A Appendix

In this section, we provide the proofs of our theoretical results. In Section A.1, we state the results we use in our analysis. Section A.2 includes the proof of estimation error bound in Lemma 2. In Sections A.3 and A.4, we provide the proofs for Theorem 3 (convex case) and Theorem 5 (non-convex case), respectively.

A.1 Preliminaries:

Theorem 7. *(Theorem 2 in Abbasi-Yadkori et al. (2011)). Let $\{\mathcal{F}_t\}_{t=0}^{\infty}$ be a filtration and $\{w_t\}_{t=1}^{\infty}$ be a real-valued stochastic process. Here, w_t is $\mathcal{F}_t$-measurable and w_t is conditionally R-sub Gaussian for some $R \geq 0$. Let $\{\mathbf{x}_t\}_{t=1}^{\infty}$ be an R^d-valued stochastic process such that $\mathbf{x}_t$ is $\mathcal{F}_{t-1}$-measurable. Let $\mathbf{V}_T \triangleq \sum_{t=1}^{T} \mathbf{x}_t\mathbf{x}_t^\top + \lambda\mathbf{I}$ where $\lambda > 0$. Define $y_t = \mathbf{a}^\top\mathbf{x}_t + w_t$, then $\hat{\mathbf{a}}_T = \mathbf{V}_T^{-1}\sum_{t=1}^{T} y_t\mathbf{x}_t$ is the ℓ_2-regularized least squares estimate of $\mathbf{a}$. Assume $\|\mathbf{a}\| \leq L_A$ and $\|\mathbf{x}_t\| \leq L,\ \forall t$. Then, for any $\delta \in (0,1)$, with probability $(1-\delta)$, the true parameter $\mathbf{a}$ lies in the following set:*

$$\left\{\mathbf{a} \in \mathrm{R}^d : \|\mathbf{a} - \hat{\mathbf{a}}_T\|_{\mathbf{V}_T} \leq R\sqrt{d\log\big(\frac{1+TL^2/\lambda}{\delta}\big)} + \sqrt{\lambda}L_A\right\},$$

for all $T \geq 1$.

Theorem 8. *(Theorem 5.1.1 in Tropp et al. (2015)). Consider a finite sequence $\{\mathbf{X}_t\}$ of independent, random and positive semi-definite matrices of dimension d. Assume that $\lambda_{\max}(\mathbf{X}_t) \leq L,\ \forall t$. Define $\mathbf{Y} \triangleq \sum_t \mathbf{X}_t$ and denote $\lambda_{\min}(\mathbb{E}[\mathbf{Y}])$ as μ. Then, we have*

$$\mathbb{P}(\lambda_{\min}(\mathbf{Y}) \leq \epsilon\mu) \leq d\exp\big(-(1-\epsilon)^2\frac{\mu}{2L}\big), \textit{ for any } \epsilon \in (0,1).$$

Now, let us define the *shrunk* version of the polytope as follows

$$\mathcal{X}_{\text{in}}^s \triangleq \{\mathbf{x} \in \mathrm{R}^d : [\mathbf{A}]_{k,:}\mathbf{x} + \tau_{\text{in}} \leq b_k, \forall k \in [n]\}, \text{ for some } \tau_{\text{in}} > 0. \tag{12}$$

Lemma 9 (Lemma 1 in Fereydounian et al. (2020))**.** *Consider a positive constant τ_{in} such that $\mathcal{X}_{\text{in}}^s$ is non-empty. Then, for any $\mathbf{x} \in \mathcal{X}^s$,*

$$\|\Pi_{\mathcal{X}_{\text{in}}^s}(\mathbf{x}) - \mathbf{x}\| \leq \frac{\sqrt{d}\tau_{\text{in}}}{C(\mathbf{A},\mathbf{b})}, \tag{13}$$

where $C(\mathbf{A},\mathbf{b})$ is a positive constant that depends only on the matrix $\mathbf{A}$ and the vector $\mathbf{b}$.

Theorem 10. *(Theorem 3.7 in Shi et al. (2015)) Let us consider the following notation for EXTRA algorithm*

$$\begin{aligned}
&\mathbf{x}_{i,k} : \textit{The iterate of agent } i \textit{ at time } k \textit{ of the EXTRA algorithm},\\
&\mathbf{X}_k = \begin{bmatrix}\mathbf{x}_{1,k}^\top\\ \vdots \\ \mathbf{x}_{m,k}^\top\end{bmatrix},\\
&\mathbf{x}^* = \textit{argmin}_{\mathbf{x}}\Big\{\sum_{i=1}^{m} f_i(\mathbf{x})\Big\},\\
&\mathbf{X}^* = \begin{bmatrix}\mathbf{x}^{*\top}\\ \vdots \\ \mathbf{x}^{*\top}\end{bmatrix},\\
&\mathbf{f}(\mathbf{X}) = \sum_{i=1}^{m} f_i(\mathbf{x}_i).
\end{aligned}$$

A convex function $h(\cdot)$ is restricted strongly convex w.r.t. a point $\mathbf{y}$ if there exists $\mu > 0$ such that

$$\langle\nabla h(\mathbf{x}) - \nabla h(\mathbf{y}), \mathbf{x} - \mathbf{y}\rangle \geq \mu\|\mathbf{x} - \mathbf{y}\|^2, \forall \mathbf{x}.$$

Suppose that the gradient of $\mathbf{f}(\mathbf{X})$ w.r.t. $\mathbf{X}$ is Lipschitz continuous with a constant L_f and $\mathbf{f}(\mathbf{X}) + \frac{1}{4\alpha}\|\mathbf{X}-\mathbf{X}^*\|_{\tilde{\mathbf{P}}-\mathbf{P}}$ is restricted strongly convex w.r.t. $\mathbf{X}^*$ with a constant μ_g. Then, with a proper step size $\alpha < \frac{2\mu_g\lambda_{min}(\tilde{\mathbf{P}})}{L_f^2}$, there exists $\varsigma > 0$ such that $\|\mathbf{X}_k - \mathbf{X}^*\|_{\tilde{\mathbf{P}}}^2$ converges to 0 at the R-linear rate of $O((1+\varsigma)^{-k})$.

A.2 Safe Distributed Set Estimation

Proof of Lemma 2. Let $\mathbf{V}_{T_0} \triangleq \sum_{i=1}^m \sum_{t=1}^{T_0} \mathbf{x}_{i,t}\mathbf{x}_{i,t}^\top$ and $\mathbf{V} = \mathbf{V}_{T_0} + \lambda\mathbf{I}$. Let $\widehat{\mathbf{A}}$ be the solution of $\text{argmin}_{\mathbf{A}} \sum_{i=1}^m l_i(\mathbf{A})$. Let $[\widehat{\mathbf{A}}]_{k,:}$ and $[\mathbf{A}]_{k,:}$ be the k-th rows of $\widehat{\mathbf{A}}$ and $\mathbf{A}$, respectively. Based on Theorem 7, we have with probability at least $(1-\delta)$,

$$\big\|[\widehat{\mathbf{A}}]_{k,:} - [\mathbf{A}]_{k,:}\big\|_{\mathbf{V}} \le R\sqrt{d\log\big(\frac{1+mT_0L^2/\lambda}{\delta/n}\big)} + \sqrt{\lambda}L_A,\ \forall k \in [n]. \tag{14}$$

Knowing that $\forall i \in [m],\ \forall t \in [T_0]$, $\mathbf{x}_{i,t} = (1-\gamma)\mathbf{x}^s + \gamma\zeta_{i,t}$, we have $\lambda_{\max}(\mathbf{x}_{i,t}\mathbf{x}_{i,t}^\top) \le L^2$ and $\mathbb{E}[\mathbf{x}_{i,t}\mathbf{x}_{i,t}^\top] = (1-\gamma)^2\mathbf{x}^s\mathbf{x}^{s\top} + \gamma^2\sigma_\zeta^2\mathbf{I} \succeq \gamma^2\sigma_\zeta^2\mathbf{I}$. Therefore, we have

$$\lambda_{\min}(\mathbb{E}[\mathbf{V}_{T_0}]) = \lambda_{\min}(\sum_{i=1}^m\sum_{t=1}^{T_0}\mathbb{E}[\mathbf{x}_{i,t}\mathbf{x}_{i,t}^\top]) \ge mT_0\gamma^2\sigma_\zeta^2. \tag{15}$$

Based on (15) and Theorem 8, we have

$$\mathbb{P}\Big(\lambda_{\min}(\mathbf{V}_{T_0}) \le \epsilon\lambda_{\min}(\mathbb{E}[\mathbf{V}_{T_0}])\Big) \le d\exp\big(-(1-\epsilon)^2\frac{mT_0\gamma^2\sigma_\zeta^2}{2L^2}\big). \tag{16}$$

By setting $\epsilon = \frac{1}{2}$ and $T_0 \ge \frac{8L^2}{m\gamma^2\sigma_\zeta^2}\log(\frac{d}{\delta})$, from (16), we have

$$\mathbb{P}\big(\lambda_{\min}(\mathbf{V}) \ge \frac{1}{2}mT_0\gamma^2\sigma_\zeta^2\big) \ge \mathbb{P}\big(\lambda_{\min}(\mathbf{V}_{T_0}) \ge \frac{1}{2}mT_0\gamma^2\sigma_\zeta^2\big) \ge (1-\delta). \tag{17}$$

