A Detailed Proofs

Proposition 1. $F_{\mathbb{A}}(\mathcal{E}) = F_{\mathbb{C}}(\mathcal{E})$.

Proof. This result is a consequence of minimal-hitting set duality between AXp's and CXp's, proved elsewhere [36]. □

Proposition 2. If a classifier and instance exhibits issue I5, then they also exhibit issue I2.

Proof. Given a classifier with classification function κ and an instance $(\mathbf{v}, c)$, it is plain that the set of features $\mathcal{F}$ represents a WAXp. Furthermore, since the classification function is assumed not to be constant, then there must exist some AXp that is not the empty set. Thus, such AXp contains at least one relevant feature, say $i_{\text{rel}} \in \mathcal{F}$. Moreover, if I5 holds, then there exists an irrelevant $i_{\text{irr}} \in \mathcal{F} \setminus \{i_{\text{rel}}\}$ with the largest absolute Shapley value. Therefore, it is the case that for feature i_{rel}, its absolute Shapley value is smaller than that of irrelevant feature i_{irr}. As a result, the function also exhibits issue I2. □

Proposition 3. For any $n \geq 3$, there exist boolean functions defined on n variables, and at least one instance, which exhibit an issue I1, i.e. there exists an irrelevant feature $i \in \mathcal{F}$, such that $\mathsf{Sv}(i) \neq 0$.

Proof. Consider two classifiers $\mathcal{M}_1$ and $\mathcal{M}_2$ implementing non-constant boolean functions κ_1 and κ_2, respectively. These functions are defined on the set of features $\mathcal{F}' = \{1, \ldots, m\}$, and such that $\kappa_1 \models \kappa_2$ but $\kappa_1 \neq \kappa_2$. Consider the set of features $\mathcal{F} = \mathcal{F}' \cup \{n\}$, we construct a new classifier $\mathcal{M}$ by combining $\mathcal{M}_1$ and $\mathcal{M}_2$. The classifier $\mathcal{M}$ is characterized by the boolean function defined as follows:

$$\kappa(x_1, \ldots, x_m, x_n) := \begin{cases} \kappa_1(x_1, \ldots, x_m) & \text{if } x_n = 0 \\ \kappa_2(x_1, \ldots, x_m) & \text{if } x_n = 1 \end{cases} \tag{9}$$

Choose a m-dimensional point $\mathbf{v}_{1..m}$ such that $\kappa_1(\mathbf{v}_{1..m}) = \kappa_2(\mathbf{v}_{1..m}) = 0$, and extend $\mathbf{v}_{1..m}$ with $v_n = 1$. Then for the n-dimensional point $\mathbf{v}_{1..n} = (\mathbf{v}_{1..m}, 1)$, we have $\kappa(\mathbf{v}_{1..n}) = 0$.

To simplify the notation, we will use $\mathbf{x}'$ to denote an arbitrary n-dimensional point $\mathbf{x}_{1..n}$. Additionally, we will use $\mathbf{y}$ to denote an arbitrary m-dimensional point $\mathbf{x}_{1..m}$. For any subset $\mathcal{S} \subseteq \mathcal{F}'$, we have:

$$\begin{aligned}
&\phi(\mathcal{S} \cup \{n\}; \mathcal{M}, \mathbf{v}_{1..n}) - \phi(\mathcal{S}; \mathcal{M}, \mathbf{v}_{1..n}) \qquad (10)\\
&= \left(\frac{1}{2^{|(\mathcal{F}' \cup \{n\}) \setminus (\mathcal{S} \cup \{n\})|}} \sum_{\mathbf{x}' \in \Upsilon(\mathcal{S} \cup \{n\}; \mathbf{v}_{1..n})} \kappa(\mathbf{x}') \right) - \left(\frac{1}{2^{|(\mathcal{F}' \cup \{n\}) \setminus \mathcal{S}|}} \sum_{\mathbf{x}' \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..n})} \kappa(\mathbf{x}') \right) \\
&= \left(\frac{1}{2^{|\mathcal{F}' \setminus \mathcal{S}|}} \sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} \kappa_2(\mathbf{y}) \right) - \frac{1}{2^{|\mathcal{F}' \setminus \mathcal{S}| + 1}} \left(\sum_{\mathbf{x}' \in \Upsilon(\mathcal{S}; (\mathbf{v}_{1..m}, 1))} \kappa(\mathbf{x}') + \sum_{\mathbf{x}' \in \Upsilon(\mathcal{S}; (\mathbf{v}_{1..m}, 0))} \kappa(\mathbf{x}') \right) \\
&= \frac{1}{2^{|\mathcal{F}' \setminus \mathcal{S}|}} \left(\sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} \kappa_2(\mathbf{y}) - \frac{1}{2} \times \sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} \kappa_2(\mathbf{y}) - \frac{1}{2} \times \sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} \kappa_1(\mathbf{y}) \right) \\
&= \frac{1}{2} \times \frac{1}{2^{|\mathcal{F}' \setminus \mathcal{S}|}} \left(\sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} \kappa_2(\mathbf{y}) - \sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} \kappa_1(\mathbf{y}) \right)
\end{aligned}$$

Given that $\kappa_1 \models \kappa_2$ but $\kappa_1 \neq \kappa_2$, it follows that for any points $\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})$, if $\kappa_1(\mathbf{y}) = 1$ then $\kappa_2(\mathbf{y}) = 1$. In other words, if $\kappa_2(\mathbf{y}) = 0$ then $\kappa_1(\mathbf{y}) = 0$. Moreover, there are cases where the following inequality holds: $\sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} \kappa_2(\mathbf{y}) - \sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} \kappa_1(\mathbf{y}) > 0$. Hence, $\mathsf{Sv}(n) \neq 0$.

To prove that the feature n is irrelevant, we assume the contrary, i.e., that n is relevant, and $\mathcal{X}$ is an AXp of $\mathcal{M}$ for the point $\mathbf{v}_{1..n}$ such that $n \in \mathcal{X}$. This means we fix the variable x_n to the value v_n, and, based on the definition of AXp, we only select the points that $\mathcal{M}_2$ predicts as 0. Since $\kappa_2(\mathbf{y}) = 0$ implies that $\kappa_1(\mathbf{y}) = 0$, removing feature n from $\mathcal{X}$ means that $\mathcal{X} \setminus n$ will not include any points predicted as 1 by either $\mathcal{M}_1$ or $\mathcal{M}_2$. Thus, $\mathcal{X} \setminus n$ remains an AXp of $\mathcal{M}$ for the point $\mathbf{v}_{1..n}$, leading to a contradiction. Thus, feature n is irrelevant. □

Proposition 4. For any odd $n \geq 3$, there exist boolean functions defined on n variables, and at least one instance, which exhibits an I3 issue, i.e. for which there exists a relevant feature $i \in \mathcal{F}$, such that $\mathsf{Sv}(i) = 0$.

Proof. Given a classifier $\mathcal{M}_1$ implementing a non-constant boolean function κ_1 defined on the set of features $\mathcal{F}_1 = \{1, \ldots, m\}$. We can replace each x_i of κ_1 with a new variable x_{m+i} to obtain a new function κ_2, defined on a new set of features $\mathcal{F}_2 = \{m+1, \ldots, 2m\}$. Importantly, κ_2 is independent of κ_1 as $\mathcal{F}_1$ and $\mathcal{F}_2$ are disjoint. Let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \{n\}$, we build a new classifier $\mathcal{M}$ characterized by the boolean function defined as follows:

$$\kappa(x_1, \ldots, x_m, x_{m+1}, \ldots, x_{2m}, x_n) := \begin{cases} \kappa_1(x_1, \ldots, x_m) & \text{if } x_n = 0 \\ \kappa_2(x_{m+1}, \ldots, x_{2m}) & \text{if } x_n = 1 \end{cases} \tag{11}$$

Choose m-dimensional points $\mathbf{v}_{1..m}$ and $\mathbf{v}_{m+1..2m}$ such that $v_i = v_{m+i}$ for any $1 \leq i \leq m$, and $\kappa_1(\mathbf{v}_{1..m}) = \kappa_2(\mathbf{v}_{m+1..2m}) = 1$. Let $\mathbf{v}_{1..n} = (\mathbf{v}_{1..m}, \mathbf{v}_{m+1..2m}, 1)$ be a n-dimensional point such that $\kappa(\mathbf{v}_{1..n}) = 1$. Moreover, let $\mathcal{F}' = \mathcal{F}_1 \cup \mathcal{F}_2$.

