Heterogeneous Matrix Factorization: When Features Differ by Datasets

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Abstract

In myriad statistical applications, data are collected from related but heterogeneous sources. These sources share some commonalities while containing idiosyncratic characteristics. One of the most fundamental challenges in such scenarios is to recover the shared and source-specific factors at scale. Despite the existence of a few heuristic approaches, a scalable algorithm with theoretical guarantees has yet to be established.

In this paper, we tackle the problem by proposing a method called **H**eterogeneous **M**atrix **F**actorization to separate the shared and unique factors for a class of problems. HMF maintains the orthogonality between the shared and unique factors by leveraging an invariance property in the objective. The algorithm is easy to implement and intrinsically distributed. On the theoretic side, we show that for the square error loss, HMF will converge into the optimal solutions, which are close to the ground truth.

HMF can be integrated auto-encoders to learn nonlinear feature mappings. Through a variety of case studies, we showcase HMF's benefits and applicability in video segmentation, time-series feature extraction, and recommender systems.

1 Introduction

In the Internet of Things (IoT), data are frequently gathered at the edge devices such as mobile phones or sensors, which often operate under different conditions such as temperature, pressure, or vibration (Kontar et al., 2021).This is an example of data collection from a diverse range of related sources. Resultingly, data exhibit both shared features, which represent common knowledge, and unique features, associated with source-specific individual characteristics.

A central challenge in analyzing the data patterns among heterogeneous sources is to decouple the shared and unique features from the observations. A natural approach to handle it is to use mutually orthogonal vectors to model the shared and unique features (Lock et al., 2013; Yang & Michailidis, 2016; Zhou et al., 2015; Sagonas et al., 2017; Gaynanova & Li, 2019; Park & Lock, 2020; Liang et al., 2023). Along this line, JIVE (Lock et al., 2013) uses an alternative minimization algorithm to optimize the joint and individual factors. While JIVE can find meaningful patterns in some genetic applications, the orthogonality is not well-respected in the algorithm. Resultingly, the separation is not ideal, and the estimates of the shared components have limited fidelities (Zhou et al., 2015). Several works attempt to improve JIVE by analyzing the singular vectors of observation matrices (Zhou et al., 2015), considering more complicated structures (Park & Lock, 2020), and many more. However, many of these algorithms focus on specific statistical problems and employ heuristic algorithms that do not have a strong convergence guarantee. Distributing these algorithms is also an arduous task.

Very recently, personalized PCA (perPCA) (Shi & Kontar, 2024) has emerged as a promising method for distributedly recovering shared and unique features with strong convergence guarantees. Despite its eminent performance on an array of applications, perPCA is limited to handling symmetric covariance matrices and cannot be readily extended to asymmetric and incomplete observation settings, where only a small subset of data is available.

Hence, there is a need for an extensible algorithm to separate shared and unique factors with a provable convergence guarantee. In this paper, we propose a framework called **H**eterogeneous **M**atrix **F**actorization (`HMF`) to capture the shared and unique factors under the orthogonality constraint. Our formulation is based on nonconvex Matrix Factorization (MF), a widely used technique in data analytics for identifying low-rank structures within high-dimensional matrices. Such low-rank structures can capture underlying physical processes or latent features that are highly informative for understanding patterns in high-dimensional observations (Wright & Ma, 2022), thus are extremely suitable to represent the shared and unique features among data. Also, the matrix factorization formulation can be extended to an auto-encoder model, which employs the representation power of neural networks.

More specifically, we consider a group of N observation matrices $\{\mathbf{M}_{(i)}\}_{i=1}^N$ from N related but heterogeneous sources. We model the common information in these matrices as low-rank components whose columns span the same subspace. Accordingly, the unique signals are modeled by low-rank components with source-specific column subspaces. The common and unique subspaces are orthogonal to represent the inductive bias that the shared and unique features are independent. In nonlinear models, we similarly use shared and source-specific nonlinear embeddings to model common and unique data patterns.

To learn the shared and unique features, `HMF` exploits an invariance property of the objective to handle the constraints by introducing two correction steps. The application of the invariance property is one of the key distinctions between `HMF` and previous works, as it allows `HMF` to maintain orthogonality between shared and unique factors without changing the objective. With the correction step, `HMF` is proved to converge linearly to an optimal solution under the SE loss with suitable stepsize and initial optimality gap. We also provide an upper bound on the statistical error between the updates and the ground truths.

We use a wide range of numerical experiments to demonstrate the effectiveness of the proposed `HMF`. The case studies on video segmentation, temporal signal analysis, and movie recommendation showcase the benefits of extracting shared and unique factors.

2 Related Work

Matrix Factorization MF has been applied to a diverse range of fields, including image processing (Lee & Seung, 1999), time series analysis (Yu et al., 2016), and many others, making it one of the most popular methods in data analytics. Numerous works analyze the theoretical properties of first-order algorithms that solve the (asymmetric) matrix factorization problem $\min_{\mathbf{U},\mathbf{V}} \left\|\mathbf{M} - \mathbf{U}\mathbf{V}^T\right\|_F^2$ or its variants (Li et al., 2018a; Ye & Du, 2021; Sun & Luo, 2016; Park et al., 2017; Tu et al., 2016). Among them, Sun & Luo (2016) analyzes the local landscape of the optimization problem and establishes the local linear convergence of a series of first-order algorithms. Park et al. (2017); Ge et al. (2017) study the global geometry of the optimization problem. Tu et al. (2016) propose the Rectangular Procrustes Flow algorithm that is proved to converge linearly into the ground truth under proper initialization and a balancing regularization of the form $\left\|\mathbf{U}^T\mathbf{U} - \mathbf{V}^T\mathbf{V}\right\|_F^2$. Despite the abundance of literature on standard matrix factorization and matrix completion, these works do not consider the case where data contain heterogeneous trends.

Distributed matrix factorization Recent development of edge computation has fueled a trend to move matrix factorization to the edge. Distributed matrix factorization (DMF) (Gemulla et al., 2011) exploits the distributed gradient descent to factorize large matrices. Chai et al. (2021) proposes a cryptographic framework where multiple clients use their local data to collaboratively factorize a matrix without leaking private information to the server. These works use one set of feature matrices $\mathbf{U}$ and $\mathbf{V}$ to fit data from all clients, and hence, they do not account for source-by-source feature differences either.

Personalized modeling As discussed, there are a few methods attempting to find the joint and individual components by heuristic algorithms. Besides the works discussed in the introduction section, some works leverage additional structural assumptions in data. Among them, SLIDE (Gaynanova & Li, 2019) analyzes the group sparsity structure. iNMF (Yang & Michailidis, 2016) adds nonnegativity constraints to the recovered matrices. RJIVE (Sagonas et al., 2017) aims to cope with the gross noise in the observations. However, to our best knowledge, none of these algorithms can guarantee that the iterates will converge into the optimal solutions. Our work improves upon the previous work by providing a flexible formulation for matrix

factorization and proposing a first-order algorithm equipped with a convergence guarantee. A comprehensive comparison between HMF and exemplary previous works is outlined in Table 1.

Table 1: Comparison of different methods. *d* stands for whether a method can be *d*istributed, *o* stands for whether the method ensures the *o*rthogonality between the shared and unique components, *c* stands for whether it has a *c*onvegence guarantee, *i* stands for whether it can handle *i*ncomplete observations.

Algorithm	d	o	c	i
DMF (Gemulla et al., 2011)	✓	✗	✓	✗
JIVE (Lock et al., 2013)	✗	✗	✗	✗
COBE (Zhou et al., 2015)	✗	✗	✗	✗
RJIVE (Sagonas et al., 2017)	✗	✓	✗	✗
perPCA (Shi & Kontar, 2024)	✓	✓	✓	✗
perDL (Liang et al., 2023)	✓	✗	✓	✗
HMF (ours)	✓	✓	✓	✓

3 Model

We consider the setting where $N \in \mathbb{N}^+$ noisy observation matrices $\mathbf{M}_{(1)}, \mathbf{M}_{(2)}, \cdots, \mathbf{M}_{(N)}$ are collected from N different but related sources. These matrices $\mathbf{M}_{(i)} \in \mathbb{R}^{n_1 \times n_{2,(i)}}$ are assumed to have the same number of rows n_1.

3.1 Linear model

The natural way to model the commonality and uniqueness among the matrices is assuming that each matrix is driven by r_1 shared factors and $r_{2,(i)}$ unique factors and also contaminated by noise. More specifically, we consider the generation model for matrix $\mathbf{M}_{(i)}$ is defined as

$$\mathbf{M}_{(i)} = \mathbf{U}^\star{}_g \mathbf{V}^{\star T}_{(i),g} + \mathbf{U}^\star{}_{(i),l} \mathbf{V}^{\star T}_{(i),l} + \mathbf{E}^\star{}_{(i)} \tag{1}$$

where $\mathbf{U}^\star{}_g \in \mathbb{R}^{n_1 \times r_1}$, $\mathbf{V}^\star{}_{(i),g} \in \mathbb{R}^{n_{2,(i)} \times r_1}$, $\mathbf{U}^\star{}_{(i),l} \in \mathbb{R}^{n_1 \times r_{2,(i)}}$, $\mathbf{V}^\star{}_{(i),l} \in \mathbb{R}^{n_{2,(i)} \times r_{2,(i)}}$, $\mathbf{E}^\star{}_{(i)} \in \mathbb{R}^{n_1 \times n_{2,(i)}}$. We use * to denote the ground truth. In the above model, r_1 is the rank of the global (shared) feature matrices, while $r_{2,(i)}$ is the rank of local (unique) feature matrices from source i. The matrix $\mathbf{U}^\star{}_g \mathbf{V}^{\star T}_{(i),g}$ models the shared low-rank components of the observation matrix, as the column space is the same across different sources. The matrix $\mathbf{U}^\star{}_{(i),l} \mathbf{V}^{\star T}_{(i),l}$ models the unique low-rank part. The rationale of the low-rank matrix factorization is to use a small number of latent vectors to explain the observations. Thus the ranks r_1 and $r_{2,(i)}$ are often much smaller than matrix dimensions. $\mathbf{E}^\star{}_{(i)}$ models the noise from source i.

In matrix factorization problems, the representations $\mathbf{U}^\star{}_g$ and $\mathbf{U}^\star{}_{(i),l}$ often correspond to latent data features. For better interpretability, it is often desirable to have the underlying features disentangled so that each feature can vary independently of others (Higgins et al., 2017). Under this rationale, we consider the model where shared and unique factors are orthogonal,

$$\mathbf{U}^{\star T}_g \mathbf{U}^\star{}_{(i),l} = 0, \ \forall i \in [N] \tag{2}$$

We use $[N]$ to denote the set $\{1, 2, \cdots, N\}$. The orthogonality of features implies that the shared and unique features span different subspaces, thus describing different patterns in the observation. The orthogonality condition is consistent with literature (Lock et al., 2013; Zhou et al., 2015).

Under the data generation model equation 1 and equation 2, our goal is to find $\mathbf{U}_g, \{\mathbf{V}_{(i),g}, \mathbf{U}_{(i),l}, \mathbf{V}_{(i),l}\}$ from observations $\{\mathbf{M}_{(i)}\}$. To achieve this goal, we propose a generic constrained optimization problem.

$$\min_{\mathbf{x}} \sum_{i=1}^{N} \tilde{f}_i(\mathbf{U}_g, \mathbf{V}_{(i),g}, \mathbf{U}_{(i),l}, \mathbf{V}_{(i),l}) \quad \text{such that } \mathbf{U}_g^T \mathbf{U}_{(i),l} = 0, \ \forall i \in [N] \tag{3}$$

where $\mathbf{x} = \left(\mathbf{U}_g, \{\mathbf{U}_{(i),l}, \mathbf{V}_{(i),g}, \mathbf{V}_{(i),l}\}_{i=1}^N\right)$ collects the decision variables and $\tilde{f}_i$ is a regularized empirical risk consisting of two parts.

$$\begin{aligned}&\tilde{f}_i(\mathbf{U}_g, \mathbf{V}_{(i),g}, \mathbf{U}_{(i),l}, \mathbf{V}_{(i),l})\\ &= \underbrace{\ell\left(\mathbf{M}_{(i)}, \mathbf{U}_g\mathbf{V}_{(i),g}^T + \mathbf{U}_{(i),l}\mathbf{V}_{(i),l}^T\right)}_{f_i} + \underbrace{\frac{\beta}{2}\left\|\mathbf{U}_g^T\mathbf{U}_g - \mathbf{I}\right\|_F^2 + \frac{\beta}{2}\left\|\mathbf{U}_{(i),l}^T\mathbf{U}_{(i),l} - \mathbf{I}\right\|_F^2}_{g_i}\end{aligned} \tag{4}$$

We will explain them respectively. Term f_i is the empirical risk between the sum of shared and unique signals and the observation matrix $\mathbf{M}_{(i)}$. ℓ is a loss metric that can incorporate a wide range of applications. For example, in matrix factorization, the squared error loss is $\ell^{se}\left(\mathbf{M}_{(i)}, \hat{\mathbf{M}}_{(i)}\right) = \frac{1}{2}\left\|\mathbf{M}_{(i)} - \hat{\mathbf{M}}_{(i)}\right\|_F^2$, i.e., the square error between the observed and reconstructed matrices. In matrix completion, only a limited number of entries are observed. We use $\Omega_{(i)}$ to denote the set of indices of the observed entries, $\mathcal{P}_{\Omega_{(i)}}(\cdot)$ to denote the projection

$$[\mathcal{P}_{\Omega_{(i)}}(\mathbf{M})]_{j,k} = \begin{cases}[\mathbf{M}]_{j,k} & \text{if entry } (j,k) \in \Omega_{(i)}\\ 0 & \text{otherwise.}\end{cases}$$

for a general matrix $\mathbf{M}$. Then the projected square error loss ℓ^{pse} is defined as $\ell^{pse}\left(\mathbf{M}_{(i)}, \hat{\mathbf{M}}_{(i)}\right) = \frac{1}{2}\left\|\mathcal{P}_{\Omega_{(i)}}\left(\mathbf{M}_{(i)} - \hat{\mathbf{M}}_{(i)}\right)\right\|_F^2$. There are also numerous other loss metrics ℓ, such as Huber loss used in robust matrix factorization (Wang & Fan, 2022) or cross-entropy loss used in community detection (Yang & Leskovec, 2013).

The constraints $\mathbf{U}_g^T\mathbf{U}_{(i),l} = 0$ reflect our prior belief about the data generation process. Since we consider the model where the true shared and unique features are orthogonal equation 2, it is natural to encode the orthogonality into constraints. Explicitly enforcing the orthogonality constraints is one of the major differences between our work and the previous works including JIVE. As we will see in numerical examples, such constraints can encourage shared and unique features to capture more distinct patterns.

Term g_i is a regularization term that pushes the $\mathbf{U}$ matrices to be orthonormal. A well-known theoretical issue in matrix factorization is that objectives like the term f_i are homogeneous with respect to the variables, which in turn leads to the non-smoothness of the loss function. Regularization terms are added to ensure the smoothness of the loss. Several works on matrix completion (e.g. (Park et al., 2017; Tu et al., 2016; Fattahi & Sojoudi, 2020)) consider balancing regularization terms of the form $\left\|\mathbf{U}_g^T\mathbf{U}_g - \mathbf{V}_{(i),g}^T\mathbf{V}_{(i),g}\right\|_F^2$ to encourage column factors $\mathbf{U}_g$ and row factors $\mathbf{V}_{(i),g}$ to have similar singular values. Under a similar rationale, we leverage the regularization terms in g_i to prevent too radical $\mathbf{U}_g$ and $\mathbf{U}_{(i),l}$'s, as such regularizations are easier to work with in the presence of the constraints $\mathbf{U}_g^T\mathbf{U}_{(i),l} = 0$. Here, the parameter β controls the strength of the regularization.

Since term f_i and g_i are both nonconvex, and the feasible set corresponding to the constraint $\mathbf{U}_g^T\mathbf{U}_{(i),l} = 0$ is also nonconvex, the problem equation 3 is nonconvex. In the next section, we will propose a gradient-based and distributed algorithm to solve the nonconvex problem.

3.2 Extension to auto-encoders

The model equation 1 is based on linear embeddings of $\mathbf{U}$ and $\mathbf{V}$. As a natural extension, we use auto-encoders (Goodfellow et al., 2016) to find the nonlinear embeddings in data. If some entries of a matrix $\mathbf{M}$ are missing, we use $\mathbf{M}^{obs}$ to denote the padded matrix $\mathbf{M}$ whose unobserved entries are simply replaced with 0. For auto-encoders, the feature matrix $\mathbf{U}$ is replaced by an encoder network $\texttt{Net}_{\mathbf{U}} : \mathbb{R}^d \to \mathbb{R}^r$ that maps a column of matrix $\mathbf{M}_{(i)}^{obs}$ into an embedding vector with dimension r, and the coefficient matrix $\mathbf{V}$ is replaced by a decoder neural network $\texttt{Net}_{\mathbf{V}} : \mathbb{R}^r \to \mathbb{R}^d$ that maps an embedding vector with dimension r to the reconstructed observation vector in $\mathbb{R}^d$. We use $\mathbf{U}$ and $\mathbf{V}$ to denote trainable weights that parametrize the neural networks. To accommodate the heterogeneity of features from N different $\mathbf{M}_{(i)}^{obs}$, we also introduce a shared encoder $\texttt{Net}_{\mathbf{U}_g}$ and N unique encoders $\{\texttt{Net}_{\mathbf{U}_{(i),l}}\}$ to generate the embeddings. All encoders have

the same architecture but are parametrized by different weights. In analogy to equation 3, we thus propose the following objective for the nonlinear version of HMF,

$$\begin{aligned}&\min_{\mathbf{U}_g,\{\mathbf{U}_{(i),l}\},\mathbf{V}} \sum_{i=1}^{N} \left[\ell\Big(\mathbf{M}_{(i)}, \texttt{Net}_{\mathbf{V}}\Big(\texttt{Net}_{\mathbf{U}_g}\left(\mathbf{M}_{(i)}^{obs}\right) + \texttt{Net}_{\mathbf{U}_{(i),l}}\left(\mathbf{M}_{(i)}^{obs}\right)\Big)\Big) + \mathcal{R}\left(\mathbf{U}_g, \mathbf{U}_{(i),l}, \mathbf{V}\right)\right] \\ &\text{such that, } \mathcal{F}(\texttt{Net}_{\mathbf{U}_g}([\mathbf{M}_{(i)}^{obs}]_{:,j}))^T \mathcal{F}(\texttt{Net}_{\mathbf{U}_{(i),l}}([\mathbf{M}_{(i)}^{obs}]_{:,j})) = 0, \quad \forall j \in [n_{(i)}], \quad i \in [N]\end{aligned} \tag{5}$$

In equation 5, we use a compact notation $\texttt{Net}_{\mathbf{U}_g}(\mathbf{M}_{(i)}^{obs})$ to denote the application of $\texttt{Net}_{\mathbf{U}_g}$ on each column of $\mathbf{M}_{(i)}^{obs}$. ℓ is still a loss metric. $\mathcal{R}\left(\mathbf{U}_g, \mathbf{U}_{(i),l}, \mathbf{V}\right)$ is a regularization term that prevents the over-fitting of the neural network. In the constraint, $\mathcal{F}(\cdot)$ is the folding operator that folds an r-dimensional vector into k vectors, each with dimension r/k. The constraint $\mathcal{F}(\texttt{Net}_{\mathbf{U}_g}([\mathbf{M}_{(i)}^{obs}]_{:,j}))^T \mathcal{F}(\texttt{Net}_{\mathbf{U}_{(i),l}}([\mathbf{M}_{(i)}^{obs}]_{:,j})) = 0$ is a "group orthogonalization condition" that requires the k folded vectors generated by the shared encoders to be orthogonal to k folded vectors generated by the unique encoders for each column of $\mathbf{M}_{(i)}^{obs}$. Such group orthogonalization constraint is a natural extension of the linear constraint in equation 3, and encourages $\texttt{Net}_{\mathbf{U}_g}$ and $\texttt{Net}_{\mathbf{U}_{(i),l}}$ to encode different embeddings.

We call this formulation heterogeneous auto-encoders (HAE).

4 Algorithm

This section will develop the heterogeneous matrix factorization (HMF) algorithm and its extenstion to optimize the heterogeneous auto-encoder equation 5.

4.1 Heterogeneous Matrix Factorization

We use $\tilde{f}$ to denote the summation of $\tilde{f}_i$'s over different sources i,

$$\tilde{f}(\mathbf{U}_g, \{\mathbf{V}_{(i),g}, \mathbf{U}_{(i),l}, \mathbf{V}_{(i),l}\}) = \sum_{i=1}^{N} \tilde{f}_i \tag{6}$$

The objective $\tilde{f}$ is differentiable. Thus to optimize equation 3, one can calculate the gradient of $\tilde{f}_i$ with respect to its variables as,

$$\begin{cases} \nabla_{\mathbf{U}_g} \tilde{f}_i = \ell' \mathbf{V}_{(i),g} + 2\beta \mathbf{U}_g \left(\mathbf{U}_g^T \mathbf{U}_g - \mathbf{I}\right) \\ \nabla_{\mathbf{V}_{(i),g}} \tilde{f}_i = \ell'^T \mathbf{U}_g \\ \nabla_{\mathbf{U}_{(i),l}} \tilde{f}_i = \ell' \mathbf{V}_{(i),l} + 2\beta \mathbf{U}_{(i),l} \left(\mathbf{U}_{(i),l}^T \mathbf{U}_{(i),l} - \mathbf{I}\right) \\ \nabla_{\mathbf{V}_{(i),l}} \tilde{f}_i = \ell'^T \mathbf{U}_{(i),l} \end{cases} \tag{7}$$

where ℓ' denotes the gradient of the empirical loss ℓ, $\ell' = \nabla_{\hat{\mathbf{M}}_{(i)}} \ell(\mathbf{M}_{(i)}, \hat{\mathbf{M}}_{(i)})$.

The constraint $\mathbf{U}_g^T \mathbf{U}_{(i),l} = 0$ poses challenges to the optimization. One naive idea is to use projected gradient descent to handle this constraint. However, the projection can not be easily implemented as the feasible region is nonconvex. A few works also introduce infeasible updates when the constraint is complicated, including ADMM (Hong & Luo, 2017) or SQP (Curtis & Overton, 2012). Such approaches often introduce additional tuning parameters to balance the objective and the constraint.

We take a different route by exploiting a special invariance property in equation 3. More specifically, for any $\mathbf{R} \in \mathbb{R}^{r_1 \times r_{2,(i)}}$, we can apply the transform $\varphi_{\mathbf{R}}$ on $\mathbf{U}_g$ and $\mathbf{U}_{(i),l}$,

$$\varphi_{\mathbf{R}} : \left(\mathbf{U}_g, \mathbf{V}_{(i),g}, \mathbf{U}_{(i),l}, \mathbf{V}_{(i),l}\right) \mapsto \left(\mathbf{U}_g, \mathbf{V}_{(i),g} + \mathbf{V}_{(i),l}\mathbf{R}^T, \mathbf{U}_{(i),l} - \mathbf{U}_g\mathbf{R}, \mathbf{V}_{(i),l}\right)$$

without changing f_i: $f_i(\mathbf{U}_g, \mathbf{V}_{(i),g}, \mathbf{U}_{(i),l}, \mathbf{V}_{(i),l}) = f_i\left(\varphi_{\mathbf{R}}\left(\mathbf{U}_g, \mathbf{V}_{(i),g}, \mathbf{U}_{(i),l}, \mathbf{V}_{(i),l}\right)\right)$.

The invariance property is a result of the special bi-linear structure in equation 4. One can use such invariance to ensure the feasibility of the iterations. By choosing $\mathbf{R} = \left(\mathbf{U}_g^T \mathbf{U}_g\right)^{-1} \mathbf{U}_g^T \mathbf{U}_{(i),l}$, the transform $\varphi_{\mathbf{R}}$ automatically corrects $\mathbf{V}_{(i),g}$ and $\mathbf{U}_{(i),l}$, such that $\mathbf{U}_{(i),l}$ is orthogonal to $\mathbf{U}_g$, without changing f_i. Based on this fact, we can propose an iterative algorithm. In each epoch, we use $\varphi_{\mathbf{R}}$ to correct the variables to ensure feasibility. Then we use gradient descent on $(\mathbf{U}_g, \mathbf{V}_{(i),g}, \mathbf{U}_{(i),l}, \mathbf{V}_{(i),l})$ to decrease the regularized objective. A pseudo-code is shown in Algorithm 1.

Algorithm 1 `HMF`

1: Input matrices $\{\mathbf{M}_{(i)}\}_{i=1}^N$, stepsize η
2: Initialize $\mathbf{U}_{g,1}, \mathbf{V}_{(i),g,\frac{1}{2}}, \mathbf{U}_{(i),l,\frac{1}{2}}, \mathbf{V}_{(i),l,1}$ to be small random matrices.
3: **for** Iteration $\tau = 1, ..., R$ **do**
4: **for** $i = 1, \cdots, N$ **do**
5: Correct $\mathbf{U}_{(i),l,\tau} = \mathbf{U}_{(i),l,\tau-\frac{1}{2}} - \mathbf{U}_{g,\tau}\left(\mathbf{U}_{g,\tau}^T \mathbf{U}_{g,\tau}\right)^{-1} \mathbf{U}_{g,\tau}^T \mathbf{U}_{(i),l,\tau-\frac{1}{2}}$
6: Correct $\mathbf{V}_{(i),g,\tau} = \mathbf{V}_{(i),g,\tau-\frac{1}{2}} + \mathbf{V}_{(i),l,\tau} \mathbf{U}_{(i),l,\tau-\frac{1}{2}}^T \mathbf{U}_{g,\tau}\left(\mathbf{U}_{g,\tau}^T \mathbf{U}_{g,\tau}\right)^{-1}$
7: Update $\mathbf{U}_{(i),g,\tau+1} = \mathbf{U}_{g,\tau} - \eta \nabla_{\mathbf{U}_g} \tilde{f}_i$.
8: Update $\mathbf{V}_{(i),g,\tau+\frac{1}{2}} = \mathbf{V}_{(i),g,\tau} - \eta \nabla_{\mathbf{V}_{(i),g}} \tilde{f}_i$.
9: Update $\mathbf{U}_{(i),l,\tau+\frac{1}{2}} = \mathbf{U}_{(i),l,\tau} - \eta \nabla_{\mathbf{U}_{(i),l}} \tilde{f}_i$.
10: Update $\mathbf{V}_{(i),l,\tau+1} = \mathbf{V}_{(i),l,\tau} - \eta \nabla_{\mathbf{V}_{(i),l}} \tilde{f}_i$.
11: **end for**
12: Calculates $\mathbf{U}_{g,\tau+1} = \frac{1}{N} \sum_{i=1}^N \mathbf{U}_{(i),g,\tau+1}$
13: **end for**
14: Return $\mathbf{U}_{g,R}, \{\mathbf{V}_{(i),g,R}\}, \{\mathbf{U}_{(i),l,R}\}, \{\mathbf{V}_{(i),l,R}\}$.

In Algorithm 1, we use τ to denote the iteration index, where a half-integer denotes that the update of the variable is half complete: it is updated by gradient descent but is not feasible yet.

One salient feature of algorithm 1 is that it is distributed in nature. Suppose there are N computation nodes, each holding one observation matrix $\mathbf{M}_{(i)}$. Then, Algorithm 1 can be run on these computation nodes with the help of a central server. In such a scenario, node i carries out all computation from line 5 to line 10 in Algorithm 1 (colored in teal) and sends the updated copies of $\mathbf{U}_{(i),g}$ to the central server. The server then takes the average in line 12 (colored in brown) and broadcasts the averaged $\mathbf{U}_g$ to all nodes.

The matrix inversion in algorithm 1 takes $O(r_1^3 + r_{2,(i)}^3)$ operations, all other matrix productions take $O(n_1 n_2 (r_1 + r_{2,(i)}))$ operations. Therefore if r_1 and $r_{2,(i)}$ is dominated by n_1 and n_2, the computational complexity of one single iteration is $O(n_1 n_2 (r_1 + r_{2,(i)}))$.

Algorithm 1 can be readily extended to train the autoencoder in equation 5. The specifics of the training algorithm is presented in Appendix.

5 Convergence Analysis

In spite of its simplicity, Algorithm 1 has provable convergence guarantees. In this section, we first show that if ℓ is the square error (SE), `HMF` can converge into the optimal solutions, which also has a statistical error upper bound. Furthermore, when ℓ takes generic forms, `HMF` will converge into KKT points.

In this section we use $\mathcal{B}(B_1, B_2)$ to denote the set of variables with bounded norms, $\mathcal{B}(B_1, B_2) = (\mathbf{x}; \|\mathbf{U}_g\|, \|\mathbf{U}_{(i),l}\| \leq B_1, \|\mathbf{V}_{(i),g}\|, \|\mathbf{V}_{(i),l}\| \leq B_2)$. We relegate the proof of all theorems in this section to the supplementary materials.

5.1 Square Error

The square error, denoted as $\ell^{se}(\mathbf{M}_{(i)}, \hat{\mathbf{M}}_{(i)}) = \left\|\mathbf{M}_{(i)} - \hat{\mathbf{M}}_{(i)}\right\|_F^2$, measures the sum of the square of the element-wise difference between $\mathbf{M}_{(i)}$ and $\hat{\mathbf{M}}_{(i)}$. Setting ℓ to ℓ^{se} naturally generates the formulation of

matrix factorization. Consequently, we thoroughly investigate the convergence behavior of HMF with respect to the SE.

Before presenting the convergence results, we briefly review the concept of misalignment proposed in Shi & Kontar (2024). Intuitively speaking, misalignment characterizes the avarage "minimal difference" among the subspace spanned by a series of vectors. More formally,

Definition 5.1 *(θ-misalignment) We say $\{\mathbf{U}^\star{}_{(i),l}\}_{i=1}^N$ are θ-misaligned if there exists a positive constant $\theta \in (0,1)$ such that, $\lambda_{\max}\left(\frac{1}{N}\sum_{i=1}^N \mathbf{P}_{\mathbf{U}^\star{}_{(i),l}}\right) \leq 1-\theta$.*

where we define the projection matrix $\mathbf{P}_{\mathbf{U}} \in \mathbb{R}^{d\times d}$ for a matrix $\mathbf{U} \in \mathbb{R}^{d\times n}$ as $\mathbf{P}_{\mathbf{U}} = \mathbf{U}\left(\mathbf{U}^T\mathbf{U}\right)^{-1}\mathbf{U}^T$.

With the definition of misalignment, we are able to show the following upper bound on the statistical error.

Theorem 1 (Statistical error) *Consider the data generation model equation 1. Suppose that (i) the signal part $\mathbf{U}^\star{}_g\mathbf{V}^{\star T}_{(i),g} + \mathbf{U}^\star{}_{(i),l}\mathbf{V}^{\star T}_{(i),l}$ has r_1+r_2 nonzero singular values upper bounded by $\sigma_{\max}$ and lower bounded by $\sigma_{\min}$, (ii) unique factors $\mathbf{U}^\star{}_{(i),l}$ are θ-misaligned, (iii) $\hat{\mathbf{U}}_g, \{\hat{\mathbf{V}}_{(i),g}, \hat{\mathbf{U}}_{(i),l}, \hat{\mathbf{V}}_{(i),l}\}_{i=1}^N$ is one set of optimal solutions to equation 3 with square error loss ℓ^{se}, then, the following holds*

$$\begin{aligned}&\sum_{i=1}^N \left\|\hat{\mathbf{U}}_g\hat{\mathbf{V}}^T_{(i),g} - \mathbf{U}^\star{}_g\mathbf{V}^{\star T}_{(i),g}\right\|_F^2 + \left\|\hat{\mathbf{U}}_{(i),l}\hat{\mathbf{V}}^T_{(i),l} - \mathbf{U}^\star{}_{(i),l}\mathbf{V}^{\star T}_{(i),l}\right\|_F^2 \\ &= O\Big(\frac{1}{\theta}\sum_{i=1}^N \left(\left\|\mathbf{E}^\star{}_{(i)}\right\|_F + \left\|\mathbf{E}^\star{}_{(i)}\right\|_F^2\right)^2 + \left(\left\|\mathbf{E}^\star{}_{(i)}\right\|_F + \left\|\mathbf{E}^\star{}_{(i)}\right\|_F^2\right)\Big)\end{aligned} \tag{8}$$

Theorem 1 provides an upper bound on the distance between the optimal solution to equation 3 and the ground truth. To the best of our knowledge, we are the first to prove such statistical error bound in the context of heterogeneous matrix factorization. Interestingly, when the misalignment parameter θ is larger, the statistical error upper bound in equation 8 is smaller, which means the features are easier to identify. In the following, we will continue to discuss the convergence into the optimal solutions.

We use $\phi_{(i),\tau}$ to denote the optimality gap at iteration τ of HMF for client i, $\phi_{(i),\tau} = \tilde{f}_i(\mathbf{U}_{g,\tau}, \mathbf{V}_{(i),g,\tau}, \mathbf{U}_{(i),l,\tau}, \mathbf{V}_{(i),l,\tau}) - \tilde{f}_i(\hat{\mathbf{U}}_{g,\tau}, \hat{\mathbf{V}}_{(i),g,\tau}, \hat{\mathbf{U}}_{(i),l,\tau}, \hat{\mathbf{V}}_{(i),l,\tau})$. Accordingly, the total optimality gap is defined as $\phi_\tau = \sum_{i=1}^N \phi_{(i),\tau}$. The following theorem establishes HMF's linear convergence.

Theorem 2 *(Convergence of HMF) If the assumptions in Theorem 1 are satisfied, and additionally, the following conditions hold: (i) $\mathbf{M}_{(i)} = \mathbf{U}^\star{}_g\mathbf{V}^{\star T}_{(i),g} + \mathbf{U}^\star{}_{(i),l}\mathbf{V}^{\star T}_{(i),l} + \mathbf{E}^\star{}_{(i)}$, where $\left\|\mathbf{E}^\star{}_{(i)}\right\|_F = O(\theta\sigma_{\min})$, (ii) the initial optimality gap satisfies $\phi_1 = O\left(\theta^{1.5}\sigma^2_{\min}\right)$, (iii) the stepsize satisfies $\eta = O\left(\frac{1}{\sigma^2_{\max}}\right)$.*

Then, there exists a constant $C > 0$ such that the iterations of HMF satisfy

$$\phi_\tau \leq (1 - C\eta)^{\tau-1}\,\phi_1 \tag{9}$$

The first condition in Theorem 2 imposes an upper bound on the noise matrix so that the benign optimization landscape is not destroyed by the gross noise. The second condition in Theorem 2 requires an upper bound on the initial optimality gap. Roughly speaking, this condition ensures that the iterates do not become trapped at a sub-optimal local solution and lie within a basin of attraction of the global solution. We note that recent results have relaxed this condition for other variants of nonconvex matrix factorization by resorting to small, random, or spectral initialization techniques (Li et al., 2018b; Stöger & Soltanolkotabi, 2021; Ma & Fattahi, 2022; Ma et al., 2022; Tu et al., 2016; Ma et al., 2018). We believe such techniques can be used in HMF to relax the aforementioned initial condition. In fact, in our simulations, we observed that HMF with a small Gaussian random initial point converges to a global solution in almost all instances.

