# Fixed-Point Reasoning: Stable and Adaptive Deep Looped Transformers

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## Abstract

Looped architectures provide an inductive bias toward learning step-by-step procedures for tasks that require compositional reasoning. The depth of effective layers reached by looping determines the quality of the solution these models find. Similar to deep architectures, looped architectures are prone to a signal propagation problem induced by depth as the halting decision is postponed. In this paper, we address the signal propagation issue by using pre-norm layers and residual scaling. Building on these architectural modifications, we propose **FPRM**: a Transformer-based **F**ixed-**P**oint **R**easoning **M**odel that uses fixed-point convergence as an end-to-end halting mechanism in a looped architecture. We show that fixed-point halting allows FPRM to adapt its compute to the difficulty of the task. FPRM proves effective on common reasoning benchmarks, namely Sudoku, Maze, state-tracking and ARC-AGI. The implementation can be found here.

## 1. Introduction

Reasoning in neural networks has increasingly been framed as a problem of scaling test-time compute: a model should be able to spend more computation on inputs it finds harder [52, 60]. However, doing so requires two ingredients. **(1) flexibility:** the possibility of spending a variable amount of compute on the problem. Once the model is flexible, the next step is **(2) adaptivity:** a way to scale the compute spent on the problem; i.e., when to halt the computation. The standard way to achieve both is through a Chain-of-Thought (CoT) mechanism [66]. With CoT, the model scales compute through verbalization, and makes halting decisions based on predicting a specialized halting token. However, this emerging behavior requires a special training regime and hand-crafted reasoning traces [33]. This makes the method complex and undermines the desirable property of end-to-end training.

An alternative approach to end-to-end training for reasoning models is emerging in the form of looped architectures [8, 15, 34, 58]. Whereas in CoT, the compute scales along the sequence dimension, in looped architectures it scales along the depth dimension ($i$):

$$\mathbf{z}_{i+1} = f_\theta(\mathbf{z}_i; \mathbf{x}), \tag{1}$$

where $f_\theta(\mathbf{z}; \mathbf{x})$ is a neural network, $\mathbf{x}$ is the input, and $\mathbf{z}_i$ is the $i^{\text{th}}$ latent representation. Thus, compute can be increased at test-time by looping in depth, naturally introducing flexibility from the architecture.

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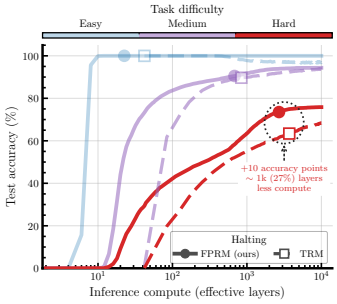
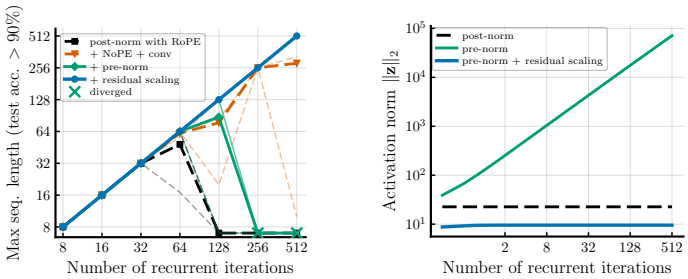



*Figure 1:* **Signal propagation and adaptivity, FPRM vs. TRM:** Sudoku-Extreme performance as a function of compute across difficulty.




**(a)** Trainability vs. context length

**(b)** Activation norm at init.

*Figure 2:* **The blessing and the curse of depth in Looped Transformers.** Increasing the number of effective layers can unlock expressivity, but also creates a stability challenge: pre-norm models without residual scaling can diverge in activation norm, while post-norm models may struggle to utilize the signal.

Deciding when to stop iterating, however, is not trivial. Most approaches either fix or sample the number of loops [8, 12, 27, 38, 58, 69], sacrificing adaptivity, or attach an Adaptive Computation Time (ACT) module [15, 32, 39, 65], which is hard to optimize as it relaxes a discrete decision and can still fail to adapt, as we show for HRM and TRM (Figures 3, 9). We instead let the model loop until its hidden state converges to a fixed-point, using the convergence itself as the halting signal. Unlike ACT, this needs no external module and lets the model spend as much compute as an input demands.

A further challenge arises as the number of loops in the model increases. This is desirable for harder tasks, but unrolling the recursions yields a deep effective layer. As a result, looped architectures suffer from the same **signal propagation** issue as deep non-looped networks. To this end, we adapt architectural techniques originally developed to optimize traditional transformers at deep effective layers, carefully modifying them for the Looped Transformer setting. Our key intuition is that *Looped Transformers are, in part, very deep transformers*.

