right)\mathbb{E}\left\|\mathbf{z}_{t,i}^j-\mathbf{z}_t\right\|^2+(16K+1)\,\eta_{x,l}^2\sigma_x^2\\
&\quad+(16K+1)\,\eta_{y,l}^2\sigma_y^2+8K\left(\eta_{x,l}^2\mathbb{E}\left\|\nabla_x f(\mathbf{z}_t)\right\|^2+\eta_{y,l}^2\mathbb{E}\left\|\nabla_y f(\mathbf{z}_t)\right\|^2\right)\\
&\le\left(1+\frac{1}{K-1}\right)\mathbb{E}\left\|\mathbf{z}_{t,i}^j-\mathbf{z}_t\right\|^2+(16K+1)\,\eta_{x,l}^2\sigma_x^2\\
&\quad+(16K+1)\,\eta_{y,l}^2\sigma_y^2+8K\left(\eta_{x,l}^2\mathbb{E}\left\|\nabla_x f(\mathbf{z}_t)\right\|^2+\eta_{y,l}^2\mathbb{E}\left\|\nabla_y f(\mathbf{z}_t)\right\|^2\right)\\
&\le\sum_{\tau=0}^{j-1}\left(1+\frac{1}{K-1}\right)^\tau\left[(16K+1)\,\eta_{x,l}^2\sigma_x^2+(16K+1)\,\eta_{y,l}^2\sigma_y^2\right.\\
&\quad\left.+8K\left(\eta_{x,l}^2\mathbb{E}\left\|\nabla_x f(\mathbf{z}_t)\right\|^2+\eta_{y,l}^2\mathbb{E}\left\|\nabla_y f(\mathbf{z}_t)\right\|^2\right)\right]\\
&\le 5K\,(16K+1)\,\eta_{x,l}^2\sigma_x^2+5K\,(16K+1)\,\eta_{y,l}^2\sigma_y^2+40K^2\left(\eta_{x,l}^2\mathbb{E}\left\|\nabla_x f(\mathbf{z}_t)\right\|^2+\eta_{y,l}^2\mathbb{E}\left\|\nabla_y f(\mathbf{z}_t)\right\|^2\right),
\end{aligned}
$$

$\bar{\mathbf{v}}_{x,t}=\frac{1}{m}\sum_{i\in S_t}\nabla_x f_i(\mathbf{z}_t,\xi_{t,i})$ and $\mathbf{v}_{x,t}^i=\nabla_x f_i(\mathbf{z}_t,\xi_{t,i})$; $\bar{\mathbf{v}}_{y,t}=\frac{1}{m}\sum_{i\in S_t}\nabla_y f_i(\mathbf{z}_t,\xi_{t,i})$ and $\bar{\mathbf{v}}_{y,t}^i=\nabla_y f_i(\mathbf{z}_t,\xi_{t,i})$; where the first inequality is due to bounded variance of stochastic gradient, the second and third inequalities follow from the fact $\|\mathbf{a}+\mathbf{b}\|^2\le\left(1+\frac{1}{\epsilon}\right)\|\mathbf{a}\|^2+(1+\epsilon)\|\mathbf{b}\|^2$, the forth inequality is due to smoothness of f in x and y, fifth inequality holds if

$$
4K\max\{L_f^2\eta_{x,l}^2,L_f^2\eta_{y,l}^2\}\le\frac{1}{2(K-1)(2K-1)},\tag{14}
$$

and the last inequality follows from the $\sum_{\tau=0}^{j-1}\left(1+\frac{1}{K-1}\right)^\tau\le(K-1)\left[\left(1+\frac{1}{K-1}\right)^K-1\right]\le 5K$.