We leave the rigorous analysis of this observation as an enticing challenge for future research. Finally, the last condition in Theorem 2 imposes an upper bound on the stepsize of our algorithm to guarantee its convergence.

5.2 General Loss

Under a generic Lipschitz continuous loss function ℓ, less geometric information is known for the loss landscape. Still, we are able to prove that Algorithm 1 will converge into a KKT point to problem equation 3. In this section, we assume $r_{2,(i)} = r_2$, without loss of generality.

Theorem 3 (Convergence of HMF under general loss) *Suppose that the following conditions are satisfied for HMF: (i) there exist constants $B_1, B_2 > 0$ that can upper bound the norm of all iterates $\left(\mathbf{U}_g, \{\mathbf{V}_{(i),g}, \mathbf{U}_{(i),l}, \mathbf{V}_{(i),l}\}\right) \in \mathcal{B}(B_1, B_2)$, (ii) ℓ is L-Lipschitz continuous, (iii) the stepsize satisfies $\eta = O\left(\frac{1}{L^2}\right)$. Then*

$$\min_{\tau \in \{1, T-1\}} \left\|\nabla \tilde{f}(\mathbf{x}_\tau)\right\|_F^2 = O\left(T^{-1}\right).$$

It is worth noting that at KKT points of problem equation 3, the gradient norm is zero $\left\|\nabla \tilde{f}\right\|_F = 0$. Hence, Theorem 3 essentially characterizes the rate for Algorithm 1 to converge into KKT solutions. Such convergence guarantees are not attainable for the previous heuristic algorithms, including JIVE (Lock et al., 2013), COBE (Zhou et al., 2015), BIDIFAC (Park & Lock, 2020), and many more, while HMF is guaranteed to converge. The improvement results from the introduced correction steps in Algorithm 1.

6 Experimental Results

In this section, we present the results of the numerical simulations. We use a synthetic example and three real-life case studies. Code for all numerical studies is available in the following repository: `https://github.com/UMDataScienceLab/hmf`.

6.1 Synthetic Data

We use synthetic data to examine the numerical convergence of HMF. We generate data according to the model equation 1, where $\mathbf{U}^\star{}_g$, $\mathbf{U}^\star{}_{(i),l}$, $\mathbf{V}^\star{}_{(i),g}$, $\mathbf{V}^\star{}_{(i),l}$ are randomly sampled from Gaussian distributions. Then we deflate $\mathbf{U}^\star{}_{(i),l}$ to satisfy the requirement $\mathbf{U}^{\star T}_g \mathbf{U}^\star{}_{(i),l} = 0$. The noise $\mathbf{E}^\star{}_{(i)}$ is set to 0 to examine the convergence behavior of our algorithm better. We fix $n_2 = 100$ and select $n_1 \in \{30, 60, 120\}$ to see the effect of different observation matrix sizes. The ranks of the global and local components are set to $r_1 = r_2 = 3$, and the number of clients is $N = 100$. We run HMF on the generated data with the decision variables $\mathbf{U}_g$, $\mathbf{U}_{(i),l}$, $\mathbf{V}_{(i),g}$, $\mathbf{V}_{(i),l}$ initialized as small Gaussian matrices. The errors $\sum_{i=1}^N \left\|\mathbf{U}_{g,\tau}\mathbf{V}^T_{(i),g,\tau} - \mathbf{U}^\star{}_g \mathbf{V}^{\star T}_{(i),g}\right\|_F^2$ and $\sum_{i=1}^N \left\|\mathbf{U}_{(i),l,\tau}\mathbf{V}^T_{(i),l,\tau} - \mathbf{U}^\star{}_{(i),l}\mathbf{V}^{\star T}_{(i),l}\right\|_F^2$ are calculated in each iteration and plotted as shared feature error and unique feature error in Figure 1. It can be observed that after a few iterations[1], the errors decrease linearly, which is consistent with our theoretical guarantee in Theorem 2.

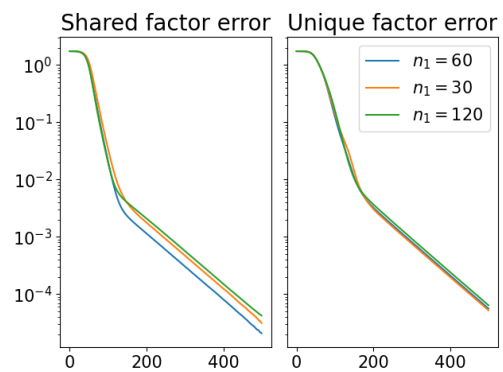

Figure 1: Shared factor error and unique factor error (log-scale) in each iteration.

We also compare HMF with three representative baseline algorithms on the synthetic dataset. To simulate the effects of missing entries, we uniformly randomly remove a subset of entries in each $\mathbf{M}_{(i)}$. Then we run HMF with ℓ^{pse}, JIVE, RJIVE, and perPCA on the partially observed $\{\mathbf{M}_{(i)}\}$ and calculate the subspace error

[1]The initial sublinear convergence of the algorithm is due to the fact that our initial point does not satisfy the condition of Theorem 2. However, once the iterations reach the basin of attraction of the global solution, the iterations converge linearly.

Table 2: Subspace error under different ratios of missing entries.

Algorithm	50%	10%	5%	1%
RJIVE	7.9 ± 0.1	5.9 ± 0.1	5.9 ± 0.1	5.9 ± 0.1
JIVE	0.52 ± 0.01	$(5.4 \pm 0.1) \times 10^{-2}$	$(2.6 \pm 0.1) \times 10^{-2}$	$(5.2 \pm 0.2) \times 10^{-3}$
perPCA	0.51 ± 0.01	$(5.4 \pm 0.1) \times 10^{-2}$	$(2.6 \pm 0.1) \times 10^{-2}$	$(5.2 \pm 0.2) \times 10^{-3}$
HMF	$\mathbf{(4.5 \pm 0.7) \times 10^{-2}}$	$\mathbf{(2.0 \pm 0.2) \times 10^{-6}}$	$\mathbf{(7.3 \pm 0.4) \times 10^{-7}}$	$\mathbf{(3.4 \pm 0.3) \times 10^{-8}}$

defined as $\left\|\mathbf{P}_{\mathbf{U}_g} - \mathbf{P}_{\mathbf{U}^\star{}_g}\right\|_F^2 + \frac{1}{N}\sum_{i=1}^N \left\|\mathbf{P}_{\mathbf{U}_{(i),l}} - \mathbf{P}_{\mathbf{U}^\star{}_{(i),l}}\right\|_F^2$. The missing entries are treated as zero in JIVE, RJIVE, and perPCA. We run the experiments from 3 different random seeds and calculate the mean and standard deviation of the subspace error. Results are reported in Table. 2. In Table. 2, HMF has consistently low subspace error, even when the ratio of missing elements is as high as 50%. The result highlights HMF's superior ability to recover shared and unique features from incomplete data.

6.2 Case study: video segmentation from incomplete entries

We use an illustrative example in video segmentation to demonstrate the application of HMF. The video comes from a simulated surveillance dataset (Vacavant et al., 2013). In the video, multiple vehicles drive through a roundabout. We uniformly randomly remove 40% pixels to simulate the effects of missing observations. We divide each video frame i into multiple 7×7 patches and flatten these patches into row vectors to construct observation matrix $\mathbf{M}_{(i)}$. Then, we apply HMF to identify the shared and unique signals from the observation matrices with $r_1 = 20$ and $r_{2,(i)} = r_2 = 30$. Naturally, the shared signals will correspond to the stationary components in the video, and unique signals correspond to the changing parts. We reconstruct the frames from the shared and unique signals and plot the results in Figure 2.

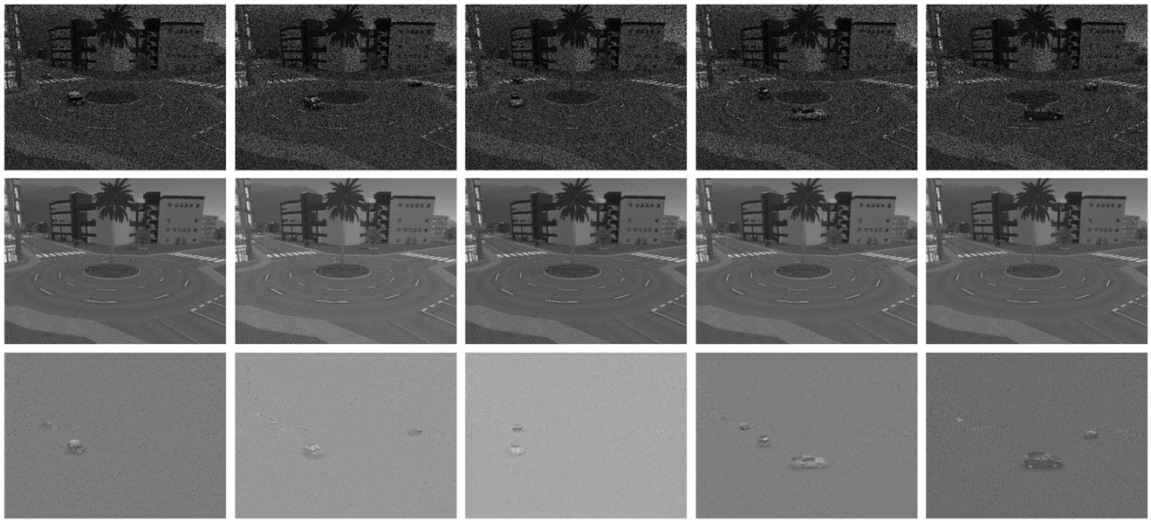

Figure 2: Video Segmentation. 5 frames from the video are plotted.

The first row of Figure 2 shows 5 sample frames from the video. They are corrupted as many pixels are missing. The second row plots the reconstructed $\mathbf{U}_g\mathbf{V}_{(i),g}^T$, one can see that HMF clearly identifies the fine details on the background. The third row of Figure 2 are the reconstructed $\mathbf{U}_{(i),l}\mathbf{V}_{(i),l}^T$, which also conspicuously show the moving cars.

As comparisons, we also run benchmark algorithms JIVE, RJIVE, and perPCA on the same dataset with the same data preprocessing procedures. To evaluate the segmentation performance, we calculate the mean squared error (MSE) and the peak signal-to-noise ratio (PSNR) for the recovered foreground and backgrounds. Results are shown in Table 3.

A higher level of quality in the reconstructed background and foreground is typically associated with a lower MSE and a higher PSNR. As indicated in Table 3, it is evident that HMF consistently achieves lower MSE and higher PSNR values for both foreground and background reconstructions in comparison to benchmark algorithms, thus confirming its superior performance.

Table 3: Separation performance for different algorithms. BMSE represents MSE for background reconstruction, FSME represents MSE for foreground reconstruction, similar for BPSNR and FPSNR.

	perPCA	JIVE	RJIVE	HMF
BMSE	0.021	0.024	0.027	**0.001**
BPSNR	16.7	16.3	15.8	**29.8**
FMSE	93.7	93.4	94.8	**69.4**
FPSNR	31.0	31.0	31.0	**31.6**

6.3 Case Study: Stocks Market Data

HMF can also be applied to data from the financial market. As a proof-of-concept, we analyze the daily stock prices of 214 stocks from January 2, 1990 to November 9, 2017. The goal is to understand the time-specific patterns in stock prices. These patterns are often related to abnormal market behaviors and provide insightful information for subsequent trading decisions.

Similar to Fattahi & Gomez (2021), we use a time window of 30 days to group the stock prices into different batches, and analyze the common and unique features among these batches. Each batched data matrix $\mathbf{M}_{(i)}$ has dimension 214×30. The unique features represent structural differences in the stock prices in each batch, thus signaling sudden changes in the market. To find the unique features, we apply HMF on the batched data to extract $\mathbf{U}_{(i),l}$'s and $\mathbf{V}_{(i),l}$'s. To measure the "heterogeneity index" in each batch, we calculate the column-wise ℓ_1 norm of the signal matrix $\mathbf{U}_{(i),l}\mathbf{V}_{(i),l}^T$ for each i. The heterogeneity indices are plotted as the blue curve in Figure 3. To provide better insight, we also plot the *SP500* closing prices in the same figure.

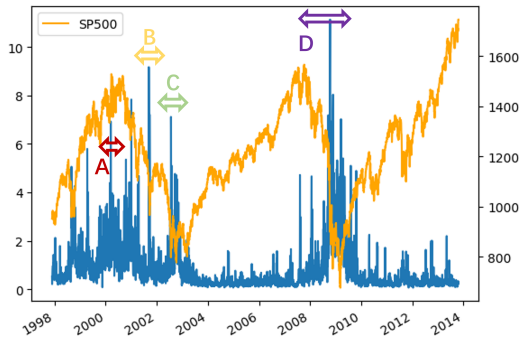

Figure 3: The blue curve denotes the heterogeneity index of 214 stock returns from 1998 to 2014. The orange curve is the *SP500* closing prices at the corresponding dates. We label 4 periods where the heterogeneity index has large peaks

From Figure 3, one can observe that almost every significant historical market crash (shown as sudden drops in the orange curve) corresponds to a peak in the heterogeneity index. We also identify 4 major periods when the heterogeneity index has several large peaks. By comparing these periods with the history of the global financial market (Wikipedia, 2023), one can see that A corresponds to the "dot-com bubble", B corresponds to the "September 11 attack", C corresponds to "stock market downturn of 2002", and D corresponds to the "2007-2008 financial crisis and the aftermath".

6.4 Case study: rating prediction on MovieLens

The ability to effectively distinguish between shared and unique features is also beneficial in recommender systems. This is exemplified in the MovieLens-100k dataset (Harper & Konstan, 2015). The dataset contains 10^5 ratings (1-5) on 1682 movies from 943 users. Additionally, each movie is labeled with genre labels, such as action, adventure, or animation. There are 19 genres in total. Notice that each movie can have more than one genre label. For example, the movie *Titanic* belongs to both the romance and drama genre.

To characterize the genre information, we first cluster movies into different groups according to the genre labels. Different groups represent different sources. More specifically, for movie m, we use multi-hot encoding to create its genre information vector $g_m \in \mathbb{R}^{19}$, whose j-th element is 1 if movie m belongs to genre j. Next, we normalize g_m and use K-means clustering to cluster the normalized g_ms into 10 groups. For each group, we construct user-movie rating matrices $\mathbf{M}_{(i)}$. Each $\mathbf{M}_{(i)}$ has 943 rows, and its number of columns is equal to the number of movies in cluster i. Thus $\mathbf{M}_{(i)}$ is highly sparse. Then we randomly split each $\Omega_{(i)}$ into 90% training set $\Omega_{(i)}^{train}$ and 10% test set $\Omega_{(i)}^{test}$ and apply HMF with ℓ^{pse} on the $\Omega_{(i)}^{train}$.

We use two methods to predict the user ratings on unseen movies: (i) collaborative filtering, which is based on linear model equation 3, and (ii) auto-encoders, which is based on the nonlinear model equation 5.

Collaborative filtering The standard collaborative filtering method pools ratings for movies from all genres and exploits matrix completion to extract the low-rank movie features and user features on $\cup_i \Omega_{(i)}^{train}$ (Koren et al., 2009). These features can then be leveraged for new rating predictions on unseen movies. Though achieving decent performance, the standard collaborative filtering only finds generic movie features and neglects the genre information. To incorporate the genre information, we apply `HMF` to the genre-clustered ratings $\mathbf{M}_{(i)}$ to identify both generic features and genre-specific features. The new predictions are given as the entries of $\hat{\mathbf{U}}_g \hat{\mathbf{V}}_{(i),g} + \hat{\mathbf{U}}_{(i),l} \hat{\mathbf{V}}_{(i),l}$.

Auto-encoder As discussed in Section 3.2, auto-encoders are nonlinear extensions to matrix factorization models and can potentially encode richer information. The auto-encoders have been employed in the rating prediction tasks and have been shown to outperform collaborative filtering based on linear features (Muller et al., 2018). The standard auto-encoder approach for rating prediction minimizes the following reconstruction loss,

$$\min_{\mathbf{U},\mathbf{V}} \left\| \mathcal{P}_{\Omega_{train}} \left(\mathbf{M} - \texttt{Net}_{\mathbf{V}} \left(\texttt{Net}_{\mathbf{U}} \left(\mathbf{M}^{obs} \right) \right) \right) \right\|_F^2 + \mathcal{R}\left(\mathbf{U}, \mathbf{V}\right) \tag{10}$$

, where $\mathbf{M}^{obs}$ is the observed rating matrix $\mathbf{M}$ padded with zero in all entries outside the training set Ω_{train}. $\mathcal{R}\left(\mathbf{U}, \mathbf{V}\right)$ is the regularization term dependent on the architecture of the neural network.

There are numerous works focusing on the architecture of the encoder and decoder network (Zhang et al., 2017; Darban & Valipour, 2022; Rashed et al., 2019). Among them, sparsely supported fully connected neural network (SparseFC) (Muller et al., 2018) achieves decent performance. We briefly introduce the main idea of sparse FC for consistency. A standard full-connected neural network consists of multiple layers. Layer l will transform the input $x^{(l-1)}$ to the output $x^{(l)}$ via a linear mapping $W^{(l)}$ followed by a nonlinear activation $\sigma^{(l)}$, i.e., $x^{(l)} = \sigma^{(l)}(W^{(l)}x^{(l-1)})$. A sparse kernel layer chooses a different parametrization of the linear mapping $W^{(l)}$. More specifically, the j-th element out the output $x^{(l)}$ is given by, $x_j^{(l)} = \sigma^{(l)}\left(\sum_i \alpha_i^{(l)} K(u_i^{(l)}, u_j^{(l)}) x_i^{(l-1)}\right)$, where $\alpha_i^{(l)}$, $u_i^{(l)}$, and $u_j^{(l)}$ are trainable parameters. K is a kernel function that can be defined as Gaussian kernel $K(x, y) = \exp\left(-\gamma \left\|x - y\right\|^2\right)$. Intuitively, such parametrization would make the weights $K(u_i^{(l)}, u_j^{(l)})$ sparse. Sparse FC networks are constructed by sequentially applying sparse kernel layers. Because of their superior generalization performance, we use sparse neural networks as the backbones for the encoder and decoder networks $\texttt{Net}_{\mathbf{U}}$ and $\texttt{Net}_{\mathbf{V}}$ in equation 10. We implement the architecture and pool $\mathbf{M}_{(i)}$ to build $\mathbf{M}$, then solve equation 10 with AdamW (Loshchilov & Hutter, 2019).

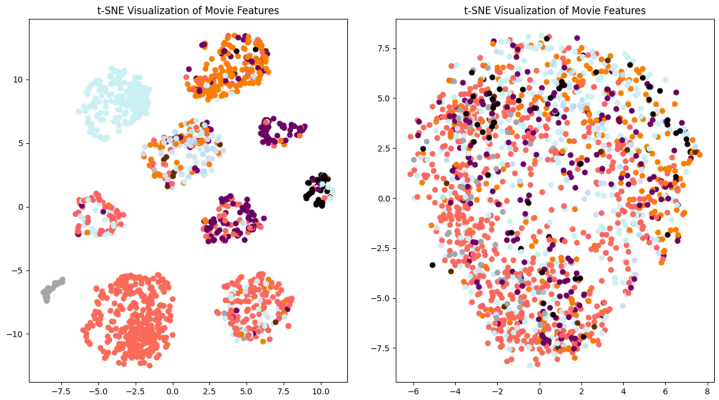

Figure 4: t-SNE plots of movie embeddings. Each dot represents a movie. Its color denotes one genre of the movie. *Left*: t-SNE of embeddings from `HAE`. *Right*: t-SNE of embeddings from standard auto-encoders

As introduced in Section 3.2, we can also employ a heterogeneous version of auto-encoders `HAE`. More specifically, we use a shared encoder network $\texttt{Net}_{\mathbf{U}_g}$ and 10 genre-specific encoders $\texttt{Net}_{\mathbf{U}_{(i),l}}$ to learn both shared and unique nonlinear embeddings of movies. Then, we use a decoder $\texttt{Net}_{\mathbf{V}}$ to predict user ratings. We still use the sparse FC network as the backbones of the encoders and decoders.

To visualize the embeddings from standard auto-encoders and heterogeneous auto-encoders, we show t-SNE plots of different movie embeddings in Figure 4.

In Figure 4, it is clear that the embeddings from HAE form clusters dependent on the genre, while the embeddings from standard auto-encoders do not have a clear structure. These observations are consistent with our intuition that movies with similar genres should have similar embeddings.

With the fitted models from HMF or HAE, we can predict user ratings on unseen movies. To evaluate the predictive performance, we calculate the predicted ratings on the test set $\Omega_{(i)}^{test}$ and compare it with the ground truth. As the movielens 100k dataset is extensively analyzed in the literature, we also report the RMSE of a few representative benchmark methods detailed as follows,

- Matrix completion (Koren et al., 2009): We implement the standard matrix completion on the pooled rating matrix $\mathbf{M}$ to extract linear features.
- AutoSVD++ (Zhang et al., 2017): AutoSVD++ is a hybrid model combining a variant of biased SVD and a 1-layer fully connected neural network.
- GHRS (Darban & Valipour, 2022): Graph-based Hybrid Recommendation System builds user similarity graph based on user ratings and side information, including age and occupation, then clusters users into different groups and uses group features to make predictions on the unseen ratings.
- GraphRec (Rashed et al., 2019): GraphRec is a nonlinear model that uses a neural network to create user and movie embeddings. It also considers the graph-based features from the user similarity graph.

The root mean square error (RMSE) of the prediction on the test set is shown in Table 4. For matrix completion, HMF, Sparse FC, and HAE, we run each experiment from 80 different random seeds and report the mean and standard deviation. For other methods, we report the performance from the literature.

Table 4: Prediction root mean square error (RMSE) on movielens 100K (10% randomly selected test set).

Method	Source	RMSE
Matrix completion	Koren et al. (2009)	0.981 ± 0.001
HMF	Ours	0.977 ± 0.001
AutoSVD++	Zhang et al. (2017)	0.901
GHRS	Darban & Valipour (2022)	0.887
GraphRec	Rashed et al. (2019)	0.883
Sparse FC	Muller et al. (2018)	0.8838 ± 0.0006
HAE	Ours	$\mathbf{0.8800} \pm 0.0006$

In Table 4, it is clear that HAE has the lowest test error, suggesting that the heterogeneous features learned by HAE are more accurate in terms of predicting the users' preferences. Therefore, the genre features bring useful information to the movie ratings prediction.

7 Conclusion and Discussion of Limitations

This work proposes HMF that solves a constrained matrix factorization problem to extract shared and unique features from heterogeneous data, and extends the model to auto-encoders. One avenue for future research is to consider scenarios where the rank of feature matrices r_1 and r_2 are unknown and must be overestimated instead (Ma & Fattahi, 2022; Stöger & Soltanolkotabi, 2021). Such a setting (also known as overparameterization) has been extensively studied for the classical matrix factorization literature. Another promising direction is to improve the initialization condition for the guaranteed convergence of our proposed algorithm. Although the theoretical guarantee of our algorithm (Theorem 2) relies on the availability of a good initial point, we hypothesize that this requirement can be relaxed, as the algorithm works well in practice with a small and random initialization.

References

Dimitri P Bertsekas. Nonlinear programming. *Journal of the Operational Research Society*, 48(3):334–334, 1997.

Di Chai, Leye Wang, Kai Chen, and Qiang Yang. Secure federated matrix factorization. *IEEE Intelligent Systems*, 36(5):11–20, 2021. doi: 10.1109/MIS.2020.3014880.

Ji Chen, Dekai Liu, and Xiaodong Li. Nonconvex rectangular matrix completion via gradient descent without $l_{2,\infty}$ regularization. *IEEE Transactions on Information Theory*, 66(9):5806–5841, 2020.

Frank E Curtis and Michael L Overton. A sequential quadratic programming algorithm for nonconvex, nonsmooth constrained optimization. *SIAM Journal on Optimization*, 22(2):474–500, 2012.

Zahra Zamanzadeh Darban and Mohammad Hadi Valipour. Ghrs: Graph-based hybrid recommendation system with application to movie recommendation. *Expert Systems with Applications*, 200:116850, 2022.

Salar Fattahi and Andres Gomez. Scalable inference of sparsely-changing gaussian markov random fields. In M. Ranzato, A. Beygelzimer, Y. Dauphin, P.S. Liang, and J. Wortman Vaughan (eds.), *Advances in Neural Information Processing Systems*, volume 34, pp. 6529–6541. Curran Associates, Inc., 2021. URL `https://proceedings.neurips.cc/paper_files/paper/2021/file/33853141e0873909be88f5c3e6144cc6-Paper.pdf`.

Salar Fattahi and Somayeh Sojoudi. Exact guarantees on the absence of spurious local minima for non-negative rank-1 robust principal component analysis. *Journal of machine learning research*, 2020.

Irina Gaynanova and Gen Li. Structural learning and integrative decomposition of multi-view data. *Biometrics*, 75(4):1121–1132, 2019.

Rong Ge, Chi Jin, and Yi Zheng. No spurious local minima in nonconvex low rank problems: A unified geometric analysis. In *International Conference on Machine Learning*, pp. 1233–1242. PMLR, 2017.

Rainer Gemulla, Erik Nijkamp, Peter J Haas, and Yannis Sismanis. Large-scale matrix factorization with distributed stochastic gradient descent. In *Proceedings of the 17th ACM SIGKDD international conference on Knowledge discovery and data mining*, pp. 69–77, 2011.

Ian Goodfellow, Yoshua Bengio, and Aaron Courville. *Deep Learning*. MIT Press, 2016. `http://www.deeplearningbook.org`.

F Maxwell Harper and Joseph A Konstan. The movielens datasets: History and context. *Acm transactions on interactive intelligent systems (tiis)*, 5(4):1–19, 2015.

Irina Higgins, Loic Matthey, Arka Pal, Christopher Burgess, Xavier Glorot, Matthew Botvinick, Shakir Mohamed, and Alexander Lerchner. beta-VAE: Learning basic visual concepts with a constrained variational framework. In *International Conference on Learning Representations*, 2017. URL `https://openreview.net/forum?id=Sy2fzU9gl`.

Mingyi Hong and Zhi-Quan Luo. On the linear convergence of the alternating direction method of multipliers. *Mathematical Programming*, 162(1-2):165–199, 2017.

Raed Kontar, Naichen Shi, Xubo Yue, Seokhyun Chung, Eunshin Byon, Mosharaf Chowdhury, Jionghua Jin, Wissam Kontar, Neda Masoud, Maher Nouiehed, et al. The internet of federated things (ioft). *IEEE Access*, 9:156071–156113, 2021.

Yehuda Koren, Robert Bell, and Chris Volinsky. Matrix factorization techniques for recommender systems. *Computer*, 42(8):30–37, 2009. doi: 10.1109/MC.2009.263.

Daniel D Lee and H Sebastian Seung. Learning the parts of objects by non-negative matrix factorization. *Nature*, 401(6755):788–791, 1999.

Xingguo Li, Jarvis Haupt, Junwei Lu, Zhaoran Wang, Raman Arora, Han Liu, and Tuo Zhao. Symmetry. saddle points, and global optimization landscape of nonconvex matrix factorization. In *2018 Information Theory and Applications Workshop (ITA)*, pp. 1–9, 2018a. doi: 10.1109/ITA.2018.8503215.

Yuanzhi Li, Tengyu Ma, and Hongyang Zhang. Algorithmic regularization in over-parameterized matrix sensing and neural networks with quadratic activations. In *Conference On Learning Theory*, pp. 2–47. PMLR, 2018b.

Geyu Liang, Naichen Shi, Raed Al Kontar, and Salar Fattahi. Personalized dictionary learning for heterogeneous datasets. In *Thirty-seventh Conference on Neural Information Processing Systems*, 2023. URL `https://openreview.net/forum?id=xw6Szwu4xz`.

Eric F Lock, Katherine A Hoadley, James Stephen Marron, and Andrew B Nobel. Joint and individual variation explained (jive) for integrated analysis of multiple data types. *The annals of applied statistics*, 7(1):523, 2013.

Ilya Loshchilov and Frank Hutter. Decoupled weight decay regularization. In *International Conference on Learning Representations*, 2019. URL `https://openreview.net/forum?id=Bkg6RiCqY7`.

Cong Ma, Kaizheng Wang, Yuejie Chi, and Yuxin Chen. Implicit regularization in nonconvex statistical estimation: Gradient descent converges linearly for phase retrieval and matrix completion. In *International Conference on Machine Learning*, pp. 3345–3354. PMLR, 2018.

Jianhao Ma and Salar Fattahi. Global convergence of sub-gradient method for robust matrix recovery: Small initialization, noisy measurements, and over-parameterization. *arXiv preprint arXiv:2202.08788*, 2022.

Jianhao Ma, Lingjun Guo, and Salar Fattahi. Behind the scenes of gradient descent: A trajectory analysis via basis function decomposition. *arXiv preprint arXiv:2210.00346*, 2022.

Lorenz Muller, Julien Martel, and Giacomo Indiveri. Kernelized synaptic weight matrices. In *International Conference on Machine Learning*, pp. 3654–3663. PMLR, 2018.

Dohyung Park, Anastasios Kyrillidis, Constantine Carmanis, and Sujay Sanghavi. Non-square matrix sensing without spurious local minima via the Burer-Monteiro approach. In Aarti Singh and Jerry Zhu (eds.), *Proceedings of the 20th International Conference on Artificial Intelligence and Statistics*, volume 54 of *Proceedings of Machine Learning Research*, pp. 65–74. PMLR, 20–22 Apr 2017. URL `https://proceedings.mlr.press/v54/park17a.html`.

Jun Young Park and Eric F Lock. Integrative factorization of bidimensionally linked matrices. *Biometrics*, 76(1):61–74, 2020.

Ahmed Rashed, Josif Grabocka, and Lars Schmidt-Thieme. Attribute-aware non-linear co-embeddings of graph features. In *Proceedings of the 13th ACM conference on recommender systems*, pp. 314–321, 2019.

Christos Sagonas, Yannis Panagakis, Alina Leidinger, and Stefanos Zafeiriou. Robust joint and individual variance explained. In *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*, pp. 5267–5276, 2017.

Naichen Shi and Raed Al Kontar. Personalized pca: Decoupling shared and unique features. *Journal of machine learning research*, 2024.

Dominik Stöger and Mahdi Soltanolkotabi. Small random initialization is akin to spectral learning: Optimization and generalization guarantees for overparameterized low-rank matrix reconstruction. *Advances in Neural Information Processing Systems*, 34:23831–23843, 2021.

Ruoyu Sun and Zhi-Quan Luo. Guaranteed matrix completion via non-convex factorization. *IEEE Transactions on Information Theory*, 62(11):6535–6579, 2016.

Stephen Tu, Ross Boczar, Max Simchowitz, Mahdi Soltanolkotabi, and Ben Recht. Low-rank solutions of linear matrix equations via procrustes flow. In *International Conference on Machine Learning*, pp. 964–973. PMLR, 2016.

Antoine Vacavant, Thierry Chateau, Alexis Wilhelm, and Laurent Lequievre. A benchmark dataset for outdoor foreground/background extraction. In *Computer Vision-ACCV 2012 Workshops: ACCV 2012 International Workshops, Daejeon, Korea, November 5-6, 2012, Revised Selected Papers, Part I 11*, pp. 291–300. Springer, 2013.

Bingyan Wang and Jianqing Fan. Robust matrix completion with heavy-tailed noise. *arXiv preprint arXiv:2206.04276*, 2022.

Wikipedia. List of stock market crashes and bear markets, 2023. URL `https://en.wikipedia.org/wiki/List_of_stock_market_crashes_and_bear_markets`. [Online; accessed 17-May-2023].

John Wright and Yi Ma. *High-dimensional data analysis with low-dimensional models: Principles, computation, and applications*. Cambridge University Press, 2022.

Jaewon Yang and Jure Leskovec. Overlapping community detection at scale: a nonnegative matrix factorization approach. In *Proceedings of the sixth ACM international conference on Web search and data mining*, pp. 587–596, 2013.

Zi Yang and George Michailidis. A non-negative matrix factorization method for detecting modules in heterogeneous omics multi-modal data. *Bioinformatics*, 32(1):1–8, 2016.

Tian Ye and Simon S Du. Global convergence of gradient descent for asymmetric low-rank matrix factorization. *Advances in Neural Information Processing Systems*, 34:1429–1439, 2021.

Hsiang-Fu Yu, Nikhil Rao, and Inderjit S Dhillon. Temporal regularized matrix factorization for high-dimensional time series prediction. *Advances in neural information processing systems*, 29, 2016.

Shuai Zhang, Lina Yao, and Xiwei Xu. Autosvd++ an efficient hybrid collaborative filtering model via contractive auto-encoders. In *Proceedings of the 40th International ACM SIGIR conference on Research and Development in Information Retrieval*, pp. 957–960, 2017.

Guoxu Zhou, Andrzej Cichocki, Yu Zhang, and Danilo P Mandic. Group component analysis for multiblock data: Common and individual feature extraction. *IEEE transactions on neural networks and learning systems*, 27(11):2426–2439, 2015.

A Appendix

This supplementary material presents the proofs of Theorems 1, 2, and 3 from the main paper.

B Proof of Theorem 1

For a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, we use $\|\mathbf{A}\|$ or $\|\mathbf{A}\|_2$ to denote $\mathbf{A}$'s operator norm and $\|\mathbf{A}\|_F$ to denote its Frobenius norm.