To simplify the notations, we will use $\mathbf{u}$ to denote $\mathbf{v}_{1..m}$ and $\mathbf{w}$ to denote $\mathbf{v}_{m+1..2m}$, furthermore, we will use $\mathbf{x}'$ to denote an arbitrary n-dimensional point $\mathbf{x}_{1..n}$, and $\mathbf{y}$ to denote an arbitrary m-dimensional point $\mathbf{x}_{1..m}$, and $\mathbf{z}$ to denote an arbitrary m-dimensional point $\mathbf{x}_{m+1..2m}$. For any subset $\mathcal{S} \subseteq \mathcal{F}'$, let $\{\mathcal{S}_1, \mathcal{S}_2\}$ be a partition of $\mathcal{S}$ such that $\mathcal{S}_1 \subseteq \mathcal{F}_1 \wedge \mathcal{S}_2 \subseteq \mathcal{F}_2$, then:

$$\begin{aligned}
&\phi(\mathcal{S} \cup \{n\}; \mathcal{M}, \mathbf{v}_{1..n}) - \phi(\mathcal{S}; \mathcal{M}, \mathbf{v}_{1..n}) \qquad (12)\\
&= \left(\frac{1}{2^{|(\mathcal{F}' \cup \{n\}) \setminus (\mathcal{S} \cup \{n\})|}} \sum_{\mathbf{x}' \in \Upsilon(\mathcal{S} \cup \{n\}; \mathbf{v}_{1..n})} \kappa(\mathbf{x}') \right) - \left(\frac{1}{2^{|(\mathcal{F}' \cup \{n\}) \setminus \mathcal{S}|}} \sum_{\mathbf{x}' \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..n})} \kappa(\mathbf{x}') \right) \\
&= \frac{1}{2^{|\mathcal{F}' \setminus \mathcal{S}|}} \left(\sum_{\mathbf{x}' \in \Upsilon(\mathcal{S}; (\mathbf{u}, \mathbf{w}, 1))} \kappa(\mathbf{x}') - \frac{1}{2} \times \sum_{\mathbf{x}' \in \Upsilon(\mathcal{S}; (\mathbf{u}, \mathbf{w}, 1))} \kappa(\mathbf{x}') - \frac{1}{2} \times \sum_{\mathbf{x}' \in \Upsilon(\mathcal{S}; (\mathbf{u}, \mathbf{w}, 0))} \kappa(\mathbf{x}') \right) \\
&= \frac{1}{2} \times \frac{1}{2^{|\mathcal{F}' \setminus \mathcal{S}|}} \left(\sum_{\mathbf{x}' \in \Upsilon(\mathcal{S}; (\mathbf{u}, \mathbf{w}, 1))} \kappa(\mathbf{x}') - \sum_{\mathbf{x}' \in \Upsilon(\mathcal{S}; (\mathbf{u}, \mathbf{w}, 0))} \kappa(\mathbf{x}') \right) \\
&= \frac{1}{2} \times \frac{1}{2^{|\mathcal{F}' \setminus \mathcal{S}|}} \left(2^{|\mathcal{F}_1 \setminus \mathcal{S}_1|} \times \sum_{\mathbf{z} \in \Upsilon(\mathcal{S}_2; \mathbf{w})} \kappa_2(\mathbf{z}) - 2^{|\mathcal{F}_2 \setminus \mathcal{S}_2|} \times \sum_{\mathbf{y} \in \Upsilon(\mathcal{S}_1; \mathbf{u})} \kappa_1(\mathbf{y}) \right)
\end{aligned}$$

For any $\{\mathcal{S}_1, \mathcal{S}_2\}$, we can construct a unique new partition $\{\mathcal{S}_1', \mathcal{S}_2'\}$ by replacing any $i \in \mathcal{S}_1$ with $m+i$ and any $m+i \in \mathcal{S}_2$ with i. Let $\mathcal{S}' = \mathcal{S}_1' \cup \mathcal{S}_2'$, then we have:

$$\begin{aligned}
&\phi(\mathcal{S}' \cup \{n\}; \mathcal{M}, \mathbf{v}_{1..n}) - \phi(\mathcal{S}'; \mathcal{M}, \mathbf{v}_{1..n}) \qquad (13)\\
&= \frac{1}{2} \times \frac{1}{2^{|\mathcal{F}' \setminus \mathcal{S}'|}} \left(2^{|\mathcal{F}_1 \setminus \mathcal{S}_2'|} \times \sum_{\mathbf{z} \in \Upsilon(\mathcal{S}_1'; \mathbf{w})} \kappa_2(\mathbf{z}) - 2^{|\mathcal{F}_2 \setminus \mathcal{S}_1'|} \sum_{\mathbf{y} \in \Upsilon(\mathcal{S}_2'; \mathbf{u})} \kappa_1(\mathbf{y}) \right)
\end{aligned}$$

Besides, we have:

$$2^{|\mathcal{F}_1 \setminus \mathcal{S}_1|} \times \sum_{\mathbf{z} \in \Upsilon(\mathcal{S}_2; \mathbf{z})} \kappa_2(\mathbf{z}) = 2^{|\mathcal{F}_2 \setminus \mathcal{S}_1'|} \sum_{\mathbf{y} \in \Upsilon(\mathcal{S}_2'; \mathbf{u})} \kappa_1(\mathbf{y})$$

and

$$2^{|\mathcal{F}_2 \setminus \mathcal{S}_2|} \times \sum_{\mathbf{y} \in \Upsilon(\mathcal{S}_1; \mathbf{u})} \kappa_1(\mathbf{y}) = 2^{|\mathcal{F}_1 \setminus \mathcal{S}_2'|} \times \sum_{\mathbf{z} \in \Upsilon(\mathcal{S}_1'; \mathbf{w})} \kappa_2(\mathbf{z})$$

which means:

$$\phi(\mathcal{S} \cup \{n\}; \mathcal{M}, \mathbf{v}_{1..n}) - \phi(\mathcal{S}; \mathcal{M}, \mathbf{v}_{1..n}) = -(\phi(\mathcal{S}' \cup \{n\}; \mathcal{M}, \mathbf{v}_{1..n}) - \phi(\mathcal{S}'; \mathcal{M}, \mathbf{v}_{1..n}))$$

note that $\frac{|\mathcal{S}|!(|\mathcal{F}|-|\mathcal{S}|-1)!}{|\mathcal{F}|!} = \frac{|\mathcal{S}'|!(|\mathcal{F}|-|\mathcal{S}'|-1)!}{|\mathcal{F}|!}$. Hence, for any subset $\mathcal{S}$, there is a unique subset $\mathcal{S}'$ that can cancel its effect, from which we can derive that $\mathsf{Sv}(n) = 0$. However, n is a relevant feature. To find an AXp containing n, we remove all features in $\mathcal{F}_1$, and keep only feature n along with all features in $\mathcal{F}_2$. This makes feature n critical to the change in the prediction of $\mathcal{M}$. Next, we compute an AXp $\mathcal{X}$ of $\mathcal{M}_2$ under the point $\mathbf{v}_{m+1..2m}$. Finally, $\mathcal{X} \cup \{n\}$ is an AXp of the classifier $\mathcal{M}$ for the point $\mathbf{v}_{1..n}$. $\square$

Proposition 5. For any even $n \geq 4$, there exist boolean functions defined on n variables, and at least one instance, for which there exists an irrelevant feature $i_1 \in \mathcal{F}$, such that $\mathsf{Sv}(i_1) \neq 0$, and a relevant feature $i_2 \in \mathcal{F} \setminus \{i_1\}$, such that $\mathsf{Sv}(i_2) = 0$.

Proof. Given a classifier $\mathcal{M}_1$ implementing a non-constant boolean function κ_1 defined on the set of features $\mathcal{F}_1 = \{1, \ldots, m\}$, we can construct a new classifier $\mathcal{M}$ characterized by the boolean function defined as follows:

$$\kappa(\mathbf{x}_{1..m}, \mathbf{x}_{m+1..2m}, x_{n-1}, x_n) := \begin{cases} \kappa_1(\mathbf{x}_{1..m}) \wedge \kappa_2(\mathbf{x}_{m+1..2m}) & \text{if } x_{n-1} = 0 \\ \kappa_1(\mathbf{x}_{1..m}) & \text{if } x_{n-1} = 1 \wedge x_n = 0 \\ \kappa_2(\mathbf{x}_{m+1..2m}) & \text{if } x_{n-1} = 1 \wedge x_n = 1 \end{cases} \tag{14}$$

where function κ_2 is obtained by replacing every x_i of κ_1 with a new variable x_{m+i}. κ_2 is defined on a new set of features $\mathcal{F}_2 = \{m+1, \ldots, 2m\}$ and is independent of κ_1. Moreover, $\mathcal{M}$ is defined on the feature set $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \{n-1, n\}$. Note that $\kappa_1 \wedge \kappa_2 \models (\neg x_n \wedge \kappa_1) \vee (x_n \wedge \kappa_2)$, this can be proved using the consensus theorem [5].

Choose m-dimensional points $\mathbf{v}_{1..m}$ and $\mathbf{v}_{m+1..2m}$ such that $v_i = v_{m+i}$ for any $1 \leq i \leq m$, and $\kappa_1(\mathbf{v}_{1..m}) = \kappa_2(\mathbf{v}_{m+1..2m}) = 0$. Let $\mathbf{v}_{1..n} = (\mathbf{v}_{1..m}, \mathbf{v}_{m+1..2m}, 1, 1)$ be a n-dimensional point such that $\kappa(\mathbf{v}_{1..n}) = 0$. Moreover, let $\mathcal{F}' = \mathcal{F}_1 \cup \mathcal{F}_2$.

To simplify the notations, we will use $\mathbf{u}$ to denote $\mathbf{v}_{1..m}$ and $\mathbf{w}$ to denote $\mathbf{v}_{m+1..2m}$, furthermore, we will use $\mathbf{x}'$ to denote an arbitrary n-dimensional point $\mathbf{x}_{1..n}$, and $\mathbf{y}$ to denote an arbitrary m-dimensional point $\mathbf{x}_{1..m}$, and $\mathbf{z}$ to denote an arbitrary m-dimensional point $\mathbf{x}_{m+1..2m}$.