Viewed this way, one design choice is surprising: looped models commonly use post-norm [15, 27, 39, 65]. Deep non-looped architectures, by contrast, prefer pre-norm, since post-norm causes unstable training [51, 67]. Yet in looped models, post-norm serves a specific purpose. It keeps the magnitude of activations bounded as the model iterates, preventing the hidden states from diverging [45] (see Figure 2). This raises the question: *can we switch the post-norm to pre-norm, while ensuring the activations stay bounded in a different way?* We answer affirmatively using residual scaling. Together with fixed-point halting, this lets FPRM, a non-hierarchical model, outperform TRM at lower cost on Sudoku-Extreme (Figure 1; described in Section E). Related work is discussed in Section A.

We summarize our contributions as follows:

1. **Successfully training a looped pre-norm Transformer.** We modify a Transformer layer to be trainable over deeper effective layers by switching post-norm to pre-norm and adding residual scaling parameters.

2. **Reaching a fixed-point as a halting mechanism.** To enable stable training, we propose a theoretically motivated modification specific to fixed-point models to limit the oscillation around the fixed-point.

3. **Proposing the framework FPRM: Fixed-Point Reasoning Model.** We show that it outperforms the baselines on Sudoku-Extreme, Maze-Hard, ARC-AGI-1, and state-tracking benchmarks $A_5$ and $S_5$ among 7M parameter models. Notably, we achieve our results without the hierarchical structure of HRM and TRM. To the best of our knowledge, FPRM is the first Transformer-based reasoning model that exhibits adaptivity of compute to task difficulty (Figures 1,3).

## 2. Method

In this section, we provide a description of our proposed method. We switch to pre-norm (Section 2.1), restore boundedness of activations via residual scaling (Section 2.2) and assemble these into FPRM with fixed-point halting (Section 2.3). An overview of the architecture can be found in Figure 5.

### 2.1. Improving signal propagation with pre-norm

We provide an overview of the pre- and post-norm Transformer layer in Section D.1 and how the placement of layer normalizations influence signal propagation. In Section C, we provide evidence for trainability issues in a toy looped model with post-norm, despite stability induced by bounded activations. Moreover, we investigate both normalization placements in Looped Transformers [15, 58] also on the state-tracking task. In Figure 1(a), we test the positive correlation between the effective layer and the expressivity (maximum sequence length with $> 90\%$ test accuracy) of a Looped Transformer for the state-tracking task $A_5$ [49]. We observe that increasing the effective layer of a Looped Transformer with post-norm does not translate into improved expressivity, while a pre-norm variant diverges at larger depth. This divergence can be attributed to the exponential growth of the activations, apparent in Figure 1(b). Therefore, to use pre-norm and improve signal propagation in deep looped models, we must first stabilize it, which we do in the following.

### 2.2. Recovering boundedness via residual scaling

While pre-norm mitigates trainability problems in looped architectures, it removes the necessary boundedness condition that motivated the use of post-norm in them. This effect can be observed in Figure 1(b), where the activations of the pre-norm model grow with deeper effective layer, causing trainability issues observable in Figure 1(a). Consequently, we propose to restore boundedness by introducing scaling parameters applied at two different scales: one over each sub-layer of the network, and one across the iterations.

**Layer-wise residual scaling.** Within a single application of $f_\theta(\mathbf{z};\mathbf{x})$, the residual stream and sub-layer output $f^\ell_{\theta^\ell}(\mathbf{z}^{\ell-1})$ are weighted by tied scalars $(\alpha_1, \beta_1)$ shared across all $L$ layers:

$$\mathbf{z}^\ell = \alpha_1\, \mathbf{z}^{\ell-1} + \beta_1\, f^\ell_{\theta^\ell}\Big(\mathrm{Norm}_{\mathrm{pre}}\Big(\mathbf{z}^{\ell-1}\Big)\Big), \qquad \ell = 1, \ldots, 2L. \tag{2}$$

**Iteration-wise input mixing.** Between consecutive applications of $f_\theta(\mathbf{z}; \mathbf{x})$, we re-inject the input $\mathbf{x}$ with tied scalars $(\alpha_2, \beta_2)$ shared across all iterations [5]:

$$\mathbf{z}_{i+1}^0 = \alpha_2 \, \mathbf{z}_i^{2L} + \beta_2 \, \mathbf{x}. \tag{3}$$

The two scaling schemes are not independent. With an appropriate coupling between them, the resulting recurrence is *bounded for any input*, resulting in a stable looping [53]. In the following statement, we formalize this claim:

**Theorem 1 (Boundedness of FPRM iterates)** *Consider the model defined by Equations 2 and 3, and assume each layer map satisfies* $\|f_{\theta^\ell}^\ell(\mathbf{u})\| \leq c_f$ *for all* $\ell$ *and input* $\mathbf{u}$. *Let* $0 \leq \alpha_1, \alpha_2 < 1$, *and set*

$$\beta_2 = 1 - \alpha_2 \alpha_1^{2L}, \qquad \beta_1 = \frac{\beta_2 \, (1 - \alpha_1)}{1 - \alpha_1^{2L}}.$$

*Then the fixed-point iterates* $\{\mathbf{z}_i^0\}_{i \geq 0}$ *from Equation (3) are bounded, and if* $\mathbf{z}_i^0 \to \mathbf{z}_\infty^0$, *then*

$$\|\mathbf{z}_\infty^0\| \leq \|\mathbf{x}\| + \alpha_2 \, c_f.$$

The proof is in Section B.1.

Note that the boundedness condition of the sequence model in Theorem 1 is satisfied when using pre-norm, as shown by Kim et al. [42]. Together, these modifications yield a pre-norm Looped Transformer that maintains performance over longer looping horizons before saturating, compared to the post-norm variant (see Figure 2). In Section D.2, we investigate the effect of residual scaling on the convergence of fixed-point iterations.

### 2.3. The fixed-point reasoning model

So far, we introduced modifications to improve signal propagation in Looped Transformers (Section 2.1) without sacrificing stability (Section 2.2). These modifications yield fixed-point iterations that can be made non-oscillatory (Section D.3) and trainable through truncated-BPTT (Section D.4), making adaptivity through fixed-points reliable. We assemble these components into FPRM, summarized in Algorithm 2 and illustrated in Figure 5. The result is a Looped Transformer that iterates until its hidden state converges, spending compute proportional to each input's difficulty—the adaptivity our halting mechanism was designed to provide. Since we observed no improvements in our experiments by using the hierarchical structure of Wang et al. [65], we opt for a classic looped architecture [15] instead. In Section D we provide further details about FPRM.

## 3. Experiments

We evaluate FPRM against looped reasoning models on puzzle, adaptivity, and signal-propagation benchmarks. Our implementation builds on the public TRM codebase [39] and adopts its deep-supervision training procedure. Additional experimental details are provided in Section G, includ-

ing the descriptions of the datasets. We first evaluate puzzle-solving performance, then adaptivity to task difficulty, and finally depth-induced signal-propagation effects.

*Table 1:* Test accuracy on Sudoku-Extreme, Maze-Hard, ARC-AGI-1, and ARC-AGI-2. For each task, the **best overall** result is bold-face, and the best result with 7M parameters is underlined. The results denoted by $^{\dagger}$ are reproduced using public checkpoints. The ARC results for URM, denoted by $^{\ddagger}$, are only reported for Pass@1.

| **Model** | **# params.** | **Single Loop (No Hier.)** | **Sudoku-Ext. Pass@1** | **Maze-Hard Pass@1** | **ARC-1 Pass@2** | **ARC-2 Pass@2** |
|---|---|---|---|---|---|---|
| *Our reproduction attempt* | | | | | | |
| Attractor Model | 27M | ✗ | 71.4% | – | – | – |
| TRM | 7M | ✗ | 72.6% | 79.0% | 40.0%$^{\dagger}$ | 6.2%$^{\dagger}$ |
| *Reported* | | | | | | |
| HRM | 27M | ✗ | 55.0% | 74.5% | 40.3% | 5.0% |
| Attractor Model | 27M | ✗ | 91.4% | **93.1%** | – | – |
| URM | 14M | ✗ | 77.6% | – | **≥53.8%**$^{\ddagger}$ | **≥16.0%**$^{\ddagger}$ |
| TRM | 7M | ✗ | 74.7% | 85.3% | 44.6% | 7.8% |
| EqR | 7M | ✗ | 93.0% | – | – | – |
| Attractor Model | 7M | ✗ | 54.3% | 46.7% | – | – |
| FPRM | 7M | ✓ | **94.2%** | 87.0% | 47.5% | 6.2% |