The proof mainly consists of a perturbation analysis of the objective (4) in the main paper. We assume that the loss metric is the square loss ℓ^{se} throughout this section. We firstly estimate the subspace error of $\left\|\mathbf{P}_{\hat{\mathbf{U}}_g} - \mathbf{P}_{\mathbf{U}^\star{}_g}\right\|_F$ and $\left\|\mathbf{P}_{\hat{\mathbf{U}}_{(i),l}} - \mathbf{P}_{\mathbf{U}^\star{}_{(i),l}}\right\|_F$. Then we estimate the reconstruction errors.

The KKT conditions for the optimal $\hat{\mathbf{V}}_{(i),g}$ and $\hat{\mathbf{V}}_{(i),l}$ are,

$$\left(\mathbf{M}_{(i)} - \hat{\mathbf{U}}_g \hat{\mathbf{V}}_{(i),g}^T - \hat{\mathbf{U}}_{(i),l} \hat{\mathbf{V}}_{(i),l}^T\right)^T \hat{\mathbf{U}}_g = 0$$

$$\left(\mathbf{M}_{(i)} - \hat{\mathbf{U}}_g \hat{\mathbf{V}}_{(i),g}^T - \hat{\mathbf{U}}_{(i),l} \hat{\mathbf{V}}_{(i),l}^T\right)^T \hat{\mathbf{U}}_{(i),l} = 0$$

Considering the constraint $\hat{\mathbf{U}}_g^T \hat{\mathbf{U}}_{(i),l} = 0$, we can solve $\hat{\mathbf{V}}_{(i),l}$ and $\hat{\mathbf{V}}_{(i),l}$ as,

$$\hat{\mathbf{V}}_{(i),g} = \mathbf{M}_{(i)} \hat{\mathbf{U}}_g \left(\hat{\mathbf{U}}_g^T \hat{\mathbf{U}}_g\right)^{-1}$$
$$\hat{\mathbf{V}}_{(i),l} = \mathbf{M}_{(i)} \hat{\mathbf{U}}_{(i),l} \left(\hat{\mathbf{U}}_{(i),l}^T \hat{\mathbf{U}}_{(i),l}\right)^{-1}$$

Then the objective becomes,

$$\begin{aligned}
&\tilde{f}_i(\hat{\mathbf{U}}_g, \{\hat{\mathbf{U}}_{(i),l}\}) = \frac{1}{2} \left\| \mathbf{M}_{(i)} - \hat{\mathbf{U}}_g \left(\hat{\mathbf{U}}_g^T \hat{\mathbf{U}}_g\right)^{-1} \hat{\mathbf{U}}_g^T \mathbf{M}_{(i)} - \hat{\mathbf{U}}_{(i),l} \left(\hat{\mathbf{U}}_{(i),l}^T \hat{\mathbf{U}}_{(i),l}\right)^{-1} \hat{\mathbf{U}}_{(i),l} \mathbf{M}_{(i)} \right\|_F^2 \\
&+ \frac{\beta}{2} \left\| \hat{\mathbf{U}}_g^T \hat{\mathbf{U}}_g - \mathbf{I} \right\|_F^2 + \frac{\beta}{2} \left\| \hat{\mathbf{U}}_{(i),l}^T \hat{\mathbf{U}}_{(i),l} - \mathbf{I} \right\|_F^2 \\
&= \frac{1}{2} \left\| \mathbf{M}_{(i)} - \mathbf{P}_{\hat{\mathbf{U}}_g} \mathbf{M}_{(i)} - \mathbf{P}_{\hat{\mathbf{U}}_{(i),l}} \mathbf{M}_{(i)} \right\|_F^2 + \frac{\beta}{2} \left\| \hat{\mathbf{U}}_g^T \hat{\mathbf{U}}_g - \mathbf{I} \right\|_F^2 + \frac{\beta}{2} \left\| \hat{\mathbf{U}}_{(i),l}^T \hat{\mathbf{U}}_{(i),l} - \mathbf{I} \right\|_F^2
\end{aligned}$$

Notice that if $\hat{\mathbf{U}}_g$ and $\{\hat{\mathbf{U}}_{(i),l}\}$ are optimal, we must have $\hat{\mathbf{U}}_g^T \hat{\mathbf{U}}_g = \mathbf{I}$ and $\hat{\mathbf{U}}_{(i),l}^T \hat{\mathbf{U}}_{(i),l} = \mathbf{I}$, since we can always use Schmidt-Gram procedure to orthonormalize the columns of $\hat{\mathbf{U}}_g$ and $\{\hat{\mathbf{U}}_{(i),l}\}$ without changing $\mathbf{P}_{\hat{\mathbf{U}}_g}$ and $\{\mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\}$. Therefore without loss of generality, we consider the problem

$$\begin{aligned}
&\min_{\hat{\mathbf{U}}_g, \{\hat{\mathbf{U}}_{(i),l}\}} \sum_{i=1}^N \left\| \mathbf{M}_{(i)} - \mathbf{P}_{\hat{\mathbf{U}}_g} \mathbf{M}_{(i)} - \mathbf{P}_{\hat{\mathbf{U}}_{(i),l}} \mathbf{M}_{(i)} \right\|_F^2 \\
&\text{such that } \hat{\mathbf{U}}_g^T \hat{\mathbf{U}}_{(i),l} = 0, \quad \forall i \in [N]
\end{aligned}$$

By our assumption on the data generation process, $\mathbf{U}^\star{}_g$ and $\{\mathbf{U}^\star{}_{(i),l}\}$'s are also feasible. Thus the objective evaluated at $\mathbf{U}^\star{}_g$ and $\{\mathbf{U}^\star{}_{(i),l}\}$'s cannot be smaller than the optimal objective,

$$\sum_{i=1}^N \left\| \mathbf{M}_{(i)} - \mathbf{P}_{\hat{\mathbf{U}}_g} \mathbf{M}_{(i)} - \mathbf{P}_{\hat{\mathbf{U}}_{(i),l}} \mathbf{M}_{(i)} \right\|_F^2 \leq \sum_{i=1}^N \left\| \mathbf{M}_{(i)} - \mathbf{P}_{\mathbf{U}^\star{}_g} \mathbf{M}_{(i)} - \mathbf{P}_{\mathbf{U}^\star{}_{(i),l}} \mathbf{M}_{(i)} \right\|_F^2$$

This is equivalent to

$$\sum_{i=1}^N \mathrm{Tr}\left(\left(\mathbf{P}_{\hat{\mathbf{U}}_g} + \mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\right) \mathbf{M}_{(i)} \mathbf{M}_{(i)}^T\right) \geq \sum_{i=1}^N \mathrm{Tr}\left(\left(\mathbf{P}_{\mathbf{U}^\star{}_g} + \mathbf{P}_{\mathbf{U}^\star{}_{(i),l}}\right) \mathbf{M}_{(i)} \mathbf{M}_{(i)}^T\right) \tag{11}$$

On the other hand, we can expand $\mathbf{M}_{(i)} \mathbf{M}_{(i)}^T$ as

$$\begin{aligned}
\mathbf{M}_{(i)} \mathbf{M}_{(i)}^T &= \left(\mathbf{U}^\star{}_g \mathbf{V}^{\star T}_{(i),g} + \mathbf{U}^\star{}_{(i),l} \mathbf{V}^{\star T}_{(i),l}\right) \left(\mathbf{U}^\star{}_g \mathbf{V}^{\star T}_{(i),g} + \mathbf{U}^\star{}_{(i),l} \mathbf{V}^{\star T}_{(i),l}\right)^T \\
&+ \left(\mathbf{U}^\star{}_g \mathbf{V}^{\star T}_{(i),g} + \mathbf{U}^\star{}_{(i),l} \mathbf{V}^{\star T}_{(i),l}\right) \mathbf{E}^{\star T}_{(i)} + \mathbf{E}^\star{}_{(i)} \left(\mathbf{U}^\star{}_g \mathbf{V}^{\star T}_{(i),g} + \mathbf{U}^\star{}_{(i),l} \mathbf{V}^{\star T}_{(i),l}\right)^T + \mathbf{E}^\star{}_{(i)} \mathbf{E}^{\star T}_{(i)}
\end{aligned}$$

For simplicity, we use $\mathbf{F}^\star{}_{(i)}$ to denote

$$\mathbf{F}^\star{}_{(i)} = \left(\mathbf{U}^\star{}_g \mathbf{V}^{\star T}_{(i),g} + \mathbf{U}^\star{}_{(i),l} \mathbf{V}^{\star T}_{(i),l}\right) \mathbf{E}^{\star T}_{(i)} + \mathbf{E}^\star{}_{(i)} \left(\mathbf{U}^\star{}_g \mathbf{V}^{\star T}_{(i),g} + \mathbf{U}^\star{}_{(i),l} \mathbf{V}^{\star T}_{(i),l}\right)^T + \mathbf{E}^\star{}_{(i)} \mathbf{E}^{\star T}_{(i)}$$

Then the following holds,

$$\begin{aligned}
&\mathrm{Tr}\left(\mathbf{M}_{(i)}\mathbf{M}_{(i)}^T\left(\left(\mathbf{P}_{\mathbf{U}^\star{}_g}+\mathbf{P}_{\mathbf{U}^\star{}_{(i),l}}\right)-\left(\mathbf{P}_{\hat{\mathbf{U}}_g}+\mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\right)\right)\right)\\
&=\mathrm{Tr}\left(\left(\mathbf{U}^\star{}_g\mathbf{V}^{\star T}_{(i),g}+\mathbf{U}^\star{}_{(i),l}\mathbf{V}^{\star T}_{(i),l}\right)\left(\mathbf{U}^\star{}_g\mathbf{V}^{\star T}_{(i),g}+\mathbf{U}^\star{}_{(i),l}\mathbf{V}^{\star T}_{(i),l}\right)^T\left(\left(\mathbf{P}_{\mathbf{U}^\star{}_g}+\mathbf{P}_{\mathbf{U}^\star{}_{(i),l}}\right)-\left(\mathbf{P}_{\hat{\mathbf{U}}_g}+\mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\right)\right)\right)\\
&+\mathrm{Tr}\left(\mathbf{F}^\star{}_{(i)}\left(\left(\mathbf{P}_{\mathbf{U}^\star{}_g}+\mathbf{P}_{\mathbf{U}^\star{}_{(i),l}}\right)-\left(\mathbf{P}_{\hat{\mathbf{U}}_g}+\mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\right)\right)\right) \qquad (12)
\end{aligned}$$

The first term in equation 12 is equal to

$$\mathrm{Tr}\left(\left(\mathbf{U}^\star{}_g\mathbf{V}^{\star T}_{(i),g}+\mathbf{U}^\star{}_{(i),l}\mathbf{V}^{\star T}_{(i),l}\right)\left(\mathbf{U}^\star{}_g\mathbf{V}^{\star T}_{(i),g}+\mathbf{U}^\star{}_{(i),l}\mathbf{V}^{\star T}_{(i),l}\right)^T\left(\mathbf{I}-\left(\mathbf{P}_{\hat{\mathbf{U}}_g}+\mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\right)\right)\right)$$

From assumption (i) in Theorem 1, we know

$$\left(\mathbf{U}^\star{}_g\mathbf{V}^{\star T}_{(i),g}+\mathbf{U}^\star{}_{(i),l}\mathbf{V}^{\star T}_{(i),l}\right)\left(\mathbf{U}^\star{}_g\mathbf{V}^{\star T}_{(i),g}+\mathbf{U}^\star{}_{(i),l}\mathbf{V}^{\star T}_{(i),l}\right)^T\succeq\left(\mathbf{P}_{\mathbf{U}^\star{}_g}+\mathbf{P}_{\mathbf{U}^\star{}_{(i),l}}\right)\sigma^2_{\min}$$

Since $\left(\mathbf{I}-\left(\mathbf{P}_{\hat{\mathbf{U}}_g}+\mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\right)\right)$ is symmetric positive semidefinite, we have,

$$\begin{aligned}
&\mathrm{Tr}\left(\left(\mathbf{U}^\star{}_g\mathbf{V}^{\star T}_{(i),g}+\mathbf{U}^\star{}_{(i),l}\mathbf{V}^{\star T}_{(i),l}\right)\left(\mathbf{U}^\star{}_g\mathbf{V}^{\star T}_{(i),g}+\mathbf{U}^\star{}_{(i),l}\mathbf{V}^{\star T}_{(i),l}\right)^T\left(\mathbf{I}-\left(\mathbf{P}_{\hat{\mathbf{U}}_g}+\mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\right)\right)\right)\\
&\geq\sigma^2_{\min}\mathrm{Tr}\left(\left(\mathbf{P}_{\mathbf{U}^\star{}_g}+\mathbf{P}_{\mathbf{U}^\star{}_{(i),l}}\right)\left(\mathbf{I}-\left(\mathbf{P}_{\hat{\mathbf{U}}_g}+\mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\right)\right)\right)
\end{aligned}$$

By Cauchy-Schwartz inequality, the second term in equation 12 is lower bounded by,

$$\begin{aligned}
&\mathrm{Tr}\left(\mathbf{F}^\star{}_{(i)}\left(\left(\mathbf{P}_{\mathbf{U}^\star{}_g}+\mathbf{P}_{\mathbf{U}^\star{}_{(i),l}}\right)-\left(\mathbf{P}_{\hat{\mathbf{U}}_g}+\mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\right)\right)\right)\\
&\geq-\left\|\mathbf{F}^\star{}_{(i)}\right\|_F\left\|\left(\left(\mathbf{P}_{\mathbf{U}^\star{}_g}+\mathbf{P}_{\mathbf{U}^\star{}_{(i),l}}\right)-\left(\mathbf{P}_{\hat{\mathbf{U}}_g}+\mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\right)\right)\right\|_F\\
&=-\sqrt{2}\left\|\mathbf{F}^\star{}_{(i)}\right\|_F\sqrt{\mathrm{Tr}\left(\left(\mathbf{P}_{\mathbf{U}^\star{}_g}+\mathbf{P}_{\mathbf{U}^\star{}_{(i),l}}\right)\left(\mathbf{I}-\left(\mathbf{P}_{\hat{\mathbf{U}}_g}+\mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\right)\right)\right)}
\end{aligned}$$

Combing them, we can derive a lower bound on equation 12 as

$$\begin{aligned}
&\mathrm{Tr}\left(\mathbf{M}_{(i)}\mathbf{M}_{(i)}^T\left(\left(\mathbf{P}_{\mathbf{U}^\star{}_g}+\mathbf{P}_{\mathbf{U}^\star{}_{(i),l}}\right)-\left(\mathbf{P}_{\hat{\mathbf{U}}_g}+\mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\right)\right)\right)\\
&\geq\sigma^2_{\min}\mathrm{Tr}\left(\left(\mathbf{P}_{\mathbf{U}^\star{}_g}+\mathbf{P}_{\mathbf{U}^\star{}_{(i),l}}\right)\left(\mathbf{I}-\left(\mathbf{P}_{\hat{\mathbf{U}}_g}+\mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\right)\right)\right)\\
&-\sqrt{2}\left\|\mathbf{F}^\star{}_{(i)}\right\|_F\sqrt{\mathrm{Tr}\left(\left(\mathbf{P}_{\mathbf{U}^\star{}_g}+\mathbf{P}_{\mathbf{U}^\star{}_{(i),l}}\right)\left(\mathbf{I}-\left(\mathbf{P}_{\hat{\mathbf{U}}_g}+\mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\right)\right)\right)}
\end{aligned}$$

Summing up both sides for i from 1 to N, and considering equation 11, we have,

$$\begin{aligned}
0&\geq\sigma^2_{\min}\sum_{i=1}^N\mathrm{Tr}\left(\left(\mathbf{P}_{\mathbf{U}^\star{}_g}+\mathbf{P}_{\mathbf{U}^\star{}_{(i),l}}\right)\left(\mathbf{I}-\left(\mathbf{P}_{\hat{\mathbf{U}}_g}+\mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\right)\right)\right)\\
&-\sqrt{2}\sum_{i=1}^N\left\|\mathbf{F}^\star{}_{(i)}\right\|_F\sqrt{\mathrm{Tr}\left(\left(\mathbf{P}_{\mathbf{U}^\star{}_g}+\mathbf{P}_{\mathbf{U}^\star{}_{(i),l}}\right)\left(\mathbf{I}-\left(\mathbf{P}_{\hat{\mathbf{U}}_g}+\mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\right)\right)\right)}\\
&\geq\sigma^2_{\min}\sum_{i=1}^N\mathrm{Tr}\left(\left(\mathbf{P}_{\mathbf{U}^\star{}_g}+\mathbf{P}_{\mathbf{U}^\star{}_{(i),l}}\right)\left(\mathbf{I}-\left(\mathbf{P}_{\hat{\mathbf{U}}_g}+\mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\right)\right)\right)\\
&-\sqrt{2}\sqrt{\sum_{i=1}^N\left\|\mathbf{F}^\star{}_{(i)}\right\|_F^2}\sqrt{\sum_{i=1}^N\mathrm{Tr}\left(\left(\mathbf{P}_{\mathbf{U}^\star{}_g}+\mathbf{P}_{\mathbf{U}^\star{}_{(i),l}}\right)\left(\mathbf{I}-\left(\mathbf{P}_{\hat{\mathbf{U}}_g}+\mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\right)\right)\right)}
\end{aligned}$$

where we applied Cauchy-Schwartz inequality in the second inequality.

Therefore, one can deduce,

$$\sum_{i=1}^{N} \mathrm{Tr}\left(\left(\mathbf{P}_{\mathbf{U}^\star{}_g} + \mathbf{P}_{\mathbf{U}^\star{}_{(i),l}}\right)\left(\mathbf{I} - \left(\mathbf{P}_{\hat{\mathbf{U}}_g} + \mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\right)\right)\right) \leq \frac{2\sum_{i=1}^{N}\left\|\mathbf{F}^\star{}_{(i)}\right\|_F^2}{\sigma_{\min}^4}$$

As $\{\mathbf{U}^\star{}_{(i),l}\}$ are θ-misaligned, from Lemma 2 in Shi & Kontar (2024), we have

$$\begin{aligned}
&\sum_{i=1}^{N} \mathrm{Tr}\left(\left(\mathbf{P}_{\mathbf{U}^\star{}_g} + \mathbf{P}_{\mathbf{U}^\star{}_{(i),l}}\right)\left(\mathbf{I} - \left(\mathbf{P}_{\hat{\mathbf{U}}_g} + \mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\right)\right)\right) \\
&\geq \frac{\theta}{2}\left(N\mathrm{Tr}\left(\mathbf{P}_{\mathbf{U}^\star{}_g} - \mathbf{P}_{\mathbf{U}^\star{}_g}\mathbf{P}_{\hat{\mathbf{U}}_g}\right) + \sum_{i=1}^{N}\mathrm{Tr}\left(\mathbf{P}_{\mathbf{U}^\star{}_{(i),l}} - \mathbf{P}_{\mathbf{U}^\star{}_{(i),l}}\mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\right)\right)
\end{aligned}$$

We can thus conclude,

$$N\left\|\mathbf{P}_{\mathbf{U}^\star{}_g} - \mathbf{P}_{\hat{\mathbf{U}}_g}\right\|_F^2 + \sum_{i=1}^{N}\left\|\mathbf{P}_{\mathbf{U}^\star{}_{(i),l}} - \mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\right\|_F^2 \leq \frac{2\sum_{i=1}^{N}\left\|\mathbf{F}^\star{}_{(i)}\right\|_F^2}{\theta\sigma_{\min}^4} \tag{13}$$

Finally, we prove an upper bound on the reconstruction error of the local and global signals. Notice that

$$\begin{aligned}
&\left\|\hat{\mathbf{U}}_g\hat{\mathbf{V}}_{(i),g}^T - \mathbf{U}^\star{}_g\mathbf{V}^{\star T}_{(i),g}\right\|_F = \left\|\mathbf{P}_{\hat{\mathbf{U}}_g}\mathbf{M}_{(i)} - \mathbf{U}^\star{}_g\mathbf{V}^{\star T}_{(i),g}\right\|_F \\
&= \left\|\mathbf{P}_{\hat{\mathbf{U}}_g}\left(\mathbf{U}^\star{}_g\mathbf{V}^{\star T}_{(i),g} + \mathbf{U}^\star{}_{(i),l}\mathbf{V}^{\star T}_{(i),i}\right) + \mathbf{P}_{\hat{\mathbf{U}}_g}\mathbf{E}^\star{}_{(i)} - \mathbf{U}^\star{}_g\mathbf{V}^{\star T}_{(i),g}\right\|_F \\
&\leq \left\|\mathbf{P}_{\hat{\mathbf{U}}_g}\left(\mathbf{U}^\star{}_g\mathbf{V}^{\star T}_{(i),g} + \mathbf{U}^\star{}_{(i),l}\mathbf{V}^{\star T}_{(i),i}\right) - \mathbf{U}^\star{}_g\mathbf{V}^{\star T}_{(i),g}\right\|_F + \left\|\mathbf{P}_{\hat{\mathbf{U}}_g}\mathbf{E}^\star{}_{(i)}\right\|_F \\
&\leq \left\|\mathbf{P}_{\hat{\mathbf{U}}_g} - \mathbf{P}_{\mathbf{U}^\star{}_g}\right\|_F\sigma_{\max} + \left\|\mathbf{E}^\star{}_{(i)}\right\|_F
\end{aligned} \tag{14}$$

We have,

$$\begin{aligned}
&\left\|\hat{\mathbf{U}}_g\hat{\mathbf{V}}_{(i),g}^T - \mathbf{U}^\star{}_g\mathbf{V}^{\star T}_{(i),g}\right\|_F^2 = \left\|\mathbf{P}_{\hat{\mathbf{U}}_g}\mathbf{M}_{(i)} - \mathbf{U}^\star{}_g\mathbf{V}^{\star T}_{(i),g}\right\|_F^2 \\
&\leq 2\left\|\mathbf{P}_{\hat{\mathbf{U}}_g} - \mathbf{P}_{\mathbf{U}^\star{}_g}\right\|_F^2\sigma_{\max}^2 + 2\left\|\mathbf{E}^\star{}_{(i)}\right\|_F^2
\end{aligned}$$

Similarly we can prove

$$\begin{aligned}
&\left\|\hat{\mathbf{U}}_{(i),l}\hat{\mathbf{V}}_{(i),l}^T - \mathbf{U}^\star{}_{(i),l}\mathbf{V}^{\star T}_{(i),l}\right\|_F^2 = \left\|\mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\mathbf{M}_{(i)} - \mathbf{U}^\star{}_{(i),l}\mathbf{V}^{\star T}_{(i),l}\right\|_F^2 \\
&\leq 2\left\|\mathbf{P}_{\hat{\mathbf{U}}_{(i),l}} - \mathbf{P}_{\mathbf{U}^\star{}_{(i),l}}\right\|_F^2\sigma_{\max}^2 + 2\left\|\mathbf{E}^\star{}_{(i)}\right\|_F^2
\end{aligned}$$

Combining the two inequalities with equation 13 and considering the fact that $\left\|\mathbf{F}^\star{}_{(i)}\right\|_F \leq 2\sigma_{\max}\left\|\mathbf{E}^\star{}_{(i)}\right\|_F + \left\|\mathbf{E}^\star{}_{(i)}\right\|_F^2$, we can prove (7) in the main paper.

C Proof of Theorem 2

In this section we will present the proof of Theorem 2 in the main paper. The proof of Theorem 2 consists of 2 stages. In the first stage, we show Algorithm 1 converges into stationary points. In the second stage, we analyze the local geometry of objective $\tilde{f}$ and show that Algorithm 1 converges into the optimal solutions

linearly. The major technical difficulties lie in the second stage, where the constraints $\mathbf{U}_g^T\mathbf{U}_{(i),l} = 0$ make the local geometry analysis complicated.

We will start with a few lemmas and prove Theorem 2 at the end. More specifically, Lemma C.1 connects the assumption on $\mathbf{E}_{(i)}$ to a few properties on the optimal solution $\left(\hat{\mathbf{U}}_g, \{\hat{\mathbf{V}}_{(i),g}, \hat{\mathbf{U}}_{(i),l}, \hat{\mathbf{V}}_{(i),l}\}\right)$. Lemma C.2 estimates the Lipschitz constant of objective $\tilde{f}$. Lemma C.3 constructs the so called sufficient descent inequality. Lemmas C.5, C.6 C.7, and C.8 are related to the geometry analysis. Finally, Lemma C.10 shows the PL inequality of the objective.

Lemma C.1 *If* $\mathbf{M}_{(i)} = \mathbf{U}^\star{}_g\mathbf{V}^{\star T}_{(i),g} + \mathbf{U}^\star{}_{(i),l}\mathbf{V}^{\star T}_{(i),l} + \mathbf{E}_{(i)}$*, where*

- $\mathbf{E}_{(i)}$*'s norm are upper bounded*

$$\left\|\mathbf{E}_{(i)}\right\|_F \leq \min\left\{\sigma_{\min}\theta^{1.5}\frac{1}{6\sqrt{2}}\frac{\sigma_{\min}}{\sigma_{\max}}, \sigma_{\min}\frac{2-\sqrt{3}}{2}\left(2+\frac{6}{\sqrt{\theta}}\left(\frac{\sigma_{\max}}{\sigma_{\min}}\right)^2\right)^{-1}, \sigma_{\min}\theta\frac{1}{64\sqrt{N}}\frac{\sigma_{\min}}{\sigma_{\max}}\right\}$$

- $\mathbf{U}^\star{}_{(i),l}$*'s are* θ*-misaligned*
- *the singular values of* $\mathbf{U}^\star{}_g\mathbf{V}^{\star T}_{(i),g} + \mathbf{U}^\star{}_{(i),l}\mathbf{V}^{\star T}_{(i),l}$ *are upper bounded by* $\sigma_{\max}$ *and lower bounded by* $\sigma_{\min}$

, we can introduce three new constants $\hat{\sigma}_{\max} = 2\sigma_{\max}$*,* $\hat{\theta} = \frac{\theta}{2}$*, and* $\hat{\sigma}^2_{gap} = \sigma^2_{\min}/2$ *such that:*

1. *The largest singular values of* $\mathbf{M}_{(i)}$*'s are upper bounded by* $\hat{\sigma}_{\max}$*.*
2. *There exists constants* $\hat{\theta}$*,* $\hat{\sigma}^2_{gap} > 0$ *such that* $\left\|\frac{1}{N}\sum_{i=1}^N \mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\right\| \leq 1-\hat{\theta}$ *,* $\sigma^2_{min}\left(\hat{\mathbf{U}}_g\hat{\mathbf{V}}^T_{(i),g} + \hat{\mathbf{U}}_{(i),l}\hat{\mathbf{V}}^T_{(i),l}\right) - \left\|\mathbf{R}_{(i)}\right\|^2 \geq \hat{\sigma}^2_{gap}$*, and* $\left\|\mathbf{R}_{(i)}\right\| \leq \frac{\hat{\theta}\hat{\sigma}^2_{gap}}{8\hat{\sigma}_{\max}}$ *for every* i*.*

Remark If the condition number $\frac{\sigma_{\max}}{\sigma_{\min}} = O(1)$, the upper bound on the norm of $\mathbf{E}_{(i)}$ can be written as $\left\|\mathbf{E}_{(i)}\right\|_F \leq O\left(\sigma_{\min}\theta^{1.5}\right)$.

Proof. We first prove Claim 1. By triangle inequality,

$$\left\|\mathbf{M}_{(i)}\right\| \leq \left\|\mathbf{U}^\star{}_g\mathbf{V}^{\star T}_{(i),g} + \mathbf{U}^\star{}_{(i),l}\mathbf{V}^{\star T}_{(i),l}\right\| + \left\|\mathbf{E}_{(i)}\right\| \leq \sigma_{\max} + \sigma_{\max} = 2\sigma_{\max}$$

where we used the condition $\left\|\mathbf{E}_{(i)}\right\| \leq \left\|\mathbf{E}_{(i)}\right\|_F \leq \sigma_{\min}\theta^{1.5}\frac{1}{6\sqrt{2}}\frac{\sigma_{\min}}{\sigma_{\max}} \leq \sigma_{\max}$.

Then we prove the first inequality of Claim 2. Notice that,

$$\begin{aligned}\left\|\frac{1}{N}\sum_{i=1}^N \mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\right\| &\leq \left\|\frac{1}{N}\sum_{i=1}^N \mathbf{P}_{\mathbf{U}^\star{}_{(i),l}}\right\| + \frac{1}{N}\sum_{i=1}^N\left\|\mathbf{P}_{\hat{\mathbf{U}}_{(i),l}} - \mathbf{P}_{\mathbf{U}^\star{}_{(i),l}}\right\| \\ &\leq 1-\theta + \frac{1}{N}\sum_{i=1}^N\left\|\mathbf{P}_{\hat{\mathbf{U}}_{(i),l}} - \mathbf{P}_{\mathbf{U}^\star{}_{(i),l}}\right\|_F\end{aligned}$$

From equation 13, we know that $\sum_{i=1}^N\left\|\mathbf{P}_{\mathbf{U}^\star{}_{(i),l}} - \mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\right\|_F^2 \leq \frac{2\sum_{i=1}^N\left\|\mathbf{F}^\star{}_{(i)}\right\|_F^2}{\theta\sigma^4_{\min}}$. Therefore,

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^N\left\|\mathbf{P}_{\hat{\mathbf{U}}_{(i),l}} - \mathbf{P}_{\mathbf{U}^\star{}_{(i),l}}\right\|_F &\leq \frac{1}{\sqrt{N}}\sqrt{\sum_{i=1}^N\left\|\mathbf{P}_{\hat{\mathbf{U}}_{(i),l}} - \mathbf{P}_{\mathbf{U}^\star{}_{(i),l}}\right\|_F^2} \\ &\leq \sqrt{\frac{2\sum_{i=1}^N\left\|\mathbf{F}^\star{}_{(i)}\right\|_F^2}{\theta\sigma^4_{\min}N}} \leq \theta/2\end{aligned}$$

where the last inequality comes from the fact that $\left\|\mathbf{E}_{(i)}\right\|_F \leq \sigma_{\min}\theta^{1.5}\frac{1}{6\sqrt{2}}\frac{\sigma_{\min}}{\sigma_{\max}}$ thus $\left\|\mathbf{F}_{(i)}\right\|_F \leq 2\sigma_{\max}\left\|\mathbf{E}_{(i)}\right\|_F + \left\|\mathbf{E}_{(i)}\right\|_F^2 \leq 3\sigma_{\max}\left\|\mathbf{E}_{(i)}\right\|_F \leq \sigma_{\min}^2\theta^{1.5}\frac{1}{2\sqrt{2}}$.

Next we will show a lower bound on $\sigma_{min}\left(\hat{\mathbf{U}}_g\hat{\mathbf{V}}_{(i),g}^T + \hat{\mathbf{U}}_{(i),l}\hat{\mathbf{V}}_{(i),l}^T\right)$. A fact is that

$$\begin{aligned}
&\sigma_{min}\left(\hat{\mathbf{U}}_g\hat{\mathbf{V}}_{(i),g}^T + \hat{\mathbf{U}}_{(i),l}\hat{\mathbf{V}}_{(i),l}^T\right)\\
&\geq \sigma_{min}\left(\mathbf{U}^\star{}_g\mathbf{V}^{\star T}_{(i),g} + \mathbf{U}^\star{}_{(i),l}\mathbf{V}^{\star T}_{(i),l}\right) - \left\|\hat{\mathbf{U}}_g\hat{\mathbf{V}}_{(i),g}^T - \mathbf{U}^\star{}_g\mathbf{V}^{\star T}_{(i),g}\right\| - \left\|\hat{\mathbf{U}}_{(i),l}\hat{\mathbf{V}}_{(i),l}^T - \mathbf{U}^\star{}_{(i),l}\mathbf{V}^{\star T}_{(i),l}\right\|
\end{aligned}$$

We know from equation 14 that

$$\begin{aligned}
&\left\|\hat{\mathbf{U}}_g\hat{\mathbf{V}}_{(i),g}^T - \mathbf{U}^\star{}_g\mathbf{V}^{\star T}_{(i),g}\right\|_F + \left\|\hat{\mathbf{U}}_{(i),l}\hat{\mathbf{V}}_{(i),l}^T - \mathbf{U}^\star{}_{(i),l}\mathbf{V}^{\star T}_{(i),l}\right\|_F\\
&\leq \left\|\mathbf{P}_{\hat{\mathbf{U}}_g} - \mathbf{P}_{\mathbf{U}^\star{}_g}\right\|_F\sigma_{\max} + \left\|\mathbf{P}_{\hat{\mathbf{U}}_{(i),l}} - \mathbf{P}_{\mathbf{U}^\star{}_{(i),l}}\right\|_F\sigma_{\max} + 2\left\|\mathbf{E}^\star{}_{(i)}\right\|_F
\end{aligned}$$

Therefore, by equation 13, the above is bounded by

$$\begin{aligned}
&\sigma_{\max}\sqrt{2}\sqrt{\left\|\mathbf{P}_{\hat{\mathbf{U}}_g} - \mathbf{P}_{\mathbf{U}^\star{}_g}\right\|_F^2 + \left\|\mathbf{P}_{\hat{\mathbf{U}}_{(i),l}} - \mathbf{P}_{\mathbf{U}^\star{}_{(i),l}}\right\|_F^2} + 2\left\|\mathbf{E}^\star{}_{(i)}\right\|_F\\
&\leq \sigma_{\max}\sqrt{2}\sqrt{\frac{2\sum_{i=1}^N\left\|\mathbf{F}^\star{}_{(i)}\right\|_F^2}{\theta\sigma_{\min}^4}} + 2\left\|\mathbf{E}^\star{}_{(i)}\right\|_F\\
&\leq \left\|\mathbf{E}^\star{}_{(i)}\right\|_F\left(2 + \frac{6\sigma_{\max}^2}{\sqrt{\theta}\sigma_{\min}^2}\right) \leq \sigma_{\min}\frac{2-\sqrt{3}}{2}
\end{aligned}$$

where we applied the condition $\left\|\mathbf{E}^\star{}_{(i)}\right\|_F \leq \sigma_{\min}\frac{2-\sqrt{3}}{2}\left(2 + \frac{6}{\sqrt{\theta}}\left(\frac{\sigma_{\max}}{\sigma_{\min}}\right)^2\right)^{-1}$ in the last inequality. Thus

$$\sigma_{min}\left(\hat{\mathbf{U}}_g\hat{\mathbf{V}}_{(i),g}^T + \hat{\mathbf{U}}_{(i),l}\hat{\mathbf{V}}_{(i),l}^T\right) \geq \sigma_{\min}\frac{\sqrt{3}}{2}$$

Also, since $\mathbf{R}_{(i)}$'s correspond to the residuals of the optimal solutions, we have,

$$\left\|\mathbf{R}_{(i)}\right\| \leq \left\|\mathbf{R}_{(i)}\right\|_F \leq \sqrt{\sum_{i=1}^N\left\|\mathbf{R}_{(i)}\right\|_F^2} \leq \sqrt{\sum_{i=1}^N\left\|\mathbf{E}^\star{}_{(i)}\right\|_F^2} \leq \sigma_{\min}/2 \tag{15}$$

where we applied the condition $\left\|\mathbf{E}^\star{}_{(i)}\right\|_F \leq \sigma_{\min}\theta\frac{1}{64\sqrt{N}}\frac{\sigma_{\min}}{\sigma_{\max}}$ in the last inequality.

