A Outline

The appendices are organized as follows:

- Notation are summarized in Appendix B.
- A detailed related work and a discussion on the limitations of Theorem 6 are given in Appendix C.
- The lower bound on the expected sample complexity under ϵ-global DP (Theorem 2) is proven in Appendix D.
- The proof of Lemma 4 is given in Appendix E.
- The proof of our concentration results are detailed in Appendix F. In particular, this includes the proof of Theorem 5.
- Appendix G gathers key properties on the (resp. modified) divergence $d_\epsilon^\pm$ (resp. $\widetilde{d}_\epsilon^\pm$), the (resp. modified) transportation costs $W_{\epsilon,a,b}$ (resp. $\widetilde{W}_{\epsilon,a,b}$) and (resp. β-)characteristic times $T_\epsilon^\star(\boldsymbol{\nu})$ (resp. $T_{\epsilon,\beta}^\star(\boldsymbol{\nu})$) and their (resp. β-)optimal allocation $w_\epsilon^\star(\boldsymbol{\nu})$ (resp. $w_{\epsilon,\beta}^\star(\boldsymbol{\nu})$). In particular, this includes the proof of Theorem 3 based on Lemmas 42 and 46.
- The proof of the upper bound on the asymptotic expected sample complexity of DP-TT (Theorem 6) is given in Appendix H.
- In Appendix I, we propose variants of algorithms to tackle ϵ-global DP BAI. We aim at providing several choices for the interested practitioners.
- Implementation details and additional experiments are presented in Appendix J.

Table 1: Notation for the setting.

Notation	Type	Description
K	$\mathbb{N}$	Number of arms
$\mathcal{F}$	$\subseteq \mathcal{P}([0,1])$	Class of Bernoulli distributions
ν_a	$\mathcal{F}$	Bernoulli distribution of arm $a \in [K]$
$\boldsymbol{\nu}$	$\mathcal{F}^K$	Vector of Bernoulli distributions, $\nu := (\nu_a)_{a\in[K]}$
μ_a	$(0,1)$	Mean of arm $a \in [K]$
μ	$(0,1)^K$	Vector of means, $\mu := (\mu_a)_{a\in[K]}$
$a^\star(\mu), a^\star(\boldsymbol{\nu})$	$\subseteq [K]$	Set of best arms, $a^\star(\boldsymbol{\nu}) = a^\star(\mu) := \arg\max_{a\in[K]} \mu_a$
$a^\star$	$[K]$	Unique best arm, i.e., $a^\star(\mu) = \{a^\star\}$
ϵ	$\mathbb{R}_+^\star$	Privacy budget for ϵ-global DP
δ	$(0,1)$	Risk for δ-correctness
$\mathrm{Alt}(\boldsymbol{\nu})$	$\subseteq \mathcal{F}^K$	Alternative instances with different best arms

B Notation

We recall some commonly used notation: the set of integers $[n] := \{1, \cdots, n\}$, the complement $X^\complement$ and interior $\mathring{X}$ of a set X, the indicator function $\mathbb{1}(X)$ of an event, the probability $\mathbb{P}_{\boldsymbol{\nu}\pi}$ and the expectation $\mathbb{E}_{\boldsymbol{\nu}\pi}$ taken over the randomness of the observations from $\boldsymbol{\nu}$ and the algorithm π, Landau's notation o, $\mathcal{O}$, Ω and Θ, the $(K-1)$-dimensional probability simplex $\triangle_K := \left\{w \in \mathbb{R}_+^K \mid w \geq 0, \sum_{i\in[K]} w_i = 1\right\}$. The functions $[x]_0^1 := \max\{0, \min\{1, x\}\}$, $k_\eta(x) := 1 + \log_{1+\eta} x$, $\overline{W}_{-1}$ in Lemma 51, h in Eq. (31), r in Eq. (33), ζ is the Riemann ζ function. Moreover, we recall the definitions: $d_\epsilon^\pm$ in Eq. (3), d_ϵ in Eq. (2), $\widetilde{d}_\epsilon^\pm$ in Eq. (32), $W_{\epsilon,a,b}^\pm$ in Eq. (4), $\widetilde{W}_{\epsilon,a,b}^\pm$ in Eq. (34), $(T_\epsilon^\star(\boldsymbol{\nu}), T_{\epsilon,\beta}^\star(\boldsymbol{\nu}), w_\epsilon^\star(\boldsymbol{\nu}), w_{\epsilon,\beta}^\star(\boldsymbol{\nu}))$ in Eq.35. While Table 1 gathers problem-specific notation, Table 2 groups notation for the algorithms.

Table 2: Notation for the algorithm.

Notation	Type	Description
B_n	$[K]$	(EB) Leader at time n
C_n	$[K]$	(TC) Challenger at time n
a_n	$[K]$	Arm sampled at time n
X_{n,a_n}	$\{0,1\}$	Sample observed at the end of time n, i.e. $X_{n,a_n} \sim \nu_{a_n}$
$Y_{k_{n,a},a}$	$\mathbb{R}$	Noisy perturbation drawn at the beginning of phase $k_{n,a}$ for arm a, i.e. $Y_{k_{n,a},a} \sim \text{Lap}(1/\epsilon)$
$\mathcal{F}_n$		History before time n
$\tilde{a}_n$	$[K]$	Arm recommended before time n
$\tau_{\epsilon,\delta}$	$\mathbb{N}$	Sample complexity (stopping time)
$c(n,\epsilon,\delta)$	$\mathbb{N} \times \mathbb{R}^\star_+ \times (0,1) \to \mathbb{R}^\star_+$	Stopping threshold function
$c_1(n,\delta)$	$\mathbb{N} \times (0,1) \to \mathbb{R}^\star_+$	Stopping threshold function
$c_2(n,\epsilon)$	$\mathbb{N} \times \mathbb{R}^\star_+ \to \mathbb{R}^\star_+$	Approximation threshold function
$N_{n,a}$	$\mathbb{N}$	Number of pulls of arm a before time n
$k_{n,a}$	$\mathbb{N}$	Current phase of arm a at time n
$T_k(a)$	$\mathbb{N}$	Time n where the arm a changes to phase k
$\tilde{S}_{k,a}$	$\mathbb{R}$	Private sum of observations for arm a at phase k
$\tilde{N}_{n,a}$	$\mathbb{N}$	Number of pulls of arm a at the beginning of phase $k_{n,a}$
$\tilde{\mu}_{n,a}$	$\mathbb{R}$	Private estimator of the empirical mean of arm a at the beginning of phase $k_{n,a}$
$L_{n,a}$	$\mathbb{N}$	Counts of $B_t = a$ before time n
$N^a_{n,a}$	$\mathbb{N}$	Counts of $(B_t, a_t) = (a,a)$ before time n
β	$(0,1)$	Fixed proportion

C Related Work and Limitations

We provide a more detailed literature review in Appendix C.1, and discuss limitations of Theorem 6 in Appendix C.2.

C.1 Related Work

Structured Bandits. While we consider unstructured bandits [6], numerous structural assumptions have been studied: linear bandits [77], generalized linear bandits [35] such as logistic bandits [53], combinatorial bandits [23], sparse bandits [46], spectral bandits [57], unimodal bandits [28], Lipschitz [65], partial monitoring [3], etc. Coping for the structural assumption while preserving ϵ-global DP is an interesting direction for future works.

Pure Exploration Problems. While we consider only BAI [33], other pure exploration problems have been studied in the literature: ϵ-BAI [66], thresholding bandits [19], Top-k identification [56], Pareto set identification [7], best partition identification [21], etc. Extending our ϵ-global DP results to answer these identification problems is an interesting research direction.

Performance Metrics. In pure exploration problems, the two major theoretical frameworks are the *fixed-confidence* setting [34, 47, 38], which is the focus of this paper, and the *fixed-budget* setting [4, 36]. In the fixed-budget setting, the objective is to minimize the probability of misidentifying a correct answer with a fixed number of samples T. Recent works have also considered the anytime setting, in which the agent aims at achieving a low probability of error at any deterministic time [88, 52]. Extending our findings to support ϵ-global DP in the fixed-budget or the anytime setting is an interesting direction for future works, see e.g., Chen et al. [24].

DP in bandits. DP has been studied for multi-armed bandits under different bandit settings: finite-armed stochastic [68, 74, 89, 44, 9, 43, 10, 85, 45], adversarial [80, 1, 82], linear [41, 60, 10], contextual linear [76, 69, 89, 10], and kernel bandits [70], among others. Most of these works were for regret minimisation, but the problem has also been explored for best-arm identification, with fixed confidence [11, 12] and fixed budget [24]. The problem has also been studied under three different DP trust models: (a) global DP where the users trust the centralised decision maker [68, 76, 74, 9, 43],

(b) local DP where each user deploys a local perturbation mechanism to send a "noisy" version of the rewards to the policy [15, 89, 40], and (c) shuffle DP where users still feed their data to a local perturbation, but now they trust an intermediary to apply a uniformly random permutation on all users' data before sending to the central servers [79, 37, 27].

In the first papers on DP for bandits, the tree-based mechanism [32, 20] was used to compute the sum of rewards privately. However, this mechanism was proven to be sub-optimal, matching the lower bounds up to logarithmic factors. Then, forgetting was first proposed by [74] to get rid of the tree-based mechanism, then adapted to UCB in [44, 9]. Finally, if the KL is the divergence that controls the complexity of bandits without privacy [58, 38], then Azize and Basu [9] were the first to show that the TV controls the complexity of private bandits, in the high privacy regime.

In this paper, *we focus on ϵ-pure DP, under a global trust model, in stochastic finite-armed bandits, for best arm identification under fixed confidence*.

Gap in the literature. This problem setting is first studied by Azize et al. [11], who proposed the first problem-dependent sample complexity lower bound, and introduced AdaP-TT, an ϵ-global DP version of the Top Algorithm. However, the sample complexity upper bound of AdaP-TT only matches the lower bound in *the high privacy regime* $\epsilon \to 0$, and for instances where the means have similar order (see Condition 1 in [11] in the discussion after Theorem 5 in [11]).

Azize et al. [12] proposes AdaP-TT*, an improved version of AdaP-TT. The improvement is achieved by using a transport inspired by the sample complexity lower bound from [11]. Using the new transport, AdaP-TT* gets rid of Condition 1 needed by AdaP-TT, and achieves the high privacy lower bound for all instances up to a multiplicative factor 48.

However, both AdaP-TT and AdaP-TT* do not match the lower bound, beyond the high privacy regime, i.e. for both the low privacy regime and transitional regimes.

C.2 Limitations of Theorem 6

Using adaptive targets $\beta_n(B_n, C_n)$ in DP-TT could replace $T^\star_{\epsilon,\beta}(\boldsymbol{\nu})$ by $T^\star_{\epsilon}(\boldsymbol{\nu})$. While we propose two adaptive choices of target based on IDS [86] or BOLD [14] (Appendix I), we leave their analysis for future work. The assumption that the means are distinct is used to prove sufficient exploration; it can be removed by using forced exploration or a fine-grained analysis [49, 52]. While it improves the asymptotic upper bound, choosing η too close to 0 negatively impacts the performance of DP-TT, due to the dependency in $\mathcal{O}(1/\log(1+\eta))$ of the stopping threshold. The suboptimal scaling in $2\log(1/\delta)$ of the stopping threshold yields the factor 2.

D Lower Bound

Let $\mathcal{M} : \mathcal{X}^n \to \mathcal{O}$ be an ϵ-DP mechanism. For $D \in \mathcal{X}^n$ an input dataset, we denote by $\mathcal{M}_D$ the distribution over outputs, when the input is D, and $\mathcal{M}_D(E)$ the probability of observing output E when the input is D.

Let $\mathbb{P}$ and $\mathbb{Q}$ be two data-generating distributions over $\mathcal{X}^n$. We denote by $\mathbb{M}_{\mathbb{P},\mathcal{M}}$ the marginal over outputs of the mechanism $\mathcal{M}$ when the input dataset is generated through $\mathbb{P}$, i.e.

$$\mathbb{M}_{\mathbb{P},\mathcal{M}}(A) := \int_{D\in\mathcal{X}^n} \mathcal{M}_D\left(A\right) \mathrm{d}\mathbb{P}\left(D\right) \ , \tag{9}$$

for any event A in the output space. We define similarly $\mathbb{M}_{\mathbb{Q},\mathcal{M}}$ the marginal over outputs of the mechanism $\mathcal{M}$ when the input dataset is generated through $\mathbb{Q}$.

The main question is to control the divergence $\mathrm{KL}\left(\mathbb{M}_{\mathbb{P},\mathcal{M}} \,\|\, \mathbb{M}_{\mathbb{Q},\mathcal{M}}\right)$ when the mechanism $\mathcal{M}$ satisfies DP. In general, for any mechanism $\mathcal{M}$, the data-processing inequality provides the following bound

$$\mathrm{KL}\left(\mathbb{M}_{\mathbb{P},\mathcal{M}} \,\|\, \mathbb{M}_{\mathbb{Q},\mathcal{M}}\right) \le \mathrm{KL}\left(\mathbb{P} \,\|\, \mathbb{Q}\right) \ . \tag{10}$$

Now, for ϵ-DP mechanisms, we want to translate the DP constraint to a tight bound on the divergence $\mathrm{KL}\left(\mathbb{M}_{\mathbb{P},\mathcal{M}} \,\|\, \mathbb{M}_{\mathbb{Q},\mathcal{M}}\right)$. To do so, let $\mathbb{L}$ be any other distribution on $\mathcal{X}^n$. Let $\mathbb{C}_{\mathbb{P},\mathbb{L}}$ be a coupling of $(\mathbb{P}, \mathbb{L})$, i.e., the marginals of $\mathbb{C}_{\mathbb{P},\mathbb{L}}$ are $\mathbb{P}$ and $\mathbb{L}$. We can now rewrite our the marginals using the definition of couplings. For $\mathbb{M}_{\mathbb{P},\mathcal{M}}$, we have

$$\mathbb{M}_{\mathbb{P},\mathcal{M}}(A) := \int_{D\in\mathcal{X}^n} \mathcal{M}_D(A)\ \mathrm{d}\mathbb{P}(D) = \int_{D,D'\in\mathcal{X}^n} \mathcal{M}_D(A)\ \mathrm{d}\mathbb{C}_{\mathbb{P},\mathbb{L}}(D,D')\ ,$$

and for $\mathbb{Q}$ we get

$$\begin{aligned}\mathbb{M}_{\mathbb{Q},\mathcal{M}}(A) := \int_{D'\in\mathcal{X}^n} \mathcal{M}_{D'}(A)\ \mathrm{d}\mathbb{Q}(D') &= \int_{D'\in\mathcal{X}^n} \mathcal{M}_{D'}(A)\ \frac{\mathrm{d}\mathbb{Q}(D')}{\mathrm{d}\mathbb{L}(D')}\,\mathrm{d}\mathbb{L}(D') \\ &= \int_{D,D'\in\mathcal{X}^n} \mathcal{M}_{D'}(A)\ \frac{\mathrm{d}\mathbb{Q}(D')}{\mathrm{d}\mathbb{L}(D')}\,\mathrm{d}\mathbb{C}_{\mathbb{P},\mathbb{L}}(D,D')\ .\end{aligned}$$

Using the data-processing inequality, we get

$$\begin{aligned}\mathrm{KL}(\mathbb{M}_{\mathbb{P},\mathcal{M}} \parallel \mathbb{M}_{\mathbb{Q},\mathcal{M}}) &\le \int_{D,D'\in\mathcal{X}^n}\int_{o\in\mathcal{O}} \log\left(\frac{\mathcal{M}_D(o)}{\mathcal{M}_{D'}(o)\,\frac{\mathrm{d}\mathbb{Q}(D')}{\mathrm{d}\mathbb{L}(D')}}\right)\mathcal{M}_D(o)\ \mathrm{d}o\,\mathrm{d}\mathbb{C}_{\mathbb{P},\mathbb{L}}(D,D') \\ &= \int_{D,D'\in\mathcal{X}^n}\left(\mathrm{KL}(\mathcal{M}_D \parallel \mathcal{M}_{D'}) + \log\left(\frac{\mathrm{d}\mathbb{L}(D')}{\mathrm{d}\mathbb{Q}(D')}\right)\right)\ \mathrm{d}\mathbb{C}_{\mathbb{P},\mathbb{L}}(D,D') \\ &= \mathbb{E}_{D,D'\sim\mathbb{C}_{\mathbb{P},\mathbb{L}}}\left[\mathrm{KL}(\mathcal{M}_D \parallel \mathcal{M}_{D'})\right] + \mathrm{KL}(\mathbb{L} \parallel \mathbb{Q})\ .\end{aligned}$$

Since this is true for any coupling $\mathbb{C}_{\mathbb{P},\mathbb{L}}$ and any distribution $\mathbb{L}$, we get the final bound

$$\mathrm{KL}(\mathbb{M}_{\mathbb{P},\mathcal{M}} \parallel \mathbb{M}_{\mathbb{Q},\mathcal{M}}) \le \inf_{\mathbb{L}\in\mathcal{P}(\mathcal{X}^n)}\left\{\inf_{\mathbb{C}_{\mathbb{P},\mathbb{L}}\in\mathcal{C}(\mathbb{P},\mathbb{L})}\left\{\mathbb{E}_{D,D'\sim\mathbb{C}_{\mathbb{P},\mathbb{L}}}\left[\mathrm{KL}(\mathcal{M}_D \parallel \mathcal{M}_{D'})\right]\right\} + \mathrm{KL}(\mathbb{L} \parallel \mathbb{Q})\right\}$$

where $\mathcal{P}(\mathcal{X}^n)$ is the set of all distributions over $\mathcal{X}^n$ and $\mathcal{C}(\mathbb{P},\mathbb{L})$ is the set of all couplings between $\mathbb{P}$ and $\mathbb{L}$. Using that the $\mathcal{M}$ is ϵ-DP, we can use the simple bound $\mathrm{KL}(\mathcal{M}_D \parallel \mathcal{M}_{D'}) \le \epsilon d_{\mathrm{Ham}}(D,D')$ which gives

$$\mathrm{KL}(\mathbb{M}_{\mathbb{P},\mathcal{M}} \parallel \mathbb{M}_{\mathbb{Q},\mathcal{M}}) \le \inf_{\mathbb{L}\in\mathcal{P}(\mathcal{X}^n)}\left\{\epsilon \inf_{\mathbb{C}_{\mathbb{P},\mathbb{L}}\in\mathcal{C}(\mathbb{P},\mathbb{L})}\left\{\mathbb{E}_{D,D'\sim\mathbb{C}_{\mathbb{P},\mathbb{L}}}\left[d_{\mathrm{Ham}}(D,D')\right]\right\} + \mathrm{KL}(\mathbb{L} \parallel \mathbb{Q})\right\}\ . \tag{11}$$

D.1 Product Distributions

Suppose that $\mathbb{P} := \bigotimes_{i=1}^n \mathbb{P}_i$ and $\mathbb{Q} := \bigotimes_{i=1}^n \mathbb{Q}_i$ are product distributions. Consider the subset of product distributions $\mathbb{L} := \bigotimes_{i=1}^n \mathbb{L}_i$, and the maximal coupling $\mathbb{C}_\infty(\mathbb{P},\mathbb{L}) := \prod_{i=1}^n \mathbb{C}_\infty(\mathbb{P}_i,\mathbb{L}_i)$. Plugging these in Equation (11), we get

$$\begin{aligned}\mathrm{KL}(\mathbb{M}_{\mathbb{P},\mathcal{M}} \parallel \mathbb{M}_{\mathbb{Q},\mathcal{M}}) &\le \inf_{\mathbb{L}_1,\dots,\mathbb{L}_n}\left\{\epsilon\sum_{i=1}^n \mathbb{E}_{D_i,D_i'\sim\mathbb{C}_\infty(\mathbb{P}_i,\mathbb{L}_i)}\left[\mathbb{1}\{D_i \neq D_i'\}\right] + \sum_{i=1}^n \mathrm{KL}(\mathbb{L}_i \parallel \mathbb{Q}_i)\right\} \\ &= \inf_{\mathbb{L}_1,\dots,\mathbb{L}_n}\left\{\sum_{i=1}^n\left(\epsilon\,\mathrm{TV}(\mathbb{P}_i \parallel \mathbb{L}_i) + \mathrm{KL}(\mathbb{L}_i \parallel \mathbb{Q}_i)\right)\right\} \\ &= \sum_{i=1}^n \inf_{\mathbb{L}_i\in\mathcal{P}(\mathcal{X})}\left\{\epsilon\,\mathrm{TV}(\mathbb{P}_i \parallel \mathbb{L}_i) + \mathrm{KL}(\mathbb{L}_i \parallel \mathbb{Q}_i)\right\} = \sum_{i=1}^n \mathrm{d}_\epsilon(\mathbb{P}_i,\mathbb{Q}_i)\ ,\end{aligned}$$

where

$$\mathrm{d}_\epsilon(\mathbb{P},\mathbb{Q}) := \inf_{\mathbb{L}\in\mathcal{P}(\mathcal{X})}\left\{\epsilon\,\mathrm{TV}(\mathbb{P} \parallel \mathbb{L}) + \mathrm{KL}(\mathbb{L} \parallel \mathbb{Q})\right\}\ . \tag{12}$$

D.2 Sequential KL decomposition for bandits under DP

In this section, we adapt the techniques from product distributions to bandit marginals.

First, we introduce the bandit canonical model.

The bandit canonical model. A stochastic bandit (or environment) is a collection of distributions $\nu \triangleq (P_a : a \in [K])$, where $[K]$ is the set of available K actions. The learner and the environment interact

Algorithm 2 Bandit interaction between a policy and an environment

```
Input: A policy π and an environment ν ≜ (P_a : a ∈ [K])
for t = 1, ... do
    The policy samples an action a_t ∼ π_t(. | a_1, r_1, ..., a_{t−1}, r_{t−1})
    The policy observes a reward r_t ∼ P_{a_t}
end for
if Regret minimisation then
    The interaction ends after T steps
else FC-BAI
    The policy decides to stop the interaction at step τ_{ε,δ} and recomends the final guess â
end if
```

sequentially over T rounds. In each round $t \in 1, \ldots, T$, the learner chooses an action $a_t \in [K]$, which is fed to the environment. The environment then samples a reward $r_t \in \mathbb{R}$ from distribution P_{a_t} and reveals r_t to the learner. The interaction between the learner (or policy) and environment induces a probability measure on the sequence of outcomes $H_T \triangleq (a_1, r_1, a_2, r_2, \ldots, a_T, r_T)$. In the following, we construct the probability space that carries these random variables.

Let $T \in \mathbb{N}^\star$ be the horizon. Let $\nu = (P_a : a \in [K])$ a bandit instance with $K \in \mathbb{N}^\star$ finite arms, and each P_a is a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with $\mathfrak{B}$ being the Borel set. For each $t \in [T]$, let $\Omega_t = ([K] \times \mathbb{R})^t \subset \mathbb{R}^{2t}$ and $\mathcal{F}_t = \mathfrak{B}(\Omega_t)$. We first formalise the definition of a policy.

Definition 2 (The policy). *A policy π is a sequence $(\pi_t)_{t=1}^T$, where π_t is a probability kernel from $(\Omega_t, \mathcal{F}_t)$ to $([K], 2^{[K]})$. Since $[K]$ is discrete, we adopt the convention that for $a \in [K]$,*

$$\pi_t(a \mid a_1, r_1, \ldots, a_{t-1}, r_{t-1}) = \pi_t(\{a\} \mid a_1, r_1, \ldots, a_{t-1}, r_{t-1})$$

We want to define a probability measure on $(\Omega_T, \mathcal{F}_T)$ that respects our understanding of the sequential nature of the interaction between the learner and a stationary stochastic bandit. Specifically, the sequence of outcomes should satisfy the following two assumptions:

(a) The conditional distribution of action a_t given $a_1, r_1, \ldots, a_{t-1}, r_{t-1}$ is $\pi(a_t \mid H_{t-1})$ almost surely.

(b) The conditional distribution of reward r_t given $a_1, r_1, \ldots, a_{t-1}, r_{t-1}, a_t$ is P_{a_t} almost surely.

The probability measure on $(\Omega_T, \mathcal{F}_T)$ depends on both the environment ν and the policy π. To construct this probability, let λ be a σ-finite measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ for which P_a is absolutely continuous with respect to λ for all $a \in [K]$. Let $p_a = dP_a/d\lambda$ be the Radon–Nikodym derivative of P_a with respect to λ. Letting ρ be the counting measure with $\rho(B) = |B|$, the density $p_{\nu\pi} : \Omega_T \to \mathbb{R}$ can now be defined with respect to the product measure $(\rho \times \lambda)^T$ by

$$p_{\nu\pi}(a_1, r_1, \ldots, a_T, r_T) \triangleq \prod_{t=1}^{T} \pi_t(a_t \mid a_1, r_1, \ldots, a_{t-1}, r_{t-1}) p_{a_t}(r_t)$$

and $\mathcal{P}_{\nu\pi}$ is defined as

$$\mathbb{P}_{\nu\pi}(B) \triangleq \int_B p_{\nu\pi}(\omega)(\rho \times \lambda)^T(\,\mathrm{d}\omega) \quad \mathrm{forall} B \in \mathcal{F}_T$$

Hence $(\Omega_T, \mathcal{F}_T, \mathbb{P}_{\nu\pi})$ is a probability space over histories induced by the interaction between π and ν. We define also a marginal distribution over the sequence of actions by

$$m_{\nu\pi}(a_1, \ldots, a_T) \triangleq \int_{r_1, \ldots, r_T} p_{\nu\pi}(a_1, r_1, \ldots, a_T, r_T)\,\mathrm{d}r_1 \ldots \mathrm{d}r_T,$$

and $\mathrm{forall} C \in \mathcal{P}([K]^T)$,

$$\mathbb{M}_{\nu\pi}(C) \triangleq \sum_{(a_1, \ldots, a_T) \in C} m_{\nu\pi}(a_1, a_2, \ldots, a_T).$$

Finally, $([K]^T, \mathcal{P}([K]^T), \mathbb{M}_{\nu\pi})$ is a probability space over sequence of actions produced when π interacts with ν for T time-steps.

The KL upper bound. Now, we adapt the techniques for the bandit marginals. Let $\nu = \{P_a, a \in [K]\}$ and $\nu' = \{P'_a, a \in [K]\}$ be two bandit instances in $\mathcal{F}^K$. We recall that, when the policy π interacts with the bandit instance ν, it induces a marginal distribution $\mathbb{M}_{\nu\pi}$ over the sequence of actions. We define $\mathbb{M}_{\nu'\pi}$ similarly.

The goal is to upper bound the quantity $\mathrm{KL}\left(\mathbb{M}_{\nu\pi} \,\|\, \mathbb{M}_{\nu'\pi}\right)$. The marginals $\mathbb{M}_{\nu\pi}$ and $\mathbb{M}_{\nu\pi}$ in the sequential setting "look like" marginals generated by "product distributions". However, the hardness of the sequential setting lies in the fact that the data-generating distributions depend on the stochastic sequential actions chosen. Thus, the results of the previous section cannot be directly applied. To adapt the proof ideas of the previous section to the bandit case, we introduce the idea of a coupled bandit instance.

Let $\nu'' = \{P''_a : a \in [K]\}$ be any “intermediary" bandit instance from $\mathcal{F}^K$. Define c_a as the maximal coupling between P_a and P''_a, i.e., $c_a := \mathbb{C}_\infty(P_a, P''_a)$. Fix a policy $\pi = \{\pi_t\}_{t=1}^T$.

Here, we build a coupled environment γ of ν and ν''. The policy π interacts with the coupled environment γ up to a given time horizon T to produce an augmented history $\{(a_t, r_t, r''_t)\}_{t=1}^T$. The iterative steps of this interaction process are:

1. The probability of choosing an action $a_t = a$ at time t is dictated only by the policy π_t and $a_1, r_1, a_2, r_2, \ldots, a_{t-1}, r_{t-1}$, *i.e.*the policy ignores $\{r''_s\}_{s=1}^{t-1}$.
2. The distribution of rewards (r_t, r''_t) is c_{a_t} and is conditionally independent of the previous observed history $\{(a_s, r_s, r''_s)\}_{t=1}^{t-1}$.

This interaction is similar to the interaction process of policy π with the first bandit instance ν, with the addition of sampling an extra r''_t from the coupling of P_{a_t} and P''_{a_t}.

The distribution of the augmented history induced by the interaction of π and the coupled environment can be defined as

$$p_{\gamma\pi}(a_1, r_1, r''_1 \ldots, a_T, r_T, r''_T) := \prod_{t=1}^{T} \pi_t(a_t \mid a_1, r_1, \ldots, a_{t-1}, r_{t-1}) c_{a_t}(r_t, r''_t)\,.$$

To simplify the notation, let $\mathbf{a} := (a_1, \ldots, a_T)$, $\mathbf{r} := (r_1, \ldots, r_T)$, $\mathbf{r'} := (r'_1, \ldots, r'_T)$ and $\mathbf{r''} := (r'_1, \ldots, r'_T)$. Also, let $c_{\mathbf{a}}(\mathbf{r}, \mathbf{r''}) := \prod_{t=1}^T c_{a_t}(r_t, r''_t)$ and $\pi(\mathbf{a} \mid \mathbf{r}) := \prod_{t=1}^T \pi_t(a_t \mid a_1, r_1, \ldots, a_{t-1}, r_{t-1})$. We put $\mathbf{h} := (\mathbf{a}, \mathbf{r}, \mathbf{r''})$. With the new notation

$$p_{\gamma\pi}(\mathbf{a}, \mathbf{r}, \mathbf{r''}) := \pi(\mathbf{a} \mid \mathbf{r}) c_{\mathbf{a}}(\mathbf{r}, \mathbf{r''})\,.$$

By the definition of the couplings, we have that $m_{\nu\pi}$ is the marginal of $p_{\gamma\pi}$ when integrated over $(\mathbf{r}, \mathbf{r''})$, i.e.,

$$m_{\nu\pi}(\mathbf{a}) = \int_{\mathbf{r}, \mathbf{r''}} p_{\gamma\pi}(\mathbf{a}, \mathbf{r}, \mathbf{r''})\, \mathrm{d}\mathbf{r}\, \mathrm{d}\mathbf{r''}\,.$$

Now, we define a new joint distribution $q_{\gamma\pi}$, inspired by the techniques used for product distributions:

$$q_{\gamma\pi}(\mathbf{a}, \mathbf{r}, \mathbf{r''}) := \pi(\mathbf{a} \mid \mathbf{r''}) \frac{p'_{\mathbf{a}}(\mathbf{r''})}{p''_{\mathbf{a}}(\mathbf{r''})} c_{\mathbf{a}}(\mathbf{r}, \mathbf{r''})\,,$$

where $p'_{\mathbf{a}}(\mathbf{r''}) := \prod_{t=1}^T p'_{a_t}(r''_t)$, and similarly, $p''_{\mathbf{a}}(\mathbf{r''}) := \prod_{t=1}^T p''_{a_t}(r''_t)$.

First, observe that it is indeed a valid joint distribution, i.e.

$$\begin{aligned}
\sum_{\mathbf{a}} \int_{\mathbf{r}, \mathbf{r''}} q_{\gamma\pi}(\mathbf{a}, \mathbf{r}, \mathbf{r''})\, \mathrm{d}\mathbf{r}\, \mathrm{d}\mathbf{r''} &= \sum_{\mathbf{a}} \int_{\mathbf{r}, \mathbf{r''}} \pi(\mathbf{a} \mid \mathbf{r''}) \frac{p'_{\mathbf{a}}(\mathbf{r''})}{p''_{\mathbf{a}}(\mathbf{r''})} c_{\mathbf{a}}(\mathbf{r}, \mathbf{r''})\, \mathrm{d}\mathbf{r}\, \mathrm{d}\mathbf{r''} \\
&= \sum_{\mathbf{a}} \int_{\mathbf{r''}} \pi(\mathbf{a} \mid \mathbf{r''}) p'_{\mathbf{a}}(\mathbf{r''})\, \mathrm{d}\mathbf{r''} = \int_{\mathbf{r''}} p'_{\mathbf{a}}(\mathbf{r''})\, \mathrm{d}\mathbf{r''} = 1\,,
\end{aligned}$$

and that $m_{\nu'\pi}$ is the marginal of $q_{\gamma\pi}$ when integrated over $(\mathbf{r}, \mathbf{r''})$, i.e.,

$$\int_{\mathbf{r}, \mathbf{r''}} q_{\gamma\pi}(\mathbf{a}, \mathbf{r}, \mathbf{r''})\, \mathrm{d}\mathbf{r}\, \mathrm{d}\mathbf{r''} = \int_{\mathbf{r}, \mathbf{r''}} \pi(\mathbf{a} \mid \mathbf{r''}) \frac{p'_{\mathbf{a}}(\mathbf{r''})}{p''_{\mathbf{a}}(\mathbf{r''})} c_{\mathbf{a}}(\mathbf{r}, \mathbf{r''})\, \mathrm{d}\mathbf{r}\, \mathrm{d}\mathbf{r''}$$

$$= \int_{\mathbf{r}''} \pi(\mathbf{a} \mid \mathbf{r}'') p'_{\mathbf{a}}(\mathbf{r}'') \, \mathrm{d}\mathbf{r}'' = m_{\nu'\pi}(\mathbf{a}) \, .$$

Using the data-processing inequality, we get

$$\mathrm{KL}\left(\mathbb{M}_{\nu\pi} \parallel \mathbb{M}_{\nu'\pi}\right) \leq \mathrm{KL}\left(p_{\gamma\pi} \parallel q_{\gamma\pi}\right) \, . \tag{13}$$

Now, we compute

$$\begin{aligned}
\mathrm{KL}\left(p_{\gamma\pi} \parallel q_{\gamma\pi}\right) &\overset{(a)}{=} \mathbb{E}_{\mathbf{h}:=(\mathbf{a},\mathbf{r},\mathbf{r}'')\sim p_{\gamma\pi}} \left[\log \left(\frac{\pi(\mathbf{a} \mid \mathbf{r}) c_{\mathbf{a}}(\mathbf{r}, \mathbf{r}'')}{\pi(\mathbf{a} \mid \mathbf{r}'') \frac{p'_{\mathbf{a}}(\mathbf{r}'')}{p''_{\mathbf{a}}(\mathbf{r}'')} c_{\mathbf{a}}(\mathbf{r}, \mathbf{r}'')} \right) \right] \\
&\overset{(b)}{\leq} \mathbb{E}_{\mathbf{h}:=(\mathbf{a},\mathbf{r},\mathbf{r}'')\sim p_{\gamma\pi}} \left[\epsilon d_{\mathrm{Ham}}(\mathbf{r}, \mathbf{r}'') + \log \left(\frac{p''_{\mathbf{a}}(\mathbf{r}'')}{p'_{\mathbf{a}}(\mathbf{r}'')} \right) \right] \\
&\overset{(c)}{=} \sum_{t=1}^{T} \mathbb{E}_{\mathbf{h}\sim p_{\gamma\pi}} \left[\epsilon \mathbb{1}\left\{ r_t \neq r''_t \right\} + \log \left(\frac{p''_{a_t}(r''_t)}{p'_{a_t}(r''_t)} \right) \right] \\
&\overset{(d)}{=} \sum_{t=1}^{T} \mathbb{E}_{\mathbf{h}\sim p_{\gamma\pi}} \left[\mathbb{E}_{\mathbf{h}\sim p_{\gamma\pi}} [\epsilon \mathbb{1}\left\{ r_t \neq r''_t \right\} + \log \left(\frac{p''_{a_t}(r''_t)}{p'_{a_t}(r''_t)} \right) \mid a_t] \right] \\
&\overset{(e)}{=} \sum_{t=1}^{T} \mathbb{E}_{\mathbf{h}\sim p_{\gamma\pi}} \left[\epsilon \mathrm{TV}\left(p_{a_t} \parallel p''_{a_t}\right) + \mathrm{KL}\left(p''_{a_t} \parallel p'_{a_t}\right) \right] \\
&\overset{(f)}{=} \mathbb{E}_{\nu\pi} \left[\sum_{t=1}^{T} \epsilon \mathrm{TV}\left(p_{a_t} \parallel p''_{a_t}\right) + \mathrm{KL}\left(p''_{a_t} \parallel p'_{a_t}\right) \right] \, .
\end{aligned}$$

where:

(a) by the definition of the KL

(b) the group privacy property, applied to the ϵ-global DP policy, we have

$$\pi(\mathbf{a} \mid \mathbf{r}) \leq e^{\epsilon d_{\mathrm{Ham}}(\mathbf{r}, \mathbf{r}''} \pi(\mathbf{a} \mid \mathbf{r}'')$$

(c) by the definition of dham

(d) by the towering property of conditional expectations

(e) given a_t, we have $r_t \sim p_{a_t}$, $r'_t \sim p'_{a_t}$ and $r" \sim p''_{a_t}$

(f) by linearity of the expectation, and the fact that the expression inside the expectation only depends on the actions a_t

Since this is true for any "intermediary" bandit instance $\nu'' \in \mathcal{F}^K$, we take $\nu''_\star$ to be the environment where the infinimum of the $d_\epsilon(P_a, P'_a)$ is attained for each arm $a \in [K]$. Specifically, let $\nu''_\star = (p^\star_a, a \in [K])$ where

$$p^\star_a = \operatorname*{arg\,min}_{\mathbb{L} \in \mathcal{F}} \left\{ \epsilon \, \mathrm{TV}\left(p_a \parallel \mathbb{L}\right) + \mathrm{KL}\left(\mathbb{L} \parallel p'_a\right) \right\}$$

Plugging $\nu''_\star$ gives

$$\mathrm{KL}\left(\mathbb{M}_{\nu\pi} \parallel \mathbb{M}_{\nu'\pi}\right) \leq \mathbb{E}_{\nu\pi} \left[\sum_{t=1}^{T} \mathrm{d}_\epsilon(p_{a_t}, p'_{a_t}) \right] \tag{14}$$

Let $N_{t,a} = \sum_{s<t} \mathbb{1}\left\{a_s = a\right\}$ be the counts of arm a before step t. Then, we can rewrite the bound as

$$\mathrm{KL}\left(\mathbb{M}_{\nu\pi} \parallel \mathbb{M}_{\nu'\pi}\right) \leq \sum_{a=1}^{K} \mathbb{E}_{\nu\pi}[N_{T+1,a}] \mathrm{d}_\epsilon(p_a, p'_a) \, , \tag{15}$$

Stopping time version of the KL decomposition for BAI under DP. Let π be an ϵ-DP BAI strategy. Let ν and λ be two bandit instances. Denote by $\mathbb{M}_{\nu\pi}$ the marginal distribution of the output of the

BAI strategy when π interacts with ν. By using Wald's lemma in the proof technique seen before for the canonical bandit setting under FC-BAI, we get that

$$\mathrm{KL}\left(\mathbb{M}_{\nu\pi} \parallel \mathbb{M}_{\lambda\pi}\right) \leq \mathbb{E}_{\nu\pi}\left(\sum_{t=1}^{\tau_{\epsilon,\delta}} \mathrm{d}_{\epsilon}(\nu_{a_t}, \lambda_{a_t})\right) = \sum_{a=1}^{K} \mathbb{E}_{\nu\pi}[N_{\tau_{\epsilon,\delta}+1,a}]\mathrm{d}_{\epsilon}(\nu_a, \lambda_a)\,, \tag{16}$$

where τ is the stopping time.

D.3 Sample Complexity Lower Bound Proof

Theorem 2 (Sample complexity lower bound for BAI under ϵ-DP)**.** *Let $(\epsilon, \delta) \in \mathbb{R}_+^\star \times (0,1)$. For any algorithm π that is δ-correct and ϵ-global DP on $\mathcal{F}^K$,*

$$\mathbb{E}_{\boldsymbol{\nu}\pi}[\tau_{\epsilon,\delta}] \geq T_\epsilon^\star(\boldsymbol{\nu}) \log(1/(3\delta))$$

for all $\boldsymbol{\nu} \in \mathcal{F}^K$ with unique best arm. The inverse of the characteristic time $T_\epsilon^\star(\boldsymbol{\nu})$ *is defined as*

$$T_\epsilon^\star(\boldsymbol{\nu})^{-1} := \sup_{w \in \triangle_K} \inf_{\boldsymbol{\kappa} \in \mathrm{Alt}(\boldsymbol{\nu})} \sum_{a=1}^{K} w_a \mathrm{d}_\epsilon(\nu_a, \kappa_a)\,, \tag{17}$$

$$\mathrm{d}_\epsilon(\nu_a, \kappa_a) := \inf_{\varphi_a \in \mathcal{F}} \left\{\mathrm{KL}\left(\varphi_a \parallel \kappa_a\right) + \epsilon \cdot \mathrm{TV}\left(\nu_a \parallel \varphi_a\right)\right\}\,. \tag{18}$$

Proof. Let π be an ϵ-global DP δ-correct BAI strategy. Let ν be a bandit instance and $\lambda \in \mathrm{Alt}(\nu)$.

Let $\mathbb{M}_{\nu\pi}$ denote the probability distribution of the output when the BAI strategy π interacts with ν. For any alternative instance $\lambda \in \mathrm{Alt}(\nu)$, the data-processing inequality gives that

$$\mathrm{KL}\left(\mathbb{M}_{\nu\pi} \parallel \mathbb{M}_{\lambda,\pi}\right) \geq \mathrm{kl}\left(\mathbb{M}_{\nu\pi}\left(\tilde{a} = a^\star(\boldsymbol{\nu})\right), \mathbb{M}_{\lambda,\pi}\left(\tilde{a} = a^\star(\boldsymbol{\nu})\right)\right) \geq \mathrm{kl}(1-\delta, \delta)\,, \tag{19}$$

where the second inequality is because π is δ-correct, i.e., $\mathbb{M}_{\nu\pi}\left(\tilde{a} = a^\star(\boldsymbol{\nu})\right) \geq 1-\delta$ and $\mathbb{M}_{\lambda,\pi}\left(\tilde{a} = a^\star(\boldsymbol{\nu})\right) \leq \delta$, and the monotonicity of the kl. Now, using the stopping time version of the KL decomposition for FC-BAI, we get that

$$\mathrm{kl}(1-\delta,\delta) \leq \mathrm{KL}\left(\mathbb{M}_{\nu,\pi} \parallel \mathbb{M}_{\lambda,\pi}\right) \leq \sum_{a=1}^{K} \mathbb{E}_{\nu\pi}[N_{\tau_{\epsilon,\delta}+1,a}]\mathrm{d}_\epsilon(\nu_a, \lambda_a)\,.$$

Since this is true for all $\lambda \in \mathrm{Alt}(\nu)$, we get

$$\begin{aligned}\mathrm{kl}(1-\delta,\delta) \leq \inf_{\lambda \in \mathrm{Alt}(\nu)} \sum_{a=1}^{K} \mathbb{E}_{\nu\pi}[N_{\tau_{\epsilon,\delta}+1,a}]\mathrm{d}_\epsilon(\nu_a,\lambda_a) &\stackrel{(a)}{=} \mathbb{E}[\tau_{\epsilon,\delta}] \inf_{\lambda \in \mathrm{Alt}(\nu)} \sum_{a=1}^{K} \frac{\mathbb{E}\left[N_{\tau_{\epsilon,\delta}+1,a}\right]}{\mathbb{E}[\tau_{\epsilon,\delta}]} \mathrm{d}_\epsilon(\nu_a,\lambda_a) \\ &\stackrel{(b)}{\leq} \mathbb{E}[\tau_{\epsilon,\delta}]\left(\sup_{\omega \in \triangle_K} \inf_{\lambda \in \mathrm{Alt}(\nu)} \sum_{a=1}^{K} \omega_a \mathrm{d}_\epsilon(\nu_a,\lambda_a)\right)\,.\end{aligned}$$

(a) is due to the fact that $\mathbb{E}[\tau_{\epsilon,\delta}]$ does not depend on λ. (b) is obtained by noting that the vector $(\omega_a)_{a \in [K]} \triangleq \left(\frac{\mathbb{E}_{\nu,\pi}\left[N_{\tau_{\epsilon,\delta}+1,a}\right]}{\mathbb{E}_{\nu,\pi}[\tau_{\epsilon,\delta}]}\right)_{a \in [K]}$ belongs to the simplex $\triangle_K$. The theorem follows by noting that for $\delta \in (0,1), \mathrm{kl}(1-\delta,\delta) \geq \log(1/3\delta)$. $\square$

E Privacy Analysis

In this section, we prove Lemma 4. First, we justify using a geometric grid for updating the means (Lemma 7). Second, we obtain Lemma 4 as a combination of Lemma 7 and the post-processing property of DP (Proposition 1).

E.1 Releasing partial sums privately

First, the following lemma justifies the use of geometric grids, and provides that the price of getting rid of forgetting is summing the Laplace noise from previous phases.

Lemma 7 (Privacy of our grid-based mean estimator)**.** *Let $T \in \{1, \dots\}$, $\ell < T$ and $t_1, \dots t_\ell, t_{\ell+1}$ be in $[1, T]$ such that $1 = t_1 < \cdots < t_\ell < t_{\ell+1} - 1 = T$.*

Let $\mathcal{M}$ be the following mechanism:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_T \end{pmatrix} \overset{\mathcal{M}}{\to} \begin{pmatrix} (x_1 + \cdots + x_{t_2-1}) + (Y_1) \\ (x_1 + \cdots + x_{t_3-1}) + (Y_1 + Y_2) \\ \vdots \\ (x_1 + \cdots + x_T) + (Y_1 + Y_2 + \cdots + Y_{\ell-1}) \end{pmatrix}$$

where $(Y_1, \dots, Y_\ell) \sim^{iid} \mathrm{Lap}(1/\epsilon)$.

Then, for any $\{x_1, \dots, x_T\} \in [0,1]^T$, $\mathcal{M}$ is ϵ-DP.

Proof. First, consider the following mechanism, that only computes the partial sums:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_T \end{pmatrix} \to \begin{pmatrix} x_1 + \cdots + x_{t_2-1} \\ x_{t_2} + \cdots + x_{t_3-1} \\ \vdots \\ x_{t_{\ell-1}} + \cdots + x_T \end{pmatrix} .$$

Because $x_t \in [0,1]$, the sensitivity of each partial sum is 1. Since each partial sum is computed over non-overlapping sequences, combining the Laplace mechanism (Theorem 1) with the parallel composition property of DP (Lemma 3) gives that the following mechanism:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_T \end{pmatrix} \overset{\mathcal{P}}{\to} \begin{pmatrix} x_1 + \cdots + x_{t_2-1} + Y_1 \\ x_{t_2} + \cdots + x_{t_3-1} + Y_2 \\ \vdots \\ x_{t_{\ell-1}} + \cdots + x_T + Y_{\ell-1} \end{pmatrix}$$

is ϵ-DP, where $(Y_1, \dots, Y_{\ell-1}) \sim^{\text{iid}} \mathrm{Lap}(1/\epsilon)$.

Consider the post-processing function $f : (x_1, \dots x_{\ell-1}) \to (x_1, x_1 + x_2, \dots, x_1 + x_2 + \cdots + x_{\ell-1})$. Then, we have that that $\mathcal{M} = f \circ \mathcal{P}$. So, by the post-processing property of DP, $\mathcal{M}$ is ϵ-DP. $\square$

Remark 1. *Mechanism $\mathcal{P}$, defined in the proof of Lemma 7, is the fundamental mechanism used by all previous bandit algorithms [74, 9, 43, 12] to justify the use of forgetting. Our mechanism $\mathcal{M}$ is just summing over the partial sums computed on each phase, and thus the price of having sums of x_i that start from the beginning (i.e. do not forget) is that we have to sum now the noise from all previous phases too.*

E.2 Proof of Lemma 4

We are now ready to prove Lemma 4, i.e. that any BAI algorithm based solely on using $\mathrm{GPE}_\eta(\epsilon)$ to access observations is ϵ-global DP on $[0,1]$.

Proof. Let π be a BAI algorithm using only $\mathrm{GPE}_\eta(\epsilon)$ to access observations. Let $R = \{x_1, \dots\}$ and $R' = \{x'_1, \dots\}$ be two neighbouring sequences of private observations, i.e. there exists a $t^\star \in \{1, \dots\}$ such that $x_t = x'_t$ for all $t \neq t^\star$, i.e. that R and R' only differ at $t^\star$.

Fix a stopping time, recommendation and sampled actions $(T+1, \tilde{a}, (a_1, \dots, a_T))$, we want to show that

$$\Pr[\pi(R) = (T+1, \tilde{a}, (a_1, \dots, a_T))] \leq e^\epsilon \Pr[\pi(R') = (T+1, \tilde{a}, (a_1, \dots, a_T))] .$$

<u>Step 1: Probability decompositions:</u> First, let us denote by τ, $\tilde{A}$ and $A_1, \dots, A_\tau$ the random variables of stopping, recommendation and sampled actions, when π interacts with R. Similarly, let τ', $\tilde{A}'$ and $A'_1, \dots, A'_\tau$ the random variables of stopping, recommendation and sampled actions, when π interacts with R'.

We have

$$\Pr[\pi(R) = (T+1, \tilde{a}, (a_1, \dots, a_T))] = \Pr[\tau = T+1, \tilde{A} = \tilde{a}, A_1 = a_1, \dots, A_T = a_T]$$
$$\Pr[\pi(R') = (T+1, \tilde{a}, (a_1, \dots, a_T))] = \Pr[\tau' = T+1, \tilde{A}' = \tilde{a}, A'_1 = a_1, \dots, A'_T = a_T]$$

Since for all $t < t^\star$, $x_t = x'_t$, the policy samples the same actions, up to step $t^\star$, i.e.

$$\Pr[A_1 = a_1, \dots, A_{t^\star} = a_{t^\star}] = \Pr[A'_1 = a_1, \dots, A'_{t^\star} = a_{t^\star}]$$

And thus

$$\frac{\Pr[\pi(R) = (T+1, \tilde{a}, (a_1, \dots, a_T))]}{\Pr[\pi(R') = (T+1, \tilde{a}, (a_1, \dots, a_T))]}$$
$$= \frac{\Pr[\tau = T+1, \tilde{A} = \tilde{a}, A_{t^\star+1} = a_{t^\star+1}, \dots, A_T = a_T \mid A_1 = a_1, \dots, A_{t^\star} = a_{t^\star}]}{\Pr[\tau' = T+1, \tilde{A}' = \tilde{a}, A'_{t^\star+1} = a_{t^\star+1}, \dots, A'_T = a_T \mid A'_1 = a_1, \dots, A'_{t^\star} = a_{t^\star}]}$$

Let us denote by $t_1, \dots, t_\ell$ the time step corresponding to the beginning of the phases when π interacts with R, and $t'_1, \dots, t'_{\ell'}$ the the time step corresponding to the beginning of the phases π interacts with **r'**.

Also, let $t_{k_\star}$ be the beginning of the phase for which $t^\star$ belongs in R phases. Similarly, let $t'_{k'_\star}$ be the beginning of the phase for which $t^\star$ belongs in R' phases.

Since the actions $a_1, \dots, a_T$ are fixed, and $r_t = r'_t$ for $t < t^\star$, $t^\star$ falls in the same phase under both R and R'. Thus, $t_{k_\star} = t'_{k'_\star}$ and $k_\star = k'_\star$.

