Global Convergence to Local Minmax Equilibrium in Classes of Nonconvex Zero-Sum Games

Tanner Fiez
University of Washington
fiezt@uw.edu

Lillian J. Ratliff
University of Washington
ratliffl@uw.edu

Eric Mazumdar
California Institute of Technology
mazumdar@caltech.edu

Evan Faulkner
University of Washington
evanjf5@uw.edu

Adhyyan Narang
University of Washington
adhyyan@uw.edu

Abstract

We study gradient descent-ascent learning dynamics with timescale separation (τ-GDA) in unconstrained continuous action zero-sum games where the minimizing player faces a nonconvex optimization problem and the maximizing player optimizes a Polyak-Łojasiewicz (PŁ) or strongly-concave (SC) objective. In contrast to past work on gradient-based learning in nonconvex-PŁ/SC zero-sum games, we assess convergence in relation to natural game-theoretic equilibria instead of only notions of stationarity. In pursuit of this goal, we prove that the only locally stable points of the τ-GDA continuous-time limiting system correspond to strict local minmax equilibria in each class of games. For these classes of games, we exploit timescale separation to construct a potential function that when combined with the stability characterization and an asymptotic saddle avoidance result gives a global asymptotic almost-sure convergence guarantee for the discrete-time gradient descent-ascent update to a set of the strict local minmax equilibrium. Moreover, we provide convergence rates for the gradient descent-ascent dynamics with timescale separation to approximate stationary points.

1 Introduction

We study continuous action zero-sum games of the form

$$\min_{x\in\mathcal{X}}\max_{y\in\mathcal{Y}} f(x,y)$$

where $f\in C^2(\mathcal{X}\times\mathcal{Y},\mathbb{R})$ and $\mathcal{X}=\mathbb{R}^{d_1}$ and $\mathcal{Y}=\mathbb{R}^{d_2}$ denote the individual action spaces and $d=d_1+d_2$. In particular, we focus on unconstrained, continuous strategy space zero-sum games in which $f(\cdot,y)$ is potentially nonconvex in $x\in\mathcal{X}$ and $f(x,\cdot)$ satisfies the Polyak-Łojasiewicz (PŁ) condition [47] or is strongly-concave (SC) in $y\in\mathcal{Y}$. We refer to these classes of games as nonconvex-PŁ and nonconvex-SC zero-sum games, respectively.

This general formulation has a broad spectrum of applications such as fair classification [45], distributionally robust optimization [44, 48, 53], and adversarial training [38]. Consequently, there has been a surge of interest in recent years toward developing methods for solving these problems efficiently. So far, existing work on gradient-based learning in nonconvex-PŁ/SC zero-sum games has exclusively focused on providing global convergence rates to approximate stationary points with no attention given to the characterization in terms of game-theoretic equilibrium concepts [33, 34, 36, 45, 48, 62].

In contrast, a common theme in the study of general nonconvex-nonconcave zero-sum games is to assess the *types* of stationary points an algorithm locally converges toward in terms of their higher

35th Conference on Neural Information Processing Systems (NeurIPS 2021).

order structure [10, 15, 16, 23, 40, 41, 58, 64]. The purpose of this analysis is generally to determine whether commonly deployed algorithms can guarantee local convergence to *only* game-theoretic equilibria or to design algorithms that achieve this objective.

The goal of this paper is to close the gap between the two problem classes and determine whether gradient-based learning algorithms in nonconvex-PŁ/SC zero-sum games can be shown to globally converge to *only* game-theoretically meaningful equilibria (local minmax/Stackelberg or Nash). We focus our attention on the canonical gradient descent-ascent learning dynamics with timescale separation between players. In this algorithm, timescale separation is manifested in the different (yet constant) learning rates of the maximizing and minimizing players. The description of this system, which we refer to as τ-GDA, where the ratio of learning rates denoted by $\tau > 0$ parameterizes the timescale separation between players, is provided in Algorithms 1–2 for deterministic and stochastic settings respectively. Simply put, τ-GDA (Algorithm 1) corresponds to each player following their individual gradient in a noiseless setting, while stochastic τ-GDA (Algorithm 2) describes each player following their individual stochastic gradient. The τ-GDA update only requires first-order gradients and thus is a computationally efficient method for machine learning problems formulated as games.

1.1 Contributions

We show that τ-GDA has *global* convergence guarantees in nonconvex-PŁ/SC zero-sum games to the natural game-theoretic solution concept for this problem class of strict local minmax equilibria.[1] The specific contributions of this work are now summarized.

1) In Theorem 1, we prove the only critical points that are locally stable with respect to the τ-GDA continuous-time limiting system are strict local minmax equilibrium in nonconvex-PŁ/SC zero-sum games. A key implication of this is that any critical point which is not a strict local minmax equilibrium is a saddle point of the continuous-time limiting system.

2) In Theorem 2, we combine Theorem 1 with a potential function construction (Lemma 1) and an asymptotic saddle avoidance result (Lemma 2) to prove that the τ-GDA update in deterministic settings (Algorithm 1) globally asymptotically converges to strict local minmax equilibria almost surely in the class of nonconvex-PŁ/SC zero-sum games.

3) In Corollary 1, using the potential function from Lemma 1, we show that τ-GDA in deterministic settings reaches an ε-critical point in $\widetilde{\mathcal{O}}(\varepsilon^{-2})$ steps in nonconvex-PŁ/SC zero-sum games. Moreover, specific to nonconvex-SC zero-sum games, Lemma 3 shows there exists learning rates for τ-GDA such that the cost function of the game itself can be made a potential. Corollary 2 then uses the potential function from Lemma 3 to show that τ-GDA reaches an ε-critical point in $\widetilde{\mathcal{O}}(\varepsilon^{-2})$ and $\widetilde{\mathcal{O}}(\varepsilon^{-6})$ steps in deterministic and stochastic problems, respectively.

Prior to moving on, we briefly comment on and provide context for each contribution of this paper. As we discuss later on in Remark 1, beyond its importance toward proving Theorem 2, Theorem 1 is potentially of independent interest given the implications it also has for the local stability of τ-GDA around critical points in the more general setting of nonconvex-nonconcave zero-sum games. Furthermore, to our knowledge, Theorem 2 provides the broadest existing global convergence guarantee for gradient-based algorithms to *game-theoretically meaningful* equilibria in zero-sum continuous games. Finally, while there exists known convergence rates for gradient-based learning algorithms to ε-critical points in nonconvex-PŁ/SC zero-sum games, we are unaware of such a result in nonconvex-PŁ zero-sum games for τ-GDA, and the fact that we derive a rate in nonconvex-SC zero-sum games using the cost function itself as a potential function has practical implications given that it can easily be monitored when running the algorithm to evaluate progress toward a solution.

1.2 Practical Motivation

The study of nonconvex-PŁ/SC zero-sum games has often been motivated by machine learning problems formulated as games. We remark that given the problem formulations, it is natural from both game-theoretic and machine learning perspectives to seek notions of minmax equilibria. Indeed, notions of stationarity are not guaranteed to reflect a meaningful solution to the underlying problem. Consequently, it is critical to give convergence guarantees to minmax equilibrium as we pursue in

[1] Strict local minmax equilibria characterized by gradient-based sufficient conditions are also known as differential Stackelberg equilibria in the literature [15, 16] and we use the terms interchangeably in this work.

this work. We now provide examples from the literature of machine learning applications that are relevant to the class of games studied in this paper. Note that the following application problems are illustrative in nature and do not immediately fall into the classes of games we study. However, as is elaborated on shortly, each of the problems can be transformed into unconstrained nonconvex-PŁ/SC zero-sum games while retaining the optimization objectives after simple, standard modifications.

Example 1. In fair classification [45] and learning from multiple distributions, the objective is to minimize the maximum loss over multiple categories. An example formulation is the problem

$$\min_{w\in\mathbb{R}^d} \max_{i\in\{1,\dots,K\}} \ell_i(w)$$

where $\ell_i(w)$ represents the loss on category i with w denoting neural network parameters. A reformulation of this problem [45] with $\mathcal{T}$ is the simplex in $\mathbb{R}^K$ is given by the zero-sum game

$$\min_{w\in\mathbb{R}^d} \max_{t\in\mathcal{T}} \sum_{i=1}^N t_i\ell_i(w). \tag{1}$$

Example 2. To train a neural network classifier robust against adversarial attacks, a common approach is to formulate training as a robust minmax optimization problem of the form

$$\min_{w\in\mathbb{R}^d} \sum_{i=1}^N \max_{\delta_i:|\delta_i|_\infty\leq\epsilon} \ell(x_i+\delta_i, y_i, w),$$

where $\ell(x_i+\delta_i, y_i, w)$ represents the loss on sample x_i perturbed by δ_i with budget ϵ and label y_i as a function of the parameter weights w [38]. A reformulation of this problem [45] is given by

$$\min_{w\in\mathbb{R}^d} \sum_{i=1}^N \max_{t\in\mathcal{T}} \sum_{j=1}^K t_j\ell(\widehat{x}_{ij}, y_i, w) \tag{2}$$

where $\widehat{x}_{ij}$ is the result of a targeted attack on the sample x_i seeking to change the output of the network to label j and $\mathcal{T}$ is the simplex in $\mathbb{R}^K$.

Example 3. Distributionally robust optimization often results in a zero-sum game of the form

$$\min_{w\in\mathbb{R}^d} \max_{t\in\mathcal{T}} \sum_{i=1}^N t_i\ell_i(w) - r(t) \tag{3}$$

where $\ell_i(x)$ is the loss of a model w on the i–th data point, $\mathcal{T}$ is the simplex in $\mathbb{R}^N$, and $r(t)$ is carefully selected convex regularizer [44, 48, 53].

The machine learning problem formulations in (1)–(3) from Examples 1–3 represent nonconvex-concave zero-sum games in which the strategy space of the minimizing player is unconstrained and the strategy space of the maximizing player is constrained to the simplex. These problems are naturally adapted to the unconstrained nonconvex-PŁ/SC zero-sum game setting considered in this paper by removing the constraint on the strategy space of the maximizing player and including a suitable PŁ/SC regularization penalty on the choice variable of the maximizing player.

1.3 Organization

The rest of the paper is organized as follows. Section 2 details related work and Section 3 presents game-theoretic, mathematical, and algorithmic preliminaries for our results. Section 4 is devoted to studying the local stability properties around critical points of the continuous-time limiting system for the τ-GDA learning dynamics. In Section 5, we present our study of the convergence properties of τ-GDA in nonconvex-PŁ/SC zero-sum games. We conclude with a discussion in Section 6. Finally, the supplementary material (appendix) contains the proofs of theoretical results.

2 Related Work

We now cover the most relevant related work with further discussion provided in Appendix A.

Nonconvex-Nonconcave Zero-Sum Games. A common theme in analyzing gradient descent-ascent with or without timescale separation in nonconvex-nonconcave zero-sum games has been to assess the local stability around critical points of the continuous-time limiting system and draw connections to local Nash and Stackelberg equilibrium notions characterized by gradient-based sufficient conditions (see Definition 4) [10, 15, 16, 23, 40, 42, 43, 63]. Importantly, it has been shown that unless the timescale separation is chosen very carefully, the stable critical points of gradient descent-ascent may not be game-theoretically meaningful and game-theoretically meaningful critical points may

Table 1: The gradient complexity of gradient descent (GD) and its perturbed variant (PGD), gradient descent-ascent with timescale separation (τ-GDA), and alternating (including multi-step) gradient descent-ascent (AGDA) in deterministic and stochastic nonconvex optimization, nonconvex-SC zero-sum games, and nonconvex-PŁ zero-sum games. We state the complexity in terms of the ε tolerance of the guarantee with the notation $\widetilde{\mathcal{O}}(\cdot)$ hiding logarithmic factors in ε.

Problem	Algorithm & Reference	Complexity		Guarantee
		Deterministic	Stochastic	
Nonconvex Optimization	GD [24]	$\widetilde{\mathcal{O}}(\varepsilon^{-2})$	$\widetilde{\mathcal{O}}(\varepsilon^{-4})$	ε–Stationarity
	PGD [24]	$\widetilde{\mathcal{O}}(\varepsilon^{-2})$	$\widetilde{\mathcal{O}}(\varepsilon^{-4})$	ε–Local Min
Nonconvex SC Zero-Sum	τ-GDA, τ-AGDA [33]	$\widetilde{\mathcal{O}}(\varepsilon^{-2})$	$\widetilde{\mathcal{O}}(\varepsilon^{-5})$	ε–Stationarity
	Theorem 2 (τ-GDA)	Asymptotic	–	**DSE**
	Corollary 1, **Corollary 2** (τ-GDA)	$\widetilde{\mathcal{O}}(\varepsilon^{-2})$	$\widetilde{\mathcal{O}}(\varepsilon^{-6})$	ε–Stationarity
Nonconvex PŁ Zero-Sum	AGDA [45],[62, Appendix D]	$\widetilde{\mathcal{O}}(\varepsilon^{-2})$	–	ε–Stationarity
	Theorem 2 (τ-GDA)	Asymptotic	–	**DSE**
	Corollary 1 (τ-GDA)	$\widetilde{\mathcal{O}}(\varepsilon^{-2})$	–	ε-Stationarity

not be stable [15, 23]. We obtain stronger stability characterizations (see Theorem 1) for τ-GDA in nonconvex-PŁ/SC zero-sum games using the structure imposed on the game cost function. The result of Theorem 1 also has novel implications for the local stability of τ-GDA around critical points in the more general setting of nonconvex-nonconcave zero-sum games (see Remark 1). Moreover, it is fundamental toward proving the global almost sure asymptotic convergence guarantee for τ-GDA in deterministic settings to strict local minmax equilibria in nonconvex-PŁ/SC zero-sum games (Theorem 2). Further discussion of this topic is given in Section 4.

Nonconvex-PŁ and Nonconvex-SC Zero-Sum Games. Table 1 provides a comprehensive comparison between our convergence results and existing convergence results for gradient descent variants in nonconvex optimization and nonconvex-PŁ/SC zero-sum games. We leave discussion of research on these classes of games with other algorithmic methods to Appendix A. The key distinction between this paper and past work is that instead of assessing convergence in terms of only reaching an approximate stationary point of the dynamics or a surrogate function, we obtain convergence guarantees in regards to differential Stackelberg equilibria (equivalently strict local minmax equilibria). Despite this being a much stricter and meaningful notion of solving the problem, Theorem 2 shows that global asymptotic convergence guarantees for τ-GDA in deterministic settings to this type of solution remain obtainable in nonconvex-PŁ/SC zero-sum games. This is the main result of the paper. While we do not prove a rate of convergence to differential Stackelberg equilibria or an approximate notion, Corollary 1 shows τ-GDA in deterministic settings with parameter choices satisfying the conditions of Theorem 2 reaches an ε-critical point in $\widetilde{\mathcal{O}}(\varepsilon^{-2})$ steps in nonconvex-PŁ/SC zero-sum games. We are unaware of an existing rate for τ-GDA in deterministic settings in nonconvex-PŁ zero-sum games. Finally, as a complementary result, Lemma 3 shows there exists learning rates for τ-GDA such that the cost function of the game itself can be made a potential in nonconvex-SC zero-sum games, which is a desirable property given that this potential function can be monitored while running the algorithm. Corollary 2 then uses this potential function to show that τ-GDA reaches an ε-critical point in $\widetilde{\mathcal{O}}(\varepsilon^{-2})$ and $\widetilde{\mathcal{O}}(\varepsilon^{-6})$ steps in deterministic and stochastic nonconvex-SC zero-sum games, respectively.

Escaping Saddle Points. Saddle avoidance results for variants of gradient descent in nonconvex optimization are asymptotic or finite-time. The former states that almost surely the algorithm does not converge to saddles points [28, 29], while the latter gives rates of escape to conclude convergence to approximate local minimum [17, 20, 24]. A primary assumption in the aforementioned works is what is known as the strict saddle property, which ensures directions of escape exists from a saddle point. The methods for showing gradient descent escapes saddle points almost surely in nonconvex optimization have more recently been extended to gradient descent-ascent in the setting of continuous games under an analogous strict saddle assumption [10, 40]. A key component of proving the global asymptotic convergence guarantee to strict local minmax for τ-GDA turns out to be ensuring that the update escapes saddle points of the continuous-time limiting system almost surely. To show this property holds for τ-GDA in nonconvex-PŁ/SC zero-sum games, we are able to invoke existing saddle avoidance results for gradient-based learning in continuous games [40].

3 Preliminaries

In the zero-sum games we study, we refer to the minimizing player controlling x as player 1 and the maximizing player controlling y as player 2. The set of players is denoted by $\mathcal{I} = \{1, 2\}$. We consider objective functions $f \in C^2(\mathcal{Z}, \mathbb{R})$ where the joint strategy space is denoted by $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$ where $\mathcal{X} = \mathbb{R}^{d_1}$ and $\mathcal{Y} = \mathbb{R}^{d_2}$ denote the individual action spaces and $d = d_1 + d_2$. We often denote a joint strategy using the shorthand notation $z = (x, y) \in \mathcal{Z} = \mathcal{X} \times \mathcal{Y}$.

Notation. We denote ∇f as the total derivative of f, $\nabla_i f$ as the derivative of f with respect to the choice variable of player i, $\nabla_{ij} f$ as the partial derivative of $\nabla_i f$ with respect to the choice variable of player j, and $\nabla_i^2 f$ as the partial derivative of $\nabla_i f$ with respect to the choice variable of player i. We let $\|\cdot\|$ denote the 2-norm of vectors unless otherwise specified, $\mathrm{spec}(\cdot)$ denote the set of eigenvalues of a matrix, $\mathrm{Re}(\cdot)$ denote the real part of a complex number, and $\mathbb{C}_-^\circ$ and $\mathbb{C}_+^\circ$ denote the open left-half and right-half complex plane, respectively. Let $\lambda_{\min}(A)$ denote the eigenvalue of A with the minimum real part, and $\lambda_{\max}(A)$ the eigenvalue of A with the maximum real part. Finally, we indicate A is positive and negative definite using the notation $A \succ 0$ and $A \prec 0$, respectively.

Classes of Games. We study and analyze both nonconvex-PŁ and nonconvex-SC zero-sum games. To begin, we state a standard smoothness assumption that is needed throughout.

Assumption 1. *Given a zero-sum game $(f, -f)$ defined by $f \in C^2(\mathcal{Z}, \mathbb{R})$, $\nabla_1 f(x, y)$ and $\nabla_2 f(x, y)$ are L_1 and L_2 Lipschitz, respectively. That is, $\forall\, x, x' \in \mathcal{X}, y, y' \in \mathcal{Y}$,*

$$\|\nabla_1 f(x, y) - \nabla_1 f(x', y')\| \leq L_1(\|x - x'\| + \|y - y'\|),$$
$$\|\nabla_2 f(x, y) - \nabla_2 f(x', y')\| \leq L_2(\|x - x'\| + \|y - y'\|).$$

This assumption immediately implies that the vector of individual gradients denoted by

$$g(x, y) := (\nabla_1 f(x, y), -\nabla_2 f(x, y))$$

is also Lipschitz with parameter $L = L_1 + L_2$.

We now define a nonconvex-PŁ zero-sum game [45]. This class of games allows for the objective to be nonconvex in $x \in \mathcal{X}$, but it needs to satisfy the Polyak-Łojasiewicz (PŁ) condition [47] in $y \in \mathcal{Y}$.

Definition 1 (Nonconvex-PŁ Game)**.** *Consider a zero-sum game $(f, -f)$ defined by $f \in C^2(\mathcal{Z}, \mathbb{R})$. The game is called nonconvex-PŁ if $f(x, \cdot)$ is μ-PŁ with respect to the argument $y \in \mathcal{Y}$. That is, for $\mu > 0$ and for all $z = (x, y) \in \mathcal{Z}$,*

$$\|\nabla_2 f(x, y)\|^2 \geq 2\mu(\max_{y' \in \mathcal{Y}} f(x, y') - f(x, y)),$$

where for any fixed $x \in \mathcal{X}$, $\max_{y' \in \mathcal{Y}} f(x, y')$ has a nonempty solution set and a finite optimal value.

In nonconvex-PŁ zero-sum games, the solution set of $\mathrm{argmax}_{y' \in \mathcal{Y}} f(x^*, y')$ for any fixed $x^* \in \mathcal{X}$ may not be a singleton and the maximizing player's objective may be nonconcave. However, by Definition 1 it follows immediately that any (x^*, y^*) such that $\|\nabla_2 f(x^*, y^*)\| = 0$ satisfies $f(x^*, y^*) = \max_{y' \in \mathcal{Y}} f(x^*, y')$ so any stationary point with respect to y is a global maximum.

The following provides a definition for what we call a nonconvex-SC zero-sum game. In this class of games, the function that defines the game may be nonconvex in $x \in \mathcal{X}$, but it must be SC in $y \in \mathcal{Y}$.

Definition 2 (Nonconvex-SC Game)**.** *Consider a zero-sum game $(f, -f)$ defined by $f \in C^2(\mathcal{Z}, \mathbb{R})$. The game is nonconvex-SC if $f(x, \cdot)$ is μ-SC with respect to the argument $y \in \mathcal{Y}$. That is, given any $x \in \mathcal{X}$ and for all $y, y' \in \mathcal{Y}$,*

$$f(x, y') \leq f(x, y) + \langle \nabla_2 f(x, y), y' - y \rangle - \frac{\mu}{2}\|y' - y\|_2^2.$$

It is important to note that nonconvex, μ-SC zero-sum games are nonconvex, μ-PŁ zero-sum games, but nonconvex-PŁ zero-sum games may not be nonconvex-SC zero-sum games, which follows from the relationship between PŁ and SC functions [25]. This is to say that the class of nonconvex-PŁ

Algorithm 1 τ-GDA

Input: $x_0 \in \mathbb{R}^{d_1}$, $y_0 \in \mathbb{R}^{d_2}$
for $k = 0, 1, \dots$ **do**
  $x_{k+1} \leftarrow x_k - \gamma \nabla_1 f(x_k, y_k)$
  $y_{k+1} \leftarrow y_k + \gamma\tau \nabla_2 f(x_k, y_k)$
end for

Algorithm 2 Stochastic τ-GDA

Input: $x_0 \in \mathbb{R}^{d_1}$, $y_0 \in \mathbb{R}^{d_2}$
for $k = 0, 1, \dots$ **do**
  $x_{k+1} \leftarrow x_k - \gamma g_1(x_k, y_k; \theta_{1,k})$
  $y_{k+1} \leftarrow y_k + \gamma\tau g_2(x_k, y_k; \theta_{2,k})$
end for

zero-sum games is a superset of the class of nonconvex-SC zero-sum games. Thus, results we state for nonconvex-PŁ zero-sum games immediately hold for nonconvex-SC zero-sum games.