According to the proof of Proposition 3, $\mathsf{Sv}(n-1) \neq 0$ but feature $n-1$ is irrelevant. Next, we show that $\mathsf{Sv}(n) = 0$ but the feature n is relevant. For any subset $\mathcal{S} \subseteq \mathcal{F}'$, let $\{\mathcal{S}_1, \mathcal{S}_2\}$ be a partition of $\mathcal{S}$ such that $\mathcal{S}_1 \subseteq \mathcal{F}_1 \wedge \mathcal{S}_2 \subseteq \mathcal{F}_2$.

1. Consider any subset $\mathcal{S} \cup \{n-1\}$, then:

$$\begin{aligned}
&\phi(\mathcal{S} \cup \{n-1, n\}; \mathcal{M}, \mathbf{v}_{1..n}) - \phi(\mathcal{S} \cup \{n-1\}; \mathcal{M}, \mathbf{v}_{1..n}) \\
&= \left(\frac{1}{2^{|\mathcal{F}' \setminus \mathcal{S}|}} \sum_{\mathbf{x}' \in \Upsilon(\mathcal{S} \cup \{n-1, n\}; \mathbf{v}_{1..n})} \kappa(\mathbf{x}') \right) - \left(\frac{1}{2^{|\mathcal{F}' \setminus \mathcal{S}|+1}} \sum_{\mathbf{x}' \in \Upsilon(\mathcal{S} \cup \{n-1\}; \mathbf{v}_{1..n})} \kappa(\mathbf{x}') \right) \\
&= \frac{1}{2} \times \frac{1}{2^{|\mathcal{F}' \setminus \mathcal{S}|}} \left(\sum_{\mathbf{x}' \in \Upsilon(\mathcal{S} \cup \{n-1, n\}; (\mathbf{u}, \mathbf{w}, 1, 1))} \kappa(\mathbf{x}') - \sum_{\mathbf{x}' \in \Upsilon(\mathcal{S} \cup \{n-1\}; (\mathbf{u}, \mathbf{w}, 1, 0))} \kappa(\mathbf{x}') \right) \\
&= \frac{1}{2} \times \frac{1}{2^{|\mathcal{F}' \setminus \mathcal{S}|}} \left(2^{|\mathcal{F}_1 \setminus \mathcal{S}_1|} \times \sum_{\mathbf{z} \in \Upsilon(\mathcal{S}_2; \mathbf{w})} \kappa_2(\mathbf{z}) - 2^{|\mathcal{F}_2 \setminus \mathcal{S}_2|} \times \sum_{\mathbf{y} \in \Upsilon(\mathcal{S}_1; \mathbf{u})} \kappa_1(\mathbf{y}) \right)
\end{aligned} \tag{15}$$

According to the proof of Proposition 4, there is a unique subset $\mathcal{S}'$ such that $|\mathcal{S}| = |\mathcal{S}'|$ and $\phi(\mathcal{S} \cup \{n-1, n\}; \mathcal{M}, \mathbf{v}_{1..n}) - \phi(\mathcal{S} \cup \{n-1\}; \mathcal{M}, \mathbf{v}_{1..n}) = -(\phi(\mathcal{S}' \cup \{n-1, n\}; \mathcal{M}, \mathbf{v}_{1..n}) - \phi(\mathcal{S}' \cup \{n-1\}; \mathcal{M}, \mathbf{v}_{1..n}))$.

[5] The consensus theorem is the identity $(x \wedge y) \vee (\neg x \wedge z) = (x \wedge y) \vee (\neg x \wedge z) \vee (y \wedge z)$, see [18] Chapter 3

2. Consider any subset $\mathcal{S} \subseteq \mathcal{F}'$, then:

$$
\begin{aligned}
&\phi(\mathcal{S} \cup \{n\}; \mathcal{M}, \mathbf{v}_{1..n}) - \phi(\mathcal{S}; \mathcal{M}, \mathbf{v}_{1..n}) \quad (16)\\
&= \left(\frac{1}{2^{|\mathcal{F}'\setminus\mathcal{S}|+1}} \sum_{\mathbf{x}'\in\Upsilon(\mathcal{S}\cup\{n\};\mathbf{v}_{1..n})} \kappa(\mathbf{x}')\right) - \left(\frac{1}{2^{|\mathcal{F}'\setminus\mathcal{S}|+2}} \sum_{\mathbf{x}'\in\Upsilon(\mathcal{S};\mathbf{v}_{1..n})} \kappa(\mathbf{x}')\right)\\
&= \frac{1}{2^{|\mathcal{F}'\setminus\mathcal{S}|+1}} \left(\sum_{\mathbf{x}'\in\Upsilon(\mathcal{S}\cup\{n\};(\mathbf{u},\mathbf{w},1,1))} \kappa(\mathbf{x}') + \sum_{\mathbf{x}'\in\Upsilon(\mathcal{S}\cup\{n\};(\mathbf{u},\mathbf{w},0,1))} \kappa(\mathbf{x}')\right)\\
&- \frac{1}{2^{|\mathcal{F}'\setminus\mathcal{S}|+2}} \left(\sum_{\mathbf{x}'\in\Upsilon(\mathcal{S};(\mathbf{u},\mathbf{w},1,1))} \kappa(\mathbf{x}') + \sum_{\mathbf{x}'\in\Upsilon(\mathcal{S};(\mathbf{u},\mathbf{w},0,1))} \kappa(\mathbf{x}')\right)\\
&- \frac{1}{2^{|\mathcal{F}'\setminus\mathcal{S}|+2}} \left(\sum_{\mathbf{x}'\in\Upsilon(\mathcal{S};(\mathbf{u},\mathbf{w},1,0))} \kappa(\mathbf{x}') + \sum_{\mathbf{x}'\in\Upsilon(\mathcal{S};(\mathbf{u},\mathbf{w},0,0))} \kappa(\mathbf{x}')\right)\\
&= \frac{1}{4} \times \frac{1}{2^{|\mathcal{F}'\setminus\mathcal{S}|}} \left(2^{|\mathcal{F}_1\setminus\mathcal{S}_1|} \times \sum_{\mathbf{z}\in\Upsilon(\mathcal{S}_2;\mathbf{w})} \kappa_2(\mathbf{z}) - 2^{|\mathcal{F}_2\setminus\mathcal{S}_2|} \times \sum_{\mathbf{y}\in\Upsilon(\mathcal{S}_1;\mathbf{u})} \kappa_1(\mathbf{y})\right)
\end{aligned}
$$

Likewise, we can find a unique subset $\mathcal{S}'$ to cancel the effect of $\phi(\mathcal{S} \cup \{n\}; \mathcal{M}, \mathbf{v}_{1..n}) - \phi(\mathcal{S}; \mathcal{M}, \mathbf{v}1..n)$.

Therefore, $\mathsf{Sv}(n) = 0$. To prove that the feature n is relevant, we compute an AXp containing the feature n. First, we free all features in $\mathcal{F}_1$ and the feature $n-1$, while keeping all features in $\mathcal{F}_2$ and the feature n. This makes feature n critical to the change in the prediction of $\mathcal{M}$. Next, we compute an AXp $\mathcal{X}$ of $\mathcal{M}_2$ under the point $\mathbf{v}_{m+1..2m}$. Finally, we can conclude that $\mathcal{X} \cup \{n\}$ is an AXp of $\mathcal{M}$ under the point $\mathbf{v}_{1..n}$. ☐

Proposition 6. For any $n \geq 4$, there exists boolean functions defined on n variables, and at least one instance, for which there exists an irrelevant feature $i \in \mathcal{F} = \{1, \ldots, n\}$, such that $|\mathsf{Sv}(i)| = \max\{|\mathsf{Sv}(j)| \mid j \in \mathcal{F}\}$.

Proof. Given a classifier $\mathcal{M}_1$ implementing a non-constant boolean function κ_1 defined on the set of variables $\mathcal{F}' = \{1, \ldots, m\}$ where $m \geq 3$, and satisfies the following conditions:

1. κ_1 predicts a specific point $\mathbf{v}_{1..m}$ as 0. Furthermore, for any point $\mathbf{x}_{1..m}$ such that $d_H(\mathbf{x}_{1..m}, \mathbf{v}_{1..m}) = 1$, where $d_H(\cdot)$ denotes the Hamming distance, we have $\kappa_1(\mathbf{x}_{1..m}) = 1$.
2. κ_1 predicts all the other points as 0.

For example, κ_1 can be the function $\sum_{i=1}^{m} \neg x_1 = 1$, which predicts the point $\mathbf{1}_{1..m}$ as 0 and all points around this point with a Hamming distance of 1 as 1. Based on κ_1, we can build a new classifier $\mathcal{M}$ characterized by the boolean function defined as follows:

$$
\kappa(x_1, \ldots, x_m, x_n) := \begin{cases} 0 & \text{if } x_n = 0 \\ \kappa_1(x_1, \ldots, x_m) & \text{if } x_n = 1 \end{cases} \tag{17}
$$

Select the m-dimensional point $\mathbf{v}_{1..m}$ from our Hamming ball such that $\kappa_1(\mathbf{v}_{1..m}) = 0$ (note that only one such point exists), and extend $\mathbf{v}_{1..m}$ with $v_n = 1$. Then for the n-dimensional point $\mathbf{v}_{1..n} = (\mathbf{v}_{1..m}, 1)$, we have $\kappa(\mathbf{v}_{1..n}) = 0$. Applying the same reasoning presented in the proof of Proposition 3, we can deduce that feature n is irrelevant.

