Supplementary Material for: Minimizing Polarization and Disagreement in Social Networks via Link Recommendation

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1 Proof of Lemma 4.1

Proof. If we perturb the network with the addition of e, we obtain the new Laplacian $\boldsymbol{L} + \boldsymbol{b}_e \boldsymbol{b}_e^\top$. By Sherman-Morrison formula [1], we obtain

$$\left(\boldsymbol{I} + \boldsymbol{L} + \boldsymbol{b}_e \boldsymbol{b}_e^\top\right)^{-1} = (\boldsymbol{I} + \boldsymbol{L})^{-1} - \frac{\boldsymbol{\Omega} \boldsymbol{b}_e \boldsymbol{b}_e^\top \boldsymbol{\Omega}}{1 + \boldsymbol{b}_e^\top \boldsymbol{\Omega} \boldsymbol{b}_e}.$$

By the definitions of P-D index, $\mathcal{I}(\mathcal{G} + e) = \boldsymbol{s}^\top (\boldsymbol{I} + \boldsymbol{L} + \boldsymbol{b}_e \boldsymbol{b}_e^\top)^{-1} \boldsymbol{s}$. We can immediately obtain $f(e) = \mathcal{I}(\mathcal{G}) - \mathcal{I}(\mathcal{G} + e) = \frac{\left(\boldsymbol{s}^\top \boldsymbol{\Omega} \boldsymbol{b}_e\right)^2}{1 + \boldsymbol{b}_e^\top \boldsymbol{\Omega} \boldsymbol{b}_e}$. Since the term $(\boldsymbol{s}^\top \boldsymbol{\Omega} \boldsymbol{b}_e)^2 = (\boldsymbol{z}_u - \boldsymbol{z}_v)^2$ is nonnegative, together with the fact that $0 \leq \boldsymbol{b}_e^\top \boldsymbol{\Omega} \boldsymbol{b}_e \leq 2$, we can conclude $f(e) \geq 0$ consequently. □

2 Proof of Remark 1

Proof. When the opinions $\boldsymbol{s}$ are mean-centered, corresponding variation of the objective could be expressed as

$$\begin{aligned} f(e) =& \overline{\boldsymbol{s}}^\top \boldsymbol{\Omega} \boldsymbol{b}_e \boldsymbol{b}_e^\top \boldsymbol{\Omega} \overline{\boldsymbol{s}} = \left((\boldsymbol{s} - \frac{\boldsymbol{s}^\top \boldsymbol{1}}{n} \boldsymbol{1})^\top \boldsymbol{\Omega} \boldsymbol{b}_e \right)^2 \\ =& \left(\boldsymbol{s}^\top \boldsymbol{\Omega} \boldsymbol{b}_e - \frac{\boldsymbol{s}^\top \boldsymbol{1}}{n} \boldsymbol{1}^\top \boldsymbol{\Omega} \boldsymbol{b}_e \right)^2 = \boldsymbol{s}^\top \boldsymbol{\Omega} \boldsymbol{b}_e \boldsymbol{b}_e^\top \boldsymbol{\Omega} \boldsymbol{s}, \end{aligned}$$

where the last equality is obtained by the fact that $\boldsymbol{1}^\top \boldsymbol{\Omega} \boldsymbol{b}_e = \boldsymbol{1}^\top \boldsymbol{b}_e = 0$.

Thus, under the perturbation of the network with a single edge e, it holds that whether the opinions $\boldsymbol{s}$ are mean-centered or not, the variation of our objective i.e. $f(e)$ are the same. The above results complete the proof. □

3 Proof of Lemma 5.1

Proof. Note, for two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, we write $\boldsymbol{A} \preceq \boldsymbol{B}$ to denote that $\boldsymbol{B} - \boldsymbol{A}$ is positive semidefinite. We use $(\boldsymbol{I} + \boldsymbol{L})_T^{-1}$ to denote the forest matrix associated with graph $\mathcal{G} + T$.

Let E_C be the candidate set, and let T, W be any two subsets of E_C. To begin with, we first derive a lower and an upper bound, respectively, for the marginal benefit function $\rho_T(W) = f(W \cup T) - f(W)$.