As a result, we have $\sigma_{min}^2\left(\hat{\mathbf{U}}_g\hat{\mathbf{V}}_{(i),g}^T + \hat{\mathbf{U}}_{(i),l}\hat{\mathbf{V}}_{(i),l}^T\right) - \left\|\mathbf{R}_{(i)}\right\|^2 \geq \hat{\sigma}_{\text{gap}}^2$.

Finally, we will prove the third condition in Claim 2. We can do so by using the condition $\left\|\mathbf{E}^\star{}_{(i)}\right\|_F \leq \sigma_{\min}\theta\frac{1}{64\sqrt{N}}\frac{\sigma_{\min}}{\sigma_{\max}}$ in equation 15, and considering the fact that $\hat{\sigma}_{\text{gap}}^2 = \sigma_{\min}^2/2$, $\hat{\theta} = \theta/2$, and $\hat{\sigma}_{\max} = 2\sigma_{\max}$. This completes our proof. □

In the following, we will always assume Claims 1 and 2 are true and prove our results under the assumptions of Lemma C.1.

The two claims are intuitively understandable. Claim 1 upper bounds the singular values of $\mathbf{M}_{(i)}$'s. This is the standard requirement in matrix factorization. Claim 2 implies $\hat{\mathbf{U}}_{(i),l}$'s should be $\hat{\theta}$-misaligned for some nonzero $\hat{\theta}$. Also, Claim 2 restricts the norm of $\mathbf{R}_{(i)}$'s to be smaller than a factor of the smallest nonzero singular value of $\hat{\mathbf{U}}_g\hat{\mathbf{V}}_{(i),g}^T + \hat{\mathbf{U}}_{(i),l}\hat{\mathbf{V}}_{(i),l}^T$. This requirement is satisfied when the norm of the noise in input $\mathbf{M}_{(i)}$ is not too large, i.e., $\mathbf{M}_{(i)}$ is not far from the linear combination of two low-rank matrices.

As introduced in the main paper, we use $\mathcal{B}(\zeta_1, \zeta_2)$ to denote the set of solutions with bounded norms:

$$\mathcal{B}(\zeta_1, \zeta_2) = \{\mathbf{U}_g, \{\mathbf{U}_{(i),l}\}, \{\mathbf{V}_{(i),g}\}, \{\mathbf{V}_{(i),l}\} \big| \, \|\mathbf{U}_g\|, \left\|\mathbf{U}_{(i),l}\right\| \leq \zeta_1, \left\|\mathbf{V}_{(i),g}\right\|, \left\|\mathbf{V}_{(i),l}\right\| \leq \zeta_2\} \tag{16}$$

The following lemma gives an upper bound on the Lipschitz constant when all iterates are bounded.

Lemma C.2 (Lipschitz continuity) *In region $\mathcal{B}(B_1, B_2)$ as defined in equation 16, g_i and $\tilde{f}_i$ are Lipschitz continuous:*

$$\begin{aligned} &\left\|\nabla g_i(\mathbf{U}_g^{'}, \mathbf{U}_{(i),l}^{'}) - \nabla g_i(\mathbf{U}_g, \mathbf{U}_{(i),l})\right\|_F \\ &\leq L_g \sqrt{\left\|\mathbf{U}_g^{'} - \mathbf{U}_g\right\|_F^2 + \left\|\mathbf{U}_{(i),l}^{'} - \mathbf{U}_{(i),l}\right\|_F^2} \end{aligned} \tag{17}$$

and

$$\begin{aligned} &\left\|\nabla \tilde{f}_i(\mathbf{U}_g^{'}, \mathbf{V}_{(i),g}^{'}, \mathbf{U}_{(i),l}^{'}, \mathbf{V}_{(i),l}^{'}) - \nabla \tilde{f}_i(\mathbf{U}_g, \mathbf{V}_{(i),g}, \mathbf{U}_{(i),l}, \mathbf{V}_{(i),l})\right\|_F \\ &\leq L \sqrt{\left\|\mathbf{U}_g^{'} - \mathbf{U}_g\right\|_F^2 + \left\|\mathbf{V}_{(i),g}^{'} - \mathbf{V}_{(i),g}\right\|_F^2 + \left\|\mathbf{U}_{(i),l}^{'} - \mathbf{U}_{(i),l}\right\|_F^2 + \left\|\mathbf{V}_{(i),l}^{'} - \mathbf{V}_{(i),l}\right\|_F^2} \end{aligned} \tag{18}$$

for $\{\mathbf{U}_g^{'}, \{\mathbf{V}_{(i),g}^{'}\}, \{\mathbf{U}_{(i),l}^{'}\}, \{\mathbf{V}_{(i),l}^{'}\}\}$,$\{\mathbf{U}_g, \{\mathbf{V}_{(i),g}\}, \{\mathbf{U}_{(i),l}\}, \{\mathbf{V}_{(i),l}\}\} \in \mathcal{B}(B_1, B_2)$, where L_g and L are constants dependent on B_1, B_2, $\hat{\sigma}_{\max}$, and β,

$$L_g = 2\beta(3B_1^2 + 1)$$

and,

$$L = \sqrt{(\hat{\sigma}_{\max} + 3B_1B_2)^2 + B_1^2B_2^2 + 2B_1^4 + B_2^4 + (B_2^2 + 2\beta(3B_1^2 + 1))^2}$$

Proof. We will firstly show the Lipschitz continuity of g_i. We know that

$$\begin{aligned} &\nabla_{\mathbf{U}_g} g_i(\mathbf{U}_g^{'}, \mathbf{U}_{(i),l}^{'}) - \nabla_{\mathbf{U}_g} g_i(\mathbf{U}_g, \mathbf{U}_{(i),l}) \\ &= 2\beta\mathbf{U}_g^{'}\left((\mathbf{U}_g^{'})^T\mathbf{U}_g^{'} - \mathbf{I}\right) - 2\beta\mathbf{U}_g\left(\mathbf{U}_g^T\mathbf{U}_g - \mathbf{I}\right) \end{aligned}$$

Therefore by some applications of triangle inequalities, we have,

$$\left\|\nabla_{\mathbf{U}_g} g_i(\mathbf{U}_g^{'}, \mathbf{U}_{(i),l}^{'}) - \nabla_{\mathbf{U}_g} g_i(\mathbf{U}_g, \mathbf{U}_{(i),l})\right\|_F \leq \left\|\mathbf{U}_g^{'} - \mathbf{U}_g\right\| 2\beta(3B_1^2 + 1)$$

This proves inequality equation 17. The Lipschitz continuity of $\tilde{f}_i$ can be proved similarly. We will calculate the gradient of $\tilde{f}_i$ over each variable, and bound the norm of the difference of the gradients.

$$\begin{aligned} &\nabla_{\mathbf{U}_g} \tilde{f}_i(\mathbf{U}_g^{'}, \mathbf{V}_{(i),g}^{'}, \mathbf{U}_{(i),l}^{'}, \mathbf{V}_{(i),l}^{'}) - \nabla_{\mathbf{U}_g} \tilde{f}_i(\mathbf{U}_g, \mathbf{V}_{(i),g}, \mathbf{U}_{(i),l}, \mathbf{V}_{(i),l}) \\ &= \left(\mathbf{U}_g^{'}(\mathbf{V}_{(i),g}^{'})^T + \mathbf{U}_{(i),l}^{'}(\mathbf{V}_{(i),l}^{'})^T - \mathbf{M}_{(i)}\right)\mathbf{V}_{(i),g}^{'} + 2\beta\mathbf{U}_g^{'}((\mathbf{U}_g^{'})^T\mathbf{U}_g^{'} - \mathbf{I}) \\ &- \left(\mathbf{U}_g(\mathbf{V}_{(i),g})^T + \mathbf{U}_{(i),l}(\mathbf{V}_{(i),l})^T - \mathbf{M}_{(i)}\right)\mathbf{V}_{(i),g} - 2\beta\mathbf{U}_g(\mathbf{U}_g^T\mathbf{U}_g - \mathbf{I}) \end{aligned}$$

Also by triangle inequalities, when $(\mathbf{U}_g^{'}, \mathbf{V}_{(i),g}^{'}, \mathbf{U}_{(i),l}^{'}, \mathbf{V}_{(i),l}^{'})$ and $(\mathbf{U}_g, \mathbf{V}_{(i),g}, \mathbf{U}_{(i),l}, \mathbf{V}_{(i),l})$ are in $\mathcal{B}(B_1, B_2)$, we have:

$$\begin{aligned} &\left\|\nabla_{\mathbf{U}_g} \tilde{f}_i(\mathbf{U}_g^{'}, \mathbf{V}_{(i),g}^{'}, \mathbf{U}_{(i),l}^{'}, \mathbf{V}_{(i),l}^{'}) - \nabla_{\mathbf{U}_g} \tilde{f}_i(\mathbf{U}_g, \mathbf{V}_{(i),g}, \mathbf{U}_{(i),l}, \mathbf{V}_{(i),l})\right\|_F \\ &\leq (B_2^2 + 2\beta(3B_1^2 + 1))\left\|\mathbf{U}_g^{'} - \mathbf{U}_g\right\|_F + (\hat{\sigma}_{\max} + 3B_1B_2)\left\|\mathbf{V}_{(i),g}^{'} - \mathbf{V}_{(i),g}\right\|_F \\ &+ B_2^2\left\|\mathbf{U}_{(i),l}^{'} - \mathbf{U}_{(i),l}\right\|_F + B_1B_2\left\|\mathbf{V}_{(i),l}^{'} - \mathbf{V}_{(i),l}\right\|_F \end{aligned}$$

Then on the derivative over $\mathbf{V}_{(i),g}$,

$$\begin{aligned} &\nabla_{\mathbf{V}_{(i),g}} \tilde{f}_i(\mathbf{U}_g^{'}, \mathbf{V}_{(i),g}^{'}, \mathbf{U}_{(i),l}^{'}, \mathbf{V}_{(i),l}^{'}) - \nabla_{\mathbf{V}_{(i),g}} \tilde{f}_i(\mathbf{U}_g, \mathbf{V}_{(i),g}, \mathbf{U}_{(i),l}, \mathbf{V}_{(i),l}) \\ &= \left(\mathbf{U}_g^{'}(\mathbf{V}_{(i),g}^{'})^T + \mathbf{U}_{(i),l}^{'}(\mathbf{V}_{(i),l}^{'})^T - \mathbf{M}_{(i)}\right)^T\mathbf{U}_g^{'} - \left(\mathbf{U}_g(\mathbf{V}_{(i),g})^T + \mathbf{U}_{(i),l}(\mathbf{V}_{(i),l})^T - \mathbf{M}_{(i)}\right)^T\mathbf{U}_g \end{aligned}$$

Thus by similar calculations, we have,

$$\begin{aligned}
&\left\|\nabla_{\mathbf{V}_{(i),g}}\tilde{f}_i(\mathbf{U}_g^{'},\mathbf{V}_{(i),g}^{'},\mathbf{U}_{(i),l}^{'},\mathbf{V}_{(i),l}^{'})-\nabla_{\mathbf{V}_{(i),g}}\tilde{f}_i(\mathbf{U}_g,\mathbf{V}_{(i),g},\mathbf{U}_{(i),l},\mathbf{V}_{(i),l})\right\|_F\\
&\leq(\hat{\sigma}_{\max}+3B_1B_2)\left\|\mathbf{U}_g^{'}-\mathbf{U}_g\right\|_F+B_1^2\left\|\mathbf{V}_{(i),g}^{'}-\mathbf{V}_{(i),g}\right\|_F\\
&+B_1B_2\left\|\mathbf{U}_{(i),l}^{'}-\mathbf{U}_{(i),l}\right\|_F+B_1^2\left\|\mathbf{V}_{(i),l}^{'}-\mathbf{V}_{(i),l}\right\|_F
\end{aligned}$$

And,

$$\begin{aligned}
&\left\|\nabla_{\mathbf{U}_{(i),l}}\tilde{f}_i(\mathbf{U}_g^{'},\mathbf{V}_{(i),g}^{'},\mathbf{U}_{(i),l}^{'},\mathbf{V}_{(i),l}^{'})-\nabla_{\mathbf{U}_{(i),l}}\tilde{f}_i(\mathbf{U}_g,\mathbf{V}_{(i),g},\mathbf{U}_{(i),l},\mathbf{V}_{(i),l})\right\|_F\\
&\leq B_2^2\left\|\mathbf{U}_g^{'}-\mathbf{U}_g\right\|_F+B_1B_2\left\|\mathbf{V}_{(i),g}^{'}-\mathbf{V}_{(i),g}\right\|_F\\
&+(B_2^2+2\beta(3B_1^2+1))\left\|\mathbf{U}_{(i),l}^{'}-\mathbf{U}_{(i),l}\right\|_F+(\hat{\sigma}_{\max}+3B_1B_2)\left\|\mathbf{V}_{(i),l}^{'}-\mathbf{V}_{(i),l}\right\|_F
\end{aligned}$$

Also,

$$\begin{aligned}
&\left\|\nabla_{\mathbf{V}_{(i),l}}\tilde{f}_i(\mathbf{U}_g^{'},\mathbf{V}_{(i),g}^{'},\mathbf{U}_{(i),l}^{'},\mathbf{V}_{(i),l}^{'})-\nabla_{\mathbf{V}_{(i),l}}\tilde{f}_i(\mathbf{U}_g,\mathbf{V}_{(i),g},\mathbf{U}_{(i),l},\mathbf{V}_{(i),l})\right\|_F\\
&\leq B_1B_2\left\|\mathbf{U}_g^{'}-\mathbf{U}_g\right\|_F+B_1^2\left\|\mathbf{V}_{(i),g}^{'}-\mathbf{V}_{(i),g}\right\|_F\\
&+(\hat{\sigma}_{\max}+3B_1B_2)\left\|\mathbf{U}_{(i),l}^{'}-\mathbf{U}_{(i),l}\right\|_F+B_1^2\left\|\mathbf{V}_{(i),l}^{'}-\mathbf{V}_{(i),l}\right\|_F
\end{aligned}$$

Combining the 4 inequalities, we have:

$$\begin{aligned}
&\left\|\nabla\tilde{f}_i(\mathbf{U}_g^{'},\mathbf{V}_{(i),g}^{'},\mathbf{U}_{(i),l}^{'},\mathbf{V}_{(i),l}^{'})-\nabla\tilde{f}_i(\mathbf{U}_g,\mathbf{V}_{(i),g},\mathbf{U}_{(i),l},\mathbf{V}_{(i),l})\right\|_F^2\\
&=\left\|\nabla_{\mathbf{U}_g}\tilde{f}_i(\mathbf{U}_g^{'},\mathbf{V}_{(i),g}^{'},\mathbf{U}_{(i),l}^{'},\mathbf{V}_{(i),l}^{'})-\nabla_{\mathbf{U}_g}\tilde{f}_i(\mathbf{U}_g,\mathbf{V}_{(i),g},\mathbf{U}_{(i),l},\mathbf{V}_{(i),l})\right\|_F^2\\
&+\left\|\nabla_{\mathbf{V}_{(i),g}}\tilde{f}_i(\mathbf{U}_g^{'},\mathbf{V}_{(i),g}^{'},\mathbf{U}_{(i),l}^{'},\mathbf{V}_{(i),l}^{'})-\nabla_{\mathbf{V}_{(i),g}}\tilde{f}_i(\mathbf{U}_g,\mathbf{V}_{(i),g},\mathbf{U}_{(i),l},\mathbf{V}_{(i),l})\right\|_F^2\\
&+\left\|\nabla_{\mathbf{U}_{(i),l}}\tilde{f}_i(\mathbf{U}_g^{'},\mathbf{V}_{(i),g}^{'},\mathbf{U}_{(i),l}^{'},\mathbf{V}_{(i),l}^{'})-\nabla_{\mathbf{U}_{(i),l}}\tilde{f}_i(\mathbf{U}_g,\mathbf{V}_{(i),g},\mathbf{U}_{(i),l},\mathbf{V}_{(i),l})\right\|_F^2\\
&+\left\|\nabla_{\mathbf{V}_{(i),l}}\tilde{f}_i(\mathbf{U}_g^{'},\mathbf{V}_{(i),g}^{'},\mathbf{U}_{(i),l}^{'},\mathbf{V}_{(i),l}^{'})-\nabla_{\mathbf{V}_{(i),l}}\tilde{f}_i(\mathbf{U}_g,\mathbf{V}_{(i),g},\mathbf{U}_{(i),l},\mathbf{V}_{(i),l})\right\|_F^2\\
&\leq L^2\left(\left\|\mathbf{U}_g^{'}-\mathbf{U}_g\right\|_F^2+\left\|\mathbf{V}_{(i),g}^{'}-\mathbf{V}_{(i),g}\right\|_F^2+\left\|\mathbf{U}_{(i),l}^{'}-\mathbf{U}_{(i),l}\right\|_F^2+\left\|\mathbf{V}_{(i),l}^{'}-\mathbf{V}_{(i),l}\right\|_F^2\right)
\end{aligned}$$

where L is a constant defined as,

$$L=\sqrt{(\hat{\sigma}_{\max}+3B_1B_2)^2+B_1^2B_2^2+2B_1^4+B_2^4+(B_2^2+2\beta(3B_1^2+1))^2}$$

□

The Lipschitz continuity allows us to prove the following sufficient descent lemma.

Lemma C.3 (Sufficient descent inequality) *For* $\{\mathbf{U}_{g,\tau},\{\mathbf{V}_{(i),g,\tau}\},\{\mathbf{U}_{(i),l,\tau}\},\{\mathbf{V}_{(i),l,\tau}\}\}\in\mathcal{B}(B_1,B_2)$*, if* $\left\|\mathbf{U}_{g,\tau}^T\mathbf{U}_{g,\tau}-\mathbf{I}\right\|\leq\frac{3}{4}$ *and* $\left\|\mathbf{U}_{(i),l,\tau}^T\mathbf{U}_{(i),l,\tau}-\mathbf{I}\right\|_F\leq\frac{3}{4}$ *for every* i*, and the stepsize* η *small enough,*

$$\begin{aligned}
\eta\leq&\min\Big\{\frac{1}{10}\Big(2B_1B_2(2B_1B_2+\hat{\sigma}_{\max})+2\beta B_1^2(B_1^2+1)+\big(B_2(2B_1B_2+\hat{\sigma}_{\max})+2\beta B_1(B_1^2+1)\big)^2\Big),\\
&\frac{1}{2\left(\frac{L}{2}+272\beta B_1^2+400L_gB_1^2\right)},1\Big\}
\end{aligned}$$

where L and L_g are defined in Lemma C.2, then the iterates of algorithm 1 satisfy the following inequality,

$$\begin{aligned}&\tilde{f}(\mathbf{U}_{g,\tau+1}, \{\mathbf{V}_{(i),g,\tau+1}, \mathbf{U}_{(i),l,\tau+1}, \mathbf{V}_{(i),l,\tau+1}\}) - \tilde{f}(\mathbf{U}_{g,\tau}, \{\mathbf{V}_{(i),g,\tau}, \mathbf{U}_{(i),l,\tau}, \mathbf{V}_{(i),l,\tau}\}) \\ &\leq -\frac{\eta}{2}\left(\left\|\frac{1}{\sqrt{N}}\nabla_{\mathbf{U}_g}\tilde{f}\right\|_F^2 + \sum_{i=1}^{N}\left\|\nabla_{\mathbf{V}_{(i),g}}\tilde{f}_i\right\|_F^2 + \left\|\nabla_{\mathbf{U}_{(i),l}}\tilde{f}_i\right\|_F^2 + \left\|\nabla_{\mathbf{V}_{(i),l}}\tilde{f}_i\right\|_F^2\right)\end{aligned}$$

Proof. We first look at the correction step. Notice that the function value f_i before and after correction is the same:

$$\begin{aligned}&f_i(\mathbf{U}_{g,\tau+1}, \mathbf{V}_{(i),g,\tau+1}, \mathbf{U}_{(i),l,\tau+1}, \mathbf{V}_{(i),l,\tau+1}) \\ &= \left\|\mathbf{M}_{(i)} - \mathbf{U}_{g,\tau+1}\mathbf{V}_{(i),g,\tau+1}^T - \mathbf{U}_{(i),l,\tau+1}\mathbf{V}_{(i),l,\tau+1}^T\right\|_F^2 \\ &= ||\mathbf{M}_{(i)} - \mathbf{U}_{g,\tau+1}\left(\mathbf{V}_{(i),g,\tau+\frac{1}{2}} + \mathbf{V}_{(i),l,\tau+1}\mathbf{U}_{(i),l,\tau+\frac{1}{2}}^T\mathbf{U}_{g,\tau+1}\left(\mathbf{U}_{g,\tau+1}^T\mathbf{U}_{g,\tau+1}\right)^{-1}\right)^T \\ &- \left(\mathbf{U}_{(i),l,\tau+\frac{1}{2}} - \mathbf{U}_{g,\tau+1}\left(\mathbf{U}_{g,\tau+1}^T\mathbf{U}_{g,\tau+1}\right)^{-1}\mathbf{U}_{g,\tau+1}^T\mathbf{U}_{(i),l,\tau+\frac{1}{2}}\right)\mathbf{V}_{(i),l,\tau+1}^T||_F^2 \\ &= \left\|\mathbf{M}_{(i),\tau+1} - \mathbf{U}_{g,\tau+1}\mathbf{V}_{(i),g,\tau+\frac{1}{2}}^T - \mathbf{U}_{(i),l,\tau+\frac{1}{2}}\mathbf{V}_{(i),l,\tau+1}^T\right\|_F^2 \\ &= f_i(\mathbf{U}_{g,\tau+1}, \mathbf{V}_{(i),g,\tau+\frac{1}{2}}, \mathbf{U}_{(i),l,\tau+\frac{1}{2}}, \mathbf{V}_{(i),l,\tau+1})\end{aligned}$$

For g_i, we have:

$$\begin{aligned}&g_i(\mathbf{U}_{g,\tau+1}, \mathbf{U}_{(i),l,\tau+1}) - g_i(\mathbf{U}_{g,\tau+1}, \mathbf{U}_{(i),l,\tau+\frac{1}{2}}) \\ &\leq \left\langle \nabla_{\mathbf{U}_{(i),l}} g_i(\mathbf{U}_{g,\tau+1}, \mathbf{U}_{(i),l,\tau+\frac{1}{2}}), \mathbf{U}_{g,\tau+1}\left(\mathbf{U}_{g,\tau+1}^T\mathbf{U}_{g,\tau+1}\right)^{-1}\mathbf{U}_{g,\tau+1}^T\mathbf{U}_{(i),l,\tau+\frac{1}{2}}\right\rangle \\ &+ \frac{L_g}{2}\left\|\mathbf{U}_{g,\tau+1}\left(\mathbf{U}_{g,\tau+1}^T\mathbf{U}_{g,\tau+1}\right)^{-1}\mathbf{U}_{g,\tau+1}^T\mathbf{U}_{(i),l,\tau+\frac{1}{2}}\right\|_F^2\end{aligned}$$

where L_g is the Lipschitz continuity constant of g_i. For the first term, we have,

$$\begin{aligned}&\left\langle \nabla_{\mathbf{U}_{(i),l}} g_i(\mathbf{U}_{g,\tau+1}, \mathbf{U}_{(i),l,\tau+\frac{1}{2}}), \mathbf{U}_{g,\tau+1}\left(\mathbf{U}_{g,\tau+1}^T\mathbf{U}_{g,\tau+1}\right)^{-1}\mathbf{U}_{g,\tau+1}^T\mathbf{U}_{(i),l,\tau+\frac{1}{2}}\right\rangle \\ &= \mathrm{Tr}\left(2\beta\mathbf{U}_{(i),l,\tau+\frac{1}{2}}^T(\mathbf{U}_{(i),l,\tau+\frac{1}{2}}^T\mathbf{U}_{(i),l,\tau+\frac{1}{2}} - \mathbf{I})\mathbf{U}_{(i),l,\tau+\frac{1}{2}}^T\mathbf{U}_{g,\tau+1}\left(\mathbf{U}_{g,\tau+1}^T\mathbf{U}_{g,\tau+1}\right)^{-1}\mathbf{U}_{g,\tau+1}^T\mathbf{U}_{(i),l,\tau+\frac{1}{2}}\right)\end{aligned}$$

Since $\left\|\mathbf{U}_{(i),l,\tau}^T\mathbf{U}_{(i),l,\tau} - \mathbf{I}\right\|_F \leq \frac{3}{4}$, we know,

$$\begin{aligned}&\left\|\mathbf{U}_{(i),l,\tau+\frac{1}{2}}^T\mathbf{U}_{(i),l,\tau+\frac{1}{2}} - \mathbf{I}\right\|_F \\ &= \left\|\mathbf{U}_{(i),l,\tau+\frac{1}{2}}^T\mathbf{U}_{(i),l,\tau+\frac{1}{2}} - \mathbf{I} - \eta\nabla_{\mathbf{U}_{(i),l}}\tilde{f}_i^T\mathbf{U}_{(i),l} - \eta\mathbf{U}_{(i),l}^T\nabla_{\mathbf{U}_{(i),l}}\tilde{f}_i + \eta^2\nabla_{\mathbf{U}_{(i),l}}\tilde{f}_i^T\nabla_{\mathbf{U}_{(i),l}}\tilde{f}_i\right\|_F \\ &\leq \left\|\mathbf{U}_{(i),l,\tau}^T\mathbf{U}_{(i),l,\tau} - \mathbf{I}\right\| + \eta\left(\left\|\nabla_{\mathbf{U}_{(i),l}}\tilde{f}_i\right\| B_1 + \left\|\nabla_{\mathbf{U}_{(i),l}}\tilde{f}_i\right\| B_1 + \eta\left\|\nabla_{\mathbf{U}_{(i),l}}\tilde{f}_i\right\|^2\right) \\ &\leq \frac{17}{20}\end{aligned}$$

where in the last inequality, we applied the condition that $\left\|\mathbf{U}_{(i),l,\tau}^T\mathbf{U}_{(i),l,\tau} - \mathbf{I}\right\| \leq \frac{3}{4}$ and η is not too large,

$$\eta \leq \frac{1}{10}\left(2B_1B_2(2B_1B_2 + \hat{\sigma}_{\max}) + 2\beta B_1^2(B_1^2+1) + \left(B_2(2B_1B_2 + \hat{\sigma}_{\max}) + 2\beta B_1(B_1^2+1)\right)^2\right)$$

As,

$$\begin{aligned}&\left\|\mathbf{U}_{g,\tau+1}^T\mathbf{U}_{g,\tau+1} - \mathbf{U}_{g,\tau}^T\mathbf{U}_{g,\tau}\right\| \\ &= \eta\left\|\mathbf{U}_{g,\tau}^T\nabla_{\mathbf{U}_g}\frac{\tilde{f}}{N} + (\nabla_{\mathbf{U}_g}\frac{\tilde{f}}{N})^T\mathbf{U}_{g,\tau} + (\nabla_{\mathbf{U}_g}\frac{\tilde{f}}{N})^T\nabla_{\mathbf{U}_g}\frac{\tilde{f}}{N}\right\| \\ &\leq \eta\left(2B_1B_2(2B_1B_2 + \hat{\sigma}_{\max}) + 2\beta B_1^2(B_1^2+1) + \eta\left(B_2(2B_1B_2 + \hat{\sigma}_{\max}) + 2\beta B_1(B_1^2+1)\right)^2\right)\end{aligned}$$

Since η is sufficiently small

$$\eta \leq \frac{3}{20}\left(2B_1B_2(2B_1B_2+\hat{\sigma}_{\max})+2\beta B_1^2(B_1^2+1)+\left(B_2(2B_1B_2+\hat{\sigma}_{\max})+2\beta B_1(B_1^2+1)\right)^2\right)$$

we know that $\left\|\mathbf{U}_{g,\tau+1}^T\mathbf{U}_{g,\tau+1}-\mathbf{U}_{g,\tau}^T\mathbf{U}_{g,\tau}\right\| \leq \frac{3}{20}$. Then, by Lemma E.4 we have,

$$\begin{aligned}
&\left\|\left(\mathbf{U}_{g,\tau+1}^T\mathbf{U}_{g,\tau+1}\right)^{-1}\right\| \\
&\leq \left\|\left(\mathbf{U}_{g,\tau}^T\mathbf{U}_{g,\tau}\right)^{-1}\right\| + \frac{\left\|\left(\mathbf{U}_{g,\tau}^T\mathbf{U}_{g,\tau}\right)^{-1}\right\|^2\left\|\mathbf{U}_{g,\tau+1}^T\mathbf{U}_{g,\tau+1}-\mathbf{U}_{g,\tau}^T\mathbf{U}_{g,\tau}\right\|}{1-\left\|\left(\mathbf{U}_{g,\tau}^T\mathbf{U}_{g,\tau}\right)^{-1}\right\|\left\|\mathbf{U}_{g,\tau+1}^T\mathbf{U}_{g,\tau+1}-\mathbf{U}_{g,\tau}^T\mathbf{U}_{g,\tau}\right\|}
\end{aligned}$$

Lemma E.1 shows that when $\left\|\mathbf{U}_{g,\tau}^T\mathbf{U}_{g,\tau}-\mathbf{I}\right\|_F \leq \frac{3}{4}$, we have $\left\|\left(\mathbf{U}_{g,\tau}^T\mathbf{U}_{g,\tau}\right)^{-1}\right\| \leq 1+3=4$. Combining these inequalities, we have $\left\|\left(\mathbf{U}_{g,\tau+1}^T\mathbf{U}_{g,\tau+1}\right)^{-1}\right\| \leq 10$.

Finally, we can look at

$$\begin{aligned}
&\left\|\mathbf{U}_{g,\tau+1}^T\mathbf{U}_{(i),l,\tau+\frac{1}{2}}\right\|_F \\
&= \left\|\mathbf{U}_{g,\tau}^T\mathbf{U}_{(i),l,\tau}-\eta(\nabla_{\mathbf{U}_g}\frac{\tilde{f}}{N})^T\mathbf{U}_{(i),l,\tau}-\eta\mathbf{U}_{g,\tau}^T\nabla_{\mathbf{U}_{(i),l}}\tilde{f}_i+\eta^2\left(\nabla_{\mathbf{U}_g}\frac{\tilde{f}}{N}\right)^T\nabla_{\mathbf{U}_{(i),l}}\tilde{f}_i\right\|_F \\
&\leq \left\|\eta(\nabla_{\mathbf{U}_g}\frac{\tilde{f}}{N})^T\mathbf{U}_{(i),l,\tau}\right\|_F+\left\|\eta\mathbf{U}_{g,\tau}^T\nabla_{\mathbf{U}_{(i),l}}\tilde{f}_i\right\|_F+\left\|\eta^2\left(\nabla_{\mathbf{U}_g}\frac{\tilde{f}}{N}\right)^T\nabla_{\mathbf{U}_{(i),l}}\tilde{f}_i\right\|_F \\
&\leq \eta 2B_1\left\|\nabla_{\mathbf{U}_g}\frac{\tilde{f}}{N}\right\|_F+\eta 2B_1\left\|\nabla_{\mathbf{U}_{(i),l}}\tilde{f}_i\right\|_F
\end{aligned}$$

where we bound the second order terms by first order terms considering the fact that $\eta\left\|\nabla_{\mathbf{U}_g}\frac{\tilde{f}}{N}\right\|_F, \left\|\eta\nabla_{\mathbf{U}_{(i),l}}\tilde{f}_i\right\|_F \leq 1$.