### 3.1. Puzzle tasks

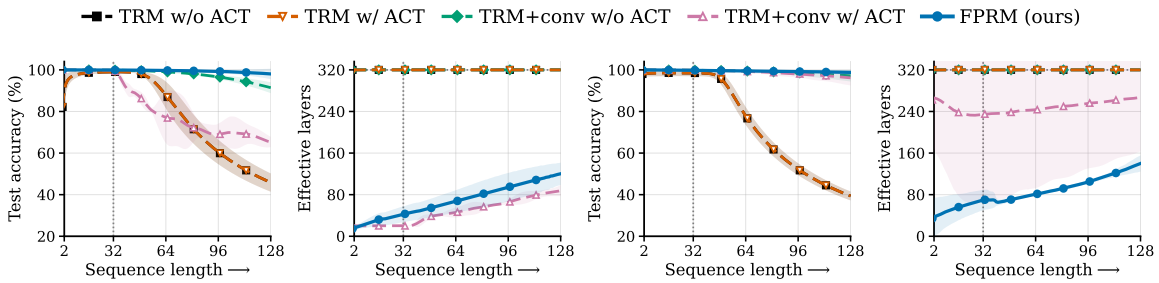
We first evaluate FPRM on puzzle problems, namely the Sudoku-Extreme, Maze-Hard [65], ARC-1, and ARC-2 [14] benchmarks. These benchmarks were designed to test whether latent recurrent reasoning models can solve symbolic search problems from limited supervision. In the following, we discuss the experimental results. For more information about the baselines, we refer the reader to Section A.

Table 1 demonstrates the effectiveness of FPRM as the best performing model with 7M parameters on Sudoku-Extreme, Maze-Hard, and ARC-1, with performance on par with TRM on ARC-2. On Sudoku-Extreme, FPRM improves upon even larger models. It is worth noting that these results are achieved without the breadth-search fixed-point method proposed by Huang et al. [37], which is orthogonal to FPRM's modifications. On Maze-Hard, FPRM underperforms compared to the larger Attractor model with ˜×4 more parameters. However, we note that we were not able to reproduce the results reported in the paper [23].

On ARC-2, FPRM performs similarly to the publicly available TRM checkpoints, but underperforms compared to the TRM results reported in [39] and the URM baseline, which has roughly twice as many parameters. However, considering the recent trend, we emphasize that the ARC benchmarks appear to be much more sensitive to parameter count compared to the other puzzle tasks considered in this section [36, 59]. Therefore, we note that the comparison may not be fair in this case.

### 3.2. Adaptivity

In this section, we demonstrate the adaptivity of FPRM, and compare it to TRM. We use three benchmarks: state-tracking benchmarks $A_5$ and $S_5$ introduced in [49] and Sudoku-Extreme from Section 3.1. The Sudoku-Extreme results are presented in Section I.2. Each benchmark provides an intuitive measure of difficulty. For Sudoku-Extreme, this is the number of empty cells, an established proxy for difficulty [55]. For state-tracking, it is the sequence length.




**(a)** Accuracy, $A_5$ **(b)** Adaptivity, $A_5$ **(c)** Accuracy, $S_5$ **(d)** Adaptivity, $S_5$

*Figure 3:* **Length generalization and adaptive compute as a function of sequence length.** Shaded bands show 95% confidence intervals over seeds. The vertical dotted line marks the training length 32. The matched compute budget is 320 effective layers.

**State tracking.** As shown in Figure 3, plain TRM fails to extrapolate: at length 128, TRM and TRM with ACT enabled both obtain $45.8\% \pm 3.9\%$ on $A_5$ and $39.4\% \pm 1.9\%$ on $S_5$. Adding a causal 1D convolution layer substantially improves length generalization, reaching $91.4\% \pm 2.3\%$ on $A_5$ and $97.2\% \pm 2.5\%$ on $S_5$. This modification is not part of the original TRM, but matches the task structure: group composition is a local left-to-right scan, $s_i = s_{i-1} \cdot g_i$, and causal convolution provides a shared translation-equivariant primitive for this operation.

However, 1D convolution does not make ACT reliable. On $A_5$, TRM+conv+ACT scales compute with sequence length but drops to $65.3\% \pm 2.2\%$, well below TRM+conv without ACT. On $S_5$, it reaches $96.2\% \pm 3.2\%$, but uses substantially more compute than FPRM; moreover, only a few seeds learn to adapt, while the others exhaust the full compute budget. In contrast, FPRM scales compute smoothly with sequence length while maintaining high accuracy, reaching $98.1\% \pm 2.2\%$ on $A_5$ and $98.8\% \pm 0.9\%$ on $S_5$ at length 128.