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On the one hand,

$$\rho_T(W) = f(W \cup T) - f(W) = \mathcal{I}(G+T) - \mathcal{I}(G+W\cup T) = \boldsymbol{s}^\top (\boldsymbol{I}+\boldsymbol{L})_T^{-1}\boldsymbol{s} - \boldsymbol{s}^\top(\boldsymbol{I}+\boldsymbol{L})_{W\cup T}^{-1}\boldsymbol{s}$$

$$= \boldsymbol{s}^\top \left(\sum_{i=1}^{n-1} \frac{1}{1+\lambda_i(\boldsymbol{L}_T)} \boldsymbol{u}_i\boldsymbol{u}_i^\top - \frac{1}{1+\lambda_i(\boldsymbol{L}_{W\cup T})}\boldsymbol{u}_i\boldsymbol{u}_i^\top \right) \boldsymbol{s}$$

$$= \boldsymbol{s}^\top \left(\sum_{i=1}^{n-1} \frac{\lambda_i(\boldsymbol{L}_{W\cup T}) - \lambda_i(\boldsymbol{L}_T)}{(1+\lambda_i(\boldsymbol{L}_T))\,(1+\lambda_i(\boldsymbol{L}_{W\cup T}))} \boldsymbol{u}_i\boldsymbol{u}_i^\top \right) \boldsymbol{s} \geq \boldsymbol{s}^\top \left(\frac{\boldsymbol{L}_{W\cup T} - \boldsymbol{L}_T}{(1+\lambda_{n-1}(\boldsymbol{L}_T))\,(1+\lambda_{n-1}(\boldsymbol{L}_{W\cup T}))} \right) \boldsymbol{s}$$

$$= \boldsymbol{s}^\top \left(\frac{\sum_{e\in W\setminus T} \boldsymbol{b}_e\boldsymbol{b}_e^\top}{(1+\lambda_{n-1}(\boldsymbol{L}_T))\,(1+\lambda_{n-1}(\boldsymbol{L}_{W\cup T}))} \right) \boldsymbol{s}.$$

On the other hand,

$$\rho_T(W) = \boldsymbol{s}^\top \left(\sum_{i=1}^{n-1} \frac{\lambda_i(\boldsymbol{L}_{W\cup T}) - \lambda_i(\boldsymbol{L}_T)}{(1+\lambda_i(\boldsymbol{L}_T))\,(1+\lambda_i(\boldsymbol{L}_{W\cup T}))} \boldsymbol{u}_i\boldsymbol{u}_i^\top \right) \boldsymbol{s}$$

$$\leq \boldsymbol{s}^\top \left(\frac{\boldsymbol{L}_{W\cup T} - \boldsymbol{L}_T}{(1+\lambda_1(\boldsymbol{L}_T))\,(1+\lambda_1(\boldsymbol{L}_{W\cup T}))} \right) \boldsymbol{s} = \boldsymbol{s}^\top \left(\frac{\sum_{e\in W\setminus T} \boldsymbol{b}_e\boldsymbol{b}_e^\top}{(1+\lambda_1(\boldsymbol{L}_T))\,(1+\lambda_1(\boldsymbol{L}_{W\cup T}))} \right) \boldsymbol{s}.$$

Putting the above two bounds together leads to

$$\frac{\sum_{e\in W\setminus T}\rho_e(T)}{\rho_T(W)} \geq \boldsymbol{s}^\top \left(\sum_{e\in W\setminus T} \frac{\boldsymbol{b}_e\boldsymbol{b}_e^\top}{(1+\lambda_{n-1}(\boldsymbol{L}_T))\,(1+\lambda_{n-1}(\boldsymbol{L}_{T+e}))} \right) \boldsymbol{s} \times \frac{(1+\lambda_1(\boldsymbol{L}_T))\,(1+\lambda_1(\boldsymbol{L}_{T\cup W}))}{\sum_{e\in W\setminus T} \boldsymbol{s}^\top \boldsymbol{b}_e\boldsymbol{b}_e^\top \boldsymbol{s}}$$

$$\geq \left(\frac{1+\lambda_1(\boldsymbol{L})}{1+\lambda_{n-1}(\boldsymbol{L}_{E_C})} \right)^2,$$

which implies the lower bounds of γ.