Combining the three, we have,

$$\begin{aligned}
&\left\langle\nabla_{\mathbf{U}_{(i),l}}g_i(\mathbf{U}_{g,\tau+1},\mathbf{U}_{(i),l,\tau+\frac{1}{2}}),\mathbf{U}_{g,\tau+1}\left(\mathbf{U}_{g,\tau+1}^T\mathbf{U}_{g,\tau+1}\right)^{-1}\mathbf{U}_{g,\tau+1}^T\mathbf{U}_{(i),l,\tau+\frac{1}{2}}\right\rangle \\
&= \mathrm{Tr}\left(2\beta\mathbf{U}_{(i),l,\tau+\frac{1}{2}}^T(\mathbf{U}_{(i),l,\tau+\frac{1}{2}}^T\mathbf{U}_{(i),l,\tau+\frac{1}{2}}-\mathbf{I})\mathbf{U}_{(i),l,\tau+\frac{1}{2}}^T\mathbf{U}_{g,\tau+1}\left(\mathbf{U}_{g,\tau+1}^T\mathbf{U}_{g,\tau+1}\right)^{-1}\mathbf{U}_{g,\tau+1}^T\mathbf{U}_{(i),l,\tau+\frac{1}{2}}\right) \\
&\leq 2\beta\left\|\mathbf{U}_{(i),l,\tau+\frac{1}{2}}^T(\mathbf{U}_{(i),l,\tau+\frac{1}{2}}^T\mathbf{U}_{(i),l,\tau+\frac{1}{2}}-\mathbf{I})\mathbf{U}_{(i),l,\tau+\frac{1}{2}}^T\mathbf{U}_{g,\tau+1}\left(\mathbf{U}_{g,\tau+1}^T\mathbf{U}_{g,\tau+1}\right)^{-1}\right\|_F\left\|\mathbf{U}_{g,\tau+1}^T\mathbf{U}_{(i),l,\tau+\frac{1}{2}}\right\|_F \\
&\leq 2\beta\left\|\mathbf{U}_{(i),l,\tau+\frac{1}{2}}\right\|\left\|\mathbf{U}_{(i),l,\tau+\frac{1}{2}}^T\mathbf{U}_{(i),l,\tau+\frac{1}{2}}-\mathbf{I}\right\|_F\left\|\left(\mathbf{U}_{g,\tau+1}^T\mathbf{U}_{g,\tau+1}\right)^{-1}\right\|_F\left\|\mathbf{U}_{g,\tau+1}^T\mathbf{U}_{(i),l,\tau+\frac{1}{2}}\right\|_F^2 \\
&\leq \eta^2\left(\left\|\nabla_{\mathbf{U}_g}\frac{\tilde{f}}{N}\right\|_F^2+\left\|\nabla_{\mathbf{U}_{(i),l}}\tilde{f}_i\right\|_F^2\right)272\beta B_1^3
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\frac{L_g}{2}\left\|\mathbf{U}_{g,\tau+1}\left(\mathbf{U}_{g,\tau+1}^T\mathbf{U}_{g,\tau+1}\right)^{-1}\mathbf{U}_{g,\tau+1}^T\mathbf{U}_{(i),l,\tau+\frac{1}{2}}\right\|_F^2 \\
&\leq \frac{L_g}{2}\left\|\mathbf{U}_{g,\tau+1}\right\|^2\left\|\left(\mathbf{U}_{g,\tau+1}^T\mathbf{U}_{g,\tau+1}\right)^{-1}\right\|^2\left\|\mathbf{U}_{g,\tau+1}^T\mathbf{U}_{(i),l,\tau+\frac{1}{2}}\right\|_F^2 \\
&\leq \eta^2\left(\left\|\nabla_{\mathbf{U}_g}\frac{\tilde{f}}{N}\right\|_F^2+\left\|\nabla_{\mathbf{U}_{(i),l}}\tilde{f}_i\right\|_F^2\right)400L_gB_1^6
\end{aligned}$$

Therefore,

$$g_i(\mathbf{U}_{g,\tau+1},\mathbf{U}_{(i),l,\tau+1})-g_i(\mathbf{U}_{g,\tau+1},\mathbf{U}_{(i),l,\tau+\frac{1}{2}}) \leq \eta^2\left(\left\|\nabla_{\mathbf{U}_g}\frac{\tilde{f}}{N}\right\|_F^2+\left\|\nabla_{\mathbf{U}_{(i),l}}\tilde{f}_i\right\|_F^2\right)\left(272\beta B_1^2+400L_gB_1^6\right)$$

Also, we know

$$
\begin{aligned}
&\tilde{f}_i(\mathbf{U}_{g,\tau+1}, \mathbf{V}_{(i),g,\tau+\frac{1}{2}}, \mathbf{U}_{(i),l,\tau+\frac{1}{2}}, \mathbf{V}_{(i),l,\tau+1}) - \tilde{f}_i(\mathbf{U}_{g,\tau}, \mathbf{V}_{(i),g,\tau}, \mathbf{U}_{(i),l,\tau}, \mathbf{V}_{(i),l,\tau}) \\
&\leq \left\langle \nabla_{\mathbf{U}_g} \tilde{f}_i, \mathbf{U}_{g,\tau+1} - \mathbf{U}_{g,\tau} \right\rangle + \left\langle \nabla_{\mathbf{V}_{(i),g}} \tilde{f}_i, \mathbf{V}_{(i),g,\tau+\frac{1}{2}} - \mathbf{V}_{(i),g,\tau} \right\rangle \\
&+ \left\langle \nabla_{\mathbf{U}_{(i),l}} \tilde{f}_i, \mathbf{U}_{(i),l,\tau+\frac{1}{2}} - \mathbf{U}_{(i),l,\tau} \right\rangle + \left\langle \nabla_{\mathbf{V}_{(i),l}} \tilde{f}_i, \mathbf{V}_{(i),l,\tau+1} - \mathbf{V}_{(i),l,\tau} \right\rangle \\
&+ \frac{L}{2} \left(\|\mathbf{U}_{g,\tau+1} - \mathbf{U}_{g,\tau}\|_F^2 + \left\| \mathbf{V}_{(i),g,\tau+\frac{1}{2}} - \mathbf{V}_{(i),g,\tau} \right\|_F^2 + \left\| \mathbf{U}_{(i),l,\tau+\frac{1}{2}} - \mathbf{U}_{(i),l,\tau} \right\|_F^2 + \left\| \mathbf{V}_{(i),l,\tau+1} - \mathbf{V}_{(i),l,\tau} \right\|_F^2 \right) \\
&\leq -\eta \left(\left\langle \nabla_{\mathbf{U}_g} \tilde{f}_i, \nabla_{\mathbf{U}_g} \frac{\tilde{f}}{N} \right\rangle + \left\| \nabla_{\mathbf{U}_{(i),l}} \tilde{f}_i \right\|_F^2 + \left\| \nabla_{\mathbf{V}_{(i),g}} \tilde{f}_i \right\|_F^2 + \left\| \nabla_{\mathbf{V}_{(i),l}} \tilde{f}_i \right\|_F^2 \right) \\
&+ \eta^2 \frac{L}{2} \left(\left\| \nabla_{\mathbf{U}_g} \frac{\tilde{f}}{N} \right\|_F^2 + \left\| \nabla_{\mathbf{U}_{(i),l}} \tilde{f}_i \right\|_F^2 + \left\| \nabla_{\mathbf{V}_{(i),g}} \tilde{f}_i \right\|_F^2 + \left\| \nabla_{\mathbf{V}_{(i),l}} \tilde{f}_i \right\|_F^2 \right)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\tilde{f}_i(\mathbf{U}_{g,\tau+1}, \mathbf{V}_{(i),g,\tau+1}, \mathbf{U}_{(i),l,\tau+1}, \mathbf{V}_{(i),l,\tau+1}) - \tilde{f}_i(\mathbf{U}_{g,\tau}, \mathbf{V}_{(i),g,\tau}, \mathbf{U}_{(i),l,\tau}, \mathbf{V}_{(i),l,\tau}) \\
&= g_i(\mathbf{U}_{g,\tau+1}, \mathbf{U}_{(i),l,\tau+1}) - g_i(\mathbf{U}_{g,\tau+1}, \mathbf{U}_{(i),l,\tau+\frac{1}{2}}) \\
&+ \tilde{f}_i(\mathbf{U}_{g,\tau+1}, \mathbf{V}_{(i),g,\tau+\frac{1}{2}}, \mathbf{U}_{(i),l,\tau+\frac{1}{2}}, \mathbf{V}_{(i),l,\tau+1}) - \tilde{f}_i(\mathbf{U}_{g,\tau}, \mathbf{V}_{(i),g,\tau}, \mathbf{U}_{(i),l,\tau}, \mathbf{V}_{(i),l,\tau}) \\
&\leq -\eta \left(\left\langle \nabla_{\mathbf{U}_g} \tilde{f}_i, \nabla_{\mathbf{U}_g} \frac{\tilde{f}}{N} \right\rangle + \left\| \nabla_{\mathbf{U}_{(i),l}} \tilde{f}_i \right\|_F^2 + \left\| \nabla_{\mathbf{V}_{(i),g}} \tilde{f}_i \right\|_F^2 + \left\| \nabla_{\mathbf{V}_{(i),l}} \tilde{f}_i \right\|_F^2 \right) \\
&+ \eta^2 \left(\frac{L}{2} + 272\beta B_1^2 + 400 L_g B_1^6 \right) \left(\left\| \nabla_{\mathbf{U}_g} \frac{\tilde{f}}{N} \right\|_F^2 + \left\| \nabla_{\mathbf{U}_{(i),l}} \tilde{f}_i \right\|_F^2 + \left\| \nabla_{\mathbf{V}_{(i),g}} \tilde{f}_i \right\|_F^2 + \left\| \nabla_{\mathbf{V}_{(i),l}} \tilde{f}_i \right\|_F^2 \right)
\end{aligned}
$$

Summing both side for i from 1 to N, we have:

$$
\begin{aligned}
&\tilde{f}(\mathbf{U}_{g,\tau+1}, \{\mathbf{V}_{(i),g,\tau+1}, \mathbf{U}_{(i),l,\tau+1}, \mathbf{V}_{(i),l,\tau+1}\}) - \tilde{f}(\mathbf{U}_{g,\tau}, \{\mathbf{V}_{(i),g,\tau}, \mathbf{U}_{(i),l,\tau}, \mathbf{V}_{(i),l,\tau}\}) \\
&= \sum_{i=1}^{N} \tilde{f}_i(\mathbf{U}_{g,\tau+1}, \mathbf{V}_{(i),g,\tau+1}, \mathbf{U}_{(i),l,\tau+1}, \mathbf{V}_{(i),l,\tau+1}) - \tilde{f}_i(\mathbf{U}_{g,\tau}, \mathbf{V}_{(i),g,\tau}, \mathbf{U}_{(i),l,\tau}, \mathbf{V}_{(i),l,\tau}) \\
&\leq -\eta \left(\left\| \frac{1}{\sqrt{N}} \nabla_{\mathbf{U}_g} \tilde{f} \right\|_F^2 + \sum_{i=1}^{N} \left\| \nabla_{\mathbf{V}_{(i),g}} \tilde{f}_i \right\|_F^2 + \left\| \nabla_{\mathbf{U}_{(i),l}} \tilde{f}_i \right\|_F^2 + \left\| \nabla_{\mathbf{V}_{(i),l}} \tilde{f}_i \right\|_F^2 \right) \\
&+ \eta^2 \left(\frac{L}{2} + 272\beta B_1^2 + 400 L_g B_1^6 \right) \Big(\left\| \frac{1}{\sqrt{N}} \nabla_{\mathbf{U}_g} \tilde{f} \right\|_F^2 + \sum_{i=1}^{N} \left\| \nabla_{\mathbf{V}_{(i),g}} \tilde{f}_i \right\|_F^2 \\
&+ \left\| \nabla_{\mathbf{U}_{(i),l}} \tilde{f}_i \right\|_F^2 + \left\| \nabla_{\mathbf{V}_{(i),l}} \tilde{f}_i \right\|_F^2 \Big)
\end{aligned}
$$

Therefore, when $\eta \leq \frac{1}{\frac{L}{2} + 272\beta B_1^2 + 400 L_g B_1^6}$, we have:

$$
\begin{aligned}
&\tilde{f}(\mathbf{U}_{g,\tau+1}, \{\mathbf{V}_{(i),g,\tau+1}, \mathbf{U}_{(i),l,\tau+1}, \mathbf{V}_{(i),l,\tau+1}\}) - \tilde{f}(\mathbf{U}_{g,\tau}, \{\mathbf{V}_{(i),g,\tau}, \mathbf{U}_{(i),l,\tau}, \mathbf{V}_{(i),l,\tau}\}) \\
&\leq -\frac{\eta}{2} \left(\left\| \frac{1}{\sqrt{N}} \nabla_{\mathbf{U}_g} \tilde{f} \right\|_F^2 + \sum_{i=1}^{N} \left\| \nabla_{\mathbf{V}_{(i),g}} \tilde{f}_i \right\|_F^2 + \left\| \nabla_{\mathbf{U}_{(i),l}} \tilde{f}_i \right\|_F^2 + \left\| \nabla_{\mathbf{V}_{(i),l}} \tilde{f}_i \right\|_F^2 \right)
\end{aligned}
$$

This completes the proof. □

The above analysis suggests that Algorithm 1 will converge to points where the gradients vanish. In other words, Algorithm 1 converges into stationary points. In the rest of this section, we will analyze the local geometry around the optimal solutions to (4) and show that such geometric properties essentially allow Algorithm 1 to converge into the optimal solutions.

To do so, we will first find one set of optimal solutions that is close to the current iterate. In particular, given an iterate $\{\mathbf{U}_{g,\tau}, \mathbf{V}_{(i),g,\tau}, \mathbf{U}_{(i),l,\tau}, \mathbf{V}_{(i),l,\tau}\}$, define:

$$\widetilde{\mathbf{U}}_{g,\tau} = \mathbf{P}_{\hat{\mathbf{U}}_g}\mathbf{U}_{g,\tau}\left(\mathbf{U}_{g,\tau}^T\mathbf{P}_{\hat{\mathbf{U}}_g}\mathbf{U}_{g,\tau}\right)^{-\frac{1}{2}} \tag{19a}$$

$$\widetilde{\mathbf{U}}_{(i),l,\tau} = \mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\mathbf{U}_{(i),l,\tau}\left(\mathbf{U}_{(i),l,\tau}^T\mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\mathbf{U}_{(i),l,\tau}\right)^{-\frac{1}{2}} \tag{19b}$$

$$\widetilde{\mathbf{V}}_{(i),g,\tau} = \mathbf{M}_{(i)}^T\widetilde{\mathbf{U}}_{(i),g,\tau} \tag{19c}$$

$$\widetilde{\mathbf{V}}_{(i),l,\tau} = \mathbf{M}_{(i)}^T\widetilde{\mathbf{U}}_{(i),l,\tau} \tag{19d}$$

Indeed, the defined solutions are optimal since $\widetilde{\mathbf{U}}_{g,\tau}^T\widetilde{\mathbf{U}}_{g,\tau} = \mathbf{I}$, $\widetilde{\mathbf{U}}_{(i),l,\tau}^T\widetilde{\mathbf{U}}_{(i),l,\tau} = \mathbf{I}$, and $\widetilde{\mathbf{U}}_{g,\tau}\widetilde{\mathbf{V}}_{(i),g,\tau}^T + \widetilde{\mathbf{U}}_{(i),l,\tau}\widetilde{\mathbf{V}}_{(i),l,\tau}^T = \hat{\mathbf{U}}_g\hat{\mathbf{V}}_{(i),g}^T + \hat{\mathbf{U}}_{(i),l}\hat{\mathbf{V}}_{(i),l}^T$. Let the difference between iterates and these optimal solutions be denoted as

$$\begin{cases} \Delta\mathbf{U}_{g,\tau} = \mathbf{U}_{g,\tau} - \widetilde{\mathbf{U}}_{g,\tau} \\ \Delta\mathbf{V}_{(i),g,\tau} = \mathbf{V}_{(i),g,\tau} - \widetilde{\mathbf{V}}_{(i),g,\tau} \\ \Delta\mathbf{U}_{(i),l,\tau} = \mathbf{U}_{(i),l,\tau} - \widetilde{\mathbf{U}}_{(i),l,\tau} \\ \Delta\mathbf{V}_{(i),l,\tau} = \mathbf{V}_{(i),l,\tau} - \widetilde{\mathbf{V}}_{(i),l,\tau} \end{cases} \tag{20}$$

Now we introduce the KKT conditions to (4). The KKT condition relies on the regularity condition that $\mathbf{M}_{(i)}$'s have rank at least $r_1 + r_2$. This assumption is easily satisfied if $\left\|\mathbf{E}_{(i)}\right\|$ is smaller than $\sigma_{\min}$.

Lemma C.4 (KKT conditions for square error) *Suppose that $\{\hat{\mathbf{U}}_g, \hat{\mathbf{U}}_{(i),l}, \hat{\mathbf{V}}_{(i),g}, \hat{\mathbf{V}}_{(i),l}\}$ is the optimal solution to problem (4) in the main paper with square error loss ℓ^{se}, and $\mathbf{M}_{(i)}$ has rank at least $r_1 + r_2$. We have*

$$\sum_{i=1}^{N}\left(\hat{\mathbf{U}}_g\hat{\mathbf{V}}_{(i),g}^T + \hat{\mathbf{U}}_{(i),l}\hat{\mathbf{V}}_{(i),l}^T - \mathbf{M}_{(i)}\right)\hat{\mathbf{V}}_{(i),g} = 0 \tag{21a}$$

$$\left(\hat{\mathbf{U}}_g\hat{\mathbf{V}}_{(i),g}^T + \hat{\mathbf{U}}_{(i),l}\hat{\mathbf{V}}_{(i),l}^T - \mathbf{M}_{(i)}\right)\hat{\mathbf{V}}_{(i),l} = 0 \tag{21b}$$

$$\left(\hat{\mathbf{U}}_g\hat{\mathbf{V}}_{(i),g}^T + \hat{\mathbf{U}}_{(i),l}\hat{\mathbf{V}}_{(i),l}^T - \mathbf{M}_{(i)}\right)^T\hat{\mathbf{U}}_{(i),l} = 0 \tag{21c}$$

$$\left(\hat{\mathbf{U}}_g\hat{\mathbf{V}}_{(i),g}^T + \hat{\mathbf{U}}_{(i),l}\hat{\mathbf{V}}_{(i),l}^T - \mathbf{M}_{(i)}\right)^T\hat{\mathbf{U}}_g = 0 \tag{21d}$$

$$\hat{\mathbf{U}}_{(i),l}^T\hat{\mathbf{U}}_{(i),l} = \mathbf{I}, \hat{\mathbf{U}}_g^T\hat{\mathbf{U}}_g = \mathbf{I}, \hat{\mathbf{U}}_{(i),l}^T\hat{\mathbf{U}}_g = 0. \tag{21e}$$

Proof. The proof is presented in two parts.

<u>*LICQ of (4).*</u> We will first show the linear independence constraint qualification (LICQ) of the optimal $\hat{\mathbf{U}}_g$, $\{\hat{\mathbf{V}}_{(i),g}, \hat{\mathbf{U}}_{(i),l}, \hat{\mathbf{V}}_{(i),l}\}$.

We begin by proving $\hat{\mathbf{U}}_g$ has full column rank. Suppose otherwise, $\hat{\mathbf{U}}_g$ has rank $r_1^{'} < r_1$. Since $\mathbf{M}_{(i)}$ has rank at least $r_1 + r_2$, the residual $\mathbf{M}_{(i)} - \hat{\mathbf{U}}_g\hat{\mathbf{V}}_{(i),g}^T - \hat{\mathbf{U}}_{(i),l}\hat{\mathbf{V}}_{(i),l}^T$ has rank at least 1. Therefore we can find $\hat{\mathbf{U}}_g^{'}$ such that $\hat{\mathbf{U}}_g^{'T}\hat{\mathbf{U}}_g^{'} = \mathbf{I}$ and $f_i(\hat{\mathbf{U}}_g^{'}, \hat{\mathbf{V}}_{(i),g}, \hat{\mathbf{U}}_{(i),l}, \hat{\mathbf{V}}_{(i),l}) < f_i(\hat{\mathbf{U}}_g, \hat{\mathbf{V}}_{(i),g}, \hat{\mathbf{U}}_{(i),l}, \hat{\mathbf{V}}_{(i),l})$. This contradicts the fact that $\hat{\mathbf{U}}_g$ is optimal.

Next we will establish the LICQ of constraints. We define h_{ijk} as the inner product between the j-th column of $\mathbf{U}_g$ and the k-th column of $\mathbf{U}_{(i),l}$, $h_{ijk}(\mathbf{x}) = [\mathbf{U}_g]_{:,j}^T[\mathbf{U}_{(i),l}]_{:,k}$. The constraints in (4) can be rewritten as $h_{ijk}(\hat{\mathbf{x}}) = 0, \forall i \in [r_1], \forall j \in [r_2], \forall k \in [N]$. LICQ requires $\nabla h_{ijk}(\hat{\mathbf{x}})$ to be linearly independent for all ijk (Bertsekas, 1997, Proposition 3.1.1).

Suppose we can find constants ψ_{ijk} such that $\sum_{i=1}^{N}\sum_{j=1}^{r_1}\sum_{k=1}^{r_2}\psi_{ijk}\nabla h_{ijk}(\hat{\mathbf{x}})=0$. We consider the partial derivative of h_{ijk} over the $k^{'}$-th column of $\mathbf{U}_{(i^{'}),l}$. It is easy to derive,

$$\frac{\partial}{\partial[\mathbf{U}_{(i^{'}),l}]_{:,k^{'}}}h_{ijk}(\hat{\mathbf{x}})=\delta_{ii^{'}}\delta_{kk^{'}}[\hat{\mathbf{U}}_g]_{:,j}$$

where $\delta_{ii^{'}}$ is the Kronecker delta function. Then the constants ψ_{ijk} should satisfy,

$$\sum_{j=1}^{r_2}\psi_{i^{'}jk^{'}}[\hat{\mathbf{U}}_g]_{:,j}=0$$

As the columns of $\hat{\mathbf{U}}_g$ are linearly independent, $\psi_{i^{'}jk^{'}}=0$ for each j. This holds for any $i^{'}$ and $k^{'}$. Therefore $\psi_{ijk}=0$ for all i,j,k. This implies ∇h_{ijk}'s are linearly independent.

Proof of C.4. The Lagrangian of the optimization problem (4) can be written as

$$\begin{aligned}\mathcal{L}=&\frac{1}{2}\sum_{i=1}^{N}\left\|\mathbf{U}_g\mathbf{V}_{(i),g}^T+\mathbf{U}_{(i),l}\mathbf{V}_{(i),l}^T-\mathbf{M}_{(i)}\right\|_F^2+\frac{\beta}{2}\left\|\mathbf{U}_g^T\mathbf{U}_g-\mathbf{I}\right\|_F^2+\frac{\beta}{2}\left\|\mathbf{U}_{(i),l}^T\mathbf{U}_{(i),l}-\mathbf{I}\right\|_F^2\\&+\mathrm{Tr}\left(\mathbf{\Lambda}_{6,(i)}\mathbf{U}_g^T\mathbf{U}_{(i),l}\right)\end{aligned}\tag{22}$$

where $\mathbf{\Lambda}_{6,(i)}$ is the dual variable for the constraint $\mathbf{U}_g^T\mathbf{U}_{(i),l}=0$.

Under the LICQ, we know that the optimality of $\hat{\mathbf{U}}_g,\{\hat{\mathbf{V}}_{(i),g},\hat{\mathbf{U}}_{(i),l},\hat{\mathbf{V}}_{(i),l}\}$ implies the KKT condition. Setting the gradient of $\mathcal{L}$ with respect to $\mathbf{V}_{(i),g}$ and $\mathbf{V}_{(i),l}$ to zero, we can prove equation 21d and equation 21c. Considering the constraint $\hat{\mathbf{U}}_g^T\hat{\mathbf{U}}_{(i),l}=0$, we can solve them as $\hat{\mathbf{V}}_{(i),g}=\mathbf{M}_{(i)}^T\hat{\mathbf{U}}_g\left(\hat{\mathbf{U}}_g^T\hat{\mathbf{U}}_g\right)^{-1}$ and $\hat{\mathbf{V}}_{(i),l}=\mathbf{M}_{(i)}^T\hat{\mathbf{U}}_{(i),l}\left(\hat{\mathbf{U}}_{(i),l}^T\hat{\mathbf{U}}_{(i),l}\right)^{-1}$. Then we examine the gradient of $\mathcal{L}$ with respect to $\mathbf{U}_{(i),l}$:

$$\frac{\partial}{\partial\mathbf{U}_{(i),l}}\mathcal{L}=\left(\mathbf{U}_g\mathbf{V}_{(i),g}^T+\mathbf{U}_{(i),l}\mathbf{V}_{(i),l}^T-\mathbf{M}_{(i)}\right)\hat{\mathbf{V}}_{(i),l}+2\beta\mathbf{U}_{(i),l}\left(\mathbf{U}_{(i),l}^T\mathbf{U}_{(i),l}-\mathbf{I}\right)+\mathbf{U}_g\mathbf{\Lambda}_{(6),i}^T$$

Substituting $\hat{\mathbf{V}}_{(i),g}=\mathbf{M}_{(i)}^T\hat{\mathbf{U}}_g\left(\hat{\mathbf{U}}_g^T\hat{\mathbf{U}}_g\right)^{-1}$ and $\hat{\mathbf{V}}_{(i),l}=\mathbf{M}_{(i)}^T\hat{\mathbf{U}}_{(i),l}\left(\hat{\mathbf{U}}_{(i),l}^T\hat{\mathbf{U}}_{(i),l}\right)^{-1}$ in the above gradient and setting it to zero, we have

$$\begin{aligned}&\left(\hat{\mathbf{U}}_g\left(\hat{\mathbf{U}}_g^T\hat{\mathbf{U}}_g\right)^{-1}\hat{\mathbf{U}}_g^T+\hat{\mathbf{U}}_{(i),l}\left(\hat{\mathbf{U}}_{(i),l}^T\hat{\mathbf{U}}_{(i),l}\right)^{-1}\hat{\mathbf{U}}_{(i),l}^T-\mathbf{I}\right)\mathbf{M}_{(i)}\hat{\mathbf{V}}_{(i),l}\\&+2\beta\hat{\mathbf{U}}_{(i),l}\left(\hat{\mathbf{U}}_{(i),l}^T\hat{\mathbf{U}}_{(i),l}-\mathbf{I}\right)+\hat{\mathbf{U}}_g\mathbf{\Lambda}_{(6),i}^T=0\end{aligned}$$

Left multiplying both sides by $\hat{\mathbf{U}}_g^T$, we have $\mathbf{\Lambda}_{6,(i)}=0$. Left multiplying both sides by $\hat{\mathbf{U}}_{(i),l}^T$, we have $\hat{\mathbf{U}}_{(i),l}^T\hat{\mathbf{U}}_{(i),l}-\mathbf{I}=0$. Therefore we also have $\left(\hat{\mathbf{U}}_g\left(\hat{\mathbf{U}}_g^T\hat{\mathbf{U}}_g\right)^{-1}\hat{\mathbf{U}}_g^T+\hat{\mathbf{U}}_{(i),l}\left(\hat{\mathbf{U}}_{(i),l}^T\hat{\mathbf{U}}_{(i),l}\right)^{-1}\hat{\mathbf{U}}_{(i),l}^T-\mathbf{I}\right)\mathbf{M}_{(i)}\mathbf{V}_{(i),l}=0$. This proves equation equation 21b. Now, setting the derivative of $\mathcal{L}$ with respect to $\mathbf{U}_g$ to zero, we have

$$\frac{\partial}{\partial\mathbf{U}_g}\mathcal{L}=\sum_{i=1}^{N}\left(\hat{\mathbf{U}}_g\hat{\mathbf{V}}_{(i),g}^T+\hat{\mathbf{U}}_{(i),l}\hat{\mathbf{V}}_{(i),l}^T-\mathbf{M}_{(i)}\right)\hat{\mathbf{V}}_{(i),g}+2N\beta\hat{\mathbf{U}}_g\left(\hat{\mathbf{U}}_g^T\hat{\mathbf{U}}_g-\mathbf{I}\right)=0$$

Left multiplying both sides by $\hat{\mathbf{U}}_g^T$, we have $\hat{\mathbf{U}}_g^T\hat{\mathbf{U}}_g-\mathbf{I}=0$. We have thus proven equation 21a. This completes the proof for equation 21.

□

We note that the KKT conditions provide a set of conditions that must be satisfied for *all* stationary points of (4). Since $\widetilde{\mathbf{U}}_{g,\tau}$, $\widetilde{\mathbf{V}}_{(i),g,\tau}$, $\widetilde{\mathbf{U}}_{(i),l,\tau}$, and $\widetilde{\mathbf{V}}_{(i),l,\tau}$ introduced in equation 19aequation 19cequation 19bequation 19d constitute *one* set of optimal solutions, and $\mathbf{R}_{(i)} = \mathbf{M}_{(i)} - \mathbf{U}^{\star}{}_{g}\mathbf{V}^{\star T}_{(i),g} - \mathbf{U}^{\star}{}_{(i),l}\mathbf{V}^{\star T}_{(i),l}$, we can rewrite the KKT conditions of the optimal solutions from Lemma C.4 as,

$$\begin{cases} \sum_{i=1}^{N} \mathbf{R}_{(i)}\widetilde{\mathbf{V}}_{(i),g} = 0 \\ \mathbf{R}_{(i)}\widetilde{\mathbf{V}}_{(i),l} = 0 \\ \mathbf{R}^T_{(i)}\widetilde{\mathbf{U}}_{(i),l} = 0, \quad \mathbf{R}^T_{(i)}\widetilde{\mathbf{U}}_{g} = 0 \\ \widetilde{\mathbf{U}}^T_{(i),l}\widetilde{\mathbf{U}}_{(i),l} = \mathbf{I}, \quad \widetilde{\mathbf{U}}^T_{g}\widetilde{\mathbf{U}}_{g} = \mathbf{I}, \widetilde{\mathbf{U}}^T_{(i),l}\widetilde{\mathbf{U}}_{g} = 0 \end{cases} \tag{23}$$

From KKT conditions, it is also straightforward to verify that,

$$\mathbf{P}_{\hat{\mathbf{U}}_g}\sum_{j=1}^{N}\mathbf{R}_{(j)} = \mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\mathbf{R}_{(i)} = 0$$

We will use these KKT conditions when analyzing the convergence of Algorithm 1.

We begin by analyzing the subspace spanned by $\mathbf{U}_g$ and $\mathbf{U}_{(i),l}$'s. The following lemma shows that the difference between the subspace spanned by the columns of $\mathbf{U}_g$ and $\mathbf{U}_{(i),l}$ and by the optimal solution cannot be too large.

Lemma C.5 *For any set of optimal solutions to (4),* $\hat{\mathbf{U}}_g, \hat{\mathbf{V}}_{(i),g}, \hat{\mathbf{U}}_{(i),l}, \hat{\mathbf{V}}_{(i),l}$*, and* $\mathbf{R}_{(i)} = \mathbf{M}_{(i)} - \hat{\mathbf{U}}_g\hat{\mathbf{V}}^T_{(i),g} - \hat{\mathbf{U}}_{(i),l}\hat{\mathbf{V}}^T_{(i),l}$*, if there exists* $\hat{\sigma}^2_{gap}, \hat{\theta} > 0$*, such that (i)* $\sigma^2_{min}\left(\hat{\mathbf{U}}_g\hat{\mathbf{V}}^T_{(i),g} + \hat{\mathbf{U}}_{(i),l}\hat{\mathbf{V}}^T_{(i),l}\right) - \left\|\mathbf{R}_{(i)}\right\|^2 \geq \hat{\sigma}^2_{gap}$*, (ii)* $\left\|\mathbf{R}_{(i)}\right\| \leq \frac{\hat{\theta}\hat{\sigma}^2_{gap}}{8\hat{\sigma}_{\max}}$ *for each i, and (iii)* $\lambda_{\max}\left(\frac{1}{N}\sum_{i=1}^{N}\mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\right) \leq 1 - \hat{\theta}$*, then we have,*

$$\begin{aligned} &\sum_{i=1}^{N}\left\|\left(\mathbf{I} - \mathbf{P}_{\mathbf{U}_{g,\tau}} - \mathbf{P}_{\mathbf{U}_{(i),l,\tau}}\right)\mathbf{M}_{(i)}\right\|^2_F - \sum_{i=1}^{N}\left\|\mathbf{R}_{(i)}\right\|^2_F \\ &\geq \frac{\hat{\theta}\hat{\sigma}^2_{gap}}{8}\left(\left\|\mathbf{P}_{\mathbf{U}_{g,\tau}} - \mathbf{P}_{\hat{\mathbf{U}}_g}\right\|^2_F + \left\|\mathbf{P}_{\mathbf{U}_{(i),l,\tau}} - \mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\right\|^2_F\right) \end{aligned} \tag{24}$$

Proof. We define $\mathbf{M}_{(i),g} = \hat{\mathbf{U}}_g\hat{\mathbf{V}}^T_{(i),g}$ and $\mathbf{M}_{(i),l} = \hat{\mathbf{U}}_{(i),l}\hat{\mathbf{V}}^T_{(i),l}$, then $\mathbf{M}_{(i)} = \mathbf{M}_{(i),g} + \mathbf{M}_{(i),l} + \mathbf{R}_{(i)}$. We have,

$$\begin{aligned} &\left\|\left(\mathbf{I} - \mathbf{P}_{\mathbf{U}_{g,\tau}} - \mathbf{P}_{\mathbf{U}_{(i),l,\tau}}\right)\mathbf{M}_{(i)}\right\|^2_F \\ &= \mathrm{Tr}\left(\left(\mathbf{I} - \mathbf{P}_{\mathbf{U}_{g,\tau}} - \mathbf{P}_{\mathbf{U}_{(i),l,\tau}}\right)\mathbf{M}_{(i)}\mathbf{M}^T_{(i)}\right) \\ &= \mathrm{Tr}\left(\left(\mathbf{I} - \mathbf{P}_{\mathbf{U}_{g,\tau}} - \mathbf{P}_{\mathbf{U}_{(i),l,\tau}}\right)\left(\mathbf{M}_{(i),g} + \mathbf{M}_{(i),l}\right)\left(\mathbf{M}_{(i),g} + \mathbf{M}_{(i),l}\right)^T\right) \\ &+ \mathrm{Tr}\left(\left(\mathbf{I} - \mathbf{P}_{\mathbf{U}_{g,\tau}} - \mathbf{P}_{\mathbf{U}_{(i),l,\tau}}\right)\mathbf{R}_{(i)}\mathbf{R}^T_{(i)}\right) \\ &+ \mathrm{Tr}\left(\left(\mathbf{I} - \mathbf{P}_{\mathbf{U}_{g,\tau}} - \mathbf{P}_{\mathbf{U}_{(i),l,\tau}}\right)\left(\mathbf{M}_{(i),g}\mathbf{R}^T_{(i)} + \mathbf{R}_{(i)}\mathbf{M}^T_{(i),g}\right)\right) \end{aligned}$$

where we applied the second equation from equation 23 that $\mathbf{R}_{(i)}\mathbf{M}^T_{(i),l} = \mathbf{R}_{(i)}\hat{\mathbf{V}}_{(i),l}\hat{\mathbf{U}}^T_{(i),l} = 0$.