$$
\begin{aligned}
&\mathbb{E}\left\|\left(\nabla_x f_i(\mathbf{z}_{t,i}^j)-\mathbf{v}_{x,t}^i+\bar{\mathbf{v}}_{x,t}\right)\right\|^2\\
&=\mathbb{E}\left\|\left(\nabla_x f_i(\mathbf{z}_{t,i}^j)-\nabla_x f_i(\mathbf{z}_t)\right)+\left(\nabla_x f_i(\mathbf{z}_t)-\mathbf{v}_{x,t}^i\right)+\left(\bar{\mathbf{v}}_{x,t}-\nabla_x f(\mathbf{z}_t)\right)+\nabla_x f(\mathbf{z}_t)\right\|^2
\end{aligned}
$$

$$\le 4\mathbb{E}\left\|\nabla_x f_i(\mathbf{z}_{t,i}^j) - \nabla_x f_i(\mathbf{z}_t)\right\|^2 + 4\mathbb{E}\left\|\nabla_x f_i(\mathbf{z}_t) - \mathbf{v}_{x,t}^i\right\|^2 + 4\mathbb{E}\left\|\bar{\mathbf{v}}_{x,t} - \nabla_x f(\mathbf{z}_t)\right\|^2 + 4\mathbb{E}\left\|\nabla_x f(\mathbf{z}_t)\right\|^2$$
$$\le 4L_f^2\mathbb{E}\left\|\mathbf{z}_{t,i}^j - \mathbf{z}_t\right\|^2 + 8\sigma_x^2 + 4\mathbb{E}\left\|\nabla_x f(\mathbf{z}_t)\right\|^2$$

□

Proof. Similar to the bound of Φ and f in (2) and (3), we have the following results:

$$\mathbb{E}\Phi(\mathbf{x}_{t+1}) - \Phi(\mathbf{x}_t) \le -\frac{1}{2}\eta_x K\|\nabla\Phi(\mathbf{x}_t)\|^2 - \frac{1}{4}\eta_x K\|\nabla_x f(\mathbf{z}_t)\|^2 + \frac{3}{2}\eta_x K\mathbb{E}\left\|\bar{\mathbf{e}}_{x,t}\right\|^2 + \eta_x K\frac{L_f^2}{\mu^2}\left\|\nabla_y f(\mathbf{z}_t)\right\|^2 + \frac{1}{2}L\eta_x^2K^2\mathbb{E}\left\|\mathbf{u}_{x,t} - \mathbf{e}_{x,t}\right\|^2.$$

$$f(\mathbf{z}_t) - \mathbb{E}f(\mathbf{z}_{t+1}) \le \frac{3}{2}\eta_x K\left\|\nabla_x f(\mathbf{z}_t)\right\|^2 + \frac{1}{2}\eta_x K\mathbb{E}\left\|\bar{\mathbf{e}}_{x,t}\right\|^2 + \frac{1}{2}\eta_y K\mathbb{E}\left\|\bar{\mathbf{e}}_{y,t}\right\|^2 - \frac{1}{2}\eta_y K\|\nabla_y f(\mathbf{z}_t)\|^2 + \frac{1}{2}L_f\eta_x^2K^2\left\|\mathbf{u}_{x,t} - \mathbf{e}_{x,t}\right\|^2 + \frac{1}{2}L_f\eta_y^2K^2\left\|\mathbf{u}_{y,t} - \mathbf{e}_{y,t}\right\|^2.$$

Define potential function $\mathcal{L}_t = \Phi(\mathbf{x}_t) - \frac{1}{10}f(\mathbf{z}_t)$,