Similarly, we derive the upper bound of the curvature α. Let j be any candidate edge in $W\setminus T$. Then,

$$\frac{\rho_j(W\setminus j \cup W)}{\rho_j(T\setminus j)} \geq \frac{\boldsymbol{s}^\top \boldsymbol{b}_j\boldsymbol{b}_j^\top \boldsymbol{s}}{(1+\lambda_{n-1}(\boldsymbol{L}_T))\,(1+\lambda_{n-1}(\boldsymbol{L}_{T+e}))} \times \frac{(1+\lambda_1(\boldsymbol{L}_T))\,(1+\lambda_1(\boldsymbol{L}_{T\cup W}))}{\boldsymbol{s}^\top \boldsymbol{b}_j\boldsymbol{b}_j^\top \boldsymbol{s}} \quad \geq \left(\frac{1+\lambda_1(\boldsymbol{L})}{1+\lambda_{n-1}(\boldsymbol{L}_{E_C})} \right)^2,$$

which combining with the definition of curvature completes the proof. □

4 Proof of Theorem 5.1

Proof. To show the non-submodularity of the function concerned, consider the graph in Figure 1, which is a 5-node path-graph with e_1 and e_2 being inexistent edges. We set initial opinion vector as $\boldsymbol{s} = (0.25, 0.5, 0.5, 0.5, 0.25)^\top$, and define two edge sets, $T = \emptyset$ and $W = \{e_2\}$. Then,

$$\mathcal{I}(T) = 0.8295, \mathcal{I}(T+e_1) = 0.8295,$$
$$\mathcal{I}(W) = 0.8227, \mathcal{I}(W+e_1) = 0.8220,$$

so that

$$\mathcal{I}(T) - \mathcal{I}(T+e_1) = 0 < 0.007 = \mathcal{I}(W) - \mathcal{I}(W+e_1),$$

which violates the definition of submodularity. Thus, it follows that the set function of our problem is non-submodular. □

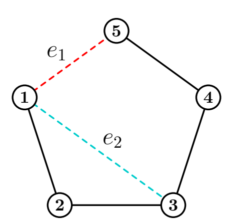

Figure 1: A 5-nodes path-graph where e_1 and e_2 are inexistent edges.

5 Proof of Lemma 6.3

Proof. According to the assumption:

$$(1-\frac{\epsilon}{12})\|\overline{\boldsymbol{X}}\boldsymbol{e}_u\|^2 \leq \|\boldsymbol{X}'\boldsymbol{e}_u\|^2 \leq (1+\frac{\epsilon}{12})\|\overline{\boldsymbol{X}}\boldsymbol{e}_u\|^2$$

holds for any node $u \in V$ and

$$(1-\frac{\epsilon}{12})\|\overline{\boldsymbol{X}}\boldsymbol{b}_e\|^2 \leq \|\boldsymbol{X}'\boldsymbol{b}_e\|^2 \leq (1+\frac{\epsilon}{12})\|\overline{\boldsymbol{X}}\boldsymbol{b}_e\|^2$$

holds for any pair of nodes u and v connecting an edge e in E.

As $\mathcal{G}$ is connected, there exists a simple path P_{uv} connecting u and v. By applying the triangle inequality twice, we obtain

$$|\|\tilde{\boldsymbol{X}}\boldsymbol{b}_e\| - \|\boldsymbol{X}'\boldsymbol{b}_e\|| \leq \|(\tilde{\boldsymbol{X}}-\boldsymbol{X}')\boldsymbol{b}_e\| \leq \sum_{(a,b)\in P_{uv}} \|(\tilde{\boldsymbol{X}}-\boldsymbol{X}')(\boldsymbol{e}_a-\boldsymbol{e}_b)\|.$$

We will upper bound the last term by considering its square:

$$\begin{aligned}\left(\sum_{(a,b)\in P_{uv}} \|(\tilde{\boldsymbol{X}}-\boldsymbol{X}')(\boldsymbol{e}_a-\boldsymbol{e}_b)\|\right)^2 \leq & n\sum_{(a,b)\in P_{uv}} \|(\tilde{\boldsymbol{X}}-\boldsymbol{X}')(\boldsymbol{e}_a-\boldsymbol{e}_b)\|^2 \leq n\sum_{(a,b)\in E} \|(\tilde{\boldsymbol{X}}-\boldsymbol{X}')(\boldsymbol{e}_a-\boldsymbol{e}_b)\|^2\\ =& n\|(\tilde{\boldsymbol{X}}-\boldsymbol{X}')\boldsymbol{B}^\top\|_F^2 = n\|\boldsymbol{B}(\tilde{\boldsymbol{X}}-\boldsymbol{X}')\|_F^2.\end{aligned}$$

