

New Properties of Clifford Prolate Spheroidal Wave Functions

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Abstract—We review a recent construction of Clifford prolate spheroidal wave functions (CPSWFs) – a multidimensional multi-channel generalization of one-dimensional PSWFs. We introduce new properties of the corresponding eigenvalues such as decay and spectral accumulation.

I. INTRODUCTION

The first version of higher dimensional prolate spheroidal wave functions has been studied in [1]. The mentioned prolates have been defined as $\psi_{n,k}(r, \theta) = R_{n,k}(r)H_k(\theta)$ where the $R_{n,k}(r)$ and $H_k(\theta)$ are radial functions and spherical harmonics, respectively. A modified version of the radial part, $\varphi_{n,k}(r) = \sqrt{r}R_{n,k}(r)$ is the eigenfunction of the following differential operator

$$M_c(u)(t) = (1 - t^2) \frac{d^2 u}{dt^2} - 2t \frac{du}{dt} + \left(\frac{1}{4} - \frac{N^2}{t^2} - c^2 t^2 \right) u = 0. \quad (1)$$

The operator has a singularity at the origin, causing instabilities. The prolates are also the eigenfunctions of the Fourier transformation truncated to the unit disc, i.e.,

$$\mathcal{F}_c(\psi_{n,k})(x) = \int_{B(1)} e^{ic\langle x, y \rangle} \psi_{n,k}(y) dy. \quad (2)$$

In recent years there have been some attempts to improve the numerical computation of the prolates [2], [3]. There are also some other versions of the higher dimensional prolates that have been developed in the Clifford analysis [4], [5]. In [6] new Clifford-type prolate spheroidal wave functions (CPSWFs) have been developed with different properties. Here we present some new properties for CPSWFs.

II. CLIFFORD ANALYSIS

Let $\{e_1, \dots, e_m\}$ be the standard basis for m -dimensional euclidean space \mathbb{R}^m . We declare the non-commutative multiplication in the Clifford algebra \mathbb{R}_m by the rules

$$\begin{aligned} e_j^2 &= -1 \quad j = 1, \dots, m \\ e_i e_j &= -e_j e_i \quad i \neq j. \end{aligned}$$

A canonical base for \mathbb{R}_m is obtained by considering for any ordered set $A = \{j_1, j_2, \dots, j_h\} \subset \{1, \dots, m\} = M$, the element $e_A = e_{j_1} e_{j_2} \dots e_{j_h}$. For example, each $\lambda \in \mathbb{R}_2$, may be written as $\lambda = \lambda_0 + \lambda_1 e_1 + \lambda_2 e_2 + \lambda_{12} e_1 e_2$, where $\lambda_i \in \mathbb{R}$.

The conjugation $\bar{\lambda}$ of $\lambda = \sum_A \lambda_A e_A \in \mathbb{R}_m$ is given by $\bar{\lambda} = \sum_A \lambda_A \bar{e}_A$ where $\bar{e}_j = -e_j$ and $\overline{\alpha\beta} = \bar{\beta}\bar{\alpha}$ for all $\alpha, \beta \in \mathbb{R}_m$. Similarly \mathbb{C}_m can be obtained if $\lambda_A \in \mathbb{C}$. The Euclidean space \mathbb{R}^m is embedded in the Clifford algebra \mathbb{R}_m by identifying the point $x = (x_1, \dots, x_m) \in \mathbb{R}^m$ with the 1-vector $x = \sum_{j=1}^m e_j x_j$. The product of two 1-vectors splits up into a scalar part and a 2-vector (also called bivector): $xy = -\langle x, y \rangle + x \wedge y$ where $\langle x, y \rangle = \sum_{j=1}^m x_j y_j$ and $x \wedge y = \sum_{i < j} e_i e_j (x_i y_j - x_j y_i)$. Note also that if x is a 1-vector, then $x^2 = -\langle x, x \rangle = -|x|^2$.

Definition 1. Let $f : \mathbb{R}^m \rightarrow \mathbb{R}_m$ be defined and continuously differentiable in an open region Ω of \mathbb{R}^m . The Dirac operator ∂_x is defined on such functions by

$$\partial_x f = \sum_{j=1}^m e_j \partial_{x_j} f.$$

We also allow the Dirac operator to act of the right in the sense that $f \partial_x = \sum_{j=1}^m \partial_{x_j} f e_j$. f is said to be left (resp. right) monogenic on Ω if $\partial_x f = 0$ (resp. $f \partial_x = 0$) on Ω . If f is left-and right monogenic, we say f is monogenic.