Learning Dynamics. We study gradient descent-ascent with timescale separation. Let γ be the learning rate of player 1 and $\tau > 0$ be the timescale parameterization so that the learning rate of player 2 is given by $\tau\gamma$. The deterministic τ-GDA dynamics are presented in Algorithm 1 and the stochastic τ-GDA dynamics are given in Algorithm 2 (see Section 5.2 for the relevant notation). The key distinction between this study of gradient descent-ascent with timescale separation and past work is how we assess convergence as we now begin to formalize.

Stationarity Notions. We call *critical points* strategy profiles at which the individual gradient of each player is equal to zero; that is $(x, y) \in \mathcal{Z}$ such that $\nabla_1 f(x, y) = 0$ and $\nabla_2 f(x, y) = 0$. Critical points correspond to stationary points of the τ-GDA dynamics. The nonconvex-PŁ/SC zero-sum game literature has generally assessed convergence in terms of the complexity of finding an ε-*critical point* (see, e.g., [45, 62]). We formally define an ε-critical point now for later reference.

Definition 3 (ε-Critical Point.)**.** *The joint strategy $(x^*, y^*) \in \mathcal{Z}$ is an ε-critical point when the conditions $\|\nabla_1 f(x^*, y^*)\| \leq \varepsilon$ and $\|\nabla_2 f(x^*, y^*) \leq \varepsilon$ hold.*

A closely related and common notion of convergence in this body of work (see e.g., [33]) is that of finding an ε-critical point of the function $\max_{y \in \mathcal{Y}} f(\cdot, y)$. This criterion amounts to seeking to achieve the condition $\|\nabla_1 \max_{y \in \mathcal{Y}} f(x, y)\| \leq \varepsilon$.

In contrast, we assess convergence with connections to the equilibrium notions that are commonly studied in the nonconvex-nonconcave zero-sum game literature. Since either stationarity notion may lack any game-theoretic meaning, we consider a strictly harder notion of solving a game.

Equilibrium Notions. The typical solution concept in game theory when an implicit or explicit order of play is present in the structure of the game is the (local) Stackelberg (equivalently minmax in zero-sum games) equilibrium concept [6].[2] Informally, in nonconvex-PŁ/SC zero-sum games, a local minmax equilibrium corresponds to a strategy pair $(x^*, y^*) \in \mathcal{Z}$ such that x^* is a local minimum of the function $f(x, y_*(x))$ where $y_*(x) \in \arg\max_{y \in \mathcal{Y}} f(x, y)$ is a local maximum of the function $f(x^*, y)$. When the function f is bounded or when $f(\cdot, y)$ is bounded and $f(x, \cdot)$ is strongly concave, a minmax equilibrium is guaranteed to exist.

We characterize the local minmax (Stackelberg) equilibrium notion in terms of sufficient conditions on player costs as is typical in learning in games (see, e.g., [15, 16, 23, 40, 58, 64]). Toward presenting this definition, we denote by $J(x, y)$ the Jacobian of the individual gradient vector $g(x, y)$ given by

$$J(x, y) = \begin{bmatrix} \nabla_1^2 f(x, y) & \nabla_{12} f(x, y) \\ -\nabla_{12}^\top f(x, y) & -\nabla_2^2 f(x, y) \end{bmatrix}. \tag{4}$$

Let $\mathtt{S}_1(\cdot)$ denote the Schur complement of $(\cdot)$ with respect to the $d_2 \times d_2$ block in $(\cdot)$. The following definition is characterized by sufficient conditions for a local minmax equilibrium in zero-sum games.

Definition 4 (Differential Stackelberg/Strict Local Minmax Equilibrium [16, 23])**.** *The joint strategy $(x^*, y^*) \in \mathcal{Z}$ is a differential Stackelberg equilibrium when the conditions $\nabla_1 f(x^*, y^*) = 0$, $\nabla_2 f(x^*, y^*) = 0$ and $\mathtt{S}_1(J(x^*, y^*)) \succ 0$, $\nabla_2^2 f(x^*, y^*) \prec 0$ hold.*

We remark that in nonconvex-PŁ/SC zero-sum games, $\mathtt{S}_1(J(x^*, y^*))$ as in Definition 4 is well-defined at any critical point (x^*, y^*) since $\det(\nabla_2^2 f(x^*, y^*)) \neq 0$ (see Lemma 8 in Appendix B.3).

[2]By implicit order of play, we mean the min and max order are not interchangeable.

4 Local Stability Analysis

To characterize the convergence of τ-GDA, we begin by studying its continuous-time limiting system

$$\dot{z} = -\Lambda_\tau g(z) \tag{5}$$

where $\dot{z} = (\dot{x}, \dot{y})$, $\Lambda_\tau = \text{blockdiag}(I_{d_1}, \tau I_{d_2})$, and $g(z)$ is the vector of individual gradients. The Jacobian of this system is given by

$$J_\tau(z) = \Lambda_\tau J(z). \tag{6}$$

We analyze the stability of the continuous-time system around critical points $z^* = (x^*, y^*)$ as a function of the timescale separation τ using the Jacobian $J_\tau(z^*)$ in this section toward drawing conclusions about the stability and convergence of the discrete time system τ-GDA. A critical point is said to be locally (exponentially) stable when the spectrum of $-J_\tau(z^*)$ is in the open left-half complex plane $\mathbb{C}_-^\circ$ (cf. Theorem 3, Appendix B.2). Simply put, a critical point z^* is locally exponentially stable if and only if the real parts of the eigenvalues of $-J_\tau(z^*)$ are strictly negative. Throughout, we use the broader term "stable" to mean the following.

Definition 5 (Stability)**.** *A critical point $z^* = (x^*, y^*) \in \mathcal{Z}$ is locally exponentially stable for $\dot{z} = -\Lambda_\tau g(z)$ if and only if $\text{spec}(-J_\tau(z^*)) \subset \mathbb{C}_-^\circ$ ($\equiv$ $\text{spec}(J_\tau(z^*)) \subset \mathbb{C}_+^\circ$).*

Stability with respect to the continuous-time τ-GDA dynamics guarantees that the system asymptotically converges at an exponential rate to the critical point in a local neighborhood. Moreover, given a suitable choice of learning rates, equivalent insights hold for the discrete-time dynamics [8].

Stability in Nonconvex-Nonconcave Zero-Sum Games. Given the implications regarding convergence, a number of papers in the past several years study the stability of τ-GDA around critical points and the connections to differential Nash and Stackelberg equilibrium in zero-sum games [10, 15, 16, 23, 40]. However, this body of research focuses on general nonconvex-nonconcave zero-sum games. In general across the spectrum of nonconvex-nonconcave zero-sum games, the stable critical points of τ-GDA coincide with the set of differential Stackelberg equilibria only when the timescale separation $\tau \to \infty$ [23]. Given that such a choice of timescale separation requires the learning rate $\gamma \to 0$ in order to retain stability of the discrete-time system, it is not clear how to derive a practical algorithm from this insight.

Toward remedying this problem, Fiez and Ratliff [15] provide stability results in terms of the timescale separation concerning a given critical point, rather than across the space of nonconvex-nonconcave zero-sum games. In particular, they prove a stability and instability result as a function of the timescale separation in the τ-GDA dynamics. The stability results say that given a differential Stackelberg equilibrium z^*, there exists a finite $\tau^* \in (0, \infty)$ that can be constructed such that z^* is stable for all $\tau \in (\tau^*, \infty)$. On the other hand, the instability results says that given a critical point which is not a differential Stackelberg equilibrium, there exists a finite $\tau_0 \in (0, \infty)$ that can be constructed such that z^* is not stable for all $\tau \in (\tau_0, \infty)$.

Stability in Nonconvex-PŁ/SC Zero-Sum Games. To our knowledge, the connection between the stability (and instability) of critical points with respect to τ-GDA dynamics and game-theoretic equilibrium notions in the semi-structured problems of nonconvex-PŁ and nonconvex-SC zero-sum games has not been fully characterized. We show in the following result that when a nonconvex-nonconcave game is specialized to a nonconvex-PŁ zero-sum game as from Definition 1, significantly more general stability characterizations can be obtained by exploiting the fact that $\nabla_2^2 f(x^*, y^*) \prec 0$ at any critical point $z^* = (x^*, y^*)$ (see Lemma 8 in Appendix B.3). Notably, any critical point $z^* = (x^*, y^*)$ that is not a differential Stackelberg equilibrium is not stable ("unstable") for all $\tau > 0$. In other words, τ-GDA does not admit spurious stable points in nonconvex-PŁ zero-sum games. This is in stark contrast to the known fact that 1-GDA admits spurious stable points in the more general class of nonconvex-nonconcave games as has been shown in previous literature [10, 23, 40]. Moreover, if z^* is a differential Stackelberg equilibrium, then z^* is stable for all τ larger than the minimum τ_* for which z^* is stable and such a finite τ_* is guaranteed to exist. This result implies that in practice, one can select a finite value of τ to run τ-GDA with, and all stable critical points (if they exist) will be differential Stackelberg equilibria, and if τ is scaled up and the set of stable points grows, then only differential Stackelberg equilibria can be introduced to the set.

Theorem 1. *Consider a nonconvex-PŁ/SC zero-sum game $(f, -f)$ where $f \in C^2(\mathcal{Z}, \mathbb{R})$. Then, the following hold: 1) Any critical point $z^* \in \mathcal{Z}$ that is not a differential Stackelberg equilibrium is unstable with respect to τ-GDA for all $\tau \in (0, \infty)$; 2) If z^* is a differential Stackelberg equilibrium,* *then* $\mathrm{spec}(-J_\tau(z^*)) \subset \mathbb{C}_-^\circ$ *for all* $\tau \in [\tau_*, \infty)$ *where* τ_* *is the minimum* $\tau \in (0, \infty)$ *such that* $\mathrm{spec}(-J_\tau(z^*)) \subset \mathbb{C}_-^\circ$ *and a finite* τ_* *is guaranteed to exist.*

In comparison to the stability results for nonconvex-nonconcave zero-sum games of Fiez and Ratliff [15], we obtain the stronger results in nonconvex-PŁ zero-sum games that (i) $\tau_0 = 0$ for any critical point that is not a differential Stackelberg equilibrium and (ii) a differential Stackelberg equilibrium is never unstable after it becomes stable as a function of the timescale separation τ.

Remark 1. The proof of Theorem 1 also reveals novel insights into the local stability of τ-GDA around critical points in nonconvex-nonconcave zero-sum games. In particular, given any critical point $z^* = (x^*, y^*)$ in a nonconvex-nonconcave zero-sum game $(f, -f)$ where $f \in C^2(\mathcal{Z}, \mathbb{R})$ such that $\nabla_2^2 f(x^*, y^*) \prec 0$, if z^* is not a differential Stackelberg equilibrium, then it is unstable with respect to τ-GDA for all $\tau \in (0, \infty)$. This observation follows from the proof of Theorem 1 relying on the fact that $\nabla_2^2 f(x^*, y^*) \prec 0$ holds at any critical point in nonconvex-PŁ/SC zero-sum games.

5 Convergence Analysis

Theorem 1 in the previous section characterizes the local behavior of the continuous-time τ-GDA dynamics around critical points. However, ultimately we are most interested in the convergence properties of the discrete-time τ-GDA dynamics. This forms the focus of this section. Specifically, Section 5.1 is devoted to showing that τ-GDA in deterministic settings almost surely asymptotically converges to strict local minmax equilibria in nonconvex-PŁ/SC zero-sum games and Section 5.2 develops convergence rates to ε-critical points in these classes of games.

5.1 Asymptotic Convergence to Strict Local Minmax Equilibrium

We prove in Theorem 2 of this subsection that the deterministic τ-GDA (Algorithm 1) update almost surely converges to a differential Stackelberg equilibrium in nonconvex-PŁ/SC zero-sum games. Despite this result being asymptotic, to our knowledge it is the most general class of zero-sum games in which a global convergence guarantee to established game-theoretic equilibria has been given.

To begin, observe that the continuous-time τ-GDA dynamics do not necessarily correspond to a gradient flow of a function, which can be observed from the fact that the Jacobian J_τ is not guaranteed to be symmetric. As a consequence, there may exist complex limiting behavior beyond convergence to a critical point such as non-trivial limit cycles or periodic orbits. To rule out this phenomenon, we prove that there exists a bounded potential function that decreases along the iterates of τ-GDA and only stops decreasing at critical points of τ-GDA. In what follows, $y_*(x)$ denotes an element of $\mathrm{argmax}_{y \in \mathcal{Y}} f(x, y)$ and we define $\kappa := L_1/\mu$. Moreover, as an implication of Assumption 1 and Definition 1, $\nabla \max_{y \in \mathcal{Y}} f(\cdot, y)$ is $L_4 = (L_1 + \kappa L_2)$ Lipschitz (see Lemma 7 in Appendix B.3).

Lemma 1. *Consider a nonconvex-PŁ/SC zero-sum game defined by* $f \in C^2(\mathcal{Z}, \mathbb{R})$ *satisfying Assumption 1. For any* $\Gamma \in (0, 1/7]$, *suppose that* $\tau \geq \Gamma^{-1}\kappa^2$ *and* $\gamma < \min\{\frac{1}{2L}, \frac{1}{2\tau L}, \frac{1}{2L_4}\}$, *then* $\Phi(x, y) = f(x, y_*(x)) - \Gamma f(x, y)$ *is a potential function for* τ-GDA.

The function $f(x, y_*(x))$ can be seen as the function the x player would minimize if the y player was playing a best-response $y_*(x)$. The potential function $\Phi(x, y)$ essentially captures that along trajectories of τ-GDA, either the value of $f(x, y_*(x))$ should decrease, or the value of $f(x, y_*(x)) - f(x, y)$ should decrease since the y-player converges at a fast rate to $y_*(x)$ given the time-scale separation. Indeed, this potential function implicitly guarantees that the maximizing player *tracks* the best response set, and that the minimizing player essentially ends up minimizing $f(x, y_*(x))$ as desired. The choice of τ and the learning rate γ allow us to guarantee that this occurs.

Given the potential function in Lemma 1, we can conclude that Algorithm 1 converges to critical points. The critical points may correspond to stable points of the continuous-time τ-GDA dynamics (Definition 5) or saddle points of the continuous-time τ-GDA dynamics, which are defined as follows.

Definition 6 (Saddle Point)**.** *The critical point* $z^* = (x^*, y^*) \in \mathcal{Z}$ *is a saddle point of the* τ-GDA *continuous-time dynamics* $\dot{z} = -\Lambda_\tau g(z)$ *if* $\mathrm{Re}(\lambda_{\max}(-J_\tau(z^*))) \geq 0$ *and a strict saddle if* $\mathrm{Re}(\lambda_{\max}(-J_\tau(z^*))) > 0$ *and* $\mathrm{Re}(\lambda) \neq 0 \; \forall \; \lambda \in \mathrm{spec}(-J_\tau(z^*))$.

Recall that Theorem 1 indicates the only stable points of the continuous-time τ-GDA dynamics are differential Stackelberg equilibria. Thus, if we can show that the discrete-time τ-GDA dynamics avoid saddle points of the continuous-time τ-GDA dynamics, then we can conclude that Algorithm 1 converges to only differential Stackelberg equilibria. The following result of Mazumdar et al. [40] states that the discrete-time τ-GDA dynamics avoid strict saddle points of the continuous-time limiting system almost surely.[3]

Lemma 2 (Theorem 4.1 [40])**.** *Consider a zero-sum game defined by the nonconvex-nonconcave function $f \in C^2(\mathcal{Z}, \mathbb{R})$ satisfying Assumption 1. The set of initial conditions $z \in \mathcal{Z}$ from which τ-GDA with $\gamma < \min\{\frac{1}{L}, \frac{1}{\tau L}\}$ converges to strict saddle points of $\dot{z} = -\Lambda_\tau g(z)$ is of measure-zero.*

Lemma 2 can be combined with the existence of a potential function (Lemma 1) and the guarantee all stable points of the τ-GDA continuous-time system are strict local minmax (Theorem 1) to arrive at a desirable global convergence guarantee for τ-GDA when given the following assumption.

Assumption 2 (Strict Saddle Property)**.** *Given a zero-sum game $(f, -f)$ defined by $f \in C^2(\mathcal{Z}, \mathbb{R})$, we assume every saddle point of the continuous-time τ-GDA dynamics given by the system $\dot{z} = -\Lambda_\tau g(z)$ from* (5) *is a strict saddle point for the choice of $\tau \in (0, \infty)$.*

Assumption 2 can be seen as an analogue for gradient dynamics in nonconvex zero-sum games to the commonly assumed strict saddle property in nonconvex optimization [17, 20, 24]. A key distinction is that in nonconvex optimization, strict saddles of the function and strict saddles of the gradient dynamics are equivalent, whereas this property does not carry over to the zero-sum game setting and instead the assumption needs to be explicitly made on saddle points of the gradient dynamics. Given this assumption, we are now ready to the state main result of this paper.

Theorem 2. *Consider a nonconvex-PŁ/SC zero-sum game $(f, -f)$ defined by $f \in C^2(\mathcal{Z}, \mathbb{R})$ that satisfies Assumptions 1–2. Then, τ-GDA with $\tau \geq \Gamma^{-1}\kappa^2$, and $\gamma < \min\{\frac{1}{2L}, \frac{1}{2\tau L}, \frac{1}{2L_4}\}$ for $\Gamma \in (0, 1/7]$ asymptotically converges to the set of strict local minmax that are stable for $\dot{z} = -\Lambda_\tau g(z)$ almost surely. That is, for almost all initial conditions, τ-GDA will converge to a strict local minmax.*

If we fix a τ satisfying the assumptions of the above theorem, then if we consider the set of stable critical points for $\dot{z} = -\Lambda_\tau g(z)$, we know that by Theorem 1 this set only contains differential Stackelberg (and hence, strict local minmax) points. It is precisely this set of strict local minmax to which τ-GDA converges almost surely. Note that a stable differential Stackelberg equilibrium is ensured to exist for the choice of τ and γ. Moreover, by Theorem 1, as we increase τ, no new spurious non-minmax points are introduced to the set of stable critical points; only additional strict local minmax points can be added to this set. Thus, this result provides a novel convergence guarantee to not just ε-stationary points but those that are game theoretically meaningful.

5.2 Convergence Rates to Approximate Stationary Points

While Theorem 2 implies a global convergence guarantee to a strict local minmax equilibrium, a global convergence rate to the same set does not easily follow. This is due to the fact that obtaining such a result requires the dynamics to escape strict saddle points not just asymptotically, but efficiently. In lieu of pursuing such a result, in this subsection, we derive global complexity bounds for τ-GDA reaching an ε-critical point. To begin, we have the following convergence result for τ-GDA, which follows rather directly from the potential function given in Lemma 1.

Corollary 1. *Consider a nonconvex-PŁ/SC zero-sum game $(f, -f)$ defined by $f \in C^2(\mathcal{Z}, \mathbb{R})$ that satisfying Assumption 1. Then, τ-GDA from any initialization with $\tau \geq \Gamma^{-1}\kappa^2$, $\gamma < \min\{\frac{1}{2L}, \frac{1}{2\tau L}, \frac{1}{2L_4}\}$ for $\Gamma \in (0, 1/7]$, has at least one iterate that is an ε-critical point after $\widetilde{\mathcal{O}}(\varepsilon^{-2})$ iterations.*

We remark that we are unaware of an existing convergence result to a ε-critical point for τ-GDA in nonconvex-PŁ zero-sum games. Moreover, it is worth noting that this convergence guarantee holds for choices of τ and γ that satisfy the conditions of Theorem 2. Thus, together, we have shown that τ-GDA finds an approximate stationary point efficiently, and then asymptotically is guaranteed to converge to a strict local minmax equilibrium. It is also known that τ-GDA locally converges

[3]The strict saddle avoidance result of Mazumdar et al. [40, Theorem 4.1] holds more generally for simultaneous gradient descent with heterogeneous learning rates in n-player general-sum nonconvex games. For simplicity, we only state the result in the context of this work. We also remark that Daskalakis and Panageas [10, Theorem 2.2] show an analogous strict saddle avoidance result for τ-GDA in nonconvex-nonconcave zero-sum games, but restricted to $\tau = 1$, hence the reason it is not referenced in Lemma 2.

exponentially fast around stable strict local minmax equilibrium in nonconvex-nonconcave zero-sum games [15] and thus in this class of games as well.

We now restrict our focus to nonconvex-SC zero-sum games and provide a complimentary convergence analysis. In the existing literature on convergence in the nonconvex-SC literature [35, 59, 62], the proposed potential function has a structure closely related to the potential function in Lemma 1; in particular, it takes the form $f(x,y)+r(x,y)$ where $r(x,y)$ is a tracking term of the form $\|y-y_*(x)\|^2$ or $f(x,y_*(x))$. Yet, perhaps surprisingly, despite the fact that the game cost depends on the sequences generated by two separate gradient updates—one from minimizing f and one from maximizing it—we show below that there are choices of τ and γ for τ-GDA such that it is decreasing along trajectories of τ-GDA. The benefit of knowing that the cost function is a potential is that it is a computable quantity that can be monitored to evaluate progress during training in machine learning problems formulated as zero-sum games in this class. For this set of results, we consider both the deterministic and stochastic settings. In the stochastic setting, for any (x,y), a gradient query to an oracle returns $g_1(x,y;\theta_1)$ and $g_2(x,y;\theta_2)$ to the two players respectively, where each θ_i is a random variable drawn from distribution $\mathcal{D}_i$ with $i\in\mathcal{I}$ and the stochastic gradients satisfy the following assumptions.[4]

Assumption 3. *For any $(x,y)\in\mathcal{Z}$ and each $i\in\mathcal{I}$, $\mathbb{E}_{\theta_i\sim\mathcal{D}_i}[g_i(x,y;\theta_i)]=\nabla_i f(x,y)$ and $\forall\, t\in\mathbb{R}$:*

$$\mathbb{P}(\|g_i(x,y;\theta)-\nabla_i f(x,y)\|\geq t)\leq 2\exp(-\tfrac{t^2}{2\sigma_i^2}).$$

Given this assumption and defining $\ell := L_1+3\kappa' L_2$ and $\kappa' := L_2/\mu$, we have the following result.

Lemma 3. *Consider a nonconvex-SC zero-sum game $(f,-f)$ defined by $f\in C^2(\mathcal{Z},\mathbb{R})$ satisfying Assumption 1 and Assumption 3. If $\gamma\leq 1/\ell$ and $\tau=\Gamma/(\mu\gamma)$ for any constant $\Gamma\geq 4$, then with probability at least $1-\delta$ for any $\delta\in(0,1)$, $f(x_k,y_k)$ is a (stochastic) potential function for τ-GDA in the stochastic and deterministic settings.*

This result then leads to the following convergence guarantee to ε-critical points, given that there is a finite minimum value f^* of the game cost function.