Note that the first inequality is derived by Cauchy-Schwarz Inequality. Below we transform the above-obtained Frobenius norm $n\|\boldsymbol{B}(\tilde{\boldsymbol{X}}-\boldsymbol{X}')\|_F^2$ into the $(\boldsymbol{I}+\boldsymbol{L})$-norm as

$$\begin{aligned}n\|\boldsymbol{B}(\tilde{\boldsymbol{X}}-\boldsymbol{X}')\|_F^2 =& n\mathrm{Tr}\left((\tilde{\boldsymbol{X}}-\boldsymbol{X}')^\top\boldsymbol{B}^\top\boldsymbol{B}(\tilde{\boldsymbol{X}}-\boldsymbol{X}')\right) = n\mathrm{Tr}\left((\tilde{\boldsymbol{X}}-\boldsymbol{X}')^\top\boldsymbol{L}(\tilde{\boldsymbol{X}}-\boldsymbol{X}')\right)\\ \leq& n\mathrm{Tr}\left((\tilde{\boldsymbol{X}}-\boldsymbol{X}')^\top(\boldsymbol{I}+\boldsymbol{L})(\tilde{\boldsymbol{X}}-\boldsymbol{X}')\right) = n\sum_{i=1}^{p}(\tilde{\boldsymbol{X}}_i-\boldsymbol{X}_i')(\boldsymbol{I}+\boldsymbol{L})(\tilde{\boldsymbol{X}}_i-\boldsymbol{X}_i')^\top\\ \leq& n\delta_1^2\sum_{i=1}^{p}\boldsymbol{X}_i'(\boldsymbol{I}+\boldsymbol{L})(\boldsymbol{X}_i')^\top.\end{aligned}$$

Applying the fact that $\boldsymbol{L} \preceq (n+1)\boldsymbol{I}$ and $\boldsymbol{\Omega}\boldsymbol{L}\boldsymbol{\Omega} \preceq \boldsymbol{\Omega} \preceq \boldsymbol{I}$, we have

$$\begin{aligned}n\delta_1^2\sum_{i=1}^{p}\boldsymbol{X}_i'(\boldsymbol{I}+\boldsymbol{L})(\boldsymbol{X}_i')^\top \leq& n\delta_1^2(n+1)\sum_{i=1}^{p}\boldsymbol{X}_i'(\boldsymbol{X}_i')^\top = n\delta_1^2(n+1)\|\boldsymbol{X}'\|_F^2\\ \leq& n\delta_1^2(n+1)\sum_{i=1}^{n}(1+\frac{\epsilon}{12})\|\overline{\boldsymbol{X}}\boldsymbol{e}_i\|^2 \leq n\delta_1^2(n+1)\sum_{i=1}^{n}(1+\frac{\epsilon}{12})\boldsymbol{e}_i^\top\boldsymbol{\Omega}\boldsymbol{e}_i\\ \leq& n\delta_1^2(n+1)(1+\frac{\epsilon}{12})n.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|\boldsymbol{X}'\boldsymbol{b}_e\|^2 \geq& (1-\frac{\epsilon}{12})\|\overline{\boldsymbol{X}}\boldsymbol{b}_e\|^2 = (1-\frac{\epsilon}{12})\boldsymbol{b}_e^\top\boldsymbol{\Omega}\boldsymbol{L}\boldsymbol{\Omega}\boldsymbol{b}_e\\ \geq& (1-\frac{\epsilon}{12})\frac{1}{n^2(n+1)^2}\|\boldsymbol{b}_e\|^2 = 2(1-\frac{\epsilon}{12})\frac{1}{n^2(n+1)^2}.\end{aligned}$$

The last inequality is obtained for the following reason. Note that $\boldsymbol{b}_e$ is orthogonal to all-one vector $\mathbf{1}$, an eigenvector of $\boldsymbol{L}$ associated with the unique eigenvalue 0. Therefore, $\boldsymbol{b}_e^\top\boldsymbol{L}\boldsymbol{b}_e \geq \lambda_{\min}\|\boldsymbol{b}_e\|^2$ holds. In addtion, $\boldsymbol{L}$ and $(\boldsymbol{I}+\boldsymbol{L})^{-1}$ share identical eigenspaces.