$$\begin{aligned}
&\mathbb{E}\mathcal{L}_{t+1} - \mathcal{L}_t = \mathbb{E}\Phi(\mathbf{x}_{t+1}) - \Phi(\mathbf{x}_t) + \frac{1}{10}\left(f(\mathbf{z}_t) - \mathbb{E}f(\mathbf{z}_{t+1})\right) \\
&\le -\frac{1}{2}\eta_x K\|\nabla\Phi(\mathbf{x}_t)\|^2 - \frac{1}{10}\eta_x K\left\|\nabla_x f(\mathbf{z}_t)\right\|^2 - \eta_y K\left(\frac{1}{20} - \frac{\eta_x}{\eta_y}\frac{L_f^2}{\mu^2}\right)\left\|\nabla_y f(\mathbf{z}_t)\right\|^2 \\
&\quad + \frac{31}{20}\eta_x K\left\|\bar{\mathbf{e}}_{x,t}\right\|^2 + \frac{1}{20}\eta_y K\left\|\bar{\mathbf{e}}_{y,t}\right\|^2 \\
&\quad + \frac{1}{2}\left(L + \frac{L_f}{10}\right)\eta_x^2K^2\mathbb{E}\left\|\mathbf{u}_{x,t} - \mathbf{e}_{x,t}\right\|^2 + \frac{1}{20}L_f\eta_y^2K^2\mathbb{E}\left\|\mathbf{u}_{y,t} - \mathbf{e}_{y,t}\right\|^2 \\
&\le -\frac{1}{2}\eta_x K\|\nabla\Phi(\mathbf{x}_t)\|^2 - \frac{1}{10}\eta_x K\left\|\nabla_x f(\mathbf{z}_t)\right\|^2 - \eta_y K\left(\frac{1}{20} - \frac{\eta_x}{\eta_y}\frac{L_f^2}{\mu^2}\right)\left\|\nabla_y f(\mathbf{z}_t)\right\|^2 \\
&\quad + \left(\frac{31}{20}\eta_x K + \frac{1}{20}\eta_y K\right)\left[\frac{L_f^2}{MK}\sum_{i\in[M],j\in[K]}\mathbb{E}\left\|\left(\mathbf{z}_t - \mathbf{z}_{t,i}^j\right)\right\|^2\right] \\
&\quad + \frac{1}{2}\left(L + \frac{L_f}{10}\right)\eta_x^2K^2\left[\frac{4}{MK}\sum_{i\in[M]}\sum_{j\in[K]}\left[L_f^2\mathbb{E}\left\|\mathbf{z}_{t,i}^j - \mathbf{z}_t\right\|^2 + \left\|\nabla_x f(\mathbf{z}_t)\right\|^2\right] + \frac{9}{mK}\sigma_x^2\right] \\
&\quad + \frac{1}{20}L_f\eta_y^2K^2\left[\frac{4}{MK}\sum_{i\in[M]}\sum_{j\in[K]}\left[L_f^2\mathbb{E}\left\|\mathbf{z}_{t,i}^j - \mathbf{z}_t\right\|^2 + \left\|\nabla_y f(\mathbf{z}_t)\right\|^2\right] + \frac{9}{mK}\sigma_y^2\right] \\
&\le -\frac{1}{2}\eta_x K\|\nabla\Phi(\mathbf{x}_t)\|^2 - \frac{1}{10}\eta_x K\left\|\nabla_x f(\mathbf{z}_t)\right\|^2 - \eta_y K\left(\frac{1}{20} - \frac{\eta_x}{\eta_y}\frac{L_f^2}{\mu^2}\right)\left\|\nabla_y f(\mathbf{z}_t)\right\|^2 \\
&\quad + \underbrace{L_f^2\left[\frac{31}{20}\eta_x K + \frac{1}{20}\eta_y K + 2\left(L + \frac{L_f}{10}\right)\eta_x^2K^2 + \frac{1}{5}L_f\eta_y^2K^2\right]}_{a_1}\left[\frac{1}{MK}\sum_{i\in[M],j\in[K]}\mathbb{E}\left\|\left(\mathbf{z}_t - \mathbf{z}_{t,i}^j\right)\right\|^2\right] \\
&\quad + \underbrace{2\left(L + \frac{L_f}{10}\right)\eta_x^2K^2}_{a_2}\left\|\nabla_x f(\mathbf{z}_t)\right\|^2 + \frac{1}{2}\left(L + \frac{L_f}{10}\right)\eta_x^2K^2\frac{9}{mK}\sigma_x^2 \\
&\quad + \underbrace{\frac{1}{5}L_f\eta_y^2K^2}_{a_3}\left\|\nabla_y f(\mathbf{z}_t)\right\|^2 + \frac{1}{20}L_f\eta_y^2K^2\frac{9}{mK}\sigma_y^2
\end{aligned}$$