For simplicity, we will use $\mathbf{x}'$ to denote an arbitrary n-dimensional point $\mathbf{x}_{1..n}$, and $\mathbf{y}$ to denote an arbitrary m-dimensional point $\mathbf{x}_{1..m}$. More importantly, for κ_1 and any subset $\mathcal{S} \subseteq \mathcal{F}'$, we have:

$$
\sum_{\mathbf{y}\in\Upsilon(\mathcal{S};\mathbf{v}_{1..m})} \kappa_1(\mathbf{y}) = m - |\mathcal{S}|
$$

1. For the feature n and an arbitrary subset $\mathcal{S} \subseteq \mathcal{F}'$, we have:

$$\begin{aligned}
&\phi(\mathcal{S} \cup \{n\}; \mathcal{M}, \mathbf{v}_{1..n}) - \phi(\mathcal{S}; \mathcal{M}, \mathbf{v}_{1..n}) \\
&= \frac{1}{2^{|(\mathcal{F}' \cup \{n\}) \setminus (\mathcal{S} \cup \{n\})|}} \sum_{\mathbf{x}' \in \Upsilon(\mathcal{S} \cup \{n\}; \mathbf{v}_{1..n})} \kappa(\mathbf{x}') - \frac{1}{2^{|(\mathcal{F}' \cup \{n\}) \setminus \mathcal{S}|}} \sum_{\mathbf{x}' \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..n})} \kappa(\mathbf{x}') \\
&= \frac{1}{2^{|\mathcal{F}' \setminus \mathcal{S}|}} \sum_{\mathbf{x}' \in \Upsilon(\mathcal{S} \cup \{n\}; \mathbf{v}_{1..n})} \kappa(\mathbf{x}') - \frac{1}{2^{|\mathcal{F}' \setminus \mathcal{S}|+1}} \sum_{\mathbf{x}' \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..n})} \kappa(\mathbf{x}') \\
&= \frac{1}{2} \times \frac{1}{2^{|\mathcal{F}' \setminus \mathcal{S}|}} \sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} \kappa_1(\mathbf{y}) \\
&= \frac{1}{2} \phi(\mathcal{S}; \mathcal{M}_1, \mathbf{v}_{1..m}) \\
&= \frac{1}{2} \times \frac{m - |\mathcal{S}|}{2^{m-|\mathcal{S}|}}
\end{aligned} \tag{18}$$

This means $\mathsf{Sv}(n) > 0$. Besides, the unique minimal value of $\phi(\mathcal{S} \cup \{n\}; \mathcal{M}, \mathbf{v}_{1..n}) - \phi(\mathcal{S}; \mathcal{M}, \mathbf{v}_{1..n})$ is 0 when $\mathcal{S} = \mathcal{F}'$.

2. For a feature $j \neq n$, consider an arbitrary subset $\mathcal{S} \subseteq \mathcal{F}' \setminus \{j\}$ and the feature n, we have:

$$\begin{aligned}
&\phi(\mathcal{S} \cup \{j, n\}; \mathcal{M}, \mathbf{v}_{1..n}) - \phi(\mathcal{S} \cup \{n\}; \mathcal{M}, \mathbf{v}_{1..n}) \\
&= \frac{1}{2^{|(\mathcal{F}' \cup \{n\}) \setminus (\mathcal{S} \cup \{j,n\})|}} \sum_{\mathbf{x}' \in \Upsilon(\mathcal{S} \cup \{j,n\}; \mathbf{v}_{1..n})} \kappa(\mathbf{x}') - \frac{1}{2^{|(\mathcal{F}' \cup \{n\}) \setminus (\mathcal{S} \cup \{n\})|}} \sum_{\mathbf{x}' \in \Upsilon(\mathcal{S} \cup \{n\}; \mathbf{v}_{1..n})} \kappa(\mathbf{x}') \\
&= \frac{1}{2^{|\mathcal{F}' \setminus (\mathcal{S} \cup \{j\})|}} \sum_{\mathbf{y} \in \Upsilon(\mathcal{S} \cup \{j\}; \mathbf{v}_{1..m})} \kappa_1(\mathbf{y}) - \frac{1}{2^{|\mathcal{F}' \setminus \mathcal{S}|}} \sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} \kappa_1(\mathbf{y}) \\
&= \phi(\mathcal{S} \cup \{j\}; \mathcal{M}_1, \mathbf{v}_{1..m}) - \phi(\mathcal{S}; \mathcal{M}_1, \mathbf{v}_{1..m}) \\
&= \frac{m - |\mathcal{S}| - 1}{2^{m-|\mathcal{S}|-1}} - \frac{m - |\mathcal{S}|}{2^{m-|\mathcal{S}|}} \\
&= \frac{m - |\mathcal{S}| - 2}{2^{m-|\mathcal{S}|}}
\end{aligned} \tag{19}$$

In this case, $\phi(\mathcal{S} \cup \{j, n\}; \mathcal{M}, \mathbf{v}_{1..n}) - \phi(\mathcal{S} \cup \{n\}; \mathcal{M}, \mathbf{v}_{1..n}) = -\frac{1}{2}$ if $|\mathcal{S}| = m - 1$, which is its unique minimal value. $\phi(\mathcal{S} \cup \{j, n\}; \mathcal{M}, \mathbf{v}_{1..n}) - \phi(\mathcal{S} \cup \{n\}; \mathcal{M}, \mathbf{v}_{1..n}) = 0$ if $|\mathcal{S}| = m - 2$, and $\phi(\mathcal{S} \cup \{j, n\}; \mathcal{M}, \mathbf{v}_{1..n}) - \phi(\mathcal{S} \cup \{n\}; \mathcal{M}, \mathbf{v}_{1..n}) > 0$ if $|\mathcal{S}| < m - 2$.

3. Moreover, for a feature $j \neq n$, consider an arbitrary subset $\mathcal{S} \subseteq \mathcal{F}' \setminus \{j\}$ and without the feature n, we have:

$$\begin{aligned}
&\phi(\mathcal{S} \cup \{j\}; \mathcal{M}, \mathbf{v}_{1..n}) - \phi(\mathcal{S}; \mathcal{M}, \mathbf{v}_{1..n}) \\
&= \frac{1}{2^{|(\mathcal{F}' \cup \{n\}) \setminus (\mathcal{S} \cup \{j\})|}} \sum_{\mathbf{x}' \in \Upsilon(\mathcal{S} \cup \{j\}; \mathbf{v}_{1..n})} \kappa(\mathbf{x}') - \frac{1}{2^{|(\mathcal{F}' \cup \{n\}) \setminus \mathcal{S}|}} \sum_{\mathbf{x}' \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..n})} \kappa(\mathbf{x}') \\
&= \frac{1}{2^{|\mathcal{F}' \setminus (\mathcal{S} \cup \{j\})|+1}} \sum_{\mathbf{y} \in \Upsilon(\mathcal{S} \cup \{j\}; \mathbf{v}_{1..m})} \kappa_1(\mathbf{y}) - \frac{1}{2^{|\mathcal{F}' \setminus \mathcal{S}|+1}} \sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} \kappa_1(\mathbf{y}) \\
&= \frac{1}{2} (\phi(\mathcal{S} \cup \{j\}; \mathcal{M}_1, \mathbf{v}_{1..m}) - \phi(\mathcal{S}; \mathcal{M}_1, \mathbf{v}_{1..m})) \\
&= \frac{1}{2} \times \frac{m - |\mathcal{S}| - 2}{2^{m-|\mathcal{S}|}}
\end{aligned} \tag{20}$$

In this case, $\phi(\mathcal{S} \cup \{j\}; \mathcal{M}, \mathbf{v}_{1..n}) - \phi(\mathcal{S}; \mathcal{M}, \mathbf{v}_{1..n}) = -\frac{1}{4}$ if $|\mathcal{S}| = m - 1$, which is its unique minimal value. $\phi(\mathcal{S} \cup \{j\}; \mathcal{M}, \mathbf{v}_{1..n}) - \phi(\mathcal{S}; \mathcal{M}, \mathbf{v}_{1..n}) = 0$ if $|\mathcal{S}| = m - 2$, and $\phi(\mathcal{S} \cup \{j\}; \mathcal{M}, \mathbf{v}_{1..n}) - \phi(\mathcal{S}; \mathcal{M}, \mathbf{v}_{1..n}) > 0$ if $|\mathcal{S}| < m - 2$.

Next, we prove $|\mathsf{Sv}(n)| > |\mathsf{Sv}(j)|$ by showing $\mathsf{Sv}(n) + \mathsf{Sv}(j) > 0$ and $\mathsf{Sv}(n) - \mathsf{Sv}(j) > 0$. Note that $\mathsf{Sv}(n) > 0$. Additionally, $\phi(\mathcal{S} \cup \{j, n\}; \mathcal{M}, \mathbf{v}_{1..n}) - \phi(\mathcal{S} \cup \{n\}; \mathcal{M}, \mathbf{v}_{1..n}) < 0$ and $\phi(\mathcal{S} \cup \{j\}; \mathcal{M}, \mathbf{v}_{1..n}) - \phi(\mathcal{S}; \mathcal{M}, \mathbf{v}_{1..n}) < 0$ only when $|\mathcal{S}| = m - 1$.