Combining the above-obtained results, it follows that

$$\frac{\left|\|\tilde{\boldsymbol{X}}\boldsymbol{b}_e\| - \|\boldsymbol{X}'\boldsymbol{b}_e\|\right|}{\|\boldsymbol{X}'\boldsymbol{b}_e\|} \leq \frac{\delta_1 n(n+1)\sqrt{(1+\epsilon/12)(n+1)}}{\sqrt{2(1-\epsilon/12)}} \leq \frac{\epsilon}{32},$$

based on which we further obtain

$$\begin{aligned}\left|\|\tilde{\boldsymbol{X}}\boldsymbol{b}_e\|^2 - \|\boldsymbol{X}'\boldsymbol{b}_e\|^2\right| =& \left|\|\tilde{\boldsymbol{X}}\boldsymbol{b}_e\| - \|\boldsymbol{X}'\boldsymbol{b}_e\|\right| \times \left|\|\tilde{\boldsymbol{X}}\boldsymbol{b}_e\| + \|\boldsymbol{X}'\boldsymbol{b}_e\|\right|\\ \leq& \frac{\epsilon}{32}(2+\frac{\epsilon}{32})\|\boldsymbol{X}'\boldsymbol{b}_e\|^2 \leq \frac{\epsilon}{12}\|\boldsymbol{X}'\boldsymbol{b}_e\|^2,\end{aligned}$$

which completes the proof. □

6 Proof of Lemma 6.5

Proof. Since $\boldsymbol{L}$ is the Laplacian of a connected graph, we can find a path P_{uv} connecting u and v. By applying the triangle inequality, we obtain

$$\begin{aligned}\boldsymbol{q}^\top \boldsymbol{b}_e^\top \boldsymbol{b}_e \boldsymbol{q} = (\boldsymbol{q}_u - \boldsymbol{q}_v)^2 &\le n \sum_{(a,b)\in P_{uv}} (\boldsymbol{q}(\boldsymbol{e}_a - \boldsymbol{e}_b))^2 \\ &\le n \sum_{(a,b)\in E} \|\boldsymbol{q}(\boldsymbol{e}_a - \boldsymbol{e}_b)\| \le n\boldsymbol{q}^\top \boldsymbol{L}\boldsymbol{q},\end{aligned}$$

which implies that

$$\|\boldsymbol{q}\|_{\boldsymbol{b}_e\boldsymbol{b}_e^\top} \le \sqrt{n}\|\boldsymbol{q}\|_{\boldsymbol{L}}.$$

We first bound the value $\left|\|\boldsymbol{q}\|_{\boldsymbol{b}_e\boldsymbol{b}_e^\top} - \|\boldsymbol{\Omega s}\|_{\boldsymbol{b}_e\boldsymbol{b}_e^\top}\right|$ by the triangle inequality

$$\begin{aligned}\left|\|\boldsymbol{q}\|_{\boldsymbol{b}_e\boldsymbol{b}_e^\top} - \|\boldsymbol{\Omega s}\|_{\boldsymbol{b}_e\boldsymbol{b}_e^\top}\right| &\le \|\boldsymbol{q} - \boldsymbol{\Omega s}\|_{\boldsymbol{b}_e\boldsymbol{b}_e^\top} \le \sqrt{n}\|\boldsymbol{q} - \boldsymbol{\Omega s}\|_{\boldsymbol{L}} \\ &\le \sqrt{n}\delta_3\|\boldsymbol{\Omega s}\|_{\boldsymbol{I}+\boldsymbol{L}} = \sqrt{n}\delta_3\sqrt{\boldsymbol{s}^\top\boldsymbol{\Omega s}} \\ &\le \delta_3\sqrt{n}\sqrt{\boldsymbol{s}^\top\boldsymbol{s}} \qquad \text{since } \|\boldsymbol{s}\|^2 \le n, \\ &\le \delta_3 n,\end{aligned}$$

based on which we proceed to bound $\left|\|\boldsymbol{q}\|^2_{\boldsymbol{b}_e\boldsymbol{b}_e^\top} - \|\boldsymbol{\Omega s}\|^2_{\boldsymbol{b}_e\boldsymbol{b}_e^\top}\right|$:

$$\begin{aligned}\left|\|\boldsymbol{q}\|^2_{\boldsymbol{b}_e\boldsymbol{b}_e^\top} - \|\boldsymbol{\Omega s}\|^2_{\boldsymbol{b}_e\boldsymbol{b}_e^\top}\right| &= \left|\|\boldsymbol{q}\|_{\boldsymbol{b}_e\boldsymbol{b}_e^\top} + \|\boldsymbol{\Omega s}\|_{\boldsymbol{b}_e\boldsymbol{b}_e^\top}\right| \times \left|\|\boldsymbol{q}\|_{\boldsymbol{b}_e\boldsymbol{b}_e^\top} - \|\boldsymbol{\Omega s}\|_{\boldsymbol{b}_e\boldsymbol{b}_e^\top}\right| \\ &\le \left(2\|\boldsymbol{\Omega s}\|_{\boldsymbol{b}_e\boldsymbol{b}_e^\top} + \delta_3 n\right)\delta_3 n \le \left(2\sqrt{n}\|\boldsymbol{\Omega s}\|_{\boldsymbol{L}} + \delta_3 n\right)\delta_3 n \\ &\le (2n + \delta_3 n)\,\delta_3 n \qquad \text{since } \|\boldsymbol{z}\|^2 \le n,\ \delta_3 \le 1 \text{ and } \boldsymbol{\Omega} \le \boldsymbol{L}^\dagger, \\ &\le 3\delta_3 n^2.\end{aligned}$$

Thus, one has

$$\left|\|\boldsymbol{q}\|^2_{\boldsymbol{b}_e\boldsymbol{b}_e^\top} - \|\boldsymbol{\Omega s}\|^2_{\boldsymbol{b}_e\boldsymbol{b}_e^\top}\right| \le 3\delta_3 n^2 \le \frac{\epsilon}{3},$$

which leads to the results directly. □

6.1 Proof of Theorem 6.1

Proof. Using Lemmas 6.3, 6.4, and 6.5, one has

$$\begin{aligned}|\hat{f}(e) - f(e)| &= \left|\frac{\|\boldsymbol{q}\|^2_{\boldsymbol{b}_e\boldsymbol{b}_e^\top}}{1 + \|\tilde{\boldsymbol{X}}\boldsymbol{b}_e\|^2 + \|\tilde{\boldsymbol{Y}}\boldsymbol{b}_e\|^2} - \frac{\|\boldsymbol{\Omega s}\|^2_{\boldsymbol{b}_e\boldsymbol{b}_e^\top}}{1 + \boldsymbol{b}_e^\top\boldsymbol{\Omega}\boldsymbol{b}_e}\right| \\ &\le \left|\frac{1}{1-\epsilon/3}\frac{\|\boldsymbol{q}\|^2_{\boldsymbol{b}_e\boldsymbol{b}_e^\top}}{1 + \boldsymbol{b}_e^\top\boldsymbol{\Omega}\boldsymbol{b}_e} - \frac{\|\boldsymbol{\Omega s}\|^2_{\boldsymbol{b}_e\boldsymbol{b}_e^\top}}{1 + \boldsymbol{b}_e^\top\boldsymbol{\Omega}\boldsymbol{b}_e}\right| \\ &\le \left|\frac{1}{1-\epsilon/3}\frac{\|\boldsymbol{\Omega s}\|^2_{\boldsymbol{b}_e\boldsymbol{b}_e^\top} + \epsilon/3}{1 + \boldsymbol{b}_e^\top\boldsymbol{\Omega}\boldsymbol{b}_e} - \frac{\|\boldsymbol{\Omega s}\|^2_{\boldsymbol{b}_e\boldsymbol{b}_e^\top}}{1 + \boldsymbol{b}_e^\top\boldsymbol{\Omega}\boldsymbol{b}_e}\right| \\ &= \left|\frac{1}{1-\epsilon/3}\frac{(\boldsymbol{z}_u - \boldsymbol{z}_v)^2 + \epsilon/3}{1 + r_{uv}} - \frac{(\boldsymbol{z}_u - \boldsymbol{z}_v)^2}{1 + r_{uv}}\right| \\ &\le \frac{2\epsilon/3}{1-\epsilon/3} \le \frac{4}{5}\epsilon, \qquad \text{since } (\boldsymbol{z}_u - \boldsymbol{z}_v)^2 \le 1 \text{ and } 0 \le r_{uv} \le 2,\end{aligned}$$

which leads to the result. □

References

[1] Carl D Meyer, Jr. Generalized inversion of modified matrices. *SIAM Journal on Applied Mathematics*, 24(3):315–323, 1973.