Combining equations (14) and (17), we have with probability at least $(1-2\delta)$,

$$\big\|[\widehat{\mathbf{A}}]_{k,:} - [\mathbf{A}]_{k,:}\big\| \le \frac{R\sqrt{d\log\big(\frac{1+mT_0L^2/\lambda}{\delta/n}\big)} + \sqrt{\lambda}L_A}{\sqrt{\frac{1}{2}m\gamma^2\sigma_\zeta^2T_0}},\ \forall k \in [n]. \tag{18}$$

Let agent i's local estimate of $\mathbf{A}$ at time $t \in [T_0+1, T_0+T_1]$ returned by the **EXTRA** algorithm (Shi et al., 2015) be denoted by $\widehat{\mathbf{A}}_i^t$. Next, we upper bound the distance between $\widehat{\mathbf{A}} = \text{argmin}_{\mathbf{A}}\sum_{i=1}^m l_i(\mathbf{A})$ and $\widehat{\mathbf{A}}_i^t$ based on Theorem 10 as follows. Based on the definition of $l_i(\mathbf{A})$, considering the vectorized version of $\mathbf{A}$, the Hessian matrix has the following expression

$$\nabla^2 l_i(\mathbf{A}) = \sum_{t=1}^{T_0} 2\begin{bmatrix}\mathbf{x}_{i,t}\mathbf{x}_{i,t}^\top & & & \\ & \mathbf{x}_{i,t}\mathbf{x}_{i,t}^\top & & \\ & & \ddots & \\ & & & \mathbf{x}_{i,t}\mathbf{x}_{i,t}^\top\end{bmatrix} + 2\frac{\lambda}{m}\mathbf{I} \preceq 2(T_0L^2 + \frac{\lambda}{m})\mathbf{I},$$

where the inequality is due to the boundedness of the baseline action and the noise vector. From above, we know $\sum_{i=1}^m l_i(\mathbf{A}_i)$ is Lipschitz smooth with the constant $2(T_0L^2 + \frac{\lambda}{m})$ and strongly convex with the constant $2\frac{\lambda}{m}$, so by selecting a step size $\alpha < \frac{(\lambda/m)\lambda_{min}(\tilde{\mathbf{P}})}{(T_0L^2+\frac{\lambda}{m})^2}$ as suggested by Theorem 10, there exists a $\tau \in (0,1)$ such that

$$\big\|[\widehat{\mathbf{A}}_i^t]_{k,:} - [\widehat{\mathbf{A}}]_{k,:}\big\| \le \nu\tau^{(t-T_0)},\ \forall i \in [m], k \in [n], t \in [T_0+1,\dots,T_0+T_1] \tag{19}$$

where $\nu > 0$ is a constant. Based on (18), (19) and our choice of T_1 ($T_1 = (-\log \tau)^{-1} \log(\nu T^\rho)$), for $k \in [n]$, $t \in [T_0 + 1, \ldots, T_0 + T_1]$ and $i, j \in [m]$, we have

$$\|[\widehat{\mathbf{A}}_i^t]_{k,:} - [\mathbf{A}]_{k,:}\| \leq \|[\widehat{\mathbf{A}}_i^t]_{k,:} - [\widehat{\mathbf{A}}]_{k,:}\| + \|[\widehat{\mathbf{A}}]_{k,:} - [\mathbf{A}]_{k,:}\| \leq \frac{1}{T^\rho} + \frac{R\sqrt{d \log\left(\frac{1+mT_0L^2/\lambda}{\delta/n}\right)} + \sqrt{\lambda} L_A}{\sqrt{\frac{1}{2} m \gamma^2 \sigma_\zeta^2 T_0}}, \tag{20}$$

and

$$\|[\widehat{\mathbf{A}}_i^t]_{k,:} - [\widehat{\mathbf{A}}_j^t]_{k,:}\| \leq \|[\widehat{\mathbf{A}}_i^t]_{k,:} - [\widehat{\mathbf{A}}]_{k,:}\| + \|[\widehat{\mathbf{A}}]_{k,:} - [\widehat{\mathbf{A}}_j^t]_{k,:}\| \leq \frac{2}{T^\rho}. \tag{21}$$

□

Lemma 11. *Define*

$$\mathcal{B}_r \triangleq \frac{1}{T^\rho} + \frac{R\sqrt{d \log\left(\frac{1+mT_0L^2/\lambda}{\delta/n}\right)} + \sqrt{\lambda} L_A}{\sqrt{\frac{1}{2} m \gamma^2 \sigma_\zeta^2 T_0}}.$$

For each agent i, construct $\widehat{\mathcal{X}}_i^s$ based on (8) *with $\mathcal{C}_{i,k}$ following from* (7)*. By running Algorithm 1 with user-specified $T_0 = \Omega(\frac{L^2}{m\gamma^2\sigma_\zeta^2} \log(\frac{d}{\delta}))$ and $T_1 = \Theta(\log T^\rho)$, there exists a mutual shrunk polytope (see the definition in* (12)*) subset $\mathcal{X}_{\text{in}}^s$ ($\tau_{\text{in}} = 2\mathcal{B}_r L$) for $\widehat{\mathcal{X}}_i^s$, $\forall i \in [m]$ with probability at least $(1 - 2\delta)$.*

Proof of Lemma 11. Consider a mutual shrunk polytope subset $\mathcal{X}_{\text{in}}^s$ ($\tau_{\text{in}} = 2\mathcal{B}_r L$). Based on Lemma 2, with probability at least $1 - 2\delta$, we have for any $\mathbf{x} \in \mathcal{X}_{\text{in}}^s$,

$$\begin{aligned}
[\widehat{\mathbf{A}}_i]_{k,:}\mathbf{x} + \mathcal{B}_r\|\mathbf{x}\| &= [\mathbf{A}]_{k,:}\mathbf{x} + ([\widehat{\mathbf{A}}_i]_{k,:} - [\mathbf{A}]_{k,:})\mathbf{x} + \mathcal{B}_r\|\mathbf{x}\| \\
&\leq [\mathbf{A}]_{k,:}\mathbf{x} + \|[\widehat{\mathbf{A}}_i]_{k,:} - [\mathbf{A}]_{k,:}\|\|\mathbf{x}\| + \mathcal{B}_r\|\mathbf{x}\| \\
&\leq [\mathbf{A}]_{k,:}\mathbf{x} + 2\mathcal{B}_r\|\mathbf{x}\| \leq [\mathbf{A}]_{k,:}\mathbf{x} + 2\mathcal{B}_r L \leq b_k, \ \forall k \in [n] \text{ and } \forall i \in [m],
\end{aligned} \tag{22}$$

which implies that $\mathcal{X}_{\text{in}}^s \subset \widehat{\mathcal{X}}_i^s$, $\forall i$. □

Lemma 12. *For each agent i, construct $\widehat{\mathcal{X}}_i^s$ based on* (8) *with $\mathcal{C}_{i,k}$ following from* (7)*. By running Algorithm 1 with user-specified $T_0 = \Omega(\frac{L^2}{m\gamma^2\sigma_\zeta^2} \log(\frac{d}{\delta}))$ and $T_1 = \Theta(\log T^\rho)$, we have for any point $\mathbf{x}$,*

$$\|\Pi_{\widehat{\mathcal{X}}_i^s}(\mathbf{x}) - \Pi_{\widehat{\mathcal{X}}_j^s}(\mathbf{x})\| \leq O(\frac{1}{T^\rho}), \ \forall i, j \in [m]. \tag{23}$$

Before we discuss the proof of Lemma 12, for the sake of completeness, we provide the formal statement of Theorem 3.1 in Bonnans et al. (1998), used in the derivation of Lemma 12.

We first define the notations used in (Bonnans et al., 1998). Note that the notations here are only locally defined for the statement of Theorem 3.1 in Bonnans et al. (1998). The work of Bonnans et al. (1998) focuses on the sensitivity analysis of parametric optimization problems of the form

$$(P_\mathbf{u}) : \min_{\mathbf{x} \in \mathcal{X}} f(\mathbf{x}, \mathbf{u}) \text{ subject to } G(\mathbf{x}, \mathbf{u}) \in \mathcal{K},$$

where $\mathcal{X}$ is a finite dimensional space, $\mathcal{U}$ is a Banach space, $\mathcal{K}$ is a closed subset of Banach space $\mathcal{Y}$ and f and G are twice continuously differentiable mappings from $\mathcal{X} \times \mathcal{U}$ to R and $\mathcal{Y}$, respectively. The optimization problem is considered to be unperturbed when $\mathbf{u} = 0$.

Given $\mathbf{u}$, the feasible set, optimal value and set of optimal solutions of $(P_\mathbf{u})$ are denoted as follows

$$\begin{aligned}
\Phi(\mathbf{u}) &\triangleq \{\mathbf{x} \in \mathcal{X} : G(\mathbf{x}, \mathbf{u}) \in \mathcal{K}\}, \\
v(\mathbf{u}) &\triangleq \inf\{f(\mathbf{x}, \mathbf{u}) : \mathbf{x} \in \Phi(\mathbf{u})\}, \\
\mathcal{S}(\mathbf{u}) &\triangleq \text{argmin}\{f(\mathbf{x}, \mathbf{u}) : \mathbf{x} \in \Phi(\mathbf{u})\}.
\end{aligned}$$

A point $\mathbf{x} \in \mathcal{X}$ is called an ϵ-optimal solution of $(P_{\mathbf{u}})$ if $\mathbf{x} \in \Phi(\mathbf{u})$ and $f(\mathbf{x}, \mathbf{u}) \leq v(\mathbf{u}) + \epsilon$.