Definition 2. A left (resp. right) monogenic homogeneous polynomial P_k of degree k ($k \geq 0$) in \mathbb{R}^m is called a left (resp. right) solid inner spherical monogenic of order k . The set of all left (resp. right) solid inner spherical monogenics of order k will be denoted by $M_l^+(k)$, respectively $M_r^+(k)$.

For the proof, the reader is referred to [7].

The \mathbb{R}_m -valued inner product of the functions $f, g : \mathbb{R}^m \rightarrow \mathbb{R}_m$ is given by $\langle f, g \rangle = \int_{\mathbb{R}^m} \overline{f(x)} g(x) dx$, where dx is Lebesgue measure on \mathbb{R}^m . The associated norm is given by $\|f\|^2 = [\langle f, f \rangle]_0$. The unitary right Clifford-module of Clifford algebra-valued measurable functions on \mathbb{R}^m for which $\|f\|^2 < \infty$ is a right Hilbert Clifford-module which we denote by $L_2(\mathbb{R}^m, \mathbb{C}_m)$. The multi-dimensional Fourier transform \mathcal{F} is given by

$$\mathcal{F}f(\xi) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp(-i\langle x, \xi \rangle) f(x) dV(x) \quad (3)$$

for $f \in L^1(\mathbb{R}^m, \mathbb{C}_m)$ and may be extended unitarily to $L^2(\mathbb{R}^m, \mathbb{C}_m)$.

Proof. For the proof see [7]. \square

Theorem 1. (Rodrigues' Formula) The Clifford Gegenbauer polynomials (CGPs) $C_n^\alpha(P_k)(x)$ are defined by

$$C_n^\alpha(Y_k)(x) = (1 - |x|^2)^{-\alpha} \partial_x^n ((1 - |x|^2)^{\alpha+n} Y_k(x)) \quad (4)$$

where $Y_k(x)$ monogenic homogenous polynomial given in Definition 2. The CGPs also satisfies in the following Clifford differential equation

$$(1 - |x|^2)^{-\alpha} \partial_x ((1 - |x|^2)^{\alpha+1} \partial_x C_{n,m}^\alpha(Y_k)(x)) = C(\alpha, n, m, k) C_{n,m}^\alpha(Y_k)(x),$$

where

$$C(\alpha, n, m, k) = \begin{cases} n(2\alpha + n + m + 2k) & \text{if } n \text{ is even,} \\ (2\alpha + n + 1)(n + m + 2k - 1) & \text{if } n \text{ is odd.} \end{cases}$$

Proof. For the proof see [7]. \square

From ([7] p. 294) we have that

$$C_{2n,m}^\alpha(Y_k)(x) = A_{n,\alpha} P_n^{\alpha, k + \frac{m}{2} - 1}(2|x|^2 - 1) Y_k(x), \quad (5)$$

where $A_{n,\alpha} = (-1)^n 2^{2n} (\alpha + n + 1)_n (n)!$, and $P_n^{(\alpha,\beta)}$ is the Jacobi polynomial. The odd version is also given in [7]. Let $f(x) = C_{2N,m}^\alpha(Y_k^j)(x) = P_{N,k,m}^\alpha(|x|^2) Y_k^j(x)$, to obtain

$$(1 - s^2) Q_N''(s) + [(k + \frac{m}{2} - 1 - \alpha) - s(k + \frac{m}{2} + \alpha + 1)] Q_N'(s) = -\frac{C(\alpha, 2N, m, k)}{4} Q_N(s), \quad (6)$$

in which $Q_N(s) = \alpha P_{N,k,m}^\alpha(2|x|^2 - 1)$ where α is any constant number. We can see the equation (6) is a Sturm-Liouville differential equation. In fact, (6) is a Jacobi differential equation.

III. COMPUTATIONS OF THE CPSWFs

Definition 3. Let $c \geq 0$ and $\alpha > -1$. The Clifford operator \mathcal{L}_c acting on $C^2(B(1), \mathbb{R}_m) \subset L_2(B(1), (1 - |x|^2)^\alpha)$ is defined as follows

$$\mathcal{L}_c f(x) = \partial_x((1 - |x|^2) \partial_x f(x)) + 4\pi^2 c^2 |x|^2 f(x). \quad (7)$$

We call the eigenfunctions of (7) CPSWFs, $\psi_{n,m}^{k,c}(x)$.