Corollary 2. *Consider a nonconvex-SC zero-sum game $(f,-f)$ defined by $f\in C^2(\mathcal{Z},\mathbb{R})$ satisfying Assumption 1 and Assumption 3. For any $\delta\in(0,1)$, there exists γ and τ satisying the conditions of Lemma 3 such that, with probability at least $1-\delta$, starting from any initialization, at least half the iterates of (stochastic) τ-GDA will be ε-critical points after $\widetilde{\mathcal{O}}(\varepsilon^{-2})$ and $\widetilde{\mathcal{O}}(\varepsilon^{-6})$ iterations in the deterministic and stochastic settings, respectively.*

6 Discussion

To the best of our knowledge, our results are the first to guarantee global convergence of gradient-based algorithms to game theoretically meaningful equilibria in nonconvex-PŁ/SC zero-sum games. We believe an interesting direction of future work in these classes of games would be to obtain an equivalent guarantee in the setting with constraints. Moreover, in the unconstrained setting, it would be desirable to show not only does a τ-GDA variant provably converge in an almost sure asymptotic sense to strict local minmax equilibria, but that there is also finite-time guarantee (preferably matching the rates from nonconvex optimization) to an appropriate approximate notion. The difficulty in showing this stems from possibility that the game dynamics can get stuck in a region around a non-game theoretically meaningful strict saddle. This problem has been observed in the nonconvex single player setting [12] with gradient descent and trivially extends to τ-GDA in the zero-sum game setting. A body of work in nonconvex optimization has developed algorithmic methods that address this issue [9, 14, 17, 20, 24] including by injecting artificial noise to perturb the updates away from strict saddle points. Yet, to our knowledge, such methods have not yet been extended to the zero-sum game setting. Notably, escaping strict saddles is more of nuanced issue in this setting since strict saddles of the dynamics are not always strict saddles of the function and vice versa.

Acknowledgments and Disclosure of Funding

This work was supported by Office of Naval Research Young Investigator Program and NSF CAREER Award 1844729. Tanner Fiez was also supported in part by a NDSEG Fellowship.

[4]Note that we are overloading notation by using $g_1(x,y;\theta_1)$ and $g_2(x,y;\theta_2)$ to denote stochastic individual gradients for each player, whereas previously $g(x,y)$ has denoted the vector of each players gradients.

References

[1] Leonard Adolphs, Hadi Daneshmand, Aurelien Lucchi, and Thomas Hofmann. Local saddle point optimization: A curvature exploitation approach. In *International Conference on Artificial Intelligence and Statistics*, pages 486–495, 2019.

[2] Naman Agarwal, Zeyuan Allen-Zhu, Brian Bullins, Elad Hazan, and Tengyu Ma. Finding approximate local minima faster than gradient descent. In *ACM SIGACT Symposium on Theory of Computing*, pages 1195–1199, 2017.

[3] Zeyuan Allen-Zhu. Natasha 2: faster non-convex optimization than sgd. In *Advances in Neural Information Processing Systems*, pages 2680–2691, 2018.

[4] Zeyuan Allen-Zhu and Yuanzhi Li. Neon2: Finding local minima via first-order oracles. In *Advances in Neural Information Processing Systems*, pages 3716–3726, 2018.

[5] Mihai Anitescu. Degenerate nonlinear programming with a quadratic growth condition. *SIAM Journal on Optimization*, 10(4):1116–1135, 2000.

[6] Tamer Başar and Geert Jan Olsder. *Dynamic noncooperative game theory*. SIAM, 1998.

[7] Yair Carmon, John C Duchi, Oliver Hinder, and Aaron Sidford. Accelerated methods for nonconvex optimization. *SIAM Journal on Optimization*, 28(2):1751–1772, 2018.

[8] Benjamin Chasnov, Lillian Ratliff, Eric Mazumdar, and Samuel Burden. Convergence analysis of gradient-based learning in continuous games. In *Conference on Uncertainty in Artificial Intelligence*, pages 935–944, 2020.

[9] Hadi Daneshmand, Jonas Kohler, Aurelien Lucchi, and Thomas Hofmann. Escaping saddles with stochastic gradients. In *International Conference on Machine Learning*, pages 1155–1164, 2018.

[10] Constantinos Daskalakis and Ioannis Panageas. The limit points of (optimistic) gradient descent in min-max optimization. In *Advances in Neural Information Processing Systems*, pages 9236–9246, 2018.

[11] Constantinos Daskalakis, Stratis Skoulakis, and Manolis Zampetakis. The complexity of constrained min-max optimization. In *ACM SIGACT Symposium on Theory of Computing*, pages 1466–1478, 2021.

[12] Simon S Du, Chi Jin, Jason D Lee, Michael I Jordan, Barnabás Póczos, and Aarti Singh. Gradient descent can take exponential time to escape saddle points. *Advances in Neural Information Processing Systems*, pages 1068–1078, 2017.

[13] Cong Fang, Chris Junchi Li, Zhouchen Lin, and Tong Zhang. Spider: Near-optimal non-convex optimization via stochastic path integrated differential estimator. In *Advances in Neural Information Processing Systems*, pages 687–697, 2018.

[14] Cong Fang, Zhouchen Lin, and Tong Zhang. Sharp analysis for nonconvex sgd escaping from saddle points. In *Conference on Learning Theory*, pages 1192–1234, 2019.

[15] Tanner Fiez and Lillian Ratliff. Local convergence analysis of gradient descent ascent with finite timescale separation. In *International Conference on Learning Representations*, 2021.

[16] Tanner Fiez, Benjamin Chasnov, and Lillian Ratliff. Implicit learning dynamics in stackelberg games: Equilibria characterization, convergence analysis, and empirical study. In *International Conference on Machine Learning*, pages 3133–3144, 2020.

[17] Rong Ge, Furong Huang, Chi Jin, and Yang Yuan. Escaping from saddle points—online stochastic gradient for tensor decomposition. In *Conference on Learning Theory*, pages 797–842, 2015.

[18] Roger A Horn and Charles R Johnson. *Matrix analysis*. Cambridge university press, 2012.

[19] Ya-Ping Hsieh, Panayotis Mertikopoulos, and Volkan Cevher. The limits of min-max optimization algorithms: Convergence to spurious non-critical sets. In *International Conference on Machine Learning*, pages 4337–4348, 2021.

[20] Chi Jin, Rong Ge, Praneeth Netrapalli, Sham M Kakade, and Michael I Jordan. How to escape saddle points efficiently. In *International Conference on Machine Learning*, pages 1724–1732, 2017.

[21] Chi Jin, Praneeth Netrapalli, and Michael I Jordan. Accelerated gradient descent escapes saddle points faster than gradient descent. In *Conference On Learning Theory*, pages 1042–1085, 2018.

[22] Chi Jin, Praneeth Netrapalli, Rong Ge, Sham M Kakade, and Michael I Jordan. A short note on concentration inequalities for random vectors with subgaussian norm. *arXiv:1902.03736*, 2019.

[23] Chi Jin, Praneeth Netrapalli, and Michael Jordan. What is local optimality in nonconvex-nonconcave minimax optimization? In *International Conference on Machine Learning*, pages 4880–4889, 2020.

[24] Chi Jin, Praneeth Netrapalli, Rong Ge, Sham M Kakade, and Michael I Jordan. On nonconvex optimization for machine learning: Gradients, stochasticity, and saddle points. *Journal of the ACM*, 68(2):1–29, 2021.

[25] Hamed Karimi, Julie Nutini, and Mark Schmidt. Linear convergence of gradient and proximal-gradient methods under the Polyak-Łojasiewicz condition. In *Machine Learning and Knowledge Discovery in Databases*, pages 795–811. Springer International Publishing, 2016.

[26] Hassan Khalil. *Nonlinear Systems*. Prentice Hall, 3rd edition, 2002.

[27] Peter Lancaster and Miron Tismenetsky. *The theory of matrices: with applications*. Elsevier, 1985.

[28] Jason D Lee, Max Simchowitz, Michael I Jordan, and Benjamin Recht. Gradient descent only converges to minimizers. In *Conference on Learning Theory*, pages 1246–1257, 2016.

[29] Jason D. Lee, Ioannis Panageas, Georgios Piliouras, Max Simchowitz, Michael I. Jordan, and Benjamin Recht. First-order methods almost always avoid strict saddle points. *Mathematical Programming*, 176(1):311–337, 2019.

[30] Alistair Letcher. On the impossibility of global convergence in multi-loss optimization. *International Conference on Learning Representations*, 2021.

[31] Kfir Y Levy. The power of normalization: Faster evasion of saddle points. *arXiv:1611.04831*, 2016.

[32] Haochuan Li, Yi Tian, Jingzhao Zhang, and Ali Jadbabaie. Complexity lower bounds for nonconvex-strongly-concave min-max optimization. *arXiv:2104.08708*, 2021.

[33] Tianyi Lin, Chi Jin, and Michael Jordan. On gradient descent ascent for nonconvex-concave minimax problems. In *International Conference on Machine Learning*, pages 6083–6093, 2020.

[34] Tianyi Lin, Chi Jin, Michael Jordan, et al. Near-optimal algorithms for minimax optimization. In *Conference on Learning Theory*, pages 2738–2779, 2020.

[35] Sijia Liu, Songtao Lu, Xiangyi Chen, Yao Feng, Kaidi Xu, Abdullah Al-Dujaili, Mingyi Hong, and Una-May O'Reilly. Min-max optimization without gradients: Convergence and applications to black-box evasion and poisoning attacks. In *International Conference on Machine Learning*, pages 6282–6293, 2020.

[36] Songtao Lu, Ioannis Tsaknakis, Mingyi Hong, and Yongxin Chen. Hybrid block successive approximation for one-sided non-convex min-max problems: algorithms and applications. *IEEE Transactions on Signal Processing*, 68:3676–3691, 2020.

[37] Zhi-Quan Luo and Paul Tseng. Error bounds and convergence analysis of feasible descent methods: a general approach. *Annals of Operations Research*, 46(1):157–178, 1993.

[38] Aleksander Madry, Aleksandar Makelov, Ludwig Schmidt, Dimitris Tsipras, and Adrian Vladu. Towards deep learning models resistant to adversarial attacks. *International Conference on Learning Representation*, 2018.

[39] Eric Mazumdar and Lillian J Ratliff. Local Nash Equilibria are Isolated, Strict Local Nash Equilibria in 'Almost All' Zero-Sum Continuous Games. In *IEEE Conference on Decision and Control*, pages 6899–6904, 2019.

[40] Eric Mazumdar, Lillian J Ratliff, and S Shankar Sastry. On gradient-based learning in continuous games. *SIAM Journal on Mathematics of Data Science*, 2(1):103–131, 2020.

[41] Eric V Mazumdar, Michael I Jordan, and S Shankar Sastry. On finding local nash equilibria (and only local nash equilibria) in zero-sum games. *arXiv:1901.00838*, 2019.

[42] Lars Mescheder, Andreas Geiger, and Sebastian Nowozin. Which training methods for gans do actually converge? In *International Conference on Machine Learning*, pages 3481–3490, 2018.

[43] Vaishnavh Nagarajan and J Zico Kolter. Gradient descent gan optimization is locally stable. In *Advances in Neural Information Processing Systems*, pages 5591–5600, 2017.

[44] Hongseok Namkoong and John C Duchi. Stochastic gradient methods for distributionally robust optimization with f-divergences. In *Advances in Neural Information Processing Systems*, pages 2208–2216, 2016.

[45] Maher Nouiehed, Maziar Sanjabi, Tianjian Huang, Jason D Lee, and Meisam Razaviyayn. Solving a class of non-convex min-max games using iterative first order methods. In *Advances in Neural Information Processing Systems*, pages 14934–14942, 2019.

[46] Ioannis Panageas and Georgios Piliouras. Gradient descent only converges to minimizers: Non-isolated critical points and invariant regions. In *Innovations in Theoretical Computer Science Conference*, 2017.

[47] Boris Teodorovich Polyak. Gradient methods for minimizing functionals. *Zhurnal vychislitel'noi matematiki i matematicheskoi fiziki*, 3(4):643–653, 1963.

[48] Hassan Rafique, Mingrui Liu, Qihang Lin, and Tianbao Yang. Weakly-convex–concave min–max optimization: provable algorithms and applications in machine learning. *Optimization Methods and Software*, pages 1–35, 2021.

[49] Lillian J Ratliff, Samuel A Burden, and S Shankar Sastry. Characterization and computation of local Nash equilibria in continuous games. In *Allerton Conference on Communication, Control, and Computing*, pages 917–924, 2013.

[50] Lillian J Ratliff, Samuel A Burden, and S Shankar Sastry. Genericity and structural stability of non-degenerate differential Nash equilibria. In *American Control Conference*, pages 3990–3995, 2014.

[51] Lillian J Ratliff, Samuel A Burden, and S Shankar Sastry. On the characterization of local Nash equilibria in continuous games. *IEEE Transactions on Automatic Control*, 61(8):2301–2307, 2016.

[52] S. Shankar Sastry. *Nonlinear Systems Theory*. Springer-Verlag, 1999.

[53] Aman Sinha, Hongseok Namkoong, and John Duchi. Certifying some distributional robustness with principled adversarial training. In *International Conference on Learning Representations*, 2018.

[54] Matthew Staib, Sashank Reddi, Satyen Kale, Sanjiv Kumar, and Suvrit Sra. Escaping saddle points with adaptive gradient methods. In *International Conference on Machine Learning*, pages 5956–5965, 2019.

[55] Gerald Teschl. *Ordinary differential equations and dynamical systems*, volume 140. American Mathematical Society, 2000.

[56] Nilesh Tripuraneni, Mitchell Stern, Chi Jin, Jeffrey Regier, and Michael I Jordan. Stochastic cubic regularization for fast nonconvex optimization. *Advances in Neural Information Processing Systems*, 2018:2899–2908, 2018.

[57] Jun-Kun Wang, Chi-Heng Lin, and Jacob Abernethy. Escaping saddle points faster with stochastic momentum. In *International Conference on Learning Representations*, 2019.

[58] Yuanhao Wang, Guodong Zhang, and Jimmy Ba. On solving minimax optimization locally: A follow-the-ridge approach. In *International Conference on Learning Representations*, 2020.

[59] Zhongruo Wang, Krishnakumar Balasubramanian, Shiqian Ma, and Meisam Razaviyayn. Zeroth-order algorithms for nonconvex minimax problems with improved complexities. *arXiv:2001.07819*, 2020.

[60] Tengyu Xu, Zhe Wang, Yingbin Liang, and H Vincent Poor. Gradient free minimax optimization: Variance reduction and faster convergence. *arXiv:2006.09361*, 2020.

[61] Yi Xu, Rong Jin, and Tianbao Yang. First-order stochastic algorithms for escaping from saddle points in almost linear time. In *Advances in Neural Information Processing Systems*, pages 5530–5540, 2018.

[62] Junchi Yang, Negar Kiyavash, and Niao He. Global convergence and variance reduction for a class of nonconvex-nonconcave minimax problems. *Advances in Neural Information Processing Systems*, 2020.

[63] Guojun Zhang, Pascal Poupart, and Yaoliang Yu. Optimality and stability in non-convex-non-concave min-max optimization. *arXiv:2002.11875*, 2020.

[64] Guojun Zhang, Kaiwen Wu, Pascal Poupart, and Yaoliang Yu. Newton-type methods for minimax optimization. *arXiv:2006.14592*, 2020.

[65] Siqi Zhang, Junchi Yang, Cristóbal Guzmán, Negar Kiyavash, and Niao He. The complexity of nonconvex-strongly-concave minimax optimization. In *Conference on Uncertainty in Artificial Intelligence*, 2021.

[66] Dongruo Zhou, Pan Xu, and Quanquan Gu. Stochastic nested variance reduction for nonconvex optimization. *Journal of machine learning research*, 21:1–63, 2020.

Checklist

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes]
   (b) Did you describe the limitations of your work? [Yes] See conclusion of paper.
   (c) Did you discuss any potential negative societal impacts of your work? [No]
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]
2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes] See the stated assumptions in the paper and the appendix.
   (b) Did you include complete proofs of all theoretical results? [Yes] See the appendix.
3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [N/A]
   (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [N/A]
   (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [N/A]

(d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [N/A]

4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...
   (a) If your work uses existing assets, did you cite the creators? [N/A]
   (b) Did you mention the license of the assets? [N/A]
   (c) Did you include any new assets either in the supplemental material or as a URL? [N/A]
   (d) Did you discuss whether and how consent was obtained from people whose data you're using/curating? [N/A]
   (e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [N/A]
5. If you used crowdsourcing or conducted research with human subjects...
   (a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]
   (b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]
   (c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]

Appendix

Below we provide a table of contents as a guide to the appendix.

Table of Contents

A Related Work

We now cover related work in further detail. We begin by discussing past work on nonconvex-nonconcave zero-sum games with a focus on papers that analyze the local stability of critical points using the continuous-time limiting system as we do in Section 4 for nonconvex-PŁ/SC zero-sum games. Then, we compare our convergence results in nonconvex-PŁ/SC zero-sum games presented in Section 5 to existing guarantees in the literature for various gradient-based learning algorithms. We remark that there are a number of papers analyzing gradient-based learning in strongly-convex-strongly-concave, strongly-convex-linear, strongly-convex-concave, convex-concave, and nonconvex-concave zero-sum games. Since each of the aforementioned problems are fundamentally different in terms of the structure compared to nonconvex-PŁ/SC zero-sum games, we do not discuss each body of work in detail and instead refer the reader to the detailed discussion of work on such problems provided in the paper of Lin et al. [34]. Finally, we review techniques and results for escaping saddle points in nonconvex optimization relevant to our analogous methods and guarantees in classes of nonconvex games.

Nonconvex-Nonconcave Zero-Sum Games. In general nonconvex-nonconcave zero-sum games, global convergence guarantees to traditional game-theoretic equilibrium notions are effectively unobtainable as a result of computational hardness results [11] and the emergence of non-trivial limit cycles and periodic orbits [19, 30]. Consequently, the analysis of gradient-based learning algorithms in nonconvex-nonconcave zero-sum games has commonly focused on determining the local stability of the continuous-time limiting system around critical points. Typically, the goal is to determine the relationship between the set of stable critical points and the critical points that correspond to game-theoretic equilibrium.

The prototypical equilibrium notions in game theory are the (local) Nash and Stackelberg equilibrium concepts [6]. The aforementioned equilibrium notions have been characterized in terms of gradient-based sufficient conditions and critical points satisfying the conditions have sometimes been termed differential Nash (strict local Nash) and differential Stackelberg (strict local Stackelberg) equilibria. The differential characterization of local Nash equilibria was reported by Ratliff et al. [49, 51], while the differential characterization of local Stackelberg equilibria was given concurrently by Fiez et al. [16] and Jin et al. [23] where the conditions presented by the former extend to general-sum games. Moreover, Fiez et al. [16] and Jin et al. [23] show that differential Nash equilibria are a subset of differential Stackelberg equilibria. A series of works analyze the properties of the equilibrium notions defined in terms of gradient-based sufficient conditions. In particular, Ratliff et al. [50] and Mazumdar and Ratliff [39] prove the genericity and structural stability of differential Nash equilibria in general-sum and zero-sum games, respectively. Analogously, Fiez et al. [16] prove the genericity and structural stability of differential Stackelberg equilibria in zero-sum games.

Several past works study the local stability of τ-GDA without timescale separation (equal learning rates, $\tau = 1$) around critical points [10, 40]. In this direction, each of the aforementioned works prove there can exist stable critical points of τ-GDA with $\tau = 1$ that are not differential Nash equilibria. Moreover, they each show that differential Nash equilibria are stable critical points of τ-GDA with $\tau = 1$.

Building on this line of work, the local stability of τ-GDA with timescale separation around critical points has been analyzed [15, 23]. We begin by commenting on the results of Jin et al. [23]. It is shown by the authors that differential Nash equilibria are stable critical points of τ-GDA with any timescale separation parameter $\tau > 0$. Moreover, they prove that given any fixed timescale separation $\tau > 0$, there exists a game with a differential Stackelberg equilibrium that is not stable with respect to the τ-GDA system. Finally, they show that as $\tau \to \infty$, the stable critical points of τ-GDA coincide with the set of differential Stackelberg equilibria. Fiez and Ratliff [15] provide complimentary results by focusing on the stability properties of any given critical point as a function of the timescale separation τ, rather than considering stability properties across the space of nonconvex-nonconcave zero-sum games. In particular, they show a stability and instability result. The stability result says that given a differential Stackelberg equilibrium (x^*, y^*), there exists a finite $\tau^* \in (0, \infty)$ that can be constructed such that (x^*, y^*) is stable for all $\tau \in (\tau^*, \infty)$ with respect to the τ-GDA dynamics. The instability results says that given a critical point (x^*, y^*) which is not a differential Stackelberg equilibrium, there exists a finite $\tau_0 \in (0, \infty)$ that can be constructed such that (x^*, y^*) is not stable for all $\tau \in (\tau_0, \infty)$ with respect to the τ-GDA dynamics. It is important to observe that the constructions of a τ^* or τ_0 depend on the properties of a given critical point.

The results we provide regarding the local stability of τ-GDA around critical points in nonconvex-PŁ/SC zero-sum games in Section 4 strengthen the stability characterizations of Fiez and Ratliff [15] by exploiting the structure of these classes of games. In comparison, we obtain the stronger results in nonconvex-PŁ zero-sum games that (i) $\tau_0 = 0$ for any critical point that is not a differential Stackelberg equilibria and (ii) a differential Stackelberg equilibria is never unstable after it becomes stable as a function of τ. That is, we obtain insights that apply to the class of games rather than specific critical points, which are easily quantifiable instead of depending on properties of a given critical point.

We remark that another line of work studies the local stability of τ-GDA without timescale separation ($\tau = 1$) in generative adversarial networks under certain assumptions [42, 43]. The results are generalized to the study of τ-GDA with timescale separation [15] to show that equivalent conclusions hold for any choice of timescale separation $\tau > 0$.

Finally, there is a line of work developing gradient-based learning algorithms using second order information in nonconvex-nonconcave zero-sum games such that the only locally stable critical points correspond to game-theoretic equilibrium notions. In particular, Adolphs et al. [1] and Mazumdar et al. [41] develop algorithms under which a critical point is locally stable if and only if it is a differential Nash equilibrium. Moreover, Fiez et al. [16], Wang et al. [58], and Zhang et al. [64] construct algorithms under which a critical point is locally stable if and only if it is a differential Stackelberg equilibria.

Nonconvex-PŁ and Nonconvex-SC Zero-Sum Games. We now expand our discussion of existing work on nonconvex-PŁ and nonconvex-SC zero-sum games. As discussed in Section 2, in comparison to past works in this area, the key distinction of this work is that we give global convergence guarantees to game-theoretic equilibria, whereas existing work only considers convergence to notions of stationarity.