1. For $\mathsf{Sv}(n)$:

$$
\begin{aligned}
\mathsf{Sv}(n) &= \sum_{\mathcal{S}\subseteq\mathcal{F}\setminus\{n\}} \frac{|\mathcal{S}|!(m-|\mathcal{S}|)!}{(m+1)!} \times \left(\phi(\mathcal{S}\cup\{n\};\mathcal{M},\mathbf{v}_{1..n}) - \phi(\mathcal{S};\mathcal{M},\mathbf{v}_{1..n})\right) \\
&= \sum_{\mathcal{S}\subseteq\mathcal{F}\setminus\{n\}} \frac{|\mathcal{S}|!(m-|\mathcal{S}|)!}{(m+1)!} \times \frac{1}{2}\phi(\mathcal{S};\mathcal{M}_1,\mathbf{v}_{1..m}) \\
&= \frac{1}{2} \times \frac{1}{m+1} \times \sum_{\mathcal{S}\subseteq\mathcal{F}\setminus\{n\}} \frac{|\mathcal{S}|!(m-|\mathcal{S}|)!}{m!}\phi(\mathcal{S};\mathcal{M}_1,\mathbf{v}_{1..m}) \\
&= \frac{1}{2} \times \frac{1}{m+1} \times \sum_{\mathcal{S}\subseteq\mathcal{F}\setminus\{n\}} \frac{|\mathcal{S}|!(m-|\mathcal{S}|)!}{m!} \times \frac{m-|\mathcal{S}|}{2^{m-|\mathcal{S}|}} \\
&= \frac{1}{2} \times \frac{1}{m+1} \times \sum_{0\leq|\mathcal{S}|\leq m} \frac{|\mathcal{S}|!(m-|\mathcal{S}|)!}{m!} \times \frac{m!}{|\mathcal{S}|!(m-|\mathcal{S}|)!} \times \frac{m-|\mathcal{S}|}{2^{m-|\mathcal{S}|}} \\
&= \frac{1}{2} \times \frac{1}{m+1} \times \sum_{0\leq|\mathcal{S}|\leq m} \frac{m-|\mathcal{S}|}{2^{m-|\mathcal{S}|}} = \frac{1}{2} \times \frac{1}{m+1} \times \sum_{k=1}^{m} \frac{k}{2^k} \\
&= \frac{1}{2} \times \frac{1}{m+1} \times \frac{2^{m+1}-m-2}{2^m} = \frac{1}{m+1} \times \frac{2^{m+1}-m-2}{2^{m+1}}
\end{aligned} \tag{21}
$$

2. For a feature $j \neq n$, consider the subset $\mathcal{S} = \mathcal{F}' \setminus \{j\}$ where $|\mathcal{S}| = m-1$ and the feature n:

$$
\begin{aligned}
&\frac{|\mathcal{S}\cup\{n\}|!(m-|\mathcal{S}\cup\{n\}|)!}{(m+1)!} \times \frac{m-|\mathcal{S}|-2}{2^{m-|\mathcal{S}|}} \\
&= \frac{m!(m-m)!}{(m+1)!} \times \frac{m-(m-1)-2}{2^{m-(m-1)}} \\
&= -\frac{1}{2} \times \frac{1}{m+1}
\end{aligned} \tag{22}
$$

3. For a feature $j \neq n$, consider the subset $\mathcal{S} = \mathcal{F}' \setminus \{j\}$ where $|\mathcal{S}| = m-1$ and without the feature n:

$$
\begin{aligned}
&\frac{|\mathcal{S}|!(m-|\mathcal{S}|)!}{(m+1)!} \times \frac{1}{2} \times \frac{m-|\mathcal{S}|-2}{2^{m-|\mathcal{S}|}} \\
&= \frac{1}{2} \times \frac{(m-1)!(m-(m-1))!}{(m+1)!} \times \frac{m-(m-1)-2}{2^{m-(m-1)}} \\
&= -\frac{1}{4} \times \frac{1}{m(m+1)}
\end{aligned} \tag{23}
$$

We consider the sum of these three values:

$$
\begin{aligned}
&\frac{1}{m+1} \times \frac{2^{m+1}-m-2}{2^{m+1}} - \frac{1}{2} \times \frac{1}{m+1} - \frac{1}{4} \times \frac{1}{m(m+1)} \\
&= \frac{1}{m+1} \times \left(\frac{(2^{m+1}-m-2)m}{m2^{m+1}} - \frac{m2^m}{m2^{m+1}} - \frac{2^{m-1}}{m2^{m+1}}\right) \\
&= \frac{1}{m(m+1)2^{m+1}} \times \left((m-\frac{1}{2})2^m - m^2 - 2m\right)
\end{aligned} \tag{24}
$$

Since $m \geq 3$, the sum of these three values is always greater than 0. Thus, we can conclude that $\mathsf{Sv}(n) + \mathsf{Sv}(j) > 0$.

To show $\mathsf{Sv}(n) - \mathsf{Sv}(j) > 0$, we focus on all subsets $\mathcal{S} \subseteq \mathcal{F}'$ where $|\mathcal{S}| < m-2$. This is because, as previously stated, $\phi(\mathcal{S}\cup\{j,n\};\mathcal{M},\mathbf{v}_{1..n}) - \phi(\mathcal{S}\cup\{n\};\mathcal{M},\mathbf{v}_{1..n}) \leq 0$ and $\phi(\mathcal{S}\cup\{j\};\mathcal{M},\mathbf{v}_{1..n}) - \phi(\mathcal{S};\mathcal{M},\mathbf{v}_{1..n}) \leq 0$ if $|\mathcal{S}| \geq m-2$.

Moreover, for all subsets $\mathcal{S} \subseteq \mathcal{F}'$ where $|\mathcal{S}| = k$ where $0 < k \leq m-3$, we compute the following three quantities:

$$
Q_1 := \sum_{\mathcal{S}\subseteq\mathcal{F}',|\mathcal{S}|=k} \phi(\mathcal{S}\cup\{n\};\mathcal{M},\mathbf{v}_{1..n}) - \phi(\mathcal{S};\mathcal{M},\mathbf{v}_{1..n})
$$

$$Q_2 := \sum_{\mathcal{S}\subseteq\mathcal{F}'\setminus\{j\},|\mathcal{S}|=k-1} \phi(\mathcal{S}\cup\{j,n\};\mathcal{M},\mathbf{v}_{1..n}) - \phi(\mathcal{S}\cup\{n\};\mathcal{M},\mathbf{v}_{1..n})$$

$$Q_3 := \sum_{\mathcal{S}\subseteq\mathcal{F}'\setminus\{j\},|\mathcal{S}|=k} \phi(\mathcal{S}\cup\{j\};\mathcal{M},\mathbf{v}_{1..n}) - \phi(\mathcal{S};\mathcal{M},\mathbf{v}_{1..n})$$

and show that $Q_1 - Q_2 - Q_3 > 0$. Note that Q_1, Q_2 and Q_3 share the same coefficient $\frac{k!(n-k-1)!}{n!}$.

1. For the feature n, we pick all possible subsets $\mathcal{S} \subseteq \mathcal{F}'$ where $|\mathcal{S}| = k$, which implies $|\mathcal{S}\cup\{n\}| = k+1$, then:
$$Q_1 = \binom{m}{|\mathcal{S}|} \times \frac{1}{2} \times \frac{m-|\mathcal{S}|}{2^{m-|\mathcal{S}|}} = \binom{m}{k} \times \frac{1}{2} \times \frac{m-k}{2^{m-k}}$$
2. For a feature $j \neq n$ and consider the feature n, we pick all possible subsets $\mathcal{S} \subseteq \mathcal{F}'$ where $|\mathcal{S}| = k-1$, which implies $|\mathcal{S}\cup\{j,n\}| = k+1$, then:
$$Q_2 = \binom{m-1}{|\mathcal{S}|} \times \frac{m-|\mathcal{S}|-2}{2^{m-|\mathcal{S}|}} = \binom{m-1}{k-1} \times \frac{m-(k-1)-2}{2^{m-(k-1)}} = \binom{m-1}{k-1} \times \frac{1}{2} \times \frac{m-k-1}{2^{m-k}}$$
3. For a feature $j \neq n$, without considering the feature n, we pick all possible subsets $\mathcal{S} \subseteq \mathcal{F}'$ where $|\mathcal{S}| = k$, which implies $|\mathcal{S}\cup\{j\}| = k+1$, then:
$$Q_3 = \binom{m-1}{|\mathcal{S}|} \times \frac{1}{2} \times \frac{m-|\mathcal{S}|-2}{2^{m-|\mathcal{S}|}} = \binom{m-1}{k} \times \frac{1}{2} \times \frac{m-k-2}{2^{m-k}}$$

Then we compute $Q_1 - Q_2 - Q_3$:

$$\begin{aligned}
&\binom{m}{k} \times \frac{1}{2} \times \frac{m-k}{2^{m-k}} - \binom{m-1}{k-1} \times \frac{1}{2} \times \frac{m-k-1}{2^{m-k}} - \binom{m-1}{k} \times \frac{1}{2} \times \frac{m-k-2}{2^{m-k}} \qquad (25)\\
&= \frac{1}{2} \times \frac{1}{2^{m-k}} \left[\binom{m}{k}(m-k) - \binom{m-1}{k-1}(m-k-1) - \binom{m-1}{k}(m-k-2)\right]\\
&= \frac{1}{2} \times \frac{1}{2^{m-k}} \left[\binom{m}{k}(m-k) - \binom{m-1}{k-1}(m-k) - \binom{m-1}{k}(m-k) + \binom{m-1}{k-1} + 2\binom{m-1}{k}\right]\\
&= \frac{1}{2} \times \frac{1}{2^{m-k}} \left[\binom{m-1}{k-1} + 2\binom{m-1}{k}\right]
\end{aligned}$$

This means that $\mathsf{Sv}(n) - \mathsf{Sv}(j) > 0$. Hence, we can conclude that $|\mathsf{Sv}(n)| > |\mathsf{Sv}(j)|$. □

Proposition 7. For any $n \geq 4$, there exist boolean functions defined on n variables, and at least one instance, for which there exists an irrelevant feature $i_1 \in \mathcal{F}$, and a relevant feature $i_2 \in \mathcal{F}\setminus\{i_1\}$, such that $|\mathsf{Sv}(i_1)| > |\mathsf{Sv}(i_2)|$.