<u>Step 2: Using the structure of $\text{GPE}_\eta(\epsilon)$</u>

Let $\tilde{S}^p_{k^\star} = \sum_{s=t_{k^\star}}^{t_{k^\star+1}-1} x_s + Y_{k_\star}$ be the noisy partial sum of observations collected at phase $k^\star$ for **r**, where $Y_{k^\star} \sim \text{Lap}(1/\epsilon)$. Similarly, let $\tilde{S'}^p_{k^\star} = \sum_{s=t_{k^\star}}^{t_{k^\star+1}-1} x'_s + Y'_{k_\star}$ be the noisy partial sum of observations collected at phase $k^\star$ for **r'**, where $Y'_{k^\star} \sim \text{Lap}(1/\epsilon)$. Using the structure of $\text{GPE}_\eta(\epsilon)$, we have that:

(a) If the value of the noisy partial sums at phase $k^\star$ is exactly the same between the neighbouring R and R', then the BAI algorithm π will sample the same sequence of actions from step $t^\star$ onward, recommend the same final guess and stop at the same time, with the same probability under R and $\mathbb{R}'$. Thus, for any $s \in \mathbb{R}$:

$$\begin{aligned}&\Pr[\tau = T+1, \tilde{A} = \tilde{a}, A_{t^\star+1} = a_{t^\star+1}, \dots, A_T = a_T \mid A_1 = a_1, \dots, A_{t^\star} = a_{t^\star}, \tilde{S}^p_{k^\star} = s] \\ &= \Pr[\tau' = T+1, \tilde{A}' = \tilde{a}, A'_{t^\star+1} = a_{t^\star+1}, \dots, A'_T = a_T \mid A'_1 = a_1, \dots, A'_{t^\star} = a_{t^\star}, \tilde{S'}^p_{k^\star} = s]\end{aligned} \tag{20}$$

This is due to the fact that, in $\text{GPE}_\eta(\epsilon)$, the reward at step $t^\star$ only affects the statistic $\tilde{S}^p_{k^\star}$, and nothing else.

(b) Since rewards are $[0, 1]$, using the Laplace mechanism, we have that

$$\Pr[\tilde{S}^p_{k^\star} = s \mid A_1 = a_1, \dots, A_{t^\star} = a_{t^\star}] \le e^\epsilon \Pr(\tilde{S'}^p_{k^\star} = s \mid A_1 = a_1, \dots, A'_{t^\star} = a_{t^\star})\,. \tag{21}$$

<u>Step 3: Combining Eq. 20 and Eq. 21, aka post-processing:</u>

We have

$$\begin{aligned}&\Pr[\tau = T+1, \tilde{A} = \tilde{a}, A_{t^\star+1} = a_{t^\star+1}, \dots, A_T = a_T \mid A_1 = a_1, \dots, A_{t^\star} = a_{t^\star}] \\ &= \int_{s\in\mathbb{R}} \Pr[\tau = T+1, \tilde{A} = \tilde{a}, A_{t^\star+1} = a_{t^\star+1}, \dots, A_T = a_T \mid A_1 = a_1, \dots, A_{t^\star} = a_{t^\star}, \tilde{S}^p_{k^\star} = s] \\ &\Pr[\tilde{S}^p_{k^\star} = s \mid A_1 = a_1, \dots, A_{t^\star} = a_{t^\star}] \\ &\le \int_{s\in\mathbb{R}} e^\epsilon \Pr[\tau' = T+1, \tilde{A}' = \tilde{a}, A'_{t^\star+1} = a_{t^\star+1}, \dots, A'_T = a_T \mid A_1 = a_1, \dots, A'_{t^\star} = a_{t^\star}, \tilde{S'}^p_{k^\star} = s] \\ &\Pr(\tilde{S'}^p_{k^\star} = s \mid A_1 = a_1, \dots, A'_{t^\star} = a_{t^\star})\end{aligned}$$

$$= e^{\epsilon} \Pr[\tau' = T+1, \tilde{A}' = \tilde{a}, A'_{t^\star+1} = a_{t^\star+1}, \dots, A'_T = a_T \mid A'_1 = a_1, \dots, A'_{t^\star} = a_{t^\star}] \, .$$

This concludes the proof:

$$\frac{\Pr[\pi(R) = (T+1, \tilde{a}, (a_1, \dots, a_T))]}{\Pr[\pi(R') = (T+1, \tilde{a}, (a_1, \dots, a_T))]} \le e^{\epsilon} \, .$$

□

E.3 Recalling the post-processing and composition properties of DP

Proposition 1 (Post-processing [30])**.** *Let $\mathcal{M}$ be a mechanism and f be an arbitrary randomised function defined on $\mathcal{M}$'s output. If $\mathcal{M}$ is ϵ-DP, then $f \circ \mathcal{M}$ is ϵ-DP.*

The post-processing property ensures that any quantity constructed only from a private output is still private, with the same privacy budget. This is a consequence of the data processing inequality.

Proposition 2 (Simple Composition)**.** *Let $\mathcal{M}^1, \dots, \mathcal{M}^k$ be k mechanisms. We define the mechanism*

$$\mathcal{G} : D \to \bigotimes_{i=1}^{k} \mathcal{M}^i_D$$

as the k composition of the mechanisms $\mathcal{M}^1, \dots, \mathcal{M}^k$.

If each $\mathcal{M}^i$ is ϵ_i-DP, then $\mathcal{G}$ is $\sum_{i=1}^k \epsilon_i$-DP.

Proposition 3 (Parallel Composition)**.** *Let $\mathcal{M}^1, \dots, \mathcal{M}^k$ be k mechanisms, such that $k < n$, where n is the size of the input dataset. Let $t_1, \dots t_k, t_{k+1}$ be indexes in $[1, n]$ such that $1 = t_1 < \cdots < t_k < t_{k+1} - 1 = n$.*
Let's define the following mechanism

$$\mathcal{G} : \{x_1, \dots, x_n\} \to \bigotimes_{i=1}^{k} \mathcal{M}^i_{\{x_{t_i}, \dots, x_{t_{i+1}-1}\}}$$

$\mathcal{G}$ is the mechanism that we get by applying each $\mathcal{M}^i$ to the i-th partition of the input dataset $\{x_1, \dots, x_n\}$ according to the indexes $t_1 < \cdots < t_k < t_{k+1}$.

If each $\mathcal{M}^i$ is ϵ-DP, then $\mathcal{G}$ is ϵ-DP.

In parallel composition, the k mechanisms are applied to different "non-overlapping" parts of the input dataset. If each mechanism is DP, then the parallel composition of the k mechanisms is DP, *with the same privacy budget*. This property will be the basis for designing private bandit algorithms.

F Concentration Results

In Appendix F, we detail the proof of all our concentration results. In Appendix F.1, we start by introducing a variant of GLR-based stopping rule using the modified transportation costs $\widetilde{W}_{\epsilon,a,b}$ (see Appendix G.2.1 for details) which are defined based on the modified divergences $\widetilde{d}_{\epsilon}^{\pm}$ (see Appendix G.1.1 for details). The proof of Theorem 5 is given in Appendix F.2. In Appendix F.3, we show tail bounds for a sum between independent Bernoulli and Laplace observations that feature the product of the tail bounds of each process. We prove time-uniform and fixed-time tails concentration for Laplace distribution in Appendix F.4, and recall existing results for Bernoulli in Appendix F.5. In Appendix F.6, we provide tail bounds for a sum between independent Bernoulli and Laplace observations that feature the modified divergence $\widetilde{d}_{\epsilon}$ defined in Eq. (32). In Appendix F.7, we give geometric grid time uniform tails concentration for the reweighted modified divergence.

F.1 Modified GLR Stopping Rule

The modified GLR stopping rule is defined as

$$\tau_{\epsilon,\delta}^{\mathrm{MGLR}} = \inf \left\{ n \mid \forall a \neq \tilde{a}_n,\ \widetilde{W}_{\epsilon,\tilde{a}_n,a}(\tilde{\mu}_n, \tilde{N}_n) > \sum_{b \in \{\tilde{a}_n, a\}} \widetilde{c}(k_{n,b}, \delta) \right\} \text{ with } \tilde{a}_n \in \arg\max_{a \in [K]} [\tilde{\mu}_{n,a}]_0^1 \,, \tag{22}$$

where $(\tilde{\mu}_n, \tilde{N}_n)$ are the outputs of $\mathrm{GPE}_\eta(\epsilon)$. The modified transportation costs $(\widetilde{W}_{\epsilon,a,b})_{(a,b)\in[K]^2}$ are defined in Eq. (34), i.e., for all $(\mu, w) \in \mathbb{R}^K \times \mathbb{R}_+^K$ and all $(a,b) \in [K]^2$ such that $a \neq b$,

$$\widetilde{W}_{\epsilon,a,b}(\mu, w) := \mathbb{1}\left([\mu_a]_0^1 > [\mu_b]_0^1\right) \inf_{u \in (0,1)} \left\{ w_a \widetilde{d}_\epsilon^-(\mu_a, u, r(w_a)) + w_b \widetilde{d}_\epsilon^+(\mu_b, u, r(w_b)) \right\} \,,$$

where $r(x) := \frac{x}{1+\log_{1+\eta} x}$ is defined in Eq. (33) for all $x \geq 1$. The modified divergence $\widetilde{d}_\epsilon^\pm$ are defined in Eq. (32), i.e., for all $(\lambda, \mu, r) \in \mathbb{R} \times (0,1) \times \mathbb{R}_+^\star$,

$$\widetilde{d}_\epsilon^-(\lambda, \mu, r) := \mathbb{1}\left(\mu < [\lambda]_0^1\right) \inf_{z \in (\mu, [\lambda]_0^1)} \left\{ \mathrm{kl}(z, \mu) + \frac{1}{r} h(r\epsilon(\lambda - z)) \right\} \,,$$
$$\widetilde{d}_\epsilon^+(\lambda, \mu, r) := \mathbb{1}\left(\mu > [\lambda]_0^1\right) \inf_{z \in ([\lambda]_0^1, \mu)} \left\{ \mathrm{kl}(z, \mu) + \frac{1}{r} h(r\epsilon(z - \lambda)) \right\} \,,$$

where $h(x) := \sqrt{1+x^2} - 1 + \log\left(\frac{2}{x^2}\left(\sqrt{1+x^2} - 1\right)\right)$ is defined in Eq. (31) for all $x > 0$.

Lemma 8 gives a stopping threshold under which the modified GLR stopping rule is δ-correct.

Lemma 8. *Let $\delta \in (0,1)$ and $\epsilon > 0$. Let $\eta > 0$. Let $s > 1$ and ζ be the Riemann ζ function. Let $\overline{W}_{-1}(x) = -W_{-1}(-e^{-x})$ for all $x \geq 1$, where W_{-1} is the negative branch of the Lambert W function. It satisfies $\overline{W}_{-1}(x) \approx x + \log x$, see Lemma 51. Given any sampling rule using the $\mathrm{GPE}_\eta(\epsilon)$, combining $\mathrm{GPE}_\eta(\epsilon)$ with the modified GLR stopping rule as in Eq.* (22) *with the stopping threshold*

$$\widetilde{c}(k, \delta) = \overline{W}_{-1}\left(\log\left(\frac{K\zeta(s)}{\delta}\right) + s\log(k) + 3 - \log 2\right) - 3 + \log 2 \,. \tag{23}$$

yields a δ-correct and ϵ-global DP algorithm for Bernoulli instances with unique best arm.

Proof. Lemma 4 yields the ϵ-global DP. Let $\mathcal{E}_\delta = \mathcal{E}_{\delta,a^\star,+} \cap \bigcap_{a \neq a^\star} \mathcal{E}_{\delta,a,-}$ with

$$\mathcal{E}_{\delta,a^\star,+} = \left\{ \forall n \in \mathbb{N},\ \tilde{N}_{n,a^\star} \widetilde{d}_\epsilon^+(\tilde{\mu}_{n,a^\star}, \mu_{a^\star}, \tilde{N}_{n,a^\star}/k_{n,a^\star}) \leq \widetilde{c}(k_{n,a^\star}, \delta) \right\} \,,$$
$$\forall a \neq a^\star, \quad \mathcal{E}_{\delta,a,-} = \left\{ \forall n \in \mathbb{N},\ \tilde{N}_{n,a} \widetilde{d}_\epsilon^-(\tilde{\mu}_{n,a}, \mu_a, \tilde{N}_{n,a}/k_{n,a}) \leq \widetilde{c}(k_{n,a}, \delta) \right\} \,,$$

where $(\tilde{\mu}_n, \tilde{N}_n, k_n)$ are given by $\mathrm{GPE}_\eta(\epsilon)$, $\widetilde{c}$ as in Eq. (23) and $\widetilde{d}_\epsilon^\pm$ as in Eq. (32).

Using Lemmas 19 and 20, we have $\mathbb{P}_{\boldsymbol{\nu}\pi}(\mathcal{E}_{\delta,a,-}^\complement) \leq \delta/K$ for all $a \neq a^\star$, and $\mathbb{P}_{\boldsymbol{\nu}\pi}(\mathcal{E}_{\delta,a^\star,+}^\complement) \leq \delta/K$. By union bound over $a \in [K]$, we obtain $\mathbb{P}_{\boldsymbol{\nu}\pi}(\mathcal{E}_\delta^\complement) \leq \delta$.

Let $\tau_{\epsilon,\delta}^{\mathrm{MGLR}}$ as in Eq. (22) and $\tilde{a}_n \in \arg\max_{a\in[K]} [\tilde{\mu}_{n,a}]_0^1$. Then, we directly have that

$$\begin{aligned}
\mathbb{P}_{\boldsymbol{\nu}\pi}\left(\tau_{\epsilon,\delta}^{\mathrm{MGLR}} < +\infty,\ \tilde{a}_{\tau_{\epsilon,\delta}^{\mathrm{MGLR}}} \neq a^\star\right) &\leq \mathbb{P}_{\boldsymbol{\nu}\pi}\left(\mathcal{E}_\delta^\complement\right) + \mathbb{P}_{\boldsymbol{\nu}\pi}\left(\mathcal{E}_\delta \cap \{\tau_{\epsilon,\delta}^{\mathrm{MGLR}} < +\infty,\ \tilde{a}_{\tau_{\epsilon,\delta}^{\mathrm{MGLR}}} \neq a^\star\}\right) \\
&\leq \delta + \mathbb{P}_{\boldsymbol{\nu}\pi}\left(\mathcal{E}_\delta \cap \{\tau_{\epsilon,\delta}^{\mathrm{MGLR}} < +\infty,\ \tilde{a}_{\tau_{\epsilon,\delta}^{\mathrm{MGLR}}} \neq a^\star\}\right) \,.
\end{aligned}$$

Under $\mathcal{E}_\delta \cap \{\tau_{\epsilon,\delta}^{\mathrm{MGLR}} < +\infty,\ \tilde{a}_{\tau_{\epsilon,\delta}^{\mathrm{MGLR}}} \neq a^\star\}$, by definition of the stopping rule as in Eq. (7) and the stopping threshold in Eq. (8), we obtain that there exists $a \neq a^\star$ and $n \in \mathbb{N}$ such that $[\tilde{\mu}_{n,a}]_0^1 > [\tilde{\mu}_{n,a^\star}]_0^1$ and

$$\sum_{b \in \{a, a^\star\}} \widetilde{c}(k_{n,b}, \delta) < \widetilde{W}_{\epsilon,a,a^\star}(\tilde{\mu}_n, \tilde{N}_n)$$

$$
\begin{aligned}
&= \inf_{u\in(0,1)} \left\{ \tilde{N}_{n,a} \widetilde{d}_{\epsilon}^{-}(\tilde{\mu}_{n,a}, u, r(\tilde{N}_{n,a})) + \tilde{N}_{n,a^\star} \widetilde{d}_{\epsilon}^{+}(\tilde{\mu}_{n,a^\star}, u, r(\tilde{N}_{n,a^\star})) \right\} \\
&= \inf_{(u_a,u_{a^\star})\in(0,1)^2,\, u_a \le u_{a^\star}} \{ \tilde{N}_{n,a} \widetilde{d}_{\epsilon}^{-}(\tilde{\mu}_{n,a}, u_a, r(\tilde{N}_{n,a})) + \tilde{N}_{n,a^\star} \widetilde{d}_{\epsilon}^{+}(\tilde{\mu}_{n,a^\star}, u_{a^\star}, r(\tilde{N}_{n,a^\star})) \} \\
&\le \tilde{N}_{n,a} \widetilde{d}_{\epsilon}^{-}(\tilde{\mu}_{n,a}, \mu_a, r(\tilde{N}_{n,a})) + \tilde{N}_{n,a^\star} \widetilde{d}_{\epsilon}^{+}(\tilde{\mu}_{n,a^\star}, \mu_{a^\star}, r(\tilde{N}_{n,a^\star})) \\
&\le \tilde{N}_{n,a} \widetilde{d}_{\epsilon}^{-}(\tilde{\mu}_{n,a}, \mu_a, \tilde{N}_{n,a}/k_{n,a}) + \tilde{N}_{n,a^\star} \widetilde{d}_{\epsilon}^{+}(\tilde{\mu}_{n,a^\star}, \mu_{a^\star}, \tilde{N}_{n,a^\star}/k_{n,a^\star}) \le \sum_{b\in\{a,a^\star\}} \widetilde{c}(k_{n,b}, \delta) \, ,
\end{aligned}
$$

where we used the definition of $\widetilde{W}_{\epsilon,a,a^\star}$ in Eq. (34) and Lemma 39 in the two equalities and $\mu_{a^\star} > \mu_a$ in the following inequality. The second to last inequality uses that $r(\tilde{N}_{n,a}) \le \tilde{N}_{n,a}/k_{n,a}$ for all $a \in [K]$ by definition of $(k_n, \tilde{N}_n)$, i.e., $k_{n,a} \le 1 + \log_{1+\eta} \tilde{N}_{n,a} \le k_{n,a} + 1$, and that $r \mapsto \widetilde{d}_{\epsilon}^{\pm}(\lambda, u, r)$ is non-decreasing, see Lemmas 30 and 31. The last inequality is obtained by the concentration event $\mathcal{E}_\delta$. Since this yields a contradiction, we obtain $\mathcal{E}_\delta \cap \{\tau_{\epsilon,\delta}^{\mathrm{MGLR}} < +\infty,\ \tilde{a}_{\tau_{\epsilon,\delta}^{\mathrm{MGLR}}} \neq a^\star\} = \emptyset$. This concludes the proof, i.e., $\mathbb{P}_{\boldsymbol{\nu}\pi}\left(\tau_{\epsilon,\delta}^{\mathrm{MGLR}} < +\infty,\ \tilde{a}_{\tau_{\epsilon,\delta}^{\mathrm{MGLR}}} \neq a^\star\right) \le \delta$. □

F.2 Proof of Theorem 5

Lemma 4 yields the ϵ-global DP. The proof of δ-correctness is the same as the one of Lemma 8 detailed above. In particular, we use the same concentration event $\mathcal{E}_\delta = \mathcal{E}_{\delta,a^\star,+} \cap \bigcap_{a\neq a^\star} \mathcal{E}_{\delta,a,-}$ that satisfies $\mathbb{P}_{\boldsymbol{\nu}\pi}(\mathcal{E}_\delta^\complement) \le \delta$.

Under $\mathcal{E}_\delta \cap \{\tau_{\epsilon,\delta} < +\infty,\ \tilde{a}_{\tau_{\epsilon,\delta}} \neq a^\star\}$, by definition of the GLR stopping rule as in Eq. (7) and the stopping threshold in Eq. (8), we obtain that there exists $a \neq a^\star$ and $n \in \mathbb{N}$ such that $[\tilde{\mu}_{n,a}]_0^1 > [\tilde{\mu}_{n,a^\star}]_0^1$,

$$
\sum_{b\in\{a,a^\star\}} \left(c_1(\tilde{N}_{n,b}, \delta) + c_2(\tilde{N}_{n,b}, \epsilon) \right) = \sum_{b\in\{a,a^\star\}} c(k_{n,b}, \epsilon, \delta) < W_{\epsilon,a,a^\star}(\tilde{\mu}_n, \tilde{N}_n) \, .
$$

Then, we obtain

$$
\begin{aligned}
W_{\epsilon,a,a^\star}(\tilde{\mu}_n, \tilde{N}_n) &= \inf_{u\in[0,1]} \left\{ \tilde{N}_{n,a} d_{\epsilon}^{-}(\tilde{\mu}_{n,a}, u) + \tilde{N}_{n,a^\star} d_{\epsilon}^{+}(\tilde{\mu}_{n,a^\star}, u) \right\} \\
&= \inf_{(u_a,u_{a^\star})\in[0,1]^2,\, u_a \le u_{a^\star}} \{ \tilde{N}_{n,a} d_{\epsilon}^{-}(\tilde{\mu}_{n,a}, u_a) + \tilde{N}_{n,a^\star} d_{\epsilon}^{+}(\tilde{\mu}_{n,a^\star}, u_{a^\star}) \} \\
&\le \tilde{N}_{n,a} d_{\epsilon}^{-}(\tilde{\mu}_{n,a}, \mu_a) + \tilde{N}_{n,a^\star} d_{\epsilon}^{+}(\tilde{\mu}_{n,a^\star}, \mu_{a^\star}) \, ,
\end{aligned}
$$

where we used the definition of $W_{\epsilon,a,a^\star}$ in Eq. (4) and Lemma 34 in the two equalities, and $(u_{a^\star}, u_a) = (\mu_{a^\star}, \mu_a) \in [0,1]^2$ that satisfies $u_{a^\star} > u_a$ in the following inequality.

Using Lemma 38 and initialization yields $\min\{r(\tilde{N}_{n,a^\star}), r(\tilde{N}_{n,a})\} > 0$ by . When $[\tilde{\mu}_{n,a}]_0^1 > \mu_a$ and $\mu_{a^\star} > [\tilde{\mu}_{n,a^\star}]_0^1$, Lemma 29 yields

$$
\begin{aligned}
&\tilde{N}_{n,a^\star}(d_{\epsilon}^{+}(\tilde{\mu}_{n,a^\star}, \mu_{a^\star}) - \widetilde{d}_{\epsilon}^{+}(\tilde{\mu}_{n,a^\star}, \mu_{a^\star}, r(\tilde{N}_{n,a^\star}))) \le k_\eta(\tilde{N}_{n,a^\star})(\log(1 + 2\epsilon \frac{\tilde{N}_{n,a^\star}}{k_\eta(\tilde{N}_{n,a^\star})}) + 1) \\
&\tilde{N}_{n,a}(d_{\epsilon}^{-}(\tilde{\mu}_{n,a}, \mu_a) - \widetilde{d}_{\epsilon}^{-}(\tilde{\mu}_{n,a}, \mu_a, r(\tilde{N}_{n,a}))) \le k_\eta(\tilde{N}_{n,a}) \left(\log \left(1 + 2\epsilon \frac{\tilde{N}_{n,a}}{k_\eta(\tilde{N}_{n,a})} \right) + 1 \right) \, ,
\end{aligned}
$$

where we used that $(\mu_a, \mu_{a^\star}) \in (0,1)^2$ and $x/r(x) = 1 + \log_{1+\eta} x = k_\eta(x)$. When $[\tilde{\mu}_{n,a}]_0^1 \le \mu_a$, we have $d_{\epsilon}^{-}(\tilde{\mu}_{n,a}, \mu_a) = 0 = \widetilde{d}_{\epsilon}^{-}(\tilde{\mu}_{n,a}, \mu_a, r(\tilde{N}_{n,a}))$. When $\mu_{a^\star} \le [\tilde{\mu}_{n,a^\star}]_0^1$, we have $d_{\epsilon}^{+}(\tilde{\mu}_{n,a^\star}, \mu_{a^\star}) = 0 = \widetilde{d}_{\epsilon}^{+}(\tilde{\mu}_{n,a^\star}, \mu_{a^\star}, r(\tilde{N}_{n,a^\star}))$. In either case, the above inequalities are still valid since the left hand side is null and the right hand side is positive. Therefore, we have

$$
\begin{aligned}
&\tilde{N}_{n,a} d_{\epsilon}^{-}(\tilde{\mu}_{n,a}, \mu_a) + \tilde{N}_{n,a^\star} d_{\epsilon}^{+}(\tilde{\mu}_{n,a^\star}, \mu_{a^\star}) \\
&\le \tilde{N}_{n,a} \widetilde{d}_{\epsilon}^{-}(\tilde{\mu}_{n,a}, \mu_a, r(\tilde{N}_{n,a})) + \tilde{N}_{n,a^\star} \widetilde{d}_{\epsilon}^{+}(\tilde{\mu}_{n,a^\star}, \mu_{a^\star}, r(\tilde{N}_{n,a^\star})) + \sum_{b\in\{a,a^\star\}} c_2(\tilde{N}_{n,b}, \epsilon)
\end{aligned}
$$

$$\leq \sum_{b \in \{a, a^\star\}} \widetilde{c}(k_{n,b}, \delta) + \sum_{b \in \{a, a^\star\}} c_2(\tilde{N}_{n,b}, \epsilon) \leq \sum_{b \in \{a, a^\star\}} \Big(c_1(\tilde{N}_{n,b}, \delta) + c_2(\tilde{N}_{n,b}, \epsilon) \Big) \ ,$$

where the second inequality uses the proof of Lemma 8, and third leverages that

$$\widetilde{c}(k_{n,a}, \delta) \leq \overline{W}_{-1} \left(\log \left(\frac{K\zeta(s)}{\delta} \right) + s \log(k_\eta(\tilde{N}_{n,a})) + 3 - \log 2 \right) - 3 + \log 2 \ ,$$

by using that $\overline{W}_{-1}$ is increasing (Lemma 51) and $k_{n,a} \leq 1 + \log_{1+\eta} \tilde{N}_{n,a} = k_\eta(\tilde{N}_{n,a})$ for all $a \in [K]$, as well as $r(x) = x/k_\eta(x)$. Combining all the above inequalities, we have shown that

$$\sum_{b \in \{a, a^\star\}} (c_1(\tilde{N}_{n,b}, \delta) + c_2(\tilde{N}_{n,b}, \epsilon)) < W_{\epsilon, a, a^\star}(\tilde{\mu}_n, \tilde{N}_n) \leq \sum_{b \in \{a, a^\star\}} (c_1(\tilde{N}_{n,b}, \delta) + c_2(\tilde{N}_{n,b}, \epsilon)) \ .$$

This yields a contradiction, hence we have $\mathcal{E}_\delta \cap \{\tau_{\epsilon,\delta} < +\infty, \ \tilde{a}_{\tau_{\epsilon,\delta}} \neq a^\star\} = \emptyset$. This concludes the proof, i.e., $\mathbb{P}_{\boldsymbol{\nu}\pi}\left(\tau_{\epsilon,\delta} < +\infty, \ \tilde{a}_{\tau_{\epsilon,\delta}} \neq a^\star\right) \leq \delta$.

F.3 Fixed Time Tails Bounds for a Convolution of Probability Distributions

We derive general upper and lower bounds on the upper and lower tails of the convolution (i.e., sum) between two independent random variables (Lemma 9). We provide upper (Lemma 11) and lower (Lemma 11) tail bounds for a sum (i.e., convolution) between independent Bernoulli and Laplace i.i.d. observations for a fixed time. The bounds are expressed as a function of the infimum over a bounded interval of a $-\frac{1}{t}\log(\cdot)$ transform of the product between the (upper or lower) tail bounds of each process. Therefore, in Lemmas 11 and 11, we can plug any bounds on the (upper or lower) tail concentration of each process. While those bounds are standard for Bernoulli distribution (Lemma 16 in Appendix F.5), we propose new bounds for Laplace distribution (Lemma 15 in Appendix F.4).

Sketch of Proof of Lemma 9 The main difficulty when studying the sum of two random variables lies in the fact that it involves the integral of the convolution of their probability measures. In all generality, it is difficult to upper bound such a quantity. The main idea behind our proof technique is to split the event of interest into a partition of carefully chosen events. Then, on each of those smaller events, we derive a "tight" upper bound on the integral of the convolution of their probability measures. It is reasonable to wonder how one could choose those events such that the upper bound is easier to obtain. When the event is defined as the intersection of two independent events, then we obtain a straightforward upper bound by the product of their respective probablities. When the event truly mixes the distributions, we need to use a smarter approach to control the integrated function. First, we upper bound a sub-component of this function by a maximum of the product of their respective probablities (on a small interval that is defined by the smaller event). Second, after this upper bound, the integrated function coincides with the hazard function, whose integral is the cumulative hazard function. To conclude the proof, it only remains to merge together the different upper bounds.

To the best of our knownledge, the proof technique closest to ours is the one used to prove Lemma 64 in Jourdan et al. [50]. They control the probability that two random variables have an unexpected empirical ranking as a function of the boundary crossing probabilities of each random variable. While tackling a distinct problem, they adopt the same proof structure. They decompose the event into carefully chosen events on which they can upper bound the integral of the convolution of their probability distributions. The upper bounds are obtained similarly as ours, with fewer events to consider.

Lemma 9 gives upper and lower bounds on the upper and lower tails of the sum of two independent random variables. This result is of independent interest.

Lemma 9. *Let θ and λ be two independent real random variables such that (i) θ has bounded support included in $[\alpha, \beta]$ and mean $\mu \in (\alpha, \beta)$ and (ii) λ has zero mean. Let*

$$\forall u \in [0,1], \ \forall v \in (0,1], \quad p(u, v) := u(1 - \log(u) + \log(v)) \ .$$

Then, for all $x > 0$, we have

$$\mathbb{P}(\theta + \lambda \geq \mu + x) \leq \mathbb{P}(\lambda \geq x)\mathbb{P}(\theta \in [\alpha, \mu]) + \mathbb{P}(\lambda \leq 0)\mathbb{P}(\theta \in [\min\{\beta, \mu + x\}, \beta])$$

$$
+ p \left(\sup_{z \in (\mu, \min\{\beta, \mu+x\})} \{\mathbb{P}(\theta \in [z, \beta])\mathbb{P}(\lambda \geq \mu + x - z)\} , \, \mathbb{P}(\theta \in [\mu, \beta])\mathbb{P}(\lambda \geq 0) \right) ,
$$

$$
\mathbb{P}(\theta + \lambda \geq \mu + x) \geq \sup_{z \in (\mu, \min\{\beta, \mu+x\})} \{\mathbb{P}(\theta \in [z, \beta])\mathbb{P}(\lambda \geq \mu + x - z)\} ,
$$

$$
\mathbb{P}(\theta + \lambda \leq \mu - x) \leq \mathbb{P}(\lambda \leq -x)\mathbb{P}(\theta \in [\mu, \beta]) + \mathbb{P}(\lambda \geq 0)\mathbb{P}(\theta \in [\alpha, \max\{\alpha, \mu - x\}])
$$

$$
p \left(\sup_{z \in (\max\{\alpha, \mu-x\}, \mu)} \{\mathbb{P}(\theta \in [\alpha, z])\mathbb{P}(\lambda \leq \mu - x - z)\} , \, \mathbb{P}(\theta \in [\alpha, \mu])\mathbb{P}(\lambda \leq 0) \right) ,
$$

$$
\mathbb{P}(\theta + \lambda \leq \mu - x) \geq \sup_{z \in (\max\{\alpha, \mu-x\}, \mu)} \{\mathbb{P}(\theta \in [\alpha, z])\mathbb{P}(\lambda \leq \mu - x - z)\} .
$$

Proof. **I. Upper Bound on Upper Tail.** We start by studying $\mathbb{P}(\theta+\lambda \geq \mu+x)$ where $x > 0$. We can suppose that there exists $y_1 \in (\max\{x + \mu - \beta, 0\}, x)$ such that $\mathbb{P}(\theta \geq x + \mu - y_1)\mathbb{P}(\lambda \geq y_1) > 0$. Otherwise, the probability of $\{\theta + \lambda \geq \mu + x\}$ is 0, and both bounds are 0 as well. Let y_1 be such a value, and

$$
y_3 \in [x, x + \mu) \quad \text{and} \quad y_2 \in (\min\{x + \mu - \beta, 0\}, 0] .
$$

First, we note that $-\log \mathbb{P}(\theta \geq x + \mu - y_1)$ and $-\log \mathbb{P}(\lambda \geq y_1)$ are finite, since $\mathbb{P}(\theta \geq x + \mu - y_1)\mathbb{P}(\lambda \geq y_1) > 0$ implies that $\min\{\mathbb{P}(\theta \geq x + \mu - y_1), \mathbb{P}(\lambda \geq y_1)\} > 0$. Second, we note that y_2 only exists when $x + \mu < \beta$, i.e., $(\min\{x + \mu - \beta, 0\}, 0] \neq \emptyset$. In order to study the cases $x + \mu < \beta$ and $x + \mu \geq \beta$ simultaneously, we adopt the convention that the maximum of a positive quantity on an empty set is defined as zero. Note that the situation $x + \mu < \beta$ has more subcases.

We partition of the event $\{\theta + \lambda \geq \mu + x\}$ into eight sets, namely

$$
\begin{aligned}
\{\theta + \lambda \geq \mu + x, \, \theta \in [\alpha, \beta], \, \lambda \in \mathbb{R}\} = \; & \{\lambda \in (\max\{x + \mu - \beta, 0\}, y_1), \, \theta \in [x + \mu - \lambda, \beta]\} \\
& \cup \{\theta \in [x + \mu - y_1, \beta], \, \lambda \geq y_1\} \\
& \cup \{\theta \in (\mu, x + \mu - y_1), \, \lambda \geq x + \mu - \theta\} \\
& \cup \{\theta \in [x + \mu - y_3, \mu], \, \lambda \geq x + \mu - \theta\} \\
& \cup \{\theta \in [\alpha, x + \mu - y_3), \lambda \geq x + \mu\} \\
& \cup \{\lambda \in [y_3, x + \mu), \, \theta \in [x + \mu - \lambda, x + \mu - y_3)\} \\
& \cup \{\lambda \in [y_2, 0], \, \theta \in [x + \mu - \lambda, \beta]\} \\
& \cup \{\lambda \in [x + \mu - \theta, y_2), \, \theta \in [x + \mu - y_2, \beta]\} .
\end{aligned}
$$

First, it is direct to see that

$$
\begin{aligned}
& \{\lambda \in [y_2, 0], \, \theta \in [x + \mu - \lambda, \beta]\} \cup \{\lambda \in [x + \mu - \theta, y_2), \, \theta \in [x + \mu - y_2, \beta]\} \\
& \quad \subseteq \{\lambda \leq 0, \theta \in [\min\{\beta, \mu + x\}, \beta]\} , \\
& \{\theta \in [x + \mu - y_3, \mu], \, \lambda \geq x + \mu - \theta\} \cup \{\theta \in [\alpha, x + \mu - y_3), \lambda \geq x + \mu\} \\
& \quad \cup \{\lambda \in [y_3, x + \mu), \, \theta \in [x + \mu - \lambda, x + \mu - y_3)\} \subseteq \{\lambda \geq x, \, \theta \in [\alpha, \mu]\} .
\end{aligned}
$$

By union bound, the probability of the union of those five events is upper bounded by the sum of the probability of those two events, i.e., $\mathbb{P}(\lambda \geq x, \, \theta \in [\alpha, \mu]) + \mathbb{P}(\lambda \leq 0, \theta \in [\min\{\beta, \mu + x\}, \beta])$.

A. Separate Conditions. Those two events and one of the three remaining do not require to control (θ, λ) simultaneously, as they separate the conditions on (θ, λ). Thanks to the independence of (θ, λ), the probability of those events can be simply upper bounded by the product of the respective probability of those conditions. Therefore, we obtain

$$
\begin{aligned}
& \mathbb{P}(\lambda \geq x, \, \theta \in [\alpha, \mu]) + \mathbb{P}(\lambda \leq 0, \theta \in [\min\{\beta, \mu + x\}, \beta]) + \mathbb{P}\left(\theta \in [x + \mu - y_1, \beta], \, \lambda \geq y_1\right) \\
& = \mathbb{P}(\lambda \geq x)\mathbb{P}(\theta \in [\alpha, \mu]) + \mathbb{P}(\lambda \leq 0)\mathbb{P}(\theta \in [\min\{\beta, \mu + x\}, \beta]) \\
& \quad + \mathbb{P}(\theta \in [x + \mu - y_1, \beta])\mathbb{P}(\lambda \geq y_1) .
\end{aligned}
$$

B. Mixed Conditions. The two remaining events truly require to control (θ, λ) simultaneously, i.e., consider their convolution. The proof idea is the following: (1) we integrate one integral to obtain one survival function, (2) we make appear the other survival function artificially, (3) we upper bound the product of their survival functions on the whole set and (4) we integrate the remaining hazard function, whose integral is the cumulative hazard function. Let $\mathrm{d}G$ and $\mathrm{d}F$ be the probability measures of θ and λ on $\mathbb{R}$.

For all $s \in (\max\{x+\mu-\beta, 0\}, y_1)$, we have $\mathbb{P}(\lambda \geq s) \geq \mathbb{P}(\lambda \geq y_1) > 0$. Then, we obtain

$$\begin{aligned}
&\mathbb{P}\left(\lambda \in (\max\{x+\mu-\beta, 0\}, y_1),\ \theta \in [x+\mu-\lambda, \beta]\right) \\
&= \int_{s \in (\max\{x+\mu-\beta,0\}, y_1)} \mathbb{P}(\theta \in [x+\mu-s, \beta]) \mathrm{d}F(s) \\
&= \int_{s \in (\max\{x+\mu-\beta,0\}, y_1)} \mathbb{P}(\lambda \geq s)\mathbb{P}(\theta \in [x+\mu-s, \beta]) \frac{1}{\mathbb{P}(\lambda \geq s)} \mathrm{d}F(s) \\
&\leq \sup_{s \in (\max\{x+\mu-\beta,0\}, y_1)} \left\{\mathbb{P}(\lambda \geq s)\mathbb{P}(\theta \in [x+\mu-s, \beta])\right\} \int_{s \in (\max\{x+\mu-\beta,0\}, y_1)} \frac{1}{\mathbb{P}(\lambda \geq s)} \mathrm{d}F(s) \\
&\leq \sup_{s \in (\max\{x+\mu-\beta,0\}, y_1)} \left\{\mathbb{P}(\lambda \geq s)\mathbb{P}(\theta \in [x+\mu-s, \beta])\right\} \left(-\log(\mathbb{P}(\lambda \geq y_1)) + \log(\mathbb{P}(\lambda \geq 0))\right)\ ,
\end{aligned}$$

where we used that $\mathbb{P}(\lambda \geq \max\{x+\mu-\beta, 0\}) \leq \mathbb{P}(\lambda \geq 0)$.

For all $z \in (\mu, x+\mu-y_1)$, we have $\mathbb{P}(\theta \in [z, \beta]) \geq \mathbb{P}(\theta \in [x+\mu-y_1, \beta]) > 0$. Then, we obtain

$$\begin{aligned}
&\mathbb{P}\left(\theta \in (\mu, x+\mu-y_1),\ \lambda \geq x+\mu-\theta\right) \\
&= \int_{z \in (\mu, x+\mu-y_1)} \mathbb{P}(\lambda \geq x+\mu-z) \mathrm{d}G(z) \\
&= \int_{z \in (\mu, x+\mu-y_1)} \mathbb{P}(\lambda \geq x+\mu-z)\mathbb{P}(\theta \in [z, \beta]) \frac{1}{\mathbb{P}(\theta \in [z, \beta])} \mathrm{d}G(z) \\
&\leq \sup_{z \in (\mu, x+\mu-y_1)} \left\{\mathbb{P}(\lambda \geq x+\mu-z)\mathbb{P}(\theta \in [z, \beta])\right\} \int_{z \in (\mu, x+\mu-y_1)} \frac{1}{\mathbb{P}(\theta \in [z, \beta])} \mathrm{d}G(z) \\
&\leq \sup_{s \in (y_1, x)} \left\{\mathbb{P}(\lambda \geq s)\mathbb{P}(\theta \in [x+\mu-s, \beta])\right\} \\
&\qquad \cdot \left(-\log(\mathbb{P}(\theta \in [x+\mu-y_1, \beta])) + \log(\mathbb{P}(\theta \in [\mu, \beta]))\right)\ .
\end{aligned}$$

C. Combining Results. Putting everything together, we have, for all $y_1 \in (\max\{x+\mu-\beta, 0\}, x)$,

$$\begin{aligned}
&\mathbb{P}\left(\theta + \lambda \geq \mu + x\right) \leq \mathbb{P}(\lambda \geq x)\mathbb{P}(\theta \in [\alpha, \mu]) + \mathbb{P}(\lambda \leq 0)\mathbb{P}(\theta \in [\min\{\beta, \mu+x\}, \beta]) \\
&+ \mathbb{P}(\theta \in [x+\mu-y_1, \beta])\mathbb{P}(\lambda \geq y_1) \\
&+ \sup_{s \in (\max\{x+\mu-\beta,0\}, y_1)} \left\{\mathbb{P}(\lambda \geq s)\mathbb{P}(\theta \in [x+\mu-s, \beta])\right\} \left(-\log(\mathbb{P}(\lambda \geq y_1)) + \log(\mathbb{P}(\lambda \geq 0))\right) \\
&+ \sup_{s \in (y_1, x)} \left\{\mathbb{P}(\lambda \geq s)\mathbb{P}(\theta \in [x+\mu-s, \beta])\right\} \left(-\log(\mathbb{P}(\theta \in [x+\mu-y_1, \beta])) + \log(\mathbb{P}(\theta \in [\mu, \beta]))\right) \\
&\leq \mathbb{P}(\lambda \geq x)\mathbb{P}(\theta \in [\alpha, \mu]) + \mathbb{P}(\lambda \leq 0)\mathbb{P}(\theta \in [\min\{\beta, \mu+x\}, \beta]) \\
&+ \mathbb{P}(\theta \in [x+\mu-y_1, \beta])\mathbb{P}(\lambda \geq y_1) \\
&+ \sup_{s \in (\max\{x+\mu-\beta,0\}, x)} \left\{\mathbb{P}(\lambda \geq s)\mathbb{P}(\theta \in [x+\mu-s, \beta])\right\} \\
&\qquad \cdot \left(-\log(\mathbb{P}(\lambda \geq y_1)\mathbb{P}(\theta \in [x+\mu-y_1, \beta])) + \log(\mathbb{P}(\theta \in [\mu, \beta])\mathbb{P}(\lambda \geq 0))\right)\ ,
\end{aligned}$$

where the second inequality is obtained by extending the two suprema to $(\max\{x+\mu-\beta, 0\}, x)$, which is possible since multiplied by a positive value, and factorizing them together. Taking

$$y_1^\star \in \operatorname*{arg\,max}_{s \in (\max\{x+\mu-\beta,0\}, x)} \left\{\mathbb{P}(\lambda \geq s)\mathbb{P}(\theta \in [x+\mu-s, \beta])\}\right\}\ ,$$

and the change of variable $z = x + \mu - s$, i.e.,

$$\begin{aligned}
&\sup_{s \in (\max\{x+\mu-\beta,0\}, x)} \left\{\mathbb{P}(\lambda \geq s)\mathbb{P}(\theta \in [x+\mu-s, \beta])\right\} \\
&= \sup_{z \in (\mu, \min\{\beta, x+\mu\})} \left\{\mathbb{P}(\theta \in [z, \beta])\mathbb{P}(\lambda \geq x+\mu-z)\right\}\ ,
\end{aligned}$$

concludes the proof of the upper bound on the upper tail.

II. Lower Bound on Upper Tail. Let $z \in (\mu, \min\{\beta, \mu+x\})$. Then, we have directly that

$$\left\{\theta \in [z, \beta],\ \lambda \geq \mu + x - z\right\} \subseteq \left\{\theta + \lambda \geq \mu + x\right\}\ .$$

Using independence, we obtain

$$\mathbb{P}(\theta \in [z, \beta])\mathbb{P}(\lambda \geq \mu + x - z) = \mathbb{P}(\theta \in [z, \beta], \lambda \geq \mu + x - z) \leq \mathbb{P}\left(\theta + \lambda \geq \mu + x\right) \ .$$

Taking the supremum over $z \in (\mu, \min\{\beta, \mu + x\})$ on the left hand side concludes the proof of the lower bound on the upper tail.

III. Upper/Lower Bound on Lower Tail. The third and forth inequalities are a direct consequence of the first and second inequalities applied to the two independent real random variables $-\theta$ and $-\lambda$ since (1) $-\theta$ has bounded support included in $[-\beta, -\alpha]$ and mean $-\mu \in (-\beta, -\alpha)$, and (2) $-\lambda$ has zero mean. Namely,

$$\begin{aligned}
&\mathbb{P}(\theta + \lambda \leq \mu - x) = \mathbb{P}(-\theta - \lambda \geq -\mu + x) \ , \\
&\mathbb{P}(\lambda \leq -x)\mathbb{P}(\theta \in [\mu, \beta]) = \mathbb{P}(-\lambda \geq x)\mathbb{P}(-\theta \in [-\beta, -\mu]) \ , \\
&\mathbb{P}(\lambda \geq 0)\mathbb{P}(\theta \in [\alpha, \max\{\alpha, \mu - x\}]) = \mathbb{P}(-\lambda \leq 0)\mathbb{P}(-\theta \in [\min\{-\alpha, x - \mu\}, -\alpha]) \ , \\
&\mathbb{P}(\theta \in [\alpha, \mu])\mathbb{P}(\lambda \leq 0) = \mathbb{P}(-\theta \in [-\mu, -\alpha])\mathbb{P}(-\lambda \geq 0) \ , \\
&\sup_{z \in (\max\{\alpha, \mu - x\}, \mu)} \left\{\mathbb{P}(\theta \in [\alpha, z])\mathbb{P}(\lambda \leq \mu - x - z)\right\} \\
&\quad = \sup_{\tilde{z} \in (-\mu, \min\{-\alpha, -\mu + x\})} \left\{\mathbb{P}(-\theta \in [\tilde{z}, -\alpha])\mathbb{P}(-\lambda \geq -\mu + x - \tilde{z})\right\} \ ,
\end{aligned}$$

where we used the change of variable $\tilde{z} = -z$. □

Properties on the Rate Function Lemma 10 gathers properties on the rate function f in Lemmas 11 and 11.

Lemma 10. *Let us define*

$$\forall x \geq 0, \quad f(x) := (x + 3 - \log 2)\exp(-x) \ . \tag{24}$$

On $\mathbb{R}_+$, the function f is twice continuously differentiable, positive, decreasing and strictly convex. It satisfies $f(0) > 1$, $\lim_{x \to +\infty} f(x) = 0$ and

$$f(x) \leq \delta \quad \iff \quad x \geq \overline{W}_{-1}(\log(1/\delta) + 3 - \log 2) - 3 + \log 2 \ ,$$

where $\overline{W}_{-1}$ is defined in Lemma 51.

Proof. Direct manipulation yields $f(0) = 3 - \log 2 > 1$, $\lim_{x \to +\infty} f(x) = 0$,

$$\forall x \geq 0, \quad f'(x) = -(x + 2 - \log 2)\exp(-x) < 0 \quad \text{and} \quad f''(x) = (x + 1 - \log 2)\exp(-x) > 0 \ .$$

Using that $f(x) = e^{3 - \log 2}\exp(-h(x + 3 - \log 2))$ where $h(x) = x - \log(x)$, Lemma 51 yields

$$\begin{aligned}
f(x) \leq \delta \quad &\iff \quad h(x + 3 - \log 2) \geq \log\left(e^{3 - \log 2}/\delta\right) \\
&\iff \quad \overline{W}_{-1}(\log(1/\delta) + 3 - \log 2) - 3 + \log 2 \leq x \ .
\end{aligned}$$

□

Fixed Time Upper and Lower Tails Concentration Lemma 11 gives an upper and lower tails bound for a sum between independent Bernoulli and Laplace i.i.d. observations for a fixed time.

Lemma 11. *Let $\mu \in (0, 1)$ and $\epsilon > 0$. Let $Z_t = \sum_{s \in [t]} X_s$ where $X_s \sim$ Ber(μ) are i.i.d. observations. Let $S_t = \sum_{s \in [n_t]} Y_s$ where $Y_s \sim$ Lap$(1/\epsilon)$ are i.i.d. observations where $(n_t)_{t \in \mathbb{N}}$ be a piece-wise constant increasing function from $\mathbb{N}$ to $\mathbb{N}$. Let f as in Eq.* (24)*. Then, for all $t \in \mathbb{N}$ and all $x > 0$,*

$$\mathbb{P}(Z_t + S_t \geq t(x + \mu)) \leq f\left(t \inf_{z \in (\mu, \min\{1, x + \mu\})} \left\{-\frac{1}{t}\log\left(\mathbb{P}(Z_t \geq tz)\mathbb{P}(S_t \geq t(x + \mu - z))\right)\right\}\right)$$

$$\mathbb{P}(Z_t + S_t \leq t(\mu - x)) \leq f\left(t \inf_{z \in (\max\{0, \mu - x\}, \mu)} \left\{-\frac{1}{t}\log\left(\mathbb{P}(Z_t \leq tz)\mathbb{P}(S_t \leq t(\mu - x - z))\right)\right\}\right)$$

Proof. Let $t \in \mathbb{N}$ and $x > 0$. Then, Z_t and S_t are two independent real random variables such that (1) Z_t has bounded support included in $[0, t]$ and mean $t\mu \in (0, t)$ and (ii) S_t has zero mean. By symmetry of Lap$(1/\epsilon)$ around 0, the cumulative sum of n_t observations (i.e., S_t) is also symmetric around 0. However, Z_t follows Bin(t, μ) which can be skewed. Therefore, we have

$$\mathbb{P}(S_t \geq 0) = 1/2 = \mathbb{P}(S_t \leq 0) \quad \text{and} \quad \max\{\mathbb{P}(Z_t \in [t\mu, t]), \mathbb{P}(Z_t \in [0, t\mu])\} \leq 1 \; ,$$
$$\forall z \in [0, 1], \quad \mathbb{P}(Z_t \in [tz, t]) = \mathbb{P}(Z_t \geq tz) \quad \text{and} \quad \mathbb{P}(Z_t \in [0, tz]) = \mathbb{P}(Z_t \leq tz) \; .$$

Using that $z \mapsto \mathbb{P}(Z_t \geq tz)$ is decreasing on $(\mu, \min\{1, x + \mu\})$ and $z \mapsto \mathbb{P}(S_t \geq t(\mu - x - z)$ is increasing on $(\mu, \min\{1, x + \mu\})$, we obtain

$$\begin{aligned}&\max\{\mathbb{P}(Z_t \geq t \min\{1, x + \mu\}))\mathbb{P}(S_t \leq 0), \mathbb{P}(S_t \geq tx)\mathbb{P}(Z_t \leq t\mu)\} \\ &\leq \sup_{z \in (\mu, \min\{1, x+\mu\})} \{\mathbb{P}(Z_t \geq tz)\mathbb{P}(S_t \geq t(x + \mu - z))\} \; .\end{aligned}$$

Let us define $g(x) := x(3 - \log(2) - \log(x))$. Using Lemma 9 for (Z_t, S_t) and considering $tx > 0$ and $z \in (\mu, \min\{\beta, \mu + x\})$ (i.e., $tz \in (t\mu, t \min\{\beta, \mu + x\})$), we obtain

$$\mathbb{P}(Z_t + S_t \geq t(x + \mu)) \leq g\left(\sup_{z \in (\mu, \min\{1, x+\mu\})} \{\mathbb{P}(Z_t \geq tz)\mathbb{P}(S_t \geq t(x + \mu - z))\} \right) \; .$$

Let f as in Eq. (24) of Lemma 10. Then, we have $f(x) = g(\exp(-x))$. This concludes the proof of the upper bound on the upper tail. The second result is obtained similarly based on Lemma 9 and the above results. □

F.4 Tails Concentration of Cumulative Laplace Distributions

We derive time-uniform (Lemma 14) and fixed-time (Lemma 15) tails concentration for the cumulative sum of i.i.d. Laplace observations. Our proof technique is based on the Chernoff method and Ville's inequality as in Eq. (26). Therefore, we need to derive the convex conjuguate of the moment generating function of a Laplace distribution (Lemma 12). While the time-uniform result requires using the peeling method, the proof of the fixed-time concentration is simpler. To use the peeling method, we need to control the deviation of the process on slices of time (Lemma 13).