In the class of nonconvex-PŁ zero-sum games, the closest works to this paper are [45] and [62]. Each of the aforementioned works show that an ε-approximate stationary point of the game function $f(\cdot, \cdot)$ can be found in $\widetilde{\mathcal{O}}(\varepsilon^{-2})$ gradient calls by variants of alternating gradient descent-ascent with timescale separation (τ-AGDA) in the deterministic setting. In comparison, we show in Theorem 2 that simultaneous gradient descent-ascent with timescale separation (τ-GDA) globally asymptotically converges to a differential Stackelberg equilibrium (local minmax equilibrium) almost surely. Moreover, we also provide a global convergence guarantee to an ε-critical point in $\widetilde{\mathcal{O}}(\varepsilon^{-2})$ gradient calls in the deterministic setting. We remark that the primary focus of [62] is on two-sided PŁ games, which is a class with a unique equilibrium and the convergence guarantees given for this class are to the equilibrium.

In the class of nonconvex-SC zero-sum games, there is an extensive amount of recent work. In terms of existing results on variants of gradient descent-ascent, Lin et al. [33] shows for both simultaneous and alternating gradient descent-ascent (τ-GDA and τ-AGDA) that with $\widetilde{\mathcal{O}}(\varepsilon^{-2})$ and $\widetilde{\mathcal{O}}(\varepsilon^{-5})$ gradient calls an ε–approximate stationary point of the function $f(\cdot,\cdot)$ or the function $\phi(\cdot) = \max_{y\in\mathcal{Y}} f(\cdot, y)$ can be obtained in the deterministic and stochastic settings, respectively. For this class of games, we show in Theorem 2 that τ-GDA globally asymptotically converges to a differential Stackelberg equilibrium (local minmax equilibrium) almost surely. Obtaining this result requires both fully characterizing the local stability around critical points (Theorem 1) and then combining asymptotic saddle escape results with a potential function. Moreover, we also show that there exists τ and γ such that the game cost function is a potential function and using this derive convergence rates to ε-critical points of $\widetilde{\mathcal{O}}(\varepsilon^{-2})$ and $\widetilde{\mathcal{O}}(\varepsilon^{-6})$ in deterministic and stochastic settings, respectively.

Before moving on, we remark that there is a number of works that develop algorithms that improve the complexity of finding stationary points in terms of the dependence on the condition number and polylogarithmic dependencies. This focus is separate from our work, but we refer the reader to [34, 36, 48] and the references therein. We believe an interesting direction of future work is strengthening the results along this direction to the stronger notions of solving the problem considered in this work. Furthermore, there is also recent work developing lower bounds for this class of games [32, 65]. Finally, there are several works in this class of games with zeroth order feedback [35, 59, 60].

Escaping Saddle Points. In this paper we present asymptotic convergence results to differential Stackelberg equilibrium in nonconvex-PŁ and nonconvex-SC zero-sum games. To obtain such results it is necessary to escape saddle points asymptotically. We build on methods from nonconvex-nonconcave games and nonconvex optimization to prove that τ-GDA has this guarantee. In what follows, we provide background on these methods.

A common assumption in the body of work on escaping saddle points in nonconvex optimization is what is known as the strict saddle property (see, e.g., [17, 20, 28]). Informally, if a function class satisfies the strict saddle property, every saddle point of the gradient descent dynamics has a strictly negative eigenvalue in the Jacobian evaluated at the critical point. This assumption ensures that there is a direction of escape from every saddle point. We make an analogous assumption adapted to the classes of nonconvex zero-sum games we analyze (Assumption 2) following the definition from the work of Mazumdar et al. [40]. For nonconvex-PŁ/SC zero-sum games, if the game satisfies the strict saddle property, every saddle point of the τ-GDA dynamics has an eigenvalue with strictly negative real part and no eigenvalue has a real part equal to zero in the Jacobian $J_\tau(x^*, y^*)$ evaluated at the critical point for the given choice of timescale separation τ. We remark that there is some distinction between the definition of a strict saddle between nonconvex optimization and nonconvex games. This stems from the fact that in nonconvex optimization, saddles of the function and saddles of the dynamics are equivalent, whereas this does not carry over to the zero-sum game setting. In this paper, the terminology of a saddle point is with respect to the dynamics.

There exists both asymptotic and finite-time saddle avoidance results in nonconvex optimization. The asymptotic saddle avoidance results in nonconvex optimization state that gradient descent dynamics almost surely avoid strict saddle points [28, 29]. This result has been extended to show that gradient descent dynamics almost surely avoid saddle points even for functions with non-isolated critical points [46] in nonconvex optimization. The drawback of this style of result is that it fails to preclude that gradient descent could spend an arbitrarily long time stuck in the neighborhood of a saddle point. In fact, it has been shown that gradient descent can take exponential time to escape from saddle points [12].

A series of works present asymptotic results on escaping saddle points in continuous action games analogous to that from nonconvex optimization. In particular, Mazumdar et al. [40] prove that in N-player general-sum games, if each player follows the gradient descent learning rule then the dynamics avoid strict saddles of the dynamics almost surely. This result allows for players to employ distinct learning rates. Translated to nonconvex-nonconcave zero-sum games, the result ensures that τ-GDA dynamics avoid strict saddles of the dynamics almost surely for any timescale separation $\tau > 0$ as is presented in Lemma 2. The aforementioned result is key to the asymptotic convergence guarantee to differential Stackelberg equilibria we provide in Theorem 2 for τ-GDA for nonconvex-PŁ/SC zero-sum games. In a related line of work, Daskalakis and Panageas [10] show that in nonconvex-nonconcave zero-sum games the τ-GDA dynamics with $\tau = 1$ (without timescale separation) avoid strict saddle points including when critical points are not isolated. We remark that the result demonstrating that gradient descent can take exponential time to escape from saddle points [12] immediately carries over to gradient descent-ascent in zero-sum games since the game could be completely decoupled an correspond to separate nonconvex-optimization problems.

The study of finite-time saddle avoidance results in nonconvex optimization generally focuses on designing variants of gradient descent that not only avoid saddle points of the dynamics almost surely, but escape from them efficiently. The methods of greatest relevance to this paper are the existing works analyzing the rates at which 'simple' variants of gradient descent escape saddle points in nonconvex optimization. This line of research dates back to the work of Ge et al. [17], who proved that stochastic gradient descent with injected noise perturbations finds an approximate local minimum in a number of gradient class that depends polynomially on the dimension. Since the initial work on escaping saddle points efficiently, there have been many follow-up works. The most relevant to our results and techniques are the works of Jin et al. [20] and Jin et al. [24]. Indeed, Jin et al. [20] prove that a perturbed variant of gradient descent with deterministic gradients finds approximate local minima under the strict saddle assumption with only a logarithmic dimension dependence, meaning the result is almost dimension free and near equivalent to the complexity needed to find a critical point using gradient descent. Moreover, Jin et al. [24] generalize such proof techniques to show that variants of gradient and stochastic gradient descent with injected noise achieve an analogous guarantee with a unified analysis. For future work, it would be interesting to see if these methods could be extended to a perturbed variant of τ-GDA toward seeking a finite-time convergence guarantee to an approximate local minmax equilibrium. There is an extensive literature on methods for escaping saddle points efficiently in nonconvex optimization beyond the works that have been mentioned thus far. This includes analysis of dynamics using normalized gradients [31], stochastic gradient descent without artificial noise injections [9], negative curvature search methods [2–4, 61], variance reduced methods [13, 66], cubic regularization [56], and adaptive acceleration methods [7, 21, 54, 57]. Finally, there is also a recent, sharper analysis of stochastic gradient descent [14].

B Preliminaries

We now review preliminaries on game-theory concepts, dynamical systems theory, PŁ functions and nonconvex-PŁ zero-sum games, linear algebra, and concentration inequalities. The sections on PŁ functions and nonconvex-PŁ zero-sum games, linear algebra, and concentration inequalities include a number of technical lemmas that are needed throughout the rest of the appendix.

B.1 Game Theory

The focus of this work is on developing convergence guarantees for gradient descent-ascent with timescale separation (τ-GDA) to game-theoretically meaningful equilibrium. In the class of games under consideration, the natural solution concept from game theory is a minmax or equivalently Stackelberg equilibrium. Given the nonconvex nature of the minimizing players objective, we consider a local refinement of this historically standard equilibrium concept. In the nonconvex-nonconcave zero-sum game literature, recent work [16, 23] develops sufficient conditions for local Stackelberg equilibrium and points satisfying the conditions correspond to strict local Stackelberg equilibrium. We refer to strict local Stackelberg equilibrium as differential Stackelberg equilibrium following the terminology of [16]. This is the the basis for the equilibrium concept we consider that is given in Definition 4. The fact we focus our attention on strict local equilibrium is standard [15, 16, 23, 40, 58]).

For completeness, we now present a formal local Stackelberg equilibrium definition characterized in terms of the costs. This definition has appeared in past work [15, 16] and it is a simple refinement to a local concept from the standard Stackelberg equilibrium definition in continuous action space games [6]. We emphasis that the equilibrium we consider in Definition 4 is characterized by sufficient conditions for the following local Stackelberg equilibrium definition.

Definition 7 (Local Minmax/Stackelberg Equilibrium)**.** *Consider $U_x \subset \mathcal{X}$ and $U_y \subset \mathcal{Y}$ where, without loss of generality, player 1 (controlling $x \in \mathcal{X}$) is minimizing player and player 2 (controlling $y \in \mathcal{Y}$) is the maximizing player. The strategy $x^* \in U_x$ is a local Stackelberg solution for the minimizing player if, $\forall x \in U_x$,*

$$\sup_{y\in r_{U_y}(x^*)} f(x^*, y) \leq \sup_{y\in r_{U_y}(x)} f(x, y),$$

where $r_{U_y}(x) = \{y' \in U_y | f(x, y') \geq f(x, y), \forall y \in U_y\}$ is the reaction curve. Moreover, for any $y^* \in r_{U_y}(x^*)$, the joint strategy profile $(x^*, y^*) \in U_x \times U_y$ is a local Stackelberg equilibrium on $U_x \times U_y$.

B.2 Dynamical Systems Theory

In this section, we provide relevant background as it pertains to the local stability analysis we presented in Section 4.

We begin by formally describing what is meant by local stability of a continuous-time dynamical system as used in this paper. In particular, we provide a formal definition for the stability definition that was presented in Definition 5. The following well-known result provides equivalent characterizations of local stability for a critical point (stationary point) of a continuous-time dynamical system of the form $\dot{z} = -g(z)$ in terms of the Jacobian matrix $J(z) = Dg(z)$.

Theorem 3 ([26, Theorem 4.6, Corollary 4.3])**.** *Consider a critical point z^* of $g(z)$. The following are equivalent: (a) z^* is a locally exponentially stable equilibrium of $\dot{z} = -g(z)$; (b) $\mathrm{spec}(-J(z^*)) \subset \mathbb{C}^\circ_-$; (c) there exists a symmetric positive-definite matrix $P = P^\top \succ 0$ such that $P\,J(z^*) + J(z^*)^\top P \succ 0$.*

Before moving on, we provide a brief discussion of the implications of determining stability using the local linearization (Jacobian of the dynamical system) around critical points. The Hartman-Grobman theorem [52, Theorem 7.3]; [55, Theorem 9.9] asserts that it is possible to continuously deform all trajectories of a nonlinear system onto trajectories of the linearization at a fixed point of the nonlinear system. Informally, the theorem states that the qualitative properties of the nonlinear system $\dot{z} = -g(z)$ in the vicinity (which is determined by the neighborhood U) of an isolated equilibrium z^* are determined by its linearization if the linearization has no eigenvalues on the imaginary axes in the complex plane. We also remark that Hartman-Grobman can also be applied to discrete time maps (cf. Sastry [52, Thm. 2.18]) with the same qualitative outcome.

In the context of this work, this means that by determining stability using the local linearization around critical points, the behavior of the nonlinear system in a neighborhood of the critical point can be inferred. In particular, given that a critical point is determined to be stable by the local linearization, then there is a neighborhood on which the dynamics converge to the critical point. This observation also applies to the discrete-time system with proper learning rates.

B.3 Polyak-Łojasiewicz Functions and Nonconvex-Polyak-Łojasiewicz Zero-Sum Games

In this section, we state properties of PŁ functions in the context of nonconvex-PŁ zero-sum games. Specifically, we state additional properties of nonconvex-PŁ zero-sum games that follow from properties of PŁ functions (see Lemma 4 and Lemma 5). Moreover, we present additional smoothness properties (see Lemma 6 and Lemma 7) that follow from properties of nonconvex-PŁ zero-sum games and Assumption 1. Finally, we characterize the curvature around critical points of PŁ functions (see Lemma 8). These properties will be used in both the proofs for the local stability and the global convergence in nonconvex-PŁ zero-sum games.

We begin by stating a known property [25] that μ-PŁ functions satisfy a quadratic growth condition [5] also with parameter μ. For clarity of presentation, we state this condition in the context of the nonconvex-PŁ zero-sum games we study. Recall from Definition 1 that given a zero-sum game $(f, -f)$ defined by $f \in C^2(\mathcal{Z}, \mathbb{R})$, the game is called nonconvex-PŁ if $f(x, \cdot)$ is μ-PŁ with respect to the argument $y \in \mathcal{Y}$. That is, for $\mu > 0$ and for all $z = (x, y) \in \mathcal{Z}$,

$$\|\nabla_2 f(x,y)\|^2 \geq 2\mu(\max_{y' \in \mathcal{Y}} f(x,y') - f(x,y)), \tag{7}$$

where for any fixed $x \in \mathcal{X}$, $\max_{y' \in \mathcal{Y}} f(x, y')$ has a nonempty solution set and a finite optimal value.

Lemma 4 ([25, Theorem 2, Appendix A])**.** *Consider a nonconvex, μ-PŁ zero-sum game defined by $f \in C^2(\mathcal{Z}, \mathbb{R})$ satisfying Assumption 1. For all $x \in \mathcal{X}$, the function $f(x, \cdot)$ satisfies the following quadratic growth condition:*

$$\max_{y' \in \mathcal{Y}} f(x,y') - f(x,y) \geq \frac{\mu}{2}\|y_p - y\|^2, \ \forall\, y \in \mathcal{Y} \tag{8}$$

where y_p is the projection onto the set $\operatorname{argmax}_{y \in \mathcal{Y}} f(x,y)$.

It is also known that μ-PŁ functions satisfy an error bound condition [37] also with parameter μ. Again, for clarity of presentation, we state this condition in the context of the nonconvex-PŁ zero-sum games we study.

Lemma 5 ([25, Theorem 2, Appendix A])**.** *Consider a nonconvex, μ-PŁ zero-sum game defined by $f \in C^2(\mathcal{Z}, \mathbb{R})$ satisfying Assumption 1. For all $x \in \mathcal{X}$, the function $f(x, \cdot)$ satisfies the following error bound condition:*

$$\|\nabla_2 f(x,y)\| \geq \mu\|y_p - y\|, \ \forall\, y \in \mathcal{Y} \tag{9}$$

where y_p is the projection onto the set $\operatorname{argmax}_{y \in \mathcal{Y}} f(x,y)$.

We now state additional smoothness properties that follow from properties of nonconvex-PŁ zero-sum games and Assumption 1. These properties are known for this class of games. For nonconvex-PŁ zero-sum games, given any fixed $x \in \mathcal{X}$, the set of maximizers of $f(x, \cdot)$ may not be a singleton. For any fixed $x \in \mathcal{X}$, we denote by $y_*(x)$ an element of $\operatorname{argmax}_{y \in \mathcal{Y}} f(x,y)$. When necessary, we point out to which specific element $y_*(x)$ corresponds. The following result shows that the mapping $y_*(x)$ satisfies a stability property.

Lemma 6 (Stability of Best-Response Map in PŁ functions [45, Lemma A.3])**.** *Consider a nonconvex, μ-PŁ zero-sum game defined by $f \in C^2(\mathcal{Z}, \mathbb{R})$ satisfying Assumption 1 and let $L_3 := L_2/\mu$. For all $x, x' \in \mathcal{X}$ and $y_*(x) \in \operatorname{argmax}_{y \in \mathcal{Y}} f(x,y)$, there exists a $y_*(x') \in \operatorname{argmax}_{y \in \mathcal{Y}} f(x,y)$ such that*

$$\|y_*(x) - y_*(x')\| \leq L_3\|x - x'\|.$$

Danskin's theorem in optimization provides conditions under which the total derivative $\nabla f(x, y_*(x))$ where

$$y_*(x) \in \operatorname*{argmax}_{y \in \mathcal{Y}} f(x,y)$$

is equivalent to $\nabla_1 f(x, y_*(x))$. That is, it gives conditions when the gradient of the function $f(x, y_*(x))$ is equal to the gradient of $f(x,y)|_{y=y_*(x)}$ evaluated directly at the optimum. Typically this requires the maximizer to be unique. However, it has been shown that for nonconvex-PŁ zero-sum games, this property carries over even without a unique solution. This is stated in the following result along with a smoothness property of the function $\max_{y \in \mathcal{Y}} f(x,y)$.

Lemma 7 (Danskin-Type Property for PŁ functions [45, Lemma A.5])**.** *Consider a nonconvex, μ-PŁ zero-sum game defined by $f \in C^2(\mathcal{Z}, \mathbb{R})$ satisfying Assumption 1. Then,*

$$\nabla f(x, y_*(x)) = \nabla_1 f(x, y_*(x)) \quad \textit{where} \quad y_*(x) \in \operatorname*{argmax}_{y \in \mathcal{Y}} f(x,y).$$

Moreover, $\nabla f(x, y_*(x))$ is $L_4 = (L_1 + \frac{L_1 L_2}{\mu})$–Lipschitz. That is, for all $x, x' \in \mathcal{X}$,

$$\|\nabla f(x, y_*(x)) - \nabla f(x', y_*(x'))\| \leq L_4\|x - x'\|.$$

We remark that L_3 and L_4 from Lemma 6 and Lemma 7 slightly deviate from that values in the work of Nouiehed et al. [45]. This is a result of the fact that a correction was made in the work of Karimi et al. [25] after the publication of the work of Nouiehed et al. [45], which showed that μ-PŁ functions satisfy the quadratic growth condition with parameter μ whereas previously it was stated that μ-PŁ functions satisfy the quadratic growth condition with parameter 4μ. Consequently, L_3 and L_4 are stated in Lemma A.3 and Lemma A.5 in the work of Nouiehed et al. [45] (in our notation) as $L_2/(2\mu)$ and $L_1 + (L_1L_2)/(2\mu)$, respectively. Incorporating the fix to the constant from the most recent version of Karimi et al. [25] into Lemma A.3 and Lemma A.5 in the work of Nouiehed et al. [45] gives the values of L_3 and L_4 stated in Lemma 6 and Lemma 7, respectively.

We now show that the quadratic growth property of PŁ functions implies that the eigenvalues of $\nabla_2^2 f(x^*, y^*)$ are upper-bounded by $-\mu$ at any critical point (x^*, y^*) of a μ-PŁ zero-sum game. This means that at any critical point of the game, player 2 must be at a maximum. It is in fact known that all critical points of PŁ functions are global minimum [25], which in the context of this work implies that all critical points of nonconvex-PŁ zero-sum games, player 2 must be not only at a local maximum, but a global maximum. This can be observed by the definitions of a critical point and nonconvex-PŁ zero-sum games we have at any critical point (x^*, y^*):

$$0 = \|\nabla_2 f(x^*, y^*)\|^2 \geq 2\mu(\max_{y' \in \mathcal{Y}} f(x^*, y') - f(x^*, y^*)) \geq 0.$$

The following property will be used in the proof of Theorem 1 in Appendix C. Note that given the inequality above it is primary needed to show that $\det(\nabla_2^2 f(x^*, y^*)) \neq 0$ at critical points (x^*, y^*).

Lemma 8. *Consider a nonconvex, μ-PŁ zero-sum game defined by $f \in C^2(\mathcal{Z}, \mathbb{R})$. At any critical point (x^*, y^*) of the game, that is where $\nabla_1 f(x^*, y^*) = 0$ and $\nabla_2 f(x^*, y^*) = 0$, the individual Hessian of the maximizing player given by $\nabla_2^2 f(x^*, y^*)$ is negative definite and eigenvalues bounded above by $-\mu$.*

Proof. Let us consider any critical point (x^*, y^*) of the game so that $\nabla_1 f(x^*, y^*) = 0$ and $\nabla_2 f(x^*, y^*) = 0$. Taking a Taylor expansion of $-f(x^*, \cdot)$ about the point y^*, we and get that

$$-f(x^*, y) = -f(x^*, y^*) - \nabla_2 f(x^*, y^*)^\top (y - y^*) - \frac{1}{2}\int_{y^*}^{z} (y - z)^\top \nabla_2^2 f(x^*, z)(y - z) dz.$$

This is equivalent to

$$\begin{aligned} -f(x^*, y) = &-f(x^*, y^*) - \nabla_2 f(x^*, y^*)^\top (y - y^*) \\ &- \frac{1}{2}\int_{\tau=0}^{1} (y^* + \tau(y - y^*) - y)^\top \nabla_2^2 f(x^*, y^* + \tau(y - y^*))(y^* + \tau(y - y^*) - y) d\tau \end{aligned}$$

Let $B_{y^*}(\|y - y^*\|)$ be a ball of radius $\|y - y^*\|$ centered at y^*. Then,

$$\begin{aligned} -f(x^*, y) &= -f(x^*, y^*) - \nabla_2 f(x^*, y^*)^\top (y - y^*) \\ &\quad - \frac{1}{2}\int_{\tau=0}^{1} (y^* + \tau(y - y^*) - y)^\top \nabla_2^2 f(x^*, y^* + \tau(y - y^*))(y^* + \tau(y - y^*) - y) d\tau \\ &\leq -f(x^*, y^*) - \nabla_2 f(x^*, y^*)^\top (y - y^*) - \frac{1}{2} \max_{z \in B_{y^*}(\|y - y^*\|)} (z - y^*)^\top \nabla_2^2 f(x^*, z)(z - y^*) \end{aligned}$$

so that

$$\begin{aligned} f(x^*, y) &\geq f(x^*, y^*) + \frac{1}{2} \max_{z \in B_{y^*}(\|y - y^*\|)} (z - y^*)^\top \nabla_2^2 f(x^*, z)(z - y^*) \\ &\geq f(x^*, y^*) + \frac{1}{2}(z - y^*)^\top \nabla_2^2 f(x^*, z)(z - y^*),\ \forall z \in B_{y^*}(\|y - y^*\|). \end{aligned}$$

Hence, by the quadratic growth property of PŁ functions from Lemma 4,

$$\frac{\mu}{2}\|y - y^*\| \leq f(x^*, y^*) - f(x^*, y) \leq -\frac{1}{2}(z - y^*)^\top \nabla_2^2 f(x^*, z)(z - y^*),\ \forall z \in B_{y^*}(\|y - y^*\|)$$

which gives the desired result.