Summing index i from 1 to N, and considering the KKT condition $\sum_{i=1}^{N} \mathbf{M}_{(i),g}\mathbf{R}_{(i)}^T = 0$, we have,

$$
\begin{aligned}
&\sum_{i=1}^{N} \left\| \left(\mathbf{I} - \mathbf{P}_{\mathbf{U}_{g,\tau}} - \mathbf{P}_{\mathbf{U}_{(i),l,\tau}}\right) \mathbf{M}_{(i)} \right\|_F^2 \\
&= \sum_{i=1}^{N} \mathrm{Tr}\left(\left(\mathbf{I} - \mathbf{P}_{\mathbf{U}_{g,\tau}} - \mathbf{P}_{\mathbf{U}_{(i),l,\tau}}\right) \mathbf{M}_{(i)}\mathbf{M}_{(i)}^T \right) \\
&= \sum_{i=1}^{N} \mathrm{Tr}\left(\left(\mathbf{I} - \mathbf{P}_{\mathbf{U}_{g,\tau}} - \mathbf{P}_{\mathbf{U}_{(i),l,\tau}}\right) \left(\mathbf{M}_{(i),g} + \mathbf{M}_{(i),l}\right) \left(\mathbf{M}_{(i),g} + \mathbf{M}_{(i),l}\right)^T \right) \\
&+ \sum_{i=1}^{N} \mathrm{Tr}\left(\left(\mathbf{I} - \mathbf{P}_{\mathbf{U}_{g,\tau}} - \mathbf{P}_{\mathbf{U}_{(i),l,\tau}}\right) \mathbf{R}_{(i)}\mathbf{R}_{(i)}^T \right) \\
&+ \sum_{i=1}^{N} \mathrm{Tr}\left(-\mathbf{P}_{\mathbf{U}_{(i),l,\tau}} \left(\mathbf{M}_{(i),g}\mathbf{R}_{(i)}^T + \mathbf{R}_{(i)}\mathbf{M}_{(i),g}^T\right) \right)
\end{aligned}
$$

We use $\hat{\sigma}_{\min}$ to denote the smallest nonzero singular value of $\mathbf{M}_{(i),g} + \mathbf{M}_{(i),l}$, then $\left(\mathbf{M}_{(i),g} + \mathbf{M}_{(i),l}\right) \left(\mathbf{M}_{(i),g} + \mathbf{M}_{(i),l}\right)^T \succeq \left(\mathbf{P}_{\hat{\mathbf{U}}_g} + \mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\right) \hat{\sigma}_{\min}{}^2$. Hence,

$$
\begin{aligned}
&\mathrm{Tr}\left(\left(\mathbf{I} - \mathbf{P}_{\mathbf{U}_{g,\tau}} - \mathbf{P}_{\mathbf{U}_{(i),l,\tau}}\right) \left(\mathbf{M}_{(i),g} + \mathbf{M}_{(i),l}\right) \left(\mathbf{M}_{(i),g} + \mathbf{M}_{(i),l}\right)^T \right) \\
&\geq \mathrm{Tr}\left(\left(\mathbf{I} - \mathbf{P}_{\mathbf{U}_{g,\tau}} - \mathbf{P}_{\mathbf{U}_{(i),l,\tau}}\right) \left(\mathbf{P}_{\hat{\mathbf{U}}_g} + \mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\right) \right) \hat{\sigma}_{\min}{}^2 \left(\mathbf{M}_{(i),g} + \mathbf{M}_{(i),l}\right) \\
&= \left(r_1 + r_2 - \left\langle \mathbf{P}_{\mathbf{U}_{g,\tau}} + \mathbf{P}_{\mathbf{U}_{(i),l,\tau}}, \mathbf{P}_{\hat{\mathbf{U}}_g} + \mathbf{P}_{\hat{\mathbf{U}}_{(i),l}} \right\rangle \right) \hat{\sigma}_{\min}{}^2 \left(\mathbf{M}_{(i),g} + \mathbf{M}_{(i),l}\right)
\end{aligned}
$$

Similarly, since $\mathbf{R}_{(i)}\mathbf{R}_{(i)}^T \preceq \left(\mathbf{I} - \mathbf{P}_{\hat{\mathbf{U}}_g} - \mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\right) \sigma_{max}^2\left(\mathbf{R}_{(i)}\right)$, we have,

$$
\begin{aligned}
&\mathrm{Tr}\left(\left(\mathbf{I} - \mathbf{P}_{\mathbf{U}_{g,\tau}} - \mathbf{P}_{\mathbf{U}_{(i),l,\tau}}\right) \mathbf{R}_{(i)}\mathbf{R}_{(i)}^T \right) \\
&= \left\|\mathbf{R}_{(i)}\right\|_F^2 - \mathrm{Tr}\left(\left(\mathbf{P}_{\mathbf{U}_{g,\tau}} + \mathbf{P}_{\mathbf{U}_{(i),l,\tau}}\right) \mathbf{R}_{(i)}\mathbf{R}_{(i)}^T \right) \\
&\geq \left\|\mathbf{R}_{(i)}\right\|_F^2 - \mathrm{Tr}\left(\left(\mathbf{P}_{\mathbf{U}_{g,\tau}} + \mathbf{P}_{\mathbf{U}_{(i),l,\tau}}\right) \left(\mathbf{I} - \mathbf{P}_{\hat{\mathbf{U}}_g} - \mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\right) \right) \sigma_{max}^2\left(\mathbf{R}_{(i)}\right)
\end{aligned}
$$

As a result,

$$
\begin{aligned}
&\sum_{i=1}^{N} \left\| \left(\mathbf{I} - \mathbf{P}_{\mathbf{U}_{g,\tau}} - \mathbf{P}_{\mathbf{U}_{(i),l,\tau}}\right) \mathbf{M}_{(i)} \right\|_F^2 - \left\| \mathbf{R}_{(i)} \right\|_F^2 \\
&\geq \sum_{i=1}^{N} \mathrm{Tr}\left(\left(\mathbf{P}_{\mathbf{U}_{g,\tau}} + \mathbf{P}_{\mathbf{U}_{(i),l,\tau}}\right) \left(\mathbf{I} - \mathbf{P}_{\hat{\mathbf{U}}_g} - \mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\right)\right) \left(\sigma_{min}^2\left(\mathbf{M}_{(i),g} + \mathbf{M}_{(i),l}\right) - \sigma_{max}^2\left(\mathbf{R}_{(i)}\right)\right) \\
&+ \sum_{i=1}^{N} \mathrm{Tr}\left(-\mathbf{P}_{\mathbf{U}_{(i),l,\tau}} \left(\mathbf{M}_{(i),g}\mathbf{R}_{(i)}^T + \mathbf{R}_{(i)}\mathbf{M}_{(i),g}^T\right)\right) \\
&\geq \sum_{i=1}^{N} \left\langle \mathbf{P}_{\mathbf{U}_{g,\tau}} + \mathbf{P}_{\mathbf{U}_{(i),l,\tau}}, \mathbf{I} - \mathbf{P}_{\hat{\mathbf{U}}_g} - \mathbf{P}_{\hat{\mathbf{U}}_{(i),l}} \right\rangle \hat{\sigma}_{\text{gap}}^2 - \sum_{i=1}^{N} 2 \left\| \mathbf{P}_{\mathbf{U}_{(i),l,\tau}} \mathbf{M}_{(i),g} \right\|_F \left\| \mathbf{P}_{\mathbf{U}_{(i),l,\tau}} \mathbf{R}_{(i)} \right\|_F \\
&\geq \sum_{i=1}^{N} \left(r_1 + r_2 - \left\langle \mathbf{P}_{\mathbf{U}_{g,\tau}}, \mathbf{P}_{\hat{\mathbf{U}}_g} \right\rangle - \left\langle \mathbf{P}_{\mathbf{U}_{(i),l,\tau}}, \mathbf{P}_{\hat{\mathbf{U}}_{(i),l}} \right\rangle \right) \frac{\hat{\theta}\hat{\sigma}_{\text{gap}}^2}{2} \\
&- \sum_{i=1}^{N} 2 \left\| \mathbf{P}_{\mathbf{U}_{(i),l,\tau}} \left(\mathbf{I} - \mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\right) \right\|_F^2 \hat{\sigma}_{\max} \left\| \mathbf{R}_i \right\| \\
&\geq \sum_{i=1}^{N} \left(r_1 + r_2 - \left\langle \mathbf{P}_{\mathbf{U}_{g,\tau}}, \mathbf{P}_{\hat{\mathbf{U}}_g} \right\rangle - \left\langle \mathbf{P}_{\mathbf{U}_{(i),l,\tau}}, \mathbf{P}_{\hat{\mathbf{U}}_{(i),l}} \right\rangle \right) \frac{\hat{\theta}\hat{\sigma}_{\text{gap}}^2}{4}
\end{aligned}
$$

where the second inequality comes the definition of $\hat{\sigma}_{\text{gap}}^2$ and Cauchy-Schwartz inequality, the third inequality comes from Lemma 2 in Shi & Kontar (2024) and Lemma E.5. Then, since $r_1 - \left\langle \mathbf{P}_{\mathbf{U}_{g,\tau}}, \mathbf{P}_{\hat{\mathbf{U}}_g} \right\rangle = \frac{1}{2} \left\| \mathbf{P}_{\mathbf{U}_{g,\tau}} - \mathbf{P}_{\hat{\mathbf{U}}_g} \right\|_F^2$ and $r_2 - \left\langle \mathbf{P}_{\mathbf{U}_{(i),l,\tau}}, \mathbf{P}_{\hat{\mathbf{U}}_{(i),l}} \right\rangle = \frac{1}{2} \left\| \mathbf{P}_{\mathbf{U}_{(i),l,\tau}} - \mathbf{P}_{\hat{\mathbf{U}}_{(i),l}} \right\|_F^2$, this completes the proof. □

The next lemma shows that $\left\| \mathbf{U}_{g,\tau}^T \mathbf{U}_{g,\tau} - \mathbf{I} \right\|$ and $\left\| \mathbf{U}_{(i),l,\tau}^T \mathbf{U}_{(i),l,\tau} - \mathbf{I} \right\|$'s are upper bounded by ϕ_τ.

Lemma C.6 *Under the same conditions as Lemma C.5, we have*

$$
\sum_{i=1}^{N} \left(\left\| \mathbf{U}_{g,\tau}^T \mathbf{U}_{g,\tau} - \mathbf{I} \right\|_F^2 + \left\| \mathbf{U}_{(i),l,\tau}^T \mathbf{U}_{(i),l,\tau} - \mathbf{I} \right\|_F^2 \right) \leq \frac{2\phi_\tau}{\beta} \tag{25}
$$

Proof. We decompose $\tilde{f}$ as,

$$
\begin{aligned}
&2\tilde{f} = 2\left(\phi_\tau + \frac{1}{2} \sum_{i=1}^{N} \left\| \mathbf{R}_{(i)} \right\|_F^2 \right) \\
&= \sum_{i=1}^{N} \left\| \mathbf{U}_{g,\tau} \mathbf{V}_{(i),g,\tau}^T + \mathbf{U}_{(i),l,\tau} \mathbf{V}_{(i),l} - \mathbf{M}_{(i)} \right\|_F^2 + \beta \left(\left\| \mathbf{U}_{g,\tau}^T \mathbf{U}_{g,\tau} - \mathbf{I} \right\|_F^2 + \left\| \mathbf{U}_{(i),l,\tau}^T \mathbf{U}_{(i),l,\tau} - \mathbf{I} \right\|_F^2 \right) \\
&= \sum_{i=1}^{N} \left\| \left(\mathbf{P}_{\mathbf{U}_{g,\tau}} + \mathbf{P}_{\mathbf{U}_{(i),l,\tau}}\right) \left(\mathbf{U}_{g,\tau} \mathbf{V}_{(i),g,\tau}^T + \mathbf{U}_{(i),l,\tau} \mathbf{V}_{(i),l} - \mathbf{M}_{(i)} \right) - \left(\mathbf{I} - \mathbf{P}_{\mathbf{U}_{g,\tau}} - \mathbf{P}_{\mathbf{U}_{(i),l,\tau}}\right) \mathbf{M}_{(i)} \right\|_F^2 \\
&+ \beta \left(\left\| \mathbf{U}_{g,\tau}^T \mathbf{U}_{g,\tau} - \mathbf{I} \right\|_F^2 + \left\| \mathbf{U}_{(i),l,\tau}^T \mathbf{U}_{(i),l,\tau} - \mathbf{I} \right\|_F^2 \right) \\
&= \sum_{i=1}^{N} \left\| \mathbf{U}_{g,\tau} \mathbf{V}_{(i),g,\tau}^T + \mathbf{U}_{(i),l,\tau} \mathbf{V}_{(i),l} - \left(\mathbf{P}_{\mathbf{U}_{g,\tau}} + \mathbf{P}_{\mathbf{U}_{(i),l,\tau}}\right) \mathbf{M}_{(i)} \right\|_F^2 \\
&+ \left\| \left(\mathbf{I} - \mathbf{P}_{\mathbf{U}_{g,\tau}} - \mathbf{P}_{\mathbf{U}_{(i),l,\tau}}\right) \mathbf{M}_{(i)} \right\|_F^2 + \beta \left(\left\| \mathbf{U}_{g,\tau}^T \mathbf{U}_{g,\tau} - \mathbf{I} \right\|_F^2 + \left\| \mathbf{U}_{(i),l,\tau}^T \mathbf{U}_{(i),l,\tau} - \mathbf{I} \right\|_F^2 \right)
\end{aligned} \tag{26}
$$

Considering Lemma C.5, we know that

$$
\begin{aligned}
&2\phi_\tau \\
&\geq \sum_{i=1}^{N} \frac{\hat{\theta}\hat{\sigma}_{\text{gap}}^2}{8} \left(\left\| \mathbf{P}_{\mathbf{U}_{g,\tau}} - \mathbf{P}_{\hat{\mathbf{U}}_g} \right\|_F^2 + \left\| \mathbf{P}_{\mathbf{U}_{(i),l,\tau}} - \mathbf{P}_{\hat{\mathbf{U}}_{(i),l}} \right\|_F^2 \right) + \beta \left\| \mathbf{U}_{g,\tau}^T \mathbf{U}_{g,\tau} - \mathbf{I} \right\|_F^2 \\
&+ \beta \left\| \mathbf{U}_{(i),l,\tau}^T \mathbf{U}_{(i),l,\tau} - \mathbf{I} \right\|_F^2 + \left\| \mathbf{U}_{g,\tau} \mathbf{V}_{(i),g,\tau}^T + \mathbf{U}_{(i),l,\tau} \mathbf{V}_{(i),l} - \left(\mathbf{P}_{\mathbf{U}_{g,\tau}} + \mathbf{P}_{\mathbf{U}_{(i),l,\tau}} \right) \mathbf{M}_{(i)} \right\|_F^2
\end{aligned} \tag{27}
$$

This completes the proof as all terms on the right hand side are nonnegative. □

Now we can prove that when the objective $\tilde{f}$ is close to the optimal value, the iterates should also be close to the set of optimal solutions defined in equation 19aequation 19cequation 19bequation 19d. We will first examine the norms of $\Delta\mathbf{U}_{g,\tau}$ and $\Delta\mathbf{U}_{(i),l,\tau}$ in Lemma C.7, then calculate the norms of $\Delta\mathbf{V}_{(i),g,\tau}$ and $\Delta\mathbf{V}_{(i),l,\tau}$ in Lemma C.8.

Lemma C.7 *Under the same conditions of Lemma C.5, if we further have* $\phi_\tau \leq \min\{\frac{9\beta}{128}, \frac{9\hat{\theta}\hat{\sigma}_{gap}^2}{1936}\}$*, we have the following,*

$$
N \left\| \Delta\mathbf{U}_{g,\tau} \right\|_F^2 + \sum_{i=1}^{N} \left\| \Delta\mathbf{U}_{(i),l,\tau} \right\|_F^2 \leq C_1 \phi_\tau \tag{28}
$$

where

$$
C_1 = \left(\frac{64}{9\beta} + 32 \left(\frac{4}{3} B_1^2 + B_1 \right)^2 \frac{1}{\hat{\theta}\hat{\sigma}_{gap}^2} \right) \tag{29}
$$

and $\Delta\mathbf{U}_{g,\tau}$ *and* $\Delta\mathbf{U}_{(i),l,\tau}$ *are introduced in equation 20. Also,*

$$
\left\| \mathbf{U}_{g,\tau} \right\|, \left\| \mathbf{U}_{(i),l,\tau} \right\| \leq \sqrt{\frac{11}{8}}, \quad \forall i \in \{1, \cdots, N\} \tag{30}
$$

Proof. We begin by calculating an upper bound of the norm of $\Delta\mathbf{U}_{g,\tau}$,

$$
\begin{aligned}
&\left\| \Delta\mathbf{U}_{g,\tau} \right\|_F \\
&= \left\| \mathbf{U}_{g,\tau} - \mathbf{P}_{\hat{\mathbf{U}}_g} \mathbf{U}_{g,\tau} \left(\mathbf{U}_{g,\tau}^T \mathbf{P}_{\hat{\mathbf{U}}_g} \mathbf{U}_{g,\tau} \right)^{-\frac{1}{2}} \right\|_F \\
&\leq \left\| \mathbf{U}_{g,\tau} - \mathbf{P}_{\hat{\mathbf{U}}_g} \mathbf{U}_{g,\tau} \right\|_F + \left\| \mathbf{P}_{\hat{\mathbf{U}}_g} \mathbf{U}_{g,\tau} - \mathbf{P}_{\hat{\mathbf{U}}_g} \mathbf{U}_{g,\tau} \left(\mathbf{U}_{g,\tau}^T \mathbf{P}_{\hat{\mathbf{U}}_g} \mathbf{U}_{g,\tau} \right)^{-\frac{1}{2}} \right\|_F \\
&\leq \left\| \mathbf{U}_{g,\tau} \right\| \left\| \mathbf{P}_{\mathbf{U}_{g,\tau}} - \mathbf{P}_{\hat{\mathbf{U}}_g} \right\|_F + \left\| \mathbf{P}_{\hat{\mathbf{U}}_g} \mathbf{U}_{g,\tau} \right\| \left\| \mathbf{I} - \left(\mathbf{U}_{g,\tau}^T \mathbf{P}_{\hat{\mathbf{U}}_g} \mathbf{U}_{g,\tau} \right)^{-\frac{1}{2}} \right\|_F
\end{aligned}
$$

Notice that

$$
\begin{aligned}
&\left\| \mathbf{I} - \left(\mathbf{U}_{g,\tau}^T \mathbf{P}_{\hat{\mathbf{U}}_g} \mathbf{U}_{g,\tau} \right)^{-\frac{1}{2}} \right\|_F \\
&\leq \left\| \mathbf{I} - \left(\mathbf{I} + \left(\mathbf{U}_{g,\tau}^T \mathbf{U}_{g,\tau} - \mathbf{I} \right) + \mathbf{U}_{g,\tau}^T \left(\mathbf{P}_{\hat{\mathbf{U}}_g} - \mathbf{P}_{\mathbf{U}_{g,\tau}} \right) \mathbf{U}_{g,\tau} \right)^{-\frac{1}{2}} \right\|_F
\end{aligned}
$$

Since $\phi_\tau \leq \frac{9\beta}{128}$, we know that $\sum_{i=1}^{N} (\left\| \mathbf{U}_{g,\tau}^T \mathbf{U}_{g,\tau} - \mathbf{I} \right\|_F^2 + \left\| \mathbf{U}_{(i),l,\tau}^T \mathbf{U}_{(i),l,\tau} - \mathbf{I} \right\|_F^2) \leq \frac{9}{64}$ from Lemma C.6. Therefore, $\left\| \mathbf{U}_{g,\tau}^T \mathbf{U}_{g,\tau} - \mathbf{I} \right\|_F \leq \frac{3}{8}$. And by triangle inequality of Frobenius norm, $\left\| \mathbf{U}_{g,\tau} \right\| \leq \left\| \mathbf{U}_{g,\tau} \right\|_F \leq \sqrt{1 + \frac{3}{8}} \leq \sqrt{\frac{11}{8}}$. Similarly, $\phi_\tau \leq \frac{9\hat{\theta}\hat{\sigma}_{\text{gap}}^2}{1936}$ and Lemma C.5 imply that $\left\| \mathbf{P}_{\hat{\mathbf{U}}_g} - \mathbf{P}_{\mathbf{U}_{g,\tau}} \right\|_F \leq \frac{3}{11}$. As a result, we

know $\left\|\left(\mathbf{U}_{g,\tau}^T\mathbf{U}_{g,\tau}-\mathbf{I}\right)+\mathbf{U}_{g,\tau}^T\left(\mathbf{P}_{\hat{\mathbf{U}}_g}-\mathbf{P}_{\mathbf{U}_{g,\tau}}\right)\mathbf{U}_{g,\tau}\right\|_F \leq \frac{3}{4}$. Thus by Lemma E.2, we have,

$$\begin{aligned}
&\left\|\mathbf{I}-\left(\mathbf{U}_{g,\tau}^T\mathbf{P}_{\hat{\mathbf{U}}_g}\mathbf{U}_{g,\tau}\right)^{-\frac{1}{2}}\right\|_F \\
&\leq \frac{4}{3}\left\|\left(\mathbf{U}_{g,\tau}^T\mathbf{U}_{g,\tau}-\mathbf{I}\right)+\mathbf{U}_{g,\tau}^T\left(\mathbf{P}_{\hat{\mathbf{U}}_g}-\mathbf{P}_{\mathbf{U}_{g,\tau}}\right)\mathbf{U}_{g,\tau}\right\|_F \\
&\leq \frac{4}{3}\left(\left\|\mathbf{U}_{g,\tau}^T\mathbf{U}_{g,\tau}-\mathbf{I}\right\|+B_1^2\left\|\mathbf{P}_{\hat{\mathbf{U}}_g}-\mathbf{P}_{\mathbf{U}_{g,\tau}}\right\|_F\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|\Delta\mathbf{U}_{g,\tau}\|_F^2 \\
&\leq \left(\frac{4}{3}B_1\left\|\mathbf{U}_{g,\tau}^T\mathbf{U}_{g,\tau}-\mathbf{I}\right\|+\left(\frac{4}{3}B_1^2+B_1\right)\left\|\mathbf{P}_{\hat{\mathbf{U}}_g}-\mathbf{P}_{\mathbf{U}_{g,\tau}}\right\|_F\right)^2 \\
&\leq \frac{32}{9}\left\|\mathbf{U}_{g,\tau}^T\mathbf{U}_{g,\tau}-\mathbf{I}\right\|^2+2\left(\frac{4}{3}B_1^2+B_1\right)^2\left\|\mathbf{P}_{\hat{\mathbf{U}}_g}-\mathbf{P}_{\mathbf{U}_{g,\tau}}\right\|_F^2
\end{aligned}$$

Similar upper bounds hold for $\Delta\mathbf{U}_{(i),l,\tau}$. Summing them up, we have,

$$\begin{aligned}
&\sum_{i=1}^N\|\Delta\mathbf{U}_{g,\tau}\|_F^2+\left\|\Delta\mathbf{U}_{(i),l,\tau}\right\|_F^2 \\
&\leq \sum_{i=1}^N\frac{32}{9}\left(\left\|\mathbf{U}_{g,\tau}^T\mathbf{U}_{g,\tau}-\mathbf{I}\right\|^2+\left\|\mathbf{U}_{(i),l,\tau}^T\mathbf{U}_{(i),l,\tau}-\mathbf{I}\right\|^2\right)+ \\
&2\left(\frac{4}{3}B_1^2+B_1\right)^2\left(\left\|\mathbf{P}_{\hat{\mathbf{U}}_g}-\mathbf{P}_{\mathbf{U}_{g,\tau}}\right\|_F^2+\left\|\mathbf{P}^\star{}_{(i)}-\mathbf{P}_{\mathbf{U}_{(i),l,\tau}}\right\|_F^2\right) \\
&\leq \left(\frac{64}{9\beta}+32\left(\frac{4}{3}B_1^2+B_1\right)^2\frac{1}{\hat{\theta}\hat{\sigma}_{\text{gap}}^2}\right)\phi_\tau
\end{aligned}$$

We applied Lemma C.5, Lemma C.6, and equation 27 in the last inequality. □

We continue to prove the upper bound of the norm of $\Delta\mathbf{V}_{(i),g,\tau}$ and $\Delta\mathbf{V}_{(i),l,\tau}$ introduced in equation 20.

Lemma C.8 *Under the same conditions of Lemma C.7, we have the following,*

$$N\|\Delta\mathbf{U}_{g,\tau}\|_F^2+\sum_{i=1}^N\left[\left\|\Delta\mathbf{V}_{(i),g,\tau}\right\|_F^2+\left\|\Delta\mathbf{U}_{(i),l,\tau}\right\|_F^2+\left\|\Delta\mathbf{V}_{(i),l,\tau}\right\|_F^2\right]\leq C_2\phi_\tau \tag{31}$$

where

$$C_2=16+\frac{704\hat{\sigma}_{\max}^2}{\beta}+352\hat{\sigma}_{\max}^2C_1+C_1 \tag{32}$$

This lemma presents a full perturbation analysis of $\tilde{f}$. It shows that if the function value is close to optimal, all iterates must also be close to one set of the optimal solutions.

Proof.

We use the decomposition equation 26 to derive upper bounds on the norm of $\Delta\mathbf{V}$'s.

By some algebra, we know that,

$$\begin{aligned}
&\left(\mathbf{P}_{\mathbf{U}_{g,\tau}}+\mathbf{P}_{\mathbf{U}_{(i),l,\tau}}\right)\mathbf{M}_{(i)}-\mathbf{U}_{g,\tau}\mathbf{V}_{(i),g,\tau}^T-\mathbf{U}_{(i),l,\tau}\mathbf{V}_{(i),l} \\
&=\mathbf{U}_{g,\tau}\Delta\mathbf{V}_{(i),g,\tau}^T+\mathbf{U}_{(i),l,\tau}\Delta\mathbf{V}_{(i),l}+\mathbf{U}_{g,\tau}\left(\left(\mathbf{U}_{g,\tau}^T\mathbf{U}_{g,\tau}\right)^{-1}-\mathbf{I}\right)\widetilde{\mathbf{V}}_{(i),g,\tau}^T \\
&+\mathbf{U}_{g,\tau}\left(\mathbf{U}_{g,\tau}^T\mathbf{U}_{g,\tau}\right)^{-1}\Delta\mathbf{U}_{g,\tau}\mathbf{M}_{(i)}+\mathbf{U}_{(i),l,\tau}\left(\left(\mathbf{U}_{(i),l,\tau}^T\mathbf{U}_{(i),l,\tau}\right)^{-1}-\mathbf{I}\right)\widetilde{\mathbf{V}}_{(i),l,\tau}^T \\
&+\mathbf{U}_{(i),l,\tau}\left(\mathbf{U}_{(i),l,\tau}^T\mathbf{U}_{(i),l,\tau}\right)^{-1}\Delta\mathbf{U}_{(i),l,\tau}\mathbf{M}_{(i)}
\end{aligned}$$

We will bound each term above. By triangle inequality, we have,

$$\begin{aligned}
&\left\|\mathbf{U}_{g,\tau}\left(\left(\mathbf{U}_{g,\tau}^T\mathbf{U}_{g,\tau}\right)^{-1}-\mathbf{I}\right)\widetilde{\mathbf{V}}_{(i),g,\tau}^T+\mathbf{U}_{g,\tau}\left(\mathbf{U}_{g,\tau}^T\mathbf{U}_{g,\tau}\right)^{-1}\Delta\mathbf{U}_{g,\tau}\mathbf{M}_{(i)}\right\|_F\\
&\leq\|\mathbf{U}_{g,\tau}\|\left\|\left(\mathbf{U}_{g,\tau}^T\mathbf{U}_{g,\tau}\right)^{-1}-\mathbf{I}\right\|_F\left\|\widetilde{\mathbf{V}}_{(i),g,\tau}^T\right\|+\|\mathbf{U}_{g,\tau}\|\left\|\left(\mathbf{U}_{g,\tau}^T\mathbf{U}_{g,\tau}\right)^{-1}\right\|\|\Delta\mathbf{U}_{g,\tau}\|_F\left\|\mathbf{M}_{(i)}\right\|\\
&\leq 4\sqrt{\frac{11}{8}}\hat{\sigma}_{\max}\left\|\mathbf{U}_{g,\tau}^T\mathbf{U}_{g,\tau}-\mathbf{I}\right\|_F+4\sqrt{\frac{11}{8}}\hat{\sigma}_{\max}\|\Delta\mathbf{U}_{g,\tau}\|_F
\end{aligned}$$

where we again used Lemma E.1 to show that when $\left\|\mathbf{U}_{(i),l,\tau}^T\mathbf{U}_{(i),l,\tau}-\mathbf{I}\right\|_F$ and $\left\|\mathbf{U}_{g,\tau}^T\mathbf{U}_{g,\tau}-\mathbf{I}\right\|_F\leq\frac{3}{4}$, we have $\left\|\left(\mathbf{U}_{g,\tau}^T\mathbf{U}_{g,\tau}\right)^{-1}\right\|\leq 1+3=4$.

A similar bound holds that,

$$\begin{aligned}
&\left\|\mathbf{U}_{(i),l,\tau}\left(\left(\mathbf{U}_{(i),l,\tau}^T\mathbf{U}_{(i),l,\tau}\right)^{-1}-\mathbf{I}\right)\widetilde{\mathbf{V}}_{(i),l,\tau}^T+\mathbf{U}_{(i),l,\tau}\left(\mathbf{U}_{(i),l,\tau}^T\mathbf{U}_{(i),l,\tau}\right)^{-1}\Delta\mathbf{U}_{(i),l,\tau}\mathbf{M}_{(i)}\right\|_F\\
&\leq 4\sqrt{\frac{11}{8}}\hat{\sigma}_{\max}\left\|\mathbf{U}_{(i),l,\tau}^T\mathbf{U}_{(i),l,\tau}-\mathbf{I}\right\|_F+4\sqrt{\frac{11}{8}}\hat{\sigma}_{\max}\left\|\Delta\mathbf{U}_{(i),l,\tau}\right\|_F
\end{aligned}$$

Then by Lemma E.3, we have,

$$\begin{aligned}
&\left\|\left(\mathbf{P}_{\mathbf{U}_{g,\tau}}+\mathbf{P}_{\mathbf{U}_{(i),l,\tau}}\right)\mathbf{M}_{(i)}-\mathbf{U}_{g,\tau}\mathbf{V}_{(i),g,\tau}^T-\mathbf{U}_{(i),l,\tau}\mathbf{V}_{(i),l}\right\|^2\\
&\geq\frac{1}{2}\left\|\mathbf{U}_{g,\tau}\Delta\mathbf{V}_{(i),g,\tau}^T+\mathbf{U}_{(i),l,\tau}\Delta\mathbf{V}_{(i),l}\right\|_F^2\\
&-\left(4\sqrt{\frac{11}{8}}\hat{\sigma}_{\max}(\left\|\mathbf{U}_{g,\tau}^T\mathbf{U}_{g,\tau}-\mathbf{I}\right\|_F+\left\|\mathbf{U}_{(i),l,\tau}^T\mathbf{U}_{(i),l,\tau}-\mathbf{I}\right\|_F)+4\sqrt{\frac{11}{8}}\hat{\sigma}_{\max}(\left\|\Delta\mathbf{U}_{(i),l,\tau}\right\|_F+\left\|\Delta\mathbf{U}_{(i),g,\tau}\right\|_F)\right)^2\\
&\geq\frac{1}{2}\left\|\mathbf{U}_{g,\tau}\Delta\mathbf{V}_{(i),g,\tau}^T\right\|_F^2+\frac{1}{2}\left\|\mathbf{U}_{(i),l,\tau}\Delta\mathbf{V}_{(i),l}\right\|_F^2\\
&-44\hat{\sigma}_{\max}^2(\left\|\mathbf{U}_{g,\tau}^T\mathbf{U}_{g,\tau}-\mathbf{I}\right\|_F^2+\left\|\mathbf{U}_{(i),l,\tau}^T\mathbf{U}_{(i),l,\tau}-\mathbf{I}\right\|_F^2)-44\hat{\sigma}_{\max}^2(\left\|\Delta\mathbf{U}_{(i),l,\tau}\right\|_F^2+\left\|\Delta\mathbf{U}_{(i),g,\tau}\right\|_F^2)\\
&\geq\frac{1}{8}\left\|\Delta\mathbf{V}_{(i),g,\tau}^T\right\|^2+\frac{1}{8}\left\|\Delta\mathbf{V}_{(i),l}\right\|_F^2\\
&-44\hat{\sigma}_{\max}^2(\left\|\mathbf{U}_{g,\tau}^T\mathbf{U}_{g,\tau}-\mathbf{I}\right\|_F^2+\left\|\mathbf{U}_{(i),l,\tau}^T\mathbf{U}_{(i),l,\tau}-\mathbf{I}\right\|_F^2)-44\hat{\sigma}_{\max}^2(\left\|\Delta\mathbf{U}_{(i),l,\tau}\right\|_F^2+\left\|\Delta\mathbf{U}_{(i),g,\tau}\right\|_F^2)
\end{aligned}$$

where we used the condition $\left\|\mathbf{U}_{g,\tau}^T\mathbf{U}_{g,\tau}-\mathbf{I}\right\|\leq\frac{3}{4}$ and $\left\|\mathbf{U}_{(i),l,\tau}^T\mathbf{U}_{(i),l,\tau}-\mathbf{I}\right\|\leq\frac{3}{4}$ in the third inequality.

Summing both sides for i from 1 to N, and considering Lemma C.5, Lemma C.7, and inequality equation 27, we have,

$$\sum_{i=1}^N\left\|\Delta\mathbf{V}_{(i),g,\tau}\right\|_F^2+\left\|\Delta\mathbf{V}_{(i),l,\tau}\right\|_F^2\leq\left(16+\frac{704\hat{\sigma}_{\max}^2}{\beta}+352\hat{\sigma}_{\max}^2C_1\right)\phi_\tau$$

We can complete the proof by adding $\sum_{i=1}^N\left\|\Delta\mathbf{U}_{g,\tau}\right\|_F^2+\left\|\Delta\mathbf{U}_{(i),l,\tau}\right\|_F^2$ on both sides. □

Now we can look at the following lemma which shows the gradient is aligned with the direction pointing from current iterate to one set of the optimal solution.

Lemma C.9 (Gradient points to an optimal solution) *Under the same conditions as Lemma C.8, if additionally* $\phi_\tau\leq\frac{1}{\left(2\sqrt{2\max\{\hat{\sigma}_{\max}^2,1\}+4\beta}\right)^2C_2^3}$, *the following inequality would hold,*

$$\begin{aligned}
&\left\langle\nabla_{\mathbf{U}_g}\tilde{f},\Delta\mathbf{U}_{g,\tau}\right\rangle+\sum_{i=1}^N\Bigg[\left\langle\nabla_{\mathbf{U}_{(i),l}}\tilde{f},\Delta\mathbf{U}_{(i),l,\tau}\right\rangle+\left\langle\nabla_{\mathbf{V}_{(i),g}}\tilde{f},\Delta\mathbf{V}_{(i),g,\tau}\right\rangle\\
&+\left\langle\nabla_{\mathbf{V}_{(i),l}}\tilde{f},\Delta\mathbf{V}_{(i),l}\right\rangle\Bigg]\geq\phi_\tau
\end{aligned}\tag{33}$$

This lemma shows that the geometry of the problem is benign.