Convex Conjuguate of the Moment Generating Function of Laplace Distribution Let $\epsilon > 0$. The moment generating function of the Laplace distribution Lap$(1/\epsilon)$ is defined as

$$\forall \lambda \in (0, \epsilon), \quad \psi_{\text{Lap},\epsilon}(\lambda) = \log \mathbb{E}_{X \sim \text{Lap}(1/\epsilon)} [\exp(\lambda X)] = -\log(1 - \lambda^2/\epsilon^2) \; . \tag{25}$$

Lemma 12 explicits the convex conjuguate of $\psi_{\text{Lap},\epsilon}$ and its associated maximizer.

Lemma 12. *Let $\psi_{Lap,\epsilon}$ as in Eq.* (25)*. Let us define*

$$\forall x > 0, \quad \psi^\star_{Lap,\epsilon}(x) := \max_{\lambda \in (0,\epsilon)} \{\lambda x - \psi_{Lap,\epsilon}(\lambda)\} \quad \textit{and} \quad \lambda(x) := \arg\max_{\lambda \in (0,\epsilon)} \{\lambda x - \psi_{Lap,\epsilon}(\lambda)\} \; .$$

Then, for all $x > 0$, we have

$$\lambda(x) = \frac{1}{x}\left(\sqrt{1 + (x\epsilon)^2} - 1\right) \in (0, \epsilon) \quad \textit{and} \quad \psi^\star_{Lap,\epsilon}(x) = h(\epsilon x) > 0 \; .$$

where h is defined in Eq. (31)*.*

Proof. Let $f(\lambda) = \lambda x - \psi_{\text{Lap},\epsilon}(\lambda)$ for all $\lambda \in (0, \epsilon)$. Direct manipulation yields that

$$\forall \lambda \in (0, \epsilon), \quad f'(\lambda) = x - \frac{2\lambda}{\epsilon^2 - \lambda^2} \quad \text{and} \quad f''(\lambda) = -2\frac{\epsilon^2 + \lambda^2}{(\epsilon^2 - \lambda^2)^2} < 0 \; .$$

Moreover, for all $\lambda \in (0, \epsilon)$, we have

$$f'(\lambda) = 0 \quad \iff \quad \lambda^2 + 2\lambda/x - \epsilon^2 = 0 \quad \iff \quad \lambda = \frac{1}{x}\left(\sqrt{1 + (x\epsilon)^2} - 1\right) \; .$$

We used that the second solution to the second order polynomial equation is negative, hence not in $(0, \epsilon)$. Moreover, it is direct to see that $\frac{1}{x}\left(\sqrt{1 + (x\epsilon)^2} - 1\right) \in (0, \epsilon)$ since $\sqrt{1 + x^2} - 1 \leq x$, as it is equivalent to $1 + x^2 \le (x+1)^2$ which is true when $x > 0$. Since f is strictly concave, the above computation gives its unique maximizer on $(0, \epsilon)$, namely we have $\lambda(x) = \frac{1}{x}\left(\sqrt{1+(x\epsilon)^2} - 1\right)$. Moreover, the convex conjuguate of $\psi_{\text{Lap},\epsilon}$ is

$$\begin{aligned}\psi^\star_{\text{Lap},\epsilon}(x) = f(\lambda(x)) &= \sqrt{1+(x\epsilon)^2} - 1 + \log\left(1 - \frac{1}{(x\epsilon)^2}\left(\sqrt{1+(x\epsilon)^2} - 1\right)^2\right) \\ &= \sqrt{1+(x\epsilon)^2} - 1 + \log\left(\frac{2}{(x\epsilon)^2}\left(\sqrt{1+(x\epsilon)^2} - 1\right)\right) .\end{aligned}$$

This concludes the proof. □

Test Martingale for Cumulative Laplace Observations Let $\epsilon > 0$ and $S_t = \sum_{s\in[t]} Y_s$ where $Y_s \sim \text{Lap}(1/\epsilon)$ are i.i.d. observations. Let us define

$$\forall \lambda \in (0,\epsilon), \quad M_t(\lambda) := \exp(\lambda S_t - t\psi_{\text{Lap},\epsilon}(\lambda)) .$$

It is direct to see that $M_0(\lambda) = 0$ almost surely and

$$\mathbb{E}[M_t(\lambda) \mid \mathcal{F}_{t-1}] = M_{t-1}(\lambda)\mathbb{E}_{X\sim\text{Lap}(1/\epsilon)}[\exp(\lambda X - \psi_{\text{Lap},\epsilon}(\lambda))] = M_{t-1}(\lambda) .$$

Therefore, $M_t(\lambda)$ is a test martingale. Using Ville's inequality [84] yields that

$$\forall \delta \in (0,1),\ \forall \lambda \in (0,\epsilon), \quad \mathbb{P}\left(\exists t \in \mathbb{N},\ \lambda S_t - t\psi_{\text{Lap},\epsilon}(\lambda) \ge \log(1/\delta)\right) \le \delta . \tag{26}$$

Time Uniform Tails Concentration Lemma 13 controls the deviation of the process on slices of time.

Lemma 13. *Let $\epsilon > 0$ and $S_t = \sum_{s\in[t]} Y_s$ where $Y_s \sim$ Lap$(1/\epsilon)$ are i.i.d. observations. Let $N > 0$. For all $x > 0$, there exists $\lambda(x)$ such that for all $t \ge N$,*

$$\{S_t \ge tx\} \subseteq \{\lambda(x)S_t - t\psi_{Lap,\epsilon}(\lambda(x)) \ge Nh(\epsilon x)\} ,$$

where $\lambda(x)$ as in Lemma 12 and h as in Eq. (31)*.*

Proof. Using Lemma 12, we obtain $\lambda(x) \in (0,\epsilon)$ and $\psi^\star_{\text{Lap},\epsilon}(x) = h(\epsilon x) > 0$, hence $t\psi^\star_{\text{Lap},\epsilon}(x) \ge N\psi^\star_{\text{Lap},\epsilon}(x)$ for $t \ge N$. Then, direct computations yield

$$\begin{aligned} S_t \ge tx &\implies \lambda(x)S_t - t\psi_{\text{Lap},\epsilon}(\lambda(x)) \ge t\left(x\lambda(x) - \psi_{\text{Lap},\epsilon}(\lambda(x))\right) = t\psi^\star_{\text{Lap},\epsilon}(x) \\ &\implies \lambda S_t - t\psi_{\text{Lap},\epsilon}(\lambda) \ge N\psi^\star_{\text{Lap},\epsilon}(x) = Nh(\epsilon x) .\end{aligned}$$

This concludes the proof. □

Lemma 14 gives time-uniform tails concentration for the cumulative sum of i.i.d. Laplace observations. It is obtained by applying Lemma 13 on slices of time with geometric growth rate.

Lemma 14. *Let $\delta \in (0,1)$. Let $\gamma > 0$, $s > 1$ and ζ be the Riemann ζ function. Let h^{-1} be the inverse of h defined as in Eq.* (31)*, which is well-defined by Lemma 27. Let $\epsilon > 0$ and $S_t = \sum_{s\in[t]} Y_s$ where $Y_s \sim$ Lap$(1/\epsilon)$ are i.i.d. observations. Then,*

$$\begin{aligned}&\mathbb{P}\left(\exists t \in \mathbb{N},\ S_t \ge \frac{t}{\epsilon}h^{-1}\left(\frac{1+\gamma}{t}\left(\log\left(\frac{\zeta(s)}{\delta}\right) + s\log\left(1 + \log_{1+\gamma} t\right)\right)\right)\right) \le \delta , \\ &\mathbb{P}\left(\exists t \in \mathbb{N},\ S_t \le -\frac{t}{\epsilon}h^{-1}\left(\frac{1+\gamma}{t}\left(\log\left(\frac{\zeta(s)}{\delta}\right) + s\log\left(1 + \log_{1+\gamma} t\right)\right)\right)\right) \le \delta .\end{aligned}$$

Proof. Let us define the geometric grid $N_i = (1+\gamma)^{i-1}$, hence we have $\mathbb{N} = \bigcup_{i\in\mathbb{N}}[N_i, N_{i+1})$. For all $i \in \mathbb{N}$, let $x_i(\delta) > 0$ to be defined later, and $\lambda(x_i(\delta))$ as in Lemma 13. For all $t \in \mathbb{N}$, let $g(t,\delta)$ to be defined later such that $g(t,\delta) \ge x_i(\delta)$ for $t \in [N_i, N_{i+1})$. Using Lemma 13 with $x_i(\delta) > 0$ and $g(t,\delta) \ge x_i(\delta)$ for $t \in [N_i, N_{i+1})$, a union bound yields that

$$\mathbb{P}\left(\exists t \in \mathbb{N},\ S_t \ge tg(t,\delta)\right)$$

$$
\begin{aligned}
&\leq \sum_{i\in\mathbb{N}} \mathbb{P}\left(\exists t \in [N_i, N_{i+1}) : S_t \geq t x_i(\delta)\right) \\
&\leq \sum_{i\in\mathbb{N}} \mathbb{P}\left(\exists t \in [N_i, N_{i+1}) : \lambda(x_i(\delta)) S_t - t\psi_{\mathrm{Lap},\epsilon}(\lambda(x_i(\delta))) \geq N_i h(\epsilon x_i(\delta))\right) \\
&\leq \sum_{i\in\mathbb{N}} e^{-N_i h(\epsilon x_i(\delta))} ,
\end{aligned}
$$

where the last inequality uses Ville's inequality as in Eq. (26) for all $i \in \mathbb{N}$. Let us define

$$
\begin{aligned}
g(t,\delta) &= \frac{1}{\epsilon} h^{-1}\left(\frac{1+\gamma}{t}\left(\log\left(\frac{\zeta(s)}{\delta}\right) + s\log\left(1 + \log_{1+\gamma}(t)\right)\right)\right) , \\
x_i(\delta) &= \frac{1}{\epsilon} h^{-1}\left(\frac{1}{N_i}\log\left(\frac{i^s\zeta(s)}{\delta}\right)\right) .
\end{aligned}
$$

Using Lemma 27, we obtain that $x_i(\delta) > 0$ and that h^{-1} is increasing on $\mathbb{R}^\star_+$. Using $t \in [N_i, N_{i+1})$ and $i = 1 + \log_{1+\gamma} N_i$, we obtain

$$
\begin{aligned}
g(t,\delta) &\geq \frac{1}{\epsilon} h^{-1}\left(\frac{1}{N_i}\left(\log\left(\frac{\zeta(s)}{\delta}\right) + s\log\left(1 + \log_{1+\gamma}(t)\right)\right)\right) \\
&\geq \frac{1}{\epsilon} h^{-1}\left(\frac{1}{N_i}\log\left(\frac{i^s\zeta(s)}{\delta}\right)\right) = x_i(\delta) .
\end{aligned}
$$

Therefore, we have

$$
\mathbb{P}\left(\exists t \in \mathbb{N},\ S_t \geq t g(t,\delta)\right) \leq \sum_{i\in\mathbb{N}} e^{-N_i h(\epsilon x_i(\delta))} \leq \frac{\delta}{\zeta(s)} \sum_{i\in\mathbb{N}} \frac{1}{i^s} = \delta .
$$

This concludes the proof of the first result.

By symmetry of the $\mathrm{Lap}(1/\epsilon)$ around zero, the cumulative sum of i.i.d. observations is symmetric around zero. Combining the first result with the symmetry around zero yields the second result. □

Fixed Time Tails Concentration When the time is fixed and not random, there is no need to consider slices of time and we can directly control the deviation of the process.

Lemma 15. *Let $\epsilon > 0$ and $S_t = \sum_{s\in[t]} Y_s$ where $Y_s \sim$ Lap$(1/\epsilon)$ are i.i.d. observations. Let h as in Eq.* (31)*. Then,*

$$
\begin{aligned}
&\forall t \in \mathbb{N}, \forall x > 0, \quad \mathbb{P}(S_t \geq tx) \leq \exp(-t h(\epsilon x)) , \\
&\forall t \in \mathbb{N}, \forall x > 0, \quad \mathbb{P}(S_t \leq -tx) \leq \exp(-t h(\epsilon x)) .
\end{aligned}
$$

Proof. The first result can be obtained with the same manipulation as in the proof of Lemma 14, i.e., combining Ville's inequality in Eq. (26) with Lemma 13 at $N = t$.

By symmetry of the $\mathrm{Lap}(1/\epsilon)$ around zero, the cumulative sum of i.i.d. observations is symmetric around zero. Combining the first result with the symmetry around zero yields the second result. □

F.5 Fixed Time Tails Concentration of Cumulative Bernoulli Distributions

The fixed time upper and lower tail concentration of cumulative Bernoulli distributions are well-studied. Using the Chernoff method yields Lemma 16, whose proof is omitted since it is a classic result.

Lemma 16 (Chernoff Tail Bound for Bernoulli Distributions [17])**.** *Let $\mu \in (0,1)$ and $Z_t = \sum_{s\in[t]} X_s$ where $X_s \sim$ Ber(μ) are i.i.d. observations. Then,*

$$
\begin{aligned}
&\forall t \in \mathbb{N}, \forall x \in (\mu, 1), \quad \mathbb{P}(Z_t \geq tx) \leq \exp(-t\,\mathrm{kl}(x,\mu)) , \\
&\forall t \in \mathbb{N}, \forall x \in (0, \mu), \quad \mathbb{P}(Z_t \leq tx) \leq \exp(-t\,\mathrm{kl}(x,\mu)) .
\end{aligned}
$$

F.6 Fixed Time Tails Concentration for a Convolution between Bernoulli and Laplace Distributions

We provide upper (Lemma 17) and lower (Lemma 18) tail concentrations for a sum (i.e., convolution) between independent Bernoulli and Laplace i.i.d. observations for a fixed time.

Fixed Time Upper Tail Concentration Lemma 17 shows an upper tail concentration on the sum (i.e., convolution) between independent Bernoulli and Laplace i.i.d. observations.

Lemma 17. *Let $\mu \in (0,1)$ and $Z_t = \sum_{s\in[t]} X_s$ where $X_s \sim Ber(\mu)$ are i.i.d. observations. Let $(n_t)_{t\in\mathbb{N}}$ be a piece-wise constant increasing function from $\mathbb{N}$ to $\mathbb{N}$. Let $\epsilon > 0$ and $S_t = \sum_{s\in[n_t]} Y_s$ where $Y_s \sim Lap(1/\epsilon)$ are i.i.d. observations. Then,*

$$\forall t \in \mathbb{N},\ \forall x > 0, \quad \mathbb{P}(Z_t + S_t \geq t(\mu + x)) \leq f\left(t\widetilde{d}_\epsilon^-(\mu + x, \mu, t/n_t)\right) ,$$

where f is defined in Eq. (24) and $\widetilde{d}_\epsilon^-$ is defined in Eq. (32).

Proof. Let $t \in \mathbb{N}$ and $x > 0$. Combining Lemmas 15 and 16, we obtain, for all $x > 0$ and all $z \in (\mu, \min\{1, x+\mu\})$,

$$-\frac{1}{t}\log\left(\bar{G}_t(tz)\bar{F}_{n_t}(t(x+\mu-z))\right) \geq \mathrm{kl}(z,\mu) + \frac{n_t}{t} h\left(\frac{t}{n_t}\epsilon(x+\mu-z)\right) ,$$

where we used that $x + \mu - z > 0$. Taking the infimum on $(\mu, \min\{1, x+\mu\})$ on both sides and using that $[x+\mu]_0^1 = \min\{1, x+\mu\} > \mu$, we obtain

$$\inf_{z\in(\mu,\min\{1,x+\mu\})} \left\{-\frac{1}{t}\log\left(\bar{G}_t(tz)\bar{F}_{n_t}(t(x+\mu-z))\right)\right\} \geq \widetilde{d}_\epsilon^-(\mu + x, \mu, t/n_t) ,$$

where $\widetilde{d}_\epsilon^-$ is defined in Eq. (32). Since f is decreasing on $\mathbb{R}_+$ (Lemma 10), using Lemma 11 yields

$$\mathbb{P}(Z_t + S_t \geq t(x+\mu)) \leq f\left(t\widetilde{d}_\epsilon^-(\mu + x, \mu, t/n_t)\right) .$$

which concludes the proof. □

Fixed Time Lower Tail Concentration Lemma 18 shows a lower tail concentration on the sum (i.e., convolution) between independent Bernoulli and Laplace i.i.d. observations.

Lemma 18. *Let $\mu \in (0,1)$ and $Z_t = \sum_{s\in[t]} X_s$ where $X_s \sim Ber(\mu)$ are i.i.d. observations. Let $(n_t)_{t\in\mathbb{N}}$ be a piece-wise constant increasing function from $\mathbb{N}$ to $\mathbb{N}$. Let $\epsilon > 0$ and $S_t = \sum_{s\in[n_t]} Y_s$ where $Y_s \sim Lap(1/\epsilon)$ are i.i.d. observations. Then,*

$$\forall t \in \mathbb{N},\ \forall x > 0, \quad \mathbb{P}(Z_t + S_t \leq t(\mu - x)) \leq f\left(t\widetilde{d}_\epsilon^+(\mu - x, \mu, t/n_t)\right) ,$$

where f is defined in Eq. (24) and $\widetilde{d}_\epsilon^+$ is defined in Eq. (32).

Proof. Let $t \in \mathbb{N}$ and $x > 0$. Combining Lemmas 15 and 16, we obtain, for all $x > 0$ and all $z \in (\max\{0, \mu - x\}, \mu)$,

$$-\frac{1}{t}\log\left(G_t(tz)F_{n_t}(t(\mu-x-z))\right) \geq \mathrm{kl}(z,\mu) + \frac{n_t}{t} h\left(\frac{t}{n_t}\epsilon(z+x-\mu)\right) ,$$

where we used that $\mu - x - z < 0$. Taking the infimum on $z \in (\max\{0, \mu-x\}, \mu)$ on both sides and using that $[\mu - x]_0^1 = \max\{0, \mu - x\} < \mu$, we obtain

$$\inf_{z\in(\max\{0,\mu-x\},\mu)} \left\{-\frac{1}{t}\log\left(G_t(tz)F_{n_t}(t(\mu-x-z))\right)\right\} \geq \widetilde{d}_\epsilon^+(\mu - x, \mu, t/n_t) ,$$

where $\widetilde{d}_\epsilon^+$ is defined in Eq. (32). Since f is decreasing on $\mathbb{R}_+$ (Lemma 10), using Lemma 11 yields

$$\mathbb{P}(Z_t + S_t \geq t(\mu - x)) \leq f\left(t\widetilde{d}_\epsilon^+(\mu - x, \mu, t/n_t)\right) ,$$

which concludes the proof. □

F.7 Geometric Grid Time Uniform Tails Concentration for a Convolution between Bernoulli and Laplace Distributions

We provide upper (Lemma 19) and lower (Lemma 20) tail concentrations for a sum (i.e., convolution) between independent Bernoulli and Laplace i.i.d. observations that holds time uniformly on a geometric grid.

Geometric Grid Time Uniform Upper Tail Concentration Lemma 19 gives a threshold ensuring that a geometric grid time uniform upper tail concentration holds with probability at least $1-\delta$.

Lemma 19. *Let $\delta \in (0,1)$. Let $(\tilde{\mu}_n, \tilde{N}_n, k_n)$ are given by GPE$_\eta(\epsilon)$. Let $s > 1$ and ζ be the Riemann ζ function. Let $\overline{W}_{-1}(x) = -W_{-1}(-e^{-x})$ for all $x \geq 1$, where W_{-1} is the negative branch of the Lambert W function. It satisfies $\overline{W}_{-1}(x) \approx x + \log x$, see Lemma 51. Let us define*

$$c(k,\delta) = \overline{W}_{-1}\left(\log(1/\delta) + s\log(k) + \log(\zeta(s)) + 3 - 2\log 2\right) - 3 + 2\log 2 \,. \tag{27}$$

For all $a \in [K]$, let us define

$$\mathcal{E}_{\delta,a,-} = \left\{\forall n \in \mathbb{N},\ \tilde{N}_{n,a}\widetilde{d}_\epsilon^-(\tilde{\mu}_{n,a}, \mu_a, \tilde{N}_{n,a}/k_{n,a}) \leq c(k_{n,a},\delta)\right\} \,, \tag{28}$$

where $\widetilde{d}_\epsilon^-$ is defined in Eq. (32). Then, we have $\mathbb{P}_{\boldsymbol{\nu}\pi}(\mathcal{E}^{\complement}_{\delta,a,-}) \leq \delta$ for all $a \in [K]$.

Proof. Let us define the geometric grid $N_i = (1+\eta)^{i-1}$, hence we have $\mathbb{N} = \bigcup_{i\in\mathbb{N}}[N_i, N_{i+1})$. Let $a \in [K]$. If $\tilde{N}_{n,a} \in [N_i, N_{i+1})$, then we have $\tilde{N}_{n,a} = \lceil N_i \rceil$ and $k_{n,a} = i$. By union bound, we obtain

$$\begin{aligned}
\mathbb{P}_{\boldsymbol{\nu}\pi}(\mathcal{E}^{\complement}_{\delta,a,-}) &= \mathbb{P}_{\boldsymbol{\nu}\pi}\left(\exists n \in \mathbb{N},\ \tilde{N}_{n,a}\widetilde{d}_\epsilon^-(\tilde{\mu}_{n,a}, \mu_a, \tilde{N}_{n,a}/k_{n,a}) \geq c(k_{n,a},\delta)\right) \\
&\leq \sum_{i\in\mathbb{N}} \mathbb{P}_{\boldsymbol{\nu}\pi}\left(\exists i \in \mathbb{N},\ (\tilde{N}_{n,a}, k_{n,a}) = (\lceil N_i \rceil, i) \ \wedge\ \tilde{N}_{n,a}\widetilde{d}_\epsilon^-(\tilde{\mu}_{n,a}, \mu_a, \tilde{N}_{n,a}/k_{n,a}) \geq c(k_{n,a},\delta)\right) \\
&= \sum_{i\in\mathbb{N}} \mathbb{P}\left(\lceil N_i \rceil \widetilde{d}_\epsilon^-((Z_{\lceil N_i \rceil} + S_i)/\lceil N_i \rceil, \mu_a, \lceil N_i \rceil/i) \geq c(i,\delta)\right) \,,
\end{aligned}$$

where $Z_{\lceil N_i \rceil}$ is the cumulative sum of $\lceil N_i \rceil$ i.i.d. observations from Ber(μ_a) and S_i is the cumulative sum of i i.i.d. observations from Lap$(1/\epsilon)$.

For all $i \in \mathbb{N}$, let $x_i > 0$ be the unique solution of $\lceil N_i \rceil \widetilde{d}_\epsilon^-(x + \mu_a, \mu_a, \lceil N_i \rceil/i) = c(i,\delta)$, which exists by Lemma 32. Then, we obtain

$$\begin{aligned}
&\mathbb{P}\left(\lceil N_i \rceil \widetilde{d}_\epsilon^-((Z_{\lceil N_i \rceil} + S_i)/\lceil N_i \rceil, \mu_a, \lceil N_i \rceil/i) \geq c(i,\delta)\right) \\
&= \mathbb{P}\left(\widetilde{d}_\epsilon^-((Z_{\lceil N_i \rceil} + S_i)/\lceil N_i \rceil, \mu_a, \lceil N_i \rceil/i) \geq \widetilde{d}_\epsilon^-(x_i + \mu_a, \mu_a, \lceil N_i \rceil/i)\right) \\
&\leq \mathbb{P}(Z_{\lceil N_i \rceil} + S_i \geq \lceil N_i \rceil(x_i + \mu_a)) \leq f\left(\lceil N_i \rceil \widetilde{d}_\epsilon^-(x_i + \mu_a, \mu_a, \lceil N_i \rceil/i)\right) = f\left(c(i,\delta)\right) \,,
\end{aligned}$$

where $f(x) := (x + 3 - \log 2)\exp(-x)$ for all $x \geq 0$. The first and the last equalities are otained by definition of x_i, i.e., $\lceil N_i \rceil \widetilde{d}_\epsilon^-(x + \mu_a, \mu_a, \lceil N_i \rceil/i) = c(i,\delta)$. The first inequality is obtained by using Lemma 33, and the second inequality is obtained by using Lemma 17. Using Lemma 10 yields

$$f(x) \leq \delta \quad \iff \quad \overline{W}_{-1}(\log(1/\delta) + 3 - \log 2) - 3 + \log 2 \leq x \,.$$

Taking

$$c(i,\delta) = \overline{W}_{-1}(\log(i^s\zeta(s)/\delta) + 3 - \log 2) - 3 + \log 2 \,,$$

we can conclude the proof since $\mathbb{P}_{\boldsymbol{\nu}\pi}(\mathcal{E}^{\complement}_{\delta,a,-}) \leq \sum_{i\in\mathbb{N}} f\left(c(i,\delta)\right) \leq \sum_{i\in\mathbb{N}} \frac{\delta}{\zeta(s)i^s} \leq \delta$. □

Geometric Grid Time Uniform Lower Tail Concentration Lemma 20 gives a threshold ensuring that a geometric grid time uniform lower tail concentration holds with probability at least $1-\delta$.

Lemma 20. *Let $\delta \in (0,1)$. Let $(\tilde{\mu}_n, \tilde{N}_n, k_n)$ are given by GPE$_\eta(\epsilon)$. Let c as in Eq. (27). For all $a \in [K]$, let us define*

$$\mathcal{E}_{\delta,a,+} = \left\{ \forall n \in \mathbb{N},\ \tilde{N}_{n,a} \widetilde{d}_\epsilon^+(\tilde{\mu}_{n,a}, \mu_a, \tilde{N}_{n,a}/k_{n,a}) \le c(k_{n,a}, \delta) \right\} \ , \tag{29}$$

where $\widetilde{d}_\epsilon^+$ is defined in Eq. (32)*. Then, we have $\mathbb{P}_{\boldsymbol{\nu}\pi}(\mathcal{E}_{\delta,a,+}^\complement) \le \delta$ for all $a \in [K]$.*

Proof. Let us define the geometric grid $N_i = (1+\eta)^{i-1}$, hence we have $\mathbb{N} = \bigcup_{i \in \mathbb{N}} [N_i, N_{i+1})$. Let $a \in [K]$. If $\tilde{N}_{n,a} \in [N_i, N_{i+1})$, then we have $\tilde{N}_{n,a} = \lceil N_i \rceil$ and $k_{n,a} = i$. By union bound, we obtain

$$\begin{aligned}
\mathbb{P}_{\boldsymbol{\nu}\pi}(\mathcal{E}_{\delta,a,+}^\complement) &= \mathbb{P}_{\boldsymbol{\nu}\pi}\left(\exists n \in \mathbb{N},\ \tilde{N}_{n,a} \widetilde{d}_\epsilon^+(\tilde{\mu}_{n,a}, \mu_a, \tilde{N}_{n,a}/k_{n,a}) \ge c(k_{n,a}, \delta) \right) \\
&\le \sum_{i \in \mathbb{N}} \mathbb{P}_{\boldsymbol{\nu}\pi}\left(\exists i \in \mathbb{N},\ (\tilde{N}_{n,a}, k_{n,a}) = (\lceil N_i \rceil, i) \ \wedge\ \tilde{N}_{n,a} \widetilde{d}_\epsilon^+(\tilde{\mu}_{n,a}, \mu_a, \tilde{N}_{n,a}/k_{n,a}) \ge c(k_{n,a}, \delta) \right) \\
&= \sum_{i \in \mathbb{N}} \mathbb{P}\left(\lceil N_i \rceil \widetilde{d}_\epsilon^+((Z_{\lceil N_i \rceil} + S_i)/\lceil N_i \rceil, \mu_a, \lceil N_i \rceil / i) \ge c(i, \delta) \right) \ ,
\end{aligned}$$

where $Z_{\lceil N_i \rceil}$ is the cumulative sum of $\lceil N_i \rceil$ i.i.d. observations from Ber(μ_a) and S_i is the cumulative sum of i i.i.d. observations from Lap$(1/\epsilon)$.

For all $i \in \mathbb{N}$, let $x_i > 0$ be the unique solution of $\lceil N_i \rceil \widetilde{d}_\epsilon^+(\mu_a - x, \mu_a, \lceil N_i \rceil / i) = c(i, \delta)$, which exists by Lemma 32. Then, we obtain

$$\begin{aligned}
&\mathbb{P}\left(\lceil N_i \rceil \widetilde{d}_\epsilon^+((Z_{\lceil N_i \rceil} + S_i)/\lceil N_i \rceil, \mu_a, \lceil N_i \rceil / i) \ge c(i, \delta) \right) \\
&= \mathbb{P}\left(\widetilde{d}_\epsilon^+((Z_{\lceil N_i \rceil} + S_i)/\lceil N_i \rceil, \mu_a, \lceil N_i \rceil / i) \ge \widetilde{d}_\epsilon^+(\mu_a - x_i, \mu_a, \lceil N_i \rceil / i) \right) \\
&\le \mathbb{P}(Z_{\lceil N_i \rceil} + S_i \le \lceil N_i \rceil (\mu_a - x_i)) \le f\left(\lceil N_i \rceil \widetilde{d}_\epsilon^+(\mu_a - x_i, \mu_a, \lceil N_i \rceil / i) \right) = f\left(c(i, \delta) \right) \le \frac{\delta}{\zeta(s) i^s}
\end{aligned}$$

where $f(x) := (x + 3 - \log 2)\exp(-x)$ for all $x \ge 0$. The first and the last equalities are otained by definition of x_i, i.e., $\lceil N_i \rceil \widetilde{d}_\epsilon^+(\mu_a - x, \mu_a, \lceil N_i \rceil / i) = c(i, \delta)$. The first inequality is obtained by using Lemma 33, and the second inequality is obtained by using Lemma 18. The last inequality uses the same derivations based on Lemma 10 as in the proof of Lemma 19 by taking

$$c(i, \delta) = \overline{W}_{-1}(\log(i^s \zeta(s)/\delta) + 3 - \log 2) - 3 + \log 2 \ .$$

This concludes the proof since $\mathbb{P}_{\boldsymbol{\nu}\pi}(\mathcal{E}_{\delta,a,-}^\complement) \le \sum_{i \in \mathbb{N}} \frac{\delta}{\zeta(s) i^s} \le \delta$. □

G Divergence, Transportation Cost and Characteristic Time

Appendix G is organized as follow. First, we derive regularity properties for the signed (modified) divergences $d_\epsilon^\pm$ (Appendix G.1) and $\widetilde{d}_\epsilon^\pm$ (Appendix G.1.1). Second, we derive regularity properties the (modified) transportation costs $W_{\epsilon,a,b}$ (Appendix G.2) and $\widetilde{W}_{\epsilon,a,b}$ (Appendix G.2.1) for a pair of arms (a,b). Third, we study the characteristic time for ϵ-global DP BAI (Appendix G.3).

G.1 Signed Divergence

Recall $[x]_0^1 := \max\{0, \min\{1, \lambda\}\}$ and

$$\forall (\lambda, \mu) \in (0,1)^2, \quad \mathrm{kl}(\lambda, \mu) := \lambda \log\left(\frac{\lambda}{\mu}\right) + (1-\lambda)\log\left(\frac{1-\lambda}{1-\mu}\right)$$

where kl is infinity when $\{\mu, \lambda\} \cap \{0,1\} \neq \emptyset$. The signed divergences $d_\epsilon^\pm$ are defined in Eq. (3), i.e.,

$$\begin{aligned}
\forall (\lambda, \mu) \in \mathbb{R} \times [0,1], \quad d_\epsilon^-(\lambda, \mu) &:= \mathbb{1}\left(\mu < [\lambda]_0^1\right) \inf_{z \in [\mu, [\lambda]_0^1]} \left\{ \mathrm{kl}(z, \mu) + \epsilon([\lambda]_0^1 - z) \right\} \ , \\
d_\epsilon^+(\lambda, \mu) &:= \mathbb{1}\left(\mu > [\lambda]_0^1\right) \inf_{z \in [[\lambda]_0^1, \mu]} \left\{ \mathrm{kl}(z, \mu) + \epsilon(z - [\lambda]_0^1) \right\} \ .
\end{aligned}$$

Lemma 21 relates d_ϵ and $d_\epsilon^\pm$.

Lemma 21. *Let $d_\epsilon^\pm$ and d_ϵ as in Eq.* (3) *and* (2)*. Let $(\kappa, \nu) \in \mathcal{F}^2$ with means $(\lambda, \mu) \in (0,1)^2$. Then,*

$$\mathrm{d}_\epsilon(\kappa, \nu) = \begin{cases} 0 & \textit{if } \lambda = \mu \\ d_\epsilon^-(\lambda, \mu) & \textit{if } \mu < \lambda \\ d_\epsilon^+(\lambda, \mu) & \textit{if } \mu > \lambda \end{cases} .$$

Proof. When $\lambda = \mu$, we have $\mathrm{d}_\epsilon(\kappa, \nu) = 0$ by taking $\varphi = \nu$ and using the non-negativity of d_ϵ.

Let $\varphi \in \mathcal{F}$ with mean $z \in (0,1)$. When $\mu < \lambda$, we have

$$\begin{aligned} \mathrm{d}_\epsilon(\kappa, \nu) &= \min\{ \inf_{z \in (0,\mu)} \{\mathrm{kl}(z, \mu) + \epsilon(\lambda - z)\}, \inf_{z \in [\mu, \lambda]} \{\mathrm{kl}(z, \mu) + \epsilon(\lambda - z)\}, \\ &\qquad \inf_{z \in (\lambda, 1)} \{\mathrm{kl}(z, \mu) + \epsilon(z - \lambda)\}\} \\ &= \inf_{z \in [\mu, \lambda]} \{\mathrm{kl}(z, \mu) + \epsilon(\lambda - z)\} = d_\epsilon^-(\lambda, \mu) , \end{aligned}$$

where we partitioned $(0,1)$ and used that (1) $z \mapsto \mathrm{kl}(z, \mu) + \epsilon(z - \lambda)$ is increasing on $(\lambda, 1)$, hence the infimum on this interval is achieved at λ, and (2) $z \mapsto \mathrm{kl}(z, \mu) + \epsilon(\lambda - z)$, is decreasing on $(0, \mu)$, hence the infimum on this interval is achieved at μ.

When $\mu > \lambda$, we have

$$\begin{aligned} \mathrm{d}_\epsilon(\kappa, \nu) &= \min\{ \inf_{z \in (0,\lambda)} \{\mathrm{kl}(z, \mu) + \epsilon(\lambda - z)\}, \inf_{z \in [\lambda, \mu]} \{\mathrm{kl}(z, \mu) + \epsilon(z - \lambda)\}, \\ &\qquad \inf_{z \in (\mu, 1)} \{\mathrm{kl}(z, \mu) + \epsilon(z - \lambda)\}\} \\ &= \inf_{z \in [\lambda, \mu]} \{\mathrm{kl}(z, \mu) + \epsilon(z - \lambda)\} = d_\epsilon^+(\lambda, \mu) , \end{aligned}$$

where we partitioned $(0,1)$ and used that (1) $z \mapsto \mathrm{kl}(z, \mu) + \epsilon(z - \lambda)$ is increasing on $(\mu, 1)$, hence the infimum on this interval is achieved at μ, and (2) $z \mapsto \mathrm{kl}(z, \mu) + \epsilon(\lambda - z)$, is decreasing on $(0, \lambda)$, hence the infimum on this interval is achieved at λ. $\square$

Lemma 22 shows a strong link between $d_\epsilon^\pm$. This symmetry property can be used to carry regularity properties from d_ϵ^+ to d_ϵ^-, and vice versa.

Lemma 22. *Let $d_\epsilon^\pm$ as in Eq. 3. For all $\mu \in [0,1]$ and all $\lambda \in \mathbb{R}$,*

$$d_\epsilon^+(1-\lambda, 1-\mu) = d_\epsilon^-(\lambda, \mu) \quad \textit{and} \quad d_\epsilon^-(1-\lambda, 1-\mu) = d_\epsilon^+(\lambda, \mu) .$$

Proof. By definitions and change of variable $\tilde{z} = 1 - z$ and $\mathrm{kl}(1-\tilde{z}, 1-\mu) = \mathrm{kl}(\tilde{z}, \mu)$, we obtain

$$\begin{aligned} d_\epsilon^+(1-\lambda, 1-\mu) &= \mathbb{1}\left(\mu < [\lambda]_0^1\right) \inf_{z \in [1-[\lambda]_0^1, 1-\mu]} \{\mathrm{kl}(z, 1-\mu) + \epsilon(\max\{0, \min\{1, \lambda\} - (1-z))\} \\ &= \mathbb{1}\left(\mu < [\lambda]_0^1\right) \inf_{\tilde{z} \in [\mu, [\lambda]_0^1]} \{\mathrm{kl}(1-\tilde{z}, 1-\mu) + \epsilon(\max\{0, \min\{1, \lambda\} - \tilde{z})\} \\ &= \mathbb{1}\left(\mu < [\lambda]_0^1\right) \inf_{\tilde{z} \in [\mu, [\lambda]_0^1]} \{\mathrm{kl}(\tilde{z}, \mu) + \epsilon(\max\{0, \min\{1, \lambda\} - \tilde{z})\} = d_\epsilon^-(\lambda, \mu) . \end{aligned}$$

The second equality is a consequence of the first. $\square$

Lemma 23 gathers regularity properties on the functions $g_\epsilon^\pm$ that appear in the explicit solutions of $d_\epsilon^\pm$, as shown below. Intuitively, those functionals govern locally the separation between the low privacy regime where $d_\epsilon^\pm$ is equals to the kl and the high privacy regime where the divergence has to be modified to account for the privacy budget ϵ.

Lemma 23. *Let $\epsilon > 0$. Let $g_\epsilon^\pm$ defined as*

$$\forall x \in [0,1], \quad g_\epsilon^+(x) := \frac{x}{x(1-e^\epsilon) + e^\epsilon} \quad \textit{and} \quad g_\epsilon^-(x) := \frac{x e^\epsilon}{x(e^\epsilon - 1) + 1} . \tag{30}$$

On $[0,1]$, the function g_ϵ^+ is twice continuously differentiable, increasing and strictly convex. It satisfies $g_\epsilon^+(0) = 0$, $g_\epsilon^+(1) = 1$ and $g_\epsilon^+(x) < x$ on $(0,1)$. On $[0,1]$, the function g_ϵ^- is twice continuously differentiable, increasing and strictly concave. It satisfies $g_\epsilon^-(0) = 0$, $g_\epsilon^-(1) = 1$ and $g_\epsilon^-(x) > x$ on $(0,1)$. For all $x \in [0,1]$, we have $g_\epsilon^+(g_\epsilon^-(x)) = x$ and $g_\epsilon^-(1-x) + g_\epsilon^+(x) = 1$. For all $x \in [0,1]$, we have $\lim_{\epsilon \to 0} g_\epsilon^+(x) = \lim_{\epsilon \to 0} g_\epsilon^-(x) = x$; it satisfies $\lim_{\epsilon \to +\infty} g_\epsilon^-(x) = 1$ if $x \neq 0$ and $\lim_{\epsilon \to +\infty} g_\epsilon^+(x) = 0$ if $x \neq 1$.

Proof. Using that $e^\epsilon > 1$, direct computations yield that, for all $x \in [0,1]$,

$$(g_\epsilon^+)'(x) = \frac{e^\epsilon}{(x(1-e^\epsilon)+e^\epsilon)^2} > 0 \quad \text{and} \quad (g_\epsilon^+)''(x) = -2\frac{e^\epsilon(1-e^\epsilon)}{(x(1-e^\epsilon)+e^\epsilon)^2} > 0 \,,$$
$$(g_\epsilon^-)'(x) = \frac{e^\epsilon}{(x(e^\epsilon-1)+1)^2} > 0 \quad \text{and} \quad (g_\epsilon^-)''(x) = -2\frac{e^\epsilon(e^\epsilon-1)}{(x(e^\epsilon-1)+1)^3} < 0 \,.$$

Therefore, g_ϵ^+ is twice continuously differentiable, increasing and strictly convex on $[0,1]$ and g_ϵ^- is twice continuously differentiable, increasing and strictly concave on $[0,1]$. It is direct to see that $g_\epsilon^+(0) = g_\epsilon^-(0) = 0$ and $g_\epsilon^+(1) = g_\epsilon^-(1) = 1$. Since they are strictly convex and strictly concave, we obtain $g_\epsilon^+(x) < x$ and $g_\epsilon^-(x) > x$ for all $x \in (0,1)$. It is direct to see that, for all $x \in [0,1]$, we have $g_\epsilon^+(g_\epsilon^-(x)) = x$ and $1 - g_\epsilon^+(x) = g_\epsilon^-(1-x)$. It is direct to see that, $\lim_{\epsilon\to 0} g_\epsilon^+(x) = \lim_{\epsilon \to 0} g_\epsilon^-(x) = x$ for all $x \in [0,1]$, and $\lim_{\epsilon\to+\infty} g_\epsilon^+(x) = 0$ if $x \neq 1$ and $\lim_{\epsilon\to+\infty} g_\epsilon^-(x) = 1$ if $x \neq 0$. □

Lemma 24 gathers regularity properties of d_ϵ^+. In particular, it gives a closed-form solution, which is a key property used in our implementation to reduce the computational cost.

Lemma 24. *Let d_ϵ^+ as in Eq.* (3)*, and $g_\epsilon^\pm$ as in Eq.* (30)*. For all $\mu \in [0,1]$ and $\lambda \in \mathbb{R}$, we have*

$$d_\epsilon^+(\lambda,\mu) = \begin{cases} 0 & \textit{if } \mu \in [0, [\lambda]_0^1] \\ -\log\left(1-\mu(1-e^{-\epsilon})\right) - \epsilon[\lambda]_0^1 & \textit{if } \mu \in (g_\epsilon^-([\lambda]_0^1), 1] \\ \mathrm{kl}\left(\lambda,\mu\right) & \textit{if } \lambda \in (0,1) \ \wedge \ \mu \in ([\lambda]_0^1, g_\epsilon^-([\lambda]_0^1)] \end{cases} .$$

The function $(\lambda,\mu) \mapsto d_\epsilon^+(\lambda,\mu)$ is jointly continuous on $\mathbb{R}\times[0,1]$. For all $\mu\in[0,1]$, the function $\lambda \mapsto d_\epsilon^+(\lambda,\mu)$ is constant on $(-\infty,0]$ and on $[1,+\infty)$. Then,

$$\forall \lambda \in (0,1), \forall \mu \in [0,1], \quad d_\epsilon^+(\lambda,\mu) = \begin{cases} 0 & \textit{if } \mu \in [0,\lambda] \\ \mathrm{kl}\left(\lambda,\mu\right) & \mu \in (\lambda, g_\epsilon^-(\lambda)] \\ -\log\left(1-\mu(1-e^{-\epsilon})\right) - \epsilon\lambda & \textit{if } \mu \in (g_\epsilon^-(\lambda),1] \end{cases} .$$

For all $\mu \in [0,1]$, the function $\lambda \mapsto d_\epsilon^+(\lambda,\mu)$ is continuously differentiable, positive, decreasing and convex on $(0,\mu)$; it is affine with negative slope $-\epsilon$ on $(0, g_\epsilon^+(\mu))$ and twice continuously differentiable and strictly convex on $(g_\epsilon^+(\mu),\mu)$.

For all $\lambda \in (0,1)$, the function $\mu \mapsto d_\epsilon^+(\lambda,\mu)$ is positive, three times differentiable with continuous first derivative, increasing and strictly convex on $(\lambda,1]$; its second derivative is discontinuous at $g_\epsilon^-(\lambda)$ with gap $\frac{\partial^2 d_\epsilon^+}{\partial\mu^2}(\lambda, g_\epsilon^-(\lambda)) - \lim_{\mu\to g_\epsilon^-(\lambda)^+}\frac{\partial^2 d_\epsilon^+}{\partial \mu^2}(\lambda,\mu) > 0$. Moreover, we have

$$\forall \mu \in (\lambda, 1], \quad \frac{\partial d_\epsilon^+}{\partial \mu}(\lambda,\mu) = \begin{cases} \frac{1-e^{-\epsilon}}{1-\mu(1-e^{-\epsilon})} & \textit{if } \mu \in (g_\epsilon^-(\lambda),1] \\ \frac{\mu-\lambda}{\mu(1-\mu)} & \textit{if } \mu \in (\lambda, g_\epsilon^-(\lambda)] \end{cases} .$$

The function d_ϵ^+ is jointly convex on $(0,1)\times[0,1]$.

Proof. Recall that $d_\epsilon^+(\lambda,\mu) = \mathbb{1}\left(\mu > [\lambda]_0^1\right)\inf_{z\in[[\lambda]_0^1,\mu]} f_\epsilon^+([\lambda]_0^1,\mu,z)$ where $f_\epsilon^+(\lambda,\mu,z) = \mathrm{kl}(z,\mu) + \epsilon(z-\lambda)$. Direct computations yield that, for all $z \in ([\lambda]_0^1,\mu)$,

$$\frac{\partial f_\epsilon^+}{\partial z}(\lambda,\mu,z) = \log\left(\frac{z(1-\mu)}{(1-z)\mu}\right) + \epsilon \quad \text{and} \quad \frac{\partial f_\epsilon^+}{\partial z}(\lambda,\mu,z) = 0 \iff z = g_\epsilon^+(\mu) \,,$$
$$\frac{\partial^2 f_\epsilon^+}{\partial z^2}(\lambda,\mu,z) = \frac{1}{z(1-z)} > 0 \,.$$

Therefore, $z \to f_\epsilon^+(\lambda,\mu,z)$ is twice continuously differentiable, positive and strictly convex on $([\lambda]_0^1,\mu)$. Moreover, $z \to f_\epsilon^+(\lambda,\mu,z)$ is decreasing on $([\lambda]_0^1, \max\{g_\epsilon^+(\mu),\lambda\})$ and increasing on $(\max\{g_\epsilon^+(\mu),\lambda\},\mu)$. Using Lemma 23, we obtain

$$f_\epsilon^+(\lambda,\mu,\lambda) = \mathrm{kl}\left(\lambda,\mu\right) \,,$$
$$\mathrm{kl}(g_\epsilon^+(\mu),\mu) = -(g_\epsilon^+(\mu) + g_\epsilon^-(1-\mu))\log\left(\mu(1-e^\epsilon)+e^\epsilon\right) + \epsilon g_\epsilon^-(1-\mu)$$

$$= -\log\left(1-\mu(1-e^{-\epsilon})\right) - \epsilon g_\epsilon^+(\mu)\,,$$
$$f_\epsilon^+(\lambda,\mu,g_\epsilon^+(\mu)) = -\log\left(1-\mu(1-e^{-\epsilon})\right) - \epsilon\lambda\,.$$

By definition of the indicator function, we have $d_\epsilon^+(\lambda,\mu) = 0$ if $\mu \in [0,[\lambda]_0^1]$. When $\lambda \le 0$, for all $\mu \in (0,1)$, we have

$$\forall \mu \in (0,1), \quad d_\epsilon^+(\lambda,\mu) = f_\epsilon^+([\lambda]_0^1,\mu,g_\epsilon^+(\mu)) = -\log\left(1-\mu(1-e^{-\epsilon})\right) - \epsilon[\lambda]_0^1\,,$$

by using the properties of $z \to f_\epsilon^+(\lambda,\mu,z)$ on $(0,1) = (g_\epsilon^-([\lambda]_0^1),1)$ by Lemma 23. This function can be extended by continuity to $\mu = 0 = g_\epsilon^-([\lambda]_0^1)$ with value $d_\epsilon^+(\lambda,0) = 0$. When $\lambda \in (0,1)$ and $\mu \in (g_\epsilon^-(\lambda),1)$, we have

$$\forall \mu \in (0,1), \quad d_\epsilon^+(\lambda,\mu) = f_\epsilon^+([\lambda]_0^1,\mu,g_\epsilon^+(\mu)) = -\log\left(1-\mu(1-e^{-\epsilon})\right) - \epsilon[\lambda]_0^1\,,$$

by using the properties of $z \to f_\epsilon^+(\lambda,\mu,z)$ on $(g_\epsilon^-(\lambda),1) = (g_\epsilon^-([\lambda]_0^1),1)$ by Lemma 23. This function can be extended by continuity to $(\lambda,\mu) = (0,0) = \lim_{\lambda\to 0^+}(\lambda, g_\epsilon^-([\lambda]_0^1))$ with value $d_\epsilon^+(0,0) = 0$. In both cases, this function can be extended by continuity to $\mu = 1$ with value $d_\epsilon^+(\lambda,1) = \epsilon(1-[\lambda]_0^1)$.

When $\lambda \in (0,1)$, i.e., $[\lambda]_0^1 = \lambda$, and $\mu \in (\lambda, g_\epsilon^-(\lambda)) \subseteq (0,1)$ by Lemma 23, we have

$$d_\epsilon^+(\lambda,\mu) = f_\epsilon^+(\lambda,\mu,\lambda) = \mathrm{kl}\left(\lambda,\mu\right)\;.$$

This function can be extended by continuity to $\mu = \lambda$ with value $d_\epsilon^+(\lambda,\lambda) = 0$ since $\mathrm{kl}\left(\lambda,\lambda\right) = 0$. Using Lemma 23, this function can be extended by continuity to $\mu = g_\epsilon^-(\lambda)$ (i.e., $\lambda = g_\epsilon^+(\mu)$) with value

$$d_\epsilon^+(\lambda,g_\epsilon^-(\lambda)) = \mathrm{kl}\left(\lambda,g_\epsilon^-(\lambda)\right) = \mathrm{kl}\left(g_\epsilon^+(\mu),\mu\right) = -\log\left(1-\mu(1-e^{-\epsilon})\right) - \epsilon g_\epsilon^+(\mu)\,.$$

Therefore, we have

$$\forall \lambda \in (0,1),\ \forall \mu \in [\lambda, g_\epsilon^-(\lambda)], \quad d_\epsilon^+(\lambda,\mu) = \mathrm{kl}\left(\lambda,\mu\right)\;.$$

Using that $\lim_{\lambda\to 0^+}[\lambda, g_\epsilon^-(\lambda)] = \{0\}$, this function can be extended by continuity to $\lambda = 0$ with value 0. Using that $\lim_{\lambda\to 1^-}[\lambda, g_\epsilon^-(\lambda)] = \{1\}$, this function can be extended by continuity to $\lambda = 1$ with value $0 = \lim_{(\mu,\lambda)\to 1^-} -\log\left(1-\mu(1-e^{-\epsilon})\right) - \epsilon[\lambda]_0^1$.