$\square$

B.4 Linear Algebra

In this section, we state a property regarding matrix inertia that is important for the proof of Theorem 1 in Appendix C. The following result from Lancaster and Tismenetsky [27, Theorem 2, Chapter 13.1] is needed for the proof of Theorem 1 given in Appendix C. We include it here for ease of reference. For a given matrix $A \in \mathbb{R}^{n\times n}$, $\upsilon_+(A)$, $\upsilon_-(A)$, and $\zeta(A)$ are the number of eigenvalues of the argument that have positive, negative and zero real parts, respectively.

Lemma 9 ([27, Theorem 2, Chapter 13.1]). *Consider a matrix $A \in \mathbb{R}^{n\times n}$.*

(a) If P is a symmetric matrix such that $AP + PA^\top = Q$ where $Q = Q^\top \succ 0$, then P is nonsingular and P and A have the same inertia, meaning that

$$\upsilon_+(A) = \upsilon_+(P),\ \upsilon_-(A) = \upsilon_-(P),\ \zeta(A) = \zeta(P). \tag{10}$$

(b) On the other hand, if $\zeta(A) = 0$, then there exists a matrix $P = P^\top$ and a matrix $Q = Q^\top \succ 0$ such that $AP + PA^\top = Q$, and P and A have the same inertia (i.e., (10) *holds).*

B.5 Concentration Inequalities

In this section, we presentation concentration inequalities for norm-subGaussian random vectors. Each of the following technical lemmas are from the works of Jin et al. [22, 24] and we reproduce them here for clarity of presentation and easy reference. These concentration inequalities will be used in Appendix E to obtain Lemma 3 and Corollary 2 from Section 5.2.

The following defines a norm-subGaussian random vector.

Definition 8. *A random vector $x \in \mathbb{R}^d$ is norm-subGaussian if there exists σ so that:*

$$\mathbb{P}(\|x - \mathbb{E}[x]\| \geq t) \leq 2e^{-\frac{-t^2}{2\sigma^2}} \quad \forall\, t \in \mathbb{R}.$$

The next result shows that a bounded random vector and a subGaussian random vector are special cases of a norm-subGaussian random vector.

Lemma 10. *There exists an absolute constant c so that the following random vectors are $c\sigma$-norm-subGaussian:*

1. *A bounded random vector $x \in \mathbb{R}^d$ such that $\|x\| \leq \sigma$.*
2. *A random vector $x \in \mathbb{R}^d$, where $x = \psi e_1$ and the random variable $\psi \in \mathbb{R}$ is σ-subGaussian.*
3. *A random vector $x \in \mathbb{R}^d$ that is $(\sigma/\sqrt{d})$-subGaussian.*

We now define the properties of norm-subGaussian martingale difference sequences.

Condition 1. *Consider random vectors $x_1, \ldots, x_n \in \mathbb{R}^d$ and the corresponding filtrations $\mathcal{F}_i = \sigma(x_1, \ldots, x_i)$ for $i \in [n]$ such that $x_i|\mathcal{F}_{i-1}$ is zero-mean σ_i-norm-subGaussian with $\sigma_i \in \mathcal{F}_{i-1}$. That is:*

$$\mathbb{E}[X_i|\mathcal{F}_{i-1}] = 0, \quad \mathbb{P}(\|x_i\| \geq t|\mathcal{F}_{i-1}) \leq 2e^{-\frac{t^2}{2\sigma_i^2}}, \quad \forall t \in \mathbb{R}, \forall i \in [n].$$

The next results give concentration inequalities for the sum of norm squares of norm-subGaussian random vectors and the sum of inner products of norm-subGaussian random vectors with another set of random vectors.

Lemma 11. *Given random vectors $x_1, \ldots, x_n \in \mathbb{R}^d$ that satisfy condition 1 with fixed $\sigma_1 = \cdots = \sigma_n = \sigma$, then for any $\iota > 0$ there exists an absolute constant c such that with probability at least $1 - e^{-\iota}$:*

$$\sum_{i=1}^n \|x_i\|^2 \leq c\sigma^2(n + \iota).$$

Lemma 12. *Given random vectors $x_1, \ldots, x_n \in \mathbb{R}^d$ that satisfy condition 1 and random vectors $\{u_i\}$ that satisfy $u_i \in \mathcal{F}_{i-1}$ for all $i \in [n]$, then for any $\iota > 0$ and $\lambda > 0$ there exists an absolute constant c such that with probability at least $1 - e^{-\iota}$:*

$$\sum_{i=1}^n \langle u_i, x_i\rangle \leq c\lambda \sum_{i=1}^n \|u_i\|^2\sigma_i^2 + \frac{1}{\lambda}\iota.$$

C Stability Analysis: Proof of Theorem 1

This appendix is devoted to providing the proof of Theorem 1. For ease of reference, we restate the result now before providing the proof.

Theorem 1. *Consider a nonconvex-PŁ/SC zero-sum game $(f, -f)$ where $f \in C^2(\mathcal{Z}, \mathbb{R})$. Then, the following hold: 1) Any critical point $z^* \in \mathcal{Z}$ that is not a differential Stackelberg equilibrium is unstable with respect to τ-*`GDA`* for all $\tau \in (0, \infty)$; 2) If z^* is a differential Stackelberg equilibrium, then $\mathrm{spec}(-J_\tau(z^*)) \subset \mathbb{C}_-^\circ$ for all $\tau \in [\tau_*, \infty)$ where τ_* is the minimum $\tau \in (0, \infty)$ such that $\mathrm{spec}(-J_\tau(z^*)) \subset \mathbb{C}_-^\circ$ and a finite τ_* is guaranteed to exist.*

Proof of Theorem 1. We begin by proving the first claim of the theorem statement.

Proof of 1. Let us first consider the case that a given a critical point z^* could be such that $\mathtt{S}_1(J_1(z^*))$ or $-\nabla_2^2 f(z^*)$ are singular. By Lemma 8 we know that for any critical point z^*, $-\nabla_2^2 f(z^*) \succ 0$ so that $-\nabla_2^2 f(z^*)$ is non-singular. Observe that at any critical point z^*, since $-\tau\nabla_2^2 f(z^*)$ is positive definite for all $\tau \in (0, \infty)$, the following identity holds for any $\tau \in (0, \infty)$:

$$\det(J_\tau(z)) = \det(\mathtt{S}_1(J(z^*))) \det(-\tau\nabla_2^2 f(z^*)).$$

From the fact that $\det(-\tau\nabla_2^2 f(z^*))$, it then easily follows that $\det(J_\tau(z^*)) = 0$ if and only if $\det(\mathtt{S}_1(J(z^*))) = 0$. Note that $\det(J_\tau(z^*)) = 0$ if and only if $0 \in \mathrm{spec}(J_\tau(z^*))$ since eigenvalues of a real square matrix are either purely real or come in complex conjugate pairs. Hence, given any critical point such that $\det(\mathtt{S}_1(J(z^*))) = 0$, then $0 \in \mathrm{spec}(J_\tau(z^*))$ and $\mathrm{spec}(-J_\tau(z^*)) \not\subset \mathbb{C}_-^\circ$ for all $\tau \in (0, \infty)$. Hence, any critical point such that $\mathtt{S}_1(J_1(z^*))$ is singular is a saddle point and thus not stable for all $\tau \in (0, \infty)$ and such a point is not a differential Stackelberg equilibrium.

Now, suppose that z^* is a critical point such that $\mathtt{S}_1(J_1(z^*))$ and $-\nabla_2^2 f(z^*)$ are non-singular. Let $\mathrm{spec}(-J_{\tau_0}(z^*)) \subset \mathbb{C}_-^\circ$ for some $\tau_0 \in (0, \infty)$. We know that $-\nabla_2^2 f(z^*) \succ 0$ by Lemma 8. We argue by contradiction that $\mathtt{S}_1(J_1(z^*)) \succ 0$. Towards this end, suppose not. That is, assume for the sake of contradiction that $\mathtt{S}_1(J_1(z^*))$ has at least one negative eigenvalue, or equivalently that $-\mathtt{S}_1(J_1(z^*))$ has at least one positive eigenvalue.

Since $\det(\mathtt{S}_1(J_1(z^*))) \neq 0$ and $\det(-\nabla_2^2 f(z^*)) \neq 0$, by Lemma 9.b, there exists non-singular Hermitian matrices P_1, P_2 and positive definite Hermitian matrices Q_1, Q_2 such that $-\mathtt{S}_1(J_1(z^*))P_1 - P_1\mathtt{S}_1(J_1(z^*)) = Q_1$ and $\nabla_2^2 f(z^*)P_2 + P_2\nabla_2^2 f(z^*) = Q_2$.

Furthermore, $-\mathtt{S}_1(J_1(z^*))$ and P_1 have the same inertia, meaning

$$\upsilon_+(-\mathtt{S}_1(J_1(z^*))) = \upsilon_+(P_1),\ \upsilon_-(-\mathtt{S}_1(J_1(z^*))) = \upsilon_-(P_1),\ \zeta(-\mathtt{S}_1(J_1(z^*))) = \zeta(P_1)$$

where for a given matrix A, $\upsilon_+(A)$, $\upsilon_-(A)$, and $\zeta(A)$ are the number of eigenvalues of the argument that have positive, negative and zero real parts, respectively. Similarly, $\nabla_2^2 f(z^*)$ and P_2 have the same inertia:

$$\upsilon_+(\nabla_2^2 f(z^*)) = \upsilon_+(P_2),\ \upsilon_-(\nabla_2^2 f(z^*)) = \upsilon_-(P_2),\ \zeta(\nabla_2^2 f(z^*)) = \zeta(P_2).$$

Since $-\mathtt{S}(J_1(z^*))$ has at least one strictly positive eigenvalue by assumption for the sake of contradiction, $\upsilon_+(P_1) = \upsilon_+(-\mathtt{S}(J_1(z^*))) \geq 1$.

Define

$$P = \begin{bmatrix} I & L_0^\top \\ 0 & I \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} I & 0 \\ L_0 & I \end{bmatrix} \tag{11}$$

where $L_0 = (\nabla_2^2 f(z^*))^{-1}\nabla_{12}^\top f(z^*)$. Since P is congruent to $\mathrm{blockdiag}(P_1, P_2)$, by Sylvester's law of inertia [18, Thm. 4.5.8], P and $\mathrm{blockdiag}(P_1, P_2)$ have the same inertia, meaning that

$$\upsilon_+(P) = \upsilon_+(\mathrm{blockdiag}(P_1, P_2)), \upsilon_-(P) = \upsilon_-(\mathrm{blockdiag}(P_1, P_2)), \zeta(P) = \zeta(\mathrm{blockdiag}(P_1, P_2)).$$

Consider the matrix equation $-PJ_{\tau_0}(z^*) - J_{\tau_0}^\top(z^*)P = Q_{\tau_0}$ for $-J_{\tau_0}(z^*)$ where

$$Q_{\tau_0} = \begin{bmatrix} I & L_0^\top \\ 0 & I \end{bmatrix} B_{\tau_0} \begin{bmatrix} I & 0 \\ L_0 & I \end{bmatrix} \tag{12}$$

with

$$B_{\tau_0} = \begin{bmatrix} Q_1 & P_1\nabla_{12} f(z^*) - \mathtt{S}(J_1(z^*))L_0^\top P_2 \\ (P_1\nabla_{12} f(z^*) - \mathtt{S}(J_1(z^*))L_0^\top P_2)^\top & P_2 L_0 \nabla_{12} f(z^*) + (P_2 L_0 \nabla_{12} f(z^*))^\top + \tau_0 Q_2 \end{bmatrix}$$

which can be verified by straightforward calculations. The matrix B_{τ_0} is a symmetric matrix, and it is positive definite. Indeed, first observe that $Q_1 \succ 0$ and $Q_2 \succ 0$. Then showing $B_{\tau_0} \succ 0$ reduces to showing

$$P_2 L_0 \nabla_{12} f(z^*) + (P_2 L_0 \nabla_{12} f(z^*))^\top \succeq 0.$$

To see this, note that it is sufficient to choose $P_2 = \nabla_2^2 f(z^*) \prec 0$ so that

$$Q_2 = P_2 \nabla_2^2 f(z^*) + \nabla_2^2 f(z^*) P_2 = 2\nabla_2^2 f(z^*)^2 \succeq 2\mu^2 I.$$

Observe that we have used the fact that the eigenvalues of $\nabla_2^2 f(z^*)^2$ are the square of the eigenvalues of $\nabla_2^2 f(z^*)$ and the eigenvalues of $\nabla_2^2 f(z^*)$ are bounded between $-\mu$ and $-L_2$ by Lemma 8 and the Lipschitz assumption.

Then, with this choice of P_2, we have that

$$P_2(\nabla_2^2 f(z^*))^{-1}\nabla_{12}^\top f(z^*)\nabla_{12} f(z^*) + (P_2(\nabla_2^2 f(z^*))^{-1}\nabla_{12}^\top f(z^*)\nabla_{12} f(z^*))^\top = 2\nabla_{12}^\top f(z^*)\nabla_{12} f(z^*) \succeq 0.$$

Now, since $B_{\tau_0} \succ 0$ so is Q_{τ_0} since they are congruent. Since $\mathrm{spec}(-J_{\tau_0}(z^*)) \subset \mathbb{C}_-^\circ$, $Q_{\tau_0} \succ 0$ implies that $P = P^\top \prec 0$ (by Lyapunov's theorem). Hence, P_1 and P_2 must be negative definite since P is congruent to $\mathrm{diag}(P_1, P_2)$, but this gives us a contradiction with the fact that P_1 has the same inertia as $-\,\mathtt{S}_1(J_1(z^*))$ which we assumed to have at least one positive eigenvalue. Hence, if $\mathrm{spec}(-J_{\tau_0}(z^*)) \subset \mathbb{C}_-^\circ$ for some $\tau_0 \in (0,\infty)$, then it must be the case that $\mathtt{S}_1(J_1(z^*)) \succ 0$ which means z^* is a differential Stackelberg equilibrium since we also have $-\nabla_2^2 f(z^*) \succ 0$ by Lemma 8.

Thus, we can finish the proof of part 1 as follows. Consider any critical point $\tilde{z}$ that is not a differential Stackelberg equilibrium and is unstable for the nominal τ_0. Then, we claim that $\mathrm{spec}(-J_\tau(\tilde{z})) \not\subset \mathbb{C}_-^\circ$ for all $\tau \geq \tau_0$. Suppose not. That is, there is some $\tau_1 \geq \tau_0$ such that $\mathrm{spec}(-J_{\tau_1}(\tilde{z})) \subset \mathbb{C}_-^\circ$. But by our argument above, since $-\nabla_2^2 f(\tilde{x}) \succ 0$, this implies that $\mathtt{S}_1(J_1(\tilde{z})) \succ 0$ which contradicts that $\tilde{z}$ is not a differential Stackelberg equilibrium. Hence, any critical point z^* that is not a differential Stackelberg equilibrium is unstable for all $\tau \in (0,\infty)$.

Proof of 2. We note that the fact there exists a finite $\tau^* \in (0,\infty)$ such that a differential Stackelberg is stable is known [15]. Let τ^* denote the minimum τ^* such that a differential Stackelberg equilibrium z^* is stable. To see that $\mathrm{spec}(-J_\tau(z^*)) \subset \mathbb{C}_-^\circ$ for all $\tau \geq \tau^*$ given that $\mathrm{spec}(-J_{\tau^*}(z^*)) \subset \mathbb{C}_-^\circ$, we can again examine the Lyapunov equation under the congruent transformation. We define the matrix P as above in equation (11) where $P_1 \prec 0$ and $P_2 \prec 0$ since $-\,\mathtt{S}(J_1(z^*)) \prec 0$ and $\nabla_2^2 f(z^*) \prec 0$ with $Q_1, Q_2 \succ 0$. With

$$Q_\tau = \begin{bmatrix} I & L_0^\top \\ 0 & I \end{bmatrix} B_\tau \begin{bmatrix} I & 0 \\ L_0 & I \end{bmatrix}$$

and

$$B_\tau = \begin{bmatrix} Q_1 & P_1\nabla_{12} f(z^*) - \mathtt{S}(J_1(z^*))L_0^\top P_2 \\ (P_1\nabla_{12} f(z^*) - \mathtt{S}(J_1(z^*))L_0^\top P_2)^\top & P_2 L_0 \nabla_{12} f(z^*) + (P_2 L_0 \nabla_{12} f(z^*))^\top + \tau Q_2 \end{bmatrix}$$

we again can see that $Q_\tau \succ 0$ for the same reason as above for any $\tau \geq \tau^*$. This, in turn, implies that $\mathrm{spec}(-J_\tau(z^*)) \subset \mathbb{C}_-^\circ$ for all $\tau \geq \tau^*$ since we constructed a Lyapunov function for z^*. Thus, we conclude that if z^* is a differential Stackelberg equilibrium, then $\mathrm{spec}(-J_\tau(z^*)) \subset \mathbb{C}_-^\circ$ for all $\tau \in [\tau_*, \infty)$ where τ_* is the minimum $\tau \in (0,\infty)$ such that $\mathrm{spec}(-J_\tau(z^*)) \subset \mathbb{C}_-^\circ$ and a finite τ_* is guaranteed to exist. □

D Convergence Analysis for Nonconvex-PŁ Zero-Sum Games

We now provide the proofs pertaining to the results presented in Section 5 for nonconvex-PŁ zero-sum games. To help the presentation, Table 2 includes the relevant game and smoothness parameters needed in the proofs.

Parameter	Meaning/Value
μ	$f(x,\cdot)$ is μ-PŁ or μ-SC for all $x \in \mathcal{X}$
L_1	$\forall\, x, x' \in \mathcal{X}, y, y' \in \mathcal{Y} : \|\nabla_1 f(x,y) - \nabla_1 f(x',y')\| \leq L_1(\|x-x'\| + \|y-y'\|)$
L_2	$\forall\, x, x' \in \mathcal{X}, y, y' \in \mathcal{Y} : \|\nabla_2 f(x,y) - \nabla_2 f(x',y')\| \leq L_2(\|x-x'\| + \|y-y'\|)$
L	$L = L_1 + L_2$ and $\forall\, x, x' \in \mathcal{X}, y, y' \in \mathcal{Y} : \|g(x,y) - g(x',y')\| \leq L(\|x-x'\| + \|y-y'\|)$
L_3	$L_3 = L_2/\mu$ and $\forall\, x, x' \in \mathcal{X} : \|y_*(x) - y_*(x')\| \leq L_3\|x-x'\|$
L_4	$L_4 = L_1 + \kappa L_2$ and $\forall\, x, x' \in \mathcal{X} : \|f(x, y_*(x)) - f(x', y_*(x'))\| \leq L_4\|x-x'\|$
κ	L_1/μ
κ'	L_2/μ
ℓ	$L_1 + 3\kappa' L_2$

Table 2: Table of Lipschitz Parameters

D.1 Proof of Lemma 1

This appendix is devoted to proving Lemma 1. To be clear, we restate the result before proving it.

Lemma 1. *Consider a nonconvex-PŁ/SC zero-sum game defined by $f \in C^2(\mathcal{Z}, \mathbb{R})$ satisfying Assumption 1. For any $\Gamma \in (0, 1/7]$, suppose that $\tau \geq \Gamma^{-1}\kappa^2$ and $\gamma < \min\{\frac{1}{2L}, \frac{1}{2\tau L}, \frac{1}{2L_4}\}$, then $\Phi(x, y) = f(x, y_*(x)) - \Gamma f(x, y)$ is a potential function for τ-*`GDA`.

Proof of Lemma 1. Let the best response be denoted by

$$y_*(x) \in \operatorname*{argmax}_{y \in \mathcal{Y}} f(x, y).$$

Since the function $f(x, \cdot)$ is PŁ, the set maximizers may not be a singleton. Hence, $y_*(x)$ is an element of the set of maximizers. Recall that

$$\Phi(x, y) := f(x, y_*(x)) - \Gamma f(x, y).$$

We claim that for any $\Gamma \in (0, 1/7]$,

$$\Phi(x_{k+1}, y_{k+1}) - \Phi(x_k, y_k) = \Gamma \underbrace{(f(x_k, y_k) - f(x_{k+1}, y_{k+1}))}_{(i)} + \underbrace{f(x_{k+1}, y_*(x_{k+1})) - f(x_k, y_*(x_k))}_{(ii)} < 0.$$

To show this, we need to bound each of the two terms (i) and (ii).