Theorem 2. Let $n = 2N$. The CPSWFs also are of two parts, i.e., $\psi_{2N,m}^{k,c,i}(x) = P_{N,m}^{k,c}(|x|^2) Y_k^i(x)$. Similarly for $n = 2N + 1$, we have that $\psi_{2N+1,m}^{k,c,i}(x) = Q_{N,m}^{k,c}(|x|^2) x Y_k^i(x)$.

For the proof see [6].

Lemma 3. By considering $|x|^2 = \frac{s+1}{2}$ we can see that

$$(1 - s^2) \frac{d^2}{ds^2} \tilde{P}_{N,m}^{k,c}(s) + [(k + \frac{m}{2} - 1) - s(k + \frac{m}{2} + 1)] \frac{d}{ds} \tilde{P}_{N,m}^{k,c}(s) - \pi^2 c^2 \frac{s+1}{2} \tilde{P}_{N,m}^{k,c}(s) = -\frac{\chi_{2N,m}^{k,c}}{4} \tilde{P}_{N,m}^{k,c}(s), \quad (8)$$

where $\tilde{P}_{N,m}^{k,c}(s) = P_{N,m}^{k,c}(2|x|^2 - 1)$ becomes a Sturm-Liouville differential equation after multiplying $y(s) = (1 + s)^{k + \frac{m}{2} - 1}$.

Proof. For the proof see [6]. \square

Equation (8) may be written as

$$\tilde{T}_c \tilde{P}_{N,m}^{k,c}(s) = (1 - s^2) \frac{d^2}{ds^2} \tilde{P}_{N,m}^{k,c}(s) + [(k + \frac{m}{2} - 1) - s(k + \frac{m}{2} + 1)] \frac{d}{ds} \tilde{P}_{N,m}^{k,c}(s) - \pi^2 c^2 \frac{s+1}{2} \tilde{P}_{N,m}^{k,c}(s). \quad (9)$$

Since by Sturm Liouville theory $\tilde{P}_{N,m}^{k,c}(s)$ is normalized in the weighted space $L^2([-1, 1], (1 + s)^{k + \frac{m}{2} - 1})$ so $\tilde{P}_{N,m}^{k,c}(s) = \sum_{n=0} a_{n,k,m} \tilde{P}_n^{(0, k + \frac{m}{2} - 1)}(s)$ as $\tilde{P}_n^{(0, k + \frac{m}{2} - 1)}(s)$ are orthonormal in the same space. So

$$\tilde{T}_c (\sum_{n=0} a_{n,k,m} \tilde{P}_n^{(0, k + \frac{m}{2} - 1)}(s)) = \sum_{n=0} a_{n,k,m} \tilde{T}_c (\tilde{P}_n^{(0, k + \frac{m}{2} - 1)}(s)),$$

where $\tilde{P}_n^{(0, k + \frac{m}{2} - 1)}(s) = \sqrt{\frac{(2n+k+\frac{m}{2})!}{2^{k+\frac{m}{2}} n!}} P_n^{(0, k + \frac{m}{2} - 1)}(s)$. Now by the use of the following identity for non-normalized Jacobi polynomials

$$(2n + \alpha + \beta + 2)(2n + \alpha + \beta + 1)(2n + \alpha + \beta)(s + 1) P_n^{(\alpha,\beta)}(s) = 2(n + 1)(n + \alpha + \beta + 1)(2n + \alpha + \beta) P_{n+1}^{(\alpha,\beta)} + [(2n + \alpha + \beta + 2)(2n + \alpha + \beta + 1)(2n + \alpha + \beta) - (2n + \alpha + \beta + 1)(\alpha^2 - \beta^2)] P_n^{(\alpha,\beta)}(s) + 2(n + \alpha)(n + \beta)(2n + \alpha + \beta + 2) P_{n-1}^{(\alpha,\beta)}(s),$$