Putting all the continuity arguments together, we have shown that $(\lambda,\mu) \to d_\epsilon^+(\lambda,\mu)$ is jointly continuous on $\mathbb{R}\times[0,1]$. Moreover, it is direct to see that, for all $\mu \in [0,1]$, the function $\lambda \to d_\epsilon^+(\lambda,\mu)$ is constant on $(-\infty,0]$ and on $[1,+\infty)$. Then,

$$\forall \lambda \in (0,1), \forall \mu \in [0,1], \quad d_\epsilon^+(\lambda,\mu) = \begin{cases} 0 & \text{if } \mu \in [0,\lambda] \\ \mathrm{kl}\left(\lambda,\mu\right) & \mu \in (\lambda, g_\epsilon^-(\lambda)] \\ -\log\left(1-\mu(1-e^{-\epsilon})\right) - \epsilon\lambda & \text{if } \mu \in (g_\epsilon^-(\lambda),1] \end{cases}\;.$$

Let $\mu \in [0,1]$ and $\lambda \in (0,\mu)$. Using that $\mu \in (g_\epsilon^-(\lambda),1]$ if and only if $\lambda \in (0,g_\epsilon^+(\mu))$. For all $\mu \in [0,1]$, the function $\lambda \to d_\epsilon^+(\lambda,\mu)$ is positive and affine with negative slope $-\epsilon$ on $(0,g_\epsilon^+(\mu))$. Let $\lambda \in (g_\epsilon^+(\mu),\mu)$. Direct computation yields that

$$\frac{\partial d_\epsilon^+}{\partial\lambda}(\lambda,\mu) = \frac{\partial \mathrm{kl}}{\partial\lambda}(\lambda,\mu) = \log\left(\frac{\lambda(1-\mu)}{(1-\lambda)\mu}\right) < 0\,,$$
$$\lim_{\lambda\to g_\epsilon^+(\mu)^+}\frac{\partial d_\epsilon^+}{\partial\lambda}(\lambda,\mu) = -\epsilon = \lim_{\lambda\to g_\epsilon^+(\mu)^-}\frac{\partial d_\epsilon^+}{\partial\lambda}(\lambda,\mu)\,,$$
$$\frac{\partial^2 d_\epsilon^+}{\partial\lambda^2}(\lambda,\mu) = \frac{\partial^2 \mathrm{kl}}{\partial\lambda^2}(\lambda,\mu) = \frac{1}{\lambda(1-\lambda)} > 0\,.$$

For all $\mu \in [0,1]$, the function $\lambda \to d_\epsilon^+(\lambda,\mu)$ is continuously differentiable, positive, decreasing and convex on $(0,\mu)$. For all $\mu \in [0,1]$, the function $\lambda \to d_\epsilon^+(\lambda,\mu)$ is twice continuously differentiable, positive and strictly convex on $(g_\epsilon^+(\mu),\mu)$. Combining the above results concludes the part of $\lambda \mapsto d_\epsilon^+(\lambda,\mu)$ on $(0,\mu)$.

Let $\lambda \in (0,1)$. Let $a > 0$ and $k \in \mathbb{N}$. The k-th derivative of $u(x) = a(1-ax)^{-1}$ on $[0,1]$ is $u^{(k)}(x) = (k-1)!a^{k+1}(1-ax)^{-(k+1)}$. Then,

$$\forall \mu \in (g_\epsilon^-(\lambda),1],\ \forall k \in \mathbb{N}, \quad \frac{\partial^k d_\epsilon^+}{\partial\mu^k}(\lambda,\mu) = \frac{(1-e^{-\epsilon})^k(k-1)!}{(1-\mu(1-e^{-\epsilon}))^k} > 0\,,$$

$$\forall \mu \in (\lambda, g_\epsilon^-(\lambda)], \quad \frac{\partial d_\epsilon^+}{\partial \mu}(\lambda,\mu) = \frac{\mu - \lambda}{\mu(1-\mu)} > 0 \,,$$
$$\frac{\partial^2 d_\epsilon^+}{\partial \mu^2}(\lambda,\mu) = \frac{(\mu-\lambda)^2 + \lambda(1-\lambda)}{\mu^2(1-\mu)^2} > 0 \,,$$
$$\frac{\partial^3 d_\epsilon^+}{\partial \mu^3}(\lambda,\mu) > 0 \,.$$

Direct computation yields

$$\lim_{\mu \to g_\epsilon^-(\lambda)} \frac{\mu - \lambda}{\mu(1-\mu)} = (1 - e^{-\epsilon})(1 + \lambda(e^\epsilon - 1)) \,,$$
$$\lim_{\mu \to g_\epsilon^-(\lambda)} \frac{1 - e^{-\epsilon}}{1 - \mu(1 - e^{-\epsilon})} = (1 - e^{-\epsilon})(1 + \lambda(e^\epsilon - 1)) \,,$$
$$\lim_{\mu \to g_\epsilon^-(\lambda)} \left\{ \frac{(\mu-\lambda)^2 + \lambda(1-\lambda)}{\mu^2(1-\mu)^2} - \frac{(1-e^{-\epsilon})^2}{(1 - \mu(1-e^{-\epsilon}))^2} \right\} = \frac{\lambda(1-\lambda)}{g_\epsilon^-(\lambda)^2(1 - g_\epsilon^-(\lambda))^2} > 0 \,.$$

For all $\lambda \in (0,1)$, the function $\mu \to d_\epsilon^+(\lambda,\mu)$ is positive, three times differentiable with continuous first derivative and increasing on $(\lambda, 1]$. For all $\lambda \in (0,1)$, the function $\mu \to d_\epsilon^+(\lambda,\mu)$ is strictly convex on $(\lambda, g_\epsilon^-(\lambda)]$ and $(g_\epsilon^-(\lambda), 1]$. The second derivative is discontinuous at $g_\epsilon^-(\lambda)$ with gap $\frac{\partial^2 d_\epsilon^+}{\partial \mu^2}(\lambda, g_\epsilon^-(\lambda)) - \lim_{\mu \to g_\epsilon^-(\lambda)^+} \frac{\partial^2 d_\epsilon^+}{\partial \mu^2}(\lambda,\mu) > 0$. Thanks to the continuity of the first derivative and the sign of the second derivative, the function $\mu \to d_\epsilon^+(\lambda,\mu)$ is strict convexity on $(\lambda, 1]$.

Let $(\mu_1, \mu_2) \in [0,1]^2$ and $(\lambda_1, \lambda_2) \in (0,1)^2$. On the convex set $\mathcal{F}_0 = \{(\lambda,\mu) \in (0,1) \times [0,1] \mid \mu \in [0,\lambda]\}$, the function d_ϵ^- is null hence jointly convex. Let $((\mu_1,\lambda_1),(\mu_2,\lambda_2)) \in (((0,1)\times[0,1]) \setminus \mathcal{F}_0)^2$. Let $(z_1, z_2) \in [\lambda_1, \mu_1] \times [\lambda_2, \mu_2]$ be the minimizers realizing $d_\epsilon^+(\lambda_1,\mu_1)$ and $d_\epsilon^+(\lambda_2,\mu_2)$. Since it is a convex set, we have $(\alpha\lambda_1 + (1-\alpha)\lambda_2, \alpha\mu_1 + (1-\alpha)\mu_2) \in ((0,1)\times[0,1]) \setminus \mathcal{F}_0$ for all $\alpha \in [0,1]$. Moreover, we have $\alpha z_1 + (1-\alpha) z_2 \in [\alpha\lambda_1 + (1-\alpha)\lambda_2, \alpha\mu_1 + (1-\alpha)\mu_2]$ for all $\alpha \in [0,1]$. Using the definition of d_ϵ^+ as an infimum, we obtain

$$\begin{aligned}
&d_\epsilon^+(\alpha\lambda_1 + (1-\alpha)\lambda_2, \alpha\mu_1 + (1-\alpha)\mu_2) \\
&\le \mathrm{kl}(\alpha z_1 + (1-\alpha)z_2, \alpha\mu_1 + (1-\alpha)\mu_2) + \epsilon(\alpha z_1 + (1-\alpha) z_2 - (\alpha\lambda_1 + (1-\alpha)\lambda_2)) \\
&\le \alpha\left(\mathrm{kl}(z_1,\mu_1) + \epsilon(z_1 - \lambda_1)\right) + (1-\alpha)\left(\mathrm{kl}(z_2,\mu_2) + \epsilon(z_2 - \lambda_2)\right) \\
&= \alpha d_\epsilon^+(\lambda_1,\mu_1) + (1-\alpha) d_\epsilon^+(\lambda_2,\mu_2)
\end{aligned}$$

where the second inequality comes from the joint convexity of the Kullback-Leibler divergence. Combining both results, we have shown that the function d_ϵ^+ is jointly convex on $(0,1)\times[0,1]$. □

Lemma 25 gather regularity properties of d_ϵ^-. In particular, it gives a closed-form solution, which is a key property used in our implementation to reduce the computational cost.

Lemma 25. *Let d_ϵ^- as in Eq.* (3)*, and $g_\epsilon^\pm$ as in Eq.* (30)*. For all $\mu \in [0,1]$ and all $\lambda \in \mathbb{R}$, we have*

$$d_\epsilon^-(\lambda,\mu) = \begin{cases} 0 & \text{if } \mu \in [[\lambda]_0^1, 1] \\ -\log\left(1 + \mu(e^\epsilon - 1)\right) + \epsilon[\lambda]_0^1 & \text{if } \mu \in [0, g_\epsilon^+([\lambda]_0^1)) \\ \mathrm{kl}(\lambda,\mu) & \text{if } \lambda \in (0,1) \text{ and } \mu \in [g_\epsilon^+([\lambda]_0^1), [\lambda]_0^1) \end{cases} .$$

The function $(\lambda,\mu) \mapsto d_\epsilon^-(\lambda,\mu)$ is jointly continuous on $\mathbb{R}\times[0,1]$. For all $\mu \in [0,1]$, the function $\lambda \mapsto d_\epsilon^-(\lambda,\mu)$ is constant on $(-\infty, 0]$ and on $[1, +\infty)$. Then,

$$\forall \lambda \in (0,1), \forall \mu \in [0,1], \quad d_\epsilon^-(\lambda,\mu) = \begin{cases} 0 & \text{if } \mu \in [\lambda, 1] \\ \mathrm{kl}(\lambda,\mu) & \text{if } \mu \in [g_\epsilon^+(\lambda), \lambda) \\ -\log\left(1 + \mu(e^\epsilon - 1)\right) + \epsilon\lambda & \text{if } \mu \in [0, g_\epsilon^+(\lambda)) \end{cases} .$$

For all $\mu \in [0,1]$, the function $\lambda \mapsto d_\epsilon^-(\lambda,\mu)$ is continuously differentiable, positive, increasing and convex on $(\mu, 1)$; it is affine with positive slope ϵ on $(g_\epsilon^-(\mu), 1)$ and twice continuously differentiable and strictly convex on $(\mu, g_\epsilon^-(\mu))$.

For all $\lambda \in (0,1)$, the function $\mu \mapsto d_\epsilon^-(\lambda,\mu)$ is positive, three times differentiable with continuous first derivative, decreasing and strictly convex on $[0,\lambda)$; its second derivative is discontinuous at $g_\epsilon^+(\lambda)$ with gap $\lim_{\mu \to g_\epsilon^+(\lambda)^-} \frac{\partial^2 d_\epsilon^-}{\partial \mu^2}(\lambda,\mu) - \frac{\partial^2 d_\epsilon^-}{\partial \mu^2}(\lambda, g_\epsilon^+(\lambda)) < 0$. Moreover, we have

$$\forall \mu \in [0,\lambda), \quad \frac{\partial d_\epsilon^-}{\partial \mu}(\lambda,\mu) = \begin{cases} -\frac{e^\epsilon - 1}{1+\mu(e^\epsilon-1)} & \text{if } \mu \in [0, g_\epsilon^+(\lambda)) \\ -\frac{\lambda-\mu}{\mu(1-\mu)} & \text{if } \mu \in [g_\epsilon^+(\lambda), \lambda) \end{cases} .$$

The function d_ϵ^- is jointly convex on $(0,1) \times [0,1]$.

Proof. Using Lemmas 22 and 23, we have

$$d_\epsilon^-(\lambda,\mu) = d_\epsilon^+(1-\lambda, 1-\mu) \quad \text{and} \quad g_\epsilon^+(\lambda) = 1 - g_\epsilon^-(1-\lambda) \,,$$
$$\frac{\partial d_\epsilon^-}{\partial \mu}(\lambda,\mu) = -\frac{\partial d_\epsilon^+}{\partial \mu}(1-\lambda,1-\mu) \quad \text{and} \quad \frac{\partial^2 d_\epsilon^-}{\partial \mu^2}(\lambda,\mu) = \frac{\partial^2 d_\epsilon^+}{\partial \mu^2}(1-\lambda,1-\mu) \,.$$

Moreover, we have $\mathrm{kl}(\lambda,\mu) = \mathrm{kl}(1-\lambda,1-\mu)$ and

$$-\log\left(1+\mu(e^\epsilon-1)\right) + \epsilon[\lambda]_0^1 = -\log\left(1-(1-\mu)(1-e^{-\epsilon})\right) - \epsilon[1-\lambda]_0^1 \,.$$

Combining the above with properties of d_ϵ^+ in Lemma 24 concludes the proof. □

G.1.1 Modified Divergence

Let us define

$$\forall x > 0, \quad h(x) := \sqrt{1+x^2} - 1 + \log\left(\frac{2}{x^2}\left(\sqrt{1+x^2}-1\right)\right) . \tag{31}$$

For all $(\lambda,\mu,r) \in \mathbb{R} \times (0,1) \times \mathbb{R}_+^\star$, we define

$$\widetilde{d}_\epsilon^-(\lambda,\mu,r) := \mathbb{1}\left(\mu < [\lambda]_0^1\right) \inf_{z \in (\mu, [\lambda]_0^1)} \left\{\mathrm{kl}(z,\mu) + \frac{1}{r} h(r\epsilon(\lambda - z))\right\} ,$$
$$\widetilde{d}_\epsilon^+(\lambda,\mu,r) := \mathbb{1}\left(\mu > [\lambda]_0^1\right) \inf_{z \in ([\lambda]_0^1, \mu)} \left\{\mathrm{kl}(z,\mu) + \frac{1}{r} h(r\epsilon(z - \lambda))\right\} . \tag{32}$$

Lemma 26 shows a strong link between $\widetilde{d}_\epsilon^\pm$. This symmetry property can be used to carry regularity properties from $\widetilde{d}_\epsilon^+$ to $\widetilde{d}_\epsilon^-$, and vice versa.

Lemma 26. *Let $\widetilde{d}_\epsilon^\pm$ as in Eq. (32). For all $(\lambda,\mu) \in \mathbb{R} \times [0,1]$, we have*

$$\widetilde{d}_\epsilon^+(1-\lambda,1-\mu,r) = \widetilde{d}_\epsilon^-(\lambda,\mu,r) \quad \text{and} \quad \widetilde{d}_\epsilon^-(1-\lambda,1-\mu,r) = \widetilde{d}_\epsilon^+(\lambda,\mu,r) \,.$$

Proof. Using the definitions, the change of variable $\tilde{z} = 1 - z$ and $\mathrm{kl}(1-\tilde{z},1-\mu) = \mathrm{kl}(\tilde{z},\mu)$, we obtain

$$\begin{aligned}
\widetilde{d}_\epsilon^+(1-\lambda,1-\mu,r) &= \mathbb{1}\left(\mu < [\lambda]_0^1\right) \inf_{z \in [1-[\lambda]_0^1, 1-\mu]} \left\{\mathrm{kl}(z,1-\mu) + \frac{1}{r} h\left(r\epsilon(\lambda - (1-z))\right)\right\} \\
&= \mathbb{1}\left(\mu < [\lambda]_0^1\right) \inf_{\tilde{z} \in [\mu, [\lambda]_0^1]} \left\{\mathrm{kl}(1-\tilde{z},1-\mu) + \frac{1}{r} h\left(r\epsilon(\lambda - \tilde{z})\right)\right\} \\
&= \mathbb{1}\left(\mu < [\lambda]_0^1\right) \inf_{\tilde{z} \in [\mu, [\lambda]_0^1]} \left\{\mathrm{kl}(\tilde{z},\mu) + \frac{1}{r} h\left(r\epsilon(\lambda - \tilde{z})\right)\right\} = \widetilde{d}_\epsilon^-(\lambda,\mu,r) \,.
\end{aligned}$$

The second equality is a consequence of the first. □

Lemma 27 gathers regularity properties of the function h defined in Eq. (31).

Lemma 27. *Let h as in Eq.* (31). *Then,*

$$\forall x > 0, \quad h'(x) = \frac{x}{\sqrt{x^2+1}+1} > 0 \quad \textit{and} \quad h''(x) = \frac{1}{1+x^2+\sqrt{1+x^2}} > 0 \,.$$

On $\mathbb{R}_+^\star$, the function h is twice continuously differentiable, increasing and strictly convex. Moreover, it satisfies

$$h(x) =_{x\to 0} x^2/4 + \mathcal{O}(x^4) \quad \textit{and} \quad h(x) =_{x\to+\infty} x - \mathcal{O}(\log(x)) \,.$$

Proof. For all $x > 0$, $h_1(x) = x + \log(x)$, $h_2(x) = \sqrt{1+x^2} - 1$ and $h_3(x) = \sqrt{1+x^2} - x$. Then

$$h_1'(x) = 1 + \frac{1}{x} \quad , \quad h_2'(x) = \frac{x}{\sqrt{1+x^2}} \quad \text{and} \quad h_3'(x) = \frac{x}{\sqrt{1+x^2}} - 1 \,,$$

Then, we have

$$\forall x > 0, \quad h(x) = h_1(h_2(x)) - 2\log(x) + \log 2 \,.$$

Therefore, we have

$$\begin{aligned} h'(x) = h_2'(x) h_1'(h_2(x)) - \frac{2}{x} &= \frac{x}{\sqrt{1+x^2}}\left(1 + \frac{1}{\sqrt{1+x^2}-1}\right) - \frac{2}{x} \\ &= \frac{x}{\sqrt{1+x^2}-1} - \frac{2}{x} = \sqrt{1+\frac{1}{x^2}} - \frac{1}{x} = h_3(1/x) \,. \end{aligned}$$

Note that

$$\sqrt{1+\frac{1}{x^2}} - \frac{1}{x} = \frac{x}{\sqrt{x^2+1}+1} \,.$$

Moreover, we have w

$$h''(x) = -\frac{1}{x^2} h_3'(1/x) = -\frac{1}{x^2}\left(\frac{1/x}{\sqrt{1+(1/x)^2}} - 1\right) = \frac{1}{1+x^2+\sqrt{1+x^2}} \,.$$

By taking the limit, we have $\lim_{x\to 0^+} h(x) = 0$. Moreover, we see that $\lim_{x\to 0^+} h'(x) = 0$ and $\lim_{x\to 0^+} h''(x) = 1/2$. Therefore, one can conclude that $h(x) =_{x\to 0} x^2/4 + \mathcal{O}(x^4)$ by Taylor expansion. The second result is obtained directly by limit. □

Lemma 28 provides upper and lower bound on the function $r \mapsto h(rx)/r$ involved in the definition of $\widetilde{d}_\epsilon^\pm$.

Lemma 28. *Let h as in Eq.* (31). *Let $\kappa(r, x) = h(rx)/r - x$ for all $r > 0$ and all $x \in \mathbb{R}_+^\star$. Then, we have*

$$\forall r > 0, \quad \frac{\partial \kappa}{\partial r}(r, x) = \frac{rxh'(rx) - h(rx)}{r^2} = \log\left(\frac{1}{2}(\sqrt{1+(rx)^2}+1)\right) > 0 \,.$$

On $\mathbb{R}_+^\star$, the function $r \mapsto \kappa(r, x)$ is increasing. Moreover, we have

$$\forall r > 0, \forall x \in \mathbb{R}_+, \quad 0 \le r\kappa(r, x) + \log(1 + 2xr) + 1 \le 1 + \log 4 \,,$$

Proof. Using Lemma 27 and the definition in Eq. (31), we obtain that

$$\forall x > 0, \quad xh'(x) - h(x) = -\log\left(\frac{2}{x^2}\left(\sqrt{1+x^2}-1\right)\right) = \log\left(\frac{1}{2}(\sqrt{1+x^2}+1)\right) > 0 \,,$$

where we used that $\sqrt{1+x^2} + 1 > 2$ for the last inequality. Let us define

$$\forall x \in \mathbb{R}_+, \quad g_1(x) = \frac{2(1+2x)}{\sqrt{1+x^2}+1} \,.$$

Then, we obtain $g_1(0) = 1$, $\lim_{x\to+\infty} g_1(x) = 4$ and

$$g_1'(x) = 2\frac{2 + 2\sqrt{1+x^2} - x}{\sqrt{1+x^2}(\sqrt{1+x^2}+1)^2} > 2\frac{2+x}{\sqrt{1+x^2}(\sqrt{1+x^2}+1)^2} > 0 \,.$$

Since g_1 is strictly increasing on $\mathbb{R}_+^\star$, we obtain $\log g_1(x) \ge \log g_1(0) = 0$ and $\log g_1(x) \le \log 4$ for all $x \in \mathbb{R}_+$.

By definition, we obtain

$$\begin{aligned} r\kappa(r,x) + \log(1+2xr) + 1 &= h(rx) - rx + 1 + \log(1+2xr) \\ &= \sqrt{1+(rx)^2} - rx + \log\left(\frac{2(1+2xr)}{\sqrt{1+r^2x^2}+1}\right) . \end{aligned}$$

Using that $0 \le \sqrt{1+x^2} - x \le 1$ on $\mathbb{R}_+$, we obtain

$$\begin{aligned} r\kappa(r,x) + \log(1+2xr) + 1 &\ge \log\left(\frac{2(1+2xr)}{\sqrt{1+r^2x^2}+1}\right) = \log(g_1(rx)) \ge 0 , \\ r\kappa(r,x) + \log(1+2xr) + 1 &\le 1 + \log(g_1(rx)) \le 1 + \log 4 . \end{aligned}$$

This concludes the proof. $\square$

Lemma 29 provides lower and upper bounds on the gap between $\widetilde{d}_\epsilon^\pm$ and $d_\epsilon^\pm$.

Lemma 29. *Let $d_\epsilon^\pm$ and $\widetilde{d}_\epsilon^\pm$ as in Eq.* (3) *and* (32)*. For all $(\lambda, \mu, r) \in \mathbb{R} \times (0,1) \times \mathbb{R}_+^\star$ such that $[\lambda]_0^1 < \mu$. Then,*

$$d_\epsilon^+(\lambda,\mu) \le \widetilde{d}_\epsilon^+(\lambda,\mu,r) + \frac{\log(1+2\epsilon r)+1}{r} .$$

For all $(\lambda, \mu, r) \in \mathbb{R} \times (0,1) \times \mathbb{R}_+^\star$ such that $[\lambda]_0^1 > \mu$. Then,

$$d_\epsilon^-(\lambda,\mu) \le \widetilde{d}_\epsilon^-(\lambda,\mu,r) + \frac{\log(1+2\epsilon r)+1}{r} .$$

For all $(\lambda, \mu, r) \in [0,1] \times (0,1) \times \mathbb{R}_+^\star$ such that $\lambda < \mu$. Then,

$$d_\epsilon^+(\lambda,\mu) \ge \widetilde{d}_\epsilon^+(\lambda,\mu,r) - \frac{\log 4}{r} .$$

For all $(\mu, \lambda, r) \in [0,1] \times \mathbb{R}_+^\star$ such that $\lambda > \mu$. Then,

$$d_\epsilon^-(\lambda,\mu) \ge \widetilde{d}_\epsilon^-(\lambda,\mu,r) - \frac{\log 4}{r} .$$

Proof. Since $\mu \in (0,1)$, we have $[\lambda]_0^1 = \max\{0, \lambda\}$. Therefore, we have $z - \lambda \ge z - [\lambda]_0^1$ and $z - [\lambda]_0^1 \in (0, \mu - [\lambda]_0^1) \subset (0,1)$ for all $z \in ([\lambda]_0^1, \mu)$. Using Lemmas 27 and 28 and $\epsilon > 0$, we obtain, for all $r > 0$ and all $z \in ([\lambda]_0^1, \mu)$,

$$\begin{aligned} \epsilon(z - [\lambda]_0^1) &\le \frac{1}{r} h(r\epsilon(z - [\lambda]_0^1)) + \frac{\log(1 + 2\epsilon(z - [\lambda]_0^1) r) + 1}{r} \\ &\le \frac{1}{r} h(r\epsilon(z-\lambda)) + \frac{\log(1+2\epsilon r)+1}{r} . \end{aligned}$$

Therefore, for all $z \in ([\lambda]_0^1, \mu)$, we obtain that

$$\mathrm{kl}(z,\mu) + \epsilon(z - [\lambda]_0^1) \le \mathrm{kl}(z,\mu) + \frac{1}{r} h(r\epsilon(z-\lambda)) + \frac{\log(1+2\epsilon r)+1}{r} .$$

Taking the infimum over $z \in ([\lambda]_0^1, \mu)$ on both sides of both inequalities and using that

$$\begin{aligned} d_\epsilon^+(\lambda,\mu) &= \inf_{z \in [[\lambda]_0^1, \mu]} \left\{ \mathrm{kl}(z,\mu) + \epsilon(z - [\lambda]_0^1) \right\} = \inf_{z \in ([\lambda]_0^1, \mu)} \left\{ \mathrm{kl}(z,\mu) + \epsilon(z - [\lambda]_0^1) \right\} , \\ \widetilde{d}_\epsilon^+(\lambda,\mu,r) &= \inf_{z \in ([\lambda]_0^1, \mu)} \left\{ \mathrm{kl}(z,\mu) + \frac{1}{r} h(r\epsilon(z-\lambda)) \right\} , \end{aligned}$$

we obtain

$$d_\epsilon^+(\lambda,\mu) \le \widetilde{d}_\epsilon^+(\lambda,\mu,r) + \frac{\log(1+2\epsilon r)+1}{r} .$$

This concludes the proof of the first result. Using Lemmas 22 and 26 yields the second result.

Suppose that $\lambda \in [0,1]$, hence $\lambda = [\lambda]_0^1$. Using Lemmas 27 and 28 and $\epsilon > 0$, we obtain, for all $r > 0$ and all $z \in ([\lambda]_0^1, \mu)$,

$$\frac{1}{r} h(r\epsilon(z-\lambda)) \leq \epsilon(z-\lambda) + \frac{\log 4 - \log(1 + 2\epsilon(z-\lambda)r)}{r} \leq \epsilon(z - [\lambda]_0^1) + \frac{\log 4}{r} \, .$$

Adding $\mathrm{kl}(z,\mu)$ on both sides and taking the infimum over $z \in ([\lambda]_0^1, \mu)$ on both sides of both inequalities yields the proof of third result. Using Lemmas 22 and 26 yields the forth result. □

Lemma 30 gathers regularity properties on the modified divergences $\widetilde{d}_\epsilon^+$. In particular, it gives a closed-form solution based on an implicit solution of a fixed-point equation. This is a key property used in our implementation to reduce the computational cost.

Lemma 30. *Let $\widetilde{d}_\epsilon^+$ as in Eq. (32), and $g_\epsilon^\pm$ as in Eq. (30). For all $\mu \in (0,1)$, $\lambda \in \mathbb{R}$ and $r > 0$, we have*

$$\widetilde{d}_\epsilon^+(\lambda, \mu, r) \\ = \begin{cases} 0 & \text{if } \mu \in (0, [\lambda]_0^1] \\ \mathrm{kl}(x_\epsilon^+(\lambda,\mu,r) + g_\epsilon^+(\mu), \mu) + \frac{1}{r} h(r\epsilon(x_\epsilon^+(\lambda,\mu,r) + g_\epsilon^+(\mu) - \lambda)) & \text{if } \mu \in ([\lambda]_0^1, 1) \end{cases} ,$$

where $x_\epsilon^+(\lambda,\mu,r) \in (\max\{0, \lambda - g_\epsilon^+(\mu)\}, \mu - g_\epsilon^+(\mu))$ is the unique solution for $x \in (\max\{0, \lambda - g_\epsilon^+(\mu)\}, \mu - g_\epsilon^+(\mu))$ of the equation

$$\log\left(1 + \frac{x}{g_\epsilon^+(\mu)(1 - x - g_\epsilon^+(\mu))}\right) + \epsilon\left(\frac{r\epsilon(x + g_\epsilon^+(\mu) - \lambda)}{\sqrt{(r\epsilon(x + g_\epsilon^+(\mu) - \lambda))^2 + 1} + 1} - 1\right) = 0 \, .$$

For all $(\mu, r) \in (0,1) \times \mathbb{R}_+^\star$, the function $\lambda \mapsto \widetilde{d}_\epsilon^+(\lambda,\mu,r)$ is positive, twice continuously differentiable, decreasing and strictly convex on $(-\infty, \mu)$; it satisfies $\lim_{\lambda \to \mu^-} \widetilde{d}_\epsilon^+(\lambda,\mu,r) = 0$ and $\lim_{\lambda \to -\infty} \widetilde{d}_\epsilon^+(\lambda,\mu,r) = +\infty$.

For all $(\lambda, r) \in \mathbb{R} \times \mathbb{R}_+^\star$, the function $\mu \mapsto \widetilde{d}_\epsilon^+(\lambda,\mu,r)$ is positive, twice continuously differentiable, increasing and strictly convex on $([\lambda]_0^1, 1)$. Moreover, we have

$$\forall \mu \in ([\lambda]_0^1, 1), \quad \frac{\partial \widetilde{d}_\epsilon^+}{\partial \mu}(\lambda,\mu,r) = \frac{\mu - g_\epsilon^+(\mu) - x_\epsilon^+(\lambda,\mu,r)}{\mu(1-\mu)} \, .$$

For all $(\lambda,\mu) \in \mathbb{R} \times (0,1)$ such that $\mu \in (0, [\lambda]_0^1]$, the function $r \mapsto \widetilde{d}_\epsilon^+(\lambda,\mu,r)$ is the zero function. For all $(\lambda,\mu) \in \mathbb{R} \times (0,1)$ such that $\mu \in ([\lambda]_0^1, 1)$, the function $r \mapsto \widetilde{d}_\epsilon^+(\lambda,\mu,r)$ is positive, continuously differentiable and increasing on $\mathbb{R}_+$.

Proof. By definition of the indicator function, we have $\widetilde{d}_\epsilon^+(\lambda,\mu,r) = 0$ if $\mu \in (0, [\lambda]_0^1]$. Let (λ,μ) such that $\mu \notin (0, [\lambda]_0^1]$, i.e., $([\lambda]_0^1, \mu)$ is non-empty. Since $\mu \in (0,1)$, this implies that $\lambda \in (-\infty, 1)$ necessarily, i.e., $[\lambda]_0^1 = \max\{0, \lambda\}$.

Recall that $\widetilde{d}_\epsilon^+(\lambda,\mu,r) = \mathbb{1}\left(\mu > [\lambda]_0^1\right) \inf_{z \in ([\lambda]_0^1, \mu)} \widetilde{f}_\epsilon^+(\lambda,\mu,r,z)$ where $\widetilde{f}_\epsilon^+(\lambda,\mu,r,z) = \mathrm{kl}(z,\mu) + \frac{1}{r} h(r\epsilon(z-\lambda))$. Using Lemma 27, direct computations yield that, for all $z \in ([\lambda]_0^1, \mu)$,

$$\begin{aligned} \frac{\partial \widetilde{f}_\epsilon^+}{\partial z}(\lambda,\mu,r,z) &= \log\left(\frac{z(1-\mu)}{(1-z)\mu}\right) + \epsilon h'(r\epsilon(z-\lambda)) \\ &= \log\left(\frac{z(1-\mu)}{(1-z)\mu}\right) + \epsilon \frac{r\epsilon(z-\lambda)}{\sqrt{(r\epsilon(z-\lambda))^2 + 1} + 1} \, , \\ \frac{\partial^2 \widetilde{f}_\epsilon^+}{\partial z^2}(\lambda,\mu,r,z) &= \frac{1}{z(1-z)} + r\epsilon^2 h''(r\epsilon(z-\lambda)) > 0 \, . \end{aligned}$$

Therefore, $z \to \widetilde{f}^{+}_{\epsilon}(\lambda,\mu,r,z)$ is twice continuously differentiable, positive and strictly convex on $([\lambda]_0^1, \mu)$. Moreover, we have

$$\begin{aligned} \lim_{z \to \mu} \frac{\partial \widetilde{f}^{+}_{\epsilon}}{\partial z}(\lambda,\mu,r,z) &= \epsilon h'(r\epsilon(\mu-\lambda)) > 0 \, , \\ \frac{\partial \widetilde{f}^{+}_{\epsilon}}{\partial z}(\lambda,\mu,r,g^{+}_{\epsilon}(\mu)) &= -\epsilon \left(1 - \frac{r\epsilon(z-\lambda)}{\sqrt{(r\epsilon(z-\lambda))^2+1}+1} \right) < 0 \, , \\ \text{When } [\lambda]_0^1 > g^{+}_{\epsilon}(\mu), \quad \frac{\partial \widetilde{f}^{+}_{\epsilon}}{\partial z}(\lambda,\mu,r,\lambda) &= \log\left(\frac{\lambda(1-\mu)}{(1-\lambda)\mu} \right) < 0 \, . \end{aligned}$$

Note that $\max\{[\lambda]_0^1, g^{+}_{\epsilon}(\mu)\} = \max\{\lambda, g^{+}_{\epsilon}(\mu)\}$ since $\mu \in (0,1)$. Using that $z \to \frac{\partial \widetilde{f}^{+}_{\epsilon}}{\partial z}(\lambda,\mu,r,z)$ is continuously differentiable and increasing on $([\lambda]_0^1, \mu)$, with negative value at $\max\{\lambda, g^{+}_{\epsilon}(\mu)\}$ and finite positive limit at μ, $z \mapsto \widetilde{f}^{+}_{\epsilon}(\lambda,\mu,r,z)$ admit a unique minimizer on $(\max\{\lambda, g^{+}_{\epsilon}(\mu)\}, \mu)$. Let $\widetilde{g}^{+}_{\epsilon}(\lambda,\mu,r) \in (\max\{\lambda, g^{+}_{\epsilon}(\mu)\}, \mu)$ be defined as this unique minimizer, defined implicitly as solution for $z \in (\max\{\lambda, g^{+}_{\epsilon}(\mu)\}, \mu)$ of the equation

$$\log\left(\frac{z(1-\mu)}{(1-z)\mu} \right) + \epsilon \frac{r\epsilon(z-\lambda)}{\sqrt{(r\epsilon(z-\lambda))^2+1}+1} = 0 \, .$$

Then, we have $\frac{\partial \widetilde{f}^{+}_{\epsilon}}{\partial z}(\lambda,\mu,r,z) = 0$ if and only if $z = \widetilde{g}^{+}_{\epsilon}(\lambda,\mu,r)$. Moreover, $z \mapsto \widetilde{f}^{+}_{\epsilon}(\lambda,\mu,r,z)$ is decreasing on $([\lambda]_0^1, \widetilde{g}^{+}_{\epsilon}(\lambda,\mu,r))$ and increasing on $(\widetilde{g}^{+}_{\epsilon}(\lambda,\mu,r), \mu)$.

Let us define $z = g^{+}_{\epsilon}(\mu) + x$ where $x \in (\max\{0, \lambda - g^{+}_{\epsilon}(\mu)\}, \mu - g^{+}_{\epsilon}(\mu))$. Then, we have

$$\begin{aligned} &\frac{\partial \widetilde{f}^{+}_{\epsilon}}{\partial z}(\lambda,\mu,r,g^{+}_{\epsilon}(\mu)+x) \\ &= \log\left(1 + \frac{x}{g^{+}_{\epsilon}(\mu)(1-x-g^{+}_{\epsilon}(\mu))} \right) + \epsilon \left(\frac{r\epsilon(x+g^{+}_{\epsilon}(\mu)-\lambda)}{\sqrt{(r\epsilon(x+g^{+}_{\epsilon}(\mu)-\lambda))^2+1}+1} - 1 \right) \end{aligned}$$

Therefore, we have $\widetilde{g}^{+}_{\epsilon}(\lambda,\mu,r) = g^{+}_{\epsilon}(\mu) + x^{+}_{\epsilon}(\lambda,\mu,r)$ where $x^{+}_{\epsilon}(\lambda,\mu,r) \in (\max\{0, \lambda - g^{+}_{\epsilon}(\mu)\}, \mu - g^{+}_{\epsilon}(\mu))$ is the solution for $x \in (\max\{0, \lambda - g^{+}_{\epsilon}(\mu)\}, \mu - g^{+}_{\epsilon}(\mu))$ of the equation

$$\log\left(1 + \frac{x}{g^{+}_{\epsilon}(\mu)(1-x-g^{+}_{\epsilon}(\mu))} \right) + \epsilon \left(\frac{r\epsilon(x+g^{+}_{\epsilon}(\mu)-\lambda)}{\sqrt{(r\epsilon(x+g^{+}_{\epsilon}(\mu)-\lambda))^2+1}+1} - 1 \right) = 0 \, .$$

When $\lambda \in (0,1)$ and $\mu \to \lambda = [\lambda]_0^1$, it is direct to see that $\widetilde{g}^{+}_{\epsilon}(\lambda,\mu,r) \to \lambda$. Then, we have

$$\lim_{\mu \to \lambda^+} \widetilde{d}^{+}_{\epsilon}(\lambda,\mu,r) = \lim_{\mu \to \lambda^+} \{\mathrm{kl}(\widetilde{g}^{+}_{\epsilon}(\lambda,\mu,r),\mu)\} + \frac{1}{r} \lim_{\widetilde{g}^{+}_{\epsilon}(\lambda,\mu,r) \to \lambda^+} \{h(r\epsilon(\widetilde{g}^{+}_{\epsilon}(\lambda,\mu,r)-\lambda))\} = 0 \, .$$

Direct computation yields that, for $z \in ([\lambda]_0^1, \mu) \subset (0,1)$,

$$\begin{aligned} \frac{\partial \widetilde{f}^{+}_{\epsilon}}{\partial \mu}(\lambda,\mu,r,z) &= \frac{\mu - z}{\mu(1-\mu)} > 0 \, , \\ \frac{\partial^2 \widetilde{f}^{+}_{\epsilon}}{\partial \mu^2}(\lambda,\mu,r,z) &= \frac{(\mu-z)^2 + z(1-z)}{\mu^2(1-\mu)^2} > 0 \, , \\ \frac{\partial^2 \widetilde{f}^{+}_{\epsilon}}{\partial z^2}(\lambda,\mu,r,z) &= \frac{1}{z(1-z)} + r\epsilon^2 h''(r\epsilon(z-\lambda)) > 0 \, , \\ \frac{\partial^2 \widetilde{f}^{+}_{\epsilon}}{\partial \mu \partial z}(\lambda,\mu,r,z) &= -\frac{1}{\mu(1-\mu)} < 0 \, . \end{aligned}$$

Since $\frac{\partial \widetilde{f}^{+}_{\epsilon}}{\partial z}(\lambda,\mu,r,g^{+}_{\epsilon}(\lambda,\mu,r)) = 0$, the implicit function theorem yields that

$$\frac{\partial \widetilde{g}^{+}_{\epsilon}}{\partial \mu}(\lambda,\mu,r) = -\frac{\frac{\partial^2 f^{+}_{\epsilon}}{\partial \mu \partial z}(\lambda,\mu,r,g^{+}_{\epsilon}(\lambda,\mu,r))}{\frac{\partial^2 \widetilde{f}^{+}_{\epsilon}}{\partial z^2}(\lambda,\mu,r,g^{+}_{\epsilon}(\lambda,\mu,r))} > 0 \, .$$

Moreover, for $\mu \in ([\lambda]_0^1, 1)$,

$$
\begin{aligned}
\frac{\partial \widetilde{d}_\epsilon^+}{\partial \mu}(\lambda, \mu, r) &= \frac{\partial \widetilde{f}_\epsilon^+}{\partial \mu}(\lambda, \mu, r, \widetilde{g}_\epsilon^+(\lambda, \mu, r)) + \frac{\partial \widetilde{g}_\epsilon^+}{\partial \mu}(\lambda, \mu, r)\frac{\partial \widetilde{f}_\epsilon^+}{\partial z}(\lambda, \mu, r, \widetilde{g}_\epsilon^+(\lambda, \mu, r)) \\
&= \frac{\partial \widetilde{f}_\epsilon^+}{\partial \mu}(\lambda, \mu, r, \widetilde{g}_\epsilon^+(\lambda, \mu, r)) = \frac{\mu - g_\epsilon^+(\mu) - x_\epsilon^+(\lambda, \mu, r)}{\mu(1-\mu)} > 0 \, , \\
\frac{\partial^2 \widetilde{d}_\epsilon^+}{\partial \mu^2}(\lambda, \mu, r) &= \frac{\partial^2 \widetilde{f}_\epsilon^+}{\partial \mu^2}(\lambda, \mu, r, \widetilde{g}_\epsilon^+(\lambda, \mu, r))\frac{\partial \widetilde{g}_\epsilon^+}{\partial \mu}(\lambda, \mu, r) > 0 \, .
\end{aligned}
$$

Therefore, for all $(\lambda, r) \in \mathbb{R} \times \mathbb{R}_+^\star$, the function $\mu \mapsto \widetilde{d}_\epsilon^+(\lambda, \mu, r)$ is positive, twice continuously differentiable, increasing and strictly convex on $([\lambda]_0^1, 1)$.

Let $(\mu, r) \in (0,1) \times \mathbb{R}_+^\star$. Direct computation yields that, for $z \in ([\lambda]_0^1, \mu) \subset (0,1)$,

$$
\begin{aligned}
\frac{\partial \widetilde{f}_\epsilon^+}{\partial \lambda}(\lambda, \mu, r, z) &= -\epsilon h'(r\epsilon(z - \lambda)) < 0 \, , \\
\frac{\partial^2 \widetilde{f}_\epsilon^+}{\partial \lambda \partial z}(\lambda, \mu, r, z) &= -r\epsilon^2 h''(r\epsilon(z - \lambda)) < 0 \, .
\end{aligned}
$$

Since $\frac{\partial \widetilde{f}_\epsilon^+}{\partial z}(\lambda, \mu, r, g_\epsilon^+(\lambda, \mu, r)) = 0$, the implicit function theorem yields that

$$
\frac{\partial \widetilde{g}_\epsilon^+}{\partial \lambda}(\lambda, \mu, r) = -\frac{\frac{\partial^2 f_\epsilon^+}{\partial \lambda \partial z}(\lambda, \mu, r, g_\epsilon^+(\lambda, \mu, r))}{\frac{\partial^2 \widetilde{f}_\epsilon^+}{\partial z^2}(\lambda, \mu, r, g_\epsilon^+(\lambda, \mu, r))} = \frac{r\epsilon^2 h''(r\epsilon(g_\epsilon^+(\lambda, \mu, r) - \lambda))}{\frac{1}{z(1-z)} + r\epsilon^2 h''(r\epsilon(g_\epsilon^+(\lambda, \mu, r) - \lambda))} < 1 \, .
$$

Direct computation yields that, for $\lambda \in (-\infty, \mu)$,

$$
\begin{aligned}
\frac{\partial \widetilde{d}_\epsilon^+}{\partial \lambda}(\lambda, \mu, r) &= \frac{\partial \widetilde{f}_\epsilon^+}{\partial \lambda}(\lambda, \mu, r, \widetilde{g}_\epsilon^+(\lambda, \mu, r)) + \frac{\partial \widetilde{g}_\epsilon^+}{\partial \lambda}(\lambda, \mu, r)\frac{\partial \widetilde{f}_\epsilon^+}{\partial z}(\lambda, \mu, r, \widetilde{g}_\epsilon^+(\lambda, \mu, r)) \\
&= \frac{\partial \widetilde{f}_\epsilon^+}{\partial \lambda}(\lambda, \mu, r, \widetilde{g}_\epsilon^+(\lambda, \mu, r)) = -\epsilon h'(r\epsilon(\widetilde{g}_\epsilon^+(\lambda, \mu, r) - \lambda)) < 0 \, , \\
\frac{\partial^2 \widetilde{d}_\epsilon^+}{\partial \lambda^2}(\lambda, \mu, r) &= r\epsilon^2 \left(1 - \frac{\partial \widetilde{g}_\epsilon^+}{\partial \lambda}(\lambda, \mu, r)\right) h''(r\epsilon(\widetilde{g}_\epsilon^+(\lambda, \mu, r) - \lambda)) > 0 \, .
\end{aligned}
$$

Therefore, for all $(\mu, r) \in (0,1) \times \mathbb{R}_+^\star$, the function $\lambda \mapsto \widetilde{d}_\epsilon^+(\lambda, \mu, r)$ is positive, twice continuously differentiable, decreasing and strictly convex on $(-\infty, \mu)$. Similarly as above, it is direct to see that $\lim_{\lambda \to \mu^-} \widetilde{d}_\epsilon^+(\lambda, \mu, r) = 0$ and $\lim_{\lambda \to -\infty} \widetilde{d}_\epsilon^+(\lambda, \mu, r) = +\infty$.

Let $(\lambda, \mu) \in \mathbb{R} \times (0,1)$. When $\mu \in (0, [\lambda]_0^1]$, we have $\widetilde{d}_\epsilon^+(\lambda, \mu, r) = 0$ for all $r \in [1, +\infty)$, hence $r \mapsto \widetilde{d}_\epsilon^+(\lambda, \mu, r)$ is non-decreasing. Let κ as in Lemma 28. Using Lemma 28, we have

$$
\forall z > \lambda, \quad \frac{\partial \widetilde{f}_\epsilon^+}{\partial r}(\lambda, \mu, r, z) = \frac{\partial \kappa}{\partial r}(r, \epsilon(z - \lambda)) > 0 \, .
$$

When $\mu \in ([\lambda]_0^1, 1)$, we have $\widetilde{g}_\epsilon^+(\lambda, \mu, r) \in (\max\{\lambda, g_\epsilon^+(\mu)\}, \mu)$ and, for all $r > 0$,

$$
\begin{aligned}
\frac{\partial \widetilde{d}_\epsilon^+}{\partial r}(\lambda, \mu, r) &= \frac{\partial \widetilde{f}_\epsilon^+}{\partial r}(\lambda, \mu, r, \widetilde{g}_\epsilon^+(\lambda, \mu, r)) + \frac{\partial \widetilde{g}_\epsilon^+}{\partial r}(\lambda, \mu, r)\frac{\partial \widetilde{f}_\epsilon^+}{\partial z}(\lambda, \mu, r, \widetilde{g}_\epsilon^+(\lambda, \mu, r)) \\
&= \frac{\partial \widetilde{f}_\epsilon^+}{\partial r}(\lambda, \mu, r, \widetilde{g}_\epsilon^+(\lambda, \mu, r)) > 0 \, ,
\end{aligned}
$$

where we used that $\widetilde{g}_\epsilon^+(\lambda, \mu, r) > \lambda$. This concludes the last part of the proof. □

Lemma 31 gathers regularity properties on the modified divergences $\widetilde{d}_\epsilon^-$. In particular, it gives a closed-form solution based on an implicit solution of a fixed-point equation. This is a key property used in our implementation to reduce the computational cost.

Lemma 31. *Let $\widetilde{d}_\epsilon^-$ as in Eq.* (32)*, and $g_\epsilon^\pm$ as in Eq.* (30)*. For all $\mu \in (0,1)$, $\lambda \in \mathbb{R}$ and $r > 0$, we have*

$$\widetilde{d}_\epsilon^-(\lambda,\mu,r) = \begin{cases} 0 & \text{if } \mu \in [[\lambda]_0^1, 1) \\ \mathrm{kl}(g_\epsilon^-(\mu) - x_\epsilon^-(\lambda,\mu,r),\mu) + \frac{1}{r} h(r\epsilon(x_\epsilon^-(\lambda,\mu,r) + \lambda - g_\epsilon^-(\mu))) & \text{if } \mu \in (0,[\lambda]_0^1) \end{cases},$$

where $x_\epsilon^-(\lambda,\mu,r) := x_\epsilon^+(1-\lambda,1-\mu,r) \in (\max\{g_\epsilon^-(\mu)-\lambda,0\}, g_\epsilon^-(\mu)-\mu)$ is the solution for $x \in (\max\{g_\epsilon^-(\mu)-\lambda,0\}, g_\epsilon^-(\mu)-\mu)$ of the equation

$$\log\left(1 + \frac{x}{(1-g_\epsilon^-(\mu))(g_\epsilon^-(\mu)-x)}\right) + \epsilon\left(\frac{r\epsilon(x - g_\epsilon^-(\mu)+\lambda)}{\sqrt{(r\epsilon(x-g_\epsilon^-(\mu)+\lambda))^2+1}+1} - 1\right) = 0\,.$$

For all $(\mu,r) \in (0,1)\times\mathbb{R}_+^\star$, the function $\lambda \mapsto \widetilde{d}_\epsilon^-(\lambda,\mu,r)$ is positive, twice continuously differentiable, increasing and strictly convex on $(\mu,+\infty)$; it satisfies $\lim_{\lambda\to\mu^+}\widetilde{d}_\epsilon^+(\lambda,\mu,r) = 0$ and $\lim_{\lambda\to+\infty}\widetilde{d}_\epsilon^-(\lambda,\mu,r) = +\infty$.

For all $(\lambda,r) \in \mathbb{R}\times\mathbb{R}_+^\star$, the function $\mu \mapsto \widetilde{d}_\epsilon^-(\lambda,\mu,r)$ is positive, twice continuously differentiable, decreasing and strictly convex on $(0,[\lambda]_0^1)$. Moreover, we have

$$\forall \mu \in (0,[\lambda]_0^1), \quad \frac{\partial \widetilde{d}_\epsilon^-}{\partial \mu}(\lambda,\mu,r) = \frac{\mu - g_\epsilon^-(\mu) + x_\epsilon^-(\lambda,\mu,r)}{\mu(1-\mu)}\,.$$

For all $(\lambda,\mu) \in \mathbb{R}\times(0,1)$ such that $\mu \in (0,[\lambda]_0^1]$, the function $r \mapsto \widetilde{d}_\epsilon^-(\lambda,\mu,r)$ is the zero function. For all $(\lambda,\mu) \in \mathbb{R}\times(0,1)$ such that $\mu \in ([\lambda]_0^1,1)$, the function $r \mapsto \widetilde{d}_\epsilon^-(\lambda,\mu,r)$ is positive, continuously differentiable and increasing on $\mathbb{R}_+$.