We also define the following notations to present the theorem statement.

$\mathcal{Y}^*$	Dual space of $\mathcal{Y}$
$\text{dist}(\mathbf{y}, \mathcal{X})$	The minimum distance from point $\mathbf{y}$ to set $\mathcal{X}$: $\inf\{\lVert\mathbf{y} - \mathbf{x}\rVert : \mathbf{x} \in \mathcal{X}\}$
$T_{\mathcal{K}}(\mathbf{y})$	The tangent cone to the set $\mathcal{K}$ at the point $\mathbf{y} \in \mathcal{K}$: $\{\mathbf{h} \in \mathcal{Y} : \text{dist}(\mathbf{y} + t\mathbf{h}, \mathcal{K}) = o(t)\}$
$N_{\mathcal{K}}(\mathbf{y})$	The normal cone to the set $\mathcal{K}$ at the point $\mathbf{y} \in \mathcal{K}$: $\{\mathbf{y}^* \in \mathcal{Y}^* : \langle \mathbf{y}^*, \mathbf{h}\rangle \leq 0, \forall \mathbf{h} \in T_{\mathcal{K}}(\mathbf{y})\}$
$Df(\mathbf{x}, \mathbf{u})$	Derivative of f
$D_{\mathbf{x}}f(\mathbf{x}, \mathbf{u})$	Partial derivative of f w.r.t. $\mathbf{x}$
$D_{\mathbf{xx}}f(\mathbf{x}, \mathbf{u})$	Second order derivative of f w.r.t. $\mathbf{x}$
$Df(\mathbf{x}', \mathbf{u}')(\mathbf{x}, \mathbf{u})$	The linear function based on the derivative at $(\mathbf{x}', \mathbf{u}')$
$L(\mathbf{x}, \lambda, \mathbf{u})$	The Lagrangian $f(\mathbf{x}, \mathbf{u}) + \langle \lambda, G(\mathbf{x}, \mathbf{u})\rangle,\ \lambda \in \mathcal{Y}^*$
$\Lambda_{\mathbf{u}}(\mathbf{x})$	$\{\lambda \in N_{\mathcal{K}}(G(\mathbf{x}, \mathbf{u})) : D_{\mathbf{x}}L(\mathbf{x}, \lambda, \mathbf{u}) = 0\}$
$\{\mathcal{X}_1 + \mathcal{X}_2\}$	$\cup\{\mathbf{x}_1 + \mathbf{x}_2\},\ \mathbf{x}_1 \in \mathcal{X}_1, \mathbf{x}_2 \in \mathcal{X}_2$
$\text{int}(\mathcal{X})$	The interior of the set $\mathcal{X}$

To study the first order differentiabilitiy of the optimal value function $v(\mathbf{u})$, for a given direction $\mathbf{d} \in \mathcal{U}$ and the optimal solution of the unperturbed problem $\mathbf{x}_0 \in \mathcal{S}(0)$, Bonnans et al. (1998) consider the linearization of the family of problems $(P_{t\mathbf{d}})$ and its dual as follows

$$
\begin{aligned}
&(PL_{\mathbf{d}}) : \min_{\mathbf{h}} f(\mathbf{x}_0, 0)(\mathbf{h}, \mathbf{d}) \text{ subject to } DG(\mathbf{x}_0, 0)(\mathbf{h}, \mathbf{d}) \in T_{\mathcal{K}}(G(\mathbf{x}_0, 0)),\\
&(DL_{\mathbf{d}}) : \max_{\lambda \in \Lambda_0(\mathbf{x}_0)} D_{\mathbf{u}}L(\mathbf{x}_0, \lambda, 0)\mathbf{d}.
\end{aligned}
$$

Theorem 13. *(Theorem 3.1 in Bonnans et al. (1998)) Let $\bar{\mathbf{x}}(t)$ be an $O(t^2)$-optimal trajectory of $(P_{t\mathbf{d}})$ converging to a point $\mathbf{x}_0 \in \Phi(0)$ as $t \to 0$. Assume $v(PL_{\mathbf{d}})$ to be finite. Suppose that the following conditions hold:*

1. *$\mathbf{x}_0$ satisfies the directional constraint qualification, which is implied if*
$$0 \in \textit{int}\{G(\mathbf{x}_0, 0) + D_{\mathbf{x}}G(\mathbf{x}_0, 0)\mathcal{X} - \mathcal{K}\}.$$
2. *$v(t\mathbf{d}) \leq v(0) + tv(PL_{\mathbf{d}}) + O(t^2),\ t \geq 0$ (Equation 3.4 in (Bonnans et al., 1998)).*
3. *The strong second order sufficient condition (Equation 3.1 in (Bonnans et al., 1998)) holds, which is implied if*
$$\sup_{\lambda \in \mathcal{S}(DL_{\mathbf{d}})} D^2_{\mathbf{xx}}L(\mathbf{x}_0, \lambda, 0)(\mathbf{h}, \mathbf{h}) > 0,\ \forall \mathbf{h} \in C(\mathbf{x}_0)\backslash\{0\},$$
where $C(\mathbf{x}_0)$ denotes the critical cone.

Then $\bar{\mathbf{x}}(t)$ is Lipschitz stable at $\mathbf{x}_0$, i.e., for $t \geq 0$, $\lVert\bar{\mathbf{x}}(t) - \mathbf{x}_0\rVert = O(t)$.

Proof. (Proof of Lemma 12) The key idea is to leverage Theorem 13, which quantifies the sensitivity of the optimal solution of a "perturbed" optimization problem. More specifically, it is shown that the distance between the original optimal solution and the optimal solution of the perturbed problem is upper-bounded by the magnitude of the perturbation.

First, we show that $\forall i \in [m]$, the projection problem $\Pi_{\widehat{\mathcal{X}}_i^s}(\mathbf{x})$ can be formulated as a quadratic programming with second-order cone constraints. The definition of $\widehat{\mathcal{X}}_i^s$ has the following equivalent expression

$$
\begin{aligned}
\widehat{\mathcal{X}}_i^s \triangleq \{\mathbf{x} \in \mathrm{R}^d : \tilde{\mathbf{a}}_k^\top \mathbf{x} \leq b_k,\ \forall \tilde{\mathbf{a}}_k \in \mathcal{C}_{i,k},\ \forall k \in [n]\} &= \{\mathbf{x} \in \mathrm{R}^d : \max_{\tilde{\mathbf{a}}_k \in \mathcal{C}_{i,k}} \tilde{\mathbf{a}}_k^\top \mathbf{x} \leq b_k,\ \forall k \in [n]\}\\
&= \{\mathbf{x} \in \mathrm{R}^d : [\widehat{\mathbf{A}}_i]_{k,:}\mathbf{x} + \mathcal{B}_r\lVert\mathbf{x}\rVert \leq b_k,\ \forall k \in [n]\},
\end{aligned}
\tag{24}
$$

where each second-order cone inequality: $[\widehat{\mathbf{A}}_i]_{k,:}\mathbf{x} + \mathcal{B}_r\|\mathbf{x}\| \leq b_k$ can be equivalently written as a linear matrix inequality (LMI):

$$[\widehat{\mathbf{A}}_i]_{k,:}\mathbf{x} + \mathcal{B}_r\|\mathbf{x}\| \leq b_k \Leftrightarrow \mathbf{G}^k(\mathbf{x}, \widehat{\mathbf{A}}_i) \triangleq \begin{bmatrix} (b_k - [\widehat{\mathbf{A}}_i]_{k,:}\mathbf{x}) & \mathcal{B}_r\mathbf{x}^\top \\ \mathcal{B}_r\mathbf{x} & (b_k - [\widehat{\mathbf{A}}_i]_{k,:}\mathbf{x})\mathbf{I} \end{bmatrix} \succeq 0. \tag{25}$$

For simplicity, we define the following matrix

$$\mathcal{G}(\mathbf{x}, \widehat{\mathbf{A}}_i) \triangleq \begin{bmatrix} \mathbf{G}^1(\mathbf{x}, \widehat{\mathbf{A}}_i) & & & \\ & \mathbf{G}^2(\mathbf{x}, \widehat{\mathbf{A}}_i) & & \\ & & \ddots & \\ & & & \mathbf{G}^n(\mathbf{x}, \widehat{\mathbf{A}}_i) \end{bmatrix}.$$

Considering the intersection of all LMIs, we have

$$\widehat{\mathcal{X}}_i^s \triangleq \{\mathbf{x} \in \mathrm{R}^d : \mathcal{G}(\mathbf{x}, \widehat{\mathbf{A}}_i) \succeq 0\}. \tag{26}$$

Based on (26), for a point $\mathbf{x} \in \mathrm{R}^d$, we have $\Pi_{\widehat{\mathcal{X}}_i^s}(\mathbf{x}) = \mathbf{x} + \xi_i$, where ξ_i is derived by solving the following optimization problem

$$\xi_i = \mathrm{argmin}_\xi\ \xi^\top\xi,\ \text{ s.t. } \begin{bmatrix} \mathbf{G}^1(\mathbf{x}+\xi, \widehat{\mathbf{A}}_i) & & & \\ & \mathbf{G}^2(\mathbf{x}+\xi, \widehat{\mathbf{A}}_i) & & \\ & & \ddots & \\ & & & \mathbf{G}^n(\mathbf{x}+\xi, \widehat{\mathbf{A}}_i) \end{bmatrix} \succeq 0. \tag{27}$$