and from (6) we have that

$$\begin{aligned} \tilde{T}_c (\tilde{P}_{N,m}^{k,c}(s)) &= \sum_{n=0} a_{n,k,m} \tilde{T}_c (\tilde{P}_n^{(0, k + \frac{m}{2} - 1)}(s)) \\ &= \sum_{n=0} a_{n,k,m} [\alpha_{n,k,m} \tilde{P}_{n-1}^{(0, k + \frac{m}{2} - 1)}(s) + \beta_{n,k,m} \tilde{P}_n^{(0, k + \frac{m}{2} - 1)}(s) \\ &\quad + \gamma_{n,k,m} \tilde{P}_{n+1}^{(0, k + \frac{m}{2} - 1)}(s)] = \sum_{n=0} [a_{n+1,k,m} \alpha_{n+1,k,m} \\ &\quad + a_{n,k,m} \beta_{n,k,m} + a_{n-1,k,m} \gamma_{n-1,k,m}] \tilde{P}_n^{(0, k + \frac{m}{2} - 1)}(s) \\ &= -\frac{\chi_{2N,m}^{k,c}}{4} \tilde{P}_{N,m}^{k,c}(s). \end{aligned}$$

Therefore,

$$a_{n+1,k,m} \alpha_{n+1,k,m} + a_{n,k,m} (\beta_{n,k,m} + \frac{\chi_{2N,m}^{k,c}}{4}) + a_{n-1,k,m} \gamma_{n-1,k,m} = 0$$

where

$$\begin{cases} \alpha_{n+1,k,m} = \frac{-\pi^2 c^2 (n+1)(n+k+\frac{m}{2})}{(2n+k+\frac{m}{2}+1)\sqrt{(2n+k+\frac{m}{2})(2n+k+\frac{m}{2}+2)}}, \\ \beta_{n,k,m} = [-n(n+k+\frac{m}{2}) - \frac{\pi^2 c^2}{2} (1 + \frac{(k+\frac{m}{2}-1)^2}{(2n+k+\frac{m}{2}-1)(2n+k+\frac{m}{2}+1)}), \\ \gamma_{n-1,k,m} = \frac{-\pi^2 c^2 n(n+k+\frac{m}{2}-1)}{(2n+k+\frac{m}{2}-1)\sqrt{(2n+k+\frac{m}{2}-2)(2n+k+\frac{m}{2})}}. \end{cases}$$

This recurrence formula holds for all $n, N \geq 0$ so the problem reduces to finding the eigenvectors $a_{n,k,m}$ and associated eigenvalues $\chi_{2N,m}^{k,c}$ of the doubly-infinite matrix $M_{k,m}^e$ with the following entries:

$$M_{k,m}^e(i, j) = \begin{cases} \gamma_{i-1,k,m}, & \text{if } i \geq 1, j = i - 1, \\ \beta_{i,k,m}, & \text{if } i = j \geq 0, \\ \alpha_{i+1,k,m}, & \text{if } i \geq 0, j = i + 1, \\ 0 & \text{else.} \end{cases}$$

The matrix is symmetric as $\gamma_{i,k,m} = \alpha_{i+1,k,m}$. The odd CPSWFs may be computed similarly. So calculation of the CPSWFs reduces to calculations of its radial parts.

IV. EIGENVALUES OF CPSWFs

Theorem 4. Let the real constants $\chi_{n,m}^{k,0}$, $\chi_{n,m}^{k,c}$, $\alpha_{n,k}$, $\beta_{n,k}$, $\gamma_{n,k}$ be given by

$$\begin{aligned} L_0 \bar{C}_{n,m}^0(Y_k^i) &= \chi_{n,m}^{k,0} \bar{C}_{n,m}^{k,0}(Y_k^i), \\ L_c \psi_{n,m}^{k,c,i} &= \chi_{n,m}^{k,c} \psi_{n,m}^{k,c,i}, \\ x^2 \bar{C}_{n,m}^0(Y_k^i) &= \alpha_{n,k,m} \bar{C}_{n+2,m}^0(Y_k^i) + \beta_{n,k,m} \bar{C}_{n,m}^0(Y_k^i) \\ &\quad + \gamma_{n,k,m} \bar{C}_{n-2,m}^0(Y_k^i). \end{aligned}$$