Bounding term (i): $f(x_k, y_k) - f(x_{k+1}, y_{k+1})$. To begin, we add and subtract $f(x_k, y_{k+1})$ to (i) and get

$$f(x_k, y_k) - f(x_{k+1}, y_{k+1}) = f(x_k, y_k) - f(x_k, y_{k+1}) + f(x_k, y_{k+1}) - f(x_{k+1}, y_{k+1}). \tag{13}$$

From a Taylor expansion of $-f(x_k, y_{k+1})$ with respect to y_{k+1} we obtain

$$\begin{aligned}
-f(x_k, y_{k+1}) &\leq -f(x_k, y_k) - \langle \nabla_2 f(x_k, y_k), y_{k+1} - y_k \rangle + \frac{L_2}{2}\|y_{k+1} - y_k\|^2 \\
&= -f(x_k, y_k) - \tau\gamma \langle \nabla_2 f(x_k, y_k), \nabla_2 f(x_k, y_k) \rangle + \frac{\tau^2\gamma^2 L_2}{2}\|\nabla_2 f(x_k, y_k)\|^2 \\
&= -f(x_k, y_k) - \Big(\tau\gamma - \frac{\tau^2\gamma^2 L_2}{2}\Big)\|\nabla_2 f(x_k, y_k)\|^2.
\end{aligned}$$

Hence, rearranging this bound and combining with (13), we get

$$f(x_k, y_k) - f(x_{k+1}, y_{k+1}) \leq -\Big(\tau\gamma - \frac{\tau^2\gamma^2 L_2}{2}\Big)\|\nabla_2 f(x_k, y_k)\|_2^2 + f(x_k, y_{k+1}) - f(x_{k+1}, y_{k+1}). \tag{14}$$

Now, from a Taylor expansion of $-f(x_{k+1}, y_{k+1})$ with respect to x_{k+1}, we get

$$\begin{aligned}
-f(x_{k+1}, y_{k+1}) &\leq -f(x_k, y_{k+1}) - \langle \nabla_1 f(x_k, y_{k+1}), x_{k+1} - x_k \rangle + \frac{L_1}{2}\|x_{k+1} - x_k\|^2 \\
&= -f(x_k, y_{k+1}) + \gamma \langle \nabla_1 f(x_k, y_{k+1}), \nabla_1 f(x_k, y_k) \rangle + \frac{L_1\gamma^2}{2}\|\nabla_1 f(x_k, y_k)\|^2.
\end{aligned}$$

We now add and subtract $\gamma\|\nabla_1 f(x_k, y_k)\|^2$ to obtain

$$\begin{aligned}
f(x_k, y_{k+1}) - f(x_{k+1}, y_{k+1}) &\leq \gamma \langle \nabla_1 f(x_k, y_{k+1}), \nabla_1 f(x_k, y_k) \rangle + \frac{L_1\gamma^2}{2}\|\nabla_1 f(x_k, y_k)\|^2 \\
&\quad - \gamma \langle \nabla_1 f(x_k, y_k), \nabla_1 f(x_k, y_k) \rangle + \gamma\|\nabla_1 f(x_k, y_k)\|_2^2 \\
&= \gamma \langle \nabla_1 f(x_k, y_{k+1}) - \nabla_1 f(x_k, y_k), \nabla_1 f(x_k, y_k) \rangle + \frac{L_1\gamma^2 + 2\gamma}{2}\|\nabla_1 f(x_k, y_k)\|^2.
\end{aligned} \tag{15}$$

Next, we combine this with (14) to get

$$\begin{aligned}
f(x_k, y_k) - f(x_{k+1}, y_{k+1}) &\leq -\Big(\tau\gamma - \frac{\tau^2\gamma^2 L_2}{2}\Big)\|\nabla_2 f(x_k, y_k)\|^2 \\
&\quad + \gamma \langle \nabla_1 f(x_k, y_{k+1}) - \nabla_1 f(x_k, y_k), \nabla_1 f(x_k, y_k) \rangle + \frac{L_1\gamma^2 + 2\gamma}{2}\|\nabla_1 f(x_k, y_k)\|^2 \\
&\leq -\Big(\tau\gamma - \frac{\tau^2\gamma^2 L_2}{2}\Big)\|\nabla_2 f(x_k, y_k)\|^2 + \frac{L_1\gamma^2 + 2\gamma}{2}\|\nabla_1 f(x_k, y_k)\|^2 \\
&\quad + \gamma\|\nabla_1 f(x_k, y_{k+1}) - \nabla_1 f(x_k, y_k)\|\|\nabla_1 f(x_k, y_k)\|.
\end{aligned}$$

Note that the final inequality is a result of applying Cauchy-Schwartz. Applying Young's inequality on the last term in the inequality above, we have that

$$\begin{aligned}
f(x_k, y_k) - f(x_{k+1}, y_{k+1}) &\leq -\Big(\tau\gamma - \frac{\tau^2\gamma^2 L_2}{2}\Big)\|\nabla_2 f(x_k, y_k)\|^2 + \frac{L_1\gamma^2 + 2\gamma + \gamma}{2}\|\nabla_1 f(x_k, y_k)\|^2 \\
&\quad + \frac{\gamma}{2}\|\nabla_1 f(x_k, y_{k+1}) - \nabla_1 f(x_k, y_k)\|^2 \\
&\leq -\Big(\tau\gamma - \frac{\tau^2\gamma^2 L_2}{2} - \frac{\gamma(\tau\gamma)^2 L_1^2}{2}\Big)\|\nabla_2 f(x_k, y_k)\|^2 + \frac{L_1\gamma^2 + 2\gamma + \gamma}{2}\|\nabla_1 f(x_k, y_k)\|^2.
\end{aligned} \tag{16}$$

Observe that the final inequality is a result of applying the Lipschitz bound

$$\frac{\gamma}{2}\|\nabla_1 f(x_k, y_{k+1}) - \nabla_1 f(x_k, y_k)\|^2 \leq \frac{\gamma L_1^2}{2}\|y_{k+1} - y_k\|^2 = \frac{\gamma(\tau^2\gamma^2)L_1^2}{2}\|\nabla_2 f(x_k, y_k)\|^2.$$

Bounding (ii): $\boldsymbol{f(x_{k+1}, y_*(x_{k+1})) - f(x_k, y_*(x_k))}$. To bound (ii), we take a Taylor expansion of $f(x_{k+1}, y_*(x_{k+1}))$ to get that

$$\begin{aligned}
f(x_{k+1}, y_*(x_{k+1})) - f(x_k, y_*(x_k)) &\leq \langle \nabla f(x_k, y_*(x_k)), x_{k+1} - x_k\rangle + \frac{L_4}{2}\|x_{k+1} - x_k\|^2 \\
&= -\gamma\langle \nabla f(x_k, y_*(x_k)), \nabla_1 f(x_k, y_k)\rangle + \frac{L_4\gamma^2}{2}\|\nabla_1 f(x_k, y_k)\|^2
\end{aligned} \tag{17}$$

where L_4 is the gradient Lipschitz bound on the total derivative of $f(x, y_*(x))$ from Lemma 7.

Combining bounds. We now combine the bounds on (i) and (ii) from (16) and (17) to give that

$$\begin{aligned}
\Phi(x_{k+1}, y_{k+1}) - \Phi(x_k, y_k) &= \Gamma\underbrace{(f(x_k, y_k) - f(x_{k+1}, y_{k+1}))}_{(i)} + \underbrace{f(x_{k+1}, y_*(x_{k+1})) - f(x_k, y_*(x_k))}_{(ii)} \\
&\leq -\Gamma\Big(\tau\gamma - \frac{\tau^2\gamma^2 L_2}{2} - \frac{\gamma(\tau\gamma)^2 L_1^2}{2}\Big)\|\nabla_2 f(x_k, y_k)\|^2 + \Gamma\Big(\frac{L_1\gamma^2 + 2\gamma + \gamma}{2}\Big)\|\nabla_1 f(x_k, y_k)\|^2 \\
&\quad - \gamma\langle \nabla f(x_k, y_*(x_k)), \nabla_1 f(x_k, y_k)\rangle + \frac{L_4\gamma^2}{2}\|\nabla_1 f(x_k, y_k)\|^2 \\
&= -\gamma\langle \nabla f(x_k, y_*(x_k)), \nabla_1 f(x_k, y_k)\rangle + \frac{\gamma}{2}(\Gamma L_1\gamma + 3\Gamma + L_4\gamma)\|\nabla_1 f(x_k, y_k)\|^2 \\
&\quad - \Gamma\Big(\tau\gamma - \frac{\tau^2\gamma^2 L_2}{2} - \frac{\gamma(\tau\gamma)^2 L_1^2}{2}\Big)\|\nabla_2 f(x_k, y_k)\|^2.
\end{aligned} \tag{18}$$

To further bound the above expression, we start by bounding the first two terms. Towards this end, define

$$V := -\gamma\langle \nabla f(x_k, y_*(x_k)), \nabla_1 f(x_k, y_k)\rangle + \frac{\gamma}{2}(\Gamma L_1\gamma + 3\Gamma + L_4\gamma)\|\nabla_1 f(x_k, y_k)\|^2. \tag{19}$$

Recall that $\gamma < 1/(2L_4)$ and $\gamma < 1/(2L)$ and observe that this implies $\gamma < 1/(2L_1)$ since $L = L_1 + L_2 > L_1$. Thus, since $\Gamma \leq 1/7$,

$$\gamma\Gamma L_1 + 3\Gamma + \gamma L_4 \leq \frac{\Gamma}{2} + 3\Gamma + \frac{1}{2} = \frac{1}{2} + \frac{7\Gamma}{2} \leq 1.$$

Thus, applying this fact, we can bound (19) as follows:

$$V \leq -\gamma\langle \nabla f(x_k, y_*(x_k)), \nabla_1 f(x_k, y_k)\rangle + \frac{\gamma}{2}\|\nabla_1 f(x_k, y_k)\|^2. \tag{20}$$

Now, by adding and subtracting $\frac{\gamma}{2}\|\nabla f(x_k, y_*(x_k))\|^2$ in (20) and then simplifying, we have that

$$\begin{aligned}
V &\leq -\frac{\gamma}{2}\|\nabla f(x_k, y_*(x_k))\|^2 + \frac{\gamma}{2}\|\nabla f(x_k, y_*(x_k))\|^2 - \gamma\langle \nabla f(x_k, y_*(x_k)), \nabla_1 f(x_k, y_k)\rangle + \frac{\gamma}{2}\|\nabla_1 f(x_k, y_k)\|^2 \\
&= -\frac{\gamma}{2}\|\nabla f(x_k, y_*(x_k))\|^2 + \frac{\gamma}{2}\big(\|\nabla f(x_k, y_*(x_k))\|^2 - 2\langle \nabla f(x_k, y_*(x_k)), \nabla_1 f(x_k, y_k)\rangle + \|\nabla_1 f(x_k, y_k)\|^2\big) \\
&= -\frac{\gamma}{2}\|\nabla f(x_k, y_*(x_k))\|^2 + \frac{\gamma}{2}\|\nabla f(x_k, y_*(x_k)) - \nabla_1 f(x_k, y_k)\|^2.
\end{aligned} \tag{21}$$

Moreover, we can continue to bound (21) in the following way that is explained below:

$$V \leq -\frac{\gamma}{2}\|\nabla f(x_k, y_*(x_k))\|^2 + \frac{\gamma}{2}\|\nabla_1 f(x_k, y_*(x_k)) - \nabla_1 f(x_k, y_k)\|^2 \tag{22}$$

$$\leq -\frac{\gamma}{2}\|\nabla f(x_k, y_*(x_k))\|^2 + \frac{\gamma L_1^2}{2}\|y_*(x_k) - y_k\|^2 \tag{23}$$

$$\leq -\frac{\gamma}{2}\|\nabla f(x_k, y_*(x_k))\|^2 + \frac{\gamma \kappa^2}{2}\|\nabla_2 f(x_k, y_k)\|^2. \tag{24}$$

Observe that (22) is a result of applying the fact $\nabla f(x_k, y_*(x_k)) = \nabla_1 f(x, y)|_{y=y_*(x)}$ by Lemma 7 and gives an equivalent formulation of (21). Then in (23) we used that $\nabla_1 f(x, y)$ is L_1-Lipschitz in y by Assumption 1. Furthermore, in (24), we used the error bound property of μ-PŁ functions from Lemma 5 to get that

$$\|y_*(x_k) - y_k\|^2 \leq \frac{1}{\mu^2}\|\nabla_2 f(x_k, y_k)\|^2.$$

Finally, also in (24), we applied the condition number definition $\kappa = L_1/\mu$.
Now, by combining the bound on V from (24) with the remaining terms in (18), we have

$$\begin{aligned}\Phi(x_{k+1}, y_{k+1}) - \Phi(x_k, y_k) &\leq -\frac{\gamma}{2}\|\nabla f(x_k, y_*(x_k))\|^2 + \frac{\gamma \kappa^2}{2}\|\nabla_2 f(x_k, y_k)\|^2 \\ &\quad - \Gamma\Big(\tau\gamma - \frac{\tau^2\gamma^2 L_2}{2} - \frac{\gamma(\tau\gamma)^2 L_1^2}{2}\Big)\|\nabla_2 f(x_k, y_k)\|^2 \\ &= -\frac{\gamma}{2}\|\nabla f(x_k, y_*(x_k))\|^2 + \frac{\tau\gamma\Gamma}{2}\Big(\frac{\kappa^2}{\tau\Gamma} + \tau\gamma L_2 + \tau\gamma^2 L_1^2 - 2\Big)\|\nabla_2 f(x_k, y_k)\|^2. \quad (25)\end{aligned}$$

Let

$$C := \frac{\kappa^2}{\tau\Gamma} + \tau\gamma L_2 + \tau\gamma^2 L_1^2 - 2.$$

As long as $C < 0$, then the function $\Phi(\cdot, \cdot)$ is decreasing along the trajectories of τ-GDA. To see this, we upper bound C in the following manner that is explained below:

$$\begin{aligned}C &= \frac{\kappa^2}{\tau\Gamma} + \tau\gamma L_2 + \tau\gamma^2 L_1^2 - 2 \\ &\leq \tau\gamma L_2 + \tau\gamma^2 L_1^2 - 1 && (26) \\ &\leq \frac{\gamma L_1}{2} - \frac{1}{2} && (27) \\ &\leq -1/4. && (28)\end{aligned}$$

The inequality in (26) is obtained using the fact that $\Gamma^{-1}\kappa^2 \leq \tau$. Moreover, (27) follows from the fact that $\gamma < 1/(2\tau L) < \min\{1/(2\tau L_1), 1/(2\tau L_2)\}$. Finally, (28) holds since $\gamma < 1/(2L) < \min\{1/(2L_1), 1/(2L_2)\}$.
Hence, combining (25) and (28) gives

$$\Phi(x_{k+1}, y_{k+1}) - \Phi(x_k, y_k) \leq -\frac{\gamma}{2}\|\nabla f(x_k, y_*(x_k))\|^2 - \frac{\tau\gamma\Gamma}{8}\|\nabla_2 f(x_k, y_k)\|^2.$$

Thus, the function $\Phi(\cdot, \cdot)$ only stops decreasing when we have both

$$\|\nabla f(x_k, y_*(x_k))\|^2 = 0 \quad \text{and} \quad \|\nabla_2 f(x_k, y_k)\|^2 = 0.$$

By the error bound property of μ-PŁ functions from Lemma 5 we have

$$\|y_*(x_k) - y_k\|^2 \leq \frac{1}{\mu^2}\|\nabla_2 f(x_k, y_k)\|^2.$$

Hence, if $\|\nabla_2 f(x_k, y_k)\|^2 \to 0$ then $y_k \to y_*(x_k)$. In particular, when $\|\nabla_2 f(x_k, y_k)\|^2 = 0$, we have that $y_k = y_*(x_k)$ so that $\|\nabla f(x_k, y_*(x_k))\|^2 = 0$ if and only if $\|\nabla_1 f(x_k, y_k)\|^2 = 0$. This implies that $\Phi(\cdot, \cdot)$ only stops decreasing along the τ-GDA iterates at critical points of τ-GDA. Moreover, observe that $\Phi(x, y) \geq 0$ for any $(x, y) \in \mathcal{X} \times \mathcal{Y}$ since by definition of $y_*(x)$ it immediately follows that $f(x, y_*(x)) \geq f(x, y)$ so that $f(x, y_*(x)) - \Gamma f(x, y) \geq 0$ owing to the fact that $\Gamma \in (0, 1/7]$. Thus, since $\Phi(\cdot, \cdot)$ is bounded below, it must stop decreasing along the τ-GDA iterates. Thus, $\Phi(\cdot, \cdot)$ is a potential function as claimed. □

D.2 Proof of Theorem 2

In this appendix, we prove Theorem 2. To be clear, we restate the result before proving it.

Theorem 2. *Consider a nonconvex-PŁ/SC zero-sum game $(f, -f)$ defined by $f \in C^2(\mathcal{Z}, \mathbb{R})$ that satisfies Assumptions 1–2. Then, τ-*`GDA`* with $\tau \geq \Gamma^{-1}\kappa^2$, and $\gamma < \min\{\frac{1}{2L}, \frac{1}{2\tau L}, \frac{1}{2L_4}\}$ for $\Gamma \in (0, 1/7]$ asymptotically converges to the set of strict local minmax that are stable for $\dot{z} = -\Lambda_\tau g(z)$ almost surely. That is, for almost all initial conditions, τ-*`GDA`* will converge to a strict local minmax.*

Proof of Theorem 2. This result follows nearly immediately from Theorem 1, Lemma 1, and Lemma 2. In particular, the potential function result from Lemma 1 guarantees that τ-`GDA` converges to a critical (stationary) point of the update for the choice of γ and τ. Then, since the only stable points of the τ-`GDA` continuous-time $\dot{z} = -\Lambda_\tau g(z)$ dynamics are strict local minmax (differential Stackelberg) equilibrium by Theorem 1, τ-`GDA` avoids strict saddle points of $\dot{z} = -\Lambda_\tau g(z)$ almost surely for the choice of γ and τ by Lemma 2, and all saddle points of $\dot{z} = -\Lambda_\tau g(z)$ are assumed to be strict saddle points by Assumption 2 for the given choice of τ, we can conclude that τ-`GDA` almost surely converges to a strict local minmax (differential Stackelberg) equilibrium. $\square$

D.3 Proof of Corollary 1

In this appendix, we prove Corollary 1. We restate the result and then provide the proof.

Corollary 1. *Consider a nonconvex-PŁ/SC zero-sum game $(f, -f)$ defined by $f \in C^2(\mathcal{Z}, \mathbb{R})$ that satisfying Assumption 1. Then, τ-*`GDA`* from any initialization with $\tau \geq \Gamma^{-1}\kappa^2$, $\gamma < \min\{\frac{1}{2L}, \frac{1}{2\tau L}, \frac{1}{2L_4}\}$ for $\Gamma \in (0, 1/7]$, has at least one iterate that is an ε-critical point after $\widetilde{\mathcal{O}}(\varepsilon^{-2})$ iterations.*

Proof of Corollary 1. For this proof, consider any $\varepsilon > 0$. Our approach will be to show that for

$$T \geq \frac{2\Phi(x_0, y_0)}{\varepsilon^2 \gamma \min\{1, \tau\Gamma/4\}},$$

we have that

$$\min_{0 \leq k \leq T-1} \max\Big\{\|\nabla f(x_k, y_*(x_k))\|, \|\nabla_2 f(x_k, y_k)\|\Big\} := \max\Big\{\|\nabla f(x_s, y_*(x_s))\|, \|\nabla_2 f(x_s, y_s)\|\Big\} \leq \varepsilon.$$

Then, we prove that given this fact

$$\|\nabla_1 f(x_s, y_s)\| \leq \Big(1 + \frac{L_1}{\mu}\Big)\varepsilon.$$

This will then allow us to conclude for

$$T \geq \frac{2\Big(1 + \frac{L_1}{\mu}\Big)^2 \Phi(x_0, y_0)}{\varepsilon^2 \gamma \min\{1, \tau\Gamma/4\}}, \tag{29}$$

we have both

$$\|\nabla_1 f(x_s, y_s)\| \leq \varepsilon \quad \text{and} \quad \|\nabla_2 f(x_s, y_s)\| \leq \varepsilon.$$

Then, by selecting the parameters to minimize the right-hand side of (29), we are able to conclude that at least one iterate of the τ-`GDA` dynamics are an ε-critical point after

$$T \geq \frac{2\Big(1 + \frac{L_1}{\mu}\Big)^2 \Phi(x_0, y_0)}{\varepsilon^2 \gamma \min\{1, \tau\Gamma/4\}}$$

iterations.

We now formally prove this. Summing the bound on the potential function from Lemma 1, we get the following that is justified below:

$$\Phi(x_0, y_0) \geq \Phi(x_0, y_0) - \Phi(x_T, y_T) \tag{30}$$

$$= \sum_{k=0}^{T-1} \Big(\Phi(x_k, y_k) - \Phi(x_{k+1}, y_{k+1}) \Big) \tag{31}$$

$$\geq \frac{\gamma}{2} \sum_{k=0}^{T-1} \|\nabla f(x_k, y_*(x_k))\|^2 + \frac{\tau\gamma\Gamma}{8} \sum_{k=0}^{T-1} \|\nabla_2 f(x_k, y_k)\|^2 \tag{32}$$

$$\geq \frac{\gamma}{2} \min\Big\{1, \frac{\tau\Gamma}{4}\Big\} \sum_{k=0}^{T-1} \Big(\|\nabla f(x_k, y_*(x_k))\|^2 + \|\nabla_2 f(x_k, y_k)\|^2 \Big) \tag{33}$$

$$\geq \frac{\gamma}{2} \min\Big\{1, \frac{\tau\Gamma}{4}\Big\} \sum_{k=0}^{T-1} \max\Big\{ \|\nabla f(x_k, y_*(x_k))\|^2, \|\nabla_2 f(x_k, y_k)\|^2 \Big\} \tag{34}$$

$$\geq \frac{\gamma T}{2} \min\Big\{1, \frac{\tau\Gamma}{4}\Big\} \min_{0\leq k\leq T-1} \max\Big\{ \|\nabla f(x_k, y_*(x_k))\|^2, \|\nabla_2 f(x_k, y_k)\|^2 \Big\}. \tag{35}$$

Note that (30) holds since $\Phi(x, y) \geq 0$ for any $(x, y) \in \mathcal{X} \times \mathcal{Y}$ since by definition of $y_*(x)$ it immediately follows that $f(x, y_*(x)) \geq f(x, y)$ so that $f(x, y_*(x)) - \Gamma f(x, y) \geq 0$ owing to the fact that $\Gamma \in (0, 1/7]$. Observe that (31) follows from telescoping of the sum, (32) is a result of applying the bound on the potential function, (33) holds since it is replacing a coefficient of a positive number with something smaller, (34) holds since the sum of positive numbers is greater than the max, and (35) is obtained from the fact that the sum of T positive numbers is greater than T times the minimum number.

From the previous steps, and also rearranging terms and then taking the square root, we have

$$\sqrt{\frac{2\Phi(x_0, y_0)}{T\gamma \min\{1, \tau\Gamma/4\}}} \geq \min_{0\leq k\leq T-1} \max\Big\{ \|\nabla f(x_k, y_*(x_k))\|, \|\nabla_2 f(x_k, y_k)\| \Big\}.$$

We now want to find T such that

$$\min_{0\leq k\leq T-1} \max\Big\{ \|\nabla f(x_k, y_*(x_k))\|, \|\nabla_2 f(x_k, y_k)\| \Big\} \leq \sqrt{\frac{2\Phi(x_0, y_0)}{T\gamma \min\{1, \tau\Gamma/4\}}} \leq \varepsilon. \tag{36}$$

By moving terms around, we find that the inequality in (36) holds for any T such that

$$T \geq T^* := \frac{2\Phi(x_0, y_0)}{\varepsilon^2 \gamma \min\{1, \tau\Gamma/4\}}. \tag{37}$$

This proves that there exists some iterate $0 \leq s \leq T - 1$ such that for $T \geq T^*$, we have both

$$\|\nabla f(x_s, y_*(x_s))\| \leq \varepsilon \quad \text{and} \quad \|\nabla_2 f(x_s, y_s)\| \leq \varepsilon.$$

We now show that this implies a bound on $\|\nabla_1 f(x_s, y_s)\|$. In particular, by the error bound property of μ-PŁ functions from Lemma 5 we have

$$\|y_s - y_*(x_s)\|^2 \leq \frac{1}{\mu^2} \|\nabla_2 f(x_s, y_s)\|^2.$$

Since $\|\nabla_2 f(x_s, y_s)\| \leq \varepsilon$, we know that $\|y_s - y_*(x_s)\| \leq \frac{\varepsilon}{\mu}$.