Proof. Using Lemmas 26 and 23, we have

$$\widetilde{d}_\epsilon^-(\lambda,\mu,r) = \widetilde{d}_\epsilon^+(1-\lambda,1-\mu,r) \quad \text{and} \quad g_\epsilon^+(\lambda) = 1 - g_\epsilon^-(1-\lambda)\,,$$
$$\frac{\partial \widetilde{d}_\epsilon^-}{\partial\mu}(\lambda,\mu,r) = -\frac{\partial \widetilde{d}_\epsilon^+}{\partial\mu}(1-\lambda,1-\mu,r) \quad \text{and} \quad \frac{\partial^2 \widetilde{d}_\epsilon^-}{\partial\mu^2}(\lambda,\mu,r) = \frac{\partial^2 \widetilde{d}_\epsilon^+}{\partial\mu^2}(1-\lambda,1-\mu,r)\,.$$

Let $x_\epsilon^+(1-\lambda,1-\mu,r) \in (\max\{0,g_\epsilon^-(\mu)-\lambda\}, g_\epsilon^-(\mu)-\mu)$ be the unique solution for $x \in (\max\{0,g_\epsilon^-(\mu)-\lambda\}, g_\epsilon^-(\mu)-\mu)$ of the equation

$$\log\left(1 + \frac{x}{(1-g_\epsilon^-(\mu))(g_\epsilon^-(\mu)-x)}\right) + \epsilon\left(\frac{r\epsilon(x - g_\epsilon^-(\mu)+\lambda)}{\sqrt{(r\epsilon(x-g_\epsilon^-(\mu)+\lambda))^2+1}+1} - 1\right) = 0\,,$$

where we used $g_\epsilon^+(1-\mu) = 1 - g_\epsilon^-(\mu)$ to simplify the formula given in Lemma 30. Therefore, we define $x_\epsilon^-(\lambda,\mu,r) = x_\epsilon^+(1-\lambda,1-\mu,r)$. Then, we have

$$\mathrm{kl}(g_\epsilon^-(\mu) - x_\epsilon^-(\lambda,\mu,r),\mu) + \frac{1}{r}h(r\epsilon(x_\epsilon^-(\lambda,\mu,r)+\lambda-g_\epsilon^-(\mu))) =$$
$$\mathrm{kl}(x_\epsilon^+(1-\lambda,1-\mu,r) + g_\epsilon^+(1-\mu), 1-\mu) + \frac{h(r\epsilon(x_\epsilon^+(1-\lambda,1-\mu,r)+g_\epsilon^+(1-\mu)-1+\lambda))}{r}$$

where we used that $\mathrm{kl}(g_\epsilon^-(\mu) - x_\epsilon^-(\lambda,\mu,r),\mu) = \mathrm{kl}(1-g_\epsilon^-(\mu)+x_\epsilon^-(\lambda,\mu,r),1-\mu)$. Combining the above with the properties on $\widetilde{d}_\epsilon^+$ in Lemma 30 concludes the proof. □

Lemma 32 shows that we can invert $\widetilde{d}_\epsilon^\pm$ with respect to their first argument, which is a key property used in Appendix F.

Lemma 32. *Let $\widetilde{d}_\epsilon^\pm$ as in Eq.* (32)*. For all $(\mu,r,c) \in (0,1)\times\mathbb{R}_+^\star\times\mathbb{R}_+^\star$, there exists $x > 0$ such that $\widetilde{d}_\epsilon^+(\mu - x,\mu,r) = c$. For all $(\mu,r,c) \in (0,1)\times\mathbb{R}_+^\star\times\mathbb{R}_+^\star$, there exists $x > 0$ such that $\widetilde{d}_\epsilon^-(\mu+x,\mu,r) = c$.*

Proof. Let us define $f(x) = \widetilde{d}_\epsilon^+(\mu - x, \mu, r)$ for all $x > 0$. Using Lemma 30, we know that f is continuous and increasing on $\mathbb{R}_+^\star$ and it satisfies $\lim_{x\to 0^+} f(x) = 0$ and $\lim_{x\to+\infty} f(x) = +\infty$. Therefore, there exists a unique $x > 0$ such that $\widetilde{d}_\epsilon^+(\mu - x, \mu, r) = c$. Using Lemma 31, we can conclude similarly for $\widetilde{d}_\epsilon^-$. □

Lemma 32 shows that $\widetilde{d}_\epsilon^\pm$ is non-decreasing with respect to their first argument, which is a key property used in Appendix F.

Lemma 33. *Let $\widetilde{d}_\epsilon^\pm$ as in Eq.* (32)*. For all $(\mu, r) \in (0, 1) \times \mathbb{R}_+^\star$ and all $(\lambda_1, \lambda_2) \in \mathbb{R} \times (-\infty, \mu)$,*

$$\widetilde{d}_\epsilon^+(\lambda_1, \mu, r) \geq \widetilde{d}_\epsilon^+(\lambda_2, \mu, r) \quad \implies \quad \lambda_1 \leq \lambda_2$$

For all $(\mu, r) \in (0, 1) \times \mathbb{R}_+^\star$ and all $(\lambda_1, \lambda_2) \in \mathbb{R} \times (\mu, +\infty)$,

$$\widetilde{d}_\epsilon^-(\lambda_1, \mu, r) \geq \widetilde{d}_\epsilon^-(\lambda_2, \mu, r) \quad \implies \quad \lambda_1 \geq \lambda_2$$

Proof. Using Lemma 30, we known that $\lambda \mapsto \widetilde{d}_\epsilon^+(\lambda, \mu, r)$ is decreasing on $(-\infty, \mu)$. Let $(\lambda_1, \lambda_2) \in \mathbb{R} \times (-\infty, \mu)$. Then, we have

$$\lambda_1 > \lambda_2 \quad \implies \quad \widetilde{d}_\epsilon^+(\lambda_1, \mu, r) < \widetilde{d}_\epsilon^+(\lambda_2, \mu, r) \, ,$$

which is equivalent to the statement of the lemma by contraposition. Using Lemma 31, we can conclude similarly for $\widetilde{d}_\epsilon^-$. □

G.2 Transportation Cost

Recall that $W_{\epsilon,a,b}$ is defined in Eq. (50), i.e., for all $(\mu, w) \in \mathbb{R}^K \times \mathbb{R}_+^K$,

$$\forall (a, b) \in [K]^2, \quad W_{\epsilon,a,b}(\mu, w) := \mathbb{1}\left([\mu_a]_0^1 > [\mu_b]_0^1\right) \inf_{u\in[0,1]} \left\{w_a d_\epsilon^-(\mu_a, u) + w_b d_\epsilon^+(\mu_b, u)\right\} \, ,$$

where $d_\epsilon^\pm$ are defined in Eq. (3).

Lemma 34 gathers regularity properties on the transportation costs.

Lemma 34. *Let $d_\epsilon^\pm$ as in Eq.* (3)*. For all $(\lambda, \mu) \in (0, 1)^2$ such that $\lambda \geq \mu$ and $w \in \mathbb{R}_+^2$.*

- *The function $u \mapsto w_1 d_\epsilon^-(\lambda, u) + w_2 d_\epsilon^+(\mu, u)$ is strictly convex on $[\mu, \lambda]$ when $\max\{w_1, w_2\} > 0$ and on $[0, 1]$ when $\min\{w_1, w_2\} > 0$. Then,*

$$\inf_{u\in[0,1]}\{w_1 d_\epsilon^-(\lambda, u) + w_2 d_\epsilon^+(\mu, u)\} = \inf_{u\in[\mu,\lambda]}\{w_1 d_\epsilon^-(\lambda, u) + w_2 d_\epsilon^+(\mu, u)\} \, .$$

- *The function $(\lambda, \mu, w) \mapsto \inf_{u\in[0,1]}\{w_1 d_\epsilon^-(\lambda, u) + w_2 d_\epsilon^+(\mu, u)\}$ is continuous on $(0, 1) \times (0, 1) \times \mathbb{R}_+^2$.*
- *If $\max\{w_1, w_2\} > 0$, $u_\star(\lambda, \mu, w) = \arg\min_{u\in[0,1]}\{w_1 d_\epsilon^-(\lambda, u) + w_2 d_\epsilon^+(\mu, u)\}$ is unique and continuous on $(0, 1) \times (0, 1) \times \mathbb{R}_+^2$.*
- *If $\min\{w_1, w_2\} > 0$ and $\lambda > \mu$, $u_\star(\lambda, \mu, w) \in (\mu, \lambda)$ and $\min\{d_\epsilon^-(\lambda, u_\star(\lambda, \mu, w)), d_\epsilon^+(\mu, u_\star(\lambda, \mu, w))\} > 0$.*

Moreover,

$$\inf_{u\in[0,1]}\{w_1 d_\epsilon^-(\lambda, u) + w_2 d_\epsilon^+(\mu, u)\} = \inf_{(u_1,u_2)\in[0,1]^2 \,:\, u_1\leq u_2}\{w_1 d_\epsilon^-(\lambda, u_1) + w_2 d_\epsilon^+(\mu, u_2)\} \, .$$

Proof. These results are obtained by leveraging Lemmas 24 and 25 at each step.

For $u \leq \mu$, the function is equal to $w_1 d_\epsilon^-(\lambda, u)$, which is decreasing and strictly convex on $[0, \lambda)$ unless $w_1 = 0$ since $u \leq \mu \leq \lambda$. Therefore, the minimum over that interval is attained at μ. For $u \geq \lambda$, the function is equal to $w_2 d_\epsilon^+(\mu, u)$, which is increasing and strictly convex on $(\mu, 1]$ unless $w_2 = 0$ since $u \geq \lambda \geq \mu$. Therefore, the minimum over that interval is attained at λ. On the interval (μ, λ), the function is equal to $w_1 d_\epsilon^-(\lambda, u) + w_2 d_\epsilon^+(\mu, u)$, hence it is the sum of two convex functions, one of which is strictly convex. Furthermore, the function is continuous at μ and λ. This concludes the first part of the proof.

As we have just shown, we can restrict the infimum to $[\mu, \lambda]$. We apply Berge's Maximum theorem [16, page 116]. Let

$$\begin{aligned}
\phi(u, \lambda, \mu, w) &= -w_1 d_\epsilon^-(\lambda, u) - w_2 d_\epsilon^+(\mu, u) \,, \\
\Gamma(\lambda, \mu, w) &= [\mu, \lambda] \,, \\
M(\lambda, \mu, w) &= \max\{\phi(u, \lambda, \mu, w) \mid u \in \Gamma(\lambda, \mu, w)\} \,, \\
\Phi(\lambda, \mu, w) &= \arg\max\{\phi(u, \lambda, \mu, w) \mid u \in \Gamma(\lambda, \mu, w)\} \,.
\end{aligned}$$

We verify the hypotheses of the theorem:

- ϕ is continuous on $[\mu, \lambda] \times (0,1) \times (0,1) \times \mathbb{R}_+^2$, by using the properties in Lemmas 24 and 25 since $(\lambda, \mu) \in (0,1)^2$.
- Γ is nonempty, compact-valued and continuous (since constant).

We obtain that M is continuous on $(0,1) \times (0,1) \times \mathbb{R}_+^2$ and that Φ is upper hemicontinuous. This concludes the second part of the proof.

When $\max\{w_1, w_2\} > 0$, we have just shown that ϕ is a strictly concave function of u. Combining this with the fact that Γ is convex, we can argue as in [78, Theorem 9.17] to prove that Φ is a single-valued upper hemicontinuous correspondence, hence a continuous function. This concludes the third part of the proof.

Suppose that $\min\{w_1, w_2\} > 0$ and $\lambda > \mu$. Using Lemmas 24 and 25, the function $u \mapsto w_1 d_\epsilon^-(\lambda, u) + w_2 d_\epsilon^+(\mu, u)$ is continuously differentiable on (μ, λ) with derivative $w_1 \frac{\partial d_\epsilon^-}{\partial u}(\lambda, u) + w_2 \frac{\partial d_\epsilon^+}{\partial u}(\mu, u)$ where

$$\begin{aligned}
\forall u \in (\mu, 1], \quad \frac{\partial d_\epsilon^+}{\partial u}(\mu, u) &= \begin{cases} \frac{1 - e^{-\epsilon}}{1 - u(1 - e^{-\epsilon})} & \text{if } u \in (g_\epsilon^-(\mu), 1] \\ \frac{u - \mu}{u(1-u)} & \text{if } u \in (\mu, g_\epsilon^-(\mu)] \end{cases} \,, \\
\forall u \in [0, \lambda), \quad \frac{\partial d_\epsilon^-}{\partial u}(\lambda, u) &= \begin{cases} -\frac{e^{\epsilon} - 1}{1 + u(e^{\epsilon} - 1)} & \text{if } u \in [0, g_\epsilon^+(\lambda)) \\ -\frac{\lambda - u}{u(1-u)} & \text{if } u \in [g_\epsilon^+(\lambda), \lambda) \end{cases} \,.
\end{aligned}$$

Since $\frac{\partial d_\epsilon^-}{\partial u}(\lambda, u) \to_{u \to \lambda^-} 0$ and $\frac{\partial d_\epsilon^+}{\partial u}(\mu, u) \to_{u \to \mu^+} 0$, we obtain

$$\begin{aligned}
\lim_{u \to \lambda^-} \left\{ w_1 \frac{\partial d_\epsilon^-}{\partial u}(\lambda, u) + w_2 \frac{\partial d_\epsilon^+}{\partial u}(\mu, u) \right\} &= w_2 \frac{\partial d_\epsilon^+}{\partial u}(\mu, \lambda) > 0 \,, \\
\lim_{u \to \mu^+} \left\{ w_1 \frac{\partial d_\epsilon^-}{\partial u}(\lambda, u) + w_2 \frac{\partial d_\epsilon^+}{\partial u}(\mu, u) \right\} &= w_1 \frac{\partial d_\epsilon^-}{\partial u}(\lambda, \mu) < 0 \,.
\end{aligned}$$

Therefore, the infimum is attained inside the open interval. Using Lemmas 24 and 25, we can conclude the proof of the first part of the fourth property.

Using the strict convexity of $u_1 \mapsto w_1 d_\epsilon^-(\lambda, u_1)$ and $u_2 \mapsto w_2 d_\epsilon^+(\mu, u_2)$ on (μ, λ), we obtain that

$$\inf_{u \in (\mu, \lambda)} \{w_1 d_\epsilon^-(\lambda, u) + w_2 d_\epsilon^+(\mu, u)\} = \inf_{(u_1, u_2) \,:\, \mu < u_1 \le u_2 < \lambda} \{w_1 d_\epsilon^-(\lambda, u_1) + w_2 d_\epsilon^+(\mu, u_2)\} \,.$$

Re-using the same arguments as above, we obtain that

$$\begin{aligned}
&\inf_{(u_1, u_2) \,:\, \mu < u_1 \le u_2 < \lambda} \{w_1 d_\epsilon^-(\lambda, u_1) + w_2 d_\epsilon^+(\mu, u_2)\} \\
&= \inf_{(u_1, u_2) \in [0,1]^2 \,:\, u_1 \le u_2} \{w_1 d_\epsilon^-(\lambda, u_1) + w_2 d_\epsilon^+(\mu, u_2)\} \,.
\end{aligned}$$

This concludes the proof of the second part of the fourth property. □

Lemma 35 relates the transportation costs $W_{\epsilon, a^\star, a}$ with the transportation costs used in Eq. (1) to define the characteristic time. Crucially, this shows the equivalence with the definitions in Eq. (35).

Lemma 35. *Let $W_{\epsilon,a^\star,a}$ and d_ϵ as in Eq.* (4) *and* (2)*. Let $\mu \in (0,1)^K$ such that $a^\star(\mu) = \{a^\star\}$. Let Alt$(\mu) = \{\lambda \in (0,1)^K \mid a^\star(\lambda) \neq \{a^\star\}\}$. Then,*

$$\forall w \in \triangle_K, \quad \inf_{\lambda \in \mathit{Alt}(\mu)} \sum_{a\in[K]} w_a \mathrm{d}_\epsilon(\mu_a, \lambda_a) = \min_{a\neq a^\star} W_{\epsilon,a^\star,a}(\mu, w) \, .$$

Proof. It is direct to see that $\mathrm{Alt}(\mu) = \bigcup_{a\neq a^\star} \mathcal{C}_a$ where $\mathcal{C}_a = \{\lambda \in (0,1)^K \mid \lambda_a \geq \lambda_{a^\star}\}$. Then,

$$\forall w \in \triangle_K, \quad \inf_{\lambda\in\mathrm{Alt}(\mu)} \sum_{a\in[K]} w_a d_\epsilon(\mu_a, \lambda_a) = \min_{a\neq a^\star} \inf_{\lambda\in\mathcal{C}_a} \sum_{c\in[K]} w_c d_\epsilon(\mu_c, \lambda_c) \, .$$

By non-negativity of $d_\epsilon(\mu_a, \lambda_a)$ for all $a \in [K]$, we obtain

$$\begin{aligned} \inf_{\lambda\in\mathcal{C}_a} \sum_{c\in[K]} w_c d_\epsilon(\mu_c, \lambda_c) &= \inf_{\lambda\in\mathcal{C}_a} \sum_{c\in\{a,a^\star\}} w_c d_\epsilon(\mu_c, \lambda_c) \\ &= \inf_{(\lambda_a,\lambda_{a^\star})\in(0,1)^2,\ \lambda_a\geq\lambda_{a^\star}} \sum_{c\in\{a,a^\star\}} w_c d_\epsilon(\mu_c, \lambda_c) \, , \end{aligned}$$

where the two equalities are obtained by choosing $\lambda(a) \in (0,1)^K$ such that $\lambda(a)_b = \mu_b$ for all $b \notin \{a, a^\star\}$ with the two other coordinates choosen freely such that $\lambda(a)_a \geq \lambda(a)_{a^\star}$. Using that $\mu_{a^\star} > \mu_a$, we can partition this set as follows

$$\begin{aligned} \mathcal{C}_{a,a^\star} = \{(\lambda_a, \lambda_{a^\star}) \in (0,1)^2 \mid \ \lambda_a \geq \lambda_{a^\star}\} = &\{(\lambda_a, \lambda_{a^\star}) \in (0,\mu_a)^2 \mid \ \lambda_a \geq \lambda_{a^\star}\} \\ &\cup \{(\lambda_a, \lambda_{a^\star}) \in [\mu_a, \mu_{a^\star}] \times (0, \mu_a)\} \\ &\cup \{(\lambda_a, \lambda_{a^\star}) \in (\mu_{a^\star}, 1)^2 \mid \ \lambda_a \geq \lambda_{a^\star}\} \\ &\cup \{(\lambda_a, \lambda_{a^\star}) \in (\mu_{a^\star}, 1) \times [\mu_a, \mu_{a^\star}]\} \\ &\cup \{(\lambda_a, \lambda_{a^\star}) \in [\mu_a, \mu_{a^\star}]^2 \mid \ \lambda_a \geq \lambda_{a^\star}\} \, . \end{aligned}$$

Using Lemma 21, $\mu_{a^\star} > \mu_a$ and Lemmas 24 and 25, we obtain

$$\begin{aligned} &\inf_{(\lambda_a,\lambda_{a^\star})\in(0,\mu_a)^2 \mid \lambda_a\geq\lambda_{a^\star}} \{w_{a^\star} d_\epsilon(\mu_{a^\star}, \lambda_{a^\star}) + w_a d_\epsilon(\mu_a, \lambda_a)\} \\ &= \inf_{(\lambda_a,\lambda_{a^\star})\in(0,\mu_a)^2 \mid \lambda_a\geq\lambda_{a^\star}} \{w_{a^\star} d_\epsilon^-(\mu_{a^\star}, \lambda_{a^\star}) + w_a d_\epsilon^-(\mu_a, \lambda_a)\} = w_{a^\star} d_\epsilon^-(\mu_{a^\star}, \mu_a) \, , \\ &\inf_{(\lambda_a,\lambda_{a^\star})\in[\mu_a,\mu_{a^\star}]\times(0,\mu_a)} \{w_{a^\star} d_\epsilon(\mu_{a^\star}, \lambda_{a^\star}) + w_a d_\epsilon(\mu_a, \lambda_a)\} \\ &= \inf_{(\lambda_a,\lambda_{a^\star})\in[\mu_a,\mu_{a^\star}]\times(0,\mu_a)} \{w_{a^\star} d_\epsilon^-(\mu_{a^\star}, \lambda_{a^\star}) + w_a d_\epsilon^+(\mu_a, \lambda_a)\} = w_{a^\star} d_\epsilon^-(\mu_{a^\star}, \mu_a) \, , \\ &\inf_{(\lambda_a,\lambda_{a^\star})\in(\mu_{a^\star},1)^2 \mid \lambda_a\geq\lambda_{a^\star}} \{w_{a^\star} d_\epsilon(\mu_{a^\star}, \lambda_{a^\star}) + w_a d_\epsilon(\mu_a, \lambda_a)\} \\ &= \inf_{(\lambda_a,\lambda_{a^\star})\in(\mu_{a^\star},1)^2 \mid \lambda_a\geq\lambda_{a^\star}} \{w_{a^\star} d_\epsilon^+(\mu_{a^\star}, \lambda_{a^\star}) + w_a d_\epsilon^+(\mu_a, \lambda_a)\} = w_a d_\epsilon^+(\mu_a, \mu_{a^\star}) \, , \\ &\inf_{(\lambda_a,\lambda_{a^\star})\in(\mu_{a^\star},1)\times[\mu_a,\mu_{a^\star}]} \{w_{a^\star} d_\epsilon(\mu_{a^\star}, \lambda_{a^\star}) + w_a d_\epsilon(\mu_a, \lambda_a)\} \\ &= \inf_{(\lambda_a,\lambda_{a^\star})\in(\mu_{a^\star},1)\times[\mu_a,\mu_{a^\star}]} \{w_{a^\star} d_\epsilon^-(\mu_{a^\star}, \lambda_{a^\star}) + w_a d_\epsilon^+(\mu_a, \lambda_a)\} = w_a d_\epsilon^+(\mu_a, \mu_{a^\star}) \, , \\ &\inf_{(\lambda_a,\lambda_{a^\star})\in[\mu_a,\mu_{a^\star}]^2 \mid \lambda_a\geq\lambda_{a^\star}} \{w_{a^\star} d_\epsilon(\mu_{a^\star}, \lambda_{a^\star}) + w_a d_\epsilon(\mu_a, \lambda_a)\} \\ &= \inf_{(\lambda_a,\lambda_{a^\star})\in[\mu_a,\mu_{a^\star}]^2 \mid \lambda_a\geq\lambda_{a^\star}} \{w_{a^\star} d_\epsilon^-(\mu_{a^\star}, \lambda_{a^\star}) + w_a d_\epsilon^+(\mu_a, \lambda_a)\} \, . \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} &\inf_{(\lambda_a,\lambda_{a^\star})\in\mathcal{C}_{a,a^\star}} \{w_{a^\star} d_\epsilon(\mu_{a^\star}, \lambda_{a^\star}) + w_a d_\epsilon(\mu_a, \lambda_a)\} \\ &= \inf_{(\lambda_a,\lambda_{a^\star})\in[\mu_a,\mu_{a^\star}]^2 \mid \lambda_a\geq\lambda_{a^\star}} \{w_{a^\star} d_\epsilon^-(\mu_{a^\star}, \lambda_{a^\star}) + w_a d_\epsilon^+(\mu_a, \lambda_a)\} \\ &= \inf_{u\in[\mu_a,\mu_{a^\star}]^2} \{w_{a^\star} d_\epsilon^-(\mu_{a^\star}, u) + w_a d_\epsilon^+(\mu_a, u)\} \\ &= \inf_{u\in[0,1]} \{w_{a^\star} d_\epsilon^-(\mu_{a^\star}, u) + w_a d_\epsilon^+(\mu_a, u)\} = W_{\epsilon,a^\star,a}(\mu, w) \, , \end{aligned}$$

where the second equality is obtained similarly as in Lemma 34 by leveraging the strict convexity of $d_\epsilon^\pm$ in their second argument (see Lemmas 24 and 25). We used Lemma 34 and the definition of $W_{\epsilon,a^\star,a}(\mu,w)$ for the last two equalities. This concludes the proof. $\square$

Lemma 36 gathers additional properties on the transportation costs.

Lemma 36. *Let $d_\epsilon^\pm$ as in Eq.* (3).

- *Let $(\lambda,\mu) \in (0,1)^2$ such that $\lambda > \mu$. When $w_2 > 0$, the function $w_1 \mapsto \min_{u\in[0,1]}\{w_1 d_\epsilon^-(\lambda,u) + w_2 d_\epsilon^+(\mu,u)\}$ is increasing on $\mathbb{R}_+$. When $w_1 > 0$, the function $w_2 \mapsto \min_{u\in[0,1]}\{w_1 d_\epsilon^-(\lambda,u) + w_2 d_\epsilon^+(\mu,u)\}$ is increasing on $\mathbb{R}_+$.*
- *Let $(\lambda,\mu) \in (0,1)^2$ and $\mu \in (0,1)^K$. The function $w \mapsto \min_{u\in[0,1]}\{w_1 d_\epsilon^-(\lambda,u) + w_2 d_\epsilon^+(\mu,u)\}$ is concave on $\mathbb{R}_+^2$. The function $w \mapsto \min_{a\in[K]\setminus\{1\}} \min_{u\in[0,1]}\{w_1 d_\epsilon^-(\mu_1,u) + w_a d_\epsilon^+(\mu_a,u)\}$ is concave on $\mathbb{R}_+^K$.*

Proof. Let $w_2 > 0$ and $w_1' > w_1 \geq 0$. Using Lemma 34, since $w_1' > 0$, there exists $u' \in [0,1]$ with $d_\epsilon^-(\lambda,u') > 0$ such that

$$\begin{aligned}
\min_{u\in[0,1]}\{w_1' d_\epsilon^-(\lambda,u) + w_2 d_\epsilon^+(\mu,u)\} &= w_1' d_\epsilon^-(\lambda,u') + w_2 d_\epsilon^+(\mu,u') \\
&> w_1 d_\epsilon^-(\lambda,u') + w_2 d_\epsilon^+(\mu,u') \\
&\geq \min_{u\in[0,1]}\{w_1 d_\epsilon^-(\lambda,u) + w_2 d_\epsilon^+(\mu,u)\} \,.
\end{aligned}$$

Let $w_1 > 0$ and $w_2' > w_2 \geq 0$. Then, we can show similarly by using Lemma 34 that

$$\min_{u\in[0,1]}\{w_1 d_\epsilon^-(\lambda,u) + w_2' d_\epsilon^+(\mu,u)\} > \min_{u\in[0,1]}\{w_1 d_\epsilon^-(\lambda,u) + w_2 d_\epsilon^+(\mu,u)\} \,.$$

This concludes the first part of the proof. The proof of the second part is direct since those functions are minimum of linear functions, hence concave. $\square$

Lemma 37 gives a closed-form solution for the transportation costs. This is a key property used in our implementation to reduce the computational cost.

Lemma 37. *Let $d_\epsilon^\pm$ and $g_\epsilon^\pm$ as in Eq.* (3) *and* (30)*. For all $(a,c) \in \mathbb{R}_+^2$ and $b \in \mathbb{R}$, let $r_{1,+}(a,b,c) := \frac{\sqrt{b^2+4ac}-b}{2a}$. For all $(\lambda,\mu) \in (0,1)^2$ and $w \in \mathbb{R}_+^2$ such that $\min\{w_1,w_2\} > 0$ and $\lambda > \mu$.*

- *When (1) $g_\epsilon^-(\mu) \geq \lambda$, or (2) $g_\epsilon^-(\mu) < \lambda$, $g_\epsilon^+(\lambda) \leq g_\epsilon^-(\mu)$ and $\frac{w_2\mu + w_1\lambda}{w_2+w_1} \in [g_\epsilon^+(\lambda), g_\epsilon^-(\mu)]$, we have $u_\star(\lambda,\mu,w) = \frac{w_2\mu+w_1\lambda}{w_2+w_1}$ and*

$$\min_{u\in[0,1]}\{w_1 d_\epsilon^-(\lambda,u) + w_2 d_\epsilon^+(\mu,u)\} = w_1 \mathrm{kl}(\lambda, u_\star(\lambda,\mu,w)) + w_2 \mathrm{kl}(\mu, u_\star(\lambda,\mu,w)) \,.$$

- *When (3) $g_\epsilon^-(\mu) < \lambda$, $g_\epsilon^+(\lambda) > g_\epsilon^-(\mu)$ and $u_{3,\star}(w) \in [g_\epsilon^-(\mu), g_\epsilon^+(\lambda)]$ where*

$$u_{3,\star}(w) := \frac{w_1(e^\epsilon - 1) - w_2(1-e^{-\epsilon})}{(w_2+w_1)(1-e^{-\epsilon})(e^\epsilon-1)} \,,$$

we have $u_\star(\lambda,\mu,w) = u_{3,\star}(w)$ and

$$\begin{aligned}
&\min_{u\in[0,1]}\{w_1 d_\epsilon^-(\lambda,u) + w_2 d_\epsilon^+(\mu,u)\} \\
&= w_1\left(-\log\left(1 + u_{3,\star}(w)(e^\epsilon - 1)\right) + \epsilon\lambda\right) + w_2\left(-\log\left(1 - u_{3,\star}(w)(1-e^{-\epsilon})\right) - \epsilon\mu\right) \,.
\end{aligned}$$

- *When (4) $g_\epsilon^-(\mu) < \lambda$, $g_\epsilon^+(\lambda) \leq g_\epsilon^-(\mu)$ and $\frac{w_2\mu+w_1\lambda}{w_2+w_1} \in (\mu, g_\epsilon^+(\lambda))$, or (5) $g_\epsilon^-(\mu) < \lambda$, $g_\epsilon^+(\lambda) > g_\epsilon^-(\mu)$ and $u_{3,\star}(w) < g_\epsilon^-(\mu)$, we have $u_\star(\lambda,\mu,w) = u_{1,\star}(\mu,w)$ and*

$$\begin{aligned}
&u_{1,\star}(\mu,w) := r_{1,+}\left((w_2+w_1)(e^\epsilon-1), \left(w_2 - (w_2\mu + w_1)(e^\epsilon - 1)\right), w_2\mu\right) \,, \\
&\min_{u\in[0,1]}\{w_1 d_\epsilon^-(\lambda,u) + w_2 d_\epsilon^+(\mu,u)\}
\end{aligned}$$

$$= w_1 \left(-\log\left(1 + u_{1,\star}(\mu, w)(e^\epsilon - 1)\right) + \epsilon\lambda\right) + w_2 \mathrm{kl}(\mu, u_{1,\star}(\mu, w)) \, .$$

- *When (6) $g_\epsilon^-(\mu) < \lambda$, $g_\epsilon^+(\lambda) \le g_\epsilon^-(\mu)$ and $\frac{w_2\mu + w_1\lambda}{w_2 + w_1} \in (g_\epsilon^-(\mu), \lambda)$, or (7) $g_\epsilon^-(\mu) < \lambda$, $g_\epsilon^+(\lambda) > g_\epsilon^-(\mu)$ and $u_{3,\star}(w) > g_\epsilon^+(\lambda)$, we have $u_\star(\lambda, \mu, w) = u_{2,\star}(\lambda, w)$ and*

$$u_{2,\star}(\lambda, w) := 1 - r_{1,+}\left((w_2 + w_1)(e^\epsilon - 1), \left(w_1 - (w_1(1-\lambda) + w_2)(e^\epsilon - 1)\right), w_1(1-\lambda)\right) \, ,$$
$$\min_{u \in [0,1]} \{w_1 d_\epsilon^-(\lambda, u) + w_2 d_\epsilon^+(\mu, u)\}$$
$$= w_1 \mathrm{kl}(\lambda, u_{2,\star}(\lambda, w)) + w_2 \left(-\log\left(1 - u_{2,\star}(\lambda, w)(1 - e^{-\epsilon})\right) - \epsilon\mu\right) \, .$$

Proof. Suppose that $g_\epsilon^-(\mu) \ge \lambda$. Using Lemma 23, we know that $g_\epsilon^-(\mu) \ge \lambda$ if and only if $\mu \ge g_\epsilon^+(\lambda)$. Therefore, for all $u \in (\mu, \lambda)$, we have

$$w_1 \frac{\partial d_\epsilon^-}{\partial u}(\lambda, u) + w_2 \frac{\partial d_\epsilon^+}{\partial u}(\mu, u) = -w_1 \frac{\lambda - u}{u(1-u)} + w_2 \frac{u - \mu}{u(1-u)} = \frac{(w_2 + w_1)u - (w_2\mu + w_1\lambda)}{u(1-u)} \, .$$

Therefore, we have

$$u_\star(\lambda, \mu, w) = \frac{w_2\mu + w_1\lambda}{w_2 + w_1} \in (\mu, \lambda) \, .$$

Suppose that $g_\epsilon^-(\mu) < \lambda$. Using Lemma 23, we know that $g_\epsilon^-(\mu) < \lambda$ if and only if $\mu < g_\epsilon^+(\lambda)$. Using strict convexity of the function on (μ, λ), it is enough to exhibit one local minimum to obtain a global minimum on (μ, λ).

Suppose that $g_\epsilon^+(\lambda) \le g_\epsilon^-(\mu)$. Similarly as above, we obtain, for all $u \in [g_\epsilon^+(\lambda), g_\epsilon^-(\mu)]$,

$$w_1 \frac{\partial d_\epsilon^-}{\partial u}(\lambda, u) + w_2 \frac{\partial d_\epsilon^+}{\partial u}(\mu, u) = \frac{(w_2 + w_1)u - (w_2\mu + w_1\lambda)}{u(1-u)} \, .$$

Suppose that $\frac{w_2\mu + w_1\lambda}{w_2 + w_1} \in [g_\epsilon^+(\lambda), g_\epsilon^-(\mu)]$. Then, we can conclude as above that

$$u_\star(\lambda, \mu, w) = \frac{w_2\mu + w_1\lambda}{w_2 + w_1} \in [g_\epsilon^+(\lambda), g_\epsilon^-(\mu)] \, ,$$

since it is a local minimum of a strictly convex function.

Suppose that $\frac{w_2\mu + w_1\lambda}{w_2 + w_1} < g_\epsilon^+(\lambda)$. Since the gradient is positive on $[g_\epsilon^+(\lambda), g_\epsilon^-(\mu)]$, we know that the minimum on (μ, λ) is achieved on $(\mu, g_\epsilon^+(\lambda))$, i.e., $u_\star(\lambda, \mu, w) \in (\mu, g_\epsilon^+(\lambda))$. Then, for all $u \in (\mu, g_\epsilon^+(\lambda))$,

$$w_1 \frac{\partial d_\epsilon^-}{\partial u}(\lambda, u) + w_2 \frac{\partial d_\epsilon^+}{\partial u}(\mu, u) = -w_1 \frac{e^\epsilon - 1}{1 + u(e^\epsilon - 1)} + w_2 \frac{u - \mu}{u(1-u)} \, .$$

Using Lemma 23, direct computation yields

$$w_1 \frac{\partial d_\epsilon^-}{\partial u}(\lambda, u) + w_2 \frac{\partial d_\epsilon^+}{\partial u}(\mu, u) > 0 \quad \iff \quad \mu < u\left(1 + \frac{w_1}{w_2}\left(1 - \frac{g_\epsilon^-(u)}{u}\right)\right) \, ,$$
$$\lim_{u \to \mu^+} u\left(1 + \frac{w_1}{w_2}\left(1 - \frac{g_\epsilon^-(u)}{u}\right)\right) = \mu\left(1 + \frac{w_1}{w_2}\left(1 - \frac{g_\epsilon^-(\mu)}{\mu}\right)\right) < \mu \, ,$$
$$\lim_{u \to g_\epsilon^+(\lambda)^-} u\left(1 + \frac{w_1}{w_2}\left(1 - \frac{g_\epsilon^-(u)}{u}\right)\right) = g_\epsilon^+(\lambda)\left(1 + \frac{w_1}{w_2}\left(1 - \frac{\lambda}{g_\epsilon^+(\lambda)}\right)\right) > \mu \, ,$$

where the second result uses that $u < g_\epsilon^-(u)$ and the last result is obtained by continuity of the differentials (Lemmas 24 and 25) and the positivity on $[g_\epsilon^+(\lambda), g_\epsilon^-(\mu)]$. For all $(a, c) \in \mathbb{R}_+^2$ and $b \in \mathbb{R}$, we define $r_{1,+}(a, b, c) = \frac{\sqrt{b^2 + 4ac} - b}{2a}$. Therefore, we have

$$\begin{aligned} & w_1 \frac{\partial d_\epsilon^-}{\partial u}(\lambda, u) + w_2 \frac{\partial d_\epsilon^+}{\partial u}(\mu, u) = 0 \\ \iff \quad & (w_2 + w_1)(e^\epsilon - 1)u^2 + \left(w_2 - (w_2\mu + w_1)(e^\epsilon - 1)\right)u - w_2\mu = 0 \\ \iff \quad & u_\star(\lambda, \mu, w) = r_{1,+}\left((w_2 + w_1)(e^\epsilon - 1), \left(w_2 - (w_2\mu + w_1)(e^\epsilon - 1)\right), w_2\mu\right) \in (\mu, g_\epsilon^+(\lambda)) \end{aligned}$$

where we used that $u_\star(\lambda, \mu, w) \in (\mu, g_\epsilon^+(\lambda))$ is unique for the last equivalence, and that the second root of the second order polynomial equation is negative. Notice that $u_\star(\lambda, \mu, w)$ is independent of λ.

Suppose that $\frac{w_2\mu + w_1\lambda}{w_2 + w_1} > g_\epsilon^-(\mu)$. Since the gradient is negative on $[g_\epsilon^+(\lambda), g_\epsilon^-(\mu)]$, we know that the minimum on (μ, λ) is achieved on $(g_\epsilon^-(\mu), \lambda)$, i.e., $u_\star(\lambda, \mu, w) \in (g_\epsilon^-(\mu), \lambda)$. Then, for all $u \in (g_\epsilon^-(\mu), \lambda)$,

$$w_1 \frac{\partial d_\epsilon^-}{\partial u}(\lambda, u) + w_2 \frac{\partial d_\epsilon^+}{\partial u}(\mu, u) = -w_1 \frac{\lambda - u}{u(1-u)} + w_2 \frac{1 - e^{-\epsilon}}{1 - u(1 - e^{-\epsilon})} \, .$$

Using Lemma 23, direct computation yields

$$\begin{aligned}
&w_1 \frac{\partial d_\epsilon^-}{\partial u}(\lambda, u) + w_2 \frac{\partial d_\epsilon^+}{\partial u}(\mu, u) < 0 \quad \iff \quad \lambda > u \left(1 + \frac{w_2}{w_1}\left(1 - \frac{g_\epsilon^+(u)}{u}\right)\right) \, , \\
&\lim_{u \to \lambda^-} u \left(1 + \frac{w_2}{w_1}\left(1 - \frac{g_\epsilon^+(u)}{u}\right)\right) = \lambda \left(1 + \frac{w_2}{w_1}\left(1 - \frac{g_\epsilon^+(\lambda)}{\lambda}\right)\right) > \lambda \, , \\
&\lim_{u \to g_\epsilon^-(\mu)^+} u \left(1 + \frac{w_2}{w_1}\left(1 - \frac{g_\epsilon^+(u)}{u}\right)\right) = g_\epsilon^-(\mu) \left(1 + \frac{w_2}{w_1}\left(1 - \frac{\mu}{g_\epsilon^-(\mu)}\right)\right) < \lambda \, ,
\end{aligned}$$

where the second result uses that $u > g_\epsilon^+(u)$ and the last result is obtained by continuity of the differentials (Lemmas 24 and 25) and the negativity on $[g_\epsilon^+(\lambda), g_\epsilon^-(\mu)]$.

Using Lemma 22, we obtain

$$\arg\min_{u \in [0,1]} \{w_1 d_\epsilon^-(\lambda, u) + w_2 d_\epsilon^+(\mu, u)\} = 1 - \arg\min_{u \in [0,1]} \{w_1 d_\epsilon^+(1-\lambda, u) + w_2 d_\epsilon^-(1-\mu, u)\} \, .$$

Using Lemma 23, we obtain

$$\begin{aligned}
&g_\epsilon^-(\mu) < \lambda \quad \iff \quad g_\epsilon^-(1-\lambda) < 1 - \mu \, , \\
&\frac{w_2\mu + w_1\lambda}{w_2 + w_1} \in (g_\epsilon^-(\mu), \lambda) \quad \iff \quad \frac{w_2(1-\mu) + w_1(1-\lambda)}{w_2 + w_1} \in (1 - \lambda, g_\epsilon^+(1-\mu)) \, .
\end{aligned}$$

Therefore, we can leverage the above case to obtain $u_\star(\lambda, \mu, w) = u_{2,\star}(\lambda, w)$ where

$$u_{2,\star}(\lambda, w) = 1 - r_{1,+}\left((w_2 + w_1)(e^\epsilon - 1), (w_1 - (w_1(1-\lambda) + w_2)(e^\epsilon - 1)), w_1(1-\lambda)\right)$$

Notice that $u_\star(\lambda, \mu, w)$ is independent of μ.

Suppose that $g_\epsilon^-(\mu) < \lambda$ and $g_\epsilon^+(\lambda) > g_\epsilon^-(\mu)$. Similarly as above, we obtain, for all $u \in [g_\epsilon^-(\mu), g_\epsilon^+(\lambda)]$,

$$w_1 \frac{\partial d_\epsilon^-}{\partial u}(\lambda, u) + w_2 \frac{\partial d_\epsilon^+}{\partial u}(\mu, u) = -w_1 \frac{e^\epsilon - 1}{1 + u(e^\epsilon - 1)} + w_2 \frac{1 - e^{-\epsilon}}{1 - u(1 - e^{-\epsilon})} \, .$$

Therefore, we obtain

$$\begin{aligned}
&w_1 \frac{\partial d_\epsilon^-}{\partial u}(\lambda, u) + w_2 \frac{\partial d_\epsilon^+}{\partial u}(\mu, u) > 0 \\
\iff \quad &w_2(1 - e^{-\epsilon})(1 + u(e^\epsilon - 1)) - w_1(e^\epsilon - 1)(1 - u(1 - e^{-\epsilon})) > 0 \\
\iff \quad &u > u_{3,\star}(w) := \frac{w_1(e^\epsilon - 1) - w_2(1 - e^{-\epsilon})}{(w_2 + w_1)(1 - e^{-\epsilon})(e^\epsilon - 1)} \, .
\end{aligned}$$

Suppose that $u_{3,\star}(w) \in [g_\epsilon^-(\mu), g_\epsilon^+(\lambda)]$. Then, we can conclude as above that

$$u_\star(\lambda, \mu, w) = u_{3,\star}(w) \in [g_\epsilon^-(\mu), g_\epsilon^+(\lambda)] \, ,$$

since it is a local minimum of a strictly convex function. Notice that $u_{3,\star}(w)$ is independent of (λ, μ).

Suppose that $u_{3,\star}(w) > g_\epsilon^+(\lambda)$. Since the gradient is negative on $[g_\epsilon^-(\mu), g_\epsilon^+(\lambda)]$, we know that the minimum on (μ, λ) is achieved on $(g_\epsilon^+(\lambda), \lambda)$, i.e., $u_\star(\lambda, \mu, w) \in (g_\epsilon^+(\lambda), \lambda)$. Then, for all $u \in (g_\epsilon^+(\lambda), \lambda)$,

$$w_1 \frac{\partial d_\epsilon^-}{\partial u}(\lambda, u) + w_2 \frac{\partial d_\epsilon^+}{\partial u}(\mu, u) = -w_1 \frac{\lambda - u}{u(1-u)} + w_2 \frac{1 - e^{-\epsilon}}{1 - u(1 - e^{-\epsilon})} \, .$$

This recovers the condition solved above. As we know that $u_\star(\lambda, \mu, w) \in (g_\epsilon^+(\lambda), \lambda)$, we obtain $u_\star(\lambda, \mu, w) = u_{2,\star}(\lambda, w)$ where

$$u_{2,\star}(\lambda, w) = 1 - r_{1,+}\left((w_2 + w_1)(e^\epsilon - 1), (w_1 - (w_1(1-\lambda) + w_2)(e^\epsilon - 1)), w_1(1-\lambda)\right)$$

Suppose that $u_{3,\star}(w) < g_\epsilon^-(\mu)$. Since the gradient is positive on $[g_\epsilon^-(\mu), g_\epsilon^+(\lambda)]$, we know that the minimum on (μ, λ) is achieved on $(\mu, g_\epsilon^-(\mu))$, i.e., $u_\star(\lambda, \mu, w) \in (\mu, g_\epsilon^-(\mu))$. Then, for all $u \in (\mu, g_\epsilon^-(\mu))$,

$$w_1 \frac{\partial d_\epsilon^-}{\partial u}(\lambda, u) + w_2 \frac{\partial d_\epsilon^+}{\partial u}(\mu, u) = -w_1 \frac{e^\epsilon - 1}{1 + u(e^\epsilon - 1)} + w_2 \frac{u - \mu}{u(1-u)} .$$

This recovers the condition solved above. As we know that $u_\star(\lambda, \mu, w) \in (\mu, g_\epsilon^-(\mu))$, we obtain $u_\star(\lambda, \mu, w) = u_{1,\star}(\mu, w)$ where

$$u_{1,\star}(\mu, w) = r_{1,+}\left((w_2 + w_1)(e^\epsilon - 1), (w_2 - (w_2\mu + w_1)(e^\epsilon - 1)), w_2\mu\right) .$$

This concludes the proof.

□

G.2.1 Modified Transportation Cost

Let $\eta > 0$ be the geometric parameter used for the geometric grid update of our private empirical mean estimator. Let us define

$$\forall x \geq 1, \quad r(x) := \frac{x}{1 + \log_{1+\eta} x} , \tag{33}$$

which is increasing if and only if $x > \frac{e}{1+\eta}$. For all $(\mu, w) \in \mathbb{R}^K \times \mathbb{R}_+^K$ and all $(a, b) \in [K]^2$ such that $a \neq b$, we define

$$\widetilde{W}_{\epsilon,a,b}(\mu, w) := \mathbb{1}\left([\mu_a]_0^1 > [\mu_b]_0^1\right) \inf_{u \in (0,1)} \left\{ w_a \widetilde{d}_\epsilon^-(\mu_a, u, r(w_a)) + w_b \widetilde{d}_\epsilon^+(\mu_b, u, r(w_b)) \right\} , \tag{34}$$

where $\widetilde{d}_\epsilon^\pm$ are defined in Eq. (32).

Lemma 38 gathers regularity properties of the function r defined in Eq. (33).

Lemma 38. *Let r as in Eq.* (33)*. Then,*

$$\forall x \geq 1, \quad r'(x) = \frac{\log(x(1+\eta)/e)}{\log(1+\eta)(1 + \log_{1+\eta} x)^2} ,$$
$$r''(x) = -\frac{1}{x(\log(1+\eta))^2} \frac{\log((1+\eta)xe^{-2})}{(1 + \log_{1+\eta} x)^3} .$$

On $[1, +\infty)$, the function r is twice continuously differentiable. It is decreasing on $[1, e/(1+\eta))$ and increasing on $(e/(1+\eta), +\infty)$; its minium is $r(e/(1+\eta)) \in (0,1)$. It is strictly convex on $[1, e^2/(1+\eta))$ and strictly concave on $(e^2/(1+\eta), +\infty)$.

Proof. The proof is obtained by direct differentiation and manipulation. We have

$$\forall \eta > 0, \quad r(e/(1+\eta)) = \frac{e \log(1+\eta)}{1+\eta} \in (0,1) .$$

□

Lemma 39 shows that the modified transportation costs can be rewritten differently, which is a key property used in Appendix F.

Lemma 39. *Let $\widetilde{d}_\epsilon^\pm$ as in Eq.* (32)*, and r as in Eq.* (33)*. For all $(\lambda, \mu) \in \mathbb{R}^2$ such that $[\lambda]_0^1 > [\mu]_0^1$ and $(w_1, w_2) \in [1, +\infty)^2$. Then,*

$$\begin{aligned}
&\inf_{u \in (0,1)} \{ w_1 \widetilde{d}_\epsilon^-(\lambda, u, r(w_1)) + w_2 \widetilde{d}_\epsilon^+(\mu, u, r(w_2)) \} \\
&= \inf_{u \in ([\mu]_0^1, [\lambda]_0^1)} \{ w_1 \widetilde{d}_\epsilon^-(\lambda, u, r(w_1)) + w_2 \widetilde{d}_\epsilon^+(\mu, u, r(w_2)) \} \\
&= \inf_{(u_1, u_2) \in (0,1)^2 :\, u_1 \leq u_2} \{ w_1 \widetilde{d}_\epsilon^-(\lambda, u_1, r(w_1)) + w_2 \widetilde{d}_\epsilon^+(\mu, u_2, r(w_2)) \} .
\end{aligned}$$

Proof. These results are obtained by leveraging Lemmas 30 and 31.

Note that the condition $[\lambda]_0^1 > [\mu]_0^1$ implies that $\mu \in (-\infty, 1)$ and $\lambda \in (0, +\infty)$, i.e., $[\mu]_0^1 = \max\{0, \mu\}$ and $[\lambda]_0^1 = \min\{1, \lambda\}$.

Suppose that $\mu \leq 0$ and $\lambda \geq 1$. Then, we have $[\mu]_0^1 = 0$ and $[\lambda]_0^1 = 1$. Therefore, the first part of the result holds by definition.

Suppose that $\mu \leq 0$ and $\lambda \in (0, 1)$. Then, we have $[\mu]_0^1 = 0$ and $[\lambda]_0^1 = \lambda$. For $u \in [\lambda, 1)$, the function is equal to $w_2 \widetilde{d}_\epsilon^{+}(\mu, u, r(w_2))$, which is increasing and strictly convex on $(0, 1)$. Therefore, the minimum over that interval is attained at λ. For $u \in (0, \lambda)$, the function is equal to $w_1 \widetilde{d}_\epsilon^{-}(\lambda, u, r(w_1)) + w_2 \widetilde{d}_\epsilon^{+}(\mu, u, r(w_2))$. Since it is the sum of two strictly convex function, the minimum over that interval is achieved in $(0, \lambda)$. This concludes the proof of the first part of the result for this case.

Suppose that $\mu \in (0, 1)$ and $\lambda \geq 1$. Then, we have $[\mu]_0^1 = \mu$ and $[\lambda]_0^1 = 1$. For $u \in (0, \mu]$, the function is equal to $w_1 \widetilde{d}_\epsilon^{-}(\lambda, u, r(w_2))$, which is decreasing and strictly convex on $(0, 1)$. Therefore, the minimum over that interval is attained at μ. For $u \in (\mu, 1)$, the function is equal to $w_1 \widetilde{d}_\epsilon^{-}(\lambda, u, r(w_1)) + w_2 \widetilde{d}_\epsilon^{+}(\mu, u, r(w_2))$. Since it is the sum of two strictly convex function, the minimum over that interval is achieved in $(\mu, 1)$. This concludes the proof of the first part of the result for this case.