Based on Lemma 2, we have $\|[\widehat{\mathbf{A}}_i]_{k,:} - [\widehat{\mathbf{A}}_j]_{k,:}\| = O(\frac{1}{T^\rho}),\ \forall i, j \in [m]$ and $\forall k \in [n]$. Therefore, $[\widehat{\mathbf{A}}_j]_{k,:}$ can be expressed as $[\widehat{\mathbf{A}}_i]_{k,:} + \psi_k$, where $\|\psi_k\| = O(\frac{1}{T^\rho})$. With this expression, the projection $\Pi_{\widehat{\mathcal{X}}_j^s}(\mathbf{x}) = \mathbf{x} + \xi_j$ can be formulated as a perturbed version of the optimization (27), where the perturbation is parameterized in terms of $\psi = [\psi_1, \ldots, \psi_n]$ as follows:

$$\xi_j = \mathrm{argmin}_\xi\ \xi^\top\xi,\ \text{ s.t. } \begin{bmatrix} \mathbf{G}^1(\mathbf{x}+\xi, \widehat{\mathbf{A}}_i + \psi) & & & \\ & \mathbf{G}^2(\mathbf{x}+\xi, \widehat{\mathbf{A}}_i + \psi) & & \\ & & \ddots & \\ & & & \mathbf{G}^n(\mathbf{x}+\xi, \widehat{\mathbf{A}}_i + \psi) \end{bmatrix} \succeq 0. \tag{28}$$

To show that $\|\Pi_{\widehat{\mathcal{X}}_i^s}(\mathbf{x}) - \Pi_{\widehat{\mathcal{X}}_j^s}(\mathbf{x})\| = \|\xi_i - \xi_j\| = O(\|\psi\|) = O(\frac{1}{T^\rho})$, we apply Theorem 13, where three conditions need to be satisfied: directional constraint qualification (DCQ), Equation 3.4 in Bonnans et al. (1998) and strong second-order sufficient conditions (we refer readers to Bonnans et al. (1998) for detailed definitions).

- **DCQ**:
  A sufficient condition for DCQ is constraint qualification (CQ) (see the definition in Bonnans et al. (1998)), which is satisfied in our problem formulation if the first-order approximation of $\mathcal{G}(\mathbf{x}+\xi, \widehat{\mathbf{A}}_i + \psi)$ w.r.t. the variable ξ can be positive-definite. Noting that $\mathcal{G}(\mathbf{x}+\xi, \widehat{\mathbf{A}}_i + \psi)$ is an affine function of ξ, the first-order approximation is exactly the original function. Now suppose that $\forall i \in [m]$, $\widehat{\mathcal{X}}_i^s$ has a strictly feasible point (this is implied by the existence of the mutual shrunk polytope), which means there exists a $\hat{\xi}$ such that $\mathcal{G}(\mathbf{x}+\hat{\xi}, \widehat{\mathbf{A}}_i + \psi)$ is positive-definite, and then CQ is satisfied.

- **Equation 3.4 in Bonnans et al. (1998)**:
  In Bonnans et al. (1998), the authors provided the sufficient conditions for Equation 3.4: DCQ and second-order regularity (Definition 2.2 in Bonnans et al. (1998)). DCQ, as mentioned previously, holds in our case, and second-order regularity holds for semi-definite optimization, which is the case for our problem setup.

- **Second-order sufficient conditions**:
  The strong second-order sufficient condition (Equation 3.1 in Bonnans et al. (1998)) has an alternative form (Equation 3.3 in Bonnans et al. (1998)), which is satisfied in our problem setup since the Hessian of the Lagrangian is $2\mathbf{I}$, which is positive-definite.

Since all the conditions above are met, the lemma is proved by applying Theorem 13. □

A.3 Convex Part

Lemma 14. *Let Algorithm 2 run with step size $\eta > 0$ and define $\mathbf{x}_t \triangleq \frac{1}{m}\sum_{i=1}^{m}\mathbf{x}_{i,t}$ and $\mathbf{y}_t \triangleq \frac{1}{m}\sum_{i=1}^{m}\mathbf{y}_{i,t}$. Under Assumptions 1 to 3 and the fact that gradients are bounded, i.e., $\left\|\nabla f_{i,t}(\mathbf{x})\right\| \leq G$ for any $\mathbf{x} \in \mathcal{X}^s$, we have that $\forall i \in [m]$*

$$\left\|\mathbf{x}_t - \mathbf{x}_{i,t}\right\| \leq \big(O(\frac{1}{T^\rho}) + 2\eta G\big)\frac{\sqrt{m}\beta}{1-\beta}.$$

Proof. For the presentation simplicity, we define the following matrices

$$\mathbf{X}_t \triangleq [\mathbf{x}_{1,t}, \ldots, \mathbf{x}_{m,t}],\ \mathbf{Y}_t \triangleq [\mathbf{y}_{1,t}, \ldots, \mathbf{y}_{m,t}],\ \mathbf{G}_t \triangleq [\nabla f_{1,t}(\mathbf{x}_{1,t}), \ldots, \nabla f_{m,t}(\mathbf{x}_{m,t})],\ \text{and}\ \mathbf{R}_t \triangleq [\mathbf{r}_{1,t}, \ldots, \mathbf{r}_{m,t}],$$

where $\mathbf{r}_{i,t} \triangleq \mathbf{y}_{i,t} - \big(\mathbf{x}_{i,t} - \eta\nabla f_{i,t}(\mathbf{x}_{i,t})\big)$. Then, the update can be expressed as $\mathbf{X}_t = \mathbf{Y}_{t-1}\mathbf{P} = \big(\mathbf{X}_{t-1} - \eta\mathbf{G}_{t-1} + \mathbf{R}_{t-1}\big)\mathbf{P}$.

Expanding the update recursively, we have

$$\mathbf{X}_t = \mathbf{X}_{T_s}\mathbf{P}^{(t-T_s)} - \eta\sum_{l=1}^{t-T_s}\mathbf{G}_{t-l}\mathbf{P}^l + \sum_{l=1}^{t-T_s}\mathbf{R}_{t-l}\mathbf{P}^l. \tag{29}$$

Since $\mathbf{P}$ is doubly stochastic, we have $\mathbf{P}^k\mathbf{1} = \mathbf{1}$ for all $k \geq 1$. Based on the geometric mixing bound of $\mathbf{P}$ and the above equation we get

$$\begin{aligned}
&\left\|\mathbf{x}_t - \mathbf{x}_{i,t}\right\| = \left\|\mathbf{X}_t(\frac{1}{m}\mathbf{1} - \mathbf{e}_i)\right\| \\
\leq&\left\|\mathbf{x}_{T_s} - \mathbf{X}_{T_s}[\mathbf{P}^{(t-T_s)}]_{:,i}\right\| + \eta\sum_{l=1}^{t-T_s}\left\|\mathbf{G}_{t-l}(\frac{1}{m}\mathbf{1} - [\mathbf{P}^l]_{:,i})\right\| + \sum_{l=1}^{t-T_s}\left\|\mathbf{R}_{t-l}(\frac{1}{m}\mathbf{1} - [\mathbf{P}^l]_{:,i})\right\| \\
\leq&\sum_{l=1}^{t-T_s}(\eta G)\sqrt{m}\beta^l + \sum_{l=1}^{t-T_s}\big(O(\frac{1}{T^\rho}) + \eta G\big)\sqrt{m}\beta^l \\
\leq&\big(O(\frac{1}{T^\rho}) + 2\eta G\big)\frac{\sqrt{m}\beta}{1-\beta},
\end{aligned}$$

where $\left\|\mathbf{x}_{T_s} - \mathbf{X}_{T_s}[\mathbf{P}^{t-T_s}]_{:,i}\right\| = 0$ by the identical initialization of all agents with the same action at T_s, and the other inequality is based on Lemma 12 as follows

$$\begin{aligned}
\left\|\mathbf{r}_{i,t}\right\| &= \left\|\mathbf{y}_{i,t} - \big(\mathbf{x}_{i,t} - \eta\nabla f_{i,t}(\mathbf{x}_{i,t})\big)\right\| \\
&\leq \Big\|\sum_j[\mathbf{P}]_{ji}\Pi_{\widehat{\mathcal{X}}_i^s}[\mathbf{y}_{j,t-1}] - \Big(\sum_j[\mathbf{P}]_{ji}\mathbf{y}_{j,t-1} - \eta\nabla f_{i,t}(\mathbf{x}_{i,t})\Big)\Big\| \\
&\leq O(T^{-\rho}) + \eta G.
\end{aligned}$$

□

Proof of Theorem 3. First, we decompose the individual regret of agent j into three terms:

$$\sum_t \sum_i f_{i,t}(\mathbf{x}_{j,t}) - \sum_t f_t(\mathbf{x}_t^*) = \underbrace{\sum_{t=1}^{T_s-1} \sum_i f_{i,t}(\mathbf{x}_{j,t}) - f_{i,t}(\mathbf{x}_t^*)}_{\text{Term I}} + \underbrace{\sum_{t=T_s}^{T} \sum_i f_{i,t}(\mathbf{x}_{j,t}) - f_{i,t}(\tilde{\mathbf{x}}_t^*)}_{\text{Term II}} + \underbrace{\sum_{t=T_s}^{T} f_t(\tilde{\mathbf{x}}_t^*) - f_t(\mathbf{x}_t^*)}_{\text{Term III}}, \tag{30}$$

where $\tilde{\mathbf{x}}_t^*$ is the projection of $\mathbf{x}_t^*$ on $\mathcal{X}_{\text{in}}^s$, which is a mutual subset of $\{\widehat{\mathcal{X}}_i^s\}_{i\in[m]}$ with $\tau_{\text{in}} = 2\mathcal{B}_r L$ based on Equation (22) in Lemma 11. We now proceed to bound each term.