Then, observe that we have the following bound explained below

$$\begin{aligned}
\|\nabla_1 f(x_s, y_s)\| &= \|\nabla_1 f(x_s, y_s) - \nabla f(x_s, y_*(x_s)) + \nabla f(x_s, y_*(x_s))\| \\
&\leq \|\nabla_1 f(x_s, y_s) - \nabla f(x_s, y_*(x_s))\| + \|\nabla f(x_s, y_*(x_s))\| && (38) \\
&= \|\nabla_1 f(x_s, y_s) - \nabla_1 f(x_s, y_*(x_s))\| + \|\nabla f(x_s, y_*(x_s))\| && (39) \\
&\leq L_1 \|y_s - y_*(x_s)\| + \|\nabla f(x_s, y_*(x_s))\| && (40) \\
&\leq \frac{L_1 \varepsilon}{\mu} + \varepsilon = \Big(1 + \frac{L_1}{\mu}\Big)\varepsilon. && (41)
\end{aligned}$$

Observe that (38) follows from the triangle inequality, (39) is a result of applying the fact $\nabla f(x_s, y_*(x_s)) = \nabla_1 f(x_s, y)|_{y=y_*(x_s)}$ by Lemma 7, in (23) we used that $\nabla_1 f(x, y)$ is L_1-Lipschitz in y by Assumption 1, and (41) applies the inequality above that $\|y_s - y_*(x_s)\| \leq \frac{\varepsilon}{\mu}$.

Thus, in order to determine the iteration complexity T needed to get that $\|\nabla_1 f(x_s, y_s)\| \leq \varepsilon$, we can consider the T^* that ensures there exists some iterate $0 \leq s \leq T-1$ such that for $T \geq T^*$, we have both

$$\|\nabla f(x_s, y_*(x_s))\| \leq \varepsilon\Big(1 + \frac{L_1}{\mu}\Big)^{-1} \quad \text{and} \quad \|\nabla_2 f(x_s, y_s)\| \leq \varepsilon\Big(1 + \frac{L_1}{\mu}\Big)^{-1}.$$

This amounts to replacing ε in (37) with $\varepsilon(1 + \frac{L_1}{\mu})^{-1}$ to get that for

$$T \geq T^* := \frac{2\Big(1 + \frac{L_1}{\mu}\Big)^2 \Phi(x_0, y_0)}{\varepsilon^2 \gamma \min\{1, \tau\Gamma/4\}}, \tag{42}$$

we have both

$$\|\nabla_1 f(x_s, y_s)\| \leq \varepsilon \quad \text{and} \quad \|\nabla_2 f(x_s, y_s)\| \leq \varepsilon.$$

This holds for $\tau > 7\kappa^2$, $\gamma < \min\{\frac{1}{2L}, \frac{1}{2\tau L}, \frac{1}{2L_4}\}$, and any $\Gamma \in (0, 1/7]$ by Lemma 1. Thus, we can see that the complexity is $\widetilde{\mathcal{O}}(\varepsilon^{-2})$ to reach an ε-critical point under the given parameter choices. To obtain an explicit bound for fixed choices of τ, γ, and Γ, we can select the parameters to minimize the right hand side of (42). In particular, selecting $\tau = 8\kappa^2$, $\Gamma = 1/8$, and $\gamma = \frac{1}{2}\min\{\frac{1}{2L}, \frac{1}{2\tau L}, \frac{1}{2L_4}\}$ gives an iteration complexity of

$$T^* = \frac{32\Big(1 + \frac{L_1}{\mu}\Big)^2 \max\{L, 8\kappa^2 L, L_4\}\Phi(x_0, y_0)}{\varepsilon^2 \min\{4, \kappa^2\}}.$$

This completes the proof. □

E Convergence Analysis for Nonconvex-SC Zero-Sum Games

This appendix contains the analysis for the results from Section 5.2 on global convergence guarantees to ε-critical points in nonconvex-SC zero-sum games. We present proofs for the stochastic descent in Appendix E.1, and convergence in Appendix E.2. Before proceeding, we restate the stochastic τ-GDA update rule from Algorithm 2 and also recall relevant notation.

Stochastic τ–GDA Dynamics. Recall from Algorithm 2 that the combined update at each time k of the τ-GDA dynamics is given by

$$\begin{aligned} x_{k+1} &= x_k - \gamma g_1(x_k, y_k; \theta_{1,k}) = x_k - \gamma(\nabla_1 f(x_k, y_k) + \zeta_{1,k}) \\ y_{k+1} &= y_k + \gamma\tau g_2(x_k, y_k; \theta_{2,k}) = y_k + \gamma\tau(\nabla_2 f(x_k, y_k) + \zeta_{2,k}), \end{aligned}$$

so that

$$\begin{aligned} x_{k+1} - x_k &= -\gamma g_1(x_k, y_k; \theta_{1,k}) = -\gamma(\nabla_1 f(x_k, y_k) + \zeta_{1,k}) \\ y_{k+1} - y_k &= \gamma\tau\gamma g_2(x_k, y_k; \theta_{2,k}) = \gamma\tau(\nabla_2 f(x_k, y_k) + \zeta_{2,k}). \end{aligned}$$

In particular, $g_i(x_k, y_k; \theta_{i,k})$ denotes the stochastic gradient at step k for player $i \in \mathcal{I} = \{1, 2\}$ with $\theta_{i,k}$ a random variable drawn from the distribution $\mathcal{D}_i$ such that (by Assumption 3)

$$\mathbb{E}_{\theta_{i,k}\sim\mathcal{D}_i}[g_i(x_k, y_k; \theta_{i,k})] = \nabla_i f(x_k, y_k),$$

and for all $t \in \mathbb{R}$,

$$\mathbb{P}(\|g_i(x_k, y_k; \theta_{i,k}) - \nabla_i f(x_k, y_k)\| \geq t) \leq 2\exp\Big(-\frac{t^2}{2\sigma_i^2}\Big).$$

That is, the stochastic gradient for each player $i \in \mathcal{I}$ is σ_i-norm-subGaussian (see Definition 8). Moreover, for each player $i \in \mathcal{I}$, the noise in the stochastic gradient is denoted by

$$\zeta_{i,k} := g_i(x_k, y_k; \theta_{i,k}) - \nabla_i f(x_k, y_k).$$

The noise in the stochastic gradient denoted by $\zeta_{i,k}$ is itself a σ_i-norm-subGaussian random vector for each player $i \in \mathcal{I}$. This observation follows immediately from the fact that the stochastic gradient $g_i(x_k, y_k; \theta_{i,k})$ is σ_i-norm-subGaussian for each player $i \in \mathcal{I}$ and from the definition of a norm-subGaussian random vector (refer to Definition 8).

E.1 Proof of Lemma 3

In this appendix, we prove Lemma 3. We restate it here more precisely. The decent lemma shows that for a nonconvex-SC zero-sum game, the function f acts as a potential function for the τ-GDA dynamics. In particular, we show that the function f can be decomposed into a component that is decreasing along trajectories of τ-GDA and a component that exhibits a possible increase due to randomness in the stochastic gradients. The primary reason for the decrease is large gradients in addition to the structure of the maximizing player's problem and the timescale separation which we exploit in the proof to ensure sufficient decrease of f along trajectories. Recall that $\ell := L_1 + 3\kappa' L_2$ where $\kappa' := \frac{L_2}{\mu}$.

Lemma 3 (Restatement of Lemma 3)**.** *Consider a nonconvex-SC zero-sum game $(f, -f)$ defined by $f \in C^2(\mathcal{Z}, \mathbb{R})$ satisfying Assumption 1 and Assumption 3. There exists a absolute constant c such that with probability at least $1 - 3e^{-\iota}$ for any $\iota > 0$, the sequence generated by (stochastic) τ-GDA with parameters $\gamma \leq 1/\ell$ and $\tau = \Gamma/(\mu\gamma)$ for an absolute constant $\Gamma \geq 4$ satisfies*

$$f(x_k, y_k) - f(x_0, y_0) \leq -\frac{\gamma}{8}\sum_{t=0}^{k-1}\left(\|\nabla_1 f(x_t, y_t)\|^2 + \tau\|\nabla_2 f(x_t, y_t)\|^2\right) + c\gamma\left(\sigma_1^2(\gamma\ell k + \iota) + \tau\gamma\sigma_2^2\iota\right).$$

Proof of Lemma 3. We first argue a one step descent bound. Towards this end, take the Taylor expansion of $f(\cdot, y_k)$ to get that

$$\begin{aligned}
f(x_{k+1}, y_k) - f(x_k, y_k) &\leq \langle \nabla_1 f(x_k, y_k), x_{k+1} - x_k\rangle + \frac{L_1}{2}\|x_{k+1} - x_k\|^2 \\
&= -\gamma\langle \nabla_1 f(x_k, y_k), \nabla_1 f(x_k, y_k) + \zeta_{1,k}\rangle + \frac{\gamma^2 L_1}{2}\|\nabla_1 f(x_k, y_k) + \zeta_{1,k}\|^2 \\
&\leq -\gamma\langle \nabla_1 f(x_k, y_k), \nabla_1 f(x_k, y_k) + \zeta_{1,k}\rangle + \frac{\gamma^2 L_1}{2}\Big(\frac{3}{2}\|\nabla_1 f(x_k, y_k)\|^2 + 3\|\zeta_{1,k}\|^2\Big) \qquad (43) \\
&= -\Big(\gamma - \frac{3\gamma^2 L_1}{4}\Big)\|\nabla_1 f(x_k, y_k)\|^2 - \gamma\langle \nabla_1 f(x_k, y_k), \zeta_{1,k}\rangle + \frac{3\gamma^2 L_1}{2}\|\zeta_{1,k}\|^2. \qquad (44)
\end{aligned}$$

Note that (43) follows from the Cauchy-Schwarz inequality and Young's inequality for products. Specifically, for any $\epsilon > 0$, we have

$$\begin{aligned}
\|\nabla_1 f(x_k, y_k) + \zeta_{1,k}\|^2 &= \|\nabla_1 f(x_k, y_k)\|^2 + 2\langle \nabla_1 f(x_k, y_k), \zeta_{1,k}\rangle + \|\zeta_{1,k}\|^2 \\
&\leq \|\nabla_1 f(x_k, y_k)\|^2 + 2\|\nabla_1 f(x_k, y_k)\|\|\zeta_{1,k}\| + \|\zeta_{1,k}\|^2 \\
&\leq \|\nabla_1 f(x_k, y_k)\|^2 + 2\Big(\frac{1}{2\epsilon}\|\nabla_1 f(x_k, y_k)\|^2 + \frac{\epsilon}{2}\|\zeta_{1,k}\|^2\Big) + \|\zeta_{1,k}\|^2 \\
&\leq \frac{1+\epsilon}{\epsilon}\|\nabla_1 f(x_k, y_k)\|^2 + (1+\epsilon)\|\zeta_{1,k}\|^2. \qquad (45)
\end{aligned}$$

Thus, selecting $\epsilon = 2$ gives rise to the inequality in (43).

Now, since $f(x_{k+1}, y)$ is μ-strongly concave with respect to y, we have

$$f(x_{k+1}, y_{k+1}) - f(x_{k+1}, y_k) \leq \langle \nabla_2 f(x_{k+1}, y_k), y_{k+1} - y_k\rangle - \frac{\mu}{2}\|y_{k+1} - y_k\|^2. \qquad (46)$$

We now add and subtract $\langle \nabla_2 f(x_k, y_k), y_{k+1} - y_k\rangle$ and then apply Cauchy-Schwarz to get that

$$\begin{aligned}
f(x_{k+1}, y_{k+1}) - f(x_{k+1}, y_k) &\leq \langle \nabla_2 f(x_{k+1}, y_k), y_{k+1} - y_k\rangle - \frac{\mu}{2}\|y_{k+1} - y_k\|^2 \pm \langle \nabla_2 f(x_k, y_k), y_{k+1} - y_k\rangle \\
&= \langle \nabla_2 f(x_{k+1}, y_k) - \nabla_2 f(x_k, y_k), y_{k+1} - y_k\rangle - \frac{\mu}{2}\|y_{k+1} - y_k\|^2 \\
&\quad + \langle \nabla_2 f(x_k, y_k), y_{k+1} - y_k\rangle \\
&\leq \|\nabla_2 f(x_{k+1}, y_k) - \nabla_2 f(x_k, y_k)\|\|y_{k+1} - y_k\| - \frac{\mu}{2}\|y_{k+1} - y_k\|^2 \\
&\quad + \langle \nabla_2 f(x_k, y_k), y_{k+1} - y_k\rangle.
\end{aligned}$$

By Young's inequality with $\epsilon > 0$ applied to the term $\|\nabla_2 f(x_{k+1}, y_k) - \nabla_2 f(x_k, y_k)\|\|y_{k+1} - y_k\|$, the fact that $\nabla_2 f(x, y)$ is L_2-Lipschitz in x by Assumption 1, and the update equation for the y-player, we have that

$$\begin{aligned}
f(x_{k+1}, y_{k+1}) - f(x_{k+1}, y_k) &\leq \frac{1}{2\epsilon}\|\nabla_2 f(x_{k+1}, y_k) - \nabla_2 f(x_k, y_k)\|^2 + \frac{\epsilon}{2}\|y_{k+1} - y_k\|^2 - \frac{\mu}{2}\|y_{k+1} - y_k\|^2 \\
&\quad + \langle \nabla_2 f(x_k, y_k), y_{k+1} - y_k\rangle \\
&\leq \frac{L_2^2}{2\epsilon}\|x_{k+1} - x_k\|^2 - \frac{\mu - \epsilon}{2}\|y_{k+1} - y_k\|^2 + \langle \nabla_2 f(x_k, y_k), y_{k+1} - y_k\rangle \\
&= \frac{\gamma^2 L_2^2}{2\epsilon}\|\nabla_1 f(x_k, y_k) + \zeta_{1,k}\|^2 - \frac{(\mu - \epsilon)(\tau\gamma)^2}{2}\|\nabla_2 f(x_k, y_k) + \zeta_{2,k}\|^2 \\
&\quad + \tau\gamma\langle \nabla_2 f(x_k, y_k), \nabla_2 f(x_k, y_k) + \zeta_{2,k}\rangle \\
&\leq \frac{3\gamma^2 L_2^2}{4\epsilon}\|\nabla_1 f(x_k, y_k)\|^2 + \frac{3\gamma^2 L_2^2}{2\epsilon}\|\zeta_{1,k}\|^2 - \frac{(\mu - \epsilon)(\tau\gamma)^2}{2}\|\nabla_2 f(x_k, y_k) + \zeta_{2,k}\|^2 \\
&\quad + \tau\gamma\|\nabla_2 f(x_k, y_k)\|^2 + \tau\gamma\langle \nabla_2 f(x_k, y_k), \zeta_{2,k}\rangle. \qquad (47)
\end{aligned}$$

Observe that the final inequality above is a result of applying the estimate

$$\|\nabla_1 f(x_k, y_k) + \zeta_{1,k}\|^2 \leq \frac{3}{2}\|\nabla_1 f(x_k, y_k)\|^2 + 3\|\zeta_{1,k}\|^2,$$

which follows from the Cauchy-Schwarz inequality and Young's inequality for products as shown in (45).

To simplify the bound on $f(x_{k+1}, y_{k+1}) - f(x_{k+1}, y_k)$ given in (47), we expand the term on the right-hand side with $\|\nabla_2 f(x_k, y_k) + \zeta_{2,k}\|^2$ and then group common terms as follows:

$$\begin{aligned}
f(x_{k+1}, y_{k+1}) - f(x_{k+1}, y_k) &\leq \frac{3\gamma^2 L_2^2}{4\epsilon}\|\nabla_1 f(x_k, y_k)\|^2 + \frac{3\gamma^2 L_2^2}{2\epsilon}\|\zeta_{1,k}\|^2 - \frac{(\mu - \epsilon)(\tau\gamma)^2}{2}\|\nabla_2 f(x_k, y_k) + \zeta_{2,k}\|^2 \\
&\quad + \tau\gamma\|\nabla_2 f(x_k, y_k)\|^2 + \tau\gamma\langle \nabla_2 f(x_k, y_k), \zeta_{2,k}\rangle \\
&= \frac{3\gamma^2 L_2^2}{4\epsilon}\|\nabla_1 f(x_k, y_k)\|^2 + \frac{3\gamma^2 L_2^2}{2\epsilon}\|\zeta_{1,k}\|^2 + \tau\gamma\|\nabla_2 f(x_k, y_k)\|^2 + \tau\gamma\langle \nabla_2 f(x_k, y_k), \zeta_{2,k}\rangle \\
&\quad - \frac{(\mu - \epsilon)(\tau\gamma)^2}{2}\Big(\|\nabla_2 f(x_k, y_k)\|^2 + 2\langle \nabla_2 f(x_k, y_k), \zeta_{2,k}\rangle + \|\zeta_{2,k}\|^2\Big) \\
&= \frac{3\gamma^2 L_2^2}{4\epsilon}\|\nabla_1 f(x_k, y_k)\|^2 + \frac{3\gamma^2 L_2^2}{2\epsilon}\|\zeta_{1,k}\|^2 + \Big(\tau\gamma - \frac{(\mu - \epsilon)(\tau\gamma)^2}{2}\Big)\|\nabla_2 f(x_k, y_k)\|^2 \\
&\quad + \big(\tau\gamma - (\mu - \epsilon)(\tau\gamma)^2\big)\langle \nabla_2 f(x_k, y_k), \zeta_{2,k}\rangle - \frac{(\mu - \epsilon)(\tau\gamma)^2}{2}\|\zeta_{2,k}\|^2. \qquad (48)
\end{aligned}$$

We next combine the bound on $f(x_{k+1}, y_{k+1}) - f(x_{k+1}, y_k)$ from (48) with the bound on $f(x_{k+1}, y_k) - f(x_k, y_k)$ from (44) to get a bound on $f(x_{k+1}, y_{k+1}) - f(x_k, y_k)$. Towards this end, let $\epsilon = \frac{1}{3}\mu$, then by combining (44) and (48), and finally simplifying by grouping terms, we have the following:

$$\begin{aligned}
f(x_{k+1},y_{k+1})-f(x_k,y_k) &\le -\Big(\gamma-\frac{3\gamma^2L_1}{4}\Big)\|\nabla_1 f(x_k,y_k)\|^2-\gamma\langle\nabla_1 f(x_k,y_k),\zeta_{1,k}\rangle+\frac{3\gamma^2L_1}{2}\|\zeta_{1,k}\|^2\\
&\quad+\frac{3\gamma^2L_2^2}{4\epsilon}\|\nabla_1 f(x_k,y_k)\|^2+\frac{3\gamma^2L_2^2}{2\epsilon}\|\zeta_{1,k}\|^2+\Big(\tau\gamma-\frac{(\mu-\epsilon)(\tau\gamma)^2}{2}\Big)\|\nabla_2 f(x_k,y_k)\|^2\\
&\quad+\big(\tau\gamma-(\mu-\epsilon)(\tau\gamma)^2\big)\langle\nabla_2 f(x_k,y_k),\zeta_{2,k}\rangle-\frac{(\mu-\epsilon)(\tau\gamma)^2}{2}\|\zeta_{2,k}\|^2\\
&= -\Big(\gamma-\frac{3\gamma^2L_1}{4}-\frac{3\gamma^2L_2^2}{4\epsilon}\Big)\|\nabla_1 f(x_k,y_k)\|^2-\gamma\langle\nabla_1 f(x_k,y_k),\zeta_{1,k}\rangle\\
&\quad+\Big(\frac{3\gamma^2L_1}{2}+\frac{3\gamma^2L_2^2}{2\epsilon}\Big)\|\zeta_{1,k}\|^2+\Big(\tau\gamma-\frac{(\mu-\epsilon)(\tau\gamma)^2}{2}\Big)\|\nabla_2 f(x_k,y_k)\|^2\\
&\quad+\big(\tau\gamma-(\mu-\epsilon)(\tau\gamma)^2\big)\langle\nabla_2 f(x_k,y_k),\zeta_{2,k}\rangle-\frac{(\mu-\epsilon)(\tau\gamma)^2}{2}\|\zeta_{2,k}\|^2\\
&\le -\Big(\gamma-\frac{3\gamma^2L_1}{4}-\frac{9\gamma^2L_2^2}{4\mu}\Big)\|\nabla_1 f(x_k,y_k)\|^2-\gamma\langle\nabla_1 f(x_k,y_k),\zeta_{1,k}\rangle\\
&\quad+\Big(\frac{3\gamma^2L_1}{2}+\frac{9\gamma^2L_2^2}{2\mu}\Big)\|\zeta_{1,k}\|^2+\Big(\tau\gamma-\frac{\mu(\tau\gamma)^2}{3}\Big)\|\nabla_2 f(x_k,y_k)\|^2\\
&\quad+\Big(\tau\gamma-\frac{2\mu(\tau\gamma)^2}{3}\Big)\langle\nabla_2 f(x_k,y_k),\zeta_{2,k}\rangle. \qquad (49)
\end{aligned}$$

Observe that the inequality in (49) follows from the inequality

$$-\frac{(\mu-\epsilon)(\tau\gamma)^2}{2}\|\zeta_{2,k}\|^2=-\frac{\mu(\tau\gamma)^2}{3}\|\zeta_{2,k}\|^2\le 0.$$

To proceed in bounding $f(x_{k+1},y_{k+1})-f(x_k,y_k)$, we derive upper bounds on coefficients on the right-hand side of (49). Define $\ell:=L_1+3\kappa' L_2$ where $\kappa':=\frac{L_2}{\mu}$ and recall that $\gamma\le\frac{1}{\ell}$ and $\tau\gamma\ge 4/\mu$. From the fact that $\gamma\le 1/\ell$, we have that

$$-\Big(\gamma-\frac{3\gamma^2L_1}{4}-\frac{9\gamma^2L_2^2}{4\mu}\Big)=-\gamma+\frac{3\gamma^2}{4}\Big(L_1+\frac{3L_2^2}{\mu}\Big)=-\gamma+\frac{3\gamma^2\ell}{4}\le-\gamma+\frac{3\gamma}{4}=-\frac{\gamma}{4}.$$

This implies

$$-\Big(\gamma-\frac{3\gamma^2L_1}{4}-\frac{9\gamma^2L_2^2}{4\mu}\Big)\|\nabla_1 f(x_k,y_k)\|^2\le-\frac{\gamma}{4}\|\nabla_1 f(x_k,y_k)\|^2. \qquad (50)$$

Moreover, from a direct simplification,

$$\frac{3\gamma^2L_1}{2}+\frac{9\gamma^2L_2^2}{2\mu}=\frac{3\gamma^2}{2}\Big(L_1+\frac{3L_2^2}{\mu}\Big)=\frac{3\gamma^2\ell}{2}.$$

Hence,

$$\Big(\frac{3\gamma^2L_1}{2}+\frac{9\gamma^2L_2^2}{2\mu}\Big)\|\zeta_{1,k}\|^2=\frac{3\gamma^2\ell}{2}\|\zeta_{1,k}\|^2. \qquad (51)$$