## 4. Conclusion

We present architectural modifications for looped fixed-point Transformers that enable the use of pre-norm, improving the model's ability to exploit deeper effective layer provided by looping. These modifications allow FPRM to outperform hierarchical baselines of similar size, such as HRM and TRM, on common symbolic reasoning benchmarks. We show that on state tracking and Sudoku-Extreme, FPRM is able to adapt its compute to the difficulty of the task. This capability stems from dynamically scaling depth through fixed-point iterations and improving signal propagation. We hope these architectural modifications and the accompanying insights will support further progress on latent reasoning models.

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**Algorithm 1** Fixed-point optimizer FPOPT: one damped step with patience-based decay

**Require:** initial damping $\eta_0$; decay $\gamma \in (0, 1)$; patience $P$

**Internal state:** $\eta \leftarrow \eta_0,\ p \leftarrow P,\ r^\star \leftarrow \infty$

**procedure** STEP($\mathbf{z}$, $\tilde{\mathbf{z}}$)
  $r \leftarrow \|\mathbf{z} - \tilde{\mathbf{z}}\|_\infty / (\|\tilde{\mathbf{z}}\|_\infty + \epsilon)$ ▷ residual
  $\mathbf{z} \leftarrow \eta\, \tilde{\mathbf{z}} + (1 - \eta)\, \mathbf{z}$ ▷ damped update
  **if** $r < r^\star$ **then**
    $r^\star \leftarrow r,\ p \leftarrow P$ ▷ progress: reset
  **else**
    $p \leftarrow p - 1$
    **if** $p \leq 0$ **and** $r > \tau$ **then**
      $\eta \leftarrow \gamma\, \eta,\ p \leftarrow P$ ▷ decay $\eta$
    **end if**
  **end if**
  **return** $\mathbf{z}$, $r$
**end procedure**

**Algorithm 2** FPRM training loop with truncated BPTT and deep supervision

**Require:** Model $f_\theta$; prediction head $h_\phi$; fixed-point optimizer FPOPT; model optimizer MODELOPT; input $\mathbf{x}$; target $\mathbf{y}$; BPTT depth $K$; initial state $\mathbf{z}_0$

$\mathbf{z} \leftarrow \mathbf{z}_0$
**while** FPOPT.CONT() **do** ▷ outer loop
  **for** $k = 1, \ldots, K$ **do** ▷ BPTT window
    $\tilde{\mathbf{z}}_k \leftarrow f_\theta(\mathbf{z}; \mathbf{x})$
    $\mathbf{z} \leftarrow$ FPOPT.STEP($\mathbf{z}$, $\tilde{\mathbf{z}}_k$)
  **end for**
  $\hat{\mathbf{y}} \leftarrow h_\phi(\mathbf{z})$ ▷ deep supervision
  $\mathcal{L} \leftarrow$ CROSSENTROPY($\hat{\mathbf{y}}$, $\mathbf{y}$)
  MODELOPT.BACKWARD($\mathcal{L}$)
  $\mathbf{z} \leftarrow \mathrm{detach}(\mathbf{z})$
**end while**

## Appendix A. Background and Related Works

**Looped models.** Looped architectures address the flexibility requirement of reasoning models by decoupling effective layer from parameter count, providing variable computation per input without scaling the number of parameters. This makes looping a natural inductive bias for tasks that can be solved by repeatedly applying local or compositional subroutines. Early examples include Neural GPUs [40] and Universal Transformers [15]; more recent works show that Looped Transformers can improve length generalization, learn algorithmic structure, and approximate much deeper untied models on reasoning tasks [22, 27, 30, 41, 44, 58, 68]. Recent recursive reasoning models such as HRM [65] and TRM [39] instantiate this principle in compact architectures. Using small networks, these models have been able to outperform prominent LLM-based reasoning models. At the core of

these methods is the idea of the need for a hierarchical looping mechanism, wherein the compute is distributed between a fast-looping component and a slow-looping component. URM [25] follows up on these works and shows that by using short-conv, performance can be further improved. However, as we describe in this paper, previous works leave two central design axes open to improvement: First, *how should a looped model decide how many iterations to run on each input at test-time?* FPRM addresses this through fixed-point halting. Second, *how to make the model utilize its deep effective layer?* FPRM does so by switching post-norm to pre-norm and adding residual scaling. Importantly, FPRM achieves better performance than TRM and HRM without using a hierarchical structure for the loops.

**Adaptive computation.** Classical ACT methods answer the adaptivity question by learning an explicit halting rule, such as halting probabilities, per-token stopping decisions, or learned distributions over computation depth [7, 15]. More recent works extend this idea by picking an adaptive depth for each token [4, 61]. Such mechanisms can, in principle, allocate more computation to harder instances, but they all add a separate learned decision process on top of the recurrent computation itself, and this process is hard to optimize because the halting decision is discrete rather than continuous. Moreover, analyses of HRM find that ACT does not always scale the inference compute with the actual difficulty of the input [26, 56], which can more generally also be the case with CoT [54], a limitation we also observe and address by instead halting when the iteration reaches a fixed-point.