The upper bound of Term I:
Note that by choosing $\gamma \leq \frac{\Delta^s}{LL_A}$, we have $\forall i \in [m]$ and $t \in [1, \dots, T_0 + T_1]$

$$[\mathbf{A}]_{k,:}\mathbf{x}_{i,t} = [\mathbf{A}]_{k,:}((1-\gamma)\mathbf{x}^s + \gamma\zeta_{i,t}) \leq (1-\gamma)b_k^s + \Delta^s \leq (1-\gamma)b_k^s + (b_k - b_k^s) < b_k, \tag{31}$$

which implies the safeness of the action.

Based on the Lipschitz property of the function sequence, we have

$$\sum_{t=1}^{T_s-1} \sum_i f_{i,t}(\mathbf{x}_{j,t}) - f_{i,t}(\mathbf{x}_t^*) \leq \sum_{t=1}^{T_s-1} \sum_i G\big\|\mathbf{x}_{j,t} - \mathbf{x}_t^*\big\| \leq 2GLm(T_0 + T_1). \tag{32}$$

The upper bound of Term II:
Based on the update rule, $\forall i \in [m]$ and $t \in [T_s, \dots, T]$ we have

$$\begin{aligned} f_{i,t}(\mathbf{x}_{i,t}) - f_{i,t}(\tilde{\mathbf{x}}_t^*) \leq& \nabla f_{i,t}(\mathbf{x}_{i,t})^\top(\mathbf{x}_{i,t} - \tilde{\mathbf{x}}_t^*) \\ =& \frac{1}{\eta}\Big[\frac{1}{2}\eta^2\big\|\nabla f_{i,t}(\mathbf{x}_{i,t})\big\|^2 + \frac{1}{2}\big\|\mathbf{x}_{i,t} - \tilde{\mathbf{x}}_t^*\big\|^2 - \frac{1}{2}\big\|\mathbf{x}_{i,t} - \tilde{\mathbf{x}}_t^* - \eta\nabla f_{i,t}(\mathbf{x}_{i,t})\big\|^2\Big] \\ \leq& \frac{1}{\eta}\Big[\frac{1}{2}\eta^2\big\|\nabla f_{i,t}(\mathbf{x}_{i,t})\big\|^2 + \frac{1}{2}\big\|\mathbf{x}_{i,t} - \tilde{\mathbf{x}}_t^*\big\|^2 - \frac{1}{2}\big\|\mathbf{y}_{i,t} - \tilde{\mathbf{x}}_t^*\big\|^2\Big] \\ =& \frac{1}{\eta}\Big[\frac{1}{2}\eta^2\big\|\nabla f_{i,t}(\mathbf{x}_{i,t})\big\|^2 + \frac{1}{2}\Big\|\sum_j [\mathbf{P}]_{ji}\mathbf{y}_{j,t-1} - \tilde{\mathbf{x}}_t^*\Big\|^2 - \frac{1}{2}\big\|\mathbf{y}_{i,t} - \tilde{\mathbf{x}}_t^*\big\|^2\Big] \\ \leq& \frac{1}{\eta}\Big[\frac{1}{2}\eta^2\big\|\nabla f_{i,t}(\mathbf{x}_{i,t})\big\|^2 + \frac{1}{2}\sum_j [\mathbf{P}]_{ji}\big\|\mathbf{y}_{j,t-1} - \tilde{\mathbf{x}}_t^*\big\|^2 - \frac{1}{2}\big\|\mathbf{y}_{i,t} - \tilde{\mathbf{x}}_t^*\big\|^2\Big], \end{aligned} \tag{33}$$

where the second inequality is due to the projection property that $\big\|\mathbf{y}_{i,t} - \tilde{\mathbf{x}}_t^*\big\| \leq \big\|\mathbf{x}_{i,t} - \eta\nabla f_{i,t}(\mathbf{x}_{i,t}) - \tilde{\mathbf{x}}_t^*\big\|$, and the third inequality is due to the convexity of the square function.

Based on Equation (33) and Lemma 14, we have

$$\begin{aligned} &f_{i,t}(\mathbf{x}_{j,t}) - f_{i,t}(\tilde{\mathbf{x}}_t^*) = f_{i,t}(\mathbf{x}_{j,t}) - f_{i,t}(\mathbf{x}_{i,t}) + f_{i,t}(\mathbf{x}_{i,t}) - f_{i,t}(\tilde{\mathbf{x}}_t^*) \\ &\leq G\big\|\mathbf{x}_{j,t} - \mathbf{x}_{i,t}\big\| + f_{i,t}(\mathbf{x}_{i,t}) - f_{i,t}(\tilde{\mathbf{x}}_t^*) \\ &\leq 2G\big(O(\frac{1}{T^\rho}) + 2\eta G\big)\frac{\sqrt{m}\beta}{1-\beta} + \frac{1}{2}\eta\big\|\nabla f_{i,t}(\mathbf{x}_{i,t})\big\|^2 + \frac{1}{2\eta}\sum_j [\mathbf{P}]_{ji}\big\|\mathbf{y}_{j,t-1} - \tilde{\mathbf{x}}_t^*\big\|^2 - \frac{1}{2\eta}\big\|\mathbf{y}_{i,t} - \tilde{\mathbf{x}}_t^*\big\|^2. \end{aligned} \tag{34}$$

Summing Equation (34) over i, we get

$$\begin{aligned}
&\sum_i \left(f_{i,t}(\mathbf{x}_{j,t}) - f_{i,t}(\tilde{\mathbf{x}}_t^*)\right) \\
\leq& 2mG\big(O(\frac{1}{T^\rho}) + 2\eta G\big)\frac{\sqrt{m}\beta}{1-\beta} + \frac{\eta}{2}\sum_i \left\|\nabla f_{i,t}(\mathbf{x}_{i,t})\right\|^2 + \frac{1}{2\eta}\sum_j \left\|\mathbf{y}_{j,t-1} - \tilde{\mathbf{x}}_t^*\right\|^2 - \frac{1}{2\eta}\sum_i \left\|\mathbf{y}_{i,t} - \tilde{\mathbf{x}}_t^*\right\|^2 \\
=& 2mG\big(O(\frac{1}{T^\rho}) + 2\eta G\big)\frac{\sqrt{m}\beta}{1-\beta} + \frac{\eta}{2}\sum_i \left\|\nabla f_{i,t}(\mathbf{x}_{i,t})\right\|^2 + \frac{1}{2\eta}\sum_i \left(\left\|\mathbf{y}_{i,t-1}\right\|^2 - \left\|\mathbf{y}_{i,t}\right\|^2 + 2(\mathbf{y}_{i,t} - \mathbf{y}_{i,t-1})^\top \tilde{\mathbf{x}}_t^*\right).
\end{aligned} \tag{35}$$

Summing Equation (35) over $t \in [T_s, \ldots, T]$, we have

$$\begin{aligned}
&\sum_{t=T_s}^{T}\sum_i \left(f_{i,t}(\mathbf{x}_{j,t}) - f_{i,t}(\tilde{\mathbf{x}}_t^*)\right) \\
\leq& \frac{\eta}{2}\sum_{t=T_s}^{T}\sum_i \left\|\nabla f_{i,t}(\mathbf{x}_{i,t})\right\|^2 + \frac{1}{2\eta}\sum_i \left\|\mathbf{y}_{i,T_s-1}\right\|^2 + \frac{1}{\eta}\big(\sum_i \mathbf{y}_{i,T}^\top \tilde{\mathbf{x}}_T^* - \sum_i \mathbf{y}_{i,T_s-1}^\top \tilde{\mathbf{x}}_{T_s-1}^*\big) \\
&+ \frac{1}{\eta}\sum_{t=T_s-1}^{T-1}\sum_i (\tilde{\mathbf{x}}_t^* - \tilde{\mathbf{x}}_{t+1}^*)^\top \mathbf{y}_{i,t} + 2TmG\big(O(\frac{1}{T^\rho}) + 2\eta G\big)\frac{\sqrt{m}\beta}{1-\beta}.
\end{aligned} \tag{36}$$

The upper bound of Term III:

Based on Lemma 9, we have for any $\mathbf{x}_t^* \in \mathcal{X}^s$ and its projection to $\mathcal{X}_{\text{in}}^s$, denoted by $\tilde{\mathbf{x}}_t^*$, that

$$\sum_{t=T_s}^{T}\sum_i \left(f_{i,t}(\tilde{\mathbf{x}}_t^*) - f_{i,t}(\mathbf{x}_t^*)\right) \leq \sum_{t=T_s}^{T}\sum_i G\left\|\tilde{\mathbf{x}}_t^* - \mathbf{x}_t^*\right\| \leq mTG\frac{2\sqrt{d}L\mathcal{B}_r}{C(\mathbf{A}, \mathbf{b})}. \tag{37}$$