Proof. Consider three classifiers $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ implementing non-constant boolean functions κ_1, κ_2 and κ_3, respectively. Actually it is possible for κ_1 to be the constant function 0. All of them are defined on the set of features $\mathcal{F}' = \{1,\ldots,m\}$ where $m \geq 2$. More importantly, κ_1, κ_2 and κ_3 satisfy the following conditions:

1. κ_2 is a function predicting exactly one point $\mathbf{v}_{1..m}$ to 1, for example, κ_2 can be $\bigwedge_{1\leq i\leq m} \neg x_i$.
2. For the point $\mathbf{v}_{1..m}$ where κ_2 predicts 1, we have $\kappa_3(\mathbf{v}_{1..m}) = 0$. This implies $\kappa_2 \wedge \kappa_3 \models \bot$, that is, the conjunction of κ_2 and κ_3 is logically inconsistent.
3. For any point $\mathbf{x}_{1..m}$ such that $d_H(\mathbf{x}_{1..m},\mathbf{v}_{1..m}) = 1$, where $d_H(\cdot)$ denotes the Hamming distance, we have $\kappa_3(\mathbf{x}_{1..m}) = 1$.
4. $\kappa_1 \wedge \kappa_2 \models \bot$ and $\kappa_1 \wedge \kappa_3 \models \bot$, indicating that the conjunction of κ_1 and κ_2 as well as the conjunction of κ_1 and κ_3 both equal to the constant function 0.
5. $\kappa_1 \vee \kappa_2 \neq 1$ and $\kappa_1 \vee \kappa_3 \neq 1$, indicating that neither the disjunction of κ_1 and κ_2 nor the disjunction of κ_1 and κ_3 equals the constant function 1.

Let $\mathcal{F} = \mathcal{F}' \cup \{n-1, n\}$, we can build a new classifier $\mathcal{M}$ from $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$. $\mathcal{M}$ is characterized by the boolean function defined as follows:

$$\kappa(\mathbf{x}_{1..m}, x_{n-1}, x_n) := \begin{cases} \kappa_1(\mathbf{x}_{1..m}) & \text{if } x_{n-1} = 0 \\ \kappa_1(\mathbf{x}_{1..m}) \vee \kappa_2(\mathbf{x}_{1..m}) & \text{if } x_{n-1} = 1 \wedge x_n = 0 \\ \kappa_1(\mathbf{x}_{1..m}) \vee \kappa_3(\mathbf{x}_{1..m}) & \text{if } x_{n-1} = 1 \wedge x_n = 1 \end{cases} \qquad (26)$$

Besides, we can derive that $(\neg x_n \wedge (\kappa_1 \vee \kappa_2)) \vee (x_n \wedge (\kappa_1 \vee \kappa_3)) = \kappa_1 \vee (\neg x_n \wedge \kappa_2) \vee (x_n \wedge \kappa_3)$. So we have $\kappa_1 \models \kappa_1 \vee (\neg x_n \wedge \kappa_2) \vee (x_n \wedge \kappa_3)$. Choose the m-dimensional point $\mathbf{v}_{1..m}$ such that $\kappa_1(\mathbf{v}_{1..m}) = \kappa_3(\mathbf{v}_{1..m}) = 0$ but $\kappa_2(\mathbf{v}_{1..m}) = 1$. Extend $\mathbf{v}_{1..m}$ with $v_{n-1} = v_n = 1$, let $\mathbf{v}_{1..n} = (\mathbf{v}_{1..m}, 1, 1)$ be the n-dimensional point, it follows that $\kappa(\mathbf{v}_{1..n}) = 0$. Based on the proof of Proposition 3, feature $n-1$ is irrelevant. To prove that feature n is relevant, we assume the contrary, i.e., that n is irrelevant. In this case, we pick the point $\mathbf{v}' = (\mathbf{v}_{1..m}, 1, 0)$ from the feature space where $\kappa_2(\mathbf{v}_{1..m}) = 1$. Clearly, for this point we have $\kappa(\mathbf{v}') = 1$, leading to a contradiction. Thus, feature n is relevant.

In the following, we prove that $|\mathsf{Sv}(n-1)| > |\mathsf{Sv}(n)|$ by showing that $\mathsf{Sv}(n-1) - \mathsf{Sv}(n) > 0$ and $\mathsf{Sv}(n-1) + \mathsf{Sv}(n) > 0$. To simplify the notations, we will use $\mathbf{x}'$ to denote an arbitrary n-dimensional point $\mathbf{x}_{1..n}$, and $\mathbf{y}$ to denote an arbitrary m-dimensional point $\mathbf{x}_{1..m}$. For any subset $\mathcal{S} \subseteq \mathcal{F}'$, we now focus on feature $n-1$.

1. For the feature $n-1$, consider an arbitrary subset $\mathcal{S} \subseteq \mathcal{F}'$ and without the feature n, then:

$$
\begin{aligned}
&\phi(\mathcal{S} \cup \{n-1\}; \mathcal{M}, \mathbf{v}_{1..n}) - \phi(\mathcal{S}; \mathcal{M}, \mathbf{v}_{1..n}) \\
&= \left(\frac{1}{2^{|\mathcal{F}'\setminus\mathcal{S}|+1}} \sum_{\mathbf{x}' \in \Upsilon(\mathcal{S}\cup\{n-1\}; \mathbf{v}_{1..n})} \kappa(\mathbf{x}') \right) - \left(\frac{1}{2^{|\mathcal{F}'\setminus\mathcal{S}|+2}} \sum_{\mathbf{x}' \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..n})} \kappa(\mathbf{x}') \right) \\
&= \frac{1}{2^{|\mathcal{F}'\setminus\mathcal{S}|+1}} \left(\sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} (\kappa_1(\mathbf{y}) \vee \kappa_3(\mathbf{y})) + \sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} (\kappa_1(\mathbf{y}) \vee \kappa_2(\mathbf{y})) \right) \\
&- \frac{1}{2^{|\mathcal{F}'\setminus\mathcal{S}|+2}} \left(\sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} (\kappa_1(\mathbf{y}) \vee \kappa_3(\mathbf{y})) + \sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} (\kappa_1(\mathbf{y}) \vee \kappa_2(\mathbf{y})) \right) \\
&- \frac{1}{2^{|\mathcal{F}'\setminus\mathcal{S}|+2}} \left(\sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} \kappa_1(\mathbf{y}) + \sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} \kappa_1(\mathbf{y}) \right) \\
&= \frac{1}{2^{|\mathcal{F}'\setminus\mathcal{S}|+2}} \left(\sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} (\kappa_1(\mathbf{y}) \vee \kappa_3(\mathbf{y})) + \sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} (\kappa_1(\mathbf{y}) \vee \kappa_2(\mathbf{y})) \right) \\
&- \frac{1}{2^{|\mathcal{F}'\setminus\mathcal{S}|+2}} \left(\sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} \kappa_1(\mathbf{y}) + \sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} \kappa_1(\mathbf{y}) \right) \\
&= \frac{1}{2^{|\mathcal{F}'\setminus\mathcal{S}|+2}} \left(\sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} \kappa_3(\mathbf{y}) + \sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} \kappa_2(\mathbf{y}) \right)
\end{aligned} \tag{27}
$$

2. For the feature $n-1$, consider an arbitrary subset $\mathcal{S} \cup \{n\}$, then:

$$
\begin{aligned}
&\phi(\mathcal{S} \cup \{n-1, n\}; \mathcal{M}, \mathbf{v}_{1..n}) - \phi(\mathcal{S} \cup \{n\}; \mathcal{M}, \mathbf{v}_{1..n}) \\
&= \left(\frac{1}{2^{|\mathcal{F}'\setminus\mathcal{S}|}} \sum_{\mathbf{x}' \in \Upsilon(\mathcal{S}\cup\{n-1,n\}; \mathbf{v}_{1..n})} \kappa(\mathbf{x}') \right) - \left(\frac{1}{2^{|\mathcal{F}'\setminus\mathcal{S}|+1}} \sum_{\mathbf{x}' \in \Upsilon(\mathcal{S}\cup\{n\}; \mathbf{v}_{1..n})} \kappa(\mathbf{x}') \right) \\
&= \frac{1}{2^{|\mathcal{F}'\setminus\mathcal{S}|+1}} \left(\sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} (\kappa_1(\mathbf{y}) \vee \kappa_3(\mathbf{y})) - \sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} \kappa_1(\mathbf{y}) \right) \\
&= \frac{1}{2^{|\mathcal{F}'\setminus\mathcal{S}|+1}} \left(\sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} \kappa_3(\mathbf{y}) \right)
\end{aligned} \tag{28}
$$

Thus, we can conclude that $\mathsf{Sv}(n-1) > 0$. For any subset $\mathcal{S} \subseteq \mathcal{F}'$, we now focus on the feature n.