$$
\begin{aligned}
&\leq -\frac{1}{2}\eta_x K\|\nabla\Phi(\mathbf{x}_t)\|^2 - \frac{1}{10}\eta_x K\left\|\nabla_x f(\mathbf{z}_t)\right\|^2 - \eta_y K\left(\frac{1}{20} - \frac{\eta_x}{\eta_y}\frac{L_f^2}{\mu^2}\right)\left\|\nabla_y f(\mathbf{z}_t)\right\|^2 \\
&\quad + \left[5K\left(16K+1\right)\eta_{x,l}^2 a_1 + \frac{1}{2}\left(L + \frac{L_f}{10}\right)\eta_x^2\frac{9K}{m}\right]\sigma_x^2 + \left[5K\left(16K+1\right)\eta_{y,l}^2 a_1 + \frac{1}{20}L_f\eta_y^2\frac{9K}{m}\right]\sigma_y^2 \\
&\quad + \left(a_2 + 40K^2\eta_{x,l}^2 a_1\right)\left\|\nabla_x f(\mathbf{z}_t)\right\|^2 + \left(a_3 + 40K^2\eta_{y,l}^2 a_1\right)\left\|\nabla_y f(\mathbf{z}_t)\right\|^2 \\
&\leq -\frac{1}{2}\eta_x K\|\nabla\Phi(\mathbf{x}_t)\|^2 + \left[5K\left(16K+1\right)\eta_{x,l}^2 a_1 + \frac{1}{2}\left(L + \frac{L_f}{10}\right)\eta_x^2\frac{9K}{m}\right]\sigma_x^2 \\
&\quad + \left[5K\left(16K+1\right)\eta_{y,l}^2 a_1 + \frac{1}{20}L_f\eta_y^2\frac{9K}{m}\right]\sigma_y^2
\end{aligned}
$$

where the last inequality follows from the conditions:

$$
\frac{1}{10}\eta_x K - \left(a_2 + 40K^2\eta_{x,l}^2 a_1\right) \geq 0, \tag{15}
$$

$$
\eta_y K\left(\frac{1}{20} - \frac{\eta_x}{\eta_y}\frac{L_f^2}{\mu^2}\right) - \left(a_3 + 40K^2\eta_{y,l}^2 a_1\right) \geq 0. \tag{16}
$$

Telescoping and rearranging, we have:

$$
\begin{aligned}
&\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}\|\nabla\Phi(\mathbf{x}_t)\|^2 \leq \frac{2\left(\mathcal{L}_0 - \mathcal{L}_*\right)}{\eta_x KT} + \left[10\left(16K+1\right)\eta_{x,l}^2\frac{a_1}{\eta_x} + \left(L + \frac{L_f}{10}\right)\frac{9\eta_x}{m}\right]\sigma_x^2 \\
&\quad + \left[10\left(16K+1\right)\eta_{y,l}^2\frac{a_1}{\eta_x} + \frac{9}{10}L_f\frac{\eta_y^2}{m\eta_x}\right]\sigma_y^2 \\
&\leq \frac{2\left(\mathcal{L}_0 - \mathcal{L}_*\right)}{\eta_x KT} + \left[\left(L + \frac{L_f}{10}\right)\frac{9\eta_x}{m}\sigma_x^2 + + \frac{9}{10}L_f\frac{\eta_y^2}{m\eta_x}\sigma_y^2\right] \\
&\quad + L_f^2\left[\frac{31}{20}K + \frac{1}{20}\frac{\eta_y}{\eta_x}K + 2\left(L + \frac{L_f}{10}\right)\eta_x K^2 + \frac{1}{5}L_f\frac{\eta_y^2}{\eta_x}K^2\right]\left[10\left(16K+1\right)\right]\left(\eta_{x,l}^2\sigma_x^2 + \eta_{y,l}^2\sigma_y^2\right).
\end{aligned}
$$

□