Substituting Equations (32), (36) and (37) into Equation (30), we get

$$\begin{aligned}
&\sum_t\sum_i \left(f_{i,t}(\mathbf{x}_{j,t}) - f_{i,t}(\mathbf{x}_t^*)\right) \\
\leq& 2GLm(T_0 + T_1) + \frac{\eta m T G^2}{2} + \frac{1}{2\eta}\sum_i \left\|\mathbf{y}_{i,T_s-1}\right\|^2 + \frac{1}{\eta}\big(\sum_i \mathbf{y}_{i,T}^\top \tilde{\mathbf{x}}_T^* - \sum_i \mathbf{y}_{i,T_s-1}^\top \tilde{\mathbf{x}}_{T_s-1}^*\big) \\
&+ \frac{1}{\eta}\sum_{t=T_s-1}^{T-1}\sum_i (\tilde{\mathbf{x}}_t^* - \tilde{\mathbf{x}}_{t+1}^*)^\top \mathbf{y}_{i,t} + 2TmG\big(O(\frac{1}{T^\rho}) + 2\eta G\big)\frac{\sqrt{m}\beta}{1-\beta} + mTG\frac{2\sqrt{d}L\mathcal{B}_r}{C(\mathbf{A}, \mathbf{b})},
\end{aligned} \tag{38}$$

which is $O(T_0 + T_1 + \frac{1}{\eta} + \frac{1}{\eta}C_T^* + \frac{T\sqrt{\log T_0}}{\sqrt{T_0}} + \frac{\beta\eta T}{(1-\beta)})$ and the final regret bound is derived by substituting the choices of η and T_0 into above. □

A.4 Non-convex Part

Lemma 15 (Lemma 4 in Ghai et al. (2022))**.** *Suppose Assumptions 5, 6, 7 hold and* $\mathbf{u}_t = q(\mathbf{x}_t)$*, then* $\left\|q(\mathbf{x}_{t+1}) - \mathbf{u}_{t+1}\right\| = O(W^4 G_F^{3/2}\eta^{3/2})$ *based on the following update rule:*

$$\begin{aligned}
\mathbf{u}_{t+1} &= \mathit{argmin}_{\mathbf{u}\in\mathcal{X}^{s\prime}}\left\{\nabla\tilde{f}_t(\mathbf{u}_t)^\top\mathbf{u} + \frac{1}{\eta}\mathcal{D}_\phi(\mathbf{u}, \mathbf{u}_t)\right\}, \\
\mathbf{x}_{t+1} &= \mathit{argmin}_{\mathbf{x}\in\mathcal{X}^s}\left\{\nabla f_t(\mathbf{x}_t)^\top\mathbf{x} + \frac{1}{2\eta}\left\|\mathbf{x} - \mathbf{x}_t\right\|^2\right\}.
\end{aligned}$$

Theorem 16 (Theorem 7 in Ghai et al. (2022))**.** *Given a convex and compact domain $\mathcal{X} \subset \mathcal{X}^s$, and not necessarily convex loss $f_t(\cdot)$ satisfying Assumption 7. When Assumption 8 is met, there exists an OMD object with convex loss $\tilde{f}_t(\cdot)$, a convex domain and a strongly convex regularization ϕ satisfying Assumption 5.*

Lemma 17. *Suppose Assumptions 5-7 hold and $\mathbf{u}_{i,t} = q(\mathbf{x}_{i,t}),\ \forall i \in [m]$; then*

$$\left\|q(\mathbf{x}_{i,t+1}) - \mathbf{u}'_{i,t+1}\right\| = O(\frac{1}{T^{2\rho}} + \eta^{3/2}),$$

based on the following update rules:

$$\begin{aligned}
\mathbf{z}_{i,t} &= \textit{argmin}_{\mathbf{u}\in\widehat{X}_i^{s\prime}} \left\{\nabla \tilde{f}_{i,t}(\mathbf{u}_{i,t})^\top \mathbf{u} + \frac{1}{\eta}\mathcal{D}_\phi(\mathbf{u}, \mathbf{u}_{i,t})\right\}, \\
\mathbf{u}'_{i,t+1} &= \sum_j [\mathbf{P}]_{ji}\mathbf{z}_{j,t}, \\
\mathbf{y}_{i,t} &= \textit{argmin}_{\mathbf{x}\in\widehat{X}_i^{s}} \left\{\nabla f_{i,t}(\mathbf{x}_{i,t})^\top \mathbf{x} + \frac{1}{2\eta}\left\|\mathbf{x}-\mathbf{x}_{i,t}\right\|^2\right\}, \\
\mathbf{x}_{i,t+1} &= \sum_j [\mathbf{P}]_{ji}\mathbf{y}_{j,t}.
\end{aligned} \tag{39}$$

Proof. We first upper bound $\left\|q(\mathbf{x}_{i,t+1}) - \mathbf{u}'_{i,t+1}\right\|$ as follows

$$\left\|q(\mathbf{x}_{i,t+1}) - \mathbf{u}'_{i,t+1}\right\| \le \Big\| \sum_j [\mathbf{P}]_{ji}\mathbf{z}_{j,t} - \sum_j [\mathbf{P}]_{ji} q(\mathbf{y}_{j,t})\Big\| + \Big\| \sum_j [\mathbf{P}]_{ji} q(\mathbf{y}_{j,t}) - q(\sum_j [\mathbf{P}]_{ji}\mathbf{y}_{j,t})\Big\|. \tag{40}$$

To bound the second term, we consider the Taylor expansion of $q(\mathbf{y})$ w.r.t. a point $\hat{\mathbf{y}}$ in the convex hull of $\{\mathbf{y}_{i,t}\}_i$:

$$\begin{aligned}
\Big\| \sum_j [\mathbf{P}]_{ji} q(\mathbf{y}_{j,t}) - q(\sum_j [\mathbf{P}]_{ji}\mathbf{y}_{j,t})\Big\| \le& \Big\| \sum_j [\mathbf{P}]_{ji}\Big(q(\hat{\mathbf{y}}) + J_q(\hat{\mathbf{y}})(\mathbf{y}_{j,t}-\hat{\mathbf{y}}) + O\big(\left\|\mathbf{y}_{j,t}-\hat{\mathbf{y}}\right\|^2\big)\Big) \\
& - \Big(q(\hat{\mathbf{y}}) + J_q(\hat{\mathbf{y}})(\sum_j [\mathbf{P}]_{ji}\mathbf{y}_{j,t}-\hat{\mathbf{y}}) + O\big(\Big\|\sum_j [\mathbf{P}]_{ji}\mathbf{y}_{j,t}-\hat{\mathbf{y}}\Big\|^2\big)\Big)\Big\| \\
\le& O\big(\sum_j [\mathbf{P}]_{ji}\left\|\mathbf{y}_{j,t}-\hat{\mathbf{y}}\right\|^2\big) + O\big(\Big\|\sum_j [\mathbf{P}]_{ji}\mathbf{y}_{j,t}-\hat{\mathbf{y}}\Big\|^2\big) \\
\le& O(D^2),
\end{aligned} \tag{41}$$

where D denotes the diameter of the convex hull of $\{\mathbf{y}_{i,t}\}$ and is upper bounded as follows

$$\begin{aligned}
D \triangleq& \max_{(i,j)} \left\|\mathbf{y}_{i,t} - \mathbf{y}_{j,t}\right\| \\
=& \max_{(i,j)} \left\|\Pi_{\widehat{X}_i^s}\big(\mathbf{x}_{i,t} - \eta\nabla f_{i,t}(\mathbf{x}_{i,t})\big) - \Pi_{\widehat{X}_j^s}\big(\mathbf{x}_{j,t} - \eta\nabla f_{j,t}(\mathbf{x}_{j,t})\big)\right\| \\
=& \max_{(i,j)} \Big\|\Pi_{\widehat{X}_i^s}\big(\mathbf{x}_{i,t} - \eta\nabla f_{i,t}(\mathbf{x}_{i,t})\big) - \Pi_{\widehat{X}_j^s}\big(\mathbf{x}_{i,t} - \eta\nabla f_{i,t}(\mathbf{x}_{i,t})\big) \\
& + \Pi_{\widehat{X}_j^s}\big(\mathbf{x}_{i,t} - \eta\nabla f_{i,t}(\mathbf{x}_{i,t})\big) - \Pi_{\widehat{X}_j^s}\big(\mathbf{x}_{j,t} - \eta\nabla f_{j,t}(\mathbf{x}_{j,t})\big)\Big\| \\
\le& \max_{(i,j)} \left\|\Pi_{\widehat{X}_i^s}\big(\mathbf{x}_{i,t} - \eta\nabla f_{i,t}(\mathbf{x}_{i,t})\big) - \Pi_{\widehat{X}_j^s}\big(\mathbf{x}_{i,t} - \eta\nabla f_{i,t}(\mathbf{x}_{i,t})\big)\right\| \\
& + \left\|\big(\mathbf{x}_{i,t} - \eta\nabla f_{i,t}(\mathbf{x}_{i,t})\big) - \big(\mathbf{x}_{j,t} - \eta\nabla f_{j,t}(\mathbf{x}_{j,t})\big)\right\| \\
\le& O(\frac{1}{T^\rho}) + 2\bigg(\big(O(\frac{1}{T^\rho}) + 2\eta G\big)\frac{\sqrt{m}\beta}{1-\beta}\bigg) + 2G\eta = O(\frac{1}{T^\rho} + \eta).
\end{aligned} \tag{42}$$

The first inequality follows from the non-expansive property of projection, where $\|\Pi_{\mathcal{X}}(\mathbf{x}) - \Pi_{\mathcal{X}}(\mathbf{y})\| \leq \|\mathbf{x} - \mathbf{y}\|$ for any $\mathbf{x}, \mathbf{y}$ and a closed convex set $\mathcal{X}$, and the last inequality is based on Lemma 12, Lemma 14 and the Lipschitz continuity of the function sequence.