**Deep equilibrium and fixed-point models.** An alternative is to use the convergence of the latent dynamics as the halting criterion. In this view, computation stops when consecutive iterates become sufficiently close $\|\mathbf{z}_{i+1} - \mathbf{z}_i\| \leq \epsilon$, which corresponds to convergence toward a fixed-point $\mathbf{z}^\star = f_\theta(\mathbf{z}^\star; \mathbf{x})$. This perspective is also supported by recent mechanistic analyses of looped language models, which observe that recurrent trajectories often converge to fixed-points, suggesting that weight-tied looped models can naturally implement an implicit fixed-point computation [11]. This view is most extensively developed in Deep Equilibrium Models [5], which replace a finite stack of layers by the equilibrium point of a weight-tied transformation. However, DEQ models usually find the fixed-point via a quasi-Newton approach such as Broyden's method [5, 13] or Anderson acceleration [1, 28] rather than fixed-point iterations, which makes them different from looped models. Moreover, DEQ models are difficult to optimize [2, 6, 39], which we address by our proposed architectural changes. Concurrent to our work, Attractor Models [23] frame the iterations of TRM as a root-finding problem similar to DEQ, which they solve with Anderson acceleration. Additionally, they propose to optionally include a larger separate network to "guess" the initial latent. Another concurrent work is Equilibrium Reasoners (EqR) [37], which also adopts a fixed-point perspective and shows that it is possible to scale the compute not only depth-wise, but also by making multiple initial guesses at training and inference time, favoring attractors with a wide basin. Our contributions are largely orthogonal: we target the signal-propagation issues that limit trainable depth, replacing post-norm with pre-norm and residual scaling, and damping the iteration to suppress oscillation around the fixed-point. Moreover, in contrast to these works we do not use the hierarchical looping of TRM.

**Score-based methods.** Diffusion Models [35, 62] and Energy-Based Models (EBMs) [46] provide another way of iterative computation. An EBM learns a scalar energy function, the core idea being that the energy represents the negative log of the unnormalized density, and solutions corre-

spond to low energy states. EBMs generate a prediction either by descending the energy landscape [9, 10] or by sampling with MCMC methods [17]. In comparison, diffusion models learn the score of the underlying distribution, which can be viewed as the gradient of a time-dependent energy function, thus the two are closely connected [19, 57]. While diffusion models typically integrate an SDE over a fixed time interval and in this sense are not adaptive, energy-based reasoning models [18, 20, 31] have an adaptive halting mechanism. In these models, the halting criterion is convergence to a local minimum of the energy landscape. Compared to these methods, we model the generative process via fixed-point iterations of a learned operator, as opposed to descending through a learned energy function. The setting of learning the operator directly is important for the theoretical analysis of our architectural modifications in Section 2.

**Signal propagation in looped models.** As unrolled looped architectures can be viewed as deep networks, they are exposed to the same signal-propagation difficulties that arise in very deep Transformers. In deep sequence models, increasing depth can make optimization harder and can prevent later layers from being effectively used, a phenomenon often discussed as the curse of depth [16, 51, 63]. Pre-norm is the standard remedy for these issues, but in a looped setting it removes a property the architecture relies on: **bounded activations** (Figure 2). Bounded activations, important for stable training, are typically enforced through post-norm [27, 39, 65]. Indeed, in our own experiments a pre-norm looped model without further modification diverges in activation norm as the iteration count grows, and fails to train at the large loop counts where this growth is most severe (Figure 2). Therefore, our FPRM combines pre-norm with residual scaling to recover bounded and stable dynamics while still allowing gradients and representations to propagate through many iterations.

## Appendix B. Proofs

### B.1. Fixed-point iterations bound

**Proof** We simplify our notation by omitting the pre-norm layer in Equation (2). We start by unrolling the computation across the $L$ layers within a single fixed-point iteration. By recursively applying the layer update, we obtain