Suppose that $(\mu, \lambda) \in (0, 1)^2$. Then, we have $[\mu]_0^1 = \mu$ and $[\lambda]_0^1 = \lambda$. For $u \in [\lambda, 1)$, the function is equal to $w_2 \widetilde{d}_\epsilon^{+}(\mu, u, r(w_2))$, which is increasing and strictly convex on $(0, 1)$. Therefore, the minimum over that interval is attained at λ. For $u \in (0, \mu]$, the function is equal to $w_1 \widetilde{d}_\epsilon^{-}(\lambda, u, r(w_2))$, which is decreasing and strictly convex on $(0, 1)$. Therefore, the minimum over that interval is attained at μ. For $u \in (\mu, \lambda)$, the function is equal to $w_1 \widetilde{d}_\epsilon^{-}(\lambda, u, r(w_1)) + w_2 \widetilde{d}_\epsilon^{+}(\mu, u, r(w_2))$. Since it is the sum of two strictly convex function, the minimum over that interval is achieved in (μ, λ). This concludes the proof of the first part of the result for this case.

In summary, we have shown that

$$\begin{aligned} &\inf_{u \in (0,1)} \{ w_1 \widetilde{d}_\epsilon^{-}(\lambda, u, r(w_1)) + w_2 \widetilde{d}_\epsilon^{+}(\mu, u, r(w_2)) \} \\ &= \inf_{u \in ([\mu]_0^1, [\lambda]_0^1)} \{ w_1 \widetilde{d}_\epsilon^{-}(\lambda, u, r(w_1)) + w_2 \widetilde{d}_\epsilon^{+}(\mu, u, r(w_2)) \} \, . \end{aligned}$$

Using the strict convexity of $u_1 \mapsto w_1 \widetilde{d}_\epsilon^{-}(\lambda, u_1, r(w_1))$ and $u_2 \mapsto w_2 \widetilde{d}_\epsilon^{+}(\mu, u_2, r(w_2))$ on $([\mu]_0^1, [\lambda]_0^1)$, we obtain that

$$\begin{aligned} &\inf_{u \in ([\mu]_0^1, [\lambda]_0^1)} \{ w_1 \widetilde{d}_\epsilon^{-}(\lambda, u, r(w_1)) + w_2 \widetilde{d}_\epsilon^{+}(\mu, u, r(w_2)) \} \\ &= \inf_{(u_1, u_2) \, : \, [\mu]_0^1 < u_1 \leq u_2 < [\lambda]_0^1} \{ w_1 \widetilde{d}_\epsilon^{-}(\lambda, u_1, r(w_1)) + w_2 \widetilde{d}_\epsilon^{+}(\mu, u_2, r(w_2)) \} \, . \end{aligned}$$

Re-using the same arguments as above, we obtain that

$$\begin{aligned} &\inf_{(u_1, u_2) \, : \, [\mu]_0^1 < u_1 \leq u_2 < [\lambda]_0^1} \{ w_1 \widetilde{d}_\epsilon^{-}(\lambda, u_1, r(w_1)) + w_2 \widetilde{d}_\epsilon^{+}(\mu, u_2, r(w_2)) \} = \\ &= \inf_{(u_1, u_2) \in (0,1)^2 \colon u_1 \leq u_2} \{ w_1 \widetilde{d}_\epsilon^{-}(\lambda, u_1, r(w_1)) + w_2 \widetilde{d}_\epsilon^{+}(\mu, u_2, r(w_2)) \} \, . \end{aligned}$$

This concludes the proof. $\square$

Lemma 40 gives a closed-form solution for the modified transportation costs based on an implicit solution of a fixed-point equation. This is a key property used in our implementation to reduce the computational cost.

Lemma 40. *Let $\widetilde{d}_\epsilon^{\pm}$ as in Eq.* (32)*, $x_\epsilon^{\pm}$ as in Lemmas 30 and 31, and r as in Eq.* (33)*. For all $(\lambda, \mu) \in \mathbb{R}^2$ such that $[\lambda]_0^1 > [\mu]_0^1$ and $w \in [1, +\infty)^2$. Then,*

$$\begin{aligned} &\inf_{u \in (0,1)} \{ w_1 \widetilde{d}_\epsilon^{-}(\lambda, u, r(w_1)) + w_2 \widetilde{d}_\epsilon^{+}(\mu, u, r(w_2)) \} \\ &= w_1 \widetilde{d}_\epsilon^{-}(\lambda, u^\star(\lambda, \mu, w), r(w_1)) + w_2 \widetilde{d}_\epsilon^{+}(\mu, u^\star(\lambda, \mu, w), r(w_2)) \, , \end{aligned}$$

where $u^\star(\lambda,\mu,w) \in ([\mu]_0^1, [\lambda]_0^1)$ is the unique solution for $u \in ([\mu]_0^1, [\lambda]_0^1)$ of the equation

$$u(w_1+w_2) - w_1 g_\epsilon^-(u) - w_2 g_\epsilon^+(u) + w_1 x_\epsilon^-(\lambda, u, r(w_1)) - w_2 x_\epsilon^+(\mu, u, r(w_2)) = 0\,.$$

Proof. Using Lemma 39, we have

$$\begin{aligned}&\inf_{u\in(0,1)} \{w_1 \widetilde{d}_\epsilon^-(\lambda,u,r(w_1)) + w_2 \widetilde{d}_\epsilon^+(\mu,u,r(w_2))\} \\ &= \inf_{u\in([\mu]_0^1,[\lambda]_0^1)} \{w_1 \widetilde{d}_\epsilon^-(\lambda,u,r(w_1)) + w_2 \widetilde{d}_\epsilon^+(\mu,u,r(w_2))\}\,.\end{aligned}$$

Using Lemmas 31 and 30, we obtain

$$\begin{aligned}\forall u \in (0,[\lambda]_0^1), \quad &\frac{\partial \widetilde{d}_\epsilon^-}{\partial u}(\lambda,u,r(w_1)) = \frac{u - g_\epsilon^-(u) + x_\epsilon^-(\lambda,u,r(w_1))}{u(1-u)}\,, \\ \forall u \in ([\mu]_0^1,1), \quad &\frac{\partial \widetilde{d}_\epsilon^+}{\partial u}(\mu,u,r(w_2)) = \frac{u - g_\epsilon^+(u) - x_\epsilon^+(\mu,u,r(w_2))}{u(1-u)}\,.\end{aligned}$$

Therefore, for all $u \in ([\mu]_0^1, [\lambda]_0^1)$,

$$\begin{aligned}&w_1 \frac{\partial \widetilde{d}_\epsilon^-}{\partial u}(\lambda,u,r(w_1)) + w_2 \frac{\partial \widetilde{d}_\epsilon^+}{\partial u}(\mu,u,r(w_2)) \\ &= \frac{w_1(u - g_\epsilon^-(u) + x_\epsilon^-(\lambda,u,r(w_1))) + w_2(u - g_\epsilon^+(u) - x_\epsilon^+(\mu,u,r(w_2)))}{u(1-u)} \\ &= \frac{u(w_1+w_2) - (w_1 g_\epsilon^-(u) + w_2 g_\epsilon^+(u)) + w_1 x_\epsilon^-(\lambda,u,r(w_1)) - w_2 x_\epsilon^+(\mu,u,r(w_2))}{u(1-u)}\,.\end{aligned}$$

For $u \in ([\mu]_0^1, [\lambda]_0^1)$, let us define

$$g_1(u) := u(w_1+w_2) - w_1 g_\epsilon^-(u) - w_2 g_\epsilon^+(u) + w_1 x_\epsilon^-(\lambda,u,r(w_1)) - w_2 x_\epsilon^+(\mu,u,r(w_2))\,.$$

Using the proof of Lemmas 31 and 30, we know that

$$\begin{aligned}&\lim_{u\to[\mu]_0^1} \frac{\partial \widetilde{d}_\epsilon^+}{\partial u}(\mu,u,r(w_1)) = 0 \quad \text{and} \quad \lim_{u\to[\lambda]_0^1} \frac{\partial \widetilde{d}_\epsilon^-}{\partial u}(\lambda,u,r(w_1)) = 0\,, \\ &\forall u \in (0,[\lambda]_0^1), \quad \frac{\partial \widetilde{d}_\epsilon^-}{\partial u}(\lambda,u,r(w_1)) < 0 \quad \text{and} \quad \forall u \in ([\mu]_0^1,1), \quad \frac{\partial \widetilde{d}_\epsilon^+}{\partial u}(\mu,u,r(w_1)) > 0\,.\end{aligned}$$

Combined with the strict convexity of $\widetilde{d}_\epsilon^\pm$ in their second argument, the equation $g_1(u) = 0$ admits a unique solution on $([\mu]_0^1, [\lambda]_0^1)$. Since $u(1-u) > 0$, we obtain the implicit equation defining $u^\star(\lambda,\mu,w)$ as above. □

G.3 Characteristic Time

Let $\boldsymbol{\nu}$ be a Bernoulli instance with means $\mu \in (0,1)^2$ and unique best arm $a^\star \in [K]$, i.e., $\arg\max_{a\in[K]} \mu_a = \{a^\star\}$. For all $\beta \in (0,1)$, we define

$$\begin{aligned}T_\epsilon^\star(\boldsymbol{\nu})^{-1} &= \sup_{w\in\triangle_K} \min_{a\neq a^\star} W_{\epsilon,a^\star,b}(\mu,w) \quad \text{and} \quad w_\epsilon^\star(\boldsymbol{\nu}) = \operatorname*{arg\,max}_{w\in\triangle_K} \min_{a\neq a^\star} W_{\epsilon,a^\star,b}(\mu,w)\,, \\ T_{\epsilon,\beta}^\star(\boldsymbol{\nu})^{-1} &= \sup_{w\in\triangle_K, w_{a^\star}=\beta} \min_{a\neq a^\star} W_{\epsilon,a^\star,b}(\mu,w) \quad \text{and} \quad w_{\epsilon,\beta}^\star(\boldsymbol{\nu}) = \operatorname*{arg\,max}_{w\in\triangle_K, w_{a^\star}=\beta} \min_{a\neq a^\star} W_{\epsilon,a^\star,b}(\mu,w)\end{aligned} \tag{35}$$

where $W_{\epsilon,a,b}$ are defined in Eq. (4).

Lemma 41 gathers regularity properties on the characteristic times and their optimal allocations.

Lemma 41. *Let $W_{\epsilon,a,b}$ as in Eq. (4). Let $(T_\epsilon^\star, T_{\epsilon,\beta}^\star)$ and $(w_\epsilon^\star, w_{\epsilon,\beta}^\star)$ as in Eq. (35). The function $(\mu,w) \mapsto \min_{a\neq a^\star(\mu)} W_{\epsilon,a^\star(\mu),a}(\mu,w)$ is continuous on $(0,1)^K \times \triangle_K$. The functions $\boldsymbol{\nu} \mapsto T_\epsilon^\star(\boldsymbol{\nu})^{-1}$ and $\boldsymbol{\nu} \mapsto T_{\epsilon,\beta}^\star(\boldsymbol{\nu})^{-1}$ are continuous on $\mathcal{F}^K$. The correspondences $\boldsymbol{\nu} \mapsto w_\epsilon^\star(\boldsymbol{\nu})$ and $\boldsymbol{\nu} \mapsto w_{\epsilon,\beta}^\star(\boldsymbol{\nu})$ are upper hemicontinuous on $\mathcal{F}^K$ with compact convex values.*

Proof. Let $\mathcal{F}_a^K = \left\{\boldsymbol{\nu} \in \mathcal{F}^K \mid a \in a^\star(\boldsymbol{\nu})\right\}$. Since $\bigcup_{a\in[K]} \mathcal{F}_a^K = \mathcal{F}^K$, it is enough to show the property for all $\mathcal{F}_a^K$ for $a \in [K]$. Let $a^\star \in [K]$.

First, the function $(w, \boldsymbol{\nu}) \mapsto \min_{a\neq a^\star} \inf_{u\in[0,1]} \{w_{a^\star} d_\epsilon^-(\mu_{a^\star}, u) + w_a d_\epsilon^+(\mu_{a^\star}, u)\}$ is continuous on $\triangle_K \times \mathcal{F}^K$ by Lemma 34 and the fact that a minimum of continuous functions is continuous. It is concave in w by Lemma 36.

The correspondence $(w, \boldsymbol{\nu}) \mapsto \triangle_K$ is nonempty compact-valued and continuous (since constant). By Berge's maximum theorem, we get that $\boldsymbol{\nu} \mapsto T_\epsilon^\star(\boldsymbol{\nu})^{-1}$ is continuous on $\mathcal{F}_{a^\star}^K$ and that $\boldsymbol{\nu} \mapsto w_\epsilon^\star(\boldsymbol{\nu})$ is upper hemicontinuous with compact values. By [78, Theorem 9.17], the concavity of the function being maximized implies that $\boldsymbol{\nu} \mapsto w_\epsilon^\star(\boldsymbol{\nu})$ is convex-valued.

The correspondence $(w, \boldsymbol{\nu}) \mapsto \triangle_K \cap \{w_{a^\star} = \beta\}$ is nonempty compact-valued and continuous (since constant). By Berge's maximum theorem, we get that $\boldsymbol{\nu} \mapsto T_{\epsilon,\beta}^\star(\boldsymbol{\nu})^{-1}$ is continuous on $\mathcal{F}_{a^\star}^K$ and that $w_\beta^\star(\boldsymbol{\nu})$ is upper hemicontinuous with compact values. By [78, Theorem 9.17], the concavity of the function being maximized implies that $\boldsymbol{\nu} \mapsto w_{\epsilon,\beta}^\star(\boldsymbol{\nu})$ is convex-valued. □

Lemma 42 provides additional properties on the characteristic times and their optimal allocations. In particular, this results show that the (β-)optimal allocations is unique, has positive allocation for each arm and that the transportation costs are equal at equilibrium. Those properties are key in the analysis of a sampling rule.

Lemma 42. *Let $W_{\epsilon,a,b}$ as in Eq. (4). Let $(T_\epsilon^\star, T_{\epsilon,\beta}^\star)$ and $(w_\epsilon^\star, w_{\epsilon,\beta}^\star)$ as in Eq. (35). Let $\beta \in (0,1)$ and $\boldsymbol{\nu} \in \mathcal{F}^K$ such that $a^\star(\boldsymbol{\nu}) = \{a^\star\}$ is a singleton.*

- $T_\epsilon^\star(\boldsymbol{\nu})^{-1} > 0$ *and* $T_{\epsilon,\beta}^\star(\boldsymbol{\nu})^{-1} > 0$.
- $\min_{a\in[K]} w_a^\star > 0$ *and* $\min_{a\in[K]} w_{\beta,a}^\star > 0$ *for all* $w^\star \in w_\epsilon^\star(\boldsymbol{\nu})$ *and* $w_\beta^\star \in w_{\epsilon,\beta}^\star(\boldsymbol{\nu})$.
- *the (β-)optimal allocations are unique and the transportation costs are all equals at equilibrium*

$$w_\epsilon^\star(\boldsymbol{\nu}) = \{w_\epsilon^\star\} \quad \textit{and} \quad \forall a \neq a^\star, \inf_{u\in[0,1]} \left\{w_{\epsilon,a^\star}^\star d_\epsilon^-(\mu_{a^\star}, u) + w_{\epsilon,a}^\star d_\epsilon^+(\mu_a, u)\right\} = T_\epsilon^\star(\boldsymbol{\nu})^{-1} ,$$

$$w_{\epsilon,\beta}^\star(\boldsymbol{\nu}) = \{w_{\epsilon,\beta}^\star\} \quad \textit{and} \quad \forall a \neq a^\star, \inf_{u\in[0,1]} \left\{w_{\epsilon,\beta,a^\star}^\star d_\epsilon^-(\mu_{a^\star}, u) + w_{\epsilon,\beta,a}^\star d_\epsilon^+(\mu_a, u)\right\} = T_{\epsilon,\beta}^\star(\boldsymbol{\nu})^{-1}$$

Proof. Using the definition of the supremum with $1_K/K \in \triangle_K$ and Lemma 34, we obtain

$$\begin{aligned} T_\epsilon^\star(\boldsymbol{\nu})^{-1} &= \sup_{w\in\triangle_K} \min_{a\neq a^\star} \inf_{u\in[0,1]} \left\{w_{a^\star} d_\epsilon^-(\mu_{a^\star}, u) + w_a d_\epsilon^+(\mu_a, u)\right\} \\ &\geq \frac{1}{K} \min_{a\neq a^\star} \inf_{u\in[0,1]} \left\{d_\epsilon^-(\mu_{a^\star}, u) + d_\epsilon^+(\mu_a, u)\right\} > 0 , \end{aligned}$$

where the last inequality strict uses Lemma 34 and $\mu_a < \mu_{a^\star}$ for all $a \neq a^\star$. Similarly, we can prove that $T_{\epsilon,\beta}^\star(\boldsymbol{\nu})^{-1} > 0$. This concludes the first part of the proof.

We proceed towards contradiction. Suppose that there exists $w^\star \in w_\epsilon^\star(\boldsymbol{\nu})$ and b with $w_b^\star = 0$. Then, we will show $T_\epsilon^\star(\boldsymbol{\nu})^{-1} = 0$, which is a contradiction with the above result. If $b = a^\star$ we have

$$T_\epsilon^\star(\boldsymbol{\nu})^{-1} = \min_{a\neq a^\star} \inf_{u\in[0,1]} w_a^\star d_\epsilon^+(\mu_a, u) \leq \min_{a\neq a^\star} w_a^\star d_\epsilon^+(\mu_a, \mu_a) = 0 .$$

If $b \neq a^\star$, we have

$$\begin{aligned} T_\epsilon^\star(\boldsymbol{\nu})^{-1} &= \min_{a\neq a^\star} \inf_{u\in[0,1]} \left\{w_{a^\star}^\star d_\epsilon^-(\mu_{a^\star}, u) + w_a^\star d_\epsilon^+(\mu_a, u)\right\} \\ &\leq \inf_{u\in[0,1]} \left\{w_{a^\star}^\star d_\epsilon^-(\mu_{a^\star}, u) + w_b^\star d_\epsilon^+(\mu_b, u)\right\} = \inf_{u\in[0,1]} w_{a^\star}^\star d_\epsilon^-(\mu_{a^\star}, u) = 0 . \end{aligned}$$

A similar proof allows to show the result for $w_{\epsilon,\beta}^\star(\boldsymbol{\nu})$ by reasoning on $T_{\epsilon,\beta}^\star(\boldsymbol{\nu})^{-1}$. This concludes the second part of the proof.

For notational simplicity, we assume without loss of generality that $a^\star = 1$ is the best arm. At the optimal allocations, all w_a are positive. Let us define $G_b(x) = \inf_{u\in[0,1]} \{d_\epsilon^-(\mu_1, u) + x d_\epsilon^+(\mu_b, u)\}$ for all $b \neq 1$. Let $w^\star \in w_\epsilon^\star(\boldsymbol{\nu})$. Then, we have

$$T_\epsilon^\star(\boldsymbol{\nu})^{-1} = \max_{w\in\triangle_K, w_1>0} w_1 \min_{b\neq 1} G_b\left(\frac{w_b}{w_1}\right) \quad \text{and} \quad w^\star \in \operatorname*{arg\,max}_{w\in\triangle_K} w_1 \min_{b\neq 1} G_b\left(\frac{w_b}{w_1}\right) .$$

Introducing $x_b^\star = \frac{w_b^\star}{w_1^\star}$ for all $b \neq 1$, using that $\sum_{b\in[K]} w_b^\star = 1$, one has

$$w_1^\star = \frac{1}{1+\sum_{c\neq 1} x_c^\star} \quad \text{and} \quad \forall b \neq 1,\ w_b^\star = \frac{x_b^\star}{1+\sum_{c\neq 1} x_c^\star} \ .$$

If $x^\star$ is unique, then so is $w^\star$. Since it is optimal, $\{x_b^\star\}_{b=2}^K \in \mathbb{R}^{K-1}$ belongs to

$$\arg\max_{\{x_b\}_{b=2}^K \in \mathbb{R}^{K-1}} \frac{\min_{b\neq 1} G_b\left(x_b\right)}{1+\sum_{c=2}^K x_c} \ . \tag{36}$$

Let's show that all the $G_b\left(x_b^\star\right)$ have to be equal. Let $\mathcal{O} = \{a \in [K]\setminus\{1\} \mid G_a\left(x_a^\star\right) = \min_{b\neq 1} G_b\left(x_b^\star\right)\}$ and $\mathcal{A} = [K]\setminus(\{1\}\cup\mathcal{O})$. Assume that $\mathcal{A} \neq \emptyset$. For all $a \in \mathcal{A}$ and $b \in \mathcal{O}$, one has $G_b\left(x_b^\star\right) > G_a\left(x_a^\star\right)$. Using the continuity of the G_b functions and the fact that they are increasing (Lemma 36), there exists $\epsilon > 0$ such that

$$\forall b \in \mathcal{A}, a \in \mathcal{O}, \quad G_b\left(x_b^\star - \epsilon/|\mathcal{A}|\right) > G_a\left(x_a^\star + \epsilon/|\mathcal{O}|\right) > G_a\left(x_a^\star\right) \ .$$

We introduce $\bar{x}_b = x_b^\star - \epsilon/|\mathcal{A}|$ for all $b \in \mathcal{A}$ and $\bar{x}_a = x_a^\star + \epsilon/|\mathcal{O}|$ for all $a \in \mathcal{O}$, hence $\sum_{b=2}^K \bar{x}_b = \sum_{b=2}^K x_b^\star$. There exists $a \in \mathcal{O}$ such that $\min_{b\neq 1} G_b\left(\bar{x}_b\right) = G_a\left(x_a^\star + \epsilon/|\mathcal{O}|\right)$, hence

$$\frac{\min_{b\neq 1} G_b\left(\bar{x}_b\right)}{1+\bar{x}_2+\ldots \bar{x}_K} = \frac{G_a\left(x_a^\star + \epsilon/|\mathcal{O}|\right)}{1+x_2^\star+\cdots+x_K^\star} > \frac{G_a\left(x_a^\star\right)}{1+x_2^\star+\cdots+x_K^\star} = \frac{\min_{b\neq 1} G_b\left(x_b^\star\right)}{1+x_2^\star+\cdots+x_K^\star} \ .$$

This is a contradiction with the fact that $x^\star$ belongs to (36). Therefore, we have $\mathcal{A} = \emptyset$.

We have proved that there is a unique value by $y^\star \in \mathbb{R}_+$, such that for all $b \neq 1$, $G_b\left(x_b^\star\right) = y^\star$. Now since G_b is increasing, this defines a unique value for $x_b^\star$, equal to $G_b^{-1}(y^\star)$.

For y in the intersection of the ranges of all G_b, let $x_b(y) = G_b^{-1}(y)$. Then, $y^\star$ belongs to

$$\arg\max_{y\in[0,\min_{b\neq 1}\lim_{+\infty} G_b(x))} \frac{y}{1+\sum_{b\neq 1} x_b(y)} \ . \tag{37}$$

For $\beta \in (0,1)$, the same results (and proof) hold for $w_{\epsilon,\beta}^\star(\boldsymbol{\nu})$ by noting that

$$T_{\epsilon,\beta}^\star(\boldsymbol{\nu})^{-1} = \max_{w\in\triangle_K : w_1 = \beta} \beta \min_{b\neq 1} G_b\left(w_b/\beta\right) \ .$$

Let $w_{\epsilon,\beta}^\star \in w_{\epsilon,\beta}^\star(\boldsymbol{\nu})$, since we have equality at the equilibrium, we obtain $\beta G_b\left(w_{\epsilon,\beta,b}^\star/\beta\right) = T_{\epsilon,\beta}^\star(\boldsymbol{\nu})^{-1}$ for all $b \neq 1$. Using the inverse mapping x_b, we obtain $w_{\epsilon,\beta,b}^\star = \beta x_b\left(T_{\epsilon,\beta}^\star(\boldsymbol{\nu})^{-1}/\beta\right)$ for all $b \neq 1$. This concludes the third part of the proof. $\square$

Lemma 43 shows that an asymptotically $1/2$-optimal algorithm has an asymptotic expected sample complexity which is at worse twice the asymptotic expected sample complexity of an asymptotically optimal algorithm. This result motivates the recommendation to the practitioner of using $\beta = 1/2$ when no prior information is available on the true instance $\boldsymbol{\nu}$.

Lemma 43. *Let $(T_\epsilon^\star, T_{\epsilon,\beta}^\star, w_\epsilon^\star)$ as in Eq.* (35)*. Let $\beta \in (0,1)$ and $\boldsymbol{\nu} \in \mathcal{F}^K$ such that $a^\star(\boldsymbol{\nu}) = \{a^\star\}$ is a singleton. Then,*

$$T_{\epsilon,1/2}^\star(\boldsymbol{\nu}) \leq 2T_\epsilon^\star(\boldsymbol{\nu}) \quad \textit{and} \quad \frac{T_\epsilon^\star(\boldsymbol{\nu})^{-1}}{T_{\epsilon,\beta}^\star(\boldsymbol{\nu})^{-1}} \leq \max\left\{\frac{\beta^\star}{\beta}, \frac{1-\beta^\star}{1-\beta}\right\} \quad \textit{with} \quad \beta^\star = w_{\epsilon,a^\star}^\star \ .$$

Proof. Define for each non-negative vector $\boldsymbol{\psi} \in \mathbb{R}_+^K$,

$$f(\boldsymbol{\psi}) := \min_{a\neq a^\star} \inf_{u\in[0,1]} \left\{\psi_{a^\star} d_\epsilon^-(\mu_{a^\star}, u) + \psi_a d_\epsilon^+(\mu_a, u)\right\} \ .$$

$T_\epsilon^\star(\boldsymbol{\nu})^{-1}$ is the maximum of $f(\boldsymbol{\psi})$ over probability vectors $\boldsymbol{\psi} \in \triangle_K$. Here, we instead define f for all non-negative vectors, and proceed by varying the total budget of measurement effort available

$\sum_{a\in[K]} \psi_a$. Using Lemma 36, f is non-decreasing in ψ_a for all a. f is homogeneous of degree 1. That is $f(c\boldsymbol{\psi}) = cf(\boldsymbol{\psi})$ for all $c \geq 1$. For each $c_1, c_2 > 0$ define

$$g\left(c_1, c_2\right) = \max\left\{ f(\boldsymbol{\psi}) \mid \boldsymbol{\psi} \in \mathbb{R}_+^K,\ \psi_{a^\star(\boldsymbol{\nu})} = c_1, \sum_{a\neq a^\star(\boldsymbol{\nu})} \psi_a \leq c_2, \right\} .$$

The function g inherits key properties of f; it is also non-decreasing and homogeneous of degree 1. We have

$$\begin{aligned} T^\star_{\epsilon,\beta}(\boldsymbol{\nu})^{-1} &= \max\left\{ f(\boldsymbol{\psi}) \mid \boldsymbol{\psi} \in \mathbb{R}_+^K,\ \psi_{a^\star} = \beta, \sum_{a\in[K]} \psi_a = 1 \right\} \\ &= \max\left\{ f(\boldsymbol{\psi}) \mid \boldsymbol{\psi} \in \mathbb{R}_+^K,\ \psi_{a^\star} = \beta, \sum_{a\neq a^\star} \psi_a \leq 1-\beta \right\} = g(\beta, 1-\beta) , \end{aligned}$$

where the second equality uses that f is non-decreasing. Similarly, $T^\star_{\epsilon}(\boldsymbol{\nu})^{-1} = g\left(\beta^\star, 1-\beta^\star\right)$ where $\beta^\star = w^\star_{\epsilon,a^\star}$. Setting $r := \max\left\{\frac{\beta^\star}{\beta}, \frac{1-\beta^\star}{1-\beta}\right\}$ implies $r\beta \geq \beta^\star$ and $r(1-\beta) \geq 1-\beta^\star$. Therefore

$$rT^\star_{\epsilon,\beta}(\boldsymbol{\nu})^{-1} = rg(\beta, 1-\beta) = g(r\beta, r(1-\beta)) \geq g\left(\beta^\star, 1-\beta^\star\right) = T^\star_{\epsilon}(\boldsymbol{\nu})^{-1} .$$

Taking $\beta = \frac{1}{2}$, yields that $T^\star_{\epsilon}(\boldsymbol{\nu})^{-1} \leq 2\max\{\beta^\star, 1-\beta^\star\} T^\star_{1/2}(\boldsymbol{\nu})^{-1} \leq 2T^\star_{1/2}(\boldsymbol{\nu})^{-1}$. $\square$

Lemma 44 gives sufficient conditions on the means and allocations in order for the transportation costs to be equals to the non-private transportation costs. Moreover, it gives sufficient conditions on the means in order for this equality to hold irrespective of the considered allocation. Taken together, this result allows to have fine and coarse understanding of the separation between the high privacy regime and the low privacy regime for ϵ-global DP BAI.

Lemma 44. *Let $W_{\epsilon,a,b}$ as in Eq. (4). Let $\mu \in (0,1)^K$ such that $a^\star = \arg\max_{a\in[K]} \mu_a$ is unique. Let $w \in (\mathbb{R}_+^\star)^K$. Let $\epsilon > 0$. For all $x \in (0,1)$, we define $f_\epsilon(x) := (1-x)\left(1 - \frac{1}{1+x(e^\epsilon-1)}\right) = (1-x)g_\epsilon^-(x)(1-e^{-\epsilon})$. Let us define $\mu^w_{a^\star,a} := \frac{w_{a^\star}\mu_{a^\star} + w_a\mu_a}{w_{a^\star}+w_a}$ for all $a \neq a^\star$. For all $a \neq a^\star$, we have*

$$\begin{aligned} & \mu_{a^\star} - \mu_a \leq \min\left\{ \left(1 + \frac{w_{a^\star}}{w_a}\right) f_\epsilon(1-\mu_{a^\star}), \left(1 + \frac{w_a}{w_{a^\star}}\right) f_\epsilon(\mu_a) \right\} \\ \implies \quad & W_{\epsilon,a^\star,a}(\mu, w) = w_{a^\star}\mathrm{kl}(\mu_{a^\star}, \mu^w_{a^\star,a}) + w_a \mathrm{kl}(\mu_a, \mu^w_{a^\star,a}) . \end{aligned}$$

Moreover, we have

$$\max_{a^\star\in[K],\ \mu\in(0,1)^K,\ a^\star(\mu)=\{a^\star\},\ w\in(\mathbb{R}_+^\star)^K} \min\left\{ \left(1 + \frac{w_{a^\star}}{w_a}\right) f_\epsilon(1-\mu_{a^\star}), \left(1 + \frac{w_a}{w_{a^\star}}\right) f_\epsilon(\mu_a) \right\} \leq \epsilon/2$$

and, for all $a \neq a^\star$, we have

$$\begin{aligned} & \epsilon \geq \log\left(\frac{\mu_{a^\star}(1-\mu_a)}{\mu_a(1-\mu_{a^\star})}\right) = \frac{\partial \mathrm{kl}}{\partial x_1}(\mu_{a^\star}, \mu_a) = \frac{\partial \mathrm{kl}}{\partial x_1}(\mu_a, \mu_{a^\star}) \\ \implies \quad & \forall w \in (\mathbb{R}_+^\star)^K, \quad W_{\epsilon,a^\star,a}(\mu, w) = w_{a^\star}\mathrm{kl}(\mu_{a^\star}, \mu^w_{a^\star,a}) + w_a \mathrm{kl}(\mu_a, \mu^w_{a^\star,a}) . \end{aligned}$$

Proof. Let us define $f_\epsilon(x) = (1-x)\left(1 - \frac{1}{1+x(e^\epsilon-1)}\right)$ for all $x \in (0,1)$. Then, we have

$$\begin{aligned} \frac{\mu_a(1-\mu_a)(e^\epsilon-1)}{1+\mu_a(e^\epsilon-1)} &= (1-\mu_a)\left(1 - \frac{1}{1+\mu_a(e^\epsilon-1)}\right) = f_\epsilon(\mu_a) , \\ \frac{\mu_{a^\star}(1-\mu_{a^\star})(e^\epsilon-1)}{e^\epsilon-\mu_{a^\star}(e^\epsilon-1)} &= \mu_{a^\star}\left(1 - \frac{1}{1+(1-\mu_{a^\star})(e^\epsilon-1)}\right) = f_\epsilon(1-\mu_{a^\star}) . \end{aligned}$$

Using Lemma 23, direct manipulation yields that

$$f_\epsilon(1-\mu_{a^\star}) < \mu_{a^\star} - \mu_a \iff g_\epsilon^+(\mu_{a^\star}) > \mu_a \iff f_\epsilon(\mu_a) < \mu_{a^\star} - \mu_a$$

$$
\begin{aligned}
g_\epsilon^-(\mu_a) < \mu^w_{a^\star,a} \quad &\iff \quad f_\epsilon(\mu_a) < \frac{w_{a^\star}}{w_{a^\star}+w_a}(\mu_{a^\star}-\mu_a) \; , \\
g_\epsilon^+(\mu_{a^\star}) > \mu^w_{a^\star,a} \quad &\iff \quad f_\epsilon(1-\mu_{a^\star}) < \frac{w_a}{w_{a^\star}+w_a}(\mu_{a^\star}-\mu_a) \; .
\end{aligned}
$$

Using that $\max\left\{\frac{w_a}{w_{a^\star}+w_a}, \frac{w_{a^\star}}{w_{a^\star}+w_a}\right\} \leq 1$, we obtain that

$$
\begin{aligned}
&\left(g_\epsilon^-(\mu_a) < \mu_{a^\star} \;\wedge\; g_\epsilon^-(\mu_a) < \mu^w_{a^\star,a}\right) \quad \iff \quad \left(1+\frac{w_a}{w_{a^\star}}\right) f_\epsilon(\mu_a) < \mu_{a^\star}-\mu_a \; , \\
&\left(g_\epsilon^-(\mu_a) < \mu_{a^\star} \;\wedge\; g_\epsilon^+(\mu_{a^\star}) > \mu^w_{a^\star,a}\right) \quad \iff \quad \left(1+\frac{w_{a^\star}}{w_a}\right) f_\epsilon(1-\mu_{a^\star}) < \mu_{a^\star}-\mu_a \; , \\
&\left(g_\epsilon^-(\mu_a) \geq \mu_{a^\star} \;\vee\; \left(g_\epsilon^-(\mu_a) < \mu_{a^\star} \;\wedge\; \mu^w_{a^\star,a} \in [g_\epsilon^+(\mu_{a^\star}), g_\epsilon^-(\mu_a)]\right)\right) \\
&\iff \quad \left(g_\epsilon^-(\mu_a) \geq \mu_{a^\star} \;\vee\; \mu^w_{a^\star,a} \in [g_\epsilon^+(\mu_{a^\star}), g_\epsilon^-(\mu_a)]\right) \\
&\iff \quad (\min\{f_\epsilon(\mu_a), f_\epsilon(1-\mu_{a^\star})\} \geq \mu_{a^\star}-\mu_a \\
&\qquad\qquad \vee\, \mu_{a^\star}-\mu_a \leq \min\left\{\left(1+\frac{w_{a^\star}}{w_a}\right) f_\epsilon(1-\mu_{a^\star}), \left(1+\frac{w_a}{w_{a^\star}}\right) f_\epsilon(\mu_a)\right\}\Bigg) \\
&\iff \quad \mu_{a^\star}-\mu_a \leq \max\{\min\{f_\epsilon(\mu_a), f_\epsilon(1-\mu_{a^\star})\}, \\
&\qquad\qquad \min\left\{\left(1+\frac{w_{a^\star}}{w_a}\right) f_\epsilon(1-\mu_{a^\star}), \left(1+\frac{w_a}{w_{a^\star}}\right) f_\epsilon(\mu_a)\right\}\Bigg\} \\
&\iff \quad \mu_{a^\star}-\mu_a \leq \min\left\{\left(1+\frac{w_{a^\star}}{w_a}\right) f_\epsilon(1-\mu_{a^\star}), \left(1+\frac{w_a}{w_{a^\star}}\right) f_\epsilon(\mu_a)\right\} \; .
\end{aligned}
$$

Combining those conditions with Lemma 37 concludes the first part of the proof.

For all $x \in (0,1)$, we have

$$
\begin{aligned}
f_\epsilon'(x) &= \frac{(1-x)(e^\epsilon-1) - x(e^\epsilon-1)(1+x(e^\epsilon-1))}{(1+x(e^\epsilon-1))^2} = -(e^\epsilon-1)\frac{x^2(e^\epsilon-1)+2x-1}{(1+x(e^\epsilon-1))^2} \; , \\
f_\epsilon'(x) &= 0 \quad \iff \quad x = \frac{e^{\epsilon/2}-1}{e^\epsilon-1} \; , \\
f_\epsilon''(x) &= -2(e^\epsilon-1)\frac{(1+x(e^\epsilon-1))^2 - (e^\epsilon-1)\left(x^2(e^\epsilon-1)+2x-1\right)}{(1+x(e^\epsilon-1))^3} \\
&= -\frac{2e^\epsilon(e^\epsilon-1)}{(1+x(e^\epsilon-1))^3} \leq 0 \; .
\end{aligned}
$$

As f_ϵ is strictly concave, the maximum is achieved at $\frac{e^{\epsilon/2}-1}{e^\epsilon-1}$ with value

$$
\max_{x\in(0,1)} f_\epsilon(x) = f_\epsilon\left(\frac{e^{\epsilon/2}-1}{e^\epsilon-1}\right) = \frac{e^\epsilon - e^{\epsilon/2}}{e^\epsilon-1}\left(1-e^{-\epsilon/2}\right) = \frac{(e^{\epsilon/2}-1)^2}{e^\epsilon-1} \; .
$$

Let $\kappa_1(x) = x(e^x-1) - 4(e^{x/2}-1)^2$ for all $x > 0$. Then, we have

$$
\frac{(e^{\epsilon/2}-1)^2}{e^\epsilon-1} \leq \epsilon/4 \quad \iff \quad \kappa_1(\epsilon) \geq 0 \; .
$$

Then, we have $\kappa_1(0) = 0$ and

$$
\kappa_1'(x) = 4e^{x/2} - 3e^x - 1 + xe^x \quad \text{and} \quad \kappa_1''(x) = e^x\left(2(e^{-x/2}-1)+x\right) \; .
$$

Using that $e^{-x/2}-1 \geq -x/2$, we obtain $\kappa_1''(x) \geq 0$. Using that $\kappa_1'(0) = 0$, we obtain $\kappa_1'(x) \geq 0$. Using that $\kappa_1(0) = 0$, we obtain $\kappa_1(x) \geq 0$. Therefore, we have shown that

$$
\forall \epsilon > 0, \quad \max_{x\in(0,1)} f_\epsilon(x) \leq \epsilon/4 \; .
$$

Direct manipulation yields that

$$
\min\left\{\left(1+\frac{w_{a^\star}}{w_a}\right) f_\epsilon(1-\mu_{a^\star}), \left(1+\frac{w_a}{w_{a^\star}}\right) f_\epsilon(\mu_a)\right\}
$$

$$\leq \left(1 + \min\left\{\frac{w_{a^\star}}{w_a}, \frac{w_a}{w_{a^\star}}\right\}\right) \max_{x\in(0,1)} f_\epsilon(x) \leq \epsilon/2 \, .$$

Taking the supremum over $w \in (\mathbb{R}_+^\star)^K$, $\mu \in (0,1)^K$ such that $a^\star = a^\star(\mu)$ and over $a^\star \in [K]$ concludes the second part of the proof.

Let $a \neq a^\star$. Direct manipulations yield that

$$\begin{aligned}
& \mu_{a^\star} - \mu_a \leq \min\{f_\epsilon(1-\mu_{a^\star}), f_\epsilon(\mu_a)\} \\
\implies\quad & \forall w \in (\mathbb{R}_+^\star)^K, \quad \mu_{a^\star} - \mu_a \leq \min\left\{\left(1+\frac{w_{a^\star}}{w_a}\right) f_\epsilon(1-\mu_{a^\star}), \left(1+\frac{w_a}{w_{a^\star}}\right) f_\epsilon(\mu_a)\right\} \\
\implies\quad & \forall w \in (\mathbb{R}_+^\star)^K, \quad W_{\epsilon,a^\star,a}(\mu, w) = w_{a^\star}\mathrm{kl}(\mu_{a^\star}, \mu^w_{a^\star,a}) + w_a \mathrm{kl}(\mu_a, \mu^w_{a^\star,a}) \, .
\end{aligned}$$

Recall that $f_\epsilon(x) = (1-x)\left(1 - \frac{1}{1+x(e^\epsilon - 1)}\right)$. Then, we have directly that

$$f_\epsilon(x) \geq y \quad \iff \quad \frac{1-x-y}{1-x} \geq \frac{1}{1+x(e^\epsilon-1)} \quad \iff \quad \frac{(y+x)(1-x)}{x(1-x-y)} \leq e^\epsilon \, .$$

Plugging this result, we obtain

$$\begin{aligned}
\mu_{a^\star} - \mu_a \leq \min\{f_\epsilon(1-\mu_{a^\star}), f_\epsilon(\mu_a)\} \quad & \iff \quad e^\epsilon \geq \frac{\mu_{a^\star}(1-\mu_a)}{\mu_a(1-\mu_{a^\star})} \\
& \iff \quad \epsilon \geq \log\left(\frac{\mu_{a^\star}(1-\mu_a)}{\mu_a(1-\mu_{a^\star})}\right) \, .
\end{aligned}$$

Recall that

$$\frac{\partial \mathrm{kl}}{\partial x_1}(\mu_{a^\star}, \mu_a) = \frac{\partial \mathrm{kl}}{\partial x_1}(\mu_a, \mu_{a^\star}) = \log\left(\frac{\mu_{a^\star}(1-\mu_a)}{\mu_a(1-\mu_{a^\star})}\right) \, .$$

This concludes the proof of the last part of the result. □

Lemma 45 shows that our lower bound is larger (hence better) than the one derived in Azize et al. [12].

Lemma 45. *Let $T^\star_g(\boldsymbol{\nu}, \epsilon)$ as in Theorem 13 in Azize et al. [12], and $T^\star_\epsilon(\boldsymbol{\nu})$ as in Eq.* (35)*. Then, we have $T^\star_g(\boldsymbol{\nu}, \epsilon) \leq T^\star_\epsilon(\boldsymbol{\nu})$.*

Proof. Let $T^\star_g(\boldsymbol{\nu}, \epsilon)$ as in Theorem 13 in Azize et al. [12]. A sufficient condition to obtain $T^\star_g(\boldsymbol{\nu}, \epsilon) \leq T^\star_\epsilon(\boldsymbol{\nu})$ is to show that, for all $\lambda \in \mathrm{Alt}(\mu)$, we have

$$\sum_{a\in[K]} w_a d_\epsilon(\mu_a, \lambda_a) \leq \min\left\{\sum_{a\in[K]} w_a \mathrm{kl}(\mu_a, \lambda_a), 6\epsilon \sum_{a\in[K]} w_a|\mu_a - \lambda_a|\right\} \, ,$$

since we can conclude by taking the infimum over $\lambda \in \mathrm{Alt}(\mu)$ and the supremum over $w \in \triangle_K$ on both sides of the inequalities. By definition of d_ϵ and evaluation the function at $z = \mu$ and $z = \lambda$ respectively, we obtain

$$d_\epsilon(\lambda, \mu) = \inf_{z\in(0,1)} \{\mathrm{kl}(z, \mu) + \epsilon|\lambda - z|\} \leq \min\{\mathrm{kl}(\lambda, \mu), \epsilon|\lambda - \mu|\} \, .$$

By summing those inequalities over arms $a \in [K]$, we obtain

$$\begin{aligned}
\sum_{a\in[K]} w_a d_\epsilon(\mu_a, \lambda_a) &\leq \sum_{a\in[K]} w_a \min\{\mathrm{kl}(\mu_a, \lambda_a), \epsilon|\mu_a - \lambda_a|\} \\
&\leq \min\left\{\sum_{a\in[K]} w_a \mathrm{kl}(\mu_a, \lambda_a), \epsilon \sum_{a\in[K]} w_a|\mu_a - \lambda_a|\right\} \, .
\end{aligned}$$

Using that $\sum_{a\in[K]} w_a|\mu_a - \lambda_a| \geq 0$ and $6\epsilon \geq \epsilon$, this concludes the proof. □

In Garivier and Kaufmann [38], the authors show how to rewrite the optimization problem underlying the characteristic time and its optimal allocation as a simpler optimization problem. Lemma 46 shows that similar properties holds for ϵ-global DP BAI. In particular, it shows that computing the characteristic time $T^\star_\epsilon(\boldsymbol{\nu})$ and their optimal allocation $w^\star_\epsilon(\boldsymbol{\nu})$ can be done explicitly based on solving nested fixed-point equations. This result is key to implement computationally tractable Track-and-Stop algorithms. Additionally, Lemma 46 gives an explicit lower bound on the characteristic time $T^\star_\epsilon(\boldsymbol{\nu})$.

Lemma 46. *Let $d^{\pm}_\epsilon$ as in Eq. (3), and $(T^\star_\epsilon, w^\star_\epsilon)$ as in Eq. (35). Let $a \neq a^\star$. For $x \in [0, +\infty)$, let*

$$G_a(x) := \inf_{u \in [0,1]} \left\{ d^-_\epsilon(\mu_{a^\star}, u) + x d^+_\epsilon(\mu_a, u) \right\} \textit{ and } u_a(x) := \operatorname*{arg\,min}_{u \in [\mu_a, \mu_{a^\star}]} \{ d^-_\epsilon(\mu_{a^\star}, u) + x d^+_\epsilon(\mu_a, u) \} .$$

- *The function G_a is an increasing and strictly concave one-to-one mapping from $[0, +\infty)$ to $[0, d^-_\epsilon(\mu_{a^\star}, \mu_a))$; it satisfies that $G_a(0) = 0$ and $\lim_{x \to +\infty} G_a(x) = d^-_\epsilon(\mu_{a^\star}, \mu_a)$.*
- *The function u_a is a decreasing one-to-one mapping from $[0, +\infty)$ to $(\mu_a, \mu_{a^\star}]$; it satisfies that $u_a(0) = \mu_{a^\star}$ and $\lim_{x \to +\infty} u_a(x) = \mu_a$.*
- *Let $x_a(y)$ be defined as the unique solution of $G_a(x) = y$ for all $y \in [0, d^-_\epsilon(\mu_{a^\star}, \mu_a))$. The function x_a is an increasing and strictly convex one-to-one mapping from $[0, d^-_\epsilon(\mu_{a^\star}, \mu_a))$ to $[0, +\infty)$; it satisfies that $x_a(0) = 0$ and $\lim_{y \to d^-_\epsilon(\mu_{a^\star}, \mu_a)} x_a(y) = +\infty$.*

For all $y \in [0, \min_{a \neq a^\star} d^-_\epsilon(\mu_{a^\star}, \mu_a))$, let us define

$$G(y) := \frac{y}{1 + \sum_{a \neq a^\star} x_a(y)} \quad \textit{and} \quad F(y) := \sum_{a \neq a^\star} \frac{d^-_\epsilon(\mu_{a^\star}, u_a(x_a(y)))}{d^+_\epsilon(\mu_a, u_a(x_a(y)))} .$$

- *The function F is an increasing one-to-one mapping from $[0, \min_{a \neq a^\star} d^-_\epsilon(\mu_{a^\star}, \mu_a))$ to $[0, +\infty)$; it satisfies that $F(0) = 0$ and $\lim_{y \to \min_{a \neq a^\star} d^-_\epsilon(\mu_{a^\star}, \mu_a)} F(y) = +\infty$.*
- *On $[0, \min_{a \neq a^\star} d^-_\epsilon(\mu_{a^\star}, \mu_a))$, the function G is maximized at the unique $y^\star$ solution in $[0, \min_{a \neq a^\star} d^-_\epsilon(\mu_{a^\star}, \mu_a))$ of the fixed-point equation $F(y) = 1$. Moreover, we have $w^\star_\epsilon(\boldsymbol{\nu})_a = w^\star_\epsilon(\boldsymbol{\nu})_{a^\star} x_a(y^\star)$ for all $a \neq a^\star$,*

$$w^\star_\epsilon(\boldsymbol{\nu})_{a^\star} = \frac{1}{1 + \sum_{a \neq a^\star} x_a(y^\star)} \quad \textit{and} \quad T^\star_\epsilon(\boldsymbol{\nu})^{-1} = \frac{y^\star}{1 + \sum_{a \neq a^\star} x_a(y^\star)} .$$

- *Moreover, we have*

$$T^\star_\epsilon(\boldsymbol{\nu}) \geq \frac{1}{\min_{a \neq a^\star} d^-_\epsilon(\mu_{a^\star}, \mu_a)} + \sum_{a \neq a^\star} \frac{1}{d^+_\epsilon(\mu_a, \mu_{a^\star})} .$$

If $\epsilon < \log\left(\frac{\mu_a(1-\mu_b)}{\mu_b(1-\mu_a)}\right)$, we have $d^+_\epsilon(\mu_a, \mu_{a^\star}) = -\log(1 - \mu_{a^\star}(1 - e^{-\epsilon})) - \epsilon \mu_a$ and $d^-_\epsilon(\mu_{a^\star}, \mu_a) = -\log(1 + \mu_a(e^\epsilon - 1)) + \epsilon \mu_{a^\star}$.

Proof. Using Lemma 36, we know that G_a is concave. Let $u_a(x) \in \arg\min_{u \in [0,1]} \{d^-_\epsilon(\mu_{a^\star}, u) + x d^+_\epsilon(\mu_a, u)\}$ for all $x \in [0, +\infty)$, whose explicit formula is given in Lemma 44. It is direct to see that $G_a(0) = 0$ and $u_a(0) = \mu_{a^\star}$. Using the optimality condition of $u_a(x)$, we obtain, for all $x \in [0, +\infty)$,

$$\begin{aligned} G'_a(x) &= u'_a(x) \left(\frac{\partial d^-_\epsilon}{\partial u}(\mu_{a^\star}, u_a(x)) + x \frac{\partial d^+_\epsilon}{\partial u}(\mu_a, u_a(x)) \right) + d^+_\epsilon(\mu_a, u_a(x)) \\ &= d^+_\epsilon(\mu_a, u_a(x)) > 0 , \end{aligned}$$

where the last inequality is obtained by Lemma 34 and using that $d^+_\epsilon(\mu_a, u_a(0)) = d^+_\epsilon(\mu_a, \mu_{a^\star}) > 0$. Therefore, G_a is an increasing one-to-one mapping from $[0, +\infty)$ to $[0, \lim_{x \to +\infty} G_a(x))$.