Substituting Equations (41) and (42) into Equation (40) and based on Lemma 15, we have

$$\begin{aligned}\|q(\mathbf{x}_{i,t+1}) - \mathbf{u}'_{i,t+1}\| \leq & \|\sum_j [\mathbf{P}]_{ji}\mathbf{z}_{j,t} - \sum_j [\mathbf{P}]_{ji} q(\mathbf{y}_{j,t})\| + \|\sum_j [\mathbf{P}]_{ji} q(\mathbf{y}_{j,t}) - q(\sum_j [\mathbf{P}]_{ji}\mathbf{y}_{j,t})\| \\ \leq & O(W^4 G_F^{3/2} \eta^{3/2}) + O(\frac{1}{T^{2\rho}} + \eta^2) = O(\frac{1}{T^{2\rho}} + \eta^{3/2}),\end{aligned} \tag{43}$$

when η is small enough. □

Proof of Theorem 5. As for the proof of Theorem 3, we decompose the individual regret into three terms:

$$\sum_t \sum_i f_{i,t}(\mathbf{x}_{j,t}) - \sum_t f_t(\mathbf{x}_t^*) = \underbrace{\sum_{t=1}^{T_s-1} \sum_i f_{i,t}(\mathbf{x}_{j,t}) - f_{i,t}(\mathbf{x}_t^*)}_{\text{Term I}} + \underbrace{\sum_{t=T_s}^{T} \sum_i f_{i,t}(\mathbf{x}_{j,t}) - f_{i,t}(\tilde{\mathbf{x}}_t^*)}_{\text{Term II}} + \underbrace{\sum_{t=T_s}^{T} f_t(\tilde{\mathbf{x}}_t^*) - f_t(\mathbf{x}_t^*)}_{\text{Term III}}, \tag{44}$$

where $\tilde{\mathbf{x}}_t^*$ is the projection of $\mathbf{x}_t^*$ on $\mathcal{X}_{\text{in}}^s$, which is a mutual subset of $\{\widehat{\mathcal{X}}_i^s\}_{i\in[m]}$ with $\tau_{\text{in}} = 2\mathcal{B}_r L$ based on Equation (22).

The upper bound of Term I:
Similar to the proof of convex part, during the estimation phase, γ is less than $\frac{\Delta^s}{LL_A}$ to ensure the safeness of each agent's action, and based on the Lipschitz property we have

$$\begin{aligned}\sum_{t=1}^{T_s-1} \sum_i f_{i,t}(\mathbf{x}_{j,t}) - f_{i,t}(\mathbf{x}_t^*) = & \sum_{t=1}^{T_s-1} \sum_i \tilde{f}_{i,t}\big(q(\mathbf{x}_{j,t})\big) - \tilde{f}_{i,t}\big(q(\mathbf{x}_t^*)\big) \\ \leq & \sum_{t=1}^{T_s-1} \sum_i G_F W \|\mathbf{x}_{j,t} - \mathbf{x}_t^*\| \leq 2 G_F W L m (T_0 + T_1).\end{aligned} \tag{45}$$

The upper bound of Term II:
Define $\widehat{\mathcal{X}}_i^{s\prime} \triangleq \{q(\mathbf{x}) | \mathbf{x} \in \widehat{\mathcal{X}}_i^s\}$, (same for $\mathcal{X}_{\text{in}}^s$ and $\mathcal{X}^s$). Then, for any $q(\tilde{\mathbf{x}}_t^*) = \tilde{\mathbf{u}}_t^* \in \mathcal{X}_{\text{in}}^{s\prime}$, based on Equation (39), we

have

$$
\begin{aligned}
\eta\left(f_{i,t}(\mathbf{x}_{i,t}) - f_{i,t}(\tilde{\mathbf{x}}_t^*)\right) &= \eta\left(\tilde{f}_{i,t}(\mathbf{u}_{i,t}) - \tilde{f}_{i,t}(\tilde{\mathbf{u}}_t^*)\right) \\
&\leq \eta\nabla\tilde{f}_{i,t}(\mathbf{u}_{i,t})^\top(\mathbf{u}_{i,t} - \tilde{\mathbf{u}}_t^*) \\
&= \left(\nabla\phi(\mathbf{u}_{i,t}) - \nabla\phi(\mathbf{z}_{i,t}) - \eta\nabla\tilde{f}_{i,t}(\mathbf{u}_{i,t})\right)^\top(\tilde{\mathbf{u}}_t^* - \mathbf{z}_{i,t}) \\
&+ \left(\nabla\phi(\mathbf{z}_{i,t}) - \nabla\phi(\mathbf{u}_{i,t})\right)^\top(\tilde{\mathbf{u}}_t^* - \mathbf{z}_{i,t}) + \eta\nabla\tilde{f}_{i,t}(\mathbf{u}_{i,t})^\top(\mathbf{u}_{i,t} - \mathbf{z}_{i,t}) \\
&\leq \left(\nabla\phi(\mathbf{z}_{i,t}) - \nabla\phi(\mathbf{u}_{i,t})\right)^\top(\tilde{\mathbf{u}}_t^* - \mathbf{z}_{i,t}) + \eta\nabla\tilde{f}_{i,t}(\mathbf{u}_{i,t})^\top(\mathbf{u}_{i,t} - \mathbf{z}_{i,t}) \\
&= \mathcal{D}_\phi(\tilde{\mathbf{u}}_t^*, \mathbf{u}_{i,t}) - \mathcal{D}_\phi(\tilde{\mathbf{u}}_t^*, \mathbf{z}_{i,t}) - \mathcal{D}_\phi(\mathbf{z}_{i,t}, \mathbf{u}_{i,t}) + \eta\nabla\tilde{f}_{i,t}(\mathbf{u}_{i,t})^\top(\mathbf{u}_{i,t} - \mathbf{z}_{i,t}) \\
&\leq \mathcal{D}_\phi(\tilde{\mathbf{u}}_t^*, \mathbf{u}_{i,t}) - \mathcal{D}_\phi(\tilde{\mathbf{u}}_t^*, \mathbf{z}_{i,t}) - \mathcal{D}_\phi(\mathbf{z}_{i,t}, \mathbf{u}_{i,t}) + \frac{1}{2}\left\|\mathbf{u}_{i,t} - \mathbf{z}_{i,t}\right\|^2 + \frac{\eta^2}{2}\left\|\nabla\tilde{f}_{i,t}(\mathbf{u}_{i,t})\right\|^2 \\
&\leq \mathcal{D}_\phi(\tilde{\mathbf{u}}_t^*, \mathbf{u}_{i,t}) - \mathcal{D}_\phi(\tilde{\mathbf{u}}_t^*, \mathbf{z}_{i,t}) + \frac{\eta^2}{2}\left\|\nabla\tilde{f}_{i,t}(\mathbf{u}_{i,t})\right\|^2 \\
&= \mathcal{D}_\phi(\tilde{\mathbf{u}}_t^*, \mathbf{u}_{i,t}) - \mathcal{D}_\phi(\tilde{\mathbf{u}}_t^*, \mathbf{u}'_{i,t}) + \mathcal{D}_\phi(\tilde{\mathbf{u}}_t^*, \mathbf{u}'_{i,t}) - \mathcal{D}_\phi(\tilde{\mathbf{u}}_t^*, \mathbf{z}_{i,t}) + \frac{\eta^2}{2}\left\|\nabla\tilde{f}_{i,t}(\mathbf{u}_{i,t})\right\|^2 \\
&\leq \mathcal{D}_\phi(\tilde{\mathbf{u}}_t^*, \mathbf{u}_{i,t}) - \mathcal{D}_\phi(\tilde{\mathbf{u}}_t^*, \mathbf{u}'_{i,t}) + \sum_j[\mathbf{P}]_{ji}\mathcal{D}_\phi(\tilde{\mathbf{u}}_t^*, \mathbf{z}_{j,t-1}) - \mathcal{D}_\phi(\tilde{\mathbf{u}}_t^*, \mathbf{z}_{i,t}) + \frac{\eta^2}{2}\left\|\nabla\tilde{f}_{i,t}(\mathbf{u}_{i,t})\right\|^2,
\end{aligned}
\tag{46}
$$

where the second inequality is based on the optimality of $\mathbf{z}_{i,t}$; the fourth inequality is due to the strong convexity of $\phi(\cdot)$ and the fifth inequality is based on Assumption 9.