$$\begin{aligned}
\mathbf{z}_i^{2L} &= \alpha_1 \cdot \mathbf{z}_i^{2L-1} + \beta_1 \cdot f_{\theta^{2L}}^{2L}\left(\mathbf{z}_i^{2L-1}\right) \\
&= \alpha_1^2 \cdot \mathbf{z}_i^{2L-2} + \alpha_1\beta_1 \cdot f_{\theta^{2L-1}}^{2L-1}\left(\mathbf{z}_i^{2L-2}\right) + \beta_1 \cdot f_{\theta^{2L}}^{2L}\left(\mathbf{z}_i^{2L-1}\right) \\
&= \alpha_1^3 \cdot \mathbf{z}_i^{2L-3} + \alpha_1^2\beta_1 \cdot f_{\theta^{2L-2}}^{2L-2}\left(\mathbf{z}_i^{2L-3}\right) + \alpha_1\beta_1 \cdot f_{\theta^{2L-1}}^{2L-1}\left(\mathbf{z}_i^{2L-2}\right) + \beta_1 \cdot f_{\theta^{2L}}^{2L}\left(\mathbf{z}_i^{2L-1}\right) \\
&\;\;\vdots \\
&= \alpha_1^{2L} \cdot \mathbf{z}_i^{0} + \beta_1 \cdot \left(\sum_{j=0}^{2L-1} \alpha_1^{j} \cdot f_{\theta^{2L-j}}^{2L-j}\left(\mathbf{z}_i^{2L-j-1}\right)\right).
\end{aligned}$$

Now, substituting this expression into the fixed-point update gives

$$\begin{aligned}\mathbf{z}_{i+1}^0 &= \alpha_2 \cdot \mathbf{z}_i^{2L} + \beta_2 \cdot \mathbf{x} \\ &= \alpha_2 \cdot \left( \alpha_1^{2L} \cdot \mathbf{z}_i^0 + \beta_1 \cdot \sum_{j=0}^{2L-1} \alpha_1^j \cdot f_{\theta^{2L-j}}^{2L-j} \left( \mathbf{z}_i^{2L-j-1} \right) \right) + \beta_2 \cdot \mathbf{x} \\ &= \alpha_2 \alpha_1^{2L} \cdot \mathbf{z}_i^0 + \alpha_2 \beta_1 \cdot \left( \sum_{j=0}^{2L-1} \alpha_1^j \cdot f_{\theta^{2L-j}}^{2L-j} \left( \mathbf{z}_i^{2L-j-1} \right) \right) + \beta_2 \cdot \mathbf{x}.\end{aligned}$$

For compactness, define $\rho = \alpha_2 \alpha_1^{2L}$ and $\mathbf{s}_i = \sum_{j=0}^{2L-1} \alpha_1^j \cdot f_{\theta^{2L-j}}^{2L-j} \left( \mathbf{z}_i^{2L-j-1} \right)$. Then the fixed-point iteration can be written as $\mathbf{z}_{i+1}^0 = \rho \cdot \mathbf{z}_i^0 + \alpha_2 \beta_1 \cdot \mathbf{s}_i + \beta_2 \cdot \mathbf{x}$. Unrolling this recursion over fixed-point iterations gives

$$\mathbf{z}_{i+1}^0 = \rho^{i+1} \cdot \mathbf{z}_0^0 + \beta_2 \left( \sum_{k=0}^{i} \rho^k \right) \cdot \mathbf{x} + \alpha_2 \beta_1 \left( \sum_{k=0}^{i} \rho^k \cdot \mathbf{s}_{i-k} \right).$$

Since $0 \leq \alpha_1, \alpha_2 < 1$, we have $0 \leq \rho < 1$. Therefore, the geometric series is convergent. Taking norms and using the boundedness of each layer map, we get

$$\left\| \mathbf{z}_{i+1}^0 \right\| \leq \rho^{i+1} \left\| \mathbf{z}_0^0 \right\| + \beta_2 \left( \sum_{k=0}^{i} \rho^k \right) \|\mathbf{x}\| + \alpha_2 \beta_1 \left( \sum_{k=0}^{i} \rho^k \right) \left( \sum_{j=0}^{L-1} \alpha_1^j \right) c_f.$$

Letting $i \to \infty$, the first term vanishes and the two geometric sums converge, which gives

$$\limsup_{i \to \infty} \left\| \mathbf{z}_i^0 \right\| \leq \frac{\beta_2}{1-\rho} \|\mathbf{x}\| + \frac{\alpha_2 \beta_1}{1-\rho} \left( \frac{1-\alpha_1^{2L}}{1-\alpha_1} \right) c_f.$$

Substituting back $\rho = \alpha_2 \alpha_1^{2L}$, we obtain

$$\limsup_{i \to \infty} \left\| \mathbf{z}_i^0 \right\| \leq \frac{\beta_2}{1-\alpha_2 \alpha_1^{2L}} \|\mathbf{x}\| + \frac{\alpha_2 \beta_1 \left(1-\alpha_1^{2L}\right)}{\left(1-\alpha_2 \alpha_1^{2L}\right)\left(1-\alpha_1\right)} c_f.$$