Finally, using that $\tau\gamma\ge 4/\mu$, we also have

$$\tau\gamma-\frac{\mu(\tau\gamma)^2}{3}=\tau\gamma-\frac{\mu\tau\gamma(\tau\gamma)}{3}\le\tau\gamma-\frac{4\tau\gamma}{3}=-\frac{\tau\gamma}{3}\le-\frac{\tau\gamma}{4}.$$

Thus,

$$\Big(\tau\gamma-\frac{\mu(\tau\gamma)^2}{3}\Big)\|\nabla_2 f(x_k,y_k)\|^2\le-\frac{\tau\gamma}{4}\|\nabla_2 f(x_k,y_k)\|^2. \qquad (52)$$

Combining (49) with (50), (51), and (52) gives the following one-step bound on the τ-GDA update rule:

$$\begin{aligned}
f(x_{k+1},y_{k+1})-f(x_k,y_k) &\le -\frac{\gamma}{4}\|\nabla_1 f(x_k,y_k)\|^2-\frac{\tau\gamma}{4}\|\nabla_2 f(x_k,y_k)\|^2\\
&\quad-\gamma\langle\nabla_1 f(x_k,y_k),\zeta_{1,k}\rangle+\Big(\tau\gamma-\frac{2\mu(\tau\gamma)^2}{3}\Big)\langle\nabla_2 f(x_k,y_k),\zeta_{2,k}\rangle+\frac{3\gamma^2\ell}{2}\|\zeta_{1,k}\|^2.
\end{aligned}$$

Summing this inequality on both sides, we have that

$$
\begin{aligned}
f(x_k, y_k) - f(x_0, y_0) \leq &-\frac{\gamma}{4}\Big(\sum_{t=0}^{k-1}\|\nabla_1 f(x_t, y_t)\|^2 + \tau\sum_{t=0}^{k-1}\|\nabla_2 f(x_t, y_t)\|^2\Big) \\
&- \gamma\sum_{t=0}^{k-1}\langle\nabla_1 f(x_t, y_t), \zeta_{1,t}\rangle + \Big(\tau\gamma - \frac{2\mu(\tau\gamma)^2}{3}\Big)\sum_{t=0}^{k-1}\langle\nabla_2 f(x_t, y_t), \zeta_{2,t}\rangle + \frac{3\gamma^2\ell}{2}\sum_{t=0}^{k-1}\|\zeta_{1,t}\|^2.
\end{aligned}
\tag{53}
$$

Now, using the fact that $\zeta_{i,0}, \ldots, \zeta_{i,k-1}$ satisfy Condition 1 with norm-subGaussian parameter σ_i for $i \in \mathcal{I}$, by Lemma 12 for each player $i \in \mathcal{I}$ and any $\iota > 0$ and $\lambda_i > 0$, there exists an absolute constant c_i such that with probability at least $1 - e^{-\iota}$ we have:

$$
\sum_{t=0}^{k-1}\langle\nabla_i f(x_t, y_t), \zeta_{i,t}\rangle \leq c_i\sigma_i^2\lambda_i\sum_{t=0}^{k-1}\|\nabla_i f(x_t, y_t)\|^2 + \frac{\iota}{\lambda_i}. \tag{54}
$$

Thus, as a result of (54), there exists an absolute constant c_1 such that with probability at least $1 - e^{-\iota}$ for any $\iota > 0$, we have

$$
\begin{aligned}
-\gamma\sum_{t=0}^{k-1} \leq \nabla_1 f(x_t, y_t), \zeta_{1,t}\rangle &\leq \gamma\Big|\sum_{t=0}^{k-1}\langle\nabla_1 f(x_t, y_t), \zeta_{1,t}\rangle\Big| \\
&\leq \gamma\Big(c_1\sigma_1^2\lambda_1\sum_{t=0}^{k-1}\|\nabla_1 f(x_t, y_t)\|^2 + \frac{\iota}{\lambda_1}\Big) \\
&= \frac{\gamma}{8}\sum_{t=0}^{k-1}\|\nabla_1 f(x_t, y_t)\|^2 + 8c_1\gamma\sigma_1^2\iota \qquad (55) \\
&= \frac{\gamma}{8}\sum_{t=0}^{k-1}\|\nabla_1 f(x_t, y_t)\|^2 + c_1'\gamma\sigma_1^2\iota. \qquad (56)
\end{aligned}
$$

Note that in (55) λ_1 is being chosen as $\lambda_1 = 1/(8c_1\sigma_1^2)$ and in (56) the constant c_1' is being defined as $c_1' := 8c_1$. Now, using that $\tau\gamma = \Gamma/\mu$ where $\Gamma > 4$ is an absolute constant, we have

$$
|\tau\gamma - \frac{2\mu(\tau\gamma)^2}{3}| \leq \tau\gamma\Big|\frac{3 - 2\Gamma}{3}\Big|.
$$

Combining this inequality with (54), there exists an absolute constant c_2 such that with probability at least $1 - e^{-\iota}$ for any $\iota > 0$, we have

$$
\begin{aligned}
\Big(\tau\gamma - \frac{2\mu(\tau\gamma)^2}{3}\Big)\sum_{t=0}^{k-1}\langle\nabla_2 f(x_t, y_t), \zeta_{2,t}\rangle &\leq \Big|\tau\gamma - \frac{2\mu(\tau\gamma)^2}{3}\Big|\Big|\sum_{t=0}^{k-1}\langle\nabla_2 f(x_t, y_t), \zeta_{2,t}\rangle\Big| \\
&\leq \tau\gamma\Big|\frac{3 - 2\Gamma}{3}\Big|\Big|\sum_{t=0}^{k-1}\langle\nabla_2 f(x_t, y_t), \zeta_{2,t}\rangle\Big| \\
&\leq \tau\gamma\Big|\frac{3 - 2\Gamma}{3}\Big|\Big(c_2\sigma_2^2\lambda_2\sum_{t=0}^{k-1}\|\nabla_2 f(x_t, y_t)\|^2 + \frac{\iota}{\lambda_2}\Big) \\
&= \frac{\tau\gamma}{8}\sum_{t=0}^{k-1}\|\nabla_2 f(x_t, y_t)\|^2 + 8c_2\Big|\frac{3 - 2\Gamma}{3}\Big|^2\tau\gamma\sigma_2^2\iota. \qquad (57) \\
&= \frac{\tau\gamma}{8}\sum_{t=0}^{k-1}\|\nabla_2 f(x_t, y_t)\|^2 + c_2'\tau\gamma\sigma_2^2\iota. \qquad (58)
\end{aligned}
$$

Note that in (57), λ_2 is being chosen as $\lambda_2 = 1/(8c_2\sigma_2^2|(3 - 2\Gamma)/3|)$, and in (58) the constant c_2' is being defined as $c_2' := 8c_2|(3 - 2\Gamma)/3|^2$.

Now, using the fact that $\zeta_{1,0}, \ldots, \zeta_{1,k-1}$ satisfy Condition 1 with norm-subGaussian parameter σ_1, by Lemma 11 there exists an absolute constant c_3 such that with probability at least $1 - e^{-\iota}$ for any $\iota > 0$,

$$\sum_{t=0}^{k-1} \|\zeta_{1,t}\|^2 \leq c_3 \sigma_1^2 (k + \iota). \tag{59}$$

Hence, by (59), there exists an absolute constant $c_3' := 3c_3/2$ such that with probability at least $1 - e^{-\iota}$ for any $\iota > 0$,

$$\frac{3\gamma^2 \ell}{2} \sum_{t=0}^{k-1} \|\zeta_{1,t}\|^2 \leq c_3' \gamma^2 \ell \sigma_1^2 (k + \iota). \tag{60}$$

Finally, using (56), (58), and (60) in (53), we have that there exists an absolute constant $c := \max\{c_1', 2c_2', 2c_3'\}$ such that with probability at least $1 - 3e^{-\iota}$ for any $\iota > 0$,

$$\begin{aligned}
f(x_k, y_k) - f(x_0, y_0) &\leq -\frac{\gamma}{4}\Big(\sum_{t=0}^{k-1} \|\nabla_1 f(x_t, y_t)\|^2 + \tau \sum_{t=0}^{k-1} \|\nabla_2 f(x_t, y_t)\|^2\Big) \\
&\quad - \gamma \sum_{t=0}^{k-1} \langle \nabla_1 f(x_t, y_t), \zeta_{1,t} \rangle + \Big(\tau\gamma - \frac{2\mu(\tau\gamma)^2}{3}\Big) \sum_{t=0}^{k-1} \langle \nabla_2 f(x_t, y_t), \zeta_{2,t} \rangle + \frac{3\gamma^2 \ell}{2} \sum_{t=0}^{k-1} \|\zeta_{1,t}\|^2 \\
&\leq -\frac{\gamma}{4}\Big(\sum_{t=0}^{k-1} \|\nabla_1 f(x_t, y_t)\|^2 + \tau \sum_{t=0}^{k-1} \|\nabla_2 f(x_t, y_t)\|^2\Big) \\
&\quad + \frac{\gamma}{8}\Big(\sum_{t=0}^{k-1} \|\nabla_1 f(x_t, y_t)\|^2 + \tau \sum_{t=0}^{k-1} \|\nabla_2 f(x_t, y_t)\|^2\Big) + c_1' \gamma \sigma_1^2 \iota + c_2' \tau \gamma \sigma_2^2 \iota + c_3' \gamma^2 \ell \sigma_1^2 (k + \iota) \\
&\leq -\frac{\gamma}{8}\Big(\sum_{t=0}^{k-1} \|\nabla_1 f(x_t, y_t)\|^2 + \tau \sum_{t=0}^{k-1} \|\nabla_2 f(x_t, y_t)\|^2\Big) + c\gamma \left(\sigma_1^2(\gamma \ell k + \iota) + \tau \sigma_2^2 \iota\right).
\end{aligned}$$

Note the final inequality is obtained using that $c := \max\{c_1', 2c_2', 2c_3'\}$ and $\gamma \leq 1/\ell$ to get

$$c_1' \gamma \sigma_1^2 \iota + c_3' \gamma^2 \ell \sigma_1^2 (k + \iota) = \max\{c_1', c_3'\} \gamma \sigma_1^2 (\iota + \gamma \ell k + \gamma \ell \iota) \leq \max\{2c_1', 2c_3'\} \gamma \sigma_1^2 (\gamma \ell k + \iota),$$

so that

$$c_2' \tau \gamma \sigma_2^2 \iota + c_1' \gamma \sigma_1^2 \iota + c_3' \gamma^2 \ell \sigma_1^2 (k + \iota) \leq c_2' \tau \gamma \sigma_2^2 \iota + \max\{2c_1', 2c_3'\} \gamma \sigma_1^2 (\gamma \ell k + \iota) \leq c\gamma \left(\sigma_1^2(\gamma \ell k + \iota) + \tau \sigma_2^2 \iota\right).$$

Finally, we remark that in the deterministic case that $\sigma_1^2 = \sigma_2^2 = 0$, the bound holds deterministically and simply reduces to

$$f(x_k, y_k) - f(x_0, y_0) \leq -\frac{\gamma}{8}\Big(\sum_{t=0}^{k-1} \|\nabla_1 f(x_t, y_t)\|^2 + \tau \sum_{t=0}^{k-1} \|\nabla_2 f(x_t, y_t)\|^2\Big).$$

This completes the proof. □

E.2 Proof of Corollary 2

In this appendix, we prove Corollary 2. We restate the result and then provide the proof.

Corollary 2. *Consider a nonconvex-SC zero-sum game $(f, -f)$ defined by $f \in C^2(\mathcal{Z}, \mathbb{R})$ satisfying Assumption 1 and Assumption 3. For any $\delta \in (0, 1)$, there exists γ and τ satisying the conditions of Lemma 3 such that, with probability at least $1 - \delta$, starting from any initialization, at least half the iterates of (stochastic) τ-*`GDA`* will be ε-critical points after $\widetilde{\mathcal{O}}(\varepsilon^{-2})$ and $\widetilde{\mathcal{O}}(\varepsilon^{-6})$ iterations in the deterministic and stochastic settings, respectively.*

Proof of Corollary 2. This proof primarily follows from Lemma 3. Specifically, Lemma 3 gives us a bound on the iterates of τ-`GDA` decreasing the function value that will be used in a proof by contradiction. In particular, we assume that more than half the iterates after running the algorithm for T steps are not ε-critical points. We then invoke Lemma 3 to conclude that if this was the case, then the function value would have decreased beyond the global minimum. Since this is not possible, it yields a contradiction, which then allows us to conclude that at least half of the iterates must be ε-critical points. We prove this result for the deterministic and stochastic settings independently following this general template.

Deterministic Setting. Suppose that γ and τ are chosen as follows:

$$\gamma = 1/\ell \quad \text{and} \quad \tau = 4/(\mu\gamma).$$

Let the number of steps that the algorithm runs for be defined by

$$T := \frac{32(f(x_0, y_0) - f^*)}{\gamma\varepsilon^2} = \frac{32\ell(f(x_0, y_0) - f^*)}{\varepsilon^2} = \widetilde{\mathcal{O}}(\varepsilon^{-2}).$$

For the sake of contradiction, suppose that within T steps, we have more than $T/2$ iterates for which $\max\{\|\nabla_1 f(x_k, y_k)\|, \|\nabla_2 f(x_k, y_k)\|\} \geq \varepsilon$. Then, by Lemma 3, taking $\sigma_1^2 = \sigma_2^2 = 0$, we have that:

$$\begin{aligned}
f(x_T, y_T) - f(x_0, y_0) &\leq -\frac{\gamma}{8}\sum_{k=0}^{T-1}\left(\|\nabla_1 f(x_k, y_k)\|^2 + \tau\|\nabla_2 f(x_k, y_k)\|^2\right) \\
&\leq -\frac{\gamma}{8}\sum_{k=0}^{T-1}\left(\|\nabla_1 f(x_k, y_k)\|^2 + \|\nabla_2 f(x_k, y_k)\|^2\right) &(61)\\
&\leq -\frac{\gamma}{8}\sum_{k=0}^{T-1}\max\{\|\nabla_1 f(x_k, y_k)\|^2, \|\nabla_2 f(x_k, y_k)\|^2\} &(62)\\
&\leq -\frac{\gamma T\varepsilon^2}{16}. &(63)
\end{aligned}$$

Observe that in (61) we have applied the fact that $\tau \geq 1$, in (62) we have used that the sum of positive numbers is greater than the max of the numbers, and in (63) we have invoked the assumption that at least $T/2$ iterates satisfy $\max\{\|\nabla_1 f(x_k, y_k)\|, \|\nabla_2 f(x_k, y_k)\|\} \geq \varepsilon$.

We now show that this implies a contradiction. In particular, the bound in (63) implies that

$$f(x_T, y_T) \leq f(x_0, y_0) - \frac{\gamma T\varepsilon^2}{16} = f(x_0, y_0) - \frac{\gamma\varepsilon^2}{16}\frac{32(f(x_0, y_0) - f^*)}{\gamma\varepsilon^2} < f(x_0, y_0) - f(x_0, y_0) - f^* = f^*.$$

This is not possible and yields a contradiction to the assumption that we have more than $T/2$ iterates for which $\max\{\|\nabla_1 f(x_k, y_k)\|, \|\nabla_2 f(x_k, y_k)\|\} \geq \varepsilon$. Hence, at most $T/2$ iterates have $\max\{\|\nabla_1 f(x_k, y_k)\|, \|\nabla_2 f(x_k, y_k)\|\} \geq \varepsilon$ and thus at least $T/2$ iterates must be ε-critical points if the algorithm runs for $T = \widetilde{\mathcal{O}}(\varepsilon^{-2})$ iterations.

Stochastic Setting. Suppose that γ and τ are chosen as follows:

$$\gamma := \frac{1}{\iota^2\ell\mathcal{R}} \quad \text{and} \quad \tau = \frac{4}{\mu\gamma} \quad \text{where} \quad \mathcal{R} := 1 + \frac{\max\{\sigma_1^2, \sigma_2^2\}}{\varepsilon^2}.$$

Moreover, $\iota > 0$ is an absolute constant to be chosen later. Let the number of steps that the algorithm runs for be defined by

$$T := \frac{\tau(f(x_0, y_0) - f^*)}{\ell\gamma\varepsilon^2} = \frac{(f(x_0, y_0) - f^*)\ell\iota^4\mathcal{R}^2}{\mu\varepsilon^2} = \widetilde{\mathcal{O}}(\varepsilon^{-6}). \tag{64}$$

For the sake of contradiction, suppose that within T steps, we have more than $T/2$ iterates for which $\max\{\|\nabla_1 f(x_k, y_k)\|, \|\nabla_2 f(x_k, y_k)\|\} \geq \varepsilon$. Then, by Lemma 3, there exists an absolute constant c such that with probability at least $1 - 3e^{-\iota}$ for any $\iota > 0$, we have that:

$$\begin{aligned}
f(x_T, y_T) - f(x_0, y_0) &\leq -\frac{\gamma}{8}\sum_{k=0}^{T-1}\left(\|\nabla_1 f(x_k, y_k)\|^2 + \tau\|\nabla_2 f(x_k, y_k)\|^2\right) + c\left(\gamma\sigma_1^2(\gamma\ell T + \iota) + \tau\gamma\sigma_2^2\iota\right) \\
&\leq -\frac{\gamma}{8}\sum_{k=0}^{T-1}\left(\|\nabla_1 f(x_k, y_k)\|^2 + \|\nabla_2 f(x_k, y_k)\|^2\right) + c\left(\gamma\sigma_1^2(\gamma\ell T + \iota) + \tau\gamma\sigma_2^2\iota\right) &(65)\\
&\leq -\frac{\gamma}{8}\sum_{k=0}^{T-1}\max\{\|\nabla_1 f(x_k, y_k)\|^2, \|\nabla_2 f(x_k, y_k)\|^2\} + c\left(\gamma\sigma_1^2(\gamma\ell T + \iota) + \tau\gamma\sigma_2^2\iota\right) &(66)\\
&\leq -\frac{\gamma T\varepsilon^2}{16} + c\gamma\left(\sigma_1^2(\gamma\ell T + \iota) + \tau\sigma_2^2\iota\right). &(67)
\end{aligned}$$

Observe that in (65) we have applied the fact that $\tau \geq 1$, in (66) we have used that the sum of positive numbers is greater than the max of the numbers, and in (67) we have invoked the assumption that at least $T/2$ iterates satisfy $\max\{\|\nabla_1 f(x_k, y_k)\|, \|\nabla_2 f(x_k, y_k)\|\} \geq \varepsilon$.
Define $\sigma^2 := \max\{\sigma_1^2, \sigma_2^2\}$. We now continue upper bounding (67) the latter term on the right-hand side. To begin, using that $\sigma^2 := \max\{\sigma_1^2, \sigma_2^2\}$ and $\tau \geq 1$, we have

$$c\gamma\left(\sigma_1^2(\gamma\ell T + \iota) + \tau\sigma_2^2\iota\right) \leq c\gamma\sigma^2(\gamma\ell T + \iota + \tau\iota) \leq c\gamma\sigma^2\tau\iota\big(\frac{\gamma\ell T}{\tau\iota} + 2\big).$$

Observe that since $\mathcal{R} = 1 + \sigma^2/\varepsilon^2 \geq \sigma^2/\varepsilon^2$, we have

$$\gamma\tau\sigma^2\iota = \frac{\tau\sigma^2\iota}{\ell\iota^2\mathcal{R}} = \frac{\tau\sigma^2}{\iota\ell\mathcal{R}} \leq \frac{\tau\varepsilon^2}{\ell\iota}.$$

Hence, combining the previous inequalities, we get

$$c\gamma\left(\sigma_1^2(\gamma\ell T + \iota) + \tau\sigma_2^2\iota\right) \leq \frac{c\tau\varepsilon^2}{\ell\iota}\big(\frac{\gamma\ell T}{\tau\iota} + 2\big) = \frac{c\gamma T\varepsilon^2}{\iota^2} + \frac{2c\tau\varepsilon^2}{\ell\iota} = \frac{c\gamma T\varepsilon^2}{\iota^2} + \frac{2c\tau\gamma\varepsilon^2}{\ell\gamma\iota}.$$

Now, since

$$\frac{\tau}{\gamma\ell} \leq \frac{\tau(f(x_0, y_0) - f^*)}{\gamma\ell\varepsilon^2} = T,$$

by choosing ι as a sufficiently large absolute constant, we get

$$c\gamma\left(\sigma_1^2(\gamma\ell T + \iota) + \tau\sigma_2^2\iota\right) \leq \frac{c\gamma T\varepsilon^2}{\iota^2} + \frac{2c\tau\gamma\varepsilon^2}{\ell\gamma\iota} \leq \frac{c\gamma T\varepsilon^2}{\iota^2} + \frac{2c\gamma T\varepsilon^2}{\iota} \leq \frac{\gamma T\varepsilon^2}{32}.$$

Thus, returning to (67), we have that

$$f(x_T, y_T) - f(x_0, y_0) \leq -\frac{\gamma T\varepsilon^2}{16} + c\gamma\left(\sigma_1^2(\gamma\ell T + \iota) + \tau\sigma_2^2\iota\right) \leq -\frac{\gamma T\varepsilon^2}{16} + \frac{\gamma T\varepsilon^2}{32} = -\frac{\gamma T\varepsilon^2}{32}.$$

We now show that this implies a contradiction. In particular, this bound implies that

$$\begin{aligned}
f(x_T, y_T) &\leq f(x_0, y_0) - \frac{\gamma T\varepsilon^2}{32} \\
&\leq f(x_0, y_0) - \frac{\gamma\varepsilon^2}{32}\frac{\tau(f(x_0, y_0) - f^*)}{\ell\gamma\varepsilon^2} \\
&= f(x_0, y_0) - \frac{4\iota^2\ell\mathcal{R}}{32\mu}(f(x_0, y_0) - f^*) \\
&< f(x_0, y_0) - f(x_0, y_0) - f^* \\
&= f^*.
\end{aligned}$$

This is not possible and yields a contradiction to the assumption that we have more than $T/2$ iterates for which $\max\{\|\nabla_1 f(x_k, y_k)\|, \|\nabla_2 f(x_k, y_k)\|\} \geq \varepsilon$. Hence, with probability at least $1 - 3e^{-\iota}$, at most $T/2$ iterates have $\max\{\|\nabla_1 f(x_k, y_k)\|, \|\nabla_2 f(x_k, y_k)\|\} \geq \varepsilon$. Thus, with probability at least $1 - 3e^{-\iota}$, at least $T/2$ iterates must be ε-critical points if the algorithm runs for $T = \widetilde{\mathcal{O}}(\varepsilon^{-6})$ iterations. This statement then holds with probability at least $1 - \delta$ for the given $\delta \in (0, 1)$ by selecting a sufficient large constant $\iota > 0$. This completes the proof. $\square$