Based on Theorem 16, Lemma 17, and the Lipschitz assumption on $\mathcal{D}_\phi$, we have

$$
\left\|\mathcal{D}_\phi(\tilde{\mathbf{u}}_t^*, \mathbf{u}_{i,t}) - \mathcal{D}_\phi(\tilde{\mathbf{u}}_t^*, \mathbf{u}'_{i,t})\right\| \leq W\left\|\mathbf{u}_{i,t} - \mathbf{u}'_{i,t}\right\| \leq O\big(W(\frac{1}{T^{2\rho}} + \eta^{3/2})\big). \tag{47}
$$

And based on Lemma 14, we get

$$
\max_{i,j\in[m]}\left\|\mathbf{u}_{i,t} - \mathbf{u}_{j,t}\right\| = \max_{i,j\in[m]}\left\|q(\mathbf{x}_{i,t}) - q(\mathbf{x}_{j,t})\right\| = O\big(W\eta\big). \tag{48}
$$

With Equations (46), (47) and (48), we derive

$$
\begin{aligned}
\tilde{f}_{i,t}(\mathbf{u}_{j,t}) - \tilde{f}_{i,t}(\tilde{\mathbf{u}}_t^*) &= \tilde{f}_{i,t}(\mathbf{u}_{j,t}) - \tilde{f}_{i,t}(\mathbf{u}_{i,t}) + \tilde{f}_{i,t}(\mathbf{u}_{i,t}) - \tilde{f}_{i,t}(\tilde{\mathbf{u}}_t^*) \\
&\leq G_F\left\|\mathbf{u}_{i,t} - \mathbf{u}_{j,t}\right\| + O\big(W(\frac{1}{\eta T^{2\rho}} + \eta^{1/2})\big) \\
&+ \frac{1}{\eta}\sum_j[\mathbf{P}]_{ji}\mathcal{D}_\phi(\tilde{\mathbf{u}}_t^*, \mathbf{z}_{j,t-1}) - \frac{1}{\eta}\mathcal{D}_\phi(\tilde{\mathbf{u}}_t^*, \mathbf{z}_{i,t}) + \frac{\eta}{2}\left\|\nabla\tilde{f}_{i,t}(\mathbf{u}_{i,t})\right\|^2 \\
&\leq O\big(G_F W\eta\big) + O\big(W(\frac{1}{\eta T^{2\rho}} + \eta^{1/2})\big) \\
&+ \frac{1}{\eta}\sum_j[\mathbf{P}]_{ji}\mathcal{D}_\phi(\tilde{\mathbf{u}}_t^*, \mathbf{z}_{j,t-1}) - \frac{1}{\eta}\mathcal{D}_\phi(\tilde{\mathbf{u}}_t^*, \mathbf{z}_{i,t}) + \frac{\eta}{2}\left\|\nabla\tilde{f}_{i,t}(\mathbf{u}_{i,t})\right\|^2.
\end{aligned}
\tag{49}
$$

Based on the definition of Bregman divergence, we have the following relationship

$$
\begin{aligned}
&\mathcal{D}_\phi(\tilde{\mathbf{u}}_t^*, \mathbf{z}_{i,t-1}) - \mathcal{D}_\phi(\tilde{\mathbf{u}}_t^*, \mathbf{z}_{i,t}) \\
&= \left(\nabla\phi(\mathbf{z}_{i,t}) - \nabla\phi(\mathbf{z}_{i,t-1})\right)^\top(\tilde{\mathbf{u}}_t^* - \mathbf{z}_{i,t}) + \mathcal{D}_\phi(\mathbf{z}_{i,t}, \mathbf{z}_{i,t-1}) \\
&= \left(\nabla\phi(\mathbf{z}_{i,t}) - \nabla\phi(\mathbf{z}_{i,t-1})\right)^\top\tilde{\mathbf{u}}_t^* + \left(\phi(\mathbf{z}_{i,t}) - \nabla\phi(\mathbf{z}_{i,t})^\top\mathbf{z}_{i,t}\right) - \left(\phi(\mathbf{z}_{i,t-1}) - \nabla\phi(\mathbf{z}_{i,t-1})^\top\mathbf{z}_{i,t-1}\right).
\end{aligned}
\tag{50}
$$

Summing Equation (49) over i, based on Equation (50) we get

$$\begin{aligned}&\sum_i \tilde{f}_{i,t}(\mathbf{u}_{j,t}) - \tilde{f}_{i,t}(\tilde{\mathbf{u}}_t^*)\\ \leq & O\big(mG_F W\eta\big) + O\big(mW(\frac{1}{\eta T^{2\rho}} + \eta^{1/2})\big) + \sum_i \frac{\eta}{2}\big\|\nabla \tilde{f}_{i,t}(\mathbf{u}_{i,t})\big\|^2\\ &+\frac{1}{\eta}\sum_i \Big[\left(\nabla\phi(\mathbf{z}_{i,t}) - \nabla\phi(\mathbf{z}_{i,t-1})\right)^\top \tilde{\mathbf{u}}_t^* + \big(\phi(\mathbf{z}_{i,t}) - \nabla\phi(\mathbf{z}_{i,t})^\top \mathbf{z}_{i,t}\big) - \big(\phi(\mathbf{z}_{i,t-1}) - \nabla\phi(\mathbf{z}_{i,t-1})^\top \mathbf{z}_{i,t-1}\big)\Big].\end{aligned} \tag{51}$$

Then, by summing Equation (51) over $[T_s, \ldots, T]$, we have

$$\begin{aligned}&\sum_{t=T_s}^{T}\sum_i \tilde{f}_{i,t}(\mathbf{u}_{j,t}) - \tilde{f}_{i,t}(\tilde{\mathbf{u}}_t^*)\\ \leq & O\big(mTG_F W\eta\big) + O\big(mTW(\frac{1}{\eta T^{2\rho}} + \eta^{1/2})\big) + \sum_{t=T_s}^{T}\sum_i \frac{\eta}{2}\big\|\nabla \tilde{f}_{i,t}(\mathbf{u}_{i,t})\big\|^2\\ &+\frac{1}{\eta}\left[\sum_{t=T_s-1}^{T-1}\sum_i (\tilde{\mathbf{u}}_t^* - \tilde{\mathbf{u}}_{t+1}^*)^\top \nabla\phi(\mathbf{z}_{i,t}) + \sum_i \nabla\phi(\mathbf{z}_{i,T})^\top \tilde{\mathbf{u}}_T^* - \sum_i \nabla\phi(\mathbf{z}_{i,T_s-1})^\top \tilde{\mathbf{u}}_{T_s-1}^*\right]\\ &+\frac{1}{\eta}\sum_i \big[\big(\phi(\mathbf{z}_{i,T}) - \nabla\phi(\mathbf{z}_{i,T})^\top \mathbf{z}_{i,T}\big) - \big(\phi(\mathbf{z}_{i,T_s-1}) - \nabla\phi(\mathbf{z}_{i,T_s-1})^\top \mathbf{z}_{i,T_s-1}\big)\big].\end{aligned} \tag{52}$$

The upper bound of Term III:
Based on Lemma 9, we have for any $\mathbf{x}_t^* \in \mathcal{X}^s$ and its projection to $\mathcal{X}_{\text{in}}^s$: $\tilde{\mathbf{x}}_t^*$

$$\sum_{t=T_s}^{T}\sum_i \big(\tilde{f}_{i,t}(q(\tilde{\mathbf{x}}_t^*)) - \tilde{f}_{i,t}(q(\mathbf{x}_t^*))\big) \leq \sum_{t=T_s}^{T}\sum_i G_F W\big\|\tilde{\mathbf{x}}_t^* - \mathbf{x}_t^*\big\| \leq mTG_F W\frac{2\sqrt{d}L\mathcal{B}_r}{C(\mathbf{A},\mathbf{b})}. \tag{53}$$

Substituting Equations (45), (52) and (53) into Equation (44), the final regret bound is as

$$\begin{aligned}&\sum_{t=1}\sum_i \left(f_{i,t}(\mathbf{x}_{j,t}) - f_{i,t}(\mathbf{x}_t^*)\right)\\ \leq & O\big(mTG_F W\eta\big) + O\big(mTW(\frac{1}{\eta T^{2\rho}} + \eta^{1/2})\big) + \sum_{t=T_s}^{T}\sum_i \frac{\eta}{2}\big\|\nabla \tilde{f}_{i,t}(\mathbf{u}_{i,t})\big\|^2\\ &+\frac{1}{\eta}\left[\sum_{t=T_s-1}^{T-1}\sum_i (\tilde{\mathbf{u}}_t^* - \tilde{\mathbf{u}}_{t+1}^*)^\top \nabla\phi(\mathbf{z}_{i,t}) + \sum_i \nabla\phi(\mathbf{z}_{i,T})^\top \tilde{\mathbf{u}}_T^* - \sum_i \nabla\phi(\mathbf{z}_{i,T_s-1})^\top \tilde{\mathbf{u}}_{T_s-1}^*\right] + 2G_F WLm(T_0+T_1)\\ &+\frac{1}{\eta}\sum_i \big[\big(\phi(\mathbf{z}_{i,T}) - \nabla\phi(\mathbf{z}_{i,T})^\top \mathbf{z}_{i,T}\big) - \big(\phi(\mathbf{z}_{i,T_s-1}) - \nabla\phi(\mathbf{z}_{i,T_s-1})^\top \mathbf{z}_{i,T_s-1}\big)\big] + mTG_F W\frac{2\sqrt{d}L\mathcal{B}_r}{C(\mathbf{A},\mathbf{b})}\\ = & O(T_0 + T_1 + T\sqrt{\eta} + \frac{T\sqrt{\log T_0}}{\sqrt{T_0}} + \frac{1}{\eta} + \frac{1}{\eta}\sum_{t=T_s}^{T}\big\|\tilde{\mathbf{u}}_t^* - \tilde{\mathbf{u}}_{t+1}^*\big\|),\end{aligned} \tag{54}$$

where the final regret bound is proved by applying the specified η and T_0. By choosing ρ as a large enough number, $\frac{1}{\eta T^{2\rho}}$ is dominated by $\eta^{1/2}$. □