Then the asymptotic behaviours of $\chi_{n,m}^{k,c}$ and $\psi_{n,m}^{k,c}$ are as follows:

$$\chi_{n,m}^{k,c} = \chi_{n,m}^{k,0} - 4\pi^2 c^2 \beta_{n,k,m} + O(c^4). \quad (10)$$

and

$$\begin{aligned} \psi_{n,m}^{k,c,i} &= \bar{C}_{n,m}^0(Y_k^i) - 4\pi^2 c^2 \left(\frac{\alpha_{n,k,m}}{\chi_{n,m}^{k,0} - \chi_{n+2,m}^{k,0}} \bar{C}_{n+2,m}^0(Y_k^i) \right. \\ &\quad \left. + \frac{\gamma_{n,k,m}}{\chi_{n,m}^{k,0} - \chi_{n-2,m}^{k,0}} \bar{C}_{n-2,m}^0(Y_k^i) \right) + O(c^4). \end{aligned} \quad (11)$$

Proof. We assume an asymptotic expansion of the form

$$\psi_{n,m}^{k,c,i}(x) = \bar{C}_{n,m}^0(Y_k^i)(x) + c^2 f(x) + O(c^4). \quad (12)$$

Since $\|\psi_{n,m}^{k,c,i}\|_2 = \|\bar{C}_{n,m}^0(Y_k^i)\|_2 = 1$, we have $\langle C_n^0(Y_k^i), f \rangle = 0$. Therefore,

$$\begin{aligned} \chi_{n,m}^{k,c} \psi_{n,m}^{k,c,i} &= L_c \bar{C}_{n,m}^0(Y_k^i) + c^2 L_c f + O(c^4) \\ &= L_0 \bar{C}_{n,m}^0(Y_k^i) + 4\pi^2 c^2 |x|^2 \bar{C}_{n,m}^0(Y_k^i) + c^2 L_0 f + O(c^4) \\ &= [\chi_{n,m}^{k,0} - 4\pi^2 c^2 \beta_{n,k}] \bar{C}_{n,m}^0(Y_k^i) - 4\pi^2 c^2 [\alpha_{n,k} \bar{C}_{n+2,m}^0(Y_k^i) \\ &\quad + \gamma_{n,k} \bar{C}_{n-2,m}^0(Y_k^i)] + c^2 L_0 f + O(c^4). \end{aligned} \quad (13)$$

On the other hand,

$$\chi_{n,m}^{k,c} \psi_{n,m}^{k,c,i} = \chi_{n,m}^{k,c} [\bar{C}_{n,m}^0(Y_k^i) + c^2 f] + O(c^4). \quad (14)$$

Subtracting (14) from (13) gives

$$\begin{aligned} (\chi_{n,m}^{k,0} - 4\pi^2 c^2 \beta_{n,k} - \chi_{n,m}^{k,c}) \bar{C}_{n,m}^0(Y_k^i) &+ c^2 (L_0 f - \chi_{n,m}^{k,c} f) \\ - 4\pi^2 c^2 [\alpha_{n,k} \bar{C}_{n+2,m}^0(Y_k^i) &+ \gamma_{n,k} \bar{C}_{n-2,m}^0(Y_k^i)] = O(c^4). \end{aligned} \quad (15)$$

Since $\langle L_0 f, \bar{C}_{n,m}^0(Y_k^i) \rangle = \langle f, L_0 \bar{C}_{n,m}^0(Y_k^i) \rangle = \chi_{n,m}^{k,0} \langle f, \bar{C}_{n,m}^0(Y_k^i) \rangle = 0$, taking the inner product of both sides on (15) against $\bar{C}_{n,m}^0(Y_k^i)$ gives (10).

We now aim to determine the function f in (12). Combining (10), (12) and (13) gives

$$L_0 f = \chi_{n,m}^{k,0} f - 4\pi^2 [\alpha_{n,k,m} \bar{C}_{n+2,m}^0(Y_k^i) + \gamma_{n,k,m} \bar{C}_{n-2,m}^0(Y_k^i)], \quad (16)$$

which has solutions of the form

$$\begin{aligned} f &= \bar{C}_{n,m}^0(Y_k^i) A + 4\pi^2 \left[\frac{\alpha_{n,k}}{\chi_{n+2,m}^{k,0} - \chi_{n,m}^{k,0}} \bar{C}_{n+2,m}^0(Y_k^i) \right. \\ &\quad \left. + \frac{\gamma_{n,k}}{\chi_{n-2,m}^{k,0} - \chi_{n,m}^{k,0}} \bar{C}_{n-2,m}^0(Y_k^i) \right], \end{aligned} \quad (17)$$