Let $\mu^x_{a^\star, a} = \frac{\mu_{a^\star} + x\mu_a}{1+x}$ for all $x \in [0, +\infty)$. It is easy to see that $G_a(0) = 0$, $u_a(0) = \mu_{a^\star}$ and $\lim_{x \to +\infty} \mu^x_{a^\star, a} = \mu_a$. Using Lemma 44, we obtain that

$$\lim_{x \to +\infty} \min \{(1 + 1/x) f_\epsilon(1 - \mu_{a^\star}), (1 + x) f_\epsilon(\mu_a)\} = f_\epsilon(1 - \mu_{a^\star}) .$$

When $\mu_{a^\star} - \mu_a \le f_\epsilon(1-\mu_{a^\star})$, we obtain

$$\begin{aligned}\lim_{x\to+\infty} G_a(x) &= \lim_{x\to+\infty} \left\{\mathrm{kl}(\mu_{a^\star}, \mu^x_{a^\star,a}) + x\mathrm{kl}(\mu_a, \mu^x_{a^\star,a})\right\} \\ &= \mathrm{kl}(\mu_{a^\star}, \mu_a) + \lim_{x\to+\infty} \left\{x\mathrm{kl}(\mu_a, \mu^x_{a^\star,a})\right\} = \mathrm{kl}(\mu_{a^\star}, \mu_a) \; ,\end{aligned}$$

where we used that

$$x\mathrm{kl}(\mu_a, \mu^x_{a^\star,a}) = x\left(\mu_a \log\left(1 - \frac{\mu_{a^\star}-\mu_a}{\mu_a(1-\mu_a)}\frac{1}{\frac{\mu_{a^\star}}{\mu_a}+x}\right) + \log\left(1 + \frac{\mu_{a^\star}-\mu_a}{1-\mu_a}\frac{1}{\frac{1-\mu_{a^\star}}{1-\mu_a}+x}\right)\right) \; ,$$

$$\lim_{x\to+\infty} \left\{x\mathrm{kl}(\mu_a, \mu^x_{a^\star,a})\right\} = \frac{(\mu_{a^\star}-\mu_a)^2}{\mu_a(1-\mu_a)^2} \lim_{x\to+\infty} \left\{\frac{x}{\left(\frac{\mu_{a^\star}}{\mu_a}+x\right)\left(\frac{1-\mu_{a^\star}}{1-\mu_a}+x\right)}\right\} = 0 \; ,$$

where we used that $\log(1+x) =_{x\to 0} x + \mathcal{O}(x^2)$. Using Lemma 25 and the proof of Lemma 44, we know that $\mu_{a^\star} - \mu_a \le f_\epsilon(1-\mu_{a^\star})$ if and only if $\mu_a \in [g^+_\epsilon(\mu_{a^\star}), \mu_{a^\star})$, hence we have $\mathrm{kl}(\mu_{a^\star}, \mu_a) = d^-_\epsilon(\mu_{a^\star}, \mu_a)$. This concludes the proof in the first case.

When $\mu_{a^\star} - \mu_a > f_\epsilon(1-\mu_{a^\star})$, we obtain

$$\lim_{x\to+\infty} G_a(x) = \lim_{x\to+\infty} \left\{-\log\left(1 + u_{1,\star}(\mu_a, x)(e^\epsilon - 1)\right) + \epsilon\mu_{a^\star} + x\mathrm{kl}(\mu_a, u_{1,\star}(\mu_a, x)\right\} \; ,$$

where

$$\begin{aligned}&u_{1,\star}(\mu_a, x) = \\ &\frac{\sqrt{(x(1-\mu_a(e^\epsilon-1)) - (e^\epsilon-1))^2 + 4(1+x)(e^\epsilon-1)x\mu_a} - (x(1-\mu_a(e^\epsilon-1)) - (e^\epsilon-1))}{2(1+x)(e^\epsilon-1)} \; .\end{aligned}$$

Direct manipulation yields that $\lim_{x\to+\infty} u_{1,\star}(\mu_a, x) = \mu_a$, hence

$$\lim_{x\to+\infty} G_a(x) = -\log\left(1 + \mu_a(e^\epsilon-1)\right) + \epsilon\mu_{a^\star} + \lim_{x\to+\infty} \left\{x\mathrm{kl}(\mu_a, u_{1,\star}(\mu_a, x)\right\} \; .$$

Let us denote $v_{1,\star}(\mu_a, x) = u_{1,\star}(\mu_a, x) - \mu_a \ge 0$, i.e., $\lim_{x\to+\infty} v_{1,\star}(\mu_a, x) = 0$. Direct manipulation yields that

$$\begin{aligned}&v_{1,\star}(\mu_a, x) \\ &= \frac{1+\mu_a(e^\epsilon-1)}{2(e^\epsilon-1)}\left(1 - \frac{1}{x+1}\right) \\ &\qquad \left(\sqrt{1 - \frac{2x(1-\mu_a(e^\epsilon+1))(e^\epsilon-1) - (e^\epsilon-1)^2}{x^2(1+\mu_a(e^\epsilon-1))^2}} - 1 + \frac{(e^\epsilon-1)(1-2\mu_a)}{x(1+\mu_a(e^\epsilon-1))}\right) \\ &= \sqrt{1 - \frac{2x(1-\mu_a(e^\epsilon+1))(e^\epsilon-1) - (e^\epsilon-1)^2}{x^2(1+\mu_a(e^\epsilon-1))^2}} - 1 + \frac{(e^\epsilon-1)(1-2\mu_a)}{x(1+\mu_a(e^\epsilon-1))} \\ &=_{x\to+\infty} \frac{(e^\epsilon-1)(1-2\mu_a)}{x(1+\mu_a(e^\epsilon-1))} - \frac{2x(1-\mu_a(e^\epsilon+1))(e^\epsilon-1) - (e^\epsilon-1)^2}{2x^2(1+\mu_a(e^\epsilon-1))^2} + \mathcal{O}(1/x^2) \\ &=_{x\to+\infty} \frac{2x(e^\epsilon-1)2(1-\mu_a)\mu_a(e^\epsilon-1) + (e^\epsilon-1)^2}{2x^2(1+\mu_a(e^\epsilon-1))^2} + \mathcal{O}(1/x^2) \; , \\ &\text{hence} \quad v_{1,\star}(\mu_a, x) =_{x\to+\infty} \frac{2(1-\mu_a)\mu_a(e^\epsilon-1)^2}{x(1+\mu_a(e^\epsilon-1))^2} + \mathcal{O}(1/x^2) \; .\end{aligned}$$

where we used that $\sqrt{1-x} - 1 =_{x\to 0} -x/2 + \mathcal{O}(x^2)$ to obtain the last result. Similarly as before, we derive

$$\begin{aligned}x\mathrm{kl}(\mu_a, u_{1,\star}(\mu_a, x)) = x\Bigg(&\mu_a \log\left(1 - \frac{1}{\mu_a(1-\mu_a)}\frac{v_{1,\star}(\mu_a, x)}{1 + v_{1,\star}(\mu_a, x)/\mu_a}\right) \\ &+ \log\left(1 + \frac{1}{1-\mu_a}\frac{v_{1,\star}(\mu_a, x)}{1 - v_{1,\star}(\mu_a, x)/(1-\mu_a)}\right)\Bigg)\end{aligned}$$

$$\lim_{x\to+\infty}\left\{x\mathrm{kl}(\mu_a,\mu^x_{a^\star,a})\right\} = \frac{1}{\mu_a(1-\mu_a)^2}\lim_{x\to+\infty}\left\{\frac{xv_{1,\star}(\mu_a,x)^2}{\left(1-\frac{v_{1,\star}(\mu_a,x)}{1-\mu_a}\right)\left(1+\frac{v_{1,\star}(\mu_a,x)}{\mu_a}\right)}\right\}$$
$$= \frac{\lim_{x\to+\infty} xv_{1,\star}(\mu_a,x)^2}{\mu_a(1-\mu_a)^2} = 0\,,$$

where we used that $v_{1,\star}(\mu_a,x) =_{x\to+\infty} \mathcal{O}(1/x)$ to conclude. Therefore, we have shown that $\lim_{x\to+\infty} G_a(x) = -\log\left(1+\mu_a(e^\epsilon-1)\right)+\epsilon\mu_{a^\star}$. Using Lemma 25 and the proof of Lemma 44, we know that $\mu_{a^\star}-\mu_a > f_\epsilon(1-\mu_{a^\star})$ if and only if $\mu_a \in [0, g^+_\epsilon(\mu_{a^\star}))$, hence we have $-\log\left(1+\mu_a(e^\epsilon-1)\right)+\epsilon\mu_{a^\star} = d^-_\epsilon(\mu_{a^\star},\mu_a)$. This concludes the proof in the second case.

Therefore, G_a is a strictly increasing one-to-one mapping from $[0,+\infty)$ to $[0, d^-_\epsilon(\mu_{a^\star},\mu_a))$. Using the implicit function theorem, we obtain

$$\forall x\in[0,+\infty),\quad u'_a(x) = -\frac{\frac{\partial d^+_\epsilon}{\partial u}(\mu_a,u_a(x))}{\frac{\partial^2 d^-_\epsilon}{\partial u^2}(\mu_{a^\star},u_a(x)) + x\frac{\partial^2 d^+_\epsilon}{\partial u^2}(\mu_a,u_a(x))} < 0\,,$$

where the strict inequality is obtained by using properties in Lemmas 24 and 25, since $u_a(x)\in(\mu_a,\mu_{a^\star})$ by Lemmas 34 and 24. Similarly, we obtain

$$\forall x\in[0,+\infty),\quad G''_a(x) = u'_a(x)\frac{\partial d^+_\epsilon}{\partial u}(\mu_a,u_a(x)) > 0\,,$$

Therefore, we have shown that G_a is strictly concave and that u_a is decreasing.

Let us define $x_a(y)$ as the unique solution of $G_a(x)=y$, which is well-defined based on our above computations. Therefore, we have

$$y = d^-_\epsilon(\mu_{a^\star},u_a(x_a(y))) + x_a(y)d^+_\epsilon(\mu_a,u_a(x_a(y)))\,.$$

Using the derivative of the inverse function, we obtain

$$\forall y\in[0,d^-_\epsilon(\mu_{a^\star},\mu_a)),\quad x'_a(y) = \frac{1}{G'_a(x_a(y))} = \frac{1}{d^+_\epsilon(\mu_a,u_a(x_a(y)))} > 0\,,$$

hence x_a is increasing on $[0,d^-_\epsilon(\mu_{a^\star},\mu_a))$. Moreover, we have

$$\forall y\in[0,d^-_\epsilon(\mu_{a^\star},\mu_a)),\quad x''_a(y) = -\frac{u'_a(x_a(y))}{d^+_\epsilon(\mu_a,u_a(x_a(y)))^3}\frac{\partial d^+_\epsilon}{\partial u}(\mu_a,u_a(x_a(y)) > 0\,,$$

hence x_a is strictly convex on $[0,d^-_\epsilon(\mu_{a^\star},\mu_a))$.

For all $y\in[0,\min_{a\neq a^\star} d^-_\epsilon(\mu_{a^\star},\mu_a))$, let us define $G(y) = \frac{y}{1+\sum_{a\neq a^\star} x_a(y)}$ and $F(y) = \sum_{a\neq a^\star}\frac{d^-_\epsilon(\mu_{a^\star},u_a(x_a(y)))}{d^+_\epsilon(\mu_a,u_a(x_a(y)))}$. Using the above results, direct manipulations yield that, for all $y\in[0,\min_{a\neq a^\star} d^-_\epsilon(\mu_{a^\star},\mu_a))$,

$$G'(y) = \frac{1+\sum_{a\neq a^\star} x_a(y) - y\sum_{a\neq a^\star} x'_a(y)}{(1+\sum_{a\neq a^\star} x_a(y))^2} = \frac{1+\sum_{a\neq a^\star} x_a(y) - \sum_{a\neq a^\star}\frac{y}{d^+_\epsilon(\mu_a,u_a(x_a(y)))}}{(1+\sum_{a\neq a^\star} x_a(y))^2}$$
$$= \frac{1-F(y)}{(1+\sum_{a\neq a^\star} x_a(y))^2}\,,$$

hence we obtain that $G'(y)=0$ if and only if $F(y)=1$. Using that $x_a(0)=0$, $u_a(0)=\mu_{a^\star}$ and $d^-_\epsilon(\mu_{a^\star},\mu_{a^\star})=0$, we obtain that $F(0)=0$.

Using that $\lim_{y\to d^-_\epsilon(\mu_{a^\star},\mu_a)} x_a(y) = +\infty$, $\lim_{x\to+\infty} u_a(x) = \mu_a$, $d^-_\epsilon(\mu_{a^\star},\mu_a) > 0$ and $d^+_\epsilon(\mu_a,\mu_a)=0$, we obtain that $\lim_{y\to\min_{a\neq a^\star} d^-_\epsilon(\mu_{a^\star},\mu_a)} F(y) = +\infty$.

Let $H(y) = \sum_{a\neq a^\star}\frac{1}{d^+_\epsilon(\mu_a,u_a(x_a(y)))}$ for all $y\in[0,\min_{a\neq a^\star} d^-_\epsilon(\mu_{a^\star},\mu_a))$. Then, we have

$$\sum_{a\neq a^\star} x_a(y) = yH(y) - F(y)\quad,\quad \sum_{a\neq a^\star} x'_a(y) = H(y)\,,$$

$$\frac{\partial d_\epsilon^-}{\partial u}(\mu_{a^\star}, u_a(x)) + x \frac{\partial d_\epsilon^+}{\partial u}(\mu_a, u_a(x)) = 0 \, ,$$
$$d_\epsilon^-(\mu_{a^\star}, u_a(x_a(y))) + x_a(y) d_\epsilon^+(\mu_a, u_a(x_a(y))) = y \, .$$

Then, for all $y \in [0, \min_{a \neq a^\star} d_\epsilon^-(\mu_{a^\star}, \mu_a))$, we have

$$\begin{aligned}
H'(y) &= -\sum_{a \neq a^\star} \frac{u_a'(x_a(y)) x_a'(y)}{(d_\epsilon^+(\mu_a, u_a(x_a(y))))^2} \frac{\partial d_\epsilon^+}{\partial u}(\mu_a, u_a(x_a(y))) \\
&= -\sum_{a \neq a^\star} \frac{u_a'(x_a(y))}{(d_\epsilon^+(\mu_a, u_a(x_a(y))))^3} \frac{\partial d_\epsilon^+}{\partial u}(\mu_a, u_a(x_a(y))) \, , \\
F'(y) &= \sum_{a \neq a^\star} \frac{u_a'(x_a(y)) x_a'(y)}{(d_\epsilon^+(\mu_a, u_a(x_a(y))))^2} \\
&\quad \left(d_\epsilon^+(\mu_a, u_a(x_a(y))) \frac{\partial d_\epsilon^-}{\partial u}(\mu_{a^\star}, u_a(x_a(y))) - d_\epsilon^-(\mu_{a^\star}, u_a(x_a(y))) \frac{\partial d_\epsilon^+}{\partial u}(\mu_a, u_a(x_a(y))) \right) \\
&= -\sum_{a \neq a^\star} \frac{u_a'(x_a(y)) x_a'(y)}{(d_\epsilon^+(\mu_a, u_a(x_a(y))))^2} \frac{\partial d_\epsilon^+}{\partial u}(\mu_a, u_a(x_a(y))) \\
&\quad \left(x_a(y) d_\epsilon^+(\mu_a, u_a(x_a(y))) + d_\epsilon^-(\mu_{a^\star}, u_a(x_a(y))) \right) \\
&= -y \sum_{a \neq a^\star} \frac{u_a'(x_a(y)) x_a'(y)}{(d_\epsilon^+(\mu_a, u_a(x_a(y))))^2} \frac{\partial d_\epsilon^+}{\partial u}(\mu_a, u_a(x_a(y))) = y H'(y) \, ,
\end{aligned}$$

Therefore, showing that H is increasing is a sufficient condition to show that F is increasing. Using the above results, we have, for all $a \neq a^\star$ and all $y \in [0, \min_{a \neq a^\star} d_\epsilon^-(\mu_{a^\star}, \mu_a))$, we have

$$\frac{1}{(d_\epsilon^+(\mu_a, u_a(x_a(y))))^3} \frac{\partial d_\epsilon^+}{\partial u}(\mu_a, u_a(x_a(y))) > 0 \quad \text{and} \quad u_a'(x_a(y)) < 0 \, .$$

Therefore, H is increasing as a summation of increasing function, hence F is increasing.

Let $y^\star$ such that $F(y^\star) = 1$. Reusing the above manipulation, we obtain

$$\begin{aligned}
G''(y) &= -\frac{F'(y)(1 + \sum_{a \neq a^\star} x_a(y)) + 2(1 - F(y)) \sum_{a \neq a^\star} x_a'(y)}{(1 + \sum_{a \neq a^\star} x_a(y))^3} \\
&= -\frac{y H'(y)(1 + y H(y) - F(y)) + 2(1 - F(y)) H(y)}{(1 + y H(y) - F(y))^3} \, , \\
G''(y^\star) &= -\frac{H'(y^\star)}{y^\star H(y^\star)^2} < 0 \, ,
\end{aligned}$$

Therefore, $y^\star$ is the unique maximum of G. We conclude this part of the proof by using the intermediate results in the proof of Lemma 42.

By strict convexity of x_a and using its properties proven above, we obtain

$$x_a(y) \geq x_a(0) + y x_a'(0) = \frac{y}{d_\epsilon^+(\mu_a, \mu_{a^\star})} \, .$$

Summing those inequalities, we obtain

$$\forall y \in [0, \min_{a \neq a^\star} d_\epsilon^-(\mu_{a^\star}, \mu_a)), \quad G(y) = \frac{y}{1 + \sum_{a \neq a^\star} x_a(y)} \leq \frac{1}{\frac{1}{y} + \sum_{a \neq a^\star} \frac{1}{d_\epsilon^+(\mu_a, \mu_{a^\star})}} \, .$$

Using that $y \mapsto 1/(1/y + \alpha)$ is increasing for $\alpha > 0$, we obtain that

$$T_\epsilon^\star(\boldsymbol{\nu})^{-1} = \max_{y \in [0, \min_{a \neq a^\star} d_\epsilon^-(\mu_{a^\star}, \mu_a))} G(y) \leq \frac{1}{\frac{1}{\min_{a \neq a^\star} d_\epsilon^-(\mu_{a^\star}, \mu_a))} + \sum_{a \neq a^\star} \frac{1}{d_\epsilon^+(\mu_a, \mu_{a^\star})}} \, .$$

This concludes the proof of the second to last result. The last result is obtained by combining Lemmas 25 and 24 and the derivation in the proof of Lemma 44. □

Lemma 47 is a technical result used in the proof of sufficient exploration of our sampling rule.

Lemma 47. *Let $d_\epsilon^{\pm}$ as in Eq.* (3)*. Let $\mu \in (0,1)^K$. There exists $\alpha > 0$ such that*

$$C_\mu := \min_{(a,b):\mu_a > \mu_b} \inf_{\substack{\lambda_a,\lambda_b: \\ \max_{c\in\{a,b\}} |\mu_c - \lambda_c| \le \alpha}} \inf_{u\in[0,1]} \left\{ d_\epsilon^-(\lambda_a, u) + d_\epsilon^+(\lambda_b, u) \right\} > 0 \, . \tag{38}$$

Proof. Using Lemma 34 for $w_1 = w_2 = 1$, the function $\mu \mapsto \inf_{u\in[0,1]} \{d_\epsilon^-(\mu_a, u) + d_\epsilon^+(\mu_b, u)\}$ is continuous on $\mathcal{F}^K$. Since it has strictly positive values when $\mu_a > \mu_b$ (Lemma 34), there exists α such that

$$\inf_{\substack{\lambda_a,\lambda_b: \\ \max_{c\in\{a,b\}} |\mu_c - \lambda_c| \le \alpha}} \inf_{u\in[0,1]} \left\{ d_\epsilon^-(\lambda_a, u) + d_\epsilon^+(\lambda_b, u) \right\} > 0 \, .$$

Further lower bounding by a finite number of strictly positive constants yields the result. □

Lemma 47 is a technical result used in the proof of convergence towards the optimal allocation of our sampling rule.

Lemma 48. *Let $d_\epsilon^{\pm}$ as in Eq.* (3)*. Let $(\phi_1, \phi_2) \in (0,1)^2$. Let $\mathcal{I}_a := \left\{ \mu \in (0,1)^K \mid a \in a^\star(\mu) \right\}$ for all $a \in [K]$. For all $a^\star \in [K]$, all $\mu \in \mathcal{I}_{a^\star}$, all $(a,b) \in ([K] \setminus \{a^\star\})^2$ such that $a \ne b$, and all $\beta \in [0,1]$, define*

$$G_{a,b}(\mu, \beta) := \inf_{u\in[0,1]} \left\{ \beta d_\epsilon^-(\mu_{a^\star}, u) + \phi_1 d_\epsilon^+(\mu_a, u) \right\} - \inf_{u\in(0,1)} \left\{ \beta d_\epsilon^-(\mu_{a^\star}, u) + \phi_2 d_\epsilon^+(\mu_b, u) \right\} \, .$$

The function $(\mu, \beta) \mapsto G_{a,b}(\mu, \beta)$ is continuous on $(0,1)^K \times [0,1]$. For all $\xi > 0$, the function $(\mu, \beta) \mapsto \inf_{\tilde\beta : |\beta - \tilde\beta| \le \xi} G_{a,b}(\mu, \beta)$ is continuous on $(0,1)^K$.

Proof. Since $\bigcup_{a\in[K]} \mathcal{I}_a^K = (0,1)^K$, it is enough to show the property for all $a \in [K]$. Let $a^\star \in [K]$, $\mu \in \mathcal{I}_{a^\star}$, $(a,b) \in ([K] \setminus \{a^\star\})^2$ such that $a \ne b$. As done in Lemma 41 by using Lemma 34, we obtain that the function $(\mu, \beta) \mapsto G_{a,b}(\mu, \beta)$ is continuous on $\mathcal{I}_{a^\star} \times [0,1]$ for all $a^\star \in [K]$, hence on $(0,1)^K \times [0,1]$. Let $\Phi : \mu \mapsto \left\{ \tilde\beta : |\beta - \tilde\beta| \le \xi \right\}$, it is a continuous (constant), compact valued and non-empty correspondence. Using the above continuity, Berge's theorem yields that $\mu \mapsto \inf_{\tilde\beta : |\beta - \tilde\beta| \le \xi} G_{a,b}(\mu, \tilde\beta)$ is continuous on $(0,1)^K$. □

H Asymptotic Upper Bound on the Expected Sample complexity

Let $\boldsymbol{\nu}$ be a Bernoulli instance with means $\mu \in (0,1)^2$ and unique best arm $a^\star \in [K]$, i.e., $\arg\max_{a\in[K]} \mu_a = \{a^\star\}$. Let $\beta \in (0,1)$. Let $w_{\epsilon,\beta}^\star(\boldsymbol{\nu}) = \{w_{\epsilon,\beta}^\star\}$ be the unique β-optimal allocation defined in Eq. (35), which satisfies $\min_{a\in[K]} w_{\epsilon,\beta,a}^\star > 0$ by Lemma 42. At equilibrium, we have equality of the transportation costs by Lemma 42, namely

$$\forall a \ne a^\star, \quad W_{\epsilon,a^\star,a}(\mu, w_{\epsilon,\beta}^\star) = T_{\epsilon,\beta}^\star(\boldsymbol{\nu})^{-1} \, , \tag{39}$$

where $W_{\epsilon,a,b}$ is defined in Eq. (4) and $T_{\epsilon,\beta}^\star$ is defined in Eq. (35).

Let $\gamma > 0$. Let $\omega \in \triangle_K$ be any allocation over arms such that $\min_a \omega_a > 0$. We denote by $T_\gamma(\omega)$ the *convergence time* towards ω, which is a random variable quantifying the number of samples required for the global empirical allocations $N_n/(n-1)$ to be γ-close to ω for any subsequent time, namely

$$T_\gamma(\omega) := \inf \left\{ T \ge 1 \mid \forall n \ge T, \ \left\| \frac{N_n}{n-1} - \omega \right\|_\infty \le \gamma \right\} \, . \tag{40}$$

The proof of Theorem 6 follows the same analysis as the unified analysis of Top Two algorithms, see, e.g., Jourdan et al. [50]. Appendix H is organised as follows. After recalling some technical results (Appendix H.1), we prove sufficient exploration of our sampling rule (Appendix H.2). Second, we prove that convergence time towards the β-optimal allocation of our sampling rule (Appendix H.3) has finite expectation. Finally, we conclude the proof of Theorems 6 (Appendix H.4).

H.1 Technical Results from the Literature

Lemma 49 relates the global counts $(N_{n,a})_{a\in[K]}$ and the local counts $(\tilde{N}_{n,a})_{a\in[K]}$.

Lemma 49. *Let $\eta > 0$ be the geometric parameter used for the geometric grid update of our private empirical mean estimator. For all $(a,k) \in [K] \times \mathbb{N}$ s.t. $\mathbb{E}_{\boldsymbol{\nu}\pi}[T_k(a)] < +\infty$, $N_{T_k(a),a} = \tilde{N}_{k,a} = \lceil (1+\eta)^{k-1} \rceil$. For all $a \in [K]$ and all $n \in \mathbb{N}$, $N_{n,a} \geq \tilde{N}_{n,a} \geq N_{n,a}/(1+\eta)$.*

Proof. Let $a \in [K]$. After initialisation, we have $k = 1$, $T_1(a) = K+1$ and $N_{T_1(a),a} = 1$. Using the definition of the phase switch, it is direct to see that $N_{T_2(a),a} = \tilde{N}_{2,a} = \lceil 1+\eta \rceil$ when $\mathbb{E}_{\boldsymbol{\nu}\pi}[T_2(a)] < +\infty$. Similarly, we obtain $N_{T_k(a),a} = \tilde{N}_{k,a} = \lceil (1+\eta)^{k-1} \rceil$ when $\mathbb{E}_{\boldsymbol{\nu}\pi}[T_k(a)] < +\infty$. The last result is a direct consequence of the definition of the per-arm geometric update grid. □

Lemma 50 controls the deviation $N^a_{n,a} - \beta L_{n,a}$ enforced by the tracking procedure.

Lemma 50 (Lemma 2.2 in [49])**.** *For all $n > K$ and all $a \in [K]$, $-1/2 \leq N^a_{n,a} - \beta L_{n,a} \leq 1$.*

Lemma 51 gathers properties on the $\overline{W}_{-1}$ function used in the stopping threshold.

Lemma 51 ([51])**.** *Let $\overline{W}_{-1}(x) := -W_{-1}(-e^{-x})$ for all $x \geq 1$, where W_{-1} is the negative branch of the Lambert W function. The function $\overline{W}_{-1}$ is increasing on $(1,+\infty)$ and strictly concave on $(1,+\infty)$. In particular, $\overline{W}'_{-1}(x) = \left(1 - \frac{1}{\overline{W}_{-1}(x)}\right)^{-1}$ for all $x > 1$. Then, for all $y \geq 1$ and $x \geq 1$,*

$$\overline{W}_{-1}(y) \leq x \quad \iff \quad y \leq x - \log(x) \; .$$

Moreover, for all $x > 1$,

$$x + \log(x) \leq \overline{W}_{-1}(x) \leq x + \log(x) + \min\left\{\frac{1}{2}, \frac{1}{\sqrt{x}}\right\} \; .$$

Lemma 52 gives an upper bound on a time define implicit as a function of $\overline{W}_{-1}$, namely it is an inversion result.

Lemma 52 (Lemma 32 in [12])**.** *Let $\overline{W}_{-1}$ defined in Lemma 51. Let $A > 0$, $B > 0$ such that $B/A + \log A > 1$ and $C(A,B) = \sup\{x \mid \; x < A \log x + B\}$. Then, $C(A,B) < h_1(A,B)$ with $h_1(z,y) = z\overline{W}_{-1}(y/z + \log z)$.*

Lemma 53 shows that upon correction the supremum of sub-exponential random variables is also a sub-exponential random variable.

Lemma 53 (Lemma 72 in [50])**.** *Suppose that $(X_n)_{n\geq 1}$ are sub-exponential random variables with constants (C_n), such that $c := \inf_n C_n > 0$. Then $\sup_n (X_n / \log(e+n))$ is sub-exponential.*

Lemma 54 gives a coarse convergence rate of the private empirical estimators of the means towards their true means.

Lemma 54. *There exist sub-exponential random variable W_μ such that almost surely, for all $a \in [K]$ and all n such that $\tilde{N}_{n,a} \geq 1$,*

$$\tilde{N}_{n,a} |\tilde{\mu}_{n,a} - \mu_a| \leq W_\mu \log(e + \tilde{N}_{n,a}) \; .$$

In particular, any random variable which is polynomial in W_μ has a finite expectation.

Proof. Let us define

$$W_\mu = \max_{a\in[K]} \sup_{n\in\mathbb{N}} \frac{\tilde{N}_{n,a} |\tilde{\mu}_{n,a} - \mu_a|}{\log(e + \tilde{N}_{n,a})} \; .$$

Let $a \in [K]$. Let us define the geometric grid $N_k = \lceil (1+\eta)^{k-1} \rceil$ for all $k \in \mathbb{N}$, on which we effectively need to control the concentration. The maximum of a finite number of sub-exponential random variables is sub-exponential. Therefore, using the geometric update grid, it suffices to show that

$$\sup_{k\in\mathbb{N}} \frac{N_k |(Z_{N_k} + S_k)/N_k - \mu_a|}{\log(e + N_k)}$$

is sub-exponential, where Z_{N_k} is the cumulative sum of N_k i.i.d. observations from $\mathrm{Ber}(\mu_a)$ and S_k is the cumulative sum of k i.i.d. observations from $\mathrm{Lap}(1/\epsilon)$.

Using that $Z_{N_k} - N_k\mu_a$ is sub-Gaussian and S_k is sub-exponential, for a fixed k, $|Z_{N_k} - N_k\mu_a + S_k|$ is sub-exponential. Applying Lemma 53, we obtain that

$$\sup_{k\in\mathbb{N}} \frac{N_k|(Z_{N_k} + S_k)/N_k - \mu_a|}{\log(e + N_k)}$$

is sub-exponential. We finally obtain that the maximum over the finitely many arms has the same property. □

H.2 Sufficient Exploration

The first step of in the generic analysis of Top Two algorithms [50] consists in showing sufficient exploration. The main idea is that, if there are still undersampled arms, either the leader or the challenger will be among them. Therefore, after a long enough time, no arm can still be undersampled. We emphasise that there are multiple ways to select the leader/challenger pair in order to ensure sufficient exploration. Therefore, other choices of leader/challenger pair would yield similar results.

Given an arbitrary phase $p \in \mathbb{N}$, we define the sampled enough set, i.e., the arms having reached phase p, and the arm with highest mean in this set (when not empty) as

$$S_n^p = \{a \in [K] \mid N_{n,a} \geq (1+\eta)^{p-1}\} \quad \text{and} \quad a_n^\star = \arg\max_{a\in S_n^p} \mu_a \,. \tag{41}$$

Since $\min_{a\neq b}|\mu_a - \mu_b| > 0$, $a_n^\star$ is unique. Let $p \in \mathbb{N}$ such that $(p-1)/4 \in \mathbb{N}$. We define the highly and the mildly under-sampled sets as

$$U_n^p := \{a \in [K] \mid N_{n,a} < (1+\eta)^{(p-1)/2}\} \quad \text{and} \quad V_n^p := \{a \in [K] \mid N_{n,a} < (1+\eta)^{3(p-1)/4}\} \,. \tag{42}$$

Those arms have not reached phase $(p-1)/2$ and phase $3(p-1)/4$, respectively.

Lemma 55 shows that, when the leader is sampled enough, it is the arm with highest true mean among the sampled enough arms.

Lemma 55. *Let S_n^p and $a_n^\star$ as in (41). There exists p_0 with $\mathbb{E}_{\boldsymbol{\nu}\pi}[\exp(\alpha p_0)] < +\infty$ for all $\alpha > 0$ such that if $p \geq p_0$, for all n such that $S_n^p \neq \emptyset$, we have*

- *For all $a \in S_n^p$, we have $\tilde{\mu}_{n,a} \in (0,1)$ and $a_n^\star = \arg\max_{a\in S_n^p} \tilde{\mu}_{n,a}$.*
- *If $B_n \in S_n^p$, then $B_n = a_n^\star$.*

Proof. Let p_0 to be specified later. Let $p \geq p_0$. Let $n \in \mathbb{N}$ such that $S_n^p \neq \emptyset$, where S_n^p and $a_n^\star$ as in Equation (41). Since $N_{n,a} \geq (1+\eta)^{p-1}$ for all $a \in S_n^p$, we have $\tilde{N}_{n,a} \geq (1+\eta)^{p-1}$. Using Lemma 54 and $x \to \log(e+x)/x$ is decreasing, we obtain that

$$\begin{aligned}
&\tilde{\mu}_{n,a_n^\star} \geq \mu_{a_n^\star} - W_\mu \frac{\log(e + (1+\eta)^{p-1})}{(1+\eta)^{p-1}} \,, \\
&\forall a \in S_n^p \setminus \{a_n^\star\}, \quad \tilde{\mu}_{n,a} \leq \mu_a + W_\mu \frac{\log(e + (1+\eta)^{p-1})}{(1+\eta)^{p-1}} \,.
\end{aligned}$$

Let $\overline{\Delta}_{\min} = \min_{a\neq b}|\mu_a - \mu_b|$ and $\Delta_0 = \min_{a\in[K]}\min\{\mu_a, 1-\mu_a\} > 0$. By assumption on the considered instances, we know that $\overline{\Delta}_{\min} > 0$. Let $p_1 = \lceil \log_{1+\eta}(X_1 - e)\rceil + 1$ with

$$\begin{aligned}
X_1 &= \sup\left\{x > 1 \mid \; x \leq 4(\min\{\overline{\Delta}_{\min}, \Delta_0\})^{-1} W_\mu \log x + e\right\} \\
&\leq h_1(4(\min\{\overline{\Delta}_{\min}, \Delta_0\})^{-1} W_\mu, \, e) \,,
\end{aligned}$$

where we used Lemma 52, and h_1 defined therein. Then, for all $p \in \mathbb{N}$ such that $p \geq p_1 + 1$ and all $n \in \mathbb{N}$ such that $S_n^p \neq \emptyset$, we have

$$\forall a \in S_n^p, \quad \mu_a - \min\{\overline{\Delta}_{\min}, \Delta_0\}/4 \leq \tilde{\mu}_{n,a} \leq \mu_a + \min\{\overline{\Delta}_{\min}, \Delta_0\}/4 \,.$$

Therefore, we have $\tilde{\mu}_{n,a} \in (0,1)$ for all $a \in S_n^p$. Since $\tilde{\mu}_{n,a_n^\star} \geq \mu_{a_n^\star} - \min\{\overline{\Delta}_{\min}, \Delta_0\}/4$ and $\tilde{\mu}_{n,a} \leq \mu_a + \min\{\overline{\Delta}_{\min}, \Delta_0\}/4$ for all $a \in S_n^p \setminus \{a_n^\star\}$, we obtain $a_n^\star = \arg\max_{a \in S_n^p} \tilde{\mu}_{n,a}$ since $\arg\max_{a \in S_n^p} \tilde{\mu}_{n,a}$ is unique. The leader is defined as $B_n = \arg\max_{a \in [K]} [\tilde{\mu}_{n,a}]_0^1$. If $B_n \in S_n^p$, we obtain

$$B_n = \arg\max_{a \in S_n^p} [\tilde{\mu}_{n,a}]_0^1 = \arg\max_{a \in S_n^p} \tilde{\mu}_{n,a} = a_n^\star \ .$$

For all $\alpha \in \mathbb{R}_+$, we have $\exp(\alpha p_1) \leq e^{3\alpha}(X_1 - e)^{\alpha/\log 2}$, hence $\mathbb{E}_{\boldsymbol{\nu}\pi}[\exp(\alpha p_1)] < +\infty$ by using Lemma 54 and $h_1(x, e) \sim_{x \to +\infty} x \log x$ to obtain that $\exp(\alpha p_1)$ is at most polynomial in W_μ. Taking $p_0 = p_1$ concludes the proof. □

Lemma 56 shows that the transportation costs between the sampled enough arms with largest true means and the other sampled enough arms are increasing fast enough.

Lemma 56. *Let S_n^p as in Eq.* (41). *There exists p_1 with $\mathbb{E}_{\boldsymbol{\nu}\pi}[\exp(\alpha p_1)] < +\infty$ for all $\alpha > 0$ such that if $p \geq p_1$, for all n such that $S_n^p \neq \emptyset$, for all $(a,b) \in (S_n^p)^2$ such that $\mu_a > \mu_b$, we have*

$$W_{\epsilon,a,b}(\tilde{\mu}_n, N_n) \geq (1+\eta)^{p-1} C_\mu \ ,$$

where $C_\mu > 0$ is a problem dependent constant.

Proof. Let p_0 as in Lemma 55. Let $p \geq p_0$. Let $n \in \mathbb{N}$ such that $S_n^p \neq \emptyset$, where S_n^p as in Eq. (41). Since $N_{n,a} \geq (1+\eta)^{p-1}$ for all $a \in S_n^p$, we have $\tilde{N}_{n,a} \geq (1+\eta)^{p-1}$ by using Lemma 49. Let $(a,b) \in (S_n^p)^2$ such that $\mu_a > \mu_b$. Using Lemma 47, there exists $\alpha_\mu > 0$ such that

$$C_\mu = \min_{(a,b): \mu_a > \mu_b} \inf_{\substack{\lambda_a, \lambda_b: \\ \max_{c \in \{a,b\}} |\mu_c - \lambda_c| \leq \alpha_\mu}} \inf_{u \in [0,1]} \left\{ d_\epsilon^-(\lambda_a, u) + d_\epsilon^+(\lambda_b, u) \right\} > 0 \ .$$

Let $\eta > 0$ s.t. $\eta < \frac{1}{4}\min\{\overline{\Delta}_{\min}, \Delta_0, \alpha_\mu\}$ where $\overline{\Delta}_{\min} = \min_{a \neq b} |\mu_a - \mu_b|$ and $\Delta_0 = \min_{a \in [K]} \min\{\mu_a, 1 - \mu_a\}$. Similarly as in the proof of Lemma 55, we can construct p_2 with $\mathbb{E}_{\boldsymbol{\nu}\pi}[\exp(\alpha p_2)] < +\infty$ for all $\alpha > 0$ such that if $p \geq p_2$, for all n such that $S_n^p \neq \emptyset$, we have $|\tilde{\mu}_{n,a} - \mu_a| \leq \eta$ for all $a \in S_n^p$. Therefore, we have $\tilde{\mu}_{n,a} = [\tilde{\mu}_{n,a}]_0^1$ and $[\tilde{\mu}_{n,b}]_0^1 = \tilde{\mu}_{n,b}$. Moreover, we have $\tilde{\mu}_{n,a} \geq \mu_a - \eta > \mu_b + \eta \geq \tilde{\mu}_{n,b}$. Then, we obtain

$$\begin{aligned}
W_{\epsilon,a,b}(\tilde{\mu}_n, N_n) &= \inf_{u \in [0,1]} \left\{ N_{n,a} d_\epsilon^-(\tilde{\mu}_{n,a}, u) + N_{n,b} d_\epsilon^+(\tilde{\mu}_{n,b}, u) \right\} \\
&\geq (1+\eta)^{p-1} \inf_{u \in [0,1]} \left\{ d_\epsilon^-(\tilde{\mu}_{n,a}, u) + d_\epsilon^+(\tilde{\mu}_{n,b}, u) \right\} \\
&\geq (1+\eta)^{p-1} \inf_{\substack{\lambda_a, \lambda_b: \\ \max_{c \in \{a,b\}} |\mu_c - \lambda_c| \leq \alpha_\mu}} \inf_{u \in [0,1]} \left\{ d_\epsilon^-(\lambda_a, u) + d_\epsilon^+(\lambda_b, u) \right\} \geq (1+\eta)^{p-1} C_\mu \ .
\end{aligned}$$

This concludes the proof. □

Lemma 57 shows that the transportation costs between sampled enough arms and undersampled arms are not increasing too fast.

Lemma 57. *Let S_n^p be as in Eq.* (41). *There exists p_2 with $\mathbb{E}_{\boldsymbol{\nu}\pi}[\exp(\alpha p_2)] < +\infty$ for all $\alpha > 0$ such that if $p \geq p_2$, for all n such that $S_n^p \neq \emptyset$, For all $p \geq p_2$ and all n such that $S_n^p \neq \emptyset$, for all $a \in S_n^p$ and $b \notin S_n^p$,*

$$W_{\epsilon,a,b}(\tilde{\mu}_n, N_n) \leq (1+\eta)^{p-1} D_\mu \ ,$$

where $D_\mu \in (0, +\infty)$ is a problem dependent constant.

Proof. Let $n \in \mathbb{N}$ such that $S_n^p \neq \emptyset$, where S_n^p as in Eq. (41). Since $N_{n,a} \geq (1+\eta)^{p-1}$ for all $a \in S_n^p$, we have $\tilde{N}_{n,a} \geq (1+\eta)^{p-1}$ by using Lemma 49. Likewise, $N_{n,b} < (1+\eta)^{p-1}$ for all $b \notin S_n^p$, we have $\tilde{N}_{n,b} < (1+\eta)^{p-1}$. Let $a \in S_n^p$ and $b \notin S_n^p$. Since the result is direct when $[\tilde{\mu}_{n,a}]_0^1 \leq [\tilde{\mu}_{n,b}]_0^1$, we assume $[\tilde{\mu}_{n,a}]_0^1 > [\tilde{\mu}_{n,b}]_0^1$ in the following.

Let $\eta > 0$ s.t. $\eta < \frac{1}{4}\min\{\overline{\Delta}_{\min}, \Delta_0\}$ where $\overline{\Delta}_{\min} = \min_{a \neq b} |\mu_a - \mu_b|$ and $\Delta_0 = \min_{a \in [K]} \min\{\mu_a, 1 - \mu_a\} > 0$. Similarly as in the proof of Lemma 55, we can construct p_2

with $\mathbb{E}_{\boldsymbol{\nu}\pi}[\exp(\alpha p_2)] < +\infty$ for all $\alpha > 0$ such that if $p \geq p_2$, for all n such that $S_n^p \neq \emptyset$, we have $|\tilde{\mu}_{n,a} - \mu_a| \leq \eta$ for all $a \in S_n^p$. Let $g_\epsilon^+(x) = \frac{x}{x(1-e^\epsilon)+e^\epsilon}$ as in Lemma 23. Using Lemma 23, for all $a \in S_n^p$, we have

$$1 > \mu_a + \min\{\overline{\Delta}_{\min}, \Delta_0\}/4 \geq \tilde{\mu}_{n,a} > g_\epsilon^+(\tilde{\mu}_{n,a}) \geq g_\epsilon^+(\mu_a - \min\{\overline{\Delta}_{\min}, \Delta_0\}/4) > 0 \, .$$

Taking $u = \tilde{\mu}_{n,a} \in [0,1]$ and using that $d_\epsilon^-(\tilde{\mu}_{n,a}, \tilde{\mu}_{n,a}) = 0$, we obtain

$$\begin{aligned} W_{\epsilon,a,b}(\tilde{\mu}_n, N_n) &= \inf_{u \in [0,1]} \left\{ N_{n,a} d_\epsilon^-(\tilde{\mu}_{n,a}, u) + N_{n,b} d_\epsilon^+(\tilde{\mu}_{n,b}, u) \right\} \\ &\leq N_{n,b} d_\epsilon^+(\tilde{\mu}_{n,b}, \tilde{\mu}_{n,a}) \leq (1+\eta)^{p-1} d_\epsilon^+(\tilde{\mu}_{n,b}, \tilde{\mu}_{n,a}) \, , \end{aligned}$$

where the last term is positive since $\tilde{\mu}_{n,a} > [\tilde{\mu}_{n,b}]_0^1$ and $\tilde{\mu}_{n,a} \in (0,1)$ by Lemma 24.

When $\tilde{\mu}_{n,b} \leq 0$, Lemma 24 yields that

$$d_\epsilon^+(\tilde{\mu}_{n,b}, \tilde{\mu}_{n,a}) = -\log\left(1 - \tilde{\mu}_{n,a}(1 - e^{-\epsilon})\right) \leq \epsilon \, ,$$

where we used that $x \to -\log\left(1 - x(1-e^{-\epsilon})\right)$ is increasing on $(0,1)$. When $\tilde{\mu}_{n,b} \in (0, g_\epsilon^+(\tilde{\mu}_{n,a}))$, Lemma 24 yields that

$$d_\epsilon^+(\tilde{\mu}_{n,b}, \tilde{\mu}_{n,a}) = -\log\left(1 - \tilde{\mu}_{n,a}(1 - e^{-\epsilon})\right) - \epsilon \tilde{\mu}_{n,b} \leq \epsilon \, .$$

When $\tilde{\mu}_{n,b} \in [g_\epsilon^+(\tilde{\mu}_{n,a}), \tilde{\mu}_{n,a})$, Lemma 24 yields that

$$\begin{aligned} d_\epsilon^+(\tilde{\mu}_{n,b}, \tilde{\mu}_{n,a}) &= \mathrm{kl}(\tilde{\mu}_{n,b}, \tilde{\mu}_{n,a}) \leq -\log\min\{\tilde{\mu}_{n,a}, 1 - \tilde{\mu}_{n,a}\} \\ &\leq -\log\min\{\mu_a - \min\{\overline{\Delta}_{\min}, \Delta_0\}/4, 1 - \mu_a - \min\{\overline{\Delta}_{\min}, \Delta_0\}/4\} \, , \end{aligned}$$

where we used the classical result that $\mathrm{kl}(q,p) \leq -\log\min\{p, 1-p\}$. Let us define

$$D_\mu = \epsilon + \max_{a \in [K]} \left\{ -\log\min\{\mu_a - \min\{\overline{\Delta}_{\min}, \Delta_0\}/4, 1 - \mu_a - \min\{\overline{\Delta}_{\min}, \Delta_0\}/4\} \right\} \, .$$

Then, we have shown that $d_\epsilon^+(\tilde{\mu}_{n,b}, \tilde{\mu}_{n,a}) \leq D_\mu$ where $D_\mu \in (0, +\infty)$. This yields the result. □

Lemma 58 shows that the challenger is mildly undersampled if the leader is not mildly undersampled.

Lemma 58. *Let V_n^p be as in Equation (42). There exists p_3 with $\mathbb{E}_{\boldsymbol{\nu}\pi}[\exp(\alpha p_3)] < +\infty$ for all $\alpha > 0$ such that if $p \geq p_3$, for all n such that $U_n^p \neq \emptyset$, $B_n \notin V_n^p$ implies $C_n \in V_n^p$.*

Proof. Let p_3 to be specified later. Let $p \geq p_3$. Let $n \in \mathbb{N}$ such that $U_n^p \neq \emptyset$ and $V_n^p \neq [K]$, where $U_n^p \subseteq V_n^p$ are defined in Eq. (42). Since the statement holds when $B_n \in V_n^p$, we suppose that $B_n \notin V_n^p$ in the following.

Let p_0 as in Lemma 55, p_1 and C_μ as in Lemma 56, and p_2 and D_μ as in Lemma 57. Let $p_4 = \max\{2p_2 - 1, \frac{4}{3}\max\{p_0, p_1\} - 1/3\}$, which satisfied that $\mathbb{E}_{\boldsymbol{\nu}\pi}[\exp(\alpha p_4)] < +\infty$ for all $\alpha > 0$ by using Lemmas 55, 56 and 57. Then, for all $p \geq p_4 = \max\{2p_2 - 1, \frac{4}{3}\max\{p_0, p_1\} - 1/3\}$ and all n such that $B_n \notin V_n^p$, we have $\tilde{\mu}_{n,a} \in (0,1)$ for all $a \notin V_n^p$, $B_n = b_n^\star := \arg\max_{a \notin V_n^p} \mu_a$, $B_n \notin U_n^p$ and

$$\begin{aligned} &\forall b \notin \{b_n^\star\} \cup V_n^p, \quad W_{\epsilon, b_n^\star, b}(\tilde{\mu}_n, N_n) + \log N_{n,b} \geq (1+\eta)^{3(p-1)/4} C_\mu + \frac{3(p-1)}{4}\log(1+\eta) \, , \\ &\forall b \in U_n^p, \quad W_{\epsilon, b_n^\star, b}(\tilde{\mu}_n, N_n) + \log N_{n,b} \leq (1+\eta)^{(p-1)/2} D_\mu + \frac{p-1}{2}\log(1+\eta) \, , \end{aligned}$$

where we used Lemmas 55, 56 and 57. Direct manipulations yield that

$$\begin{aligned} &(1+\eta)^{3(p-1)/4} C_\mu + \frac{3(p-1)}{4}\log(1+\eta) \geq (1+\eta)^{(p-1)/2} D_\mu + \frac{p-1}{2}\log(1+\eta) \\ &\Longleftarrow \quad p \geq p_5 = 4\lceil \log_{1+\eta}(D_\mu / C_\mu) \rceil + 1 \, , \end{aligned}$$

where $\mathbb{E}_{\boldsymbol{\nu}\pi}[\exp(\alpha p_5)] < +\infty$ for all $\alpha > 0$ since it is a deterministic constant. Let $p_3 = \max\{p_4, p_5\}$ which satisfies $\mathbb{E}_{\boldsymbol{\nu}\pi}[\exp(\alpha p_3)] < +\infty$ for all $\alpha > 0$. Then, we have shown that for all $p \geq p_3$, for all n such that $B_n \notin V_n^p$, we have $B_n = b_n^\star$ and

$$\min_{b \notin \{b_n^\star\} \cup V_n^p} \{W_{\epsilon, b_n^\star, b}(\tilde{\mu}_n, N_n) + \log N_{n,b}\} > \max_{b \in U_n^p} \{W_{\epsilon, b_n^\star, b}(\tilde{\mu}_n, N_n) + \log N_{n,b}\} \, .$$

By definition of the TC challenger, i.e., $C_n \in \arg\min_{b \neq B_n} \{W_{\epsilon, B_n, b}(\tilde{\mu}_n, N_n) + \log N_{n,b}\}$, we obtain that $C_n \in V_n^p$. Otherwise, there would be a contradiction since we assumed $U_n^p \neq \emptyset$. This concludes the proof. □

Lemma 59 shows that all the arms are sufficient explored for large enough n.

Lemma 59. *There exists N_0 with $\mathbb{E}_{\boldsymbol{\nu}\pi}[N_0] < +\infty$ such that, for all $n \geq N_0$ and all $a \in [K]$, $N_{n,a} \geq \sqrt{n/K}$.*

Proof. Let p_0 and p_3 as in Lemmas 55 and 58. Combining Lemmas 55 and 58 yields that, for all $p \geq p_4 = \max\{p_3, 4p_0/3 - 1/3\}$ and all n such that $U_n^p \neq \emptyset$, we have $B_n \in V_n^p$ or $C_n \in V_n^p$. We have $\mathbb{E}_{\boldsymbol{\nu}\pi}[(1+\eta)^{p_2}] < +\infty$. We have $(1+\eta)^{p-1} \geq K(1+\eta)^{3(p-1)/4}$ for all $p \geq p_5 = 4\lceil \log_{1+\eta} K \rceil + 1$. Let $p \geq \max\{p_5, p_4\}$. For notational simplicity, we conduct the proof as if that $k(1+\eta)^{p-1} \in \mathbb{N}$ for all $k \in [K]$. It is direct to adapt the proof by using the operator $\lceil \cdot \rceil$.

Suppose towards contradiction that $U^p_{K(1+\eta)^{p-1}}$ is not empty. Then, for any $1 \leq t \leq K(1+\eta)^{p-1}$, U_t^p and V_t^p are non empty as well. Using the pigeonhole principle, there exists some $a \in [K]$ such that $N_{(1+\eta)^{p-1},a} \geq (1+\eta)^{3(p-1)/4}$. Thus, we have $\left|V^p_{(1+\eta)^{p-1}}\right| \leq K-1$. Our goal is to show that $\left|V^p_{2(1+\eta)^{p-1}}\right| \leq K-2$. A sufficient condition is that one arm in $V^p_{(1+\eta)^{p-1}}$ is pulled at least $(1+\eta)^{3(p-1)/4}$ times between $(1+\eta)^{p-1}$ and $2(1+\eta)^{p-1}-1$.