Proof. Similar to Sun & Luo (2016), we define two notations $a_{i,\tau}$, $b_{i,\tau}$ as,

$$\begin{aligned} a_{i,\tau} &= \widetilde{\mathbf{U}}_{g,\tau}\Delta\mathbf{V}_{(i),g,\tau}^T + \Delta\mathbf{U}_{g,\tau}\widetilde{\mathbf{V}}_{(i),g,\tau}^T \\ &+ \widetilde{\mathbf{U}}_{(i),l,\tau}\Delta\mathbf{V}_{(i),l,\tau}^T + \Delta\mathbf{U}_{(i),l,\tau}\widetilde{\mathbf{V}}_{(i),l,\tau}^T \end{aligned} \tag{34}$$

and

$$b_{i,\tau} = \Delta\mathbf{U}_{g,\tau}\Delta\mathbf{V}_{(i),g,\tau}^T + \Delta\mathbf{U}_{(i),l,\tau}\Delta\mathbf{V}_{(i),l,\tau}^T \tag{35}$$

Then we can calculate the inner product between the derivatives of f_i and the difference between iterates and optimal values,

$$\begin{aligned} &\left\langle \nabla_{\mathbf{U}_g} f_i, \Delta\mathbf{U}_{g,\tau}\right\rangle + \left\langle \nabla_{\mathbf{V}_{(i),g}} f_i, \Delta\mathbf{V}_{(i),g,\tau}\right\rangle \\ &+ \left\langle \nabla_{\mathbf{U}_{(i),l}} f_i, \Delta\mathbf{U}_{(i),l,\tau}\right\rangle + \left\langle \nabla_{\mathbf{V}_{(i),l,\tau}} f_i, \Delta\mathbf{V}_{(i),l,\tau}\right\rangle \\ &= \left\langle \mathbf{U}_{g,\tau}\mathbf{V}_{(i),g,\tau}^T + \mathbf{U}_{(i),l,\tau}\mathbf{V}_{(i),l,\tau}^T - \mathbf{M}_{(i)}, a_{i,\tau} + 2b_{i,\tau}\right\rangle \\ &= \left\langle a_{i,\tau} + b_{i,\tau} + \mathbf{R}_{(i)}, a_{i,\tau} + 2b_{i,\tau}\right\rangle \\ &= \left\|a_{i,\tau}\right\|_F^2 + 2\left\|b_{i,\tau}\right\|_F^2 + 3\left\langle a_{i,\tau}, b_{i,\tau}\right\rangle + \left\langle \mathbf{R}_{(i)}, a_{i,\tau}\right\rangle + 2\left\langle \mathbf{R}_{(i)}, b_{i,\tau}\right\rangle \end{aligned}$$

As

$$\begin{aligned} &2f_i - \left\|\mathbf{R}_{(i)}\right\|_F^2 = \left\|\mathbf{U}_{g,\tau}\mathbf{V}_{(i),g,\tau}^T + \mathbf{U}_{(i),l,\tau}\mathbf{V}_{(i),l,\tau}^T - \mathbf{M}_{(i)}\right\|_F^2 - \left\|\mathbf{R}_{(i)}\right\|_F^2 \\ &= \left\|a_{i,\tau} + b_{i,\tau} + \mathbf{R}_{(i)}\right\|_F^2 - \left\|\mathbf{R}_{(i)}\right\|_F^2 \\ &= \left\|a_{i,\tau}\right\|_F^2 + \left\|b_{i,\tau}\right\|_F^2 + 2\left\langle a_{i,\tau}, b_{i,\tau}\right\rangle + 2\left\langle \mathbf{R}_{(i)}, a_{i,\tau}\right\rangle + 2\left\langle \mathbf{R}_{(i)}, b_{i,\tau}\right\rangle \end{aligned}$$

We know that,

$$\begin{aligned} &\left\langle \nabla_{\mathbf{U}_g} f_i, \Delta\mathbf{U}_{g,\tau}\right\rangle + \left\langle \nabla_{\mathbf{V}_{(i),g}} f_i, \Delta\mathbf{V}_{(i),g,\tau}\right\rangle \\ &+ \left\langle \nabla_{\mathbf{U}_{(i),l}} f_i, \Delta\mathbf{U}_{(i),l,\tau}\right\rangle + \left\langle \nabla_{\mathbf{V}_{(i),l,\tau}} f_i, \Delta\mathbf{V}_{(i),l,\tau}\right\rangle \\ &= 2f_i + \left\|b_{i,\tau}\right\|_F^2 + \left\langle a_{i,\tau}, b_{i,\tau}\right\rangle + \left\langle \mathbf{R}_{(i)}, a_{i,\tau}\right\rangle - \left\|\mathbf{R}_{(i)}\right\|_F^2 \end{aligned}$$

Similarly, we can calculate the inner product

$$\begin{aligned} &\left\langle \nabla_{\mathbf{U}_g} g_i, \Delta\mathbf{U}_{g,\tau}\right\rangle \\ &= 2\beta\mathrm{Tr}\left(\widetilde{\mathbf{U}}_{g,\tau}^T\Delta\mathbf{U}_{g,\tau}\widetilde{\mathbf{U}}_{g,\tau}^T\Delta\mathbf{U}_{g,\tau} + \widetilde{\mathbf{U}}_{g,\tau}^T\Delta\mathbf{U}_{g,\tau}\Delta\mathbf{U}_{g,\tau}^T\widetilde{\mathbf{U}}_{g,\tau}\right) \\ &+ 2\beta\mathrm{Tr}\left(3\Delta\mathbf{U}_{g,\tau}^T\Delta\mathbf{U}_{g,\tau}\widetilde{\mathbf{U}}_{g,\tau}^T\Delta\mathbf{U}_{g,\tau}\right) + 2\beta\mathrm{Tr}\left(\Delta\mathbf{U}_{g,\tau}^T\Delta\mathbf{U}_{g,\tau}\Delta\mathbf{U}_{g,\tau}^T\Delta\mathbf{U}_{g,\tau}\right) \end{aligned}$$

Since

$$\begin{aligned} &\left\|\mathbf{U}_{g,\tau}^T\mathbf{U}_{g,\tau} - \mathbf{I}\right\|_F^2 \\ &= \left\|\widetilde{\mathbf{U}}_{g,\tau}^T\Delta\mathbf{U}_{g,\tau} + \Delta\mathbf{U}_{g,\tau}^T\widetilde{\mathbf{U}}_{g,\tau} + \Delta\mathbf{U}_{g,\tau}^T\Delta\mathbf{U}_{g,\tau}\right\|_F^2 \\ &= 2\left\|\widetilde{\mathbf{U}}_{g,\tau}^T\Delta\mathbf{U}_{g,\tau}\right\|_F^2 + 2\mathrm{Tr}\left(\Delta\mathbf{U}_{g,\tau}^T\widetilde{\mathbf{U}}_{g,\tau}\Delta\mathbf{U}_{g,\tau}^T\widetilde{\mathbf{U}}_{g,\tau}\right) \\ &+ 2\mathrm{Tr}\left(\widetilde{\mathbf{U}}_{g,\tau}^T\Delta\mathbf{U}_{g,\tau}\Delta\mathbf{U}_{g,\tau}^T\Delta\mathbf{U}_{g,\tau}\right) + 2\mathrm{Tr}\left(\Delta\mathbf{U}_{g,\tau}^T\widetilde{\mathbf{U}}_{g,\tau}\Delta\mathbf{U}_{g,\tau}^T\Delta\mathbf{U}_{g,\tau}\right) \\ &+ \left\|\Delta\mathbf{U}_{g,\tau}^T\Delta\mathbf{U}_{g,\tau}\right\|_F^2 \end{aligned}$$

We know that

$$\begin{aligned} &\left\langle \nabla_{\mathbf{U}_g} g_i, \Delta\mathbf{U}_{g,\tau}\right\rangle \\ &= \beta\left\|\mathbf{U}_{g,\tau}^T\mathbf{U}_{g,\tau} - \mathbf{I}\right\|_F^2 + 2\beta\mathrm{Tr}\left(\widetilde{\mathbf{U}}_{g,\tau}^T\Delta\mathbf{U}_{g,\tau}\Delta\mathbf{U}_{g,\tau}^T\Delta\mathbf{U}_{g,\tau}\right) + \beta\left\|\Delta\mathbf{U}_{g,\tau}^T\Delta\mathbf{U}_{g,\tau}\right\|_F^2 \end{aligned}$$

As a result,

$$
\begin{aligned}
&\left\langle \nabla_{\mathbf{U}_g} g_i, \Delta \mathbf{U}_{g,\tau} \right\rangle + \left\langle \nabla_{\mathbf{U}_{(i),l}} g_i, \Delta \mathbf{U}_{(i),l,\tau} \right\rangle \\
&= 2g_i + 2\beta \mathrm{Tr}\left(\widetilde{\mathbf{U}}_{g,\tau}^T \Delta \mathbf{U}_{g,\tau} \Delta \mathbf{U}_{g,\tau}^T \Delta \mathbf{U}_{g,\tau} \right) + 2\beta \mathrm{Tr}\left(\widetilde{\mathbf{U}}_{(i),l,\tau}^T \Delta \mathbf{U}_{(i),l,\tau} \Delta \mathbf{U}_{(i),l,\tau}^T \Delta \mathbf{U}_{(i),l,\tau} \right) \\
&+ \beta \left\| \Delta \mathbf{U}_{g,\tau} \right\|_F^4 + \beta \left\| \Delta \mathbf{U}_{(i),l,\tau} \right\|_F^4 \\
&\geq 2g_i - 2\beta \left\| \Delta \mathbf{U}_{g,\tau} \right\|_F^3 - 2\beta \left\| \Delta \mathbf{U}_{(i),l,\tau} \right\|_F^3 + \beta \left\| \Delta \mathbf{U}_{g,\tau} \right\|_F^4 + \beta \left\| \Delta \mathbf{U}_{(i),l,\tau} \right\|_F^4
\end{aligned}
$$

Therefore, we can sum up the two inequalities and derive,

$$
\begin{aligned}
&\left\langle \nabla_{\mathbf{U}_g} \tilde{f}_i, \mathbf{U}_{g,\tau} - \widetilde{\mathbf{U}}_{g,\tau} \right\rangle + \left\langle \nabla_{\mathbf{V}_{(i),g}} \tilde{f}_i, \mathbf{V}_{(i),g,\tau} - \widetilde{\mathbf{V}}_{(i),g,\tau} \right\rangle \\
&+ \left\langle \nabla_{\mathbf{U}_{(i),l}} \tilde{f}_i, \mathbf{U}_{(i),l,\tau} - \widetilde{\mathbf{U}}_{(i),l,\tau} \right\rangle + \left\langle \nabla_{\mathbf{V}_{(i),l,\tau}} \tilde{f}_i, \mathbf{V}_{(i),l,\tau} - \widetilde{\mathbf{V}}_{(i),l,\tau} \right\rangle \\
&\geq 2\tilde{f}_i - \left\| \mathbf{R}_{(i)} \right\|_F^2 \\
&+ \left\| b_{i,\tau} \right\|_F^2 + \left\langle a_{i,\tau}, b_{i,\tau} \right\rangle + \left\langle \mathbf{R}_{(i)}, a_{i,\tau} \right\rangle \\
&- 2\beta \left\| \Delta \mathbf{U}_{g,\tau} \right\|_F^3 - 2\beta \left\| \Delta \mathbf{U}_{(i),l,\tau} \right\|_F^3 + \beta \left\| \Delta \mathbf{U}_{g,\tau} \right\|_F^4 + \beta \left\| \Delta \mathbf{U}_{(i),l,\tau} \right\|_F^4
\end{aligned}
$$

We can sum up both sides for i from 1 to N, considering the KKT condition that $\sum_{i=1}^N \left\langle \mathbf{R}_{(i)}, a_{i,\tau} \right\rangle = 0$, and the definition of ϕ_τ that $\sum_{i=1}^N 2\tilde{f}_i - \left\| \mathbf{R}_{(i)} \right\|_F^2 = 2\phi_\tau$, we have,

$$
\begin{aligned}
&\sum_{i=1}^N \Big[\left\langle \nabla_{\mathbf{U}_g} \tilde{f}_i, \mathbf{U}_{g,\tau} - \widetilde{\mathbf{U}}_{g,\tau} \right\rangle + \left\langle \nabla_{\mathbf{V}_{(i),g}} \tilde{f}_i, \mathbf{V}_{(i),g,\tau} - \widetilde{\mathbf{V}}_{(i),g,\tau} \right\rangle \\
&+ \left\langle \nabla_{\mathbf{U}_{(i),l}} \tilde{f}_i, \mathbf{U}_{(i),l,\tau} - \widetilde{\mathbf{U}}_{(i),l,\tau} \right\rangle + \left\langle \nabla_{\mathbf{V}_{(i),l,\tau}} \tilde{f}_i, \mathbf{V}_{(i),l,\tau} - \widetilde{\mathbf{V}}_{(i),l,\tau} \right\rangle \Big] \\
&\geq 2\phi_\tau \\
&+ \sum_{i=1}^N \Big[\left\| b_{i,\tau} \right\|_F^2 + \left\langle a_{i,\tau}, b_{i,\tau} \right\rangle - 2\beta \left\| \Delta \mathbf{U}_{g,\tau} \right\|_F^3 - 2\beta \left\| \Delta \mathbf{U}_{(i),l,\tau} \right\|_F^3 + \beta \left\| \Delta \mathbf{U}_{g,\tau} \right\|_F^4 + \beta \left\| \Delta \mathbf{U}_{(i),l,\tau} \right\|_F^4 \Big] \\
&\geq 2\phi_\tau + \sum_{i=1}^N \Big[- \left\| a_{i,\tau} \right\|_F \left\| b_{i,\tau} \right\|_F - 2\beta \left\| \Delta \mathbf{U}_{g,\tau} \right\|_F^3 - 2\beta \left\| \Delta \mathbf{U}_{(i),l,\tau} \right\|_F^3 \Big]
\end{aligned}
\tag{36}
$$

For the higher order terms,

$$
\begin{aligned}
&\sum_{i=1}^N \left\| a_{i,\tau} \right\|_F \left\| b_{i,\tau} \right\|_F \\
&\leq \sqrt{\sum_{i=1}^N \left\| a_{i,\tau} \right\|_F^2} \sqrt{\sum_{i=1}^N \left\| b_{i,\tau} \right\|_F^2} \\
&\leq \sqrt{\sum_{i=1}^N \left(\hat{\sigma}_{\max} (\left\| \Delta \mathbf{U}_{g,\tau} \right\|_F + \left\| \Delta \mathbf{U}_{(i),l,\tau} \right\|_F) + \left\| \Delta \mathbf{V}_{(i),g,\tau} \right\|_F + \left\| \Delta \mathbf{V}_{(i),l,\tau} \right\|_F \right)^2} \\
&\sqrt{\sum_{i=1}^N \left(\left\| \Delta \mathbf{U}_{g,\tau} \right\|_F \left\| \Delta \mathbf{V}_{(i),g,\tau} \right\|_F + \left\| \Delta \mathbf{U}_{(i),l,\tau} \right\|_F \left\| \Delta \mathbf{V}_{(i),l,\tau} \right\|_F \right)^2} \\
&\leq \sqrt{2 \sum_{i=1}^N \left(\hat{\sigma}_{\max}^2 (\left\| \Delta \mathbf{U}_{g,\tau} \right\|_F^2 + \left\| \Delta \mathbf{U}_{(i),l,\tau} \right\|_F^2) + \left\| \Delta \mathbf{V}_{(i),g,\tau} \right\|_F^2 + \left\| \Delta \mathbf{V}_{(i),l,\tau} \right\|_F^2 \right)} \\
&\sqrt{2 \sum_{i=1}^N \left(\left\| \Delta \mathbf{U}_{g,\tau} \right\|_F^2 \left\| \Delta \mathbf{V}_{(i),g,\tau} \right\|_F^2 + \left\| \Delta \mathbf{U}_{(i),l,\tau} \right\|_F^2 \left\| \Delta \mathbf{V}_{(i),l,\tau} \right\|_F^2 \right)}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sqrt{2\max\{\hat{\sigma}_{\max}^2, 1\}}\sqrt{\sum_{i=1}^{N}\left(\|\Delta\mathbf{U}_{g,\tau}\|_F^2 + \left\|\Delta\mathbf{U}_{(i),l,\tau}\right\|_F^2 + \left\|\Delta\mathbf{V}_{(i),g,\tau}\right\|_F^2 + \left\|\Delta\mathbf{V}_{(i),l,\tau}\right\|_F^2\right)} \\
&\sqrt{2}\sqrt{\left(\sum_{i=1}^{N}\|\Delta\mathbf{U}_{g,\tau}\|_F^2\right)\left(\sum_{i=1}^{N}\left\|\Delta\mathbf{V}_{(i),g,\tau}\right\|_F^2\right) + \left(\sum_{i=1}^{N}\left\|\Delta\mathbf{U}_{(i),l,\tau}\right\|_F^2\right)\left(\sum_{i=1}^{N}\left\|\Delta\mathbf{V}_{(i),l,\tau}\right\|_F^2\right)} \\
&\leq 2\sqrt{2\max\{\hat{\sigma}_{\max}^2, 1\}}\sqrt{\sum_{i=1}^{N}\left(\|\Delta\mathbf{U}_{g,\tau}\|_F^2 + \left\|\Delta\mathbf{U}_{(i),l,\tau}\right\|_F^2 + \left\|\Delta\mathbf{V}_{(i),g,\tau}\right\|_F^2 + \left\|\Delta\mathbf{V}_{(i),l,\tau}\right\|_F^2\right)}^3 \\
&\leq 2\sqrt{2\max\{\hat{\sigma}_{\max}^2, 1\}}\left(\phi_\tau C_2\right)^{3/2}
\end{aligned}
$$

where we applied Lemma C.8 in the last inequality.

And

$$
\begin{aligned}
&2\beta\sum_{i=1}^{N}\left(\|\Delta\mathbf{U}_{g,\tau}\|_F^3 + \left\|\Delta\mathbf{U}_{(i),l,\tau}\right\|_F^3\right) \\
&\leq 4\beta\sqrt{\sum_{i=1}^{N}\left(\|\Delta\mathbf{U}_{g,\tau}\|_F^2 + \left\|\Delta\mathbf{U}_{(i),l,\tau}\right\|_F^2 + \left\|\Delta\mathbf{V}_{(i),g,\tau}\right\|_F^2 + \left\|\Delta\mathbf{V}_{(i),l,\tau}\right\|_F^2\right)}^3 \\
&\leq 4\beta\left(\phi_\tau C_2\right)^{3/2}
\end{aligned}
$$

where we also applied Lemma C.8 in the last inequality.

Thus when $\phi_\tau \leq \frac{1}{\left(2\sqrt{2\max\{\hat{\sigma}_{\max}^2,1\}}+4\beta\right)^2 C_2^3}$, the higher order $O(\phi_\tau^{3/2})$ terms in equation 36 are dominated by the leading order term $2\phi_\tau$. As a result, we have,

$$
\begin{aligned}
&\sum_{i=1}^{N}\Big[\left\langle\nabla_{\mathbf{U}_g}\tilde{f}_i, \mathbf{U}_{g,\tau} - \widetilde{\mathbf{U}}_{g,\tau}\right\rangle + \left\langle\nabla_{\mathbf{V}_{(i),g}}\tilde{f}_i, \mathbf{V}_{(i),g,\tau} - \widetilde{\mathbf{V}}_{(i),g,\tau}\right\rangle \\
&+ \left\langle\nabla_{\mathbf{U}_{(i),l}}\tilde{f}_i, \mathbf{U}_{(i),l,\tau} - \widetilde{\mathbf{U}}_{(i),l,\tau}\right\rangle + \left\langle\nabla_{\mathbf{V}_{(i),l,\tau}}\tilde{f}_i, \mathbf{V}_{(i),l,\tau} - \widetilde{\mathbf{V}}_{(i),l,\tau}\right\rangle\Big] \\
&\geq \phi_\tau
\end{aligned}
$$

This completes the proof. □

Finally, we are able to prove the PL inequality,

Lemma C.10 (PL inequality) *Under the same conditions as Lemma C.9, we have the following PL inequality,*

$$
\left\|\frac{1}{\sqrt{N}}\nabla_{\mathbf{U}_g}\tilde{f}\right\|_F^2 + \sum_{i=1}^{N}\left\|\nabla_{\mathbf{V}_{(i),g}}\tilde{f}_i\right\|_F^2 + \left\|\nabla_{\mathbf{U}_{(i),l}}\tilde{f}_i\right\|_F^2 + \left\|\nabla_{\mathbf{V}_{(i),l}}\tilde{f}_i\right\|_F^2 \geq \frac{1}{C_2}\phi_\tau \tag{37}
$$

Proof. The proof is straightforward. We first combine Lemma C.9 with Cauchy-Schwartz inequality,

$$\sqrt{\left\|\frac{1}{\sqrt{N}}\nabla_{\mathbf{U}_g}\tilde{f}\right\|_F^2+\sum_{i=1}^N\left\|\nabla_{\mathbf{V}_{(i),g}}\tilde{f}_i\right\|_F^2+\left\|\nabla_{\mathbf{U}_{(i),l}}\tilde{f}_i\right\|_F^2+\left\|\nabla_{\mathbf{V}_{(i),l}}\tilde{f}_i\right\|_F^2}$$
$$\sqrt{N\left\|\Delta\mathbf{U}_{g,\tau}\right\|_F^2+\sum_{i=1}^N\left\|\Delta\mathbf{V}_{(i),g,\tau}\right\|_F^2+\left\|\Delta\mathbf{U}_{(i),l,\tau}\right\|_F^2+\left\|\nabla_{\mathbf{V}_{(i),l}}\tilde{f}_i\right\|_F^2}$$
$$\geq\left\|\frac{1}{\sqrt{N}}\nabla_{\mathbf{U}_g}\tilde{f}\right\|_F\left\|\sqrt{N}\Delta\mathbf{U}_{g,\tau}\right\|_F+\sum_{i=1}^N\Big[\left\|\nabla_{\mathbf{V}_{(i),g}}\tilde{f}_i\right\|_F\left\|\Delta\mathbf{V}_{(i),g,\tau}\right\|_F$$
$$\left\|\nabla_{\mathbf{U}_{(i),l}}\tilde{f}_i\right\|_F\left\|\Delta\mathbf{U}_{(i),l,\tau}\right\|_F+\left\|\nabla_{\mathbf{V}_{(i),l}}\tilde{f}_i\right\|_F\left\|\Delta\mathbf{V}_{(i),l,\tau}\right\|_F\Big]$$
$$\geq\sum_{i=1}^N\Big[\left\langle\nabla_{\mathbf{U}_g}\tilde{f}_i,\mathbf{U}_{g,\tau}-\widetilde{\mathbf{U}}_{g,\tau}\right\rangle+\left\langle\nabla_{\mathbf{V}_{(i),g}}\tilde{f}_i,\mathbf{V}_{(i),g,\tau}-\widetilde{\mathbf{V}}_{(i),g,\tau}\right\rangle$$
$$+\left\langle\nabla_{\mathbf{U}_{(i),l}}\tilde{f}_i,\mathbf{U}_{(i),l,\tau}-\widetilde{\mathbf{U}}_{(i),l,\tau}\right\rangle+\left\langle\nabla_{\mathbf{V}_{(i),l,\tau}}\tilde{f}_i,\mathbf{V}_{(i),l,\tau}-\widetilde{\mathbf{V}}_{(i),l,\tau}\right\rangle\Big]$$
$$\geq\phi_\tau$$

Dividing both sides by $\sqrt{N\left\|\Delta\mathbf{U}_{g,\tau}\right\|_F^2+\sum_{i=1}^N\left\|\Delta\mathbf{V}_{(i),g,\tau}\right\|_F^2+\left\|\Delta\mathbf{U}_{(i),l,\tau}\right\|_F^2+\left\|\nabla_{\mathbf{V}_{(i),l}}\tilde{f}_i\right\|_F^2}$ and applying Lemma C.8 will give us the desired result. □

Finally, we will combine the derived results and show the linear convergence of Algorithm 1. Theorem 4 is a formal statement of the convergence guarantee.

Theorem 4 (Formal version of Theorem 2 in the main paper) *Under the following conditions,*

1. *The largest singular values of $\mathbf{M}_{(i)}$'s are upper bounded by $\hat{\sigma}_{\max}$.*
2. *There exists constants $\hat{\theta}$, $\hat{\sigma}_{gap}^2>0$ such that $\left\|\frac{1}{N}\sum_{i=1}^N\mathbf{P}_{\hat{\mathbf{U}}_{(i),l}}\right\|\leq 1-\hat{\theta}$, $\sigma_{min}^2\left(\hat{\mathbf{U}}_g\hat{\mathbf{V}}_{(i),g}^T+\hat{\mathbf{U}}_{(i),l}\hat{\mathbf{V}}_{(i),l}^T\right)-\left\|\mathbf{R}_{(i)}\right\|^2\geq\hat{\sigma}_{gap}^2$, and $\left\|\mathbf{R}_{(i)}\right\|\leq\frac{\hat{\theta}\hat{\sigma}_{gap}^2}{8\hat{\sigma}_{\max}}$ for every i.*
3. *There exist constants $B_1>1$ and $B_2>\hat{\sigma}_{\max}$ such that $(\mathbf{U}_{g,1},\{\mathbf{U}_{(i),l,1}\},\{\mathbf{V}_{(i),g,1}\},\{\mathbf{V}_{(i),l,1}\})\in\mathcal{B}(B_1,B_2)$ and the constant stepsize η is upper bounded by $\eta\leq\min\{\frac{1}{10}\Big(2B_1B_2(2B_1B_2+\sigma_{\max})+2\beta B_1^2(B_1^2+1)+\big(B_2(2B_1B_2+\sigma_{\max})+2\beta B_1(B_1^2+1)\big)^2\Big),\frac{1}{2\left(\frac{L}{2}+272\beta B_1^2+400L_gB_1^2\right)},1\}$.*
4. *The iterates are initialized properly $\phi_1\leq\min\{\frac{9\beta}{128},\frac{9\hat{\theta}\hat{\sigma}_{gap}^2}{1936},\frac{1}{\left(2\sqrt{2\max\{\sigma_{\max}^2,1\}+4\beta}\right)^2C_2^3},\frac{(B_1-1)^2}{C_1},\frac{(B_2-\sigma_{\max})^2}{C_2}\}$.*

then all iterates reside in $\mathcal{B}(B_1,B_2)$ and the following holds,

$$\phi_\tau\leq(1-\eta B_3)^{\tau-1}\phi_1 \tag{38}$$

where B_3 is a constant

$$B_3=\frac{1}{2C_2}$$

Notice that by combining Theorem 4 with Lemma C.1, we can immediately prove Theorem 2.

We shall emphasize that the conditions in Theorem 4 are more general than that in Theorem 2, as we do not make assumptions on the input noise structure. Most works on nonconvex matrix factorization (e.g., Sun & Luo (2016); Chen et al. (2020); Ye & Du (2021)) assume the input data are exactly low-rank, i.e., $\mathbf{R}_{(i)}=0$. Theorem 4 relaxes such assumption by allowing $\mathbf{R}_{(i)}$ to be small nonzero matrices.

Proof. We will prove that the following claims hold for every $\tau\geq 1$ by induction.

1. $\|\mathbf{U}_{g,\tau}\|, \left\|\mathbf{U}_{(i),l,\tau}\right\| \leq B_1$ and $\left\|\mathbf{V}_{(i),g,\tau}\right\|, \left\|\mathbf{V}_{(i),l,\tau}\right\| \leq B_2$.
2. $\phi_\tau \leq (1 - \frac{\eta}{2C_2})^\tau \phi_0$

<u>*Base case*</u> At initialization $\tau = 1$, Claim 2 is simply true. Claim 1 is true because we assume at initialization, the norms of $\mathbf{U}_{g,1}$, $\mathbf{U}_{(i),l,1}$'s, $\mathbf{V}_{(i),g,1}$'s, $\mathbf{V}_{(i),l,1}$'s are upper bounded.

<u>*Induction step*</u> Now we assume Claim 1 and 2 are true for $\tau = 1, \cdots, t$ and prove that they still hold for $\tau = t + 1$.

Since Claim 2 is true for $\tau = t$, we know $\phi_t = (1 - \frac{\eta}{2C_2})^{t-1}\phi_1 \leq \phi_1$. As $\phi_1 \leq \frac{1}{(2\sqrt{2\max\{\hat{\sigma}_{\max}^2, 1\} + 4\beta})^2 C_3^3}$ and $\phi_1 \leq \min\{\frac{9\beta}{128}, \frac{9\hat{\theta}\hat{\sigma}_{\text{gap}}^2}{1936}\}$, we know that the result of Lemma C.10 holds.

Also from Lemma C.6 and the fact that $\phi_\tau \leq \frac{9\beta}{128}$, we know $\left\|\mathbf{U}_{g,\tau}^T \mathbf{U}_{g,\tau} - \mathbf{I}\right\|_F \leq \frac{3}{4}$ and $\left\|\mathbf{U}_{(i),l,\tau}^T \mathbf{U}_{(i),l,\tau} - \mathbf{I}\right\|_F \leq \frac{3}{4}$.

Since Claim 1 is true for $\tau = t$, and the stepsize is upper bounded by $\eta \leq \min\{\frac{1}{10}\Big(2B_1B_2(2B_1B_2 + \sigma_{\max}) + 2\beta B_1^2(B_1^2 + 1) + \big(B_2(2B_1B_2 + \sigma_{\max}) + 2\beta B_1(B_1^2 + 1)\big)^2\Big), \frac{1}{2\left(\frac{L}{2} + 272\beta B_1^2 + 400 L_g B_1^2\right)}, 1\}$, all the conditions of Lemma C.3 are satisfied. Thus the result of Lemma C.3 holds for $\tau = t$.

Therefore we can combine the results of Lemma C.10 and Lemma C.3 to derive,

$$
\begin{aligned}
\phi_{t+1} &\leq \phi_t - \frac{\eta}{2}\left(\left\|\frac{1}{\sqrt{N}}\nabla_{\mathbf{U}_g}\tilde{f}\right\|_F^2 + \sum_{i=1}^N \left\|\nabla_{\mathbf{V}_{(i),g}}\tilde{f}_i\right\|_F^2 + \left\|\nabla_{\mathbf{U}_{(i),l}}\tilde{f}_i\right\|_F^2 + \left\|\nabla_{\mathbf{V}_{(i),l}}\tilde{f}_i\right\|_F^2\right) \\
&\leq \left(1 - \frac{\eta}{2}\frac{1}{C_2}\right)\phi_t \\
&\leq (1 - \frac{\eta}{2C_2})^{t+1}\phi_0
\end{aligned}
$$

. We thus prove Claim 2 for $\tau = t + 1$.

We then show that Claim 1 is true for $t + 1$. As $\phi_{t+1} < \phi_1 \leq \frac{(B_1 - 1)^2}{C_1}$, by Lemma C.7, we have,

$$\|\Delta\mathbf{U}_{g,t+1}\|_F \leq \sqrt{C_1\phi_{t+1}} \leq \sqrt{C_1\phi_0} \leq B_1 - 1$$

Thus by triangle inequality,

$$\|\mathbf{U}_{g,t+1}\| \leq \left\|\widetilde{\mathbf{U}}_{g,t+1}\right\| + \|\Delta\mathbf{U}_{g,t+1}\| \leq B_1$$

Similar bounds hold for $\mathbf{U}_{(i),l,t+1}$'s. Also, as $\phi_{t+1} < \phi_1 \leq \frac{(B_2 - \hat{\sigma}_{\max})^2}{C_1}$, by Lemma C.7, we have,

$$\left\|\Delta\mathbf{V}_{(i),g,t+1}\right\|_F \leq \sqrt{C_2\phi_{t+1}} \leq \sqrt{C_2\phi_0} \leq B_2 - \hat{\sigma}_{\max}$$

Also, by triangle inequality,

$$\left\|\mathbf{V}_{(i),g,t+1}\right\| \leq \left\|\widetilde{\mathbf{V}}_{(i),g,t+1}\right\| + \left\|\Delta\mathbf{U}_{(i),g,t+1}\right\| \leq B_2$$

Similar bounds hold for $\mathbf{V}_{(i),l,t+1}$'s.

This completes the proof. □

D Proof of Theorem 3

The procedures to prove Theorem 3 are close to the first stage of the proof of Theorem 2. We first establish the KKT condition, then derive the sufficient decrease inequality for **.**

The following lemma presents the KKT conditions to problem (4) in the main paper.