where A is an arbitrary Clifford constant. Substituting (17) into (12) gives

$$\begin{aligned} \psi_{n,m}^{k,c,i} &= \bar{C}_{n,m}^0(Y_k^i) (1 + A c^2) \\ &\quad - 4\pi^2 c^2 \left(\frac{\alpha_{n,k}}{\chi_{n,m}^{k,0} - \chi_{n+2,m}^{k,0}} \bar{C}_{n+2,m}^0(Y_k^i) \right. \\ &\quad \left. + \frac{\gamma_{n,k}}{\chi_{n,m}^{k,0} - \chi_{n-2,m}^{k,0}} \bar{C}_{n-2,m}^0(Y_k^i) \right) + O(c^4), \end{aligned} \quad (18)$$

However, applying L_c to both sides of (18) and applying (10) gives

$$L_c \psi_{n,m}^{k,c,i} - \chi_{n,m}^{k,c} \psi_{n,m}^{k,c,i} = \bar{C}_{n,m}^0(Y_k^i) A c^2 + O(c^4),$$

from which we conclude that $A = 0$. Putting $A = 0$ in (18) gives (11). \square

Definition 4. We define $\mathcal{G}_c : L^2(B(1), \mathbb{C}_m) \rightarrow L^2(\mathbb{R}^m, \mathbb{C}_m)$ by

$$\mathcal{G}_c f(x) = \chi_{B(1)}(x) \int_{B(1)} e^{2\pi i c \langle x, y \rangle} f(y) dy, \quad (19)$$

where $\chi_{B(1)}$ is the characteristic function of the unit ball $B(1)$ in \mathbb{R}^m . The adjoint \mathcal{G}_c^* of \mathcal{G}_c is given by

$$\mathcal{G}_c^* f(x) = \chi_{B(1)}(x) \int_{\mathbb{R}^m} e^{-2\pi i c \langle x, y \rangle} f(y) dy.$$

Definition 5. The ‘‘space-limiting’’ operator $Q : L^2(\mathbb{R}^m, \mathbb{C}_m) \rightarrow L^2(B(1), \mathbb{C}_m)$ is given by

$$Qf(x) = \chi_{B(1)}(x) f(x),$$

and the ‘‘bandlimiting’’ operator $P_c : L^2(\mathbb{R}^m, \mathbb{C}_m) \rightarrow PW_c$ is given by

$$P_c f(x) = \int_{B(c)} \mathcal{F}f(\xi) e^{2\pi i \langle \xi, x \rangle} d\xi.$$

Here, PW_c is the Paley-Wiener space of functions with bandlimit c , i.e.,

$$PW_c = \{f \in L^2(\mathbb{R}^m, \mathbb{C}_m) : \hat{f}(\xi) = 0 \text{ if } |\xi| > c\}. \quad (20)$$

Theorem 5. The CPSWFs $\psi_{n,m}^{k,c,i}$ are eigenfunctions of \mathcal{G}_c and QP_c . In other words, $L_c \psi_{n,m}^{k,c,i}(x) = \mu_{n,m}^{k,c} \psi_{n,m}^{k,c,i}(x)$, and $QP_c \psi_{n,m}^{k,c,i}(x) = \lambda_{n,m}^{k,c} \psi_{n,m}^{k,c,i}(x)$.