1. For the feature n, consider an arbitrary subset $\mathcal{S} \subseteq \mathcal{F}'$ and without the feature $n-1$, then:

$$
\begin{aligned}
&\phi(\mathcal{S} \cup \{n\}; \mathcal{M}, \mathbf{v}_{1..n}) - \phi(\mathcal{S}; \mathcal{M}, \mathbf{v}_{1..n}) \\
&= \left(\frac{1}{2^{|\mathcal{F}' \setminus \mathcal{S}|+1}} \sum_{\mathbf{x}' \in \Upsilon(\mathcal{S} \cup \{n\}; \mathbf{v}_{1..n})} \kappa(\mathbf{x}') \right) - \left(\frac{1}{2^{|\mathcal{F}' \setminus \mathcal{S}|+2}} \sum_{\mathbf{x}' \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..n})} \kappa(\mathbf{x}') \right) \\
&= \frac{1}{2^{|\mathcal{F}' \setminus \mathcal{S}|+1}} \left(\sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} (\kappa_1(\mathbf{y}) \vee \kappa_3(\mathbf{y})) + \sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} \kappa_1(\mathbf{y}) \right) \\
&- \frac{1}{2^{|\mathcal{F}' \setminus \mathcal{S}|+2}} \left(\sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} (\kappa_1(\mathbf{y}) \vee \kappa_3(\mathbf{y})) + \sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} \kappa_1(\mathbf{y}) \right) \\
&- \frac{1}{2^{|\mathcal{F}' \setminus \mathcal{S}|+2}} \left(\sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} (\kappa_1(\mathbf{y}) \vee \kappa_2(\mathbf{y})) + \sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} \kappa_1(\mathbf{y}) \right) \\
&= \frac{1}{2^{|\mathcal{F}' \setminus \mathcal{S}|+2}} \left(\sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} (\kappa_1(\mathbf{y}) \vee \kappa_3(\mathbf{y})) - \sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} (\kappa_1(\mathbf{y}) \vee \kappa_2(\mathbf{y})) \right) \\
&= \frac{1}{2^{|\mathcal{F}' \setminus \mathcal{S}|+2}} \left(\sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} \kappa_3(\mathbf{y}) - \sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} \kappa_2(\mathbf{y}) \right)
\end{aligned} \tag{29}
$$

2. For the feature n, consider an arbitrary subset $\mathcal{S} \cup \{n-1\}$, then:

$$
\begin{aligned}
&\phi(\mathcal{S} \cup \{n-1, n\}; \mathcal{M}, \mathbf{v}_{1..n}) - \phi(\mathcal{S} \cup \{n-1\}; \mathcal{M}, \mathbf{v}_{1..n}) \\
&= \left(\frac{1}{2^{|\mathcal{F}' \setminus \mathcal{S}|}} \sum_{\mathbf{x}' \in \Upsilon(\mathcal{S} \cup \{n-1, n\}; \mathbf{v}_{1..n})} \kappa(\mathbf{x}') \right) - \left(\frac{1}{2^{|\mathcal{F}' \setminus \mathcal{S}|+1}} \sum_{\mathbf{x}' \in \Upsilon(\mathcal{S} \cup \{n-1\}; \mathbf{v}_{1..n})} \kappa(\mathbf{x}') \right) \\
&= \frac{1}{2^{|\mathcal{F}' \setminus \mathcal{S}|+1}} \left(\sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} (\kappa_1(\mathbf{y}) \vee \kappa_3(\mathbf{y})) - \sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} (\kappa_1(\mathbf{y}) \vee \kappa_2(\mathbf{y})) \right) \\
&= \frac{1}{2^{|\mathcal{F}' \setminus \mathcal{S}|+1}} \left(\sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} \kappa_3(\mathbf{y}) - \sum_{\mathbf{y} \in \Upsilon(\mathcal{S}; \mathbf{v}_{1..m})} \kappa_2(\mathbf{y}) \right)
\end{aligned} \tag{30}
$$

Note that $\phi(\mathcal{S} \cup \{n\}; \mathcal{M}, \mathbf{v}_{1..n}) - \phi(\mathcal{S}; \mathcal{M}, \mathbf{v}_{1..n}) < 0$ and $\phi(\mathcal{S} \cup \{n-1, n\}; \mathcal{M}, \mathbf{v}_{1..n}) - \phi(\mathcal{S} \cup \{n-1\}; \mathcal{M}, \mathbf{v}_{1..n}) < 0$ if and only if $\mathcal{S} = \mathcal{F}'$. For any other proper subset $\mathcal{S} \subset \mathcal{F}'$, $\phi(\mathcal{S} \cup \{n\}; \mathcal{M}, \mathbf{v}_{1..n}) - \phi(\mathcal{S}; \mathcal{M}, \mathbf{v}_{1..n}) \geq 0$ and $\phi(\mathcal{S} \cup \{n-1, n\}; \mathcal{M}, \mathbf{v}_{1..n}) - \phi(\mathcal{S} \cup \{n-1\}; \mathcal{M}, \mathbf{v}_{1..n}) \geq 0$.

Moreover, for a fixed set $\mathcal{S} \subseteq \mathcal{F}'$, we have $(\phi(\mathcal{S} \cup \{n-1\}; \mathcal{M}, \mathbf{v}_{1..n}) - \phi(\mathcal{S}; \mathcal{M}, \mathbf{v}_{1..n})) > (\phi(\mathcal{S} \cup \{n\}; \mathcal{M}, \mathbf{v}_{1..n}) - \phi(\mathcal{S}; \mathcal{M}, \mathbf{v}_{1..n}))$, and $(\phi(\mathcal{S} \cup \{n-1, n\}; \mathcal{M}, \mathbf{v}_{1..n}) - \phi(\mathcal{S} \cup \{n\}; \mathcal{M}, \mathbf{v}_{1..n})) > (\phi(\mathcal{S} \cup \{n-1, n\}; \mathcal{M}, \mathbf{v}_{1..n}) - \phi(\mathcal{S} \cup \{n-1\}; \mathcal{M}, \mathbf{v}_{1..n}))$. Therefore, $\mathsf{Sv}(n-1) > \mathsf{Sv}(n)$.

In the following, we prove that $\mathsf{Sv}(n-1)+\mathsf{Sv}(n)>0$ by focusing on all subsets $\mathcal{S}\subseteq\mathcal{F}$ where $m-2\leq|\mathcal{S}|\leq m+1$. For the feature $n-1$, we have:

$$\begin{aligned}
&\sum_{\substack{\mathcal{S}\subseteq\mathcal{F}\setminus\{n-1\}\\ m-2\leq|\mathcal{S}|\leq m+1}} \frac{|\mathcal{S}|!(m+2-|\mathcal{S}|-1)!}{(m+2)!}\times\left(\phi(\mathcal{S}\cup\{n-1\};\mathcal{M},\mathbf{v}_{1..n})-\phi(\mathcal{S};\mathcal{M},\mathbf{v}_{1..n})\right) \quad (31)\\
&=\sum_{|\mathcal{S}|=m+1,n\in\mathcal{S}}\frac{(m+1)!(m+1-(m+1))!}{(m+2)!}\times\frac{1}{2^{m-m+1}}\times 0\\
&+\sum_{|\mathcal{S}|=m,n\in\mathcal{S}}\frac{m!(m+1-m)!}{(m+2)!}\times\frac{1}{2^{m-(m-1)+1}}\times 1\\
&+\sum_{|\mathcal{S}|=m,n\notin\mathcal{S}}\frac{m!(m+1-m)!}{(m+2)!}\times\frac{1}{2^{m-m+2}}\times(0+1)\\
&+\sum_{|\mathcal{S}|=m-1,n\in\mathcal{S}}\frac{(m-1)!(m+1-(m-1))!}{(m+2)!}\times\frac{1}{2^{m-(m-2)+1}}\times 2\\
&+\sum_{|\mathcal{S}|=m-1,n\notin\mathcal{S}}\frac{(m-1)!(m+1-(m-1))!}{(m+2)!}\times\frac{1}{2^{m-(m-1)+2}}\times(1+1)\\
&+\sum_{|\mathcal{S}|=m-2,n\notin\mathcal{S}}\frac{(m-2)!(m+1-(m-2))!}{(m+2)!}\times\frac{1}{2^{m-(m-2)+2}}\times(2+1)\\
&=\frac{8m+13}{16(m+2)(m+1)}
\end{aligned}$$