Case 1. Suppose there exists $a \in V^p_{(1+\eta)^{p-1}}$ such that $L_{2(1+\eta)^{p-1},a} - L_{(1+\eta)^{p-1},a} \geq \beta^{-1}\left((1+\eta)^{3(p-1)/4} + 3/2\right)$. Using Lemma 50, we obtain

$$N^a_{2(1+\eta)^{p-1},a} - N^a_{(1+\eta)^{p-1},a} \geq \beta(L_{2(1+\eta)^{p-1},a} - L_{(1+\eta)^{p-1},a}) - 3/2 \geq (1+\eta)^{3(p-1)/4} ,$$

hence a is sampled $(1+\eta)^{3(p-1)/4}$ times between $(1+\eta)^{p-1}$ and $2(1+\eta)^{p-1}-1$.

Case 2. Suppose that for all $a \in V^p_{(1+\eta)^{p-1}}$, we have $L_{2(1+\eta)^{p-1},a} - L_{(1+\eta)^{p-1},a} < \beta^{-1}\left((1+\eta)^{3(p-1)/4} + 3/2\right)$. Then,

$$\sum_{a \notin V^p_{(1+\eta)^{p-1}}} (L_{2(1+\eta)^{p-1},a} - L_{(1+\eta)^{p-1},a}) \geq (1+\eta)^{p-1} - K\beta^{-1}\left((1+\eta)^{3(p-1)/4} + 3/2\right) .$$

Using Lemma 50, we obtain

$$\left| \sum_{a \notin V^p_{(1+\eta)^{p-1}}} (N^a_{2(1+\eta)^{p-1},a} - N^a_{(1+\eta)^{p-1},a}) - \beta \sum_{a \notin V^p_{(1+\eta)^{p-1}}} (L_{2(1+\eta)^{p-1},a} - L_{(1+\eta)^{p-1},a}) \right| \\ \leq 3(K-1)/2 .$$

Combining all the above, we obtain

$$\begin{aligned} &\sum_{a \notin V^p_{(1+\eta)^{p-1}}} (L_{2(1+\eta)^{p-1},a} - L_{(1+\eta)^{p-1},a}) - \sum_{a \notin V^p_{(1+\eta)^{p-1}}} (N^a_{2(1+\eta)^{p-1},a} - N^a_{(1+\eta)^{p-1},a}) \\ &\geq (1-\beta) \sum_{a \notin V^p_{(1+\eta)^{p-1}}} (L_{2(1+\eta)^{p-1},a} - L_{(1+\eta)^{p-1},a}) - 3(K-1)/2 \\ &\geq (1-\beta)\left((1+\eta)^{p-1} - K\beta^{-1}\left((1+\eta)^{3(p-1)/4} + 3/2\right)\right) - 3(K-1)/2 \geq K(1+\eta)^{3(p-1)/4} \end{aligned}$$

where the last inequality is obtained for $p \geq p_6 + 1$ with

$$p_6 = \sup\left\{p \in \mathbb{N} \mid (1-\beta)\left((1+\eta)^{p-1} - K\beta^{-1}\left((1+\eta)^{3(p-1)/4} + 3/2\right)\right) - \frac{3}{2}(K-1) \right. \\ \left. < K(1+\eta)^{3(p-1)/4}\right\} .$$

The left hand side summation is exactly the number of times where an arm $a \notin V^p_{(1+\eta)^{p-1}}$ was leader but wasn't sampled, hence we have shown that

$$\sum_{t=(1+\eta)^{p-1}}^{2(1+\eta)^{p-1}-1} \mathbb{1}\left(B_t \notin V^p_{(1+\eta)^{p-1}},\ a_t = C_t\right) \geq K(1+\eta)^{3(p-1)/4} .$$

For any $(1+\eta)^{p-1} \le t \le 2(1+\eta)^{p-1}-1$, U_t^p is non-empty, hence we have $B_t \notin V^p_{(1+\eta)^{p-1}}$ (hence $B_t \notin V_t^p$) implies $C_t \in V_t^p \subseteq V^p_{(1+\eta)^{p-1}}$. Therefore, we have shown that

$$\sum_{t=(1+\eta)^{p-1}}^{2(1+\eta)^{p-1}-1} \mathbb{1}\left(a_t \in V^p_{(1+\eta)^{p-1}}\right) \ge \sum_{t=(1+\eta)^{p-1}}^{2(1+\eta)^{p-1}-1} \mathbb{1}\left(B_t \notin V^p_{(1+\eta)^{p-1}},\, a_t = C_t\right) \ge K(1+\eta)^{3(p-1)/4}$$

Therefore, there is at least one arm in $V^p_{(1+\eta)^{p-1}}$ that is sampled $(1+\eta)^{3(p-1)/4}$ times between $(1+\eta)^{p-1}$ and $2(1+\eta)^{p-1}-1$.

In summary, we have shown $\left|V^p_{2(1+\eta)^{p-1}}\right| \le K-2$ for all $p \ge p_7 = \max\{p_6, p_4, p_5\}$. By induction, for any $1 \le k \le K$, we have $\left|V^p_{k(1+\eta)^{p-1}}\right| \le K-k$, and finally $U^p_{K(1+\eta)^{p-1}} = \emptyset$ for all $p \ge p_7$. Defining $N_0 = K(1+\eta)^{p_7-1}$, we have $\mathbb{E}_{\boldsymbol{\nu}\pi}[N_0] < +\infty$ by using Lemmas 55 and 58 for $p_4 = \max\{p_3, 4p_0/3-1/3\}$ and p_6 and p_5 are deterministic. For all $n \ge N_0$, we let $(1+\eta)^{p-1} = \frac{n}{K}$. Then, by applying the above, we have $U^p_{K(1+\eta)^{p-1}} = U_n^{\log_{1+\eta}(n/K)+1}$ is empty, which shows that $N_{n,a} \ge \sqrt{n/K}$ for all $a \in [K]$. $\square$

H.3 Convergence Towards β-Optimal Allocation

The second step of in the generic analysis of Top Two algorithms [50] is to show the convergence of the empirical proportions towards the β-optimal allocation. First, we show that the leader coincides with the best arm. Hence, the tracking procedure will ensure that the empirical proportion of time we sample it is exactly β. Second, we show that a sub-optimal arm whose empirical proportion overshoots its β-optimal allocation will not be sampled next as challenger. Therefore, this "overshoots implies not sampled" mechanism will ensure the convergence towards the β-optimal allocation. We emphasise that there are multiple ways to select the leader/challenger pair in order to ensure convergence towards the β-optimal allocation. Therefore, other choices of leader/challenger pair would yield similar results. Note that our results heavily rely on having obtained sufficient exploration first.

Lemma 60 shows the leader and the candidate answer are equal to the best arm for large enough n.

Lemma 60. *Let N_0 be as in Lemma 59. There exists $N_1 \ge N_0$ with $\mathbb{E}_{\boldsymbol{\nu}\pi}[N_1] < +\infty$ such that, for all $n \ge N_1$, we have $\tilde{\mu}_n \in (0,1)^K$ and $\tilde{a}_n = B_n = a^\star$.*

Proof. Let $\Delta_{\min} = \min_{a \neq a^\star}(\mu_{a^\star} - \mu_a)$ and $\Delta_0 = \min_{a \in [K]} \min\{\mu_a, 1-\mu_a\} > 0$. Using Lemma 54, we obtain, for all $n \ge N_0$,

$$\tilde{\mu}_{n,a^\star} \ge \mu_{a^\star} - W_\mu \frac{\log(e + \sqrt{n/K}/(1+\eta))}{\sqrt{n/K}/(1+\eta)}$$
$$\forall a \neq a^\star, \quad \tilde{\mu}_{n,a} \le \mu_a + W_\mu \frac{\log(e + \sqrt{n/K}/(1+\eta))}{\sqrt{n/K}/(1+\eta)} ,$$

where we used that $x \to \log(e+x)/x$ is decreasing and $\tilde{N}_{n,a} \ge N_{n,a}/(1+\eta) \ge \sqrt{n/K}/(1+\eta)$. Let $N_1 = \max\{N_0, \lceil K(1+\eta)^2 X_1^2 \rceil\}$ where

$$X_1 = \sup\left\{x > 1 \mid \ x \le 4(\Delta_{\min}, \Delta_0)^{-1} W_\mu \log x + e\right\} \le h_1(4(\Delta_{\min}, \Delta_0)^{-1} W_\mu,\ e) ,$$

where we used Lemma 52, and h_1 defined therein. Using Lemmas 54 and 59, we obtain $\mathbb{E}_{\boldsymbol{\nu}\pi}[N_1] < +\infty$. Then, we have $0 < \mu_a - \Delta_0/4 \le \tilde{\mu}_{n,a} \le \mu_a + \Delta_0/4 < 1$ for all $a \in [K]$. Moreover, for all $n \ge N_1$, we have $\tilde{\mu}_{n,a^\star} \ge \mu_{a^\star} - \Delta_{\min}/4$ and $\tilde{\mu}_{n,a} \le \mu_a + \Delta_{\min}/4$ for all $a \neq a^\star$, hence

$$a^\star = \arg\max_{a \in [K]} \tilde{\mu}_{n,a} = \arg\max_{a \in [K]} [\tilde{\mu}_{n,a}]_0^1 = \tilde{a}_n = B_n .$$

This concludes the proof. $\square$

Lemma 61 shows that that the pulling proportion of the best arm converges towards β. It is a direct consequence of Lemma 60 by using the same proof as Lemma 39 in Azize et al. [12], hence we omit the proof.

Lemma 61 (Lemma 39 in Azize et al. [12]). *Let $\gamma > 0$, and N_1 be as in Lemma 60. There exists a deterministic constant $C_0 \geq 1$ such that, for all $n \geq C_0 N_1$, we have $\left|\frac{N_{n,a^\star}}{n-1} - \beta\right| \leq \gamma$.*

Lemma 62 shows that if a sub-optimal arm overshoots its β-optimal allocation then it cannot be selected as challenger for large enough n.

Lemma 62. *Let $\gamma \in (0, \gamma_\mu)$ where γ_μ is a problem dependent constant. Let N_1 and C_0 be as in Lemma 60 and 61. There exists $N_2 \geq C_0 N_1$ with $\mathbb{E}_{\boldsymbol{\nu}\pi}[N_2] < +\infty$ such that, for all $n \geq N_2$,*

$$\exists a \neq a^\star, \quad \frac{N_{n,a}}{n-1} \geq \gamma + \omega^\star_{\epsilon,\beta,a} \quad \implies \quad C_n \neq a \, .$$

Proof. Let $\eta > 0$ and $\gamma > 0$ be small enough, which we will specify below. Let $\tilde{\gamma} \in (0, \gamma)$. Let N_1 as in Lemma 60 and C_0 as in Lemma 61 for $\tilde{\gamma}$. Let $n \geq C_0 N_1$. Therefore, we have $\tilde{\mu}_n \in (0,1)^K$ and $\tilde{a}_n = B_n = a^\star$ and $\left|\frac{N_{n,a^\star}}{n-1} - \beta\right| \leq \tilde{\gamma}$. Using the same proof as in Lemma 60, there exists N_3 with $\mathbb{E}_{\boldsymbol{\nu}\pi}[N_3] < +\infty$ such that, for all $n \geq N_3$, we have $\|\tilde{\mu}_n - \mu\|_\infty \leq \eta$. Let $n \geq \max\{C_0 N_1, N_3\}$.

Let $a \neq a^\star$ such that $\frac{N_{n,a}}{n-1} \geq \omega^\star_{\epsilon,\beta,a} + \gamma$. Suppose towards contradiction that $\frac{N_{n,b}}{n-1} > \omega^\star_{\epsilon,\beta,b}$ for all $b \notin \{a^\star, a\}$. Then, for all $n \geq C_0 N_1$, we have

$$1 - \beta + \tilde{\gamma} \geq 1 - \frac{N_{n,a^\star}}{n-1} = \sum_{b \neq a^\star} \frac{N_{n,b}}{n-1} > \gamma + \sum_{b \neq a^\star} \omega^\star_{\epsilon,\beta,b} = 1 - \beta + \gamma \, ,$$

which yields a contradiction since $\tilde{\gamma} < \gamma$. Therefore, for all $n \geq C_0 N_1$, we have

$$\exists a \neq a^\star, \quad \frac{N_{n,a}}{n-1} \geq \omega^\star_{\epsilon,\beta,a} + \gamma \quad \implies \quad \exists b \notin \{a^\star, a\}, \quad \frac{N_{n,b}}{n-1} \leq \omega^\star_{\epsilon,\beta,b} \, .$$

Let $b \notin \{a^\star, a\}$ such that $\frac{N_{n,b}}{n-1} \leq \omega^\star_{\epsilon,\beta,b}$. By definition of the TC challenger, we obtain

$$\begin{aligned} C_n \neq a \impliedby \; & W_{\epsilon,a^\star,a}(\tilde{\mu}_n, N_n) + \log N_{n,a} > W_{\epsilon,a^\star,b}(\tilde{\mu}_n, N_n) + \log N_{n,b} \\ \impliedby \; & \frac{1}{n-1}\left(W_{\epsilon,a^\star,a}(\tilde{\mu}_n, N_n) - W_{\epsilon,a^\star,b}(\tilde{\mu}_n, N_n)\right) > \frac{1}{n-1} \log \frac{\omega^\star_{\epsilon,\beta,b}}{\omega^\star_{\epsilon,\beta,a} + \gamma} \\ \impliedby \; & \frac{1}{n-1}\left(W_{\epsilon,a^\star,a}(\tilde{\mu}_n, N_n) - W_{\epsilon,a^\star,b}(\tilde{\mu}_n, N_n)\right) > \frac{1}{n-1} \max_{a \neq b} \left|\log \frac{\omega^\star_{\epsilon,\beta,b}}{\omega^\star_{\epsilon,\beta,a}}\right| \, , \end{aligned}$$

where we used the positivity of the β-optimal allocation (Lemma 42) to ensure that $\max_{a \neq b} \left|\log \frac{\omega^\star_{\epsilon,\beta,b}}{\omega^\star_{\epsilon,\beta,a}}\right| \in (0, +\infty)$. Using that $\tilde{\mu}_{n,a^\star} > \max\{\tilde{\mu}_{n,a}, \tilde{\mu}_{n,b}\}$, we obtain

$$\begin{aligned} & \frac{1}{n-1}\left(W_{\epsilon,a^\star,a}(\tilde{\mu}_n, N_n) - W_{\epsilon,a^\star,b}(\tilde{\mu}_n, N_n)\right) \\ & \geq \inf_{u \in [0,1]} \left\{ \frac{N_{n,a^\star}}{n-1} d^-_\epsilon(\tilde{\mu}_{n,a^\star}, u) + (\omega^\star_{\epsilon,\beta,a} + \gamma) d^+_\epsilon(\tilde{\mu}_{n,a}, u) \right\} \\ & \qquad - \inf_{u \in [0,1]} \left\{ \frac{N_{n,a^\star}}{n-1} d^-_\epsilon(\tilde{\mu}_{n,a^\star}, u) + \omega^\star_{\epsilon,\beta,b} d^+_\epsilon(\tilde{\mu}_{n,b}, u) \right\} \\ & \geq \inf_{\tilde{\beta}: |\tilde{\beta} - \beta| \leq \tilde{\gamma}} G_{a,b}(\tilde{\mu}_n, \tilde{\beta}) \geq \inf_{\lambda: \|\lambda - \mu\|_\infty \leq \eta} \inf_{\tilde{\beta}: |\tilde{\beta} - \beta| \leq \tilde{\gamma}} G_{a,b}(\lambda, \tilde{\beta}) \, , \end{aligned}$$

where, for all $(a,b) \in ([K] \setminus \{a^\star\})^2$ such that $a \neq b$,

$$\begin{aligned} G_{a,b}(\lambda, \tilde{\beta}) = & \inf_{u \in [0,1]} \left\{ \tilde{\beta} d^-_\epsilon(\lambda_{a^\star}, u) + (\omega^\star_{\epsilon,\beta,a} + \gamma) d^+_\epsilon(\lambda_a, u) \right\} \\ & \qquad - \inf_{u \in [0,1]} \left\{ \tilde{\beta} d^-_\epsilon(\lambda_{a^\star}, u) + \omega^\star_{\epsilon,\beta,b} d^+_\epsilon(\lambda_b, u) \right\} \, . \end{aligned}$$

Using the equality at equilibrium from (39) (see Lemma 42) and the fact that the transportation costs are increasing in their allocation argument (see Lemma 36), we obtain $G_{a,b}(\mu, \beta) > 0$ for all $(a,b) \in ([K] \setminus \{a^\star\})^2$ such that $a \neq b$, since

$$\inf_{u \in [0,1]} \left\{ \beta d^-_\epsilon(\mu_{a^\star}, u) + (\omega^\star_{\epsilon,\beta,a} + \gamma) d^+_\epsilon(\mu_a, u) \right\} > W_{\epsilon,a^\star,a}(\mu, w^\star_{\epsilon,\beta}) = W_{\epsilon,a^\star,b}(\mu, w^\star_{\epsilon,\beta}) \, .$$

By Lemma 48, the functions $(\lambda, \tilde{\beta}) \to G_{a,b}(\lambda, \tilde{\beta})$ and $\lambda \to \inf_{\tilde{\beta}:|\tilde{\beta}-\beta|\le\tilde{\gamma}} G_{a,b}(\lambda, \tilde{\beta})$ are continuous. Therefore, there exists η_μ and γ_μ small enough such that

$$\inf_{\lambda:\|\lambda-\mu\|_\infty\le\eta} \inf_{\tilde{\beta}:|\tilde{\beta}-\beta|\le\tilde{\gamma}} G_{a,b}(\lambda, \tilde{\beta}) \ge G_{a,b}(\mu, \beta)/2 \ge \frac{1}{2} \min_{a\neq b, a\neq a^\star, b\neq a^\star} G_{a,b}(\mu, \beta) > 0 ,$$

where the last strict inequality uses that the minimum of a finite number of positive constants is also positive. Considering such (η_μ, γ_μ) at the beginning of the proof and taking $N_2 = \max\{C_0 N_1, N_3, \kappa_\mu\}$ where

$$\kappa_\mu = 2 + \frac{2\max_{a\neq b}\left|\log \frac{\omega^\star_{\epsilon,\beta,b}}{\omega^\star_{\epsilon,\beta,a}}\right|}{\min_{a\neq b, a\neq a^\star, b\neq a^\star} G_{a,b}(\mu, \beta)} < +\infty ,$$

As it satisfies $\mathbb{E}_{\boldsymbol{\nu}\pi}[N_2] < +\infty$, this concludes the proof. □

Lemma 63 shows that the convergence time towards the β-optimal allocation has finite expectation. It is a direct consequence of Lemmas 60, 61 and 62 by using the same proof as Lemma 41 in Azize et al. [12], hence we omit the proof.

Lemma 63 (Lemma 41 in Azize et al. [12])**.** *Let $\gamma \in (0, \gamma_\mu)$ where γ_μ is a problem dependent constant, and $T_\gamma(w)$ as in Eq.* (40)*. Then, we have $\mathbb{E}_{\boldsymbol{\nu}\pi}[T_\gamma(\omega^\star_{\epsilon,\beta})] < +\infty$.*

H.4 Asymptotic Upper Bound

The final step of the generic analysis of Top Two algorithms [50] is to invert the GLR stopping rule in Eq. (7) by leveraging the convergence of the empirical proportions towards the β-optimal allocation. Provided this convergence is shown, the asymptotic upper bound on the expected sample complexity only depends on the dependence in $\log(1/\delta)$ of the threshold that ensures δ-correctness. Compared to the non-private GLR stopping rule, the GLR stopping rule in Eq. (7) pay an extra cost to ensure privacy.

Lemma 64. *Let $\epsilon > 0$, $\eta > 0$ and $(\delta, \beta) \in (0, 1)^2$. Let $T^\star_{\epsilon,\beta}(\boldsymbol{\nu})$ as in Eq.* (35) *and $\omega^\star_{\epsilon,\beta}$ be its associated β-optimal allocation. Assume that there exists $\gamma_\mu > 0$ such that $\mathbb{E}_{\boldsymbol{\nu}\pi}[T_\gamma(\omega^\star_{\epsilon,\beta})] < +\infty$ for all $\gamma \in (0, \gamma_\mu)$, where $T_\gamma(w)$ is defined in Eq.* (40)*. Combining such a sampling rule, using the GPE$_\eta(\epsilon)$ update, with the GLR stopping rule as in Eq.* (7) *and the stopping threshold c as in Eq.* (8) *yields an ϵ-global DP and δ-correct algorithm which satisfies that, for all $\boldsymbol{\nu}$ with mean μ such that $|a^\star(\mu)| = 1$,*

$$\limsup_{\delta\to 0} \frac{\mathbb{E}_{\boldsymbol{\nu}\pi}\left[\tau_{\epsilon,\delta}\right]}{\log(1/\delta)} \le 2(1+\eta) T^\star_{\epsilon,\beta}(\boldsymbol{\nu}) .$$

Proof. Lemma 4 yields the ϵ-global DP. Theorem 5 yields the δ-correctness.

Let $\zeta > 0$ and $a^\star$ be the unique best arm. Using the equality at equilibrium from (39) (see Lemma 42) and the continuity of $(\mu, w) \mapsto \min_{a\neq a^\star(\mu)} W_{\epsilon,a^\star(\mu),a}(\mu, w)$ (see Lemma 41), there exists $\gamma_\zeta > 0$ such that $\left\|\frac{N_n}{n-1} - \omega^\star_{\epsilon,\beta}\right\|_\infty \le \gamma_\zeta$ and $\|\tilde{\mu}_n - \mu\|_\infty \le \gamma_\zeta$ implies that

$$\forall a \neq a^\star, \quad W_{\epsilon,a^\star,a}(\tilde{\mu}_n, N_n/(n-1)) \ge \frac{(1-\zeta)}{T^\star_{\epsilon,\beta}(\boldsymbol{\nu})} .$$

We choose such a γ_ζ. Let $\gamma_\mu > 0$ be such that for $\mathbb{E}_{\boldsymbol{\nu}\pi}[T_\gamma(\omega^\star_{\epsilon,\beta})] < +\infty$ for all $\gamma \in (0, \gamma_\mu)$, where $T_\gamma(\omega)$ is defined in Eq. (40). Let $\gamma \in (0, \min\{\gamma_\mu, \gamma_\zeta, \min_{a\in[K]} \omega^\star_{\epsilon,\beta,a}/4, \Delta_{\min}/4, \Delta_0/4\})$ where $\Delta_{\min} = \min_{a\neq a^\star}(\mu_{a^\star} - \mu_a)$ and $\Delta_0 = \min_{a\in[K]} \min\{\mu_a, 1-\mu_a\}$. For all $n \ge T_\gamma(\omega^\star_{\epsilon,\beta})$, we have

$$\tilde{N}_{n,a} \ge N_{n,a}/(1+\eta) \ge (n-1)(\omega^\star_{\epsilon,\beta,a} - \gamma)/(1+\eta) \ge (n-1)\frac{3}{4(1+\eta)} \min_{a\in[K]} \omega^\star_{\epsilon,\beta,a} > 0 ,$$

where the last inequality used the positivity of the β-optimal allocation (Lemma 42). Since arms are sampled linearly, it is direct to construct $N_3 \ge T_\gamma(\omega^\star_{\epsilon,\beta})$ with $\mathbb{E}_{\boldsymbol{\nu}\pi}[N_3] < +\infty$ such that $\|\tilde{\mu}_n - \mu\|_\infty \le \gamma$ and $\left\|\frac{N_n}{n-1} - \omega^\star_{\epsilon,\beta}\right\|_\infty \le \gamma$ (as well as $\min_{a\in[K]} N_{n,a} > e$ trivially).

Recall that $c(n,\epsilon,\delta) = c_1(n,\delta) + c_2(n,\epsilon)$ where $n \mapsto c_1(n,\delta)$ and $n \mapsto c_1(n,\delta)$ are increasing (see Lemmas 51 and 38). Since $\tilde{N}_{n,a} \le N_{n,a} \le n$, we obtain

$$\sum_{b \in \{a^\star, a\}} c(\tilde{N}_n, \epsilon, \delta) \le 2(c_1(n,\delta) + c_2(n,\epsilon)) \, .$$

Using Lemma 36 and $\tilde{N}_{n,a} \ge N_{n,a}/(1+\eta)$ for all $a \in [K]$ (Lemma 49), we obtain

$$W_{\epsilon,a^\star,a}(\tilde{\mu}_n, \tilde{N}_n) \ge \frac{n-1}{1+\eta} W_{\epsilon,a^\star,a}\left(\tilde{\mu}_n, \frac{N_n}{n-1}\right) \, .$$

Let $\kappa \in (0,1)$ and $T > N_3/\kappa$. For all $n \in [\kappa T, T]$, we have $\tilde{a}_n = a^\star$ and, for all $a \neq a^\star$,

$$\begin{aligned}
& \tau_{\epsilon,\delta} > n \\
\implies \quad & \exists a \neq a^\star, \quad W_{\epsilon,a^\star,a}(\tilde{\mu}_n, \tilde{N}_n) \le \sum_{b \in \{a^\star,a\}} c(\tilde{N}_n, \epsilon, \delta) \\
\implies \quad & \exists a \neq a^\star, \quad \frac{n-1}{1+\eta} W_{\epsilon,a^\star,a}\left(\tilde{\mu}_n, \frac{N_n}{n-1}\right) \le 2(c_1(n,\delta) + c_2(n,\epsilon)) \\
\implies \quad & \exists a \neq a^\star, \quad \frac{n-1}{1+\eta} \frac{(1-\zeta)}{T^\star_{\epsilon,\beta}(\boldsymbol{\nu})} \le 2c_1(T,\delta) + 2c_2(T,\epsilon) \, ,
\end{aligned}$$

where we used that $n \mapsto c_1(n,\delta)$ and $n \mapsto c_2(n,\epsilon)$ are increasing and $n \le T$. Therefore, we obtain

$$\begin{aligned}
\min\{\tau_{\epsilon,\delta}, T\} &\le \kappa T + \sum_{n=\kappa T}^{T} \mathbb{1}\left(\tau_\delta > n\right) \\
&\le \kappa T + \sum_{n=\kappa T}^{T} \mathbb{1}\left(\frac{n-1}{1+\eta} \frac{(1-\zeta)}{T^\star_{\epsilon,\beta}(\boldsymbol{\nu})} \le 2c_1(T,\delta) + 2c_2(T,\epsilon)\right) \\
&\le \kappa T + 1 + \frac{2(1+\eta) T^\star_{\epsilon,\beta}(\boldsymbol{\nu})}{1-\zeta} \left(c_1(T,\delta) + c_2(T,\epsilon)\right) \, .
\end{aligned}$$

Let $T_\zeta(\delta)$ defined as

$$T_\zeta(\delta) := \inf\left\{T \ge 1 \mid \frac{1}{1-\kappa}\left(1 + \frac{2(1+\eta) T^\star_{\epsilon,\beta}(\boldsymbol{\nu})}{1-\zeta} \left(c_1(T,\delta) + c_2(T,\epsilon)\right)\right) \le T\right\} \, .$$

Using Lemma 51, we know that $\overline{W}_{-1}(x) =_{x \to \infty} x + \log x$, hence we have $\limsup_{\delta \to 0} c_1(T,\delta)/\log(1/\delta) \le 1$. Since $\lim_{\delta \to 0} c_2(T,\epsilon)/\log(1/\delta) = 0$, we obtain $\limsup_{\delta \to 0} \frac{T_\zeta(\delta)}{\log(1/\delta)} \le \frac{2(1+\eta)T^\star_{\epsilon,\beta}(\boldsymbol{\nu})}{(1-\zeta)(1-\kappa)}$. For every $T \ge \max\{T_\zeta(\delta), N_3/\kappa\}$, we have $\tau_{\epsilon,\delta} \le T$, hence $\mathbb{E}_{\boldsymbol{\nu}\pi}\left[\tau_{\epsilon,\delta}\right] \le T_\zeta(\delta) + \mathbb{E}_{\boldsymbol{\nu}\pi}\left[N_3\right]/\kappa < +\infty$. Therefore, for all $\zeta, \kappa > 0$, we obtain

$$\limsup_{\delta \to 0} \frac{\mathbb{E}_{\boldsymbol{\nu}\pi}\left[\tau_{\epsilon,\delta}\right]}{\log(1/\delta)} \le \limsup_{\delta \to 0} \frac{T_\zeta(\delta)}{\log(1/\delta)} \le \frac{2(1+\eta)T^\star_{\epsilon,\beta}(\boldsymbol{\nu})}{(1-\zeta)(1-\kappa)} \, .$$

Letting ζ and κ go to zero concludes the proof. □

Proof of Theorem 6 The proof is obtained by combining Theorem 5 and Lemmas 4, 59, 63 and 64.

I Variants of Algorithms

In Appendix I, we propose several variants of the algorithmic components used in our algorithm. The objective is to give freedom of choice for the practitioners interested in solving ϵ-global DP BAI. Given the rich literature on BAI, it is unreasonable to provide details for the ϵ-global DP version of all the existing BAI algorithms. Therefore, we settle for a few instances that has received increased scrutiny in the BAI literature.

First, we adapt the Track-and-Stop sampling rule [38] to solve ϵ-global DP BAI (Appendix I.1). This leverages the computational tractable procedure to compute the optimal allocation $w^\star_\epsilon$ derived in Lemma 46. Second, we explore some alternative choices of components of the Top Two sampling rule for ϵ-global DP BAI (Appendix I.2). This includes adaptive choice of target for the leader, hence aiming at achieving $T^\star_\epsilon(\boldsymbol{\nu})$ instead of $T^\star_{\epsilon,\beta}(\boldsymbol{\nu})$. Third, we adapt the LUCB sampling rule [55] for ϵ-global DP BAI (Appendix I.3).

I.1 Track-and-Stop Sampling Rule

The Track-and-Stop (TaS) sampling rule was introduced in the seminal paper [38]. At each time n, it solves the optimization problem defining the characteristic time for the current empirical estimator $\tilde{\mu}_n$. When $\tilde{\mu}_n \in (0,1)^K$, we define $\widetilde{w}_n = w^\star_\epsilon(\tilde{\nu}_n)$ where $\tilde{\nu}_n$ is the Bernoulli instance with means $\tilde{\mu}_n$. When $\tilde{\mu}_n \notin (0,1)^K$, $[\tilde{\mu}_n]_0^1$ corresponds to a degenerate Bernoulli instance, hence we define $\widetilde{w}_n = 1_K/K$. Since $\tilde{\mu}_n$ is updated on a per-arm geometric grid governed by η, the optimal allocation $\widetilde{w}_n$ is updated on the same per-arm geometric grid. Therefore, choosing a larger η yields lower computational cost of TaS at the cost of larger expected sampled complexity, i.e., asymptotic multiplicative factor $1+\eta$ due to the update grid.

Given the vector $\widetilde{w}_n \in \triangle_K$, the next arm a_n to sample is obtained by using C-Tracking [38] with forced exploration in order to ensure that sufficient exploration holds. This is done here by projecting on $\triangle_K^\epsilon = \{w \in [\epsilon,1]^K \mid \sum_{a\in[K]} w_a = 1\}$ for a well chosen $\epsilon \in (0, 1/K]$. Let $\widetilde{w}_n^{\epsilon_n}$ be the ℓ_∞ projection of $\widetilde{w}_n$ on $\triangle_K^{\epsilon_n}$ with $\epsilon_n = (K^2+n)^{-1/2}/2$. While we consider a projection that changes at each time n (due to ϵ_n), $\widetilde{w}_n^{\epsilon_n}$ could also be updated on a per-arm geometric grid, i.e., when $\widetilde{w}_n$ is updated itself. For all $n \ge K+1$, the TaS sampling rule defines

$$a_n \in \underset{a\in[K]}{\arg\max} \left\{ \sum_{t\in[n]} \widetilde{w}_{t,a}^{\epsilon_t} - N_{n,a} \right\} . \tag{43}$$

In summary, our proposed Track-and-Stop algorithm is defined as in DP-TT with the sole modification that Lines 13-14 are replaced by the sampling rule defined in Eq. (43).

Optimal Allocation Oracle In Lemma 46, we show that $w^\star_\epsilon(\boldsymbol{\nu})$ can be computed explicitly based on the unique fixed-point solution $F_\mu(y) = 1$ for $y \in [0, \min_{a\neq a^\star(\mu)} d_\epsilon^-(\mu_{a^\star(\mu)}, \mu_a))$, where F_μ is an increasing one-to-one mapping from $[0, \min_{a\neq a^\star(\mu)} d_\epsilon^-(\mu_{a^\star(\mu)}, \mu_a))$ to $[0,+\infty)$ defined as

$$F_\mu(y) = \sum_{a\neq a^\star(\mu)} \frac{d_\epsilon^-(\mu_{a^\star(\mu)}, u_a(x_a(y)))}{d_\epsilon^+(\mu_a, u_a(x_a(y)))} . \tag{44}$$

The definitions of u_a and x_a is defered to Lemma 46, u_a is decreasing and x_a is increasing and strictly convex.

Asymptotic Expected Sample Complexity Combining the TaS sampling rule a_n as in Eq. (43) with the $\text{GPE}_\eta(\epsilon)$ update and the GLR stopping rule as in Eq. (7) for the stopping threshold as in Eq. (8) yields a δ-correct and ϵ-global DP algorithm (see Lemma 4 and Theorem 5). Moreover, we conjecture that its satisfies that, for all $\boldsymbol{\nu} \in \mathcal{F}^K$ with unique best arm,

$$\limsup_{\delta\to 0} \frac{\mathbb{E}_{\boldsymbol{\nu}\pi}[\tau_{\epsilon,\delta}]}{\log(1/\delta)} \le 2(1+\eta) T^\star_\epsilon(\boldsymbol{\nu}) .$$

The multiplicative factor $1+\eta$ comes from the per-arm geometric update grid, and the factor 2 comes from the asymptotic scaling in $2\log(1/\delta)$ of the stopping threshold. Using Theorem 6 for $\beta = 1/2$ and $T^\star_{\epsilon,1/2}(\boldsymbol{\nu}) \le 2T^\star_\epsilon(\boldsymbol{\nu})$ (Lemma 43), proving this conjecture would only yield an asymptotic improvement by a factor of at most 2. However, this would come at the price of a significantly higher computational cost.

Proof Sketch of Conjecture While the detailed proof of this conjecture is beyond the scope of this work, an astute reader could notice that all the necessary steps were proven to derive Theorem 6 for DP-TT. At a high level, it is intuitive that the asymptotic analysis of Track-and-Stop is simpler than the one of DP-TT.

First, the forced exploration is enforced algorithmically, hence an equivalent of Lemma 59 can be shown for the Track-and-Stop sampling rule. In contrast, the proof of sufficient exploration for DP-TT is more challenging and involves a subbtle reasoning towards contradiction, see Appendix H.2 for more details.

Second, the convergence towards the optimal allocation is also enforced algorithmically. Thanks to the forced exploration and due to the continuity of $\boldsymbol{\nu} \mapsto w^\star_\epsilon(\boldsymbol{\nu})$ (Lemma 41) and the convergence $\tilde{\mu}_n \to_{n \to +\infty} \mu$, the empirical optimal allocation $\widetilde{w}_n$ converges towards the true optimal allocation $w^\star_\epsilon(\boldsymbol{\nu})$. Therefore, an equivalent of Lemma 63 can be shown for the Track-and-Stop sampling rule. In contrast, the proof of convergence towards β-optimal allocation for DP-TT is more challenging and leverage subbtle regularity properties of the β characteristic time and its optimal allocation, e.g., the equality at equilibrium of all the transportations costs in Eq. (39), see Appendix H.3 for more details.

Third, the invertion of the GLR stopping rule can be done similarly as for DP-TT. The sole modification lies in using our derived regularity properties for $w^\star_\epsilon(\boldsymbol{\nu})$ instead of $w^\star_{\epsilon,\beta}(\boldsymbol{\nu})$, e.g., the equality at equilibrium of all the transportations costs in Lemma 42. Therefore, an equivalent of Lemma 64 can be shown for the Track-and-Stop sampling rule with $2(1+\eta)T^\star_\epsilon(\boldsymbol{\nu})$ instead of $2(1+\eta)T^\star_{\epsilon,\beta}(\boldsymbol{\nu})$, see Appendix H.4 for more details.

I.2 Top Two Sampling Rule

As detailed in Chapter 2.2 in [48], a Top Two sampling rule is defined by four choices: a leader arm $B_n \in [K]$, a challenger arm $C_n \in [K] \setminus \{B_n\}$, a target $\beta_n(B_n, C_n) \in [0,1]$ and a mechanism to reach the target, i.e., $a_n \in \{B_n, C_n\}$ by using $\beta_n(B_n, C_n)$. For instance, the sampling rule in DP-TT uses the EB leader, the TCI challenger, a fixed target $\beta \in (0,1)$ and K independent β-tracking procedures (one per leader). We propose adaptive choice of target (Appendix I.2.1), as well as leader fostering implicit exploration (Appendix I.2.2).

I.2.1 Adaptive Target

When the target is fixed to β beforehand, the Top Two sampling rule can achieve $T^\star_{\epsilon,\beta}(\boldsymbol{\nu})$ at best. We propose adaptive choices of the target inspired by the recent literature on asymptotically optimal Top Two algorithms [86, 14].

BOLD Target Given the EB-TCI leader/challenger pair (B_n, C_n) defined in DP-TT, we adapt the BOLD target from Bandyopadhyay et al. [14]. Let us define

$$u_{\epsilon,B_n,a}(\tilde{\mu}_n, N_n) = \operatorname*{arg\,min}_{u \in [0,1]} \left\{ N_{n,B_n} d^-_\epsilon(\tilde{\mu}_{n,B_n}, u) + N_{n,a} d^+_\epsilon(\tilde{\mu}_{n,b}, u) \right\} \ , \tag{45}$$

whose closed-form solution is given in Lemma 44. Then, the deterministic BOLD target defines the next arm to pull as

$$a_n = B_n \quad \text{if} \quad \sum_{a \neq B_n} \frac{d^-_\epsilon(\tilde{\mu}_{n,B_n}, u_{\epsilon,B_n,a}(\tilde{\mu}_n, N_n))}{d^+_\epsilon(\tilde{\mu}_{n,a}, u_{\epsilon,B_n,a}(\tilde{\mu}_n, N_n))} > 1 \quad \text{and} \quad a_n = C_n \quad \text{otherwise.} \tag{46}$$

In summary, the sole modification in DP-TT is Line 14 that is replaced by the sampling rule defined in Eq. (46).

For any single-parameter exponential family of distributions, Bandyopadhyay et al. [14] shows that the BOLD target allows to reach asymptotic optimality. Forced exploration is added by Bandyopadhyay et al. [14] to ensure that sufficient exploration holds. Showing that the BOLD target can achieve asymptotic optimality without forced exploration, i.e., meaning that it ensures sufficient exploration on its own, is an open problem.

IDS Target Given the EB-TCI leader/challenger pair (B_n, C_n) defined in DP-TT, we adapt the IDS target from You et al. [86]. Namely, the randomized IDS target defines the next arm to pull from as

$$a_n = \begin{cases} B_n & \text{with proba } \beta_n(B_n, C_n) \\ C_n & \text{otherwise} \end{cases} \quad \text{where } \beta_n(B_n, C_n) = \frac{N_{n,B_n} d^-_\epsilon(\tilde{\mu}_{n,B_n}, u_{\epsilon,B_n,C_n}(\tilde{\mu}_n, N_n)}{W_{\epsilon,B_n,C_n}(\tilde{\mu}_n, N_n)} \ , \tag{47}$$

where $u_{\epsilon,B_n,C_n}(\tilde{\mu}_n, N_n)$ is defined in Eq. (45). In summary, the sole modification in DP-TT is Line 14 that is replaced by the sampling rule defined in Eq. (47).

While we could use $K(K-1)$ tracking procedures to select $a_n \in \{B_n, C_n\}$, we use randomization above for the sake of simplicity. For Gaussian distributions with known variance, You et al. [86] shows that the IDS target allows to reach asymptotic optimality. Showing that the IDS target can achieve optimality for other classes of distributions is an open problem.

Asymptotic Expected Sample Complexity Sampling a_n as in Eq. (46) or (47) for the EB-TCI leader/challenger pair (B_n, C_n) defined in DP-TT based on the GPE$_\eta(\epsilon)$ update and the GLR stopping rule as in Eq. (7) for the stopping threshold as in Eq. (8) yields a δ-correct and ϵ-global DP algorithm (see Lemma 4 and Theorem 5).

While we conjecture that their asymptotic expected sample complexities $\limsup_{\delta\to 0} \frac{\mathbb{E}_{\boldsymbol{\nu}\pi}[\tau_{\epsilon,\delta}]}{\log(1/\delta)}$ are both upper bounded by $2(1+\eta)T^\star_\epsilon(\boldsymbol{\nu})$, we emphasize that our analysis doesn't provide the necessary steps for this result to hold. This is an interesting research direction left for future work.

I.2.2 Implicit Exploring Leaders and TC Chalenger

The empirical best (EB) leader is a greedy choice of leader that doesn't foster implicit exploration. Without additional exploration mechanism, it can suffer from large empirical stopping time despite being enough for an asymptotic analysis, see [50]. This motivated the choice of the TCI challenger for DP-TT, since it fosters additional implicit exploration by penalizing over sampled challengers with the $\log N_{n,a}$ term. We propose other choices of leaders that foster implicit exploration, and define the TC challenger that removes this penalization.

The UCB leader is defined as

$$B_n^{\mathrm{UCB}} \in \arg\max_{a\in[K]} U_{n,a} \quad \text{where} \quad U_{n,a} = \max\left\{u \in [0,1] \mid N_{n,a} d_\epsilon^+([\tilde{\mu}_{n,a}]_0^1, u) \le \log(n)\right\} . \tag{48}$$

By adding a bonus to the empirical mean, we are optimistic since we consider that the means are better than suggested by our observations.

The IMED leader builds on the IMED algorithm [42] is defined as

$$B_n^{\mathrm{IMED}} \in \arg\min_{a\in[K]} \left\{N_{n,a} d_\epsilon^+([\tilde{\mu}_{n,a}]_0^1, \tilde{\mu}_n^\star) + \log N_{n,a}\right\} \quad \text{where} \quad \tilde{\mu}_n^\star = \max_{a\in[K]} [\tilde{\mu}_{n,a}]_0^1 . \tag{49}$$

The TC challenger is defined as

$$C_n^{\mathrm{TC}} \in \arg\min_{a\neq B_n} W_{\epsilon,B_n,b}(\tilde{\mu}_n, N_n) , \tag{50}$$

where $W_{\epsilon,a,b}$ is defined as in Eq. (4).

In summary, the sole modification in DP-TT is Line 13 which can be replaced by choosing the leader as in Eq. (48) or Eq. (49), or choosing the challenger as in Eq. (50).

Asymptotic Expected Sample Complexity Choosing the leader as in Eq. (48) or Eq. (49) or the challenger as in Eq. (50) based on the β-tracking as in DP-TT, the GPE$_\eta(\epsilon)$ update and the GLR stopping rule as in Eq. (7) for the stopping threshold as in Eq. (8) yields a δ-correct and ϵ-global DP algorithm (see Lemma 4 and Theorem 5). Moreover, we conjecture that its satisfies that, for all $\boldsymbol{\nu} \in \mathcal{F}^K$ with distinct means,

$$\limsup_{\delta\to 0} \frac{\mathbb{E}_{\boldsymbol{\nu}\pi}[\tau_{\epsilon,\delta}]}{\log(1/\delta)} \le 2(1+\eta)T^\star_{\epsilon,\beta}(\boldsymbol{\nu}) .$$

While the detailed proof of this conjecture is beyond the scope of this work, an astute reader could notice that all the necessary steps were proven to derive Theorem 6 for DP-TT. When using the TC challenger as in Eq. (50), the proofs of Lemmas 58 and 62 can be readily adapted. When using the UCB leader as in Eq. (48) or the IMED leader as in Eq. (49), the proofs of Lemmas 55 and 60 could also be adapted.

I.3 LUCB Sampling Rule

While the Top Two terminology was introduced in Russo [73], the first sampling rule having a Top Two structure is the greedy sampling strategy in LUCB1 introduced by Kalyanakrishnan et al. [55]. At each time n, it selects the EB leader $B_n^{\mathrm{EB}} = \tilde{a}_n$ and the UCB challenger defined as

$$C_n^{\mathrm{UCB}} \in \arg\max_{a \neq B_n^{\mathrm{EB}}} U_{n,a} \quad \text{where} \quad U_{n,a} \quad \text{as in Eq. (48)} . \tag{51}$$

Then, it samples both B_n^{EB} and C_n^{UCB}. Instead of using the GLR stopping rule as in Eq.(7), LUCB1 stops when the LCB (lower confidence bound) of the leader exceeds the UCB of the challenger, i.e.,

$$\tau_{\epsilon,\delta}^{\mathrm{LUCB1}} = \inf \left\{ n \mid \widetilde{L}_{n,B_n^{\mathrm{EB}}} > U_{n,C_n^{\mathrm{UCB}}} \right\} , \tag{52}$$

where

$$\widetilde{L}_{n,a} = \max \left\{ u \in [0,1] \mid N_{n,a} d_\epsilon^-([\tilde{\mu}_{n,a}]_0^1, u) \leq \log(n) \right\} . \tag{53}$$

In summary, the modifications in DP-TT are: (1) the sampling rule in Lines 13-15 is replaced by sampling both B_n^{EB} and C_n^{UCB}, and (2) the stopping rule in Line 10 is replaced by Eq. (52). While studying this algorithm is beyond the scope of this work, we emphasize that LUCB is known to not reach asymptotic (β-)optimality.

J Implementation Details and Supplementary Experiments

Appendix J is organized as follows. First, we provide additional detail on the implementation details for our algorithm (Appendix J.1). Second, we provide supplementary experiments to illustrate the good performance of our algorithm (Appendix J.2).

J.1 Implementation Details

We present additional experiments comparing the algorithms in different bandit instances with Bernoulli distributions, as defined by Sajed and Sheffet [74], namely

$$\begin{aligned} &\mu_1 = (0.95, 0.9, 0.9, 0.9, 0.5), && \mu_2 = (0.75, 0.7, 0.7, 0.7, 0.7), \\ &\mu_3 = (0.1, 0.3, 0.5, 0.7, 0.9), && \mu_4 = (0.75, 0.625, 0.5, 0.375, 0.25)\}, \\ &\mu_5 = (0.75, 0.53125, 0.375, 0.28125, 0.25), && \mu_6 = (0.75, 0.71875, 0.625, 0.46875, 0.25)\}. \end{aligned}$$

For each Bernoulli instance, we implement the algorithms with

$$\epsilon \in \{0.001, 0.005, 0.01, 0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1, 10, 100, 125\} .$$

The risk level is set at $\delta = 0.01$. We verify empirically that the algorithms are δ-correct by running each algorithm 1000 times.

We implement all the algorithms in Python (version 3.8) and on an 8 core 64-bits Intel i5@1.6 GHz CPU.

Remark 2. *To implement the thresholds of* AdaP-TT*,* AdaP-TT* *and DP-TT, we use empirical thresholds that we get by approximating the theoretical thresholds. The expressions of the empirical thresholds used can be found in the code in the supplementary material.*

J.2 Supplementary Experiments

Figure 2 confirms our experimental findings from Section 6. DP-TT outperforms all the other δ-correct and ϵ-global DP BAI algorithms, for different values of ϵ and in all the instances tested. The empirical performance of DP-TT demonstrates two regimes. A high-privacy regime, where the stopping time depends on the privacy budget ϵ, and a low privacy regime, where the performance of DP-TT is independent of ϵ, and requires twice the number of samples used by the non-private EB-TCI-β.

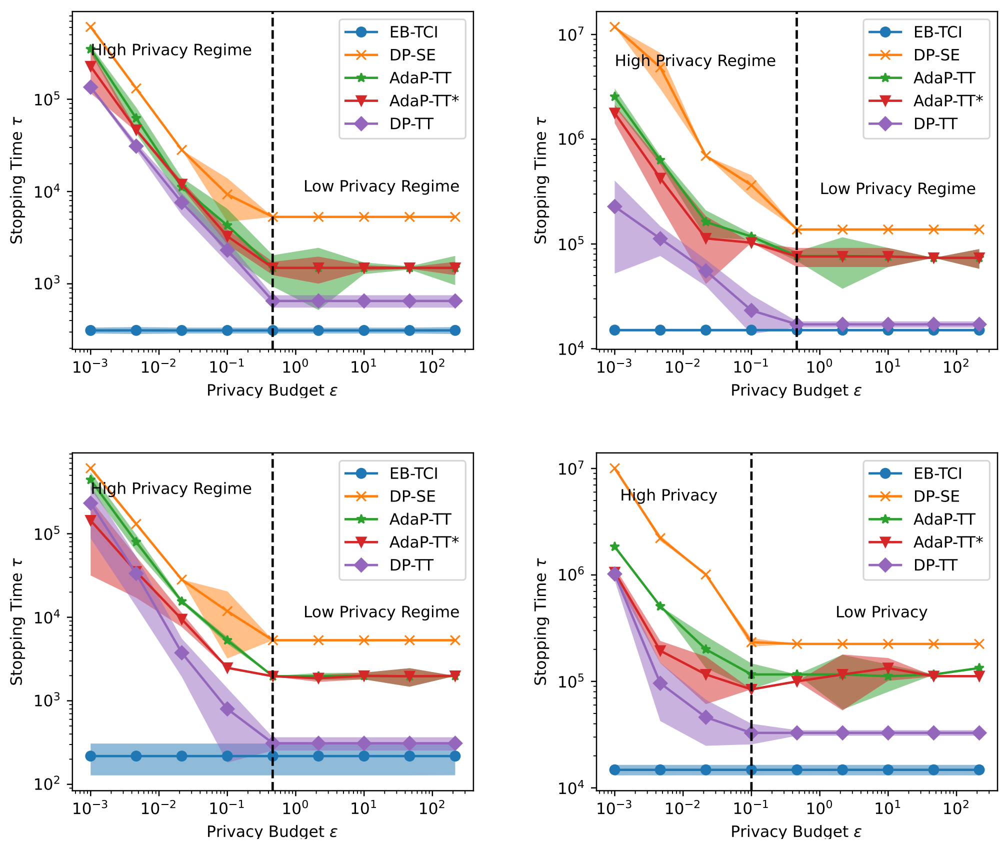

Figure 2: Empirical stopping time $\tau_{\epsilon,\delta}$ (mean ± 2 std. over 1000 runs) for $\delta = 10^{-2}$ with respect to the privacy budget ϵ for ϵ-global DP on Bernoulli instances μ_3, μ_4, μ_5 and μ_6 (top left to bottom right). The shaded vertical line separates the two privacy regimes.