Proof. See the [6] for the proof. \square

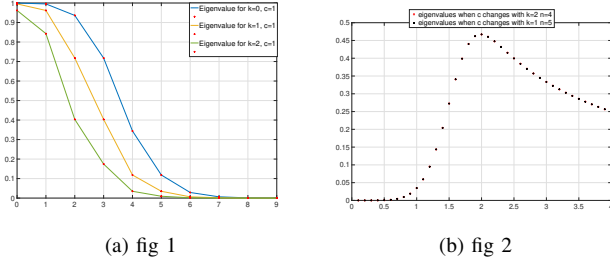


Fig. 1. Behaviour of the eigenvalues $\lambda_{n,m}^{k,c}$

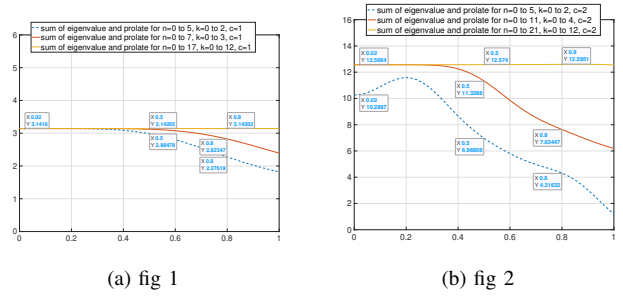


Fig. 2. Plotting $G(r^2)$ for $c = 1, 2$

Theorem 6. The eigenvalues $\{\mu_{n,m}^{k,c}; n, k \geq 0\}$ of \mathcal{G}_c , enjoy the following relationship

$$\mu_{2N,m}^{k,c} = \mu_{2N+1,m}^{k-1,c}. \quad (21)$$

and for a fixed k the eigenvalues $\lambda_{n,m}^{k,c}$ of QP_c , are non-degenerate, and

$$\lambda_{0,m}^{k,c} > \lambda_{1,m}^{k,c} > \dots > \lambda_{n,m}^{k,c} > \lambda_{n+1,m}^{k,c} > \dots.$$

Proof. See [6] for the proof. \square

The behaviour of the eigenvalues $\lambda_{n,m}^{k,c}$ is displayed in Figure 1.

From Theorem 5 we see that

$$\lambda_{n,m}^{k,c} \psi_{n,m}^{k,c,i}(x) = \int_{B(1)} K_c(x-y) \psi_{n,m}^{k,c,i}(y) dy = QP_c \psi_{n,m}^{k,c,i}(x)$$

where $K_c(x) = c^m \int_{B(1)} e^{2\pi i c(\omega, x)} d\omega$. Then

$$\begin{aligned} & \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \sum_{i=1}^{d_{k,m}} \lambda_{n,m}^{k,c} |\psi_{n,m}^{k,c,i}(x)|^2 \\ &= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \sum_{i=1}^{d_{k,m}} \lambda_{n,m}^{k,c} \overline{[\psi_{n,m}^{k,c,i}(x) \psi_{n,m}^{k,c,i}(x)]}_0 \\ &= \left[\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \sum_{i=1}^{d_{k,m}} \int_{B(1)} \overline{\psi_{n,m}^{k,c,i}(y)} K_c(x-y) dy \psi_{n,m}^{k,c,i}(x) \right]_0 \\ &= K_c(0) = c^m |B(1)|. \end{aligned}$$

On the other hand, if instead of summing over all $n, k \geq 0$ in the above calculation, we perform a truncated sum by restricting the values of n and k so that $0 \leq n \leq 2N+1$ and $0 \leq k \leq K$, then we have

$$\begin{aligned} & \sum_{k=0}^K \sum_{n=0}^{2N+1} \sum_{i=1}^{d_{k,m}} \lambda_{n,m}^{k,c} |\psi_{n,m}^{k,c,i}(x)|^2 \\ &= \sum_{k=0}^K \sum_{n=0}^N \lambda_{2n,m}^{k,c} |P_{n,m}^{k,c}(|x|^2)|^2 |x|^{2k} \frac{(k+m-2)}{|S^{m-1}|(m-2)} \\ &+ \sum_{k=0}^K \sum_{n=0}^N \lambda_{2n+1,m}^{k,c} |Q_{n,m}^{k,c}(|x|^2)|^2 |x|^{2k+2} \frac{(k+m-2)}{|S^{m-1}|(m-2)} \\ &= G(|x|^2), \end{aligned} \quad (22)$$

where we have used the reproducing kernel for monogenic functions, Theorem 3.3 in [8]. In Figures 2, we see numerical computations of the partial sums (22) demonstrating the convergence of the partial sums to the constant $c^2|B(1)|$.

V. CONCLUSION AND FUTURE WORK

Through consideration of Clifford Gegenbauer polynomials, we developed in this paper a new method for computing multidimensional Clifford-valued PSWF's, defined as the eigenfunctions the differential operator L_c involving the Dirac operator. We then introduced a relationship between the eigenvalues of the differential operators L_c and L_0 . We defined time and frequency projections Q and P_c as CPSWFs and the eigenfunctions of QP_c . We then developed some properties for the eigenvalues of the QP_c . The spectrum accumulations property is also introduced.

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