For the feature n, we have:

$$\begin{aligned}
&\sum_{\substack{\mathcal{S}\subseteq\mathcal{F}\setminus\{n\}\\ m-2\leq|\mathcal{S}|\leq m+1}} \frac{|\mathcal{S}|!(m+2-|\mathcal{S}|-1)!}{(m+2)!}\times\left(\phi(\mathcal{S}\cup\{n\};\mathcal{M},\mathbf{v}_{1..n})-\phi(\mathcal{S};\mathcal{M},\mathbf{v}_{1..n})\right) \quad (32)\\
&=\sum_{|\mathcal{S}|=m+1,n-1\in\mathcal{S}}\frac{(m+1)!(m+1-(m+1))!}{(m+2)!}\times\frac{1}{2^{m-m+1}}\times(0-1)\\
&+\sum_{|\mathcal{S}|=m,n-1\in\mathcal{S}}\frac{m!(m+1-m)!}{(m+2)!}\times\frac{1}{2^{m-(m-1)+1}}\times(1-1)\\
&+\sum_{|\mathcal{S}|=m,n-1\notin\mathcal{S}}\frac{m!(m+1-m)!}{(m+2)!}\times\frac{1}{2^{m-m+2}}\times(0-1)\\
&+\sum_{|\mathcal{S}|=m-1,n-1\in\mathcal{S}}\frac{(m-1)!(m+1-(m-1))!}{(m+2)!}\times\frac{1}{2^{m-(m-2)+1}}\times(2-1)\\
&+\sum_{|\mathcal{S}|=m-1,n-1\notin\mathcal{S}}\frac{(m-1)!(m+1-(m-1))!}{(m+2)!}\times\frac{1}{2^{m-(m-1)+2}}\times(1-1)\\
&+\sum_{|\mathcal{S}|=m-2,n-1\notin\mathcal{S}}\frac{(m-2)!(m+1-(m-2))!}{(m+2)!}\times\frac{1}{2^{m-(m-2)+2}}\times(2-1)\\
&=\frac{-6m-11}{16(m+2)(m+1)}
\end{aligned}$$

Their summation is $\frac{m+1}{8(m+2)(m+1)}$, since $m\geq 2$, $\mathsf{Sv}(n-1)+\mathsf{Sv}(n)>0$. Thus, it can be concluded that for the irrelevant feature $n-1$ and the relevant feature n, $|\mathsf{Sv}(n-1)|>|\mathsf{Sv}(n)|$. $\square$

Corollary 1. For any $n\geq 7$, there exist boolean functions defined on n variables, and at least one instance, for which there exists an irrelevant feature $i_1\in\mathcal{F}$, and a relevant feature $i_2\in\mathcal{F}\setminus\{i_1\}$, such that $\mathsf{Sv}(i_1)>\mathsf{Sv}(i_2)>0$.

Proof. We utilize the function constructed in Proposition 7, which is given by:

$$\kappa(\mathbf{x}_{1..m}, x_{n-1}, x_n) := \begin{cases} \kappa_1(\mathbf{x}_{1..m}) & \text{if } x_{n-1} = 0 \\ \kappa_1(\mathbf{x}_{1..m}) \vee \kappa_2(\mathbf{x}_{1..m}) & \text{if } x_{n-1} = 1 \wedge x_n = 0 \\ \kappa_1(\mathbf{x}_{1..m}) \vee \kappa_3(\mathbf{x}_{1..m}) & \text{if } x_{n-1} = 1 \wedge x_n = 1 \end{cases} \tag{33}$$

However, we choose a different function κ_3 that satisfies the following condition: for any point $\mathbf{x}_{1..m}$ such that $d_H(\mathbf{x}_{1..m}, \mathbf{v}_{1..m}) \leq 2$, where $d_H(\cdot)$ represents the Hamming distance, $\kappa_3(\mathbf{x}_{1..m}) = 1$, According to the proof of Proposition 7, it can be derived that $\mathsf{Sv}(n-1) > 0$ and $\mathsf{Sv}(n-1) > \mathsf{Sv}(n)$. In the following, we prove that $\mathsf{Sv}(n) > 0$ by focusing on all subsets $\mathcal{S} \subseteq \mathcal{F} \setminus \{n\}$ where $m - 4 \leq |\mathcal{S}| \leq m + 1$, and show that the sum of their values is greater than 0, which implies that $\mathsf{Sv}(n) > 0$ when considering all possible subsets $\mathcal{S}$.

$$\begin{aligned}
&\sum_{\substack{\mathcal{S}\subseteq\mathcal{F}\setminus\{n\} \\ m-4\leq|\mathcal{S}|\leq m+1}} \frac{|\mathcal{S}|!(m+2-|\mathcal{S}|-1)!}{(m+2)!} \times (\phi(\mathcal{S}\cup\{n\};\mathcal{M},\mathbf{v}_{1..n}) - \phi(\mathcal{S};\mathcal{M},\mathbf{v}_{1..n})) \qquad (34) \\
&= \sum_{|\mathcal{S}|=m+1, n-1\in\mathcal{S}} \frac{(m+1)!(m+1-(m+1))!}{(m+2)!} \times \frac{1}{2^{m-m+1}} \times (0-1) \\
&+ \sum_{|\mathcal{S}|=m, n-1\in\mathcal{S}} \frac{m!(m+1-m)!}{(m+2)!} \times \frac{1}{2^{m-(m-1)+1}} \times (1-1) \\
&+ \sum_{|\mathcal{S}|=m, n-1\notin\mathcal{S}} \frac{m!(m+1-m)!}{(m+2)!} \times \frac{1}{2^{m-m+2}} \times (0-1) \\
&+ \sum_{|\mathcal{S}|=m-1, n-1\in\mathcal{S}} \frac{(m-1)!(m+1-(m-1))!}{(m+2)!} \times \frac{1}{2^{m-(m-2)+1}} \times (\binom{2}{1} + \binom{2}{2} - 1) \\
&+ \sum_{|\mathcal{S}|=m-1, n-1\notin\mathcal{S}} \frac{(m-1)!(m+1-(m-1))!}{(m+2)!} \times \frac{1}{2^{m-(m-1)+2}} \times (1-1) \\
&+ \sum_{|\mathcal{S}|=m-2, n-1\in\mathcal{S}} \frac{(m-2)!(m+1-(m-2))!}{(m+2)!} \times \frac{1}{2^{m-(m-3)+1}} \times (\binom{3}{1} + \binom{3}{2} - 1) \\
&+ \sum_{|\mathcal{S}|=m-2, n-1\notin\mathcal{S}} \frac{(m-2)!(m+1-(m-2))!}{(m+2)!} \times \frac{1}{2^{m-(m-2)+2}} \times (\binom{2}{1} + \binom{2}{2} - 1) \\
&+ \sum_{|\mathcal{S}|=m-3, n-1\in\mathcal{S}} \frac{(m-3)!(m+1-(m-3))!}{(m+2)!} \times \frac{1}{2^{m-(m-4)+1}} \times (\binom{4}{1} + \binom{4}{2} - 1) \\
&+ \sum_{|\mathcal{S}|=m-3, n-1\notin\mathcal{S}} \frac{(m-3)!(m+1-(m-3))!}{(m+2)!} \times \frac{1}{2^{m-(m-3)+2}} \times (\binom{3}{1} + \binom{3}{2} - 1) \\
&= \frac{11m-47}{32(m+2)(m+1)}
\end{aligned}$$

Since $m \geq 5$, for all subsets $\mathcal{S} \subseteq \mathcal{F} \setminus \{n\}$ where $m - 4 \leq |\mathcal{S}| \leq m + 1$, their summation is greater than 0. This implies that $\mathsf{Sv}(n) > 0$. Thus, it can be concluded that for the irrelevant feature $n - 1$ and the relevant feature n, we have $\mathsf{Sv}(n-1) > \mathsf{Sv}(n) > 0$. $\square$

Proposition 8. For Propositions 3 to 5,and Proposition 7 the following are lower bounds on the numbers issues exhibiting the respective issues:

1. For Proposition 3, a lower bound on the number of functions exhibiting I1 is $2^{2^{(n-1)}} - n - 3$.
2. For Proposition 4, a lower bound on the number of functions exhibiting I3 is $2^{2^{(n-1)/2}} - 2$.
3. For Proposition 5, a lower bound on the number of functions exhibiting I4 is $2^{2^{(n-2)/2}} - 2$.
4. For Proposition 7, a lower bound on the number of functions exhibiting I2 is $2^{2^{n-2}-(n-2)-1} - 1$.

Sketch. For Proposition 3, there exist $2^{2^{(n-1)}} - n - 3$ distinct non-constant functions κ_2. For each such function κ_2, κ_1 can be defined by changing the prediction of some points predicted as 1 by κ_2 to 0. It is evident that $\kappa_1 \models \kappa_2$ but $\kappa_1 \neq \kappa_2$.

For Propositions 4 and 5, there exist $2^{2^{(n\,-\,1)/2}} - 2$ distinct non-constant functions κ_1. We can then define κ_2 by renaming each variable x_i of κ_1 with a new variable x_{m+i}.

For Proposition 7, the functions κ_2 and κ_3 are assumed to be fixed, while the flexibility lies in the choice of κ_1 (κ_1 can be 0 but cannot be 1). As κ_2 covers 1 point and κ_3 covers $n-2$ points, the remaining points in the feature space can be used to define the function κ_1. Thus, there are $2^{2^{n-2}-(n-2)-1} - 1$ possible functions for κ_1. □