Lemma D.1 (KKT conditions for general loss) ***Suppose that*** $\{\hat{\mathbf{U}}_g, \hat{\mathbf{U}}_{(i),l}, \hat{\mathbf{V}}_{(i),g}, \hat{\mathbf{V}}_{(i),l}\}$ ***is the optimal solution to problem (4) in the main paper with general loss metric*** ℓ***,*** $\hat{\mathbf{U}}_g$ ***and*** $\hat{\mathbf{U}}_{(i),l}$***'s are non-singular, and*** $\mathbf{M}_{(i)}$ ***has rank at least*** $r_1 + r_2$***. We have***

$$\sum_{i=1}^{N} \ell' \left(\mathbf{M}_{(i)}, \hat{\mathbf{U}}_g \hat{\mathbf{V}}_{(i),g}^T + \hat{\mathbf{U}}_{(i),l} \hat{\mathbf{V}}_{(i),l}^T\right) \hat{\mathbf{V}}_{(i),g} = 0 \tag{39a}$$

$$\ell' \left(\mathbf{M}_{(i)}, \hat{\mathbf{U}}_g \hat{\mathbf{V}}_{(i),g}^T + \hat{\mathbf{U}}_{(i),l} \hat{\mathbf{V}}_{(i),l}^T\right) \hat{\mathbf{V}}_{(i),l} = 0 \tag{39b}$$

$$\ell' \left(\mathbf{M}_{(i)}, \hat{\mathbf{U}}_g \hat{\mathbf{V}}_{(i),g}^T + \hat{\mathbf{U}}_{(i),l} \hat{\mathbf{V}}_{(i),l}^T\right)^T \hat{\mathbf{U}}_{(i),l} = 0 \tag{39c}$$

$$\ell' \left(\mathbf{M}_{(i)}, \hat{\mathbf{U}}_g \hat{\mathbf{V}}_{(i),g}^T + \hat{\mathbf{U}}_{(i),l} \hat{\mathbf{V}}_{(i),l}^T\right)^T \hat{\mathbf{U}}_g = 0 \tag{39d}$$

$$\hat{\mathbf{U}}_{(i),l}^T \hat{\mathbf{U}}_{(i),l} = \mathbf{I}, \hat{\mathbf{U}}_g^T \hat{\mathbf{U}}_g = \mathbf{I}, \hat{\mathbf{U}}_{(i),l}^T \hat{\mathbf{U}}_g = 0. \tag{39e}$$

Proof. **The proof of the above lemma is very similar to the proof of Lemma C.4. The Lagrangian of the optimization problem (4) in the main paper can be written as**

$$\begin{aligned}\mathcal{L} = &\sum_{i=1}^{N} \ell \left(\mathbf{M}_{(i)}, \mathbf{U}_g \mathbf{V}_{(i),g}^T + \mathbf{U}_{(i),l} \mathbf{V}_{(i),l}^T\right) + \frac{\beta}{2} \left\|\mathbf{U}_g^T \mathbf{U}_g - \mathbf{I}\right\|_F^2 + \frac{\beta}{2} \left\|\mathbf{U}_{(i),l}^T \mathbf{U}_{(i),l} - \mathbf{I}\right\|_F^2 \\ &+ \mathbf{Tr}\left(\mathbf{\Lambda}_{7,(i)} \mathbf{U}_g^T \mathbf{U}_{(i),l}\right)\end{aligned} \tag{40}$$

where $\mathbf{\Lambda}_{7,(i)}$ **is the dual variable for the constraint** $\mathbf{U}_g^T \mathbf{U}_{(i),l} = 0$**.**

Similar to Lemma C.4, under the LICQ, we know that the optimality of $\hat{\mathbf{U}}_g, \{\hat{\mathbf{V}}_{(i),g}, \hat{\mathbf{U}}_{(i),l}, \hat{\mathbf{V}}_{(i),l}\}$ **implies the KKT condition. Setting the gradient of** $\mathcal{L}$ **with respect to** $\mathbf{V}_{(i),g}$ **and** $\mathbf{V}_{(i),l}$ **to zero, we can prove equation 39d and equation 39c. Then we examine the gradient of** $\mathcal{L}$ **with respect to** $\mathbf{U}_{(i),l}$**:**

$$\frac{\partial}{\partial \mathbf{U}_{(i),l}} \mathcal{L} = \ell' \left(\mathbf{M}_{(i)}, \mathbf{U}_g \mathbf{V}_{(i),g}^T + \mathbf{U}_{(i),l} \mathbf{V}_{(i),l}^T\right) \hat{\mathbf{V}}_{(i),l} + 2\beta \mathbf{U}_{(i),l} \left(\mathbf{U}_{(i),l}^T \mathbf{U}_{(i),l} - \mathbf{I}\right) + \mathbf{U}_g \mathbf{\Lambda}_{(7),i}^T$$

Left multiplying both sides by $\hat{\mathbf{U}}_g^T$**, we have** $\mathbf{\Lambda}_{7,(i)} = 0$**. Left multiplying both sides by** $\hat{\mathbf{U}}_{(i),l}^T$**, we have** $\hat{\mathbf{U}}_{(i),l}^T \hat{\mathbf{U}}_{(i),l} - \mathbf{I} = 0$**. Therefore we have** $\ell' \left(\mathbf{M}_{(i)}, \mathbf{U}_g \mathbf{V}_{(i),g}^T + \mathbf{U}_{(i),l} \mathbf{V}_{(i),l}^T\right) \hat{\mathbf{V}}_{(i),l} = 0$**. This proves equation equation 39b.**

Now, setting the derivative of $\mathcal{L}$ **with respect to** $\mathbf{U}_g$ **to zero, we have**

$$\frac{\partial}{\partial \mathbf{U}_g} \mathcal{L} = \sum_{i=1}^{N} \ell' \left(\mathbf{M}_{(i)}, \hat{\mathbf{U}}_g \hat{\mathbf{V}}_{(i),g}^T + \hat{\mathbf{U}}_{(i),l} \hat{\mathbf{V}}_{(i),l}^T\right) \hat{\mathbf{V}}_{(i),g} + 2N\beta \hat{\mathbf{U}}_g \left(\hat{\mathbf{U}}_g^T \hat{\mathbf{U}}_g - \mathbf{I}\right) = 0$$

Left multiplying both sides by $\hat{\mathbf{U}}_g^T$**, we have** $\hat{\mathbf{U}}_g^T \hat{\mathbf{U}}_g - \mathbf{I} = 0$**. We have thus proven equation 39a. This completes the proof for equation 39.** □

The following lemma gives an upper bound on the Lipschitz constant when all the norm of iterates and gradients are bounded.

Lemma D.2 (Lipschitz continuity for general loss metrics) ***In region*** $\mathcal{B}(B_1, B_2)$ ***as defined in equation 16, if there exists a constant*** $B_4 > 0$ ***such that*** $\left\|\ell'(\mathbf{M}_{(i)}, \mathbf{U}_g \mathbf{V}_{(i),g}^T + \mathbf{U}_{(i),l} \mathbf{V}_{(i),l}^T)\right\| \leq B_4$***, and loss metric*** ℓ ***is*** L_ℓ ***Lipschitz continuous in the region, then the objectives*** $\tilde{f}_i$ ***with*** $\beta = 0$ ***are Lipschitz continuous in the region,***

$$\begin{aligned}&\left\|\nabla \tilde{f}_i(\mathbf{U}_g', \mathbf{V}_{(i),g}', \mathbf{U}_{(i),l}', \mathbf{V}_{(i),l}') - \nabla \tilde{f}_i(\mathbf{U}_g, \mathbf{V}_{(i),g}, \mathbf{U}_{(i),l}, \mathbf{V}_{(i),l})\right\|_F \\ &\leq L_{gen} \sqrt{\left\|\mathbf{U}_g' - \mathbf{U}_g\right\|_F^2 + \left\|\mathbf{V}_{(i),g}' - \mathbf{V}_{(i),g}\right\|_F^2 + \left\|\mathbf{U}_{(i),l}' - \mathbf{U}_{(i),l}\right\|_F^2 + \left\|\mathbf{V}_{(i),l}' - \mathbf{V}_{(i),l}\right\|_F^2}\end{aligned} \tag{41}$$

for $\{\mathbf{U}_g^{'}, \{\mathbf{V}_{(i),g}^{'}\}, \{\mathbf{U}_{(i),l}^{'}\}, \{\mathbf{V}_{(i),l}^{'}\}\}$,$\{\mathbf{U}_g, \{\mathbf{V}_{(i),g}\}, \{\mathbf{U}_{(i),l}\}, \{\mathbf{V}_{(i),l}\}\} \in \mathcal{B}(B_1, B_2)$***, where*** L_{gen} ***is a constant dependent on*** B_1***,*** B_2***,*** L_ℓ***, and*** B_4***,***

$$L_{gen} = \sqrt{8\left(B_2^2 + B_1^2\right) L_\ell^2 \max\{B_1^2, B_2^2\} + 2B_4^2}$$

Proof. The proof is similar to the proof of Lemma C.2. We will calculate the gradient of $\tilde{f}_i$ over each variable, and bound the norm of the difference of the gradients. For simplicity, we use $\Delta\mathbf{M}_{(i)}^{'}$ to denote $\Delta\mathbf{M}_{(i)}^{'} = \mathbf{U}_g^{'}(\mathbf{V}_{(i),g}^{'})^T + \mathbf{U}_{(i),l}^{'}(\mathbf{V}_{(i),l}^{'})^T - \mathbf{U}_g\mathbf{V}_{(i),g}^T - \mathbf{U}_{(i),l}\mathbf{V}_{(i),l}^T$

$$\begin{aligned}
&\nabla_{\mathbf{U}_g}\tilde{f}_i(\mathbf{U}_g^{'}, \mathbf{V}_{(i),g}^{'}, \mathbf{U}_{(i),l}^{'}, \mathbf{V}_{(i),l}^{'}) - \nabla_{\mathbf{U}_g}\tilde{f}_i(\mathbf{U}_g, \mathbf{V}_{(i),g}, \mathbf{U}_{(i),l}, \mathbf{V}_{(i),l}) \\
&= \ell'\left(\mathbf{M}_{(i)}, \mathbf{U}_g^{'}(\mathbf{V}_{(i),g}^{'})^T + \mathbf{U}_{(i),l}^{'}(\mathbf{V}_{(i),l}^{'})^T\right)\mathbf{V}_{(i),g}^{'} \\
&- \ell'\left(\mathbf{M}_{(i)}, \mathbf{U}_g(\mathbf{V}_{(i),g})^T + \mathbf{U}_{(i),l}(\mathbf{V}_{(i),l})^T\right)\mathbf{V}_{(i),g}
\end{aligned}$$

Also by triangle inequalities, when $(\mathbf{U}_g^{'}, \mathbf{V}_{(i),g}^{'}, \mathbf{U}_{(i),l}^{'}, \mathbf{V}_{(i),l}^{'})$ and $(\mathbf{U}_g, \mathbf{V}_{(i),g}, \mathbf{U}_{(i),l}, \mathbf{V}_{(i),l})$ are in $\mathcal{B}(B_1, B_2)$, we have:

$$\begin{aligned}
&\left\|\nabla_{\mathbf{U}_g}\tilde{f}_i(\mathbf{U}_g^{'}, \mathbf{V}_{(i),g}^{'}, \mathbf{U}_{(i),l}^{'}, \mathbf{V}_{(i),l}^{'}) - \nabla_{\mathbf{U}_g}\tilde{f}_i(\mathbf{U}_g, \mathbf{V}_{(i),g}, \mathbf{U}_{(i),l}, \mathbf{V}_{(i),l})\right\|_F \\
&\leq L_\ell B_2\left\|\Delta\mathbf{M}_{(i)}^{'}\right\|_F + B_4\left\|\mathbf{V}_{(i),g}^{'} - \mathbf{V}_{(i),g}\right\|_F
\end{aligned}$$

Then on the derivative over $\mathbf{V}_{(i),g}$,

$$\begin{aligned}
&\nabla_{\mathbf{V}_{(i),g}}\tilde{f}_i(\mathbf{U}_g^{'}, \mathbf{V}_{(i),g}^{'}, \mathbf{U}_{(i),l}^{'}, \mathbf{V}_{(i),l}^{'}) - \nabla_{\mathbf{V}_{(i),g}}\tilde{f}_i(\mathbf{U}_g, \mathbf{V}_{(i),g}, \mathbf{U}_{(i),l}, \mathbf{V}_{(i),l}) \\
&= \ell'\left(\mathbf{M}_{(i)}, \mathbf{U}_g^{'}(\mathbf{V}_{(i),g}^{'})^T + \mathbf{U}_{(i),l}^{'}(\mathbf{V}_{(i),l}^{'})^T\right)^T\mathbf{U}_g^{'} - \ell'\left(\mathbf{M}_{(i)}, \mathbf{U}_g(\mathbf{V}_{(i),g})^T + \mathbf{U}_{(i),l}(\mathbf{V}_{(i),l})^T\right)^T\mathbf{U}_g
\end{aligned}$$

Thus by similar calculations, we have,

$$\begin{aligned}
&\left\|\nabla_{\mathbf{V}_{(i),g}}\tilde{f}_i(\mathbf{U}_g^{'}, \mathbf{V}_{(i),g}^{'}, \mathbf{U}_{(i),l}^{'}, \mathbf{V}_{(i),l}^{'}) - \nabla_{\mathbf{V}_{(i),g}}\tilde{f}_i(\mathbf{U}_g, \mathbf{V}_{(i),g}, \mathbf{U}_{(i),l}, \mathbf{V}_{(i),l})\right\|_F \\
&\leq L_\ell B_1\left\|\Delta\mathbf{M}_{(i)}^{'}\right\|_F + B_4\left\|\mathbf{U}_{(i),g}^{'} - \mathbf{U}_{(i),g}\right\|_F
\end{aligned}$$

And,

$$\begin{aligned}
&\left\|\nabla_{\mathbf{U}_{(i),l}}\tilde{f}_i(\mathbf{U}_g^{'}, \mathbf{V}_{(i),g}^{'}, \mathbf{U}_{(i),l}^{'}, \mathbf{V}_{(i),l}^{'}) - \nabla_{\mathbf{U}_{(i),l}}\tilde{f}_i(\mathbf{U}_g, \mathbf{V}_{(i),g}, \mathbf{U}_{(i),l}, \mathbf{V}_{(i),l})\right\|_F \\
&\leq L_\ell B_2\left\|\Delta\mathbf{M}_{(i)}^{'}\right\|_F + B_4\left\|\mathbf{V}_{(i),l}^{'} - \mathbf{V}_{(i),l}\right\|_F
\end{aligned}$$

Also,

$$\begin{aligned}
&\left\|\nabla_{\mathbf{V}_{(i),l}}\tilde{f}_i(\mathbf{U}_g^{'}, \mathbf{V}_{(i),g}^{'}, \mathbf{U}_{(i),l}^{'}, \mathbf{V}_{(i),l}^{'}) - \nabla_{\mathbf{V}_{(i),l}}\tilde{f}_i(\mathbf{U}_g, \mathbf{V}_{(i),g}, \mathbf{U}_{(i),l}, \mathbf{V}_{(i),l})\right\|_F \\
&\leq L_\ell B_1\left\|\Delta\mathbf{M}_{(i)}^{'}\right\|_F + B_4\left\|\mathbf{U}_{(i),l}^{'} - \mathbf{U}_{(i),l}\right\|_F
\end{aligned}$$

Combining the 4 inequalities and the fact that $\left\|\Delta \mathbf{M}'_{(i)}\right\|_F \leq B_2 \left\|\mathbf{U}_g - \mathbf{U}'_g\right\|_F + B_1 \left\|\mathbf{V}_{(i),g} - \mathbf{V}'_{(i),g}\right\|_F + B_2 \left\|\mathbf{U}_{(i),l} - \mathbf{U}'_{(i),l}\right\|_F + B_1 \left\|\mathbf{V}_{(i),l} - \mathbf{V}'_{(i),l}\right\|_F$**, we have:**

$$
\begin{aligned}
&\left\|\nabla \tilde{f}_i(\mathbf{U}'_g, \mathbf{V}'_{(i),g}, \mathbf{U}'_{(i),l}, \mathbf{V}'_{(i),l}) - \nabla \tilde{f}_i(\mathbf{U}_g, \mathbf{V}_{(i),g}, \mathbf{U}_{(i),l}, \mathbf{V}_{(i),l})\right\|_F^2 \\
&= \left\|\nabla_{\mathbf{U}_g} \tilde{f}_i(\mathbf{U}'_g, \mathbf{V}'_{(i),g}, \mathbf{U}'_{(i),l}, \mathbf{V}'_{(i),l}) - \nabla_{\mathbf{U}_g} \tilde{f}_i(\mathbf{U}_g, \mathbf{V}_{(i),g}, \mathbf{U}_{(i),l}, \mathbf{V}_{(i),l})\right\|_F^2 \\
&+ \left\|\nabla_{\mathbf{V}_{(i),g}} \tilde{f}_i(\mathbf{U}'_g, \mathbf{V}'_{(i),g}, \mathbf{U}'_{(i),l}, \mathbf{V}'_{(i),l}) - \nabla_{\mathbf{V}_{(i),g}} \tilde{f}_i(\mathbf{U}_g, \mathbf{V}_{(i),g}, \mathbf{U}_{(i),l}, \mathbf{V}_{(i),l})\right\|_F^2 \\
&+ \left\|\nabla_{\mathbf{U}_{(i),l}} \tilde{f}_i(\mathbf{U}'_g, \mathbf{V}'_{(i),g}, \mathbf{U}'_{(i),l}, \mathbf{V}'_{(i),l}) - \nabla_{\mathbf{U}_{(i),l}} \tilde{f}_i(\mathbf{U}_g, \mathbf{V}_{(i),g}, \mathbf{U}_{(i),l}, \mathbf{V}_{(i),l})\right\|_F^2 \\
&+ \left\|\nabla_{\mathbf{V}_{(i),l}} \tilde{f}_i(\mathbf{U}'_g, \mathbf{V}'_{(i),g}, \mathbf{U}'_{(i),l}, \mathbf{V}'_{(i),l}) - \nabla_{\mathbf{V}_{(i),l}} \tilde{f}_i(\mathbf{U}_g, \mathbf{V}_{(i),g}, \mathbf{U}_{(i),l}, \mathbf{V}_{(i),l})\right\|_F^2 \\
&\leq L_{gen}^2 \left(\left\|\mathbf{U}'_g - \mathbf{U}_g\right\|_F^2 + \left\|\mathbf{V}'_{(i),g} - \mathbf{V}_{(i),g}\right\|_F^2 + \left\|\mathbf{U}'_{(i),l} - \mathbf{U}_{(i),l}\right\|_F^2 + \left\|\mathbf{V}'_{(i),l} - \mathbf{V}_{(i),l}\right\|_F^2 \right)
\end{aligned}
$$

where L_{gen} **is a constant defined as,**

$$
L_{gen} = \sqrt{8\left(B_2^2 + B_1^2\right) L_\ell^2 \max\{B_1^2, B_2^2\} + 2B_4^2} = O(L_\ell)
$$

□

With the established Lipschits continuity of the objective, we can prove the convergence of Algorithm 1. Theorem 5 is a formal statement of such convergence guarantee.

Theorem 5 (Formal version of Theorem 3 in the main paper) ***Under the following conditions,***

1. ***There exists conctants*** $B_1, B_2, B_4 > 0$ ***such that the iterates are bounded,*** $(\mathbf{U}_g\{\mathbf{V}_{(i),g}, \mathbf{U}_{(i),l}, \mathbf{V}_{(i),l}\} \in \mathcal{B}\left(B_1, B_2\right)$ ***and the gradient norm is alsop upper bounded*** $\ell' \leq B_4$***.***

2. ℓ ***is*** L_ℓ***-Lipschitz continuous and lower bounded by a constant.***

3. ***The constant stepsize*** η ***is upper bounded by*** $\eta \leq \frac{1}{L_{gen}} = O(\frac{1}{L_\ell})$***.***

then for Algorithm **with** $\beta = 0$**, the following holds,**

$$
\min_{\tau \in [1, \cdots, T]} \left\|\nabla \tilde{f}\left(\mathbf{U}_{g,\tau}, \{\mathbf{V}_{(i),g,\tau}, \mathbf{U}_{(i),l,\tau}, \mathbf{V}_{(i),l,\tau}\}\right)\right\|_F^2 = O\left(\frac{1}{T}\right) \tag{42}
$$

Proof. The proof is straightforward given Lemma D.2. As $\beta = 0$**, we have,**

$$
\begin{aligned}
&\tilde{f}_i(\mathbf{U}_{g,\tau+1}, \mathbf{V}_{(i),g,\tau+1}, \mathbf{U}_{(i),l,\tau+1}, \mathbf{V}_{(i),l,\tau+1}) \\
&= \ell\left(\mathbf{M}_{(i)}, \mathbf{U}_{g,\tau+1}\mathbf{V}_{(i),g,\tau+1}^T + \mathbf{U}_{(i),l,\tau+1}\mathbf{V}_{(i),l,\tau+1}^T\right) \\
&= \ell\Big(\mathbf{M}_{(i)}, \mathbf{U}_{g,\tau+1}\left(\mathbf{V}_{(i),g,\tau+\frac{1}{2}} + \mathbf{V}_{(i),l,\tau+1}\mathbf{U}_{(i),l,\tau+\frac{1}{2}}^T \mathbf{U}_{g,\tau+1}\left(\mathbf{U}_{g,\tau+1}^T\mathbf{U}_{g,\tau+1}\right)^{-1}\right)^T \\
&+ \left(\mathbf{U}_{(i),l,\tau+\frac{1}{2}} - \mathbf{U}_{g,\tau+1}\left(\mathbf{U}_{g,\tau+1}^T\mathbf{U}_{g,\tau+1}\right)^{-1}\mathbf{U}_{g,\tau+1}^T\mathbf{U}_{(i),l,\tau+\frac{1}{2}}\right)\mathbf{V}_{(i),l,\tau+1}^T\Big) \\
&= \ell\left(\mathbf{M}_{(i)}, \mathbf{U}_{g,\tau+1}\mathbf{V}_{(i),g,\tau+\frac{1}{2}}^T + \mathbf{U}_{(i),l,\tau+\frac{1}{2}}\mathbf{V}_{(i),l,\tau+1}^T\right) \\
&= \tilde{f}_i(\mathbf{U}_{g,\tau+1}, \mathbf{V}_{(i),g,\tau+\frac{1}{2}}, \mathbf{U}_{(i),l,\tau+\frac{1}{2}}, \mathbf{V}_{(i),l,\tau+1})
\end{aligned}
$$

Combining this and Lemma D.2, we have

$$
\begin{aligned}
&\tilde{f}_i(\mathbf{U}_{g,\tau+1}, \mathbf{V}_{(i),g,\tau+1}, \mathbf{U}_{(i),l,\tau+1}, \mathbf{V}_{(i),l,\tau+1}) - \tilde{f}_i(\mathbf{U}_{g,\tau}, \mathbf{V}_{(i),g,\tau}, \mathbf{U}_{(i),l,\tau}, \mathbf{V}_{(i),l,\tau}) \\
&= \tilde{f}_i(\mathbf{U}_{g,\tau+1}, \mathbf{V}_{(i),g,\tau+\frac{1}{2}}, \mathbf{U}_{(i),l,\tau+\frac{1}{2}}, \mathbf{V}_{(i),l,\tau+1}) - \tilde{f}_i(\mathbf{U}_{g,\tau}, \mathbf{V}_{(i),g,\tau}, \mathbf{U}_{(i),l,\tau}, \mathbf{V}_{(i),l,\tau}) \\
&\leq \left\langle \nabla_{\mathbf{U}_g} \tilde{f}_i, \mathbf{U}_{g,\tau+1} - \mathbf{U}_{g,\tau} \right\rangle + \left\langle \nabla_{\mathbf{V}_{(i),g}} \tilde{f}_i, \mathbf{V}_{(i),g,\tau+\frac{1}{2}} - \mathbf{V}_{(i),g,\tau} \right\rangle \\
&+ \left\langle \nabla_{\mathbf{U}_{(i),l}} \tilde{f}_i, \mathbf{U}_{(i),l,\tau+\frac{1}{2}} - \mathbf{U}_{(i),l,\tau} \right\rangle + \left\langle \nabla_{\mathbf{V}_{(i),l}} \tilde{f}_i, \mathbf{V}_{(i),l,\tau+1} - \mathbf{V}_{(i),l,\tau} \right\rangle \\
&+ \frac{L_{gen}}{2} \left(\|\mathbf{U}_{g,\tau+1} - \mathbf{U}_{g,\tau}\|_F^2 + \left\| \mathbf{V}_{(i),g,\tau+\frac{1}{2}} - \mathbf{V}_{(i),g,\tau} \right\|_F^2 + \left\| \mathbf{U}_{(i),l,\tau+\frac{1}{2}} - \mathbf{U}_{(i),l,\tau} \right\|_F^2 + \left\| \mathbf{V}_{(i),l,\tau+1} - \mathbf{V}_{(i),l,\tau} \right\|_F^2 \right) \\
&\leq -\eta \left(\left\langle \nabla_{\mathbf{U}_g} \tilde{f}_i, \nabla_{\mathbf{U}_g} \frac{\tilde{f}}{N} \right\rangle + \left\| \nabla_{\mathbf{U}_{(i),l}} \tilde{f}_i \right\|_F^2 + \left\| \nabla_{\mathbf{V}_{(i),g}} \tilde{f}_i \right\|_F^2 + \left\| \nabla_{\mathbf{V}_{(i),l}} \tilde{f}_i \right\|_F^2 \right) \\
&+ \eta^2 \frac{L_{gen}}{2} \left(\left\| \nabla_{\mathbf{U}_g} \frac{\tilde{f}}{N} \right\|_F^2 + \left\| \nabla_{\mathbf{U}_{(i),l}} \tilde{f}_i \right\|_F^2 + \left\| \nabla_{\mathbf{V}_{(i),g}} \tilde{f}_i \right\|_F^2 + \left\| \nabla_{\mathbf{V}_{(i),l}} \tilde{f}_i \right\|_F^2 \right)
\end{aligned}
$$

Summing both side for i from 1 to N, we have:

$$
\begin{aligned}
&\tilde{f}(\mathbf{U}_{g,\tau+1}, \{\mathbf{V}_{(i),g,\tau+1}, \mathbf{U}_{(i),l,\tau+1}, \mathbf{V}_{(i),l,\tau+1}\}) - \tilde{f}(\mathbf{U}_{g,\tau}, \{\mathbf{V}_{(i),g,\tau}, \mathbf{U}_{(i),l,\tau}, \mathbf{V}_{(i),l,\tau}\}) \\
&= \sum_{i=1}^{N} \tilde{f}_i(\mathbf{U}_{g,\tau+1}, \mathbf{V}_{(i),g,\tau+1}, \mathbf{U}_{(i),l,\tau+1}, \mathbf{V}_{(i),l,\tau+1}) - \tilde{f}_i(\mathbf{U}_{g,\tau}, \mathbf{V}_{(i),g,\tau}, \mathbf{U}_{(i),l,\tau}, \mathbf{V}_{(i),l,\tau}) \\
&\leq -\eta \left(\left\| \frac{1}{\sqrt{N}} \nabla_{\mathbf{U}_g} \tilde{f} \right\|_F^2 + \sum_{i=1}^{N} \left\| \nabla_{\mathbf{V}_{(i),g}} \tilde{f}_i \right\|_F^2 + \left\| \nabla_{\mathbf{U}_{(i),l}} \tilde{f}_i \right\|_F^2 + \left\| \nabla_{\mathbf{V}_{(i),l}} \tilde{f}_i \right\|_F^2 \right) \\
&+ \eta^2 \frac{L_{gen}}{2} \Big(\left\| \frac{1}{\sqrt{N}} \nabla_{\mathbf{U}_g} \tilde{f} \right\|_F^2 + \sum_{i=1}^{N} \left\| \nabla_{\mathbf{V}_{(i),g}} \tilde{f}_i \right\|_F^2 \\
&+ \left\| \nabla_{\mathbf{U}_{(i),l}} \tilde{f}_i \right\|_F^2 + \left\| \nabla_{\mathbf{V}_{(i),l}} \tilde{f}_i \right\|_F^2 \Big)
\end{aligned}
$$

Therefore, when $\eta \leq \frac{1}{L_{gen}}$, we have:

$$
\begin{aligned}
&\tilde{f}(\mathbf{U}_{g,\tau+1}, \{\mathbf{V}_{(i),g,\tau+1}, \mathbf{U}_{(i),l,\tau+1}, \mathbf{V}_{(i),l,\tau+1}\}) - \tilde{f}(\mathbf{U}_{g,\tau}, \{\mathbf{V}_{(i),g,\tau}, \mathbf{U}_{(i),l,\tau}, \mathbf{V}_{(i),l,\tau}\}) \\
&\leq -\frac{\eta}{2} \left(\left\| \frac{1}{\sqrt{N}} \nabla_{\mathbf{U}_g} \tilde{f} \right\|_F^2 + \sum_{i=1}^{N} \left\| \nabla_{\mathbf{V}_{(i),g}} \tilde{f}_i \right\|_F^2 + \left\| \nabla_{\mathbf{U}_{(i),l}} \tilde{f}_i \right\|_F^2 + \left\| \nabla_{\mathbf{V}_{(i),l}} \tilde{f}_i \right\|_F^2 \right)
\end{aligned}
$$

Summing up both sides for τ from 1 to T, we have:

$$
\begin{aligned}
&\sum_{\tau=1}^{T} \left(\left\| \frac{1}{\sqrt{N}} \nabla_{\mathbf{U}_g} \tilde{f} \right\|_F^2 + \sum_{i=1}^{N} \left\| \nabla_{\mathbf{V}_{(i),g}} \tilde{f}_i \right\|_F^2 + \left\| \nabla_{\mathbf{U}_{(i),l}} \tilde{f}_i \right\|_F^2 + \left\| \nabla_{\mathbf{V}_{(i),l}} \tilde{f}_i \right\|_F^2 \right) \\
&\leq \frac{2}{\eta} \left(\tilde{f}(\mathbf{U}_{g,1}, \{\mathbf{V}_{(i),g,1}, \mathbf{U}_{(i),l,1}, \mathbf{V}_{(i),l,1}\}) - \tilde{f}(\mathbf{U}_{g,T+1}, \{\mathbf{V}_{(i),g,T+1}, \mathbf{U}_{(i),l,T+1}, \mathbf{V}_{(i),l,T+1}\}) \right)
\end{aligned}
$$

As $\tilde{f}(\mathbf{U}_{g,T+1}, \{\mathbf{V}_{(i),g,T+1}, \mathbf{U}_{(i),l,T+1}, \mathbf{V}_{(i),l,T+1}\})$ is lower bounded by a constant, the right hand $\frac{2}{\eta} \left(\tilde{f}(\mathbf{U}_{g,1}, \{\mathbf{V}_{(i),g,1}, \mathbf{U}_{(i),l,1}, \mathbf{V}_{(i),l,1}\}) - \tilde{f}(\mathbf{U}_{g,T+1}, \{\mathbf{V}_{(i),g,T+1}, \mathbf{U}_{(i),l,T+1}, \mathbf{V}_{(i),l,T+1}\}) \right)$ is upper bounded by a constant. Hence,

$$
\min_{\tau \in \{1, \cdots, T\}} \left(\left\| \frac{1}{\sqrt{N}} \nabla_{\mathbf{U}_g} \tilde{f} \right\|_F^2 + \sum_{i=1}^{N} \left\| \nabla_{\mathbf{V}_{(i),g}} \tilde{f}_i \right\|_F^2 + \left\| \nabla_{\mathbf{U}_{(i),l}} \tilde{f}_i \right\|_F^2 + \left\| \nabla_{\mathbf{V}_{(i),l}} \tilde{f}_i \right\|_F^2 \right) = O\left(\frac{1}{T} \right)
$$

This completes the proof. □

E Auxiliary Lemmas

This section discusses some helper lemmas useful for our main proofs. These lemmas are mostly derived from basic linear algebra.

Lemma E.1 ***For a symmetric matrix*** $\mathbf{A} \in \mathbb{R}^{r\times r}$***, if*** $\|\mathbf{A}\|_F \leq \frac{3}{4}$***, we have,***

$$\left\|\mathbf{I} - (\mathbf{I}+\mathbf{A})^{-1}\right\|_F \leq 4\|\mathbf{A}\|_F$$

Proof. We have

$$\left\|\mathbf{I} - (\mathbf{I}+\mathbf{A})^{-1}\right\|_F = \left\|(\mathbf{I}+\mathbf{A})^{-1}(-\mathbf{A})\right\|_F \leq (1-\|\mathbf{A}\|_F)^{-1}\|\mathbf{A}\|_F \leq 4\|\mathbf{A}\|_F$$

□

The next lemma presents a similar result.

Lemma E.2 ***For a symmetric matrix*** $\mathbf{A} \in \mathbb{R}^{r\times r}$***, if*** $\|\mathbf{A}\|_F \leq \frac{3}{4}$***, we have,***

$$\left\|\mathbf{I} - (\mathbf{I}+\mathbf{A})^{-\frac{1}{2}}\right\|_F \leq \frac{4\|\mathbf{A}\|_F}{3}$$

Proof. Since $\|\mathbf{A}\|_F \leq \frac{3}{4} < 1$, we can use the series

$$\begin{aligned}
\left\|\mathbf{I} - (\mathbf{I}+\mathbf{A})^{-\frac{1}{2}}\right\|_F &= \left\|\sum_{n=1}^{\infty}\frac{(2n-1)!!(-1)^n}{2^n n!}\mathbf{A}^n\right\|_F \\
&\leq \sum_{n=1}^{\infty}\frac{(2n-1)!!(-1)^n}{2^n n!}\|\mathbf{A}\|_F^n = (1-\|\mathbf{A}\|_F)^{-\frac{1}{2}} - 1 \\
&\leq \frac{\|\mathbf{A}\|_F}{\sqrt{1-\|\mathbf{A}\|_F}\left(\sqrt{1-\|\mathbf{A}\|_F}+1\right)} \leq \frac{4\|\mathbf{A}\|_F}{3}
\end{aligned}$$

□

Lemma E.3 ***For two matrices*** $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{r\times r}$***, we have,***

$$\|\mathbf{A}-\mathbf{B}\|_F^2 \geq \frac{1}{2}\|\mathbf{A}\|_F^2 - \|\mathbf{B}\|_F^2$$

Proof. We have

$$\|\mathbf{A}-\mathbf{B}\|_F^2 = \|\mathbf{A}\|_F^2 + \|\mathbf{B}\|_F^2 + 2\langle\mathbf{A},\mathbf{B}\rangle \geq \|\mathbf{A}\|_F^2 + \|\mathbf{B}\|_F^2 - \left(\frac{1}{2}\|\mathbf{A}\|_F^2 + 2\|\mathbf{B}\|_F^2\right) \geq \frac{1}{2}\|\mathbf{A}\|_F^2 - \|\mathbf{B}\|_F^2$$

□

Lemma E.4 ***For two matrices*** $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{r\times r}$***, if*** A ***is invertable and*** $\left\|\mathbf{A}^{-1}\right\|\|\mathbf{B}\| < 1$***, we have,***

$$\left\|(\mathbf{A}+\mathbf{B})^{-1}\right\| \leq \left\|\mathbf{A}^{-1}\right\| + \frac{\left\|\mathbf{A}^{-1}\right\|^2\|\mathbf{B}\|}{1-\|\mathbf{A}^{-1}\|\|\mathbf{B}\|}$$

Proof. We have

$$\left\|(\mathbf{A}+\mathbf{B})^{-1} - \mathbf{A}^{-1}\right\| = \left\|\left(\left(\mathbf{I}+\mathbf{B}\mathbf{A}^{-1}\right)^{-1} - \mathbf{I}\right)\mathbf{A}^{-1}\right\| \leq \left\|\mathbf{A}^{-1}\right\|\frac{\left\|\mathbf{A}^{-1}\right\|\|\mathbf{B}\|}{1-\|\mathbf{A}^{-1}\|\|\mathbf{B}\|}.$$

The proof is completed by invoking triangle inequality. □

The following lemma is a well-known result and provides an upper bound on the norm of product matrices.

Lemma E.5 ***For two matrices*** $\mathbf{A} \in \mathbb{R}^{m \times n}$ ***and*** $\mathbf{B} \in \mathbb{R}^{n \times p}$***, we have,***

$$\|\mathbf{AB}\|_F \leq \|\mathbf{A}\|_2 \|\mathbf{B}\|_F$$

and

$$\|\mathbf{AB}\|_2 \leq \|\mathbf{A}\|_2 \|\mathbf{B}\|_2$$

Proof.

The proof can be found in Sun & Luo (2016). □