We now set $\beta_2 = 1 - \alpha_2 \alpha_1^{2L}$. This makes the coefficient of $\|\mathbf{x}\|$ equal to 1. Furthermore, setting $\beta_1 = \frac{\beta_2 (1-\alpha_1)}{\left(1-\alpha_1^{2L}\right)}$ makes the coefficient of $c_f$ equal to $\alpha_2$. Therefore,

$$\limsup_{i \to \infty} \left\| \mathbf{z}_i^0 \right\| \leq \|\mathbf{x}\| + \alpha_2 \cdot c_f.$$

In particular, if the fixed-point iteration converges to $\mathbf{z}_\infty^0$, then

$$\left\| \mathbf{z}_\infty^0 \right\| \leq \|\mathbf{x}\| + \alpha_2 \cdot c_f.$$

This completes the proof. ■

### B.2. Contractive mapping

**Proof** We first show that $f_\theta(.;\mathbf{x})$ is a contraction with respect to $\mathbf{z}$. For any $\mathbf{z}, \mathbf{z}'$, using the Lipschitzness of the $L$-layer model, we have

$$\begin{aligned}\left\|f_\theta(\mathbf{z};\mathbf{x}) - f_\theta(\mathbf{z}';\mathbf{x})\right\| &\leq \lambda_f \left\|(\alpha_2 \cdot \mathbf{z} + \beta_2 \cdot \mathbf{x}) - \big(\alpha_2 \cdot \mathbf{z}' + \beta_2 \cdot \mathbf{x}\big)\right\| \\ &= \alpha_2 \lambda_f \left\|\mathbf{z} - \mathbf{z}'\right\|.\end{aligned}$$

Therefore, if $0 \leq \alpha_2\lambda_f < 1$, the map $f_\theta(.;\mathbf{x})$ is strictly contractive. By the Banach fixed-point theorem, it has a unique fixed-point $\mathbf{z}^\star$, and the iteration $\mathbf{z}_{i+1} = f_\theta(\mathbf{z}_i;\mathbf{x})$ converges to $\mathbf{z}^\star$.

We now prove the residual bound. Since $\mathbf{z}_{i+1} = f_\theta(\mathbf{z}_i;\mathbf{x})$, the residual at iteration $i$ can be written as

$$\|f_\theta(\mathbf{z}_i;\mathbf{x}) - \mathbf{z}_i\| = \|\mathbf{z}_{i+1} - \mathbf{z}_i\|.$$

Using the contraction property of $f_\theta(.;\mathbf{x})$, we get

$$\begin{aligned}\|\mathbf{z}_{i+1} - \mathbf{z}_i\| &= \|f_\theta(\mathbf{z}_i;\mathbf{x}) - f_\theta(\mathbf{z}_{i-1};\mathbf{x})\| \\ &\leq \alpha_2\lambda_f \|\mathbf{z}_i - \mathbf{z}_{i-1}\|.\end{aligned}$$

Applying this inequality recursively gives

$$\|\mathbf{z}_{i+1} - \mathbf{z}_i\| \leq (\alpha_2\lambda_f)^i \|\mathbf{z}_1 - \mathbf{z}_0\|.$$

Since $\mathbf{z}_1 = f_\theta(\mathbf{z}_0;\mathbf{x})$, we obtain

$$\|f_\theta(\mathbf{z}_i;\mathbf{x}) - \mathbf{z}_i\| \leq (\alpha_2\lambda_f)^i \|f_\theta(\mathbf{z}_0;\mathbf{x}) - \mathbf{z}_0\|.$$

This completes the proof. ■

### B.3. Mitigating oscillation through damping

**Proof** We first show that the fixed-points of $g_{\eta,\theta}(\,\cdot\,;\mathbf{x})$ and $f_\theta(\,\cdot\,;\mathbf{x})$ coincide. Since

$$g_{\eta,\theta}(\mathbf{z};\mathbf{x}) - \mathbf{z} \;=\; \eta\big(f_\theta(\mathbf{z};\mathbf{x}) - \mathbf{z}\big),$$

and $\eta > 0$, we have $g_{\eta,\theta}(\mathbf{z};\mathbf{x}) = \mathbf{z}$ if and only if $f_\theta(\mathbf{z};\mathbf{x}) = \mathbf{z}$.

We now study the local stability of the damped iteration around $\mathbf{z}^\star$. The Jacobian of $g_{\eta,\theta}(\,\cdot\,;\mathbf{x})$ at $\mathbf{z}^\star$ is

$$\frac{\partial g_{\eta,\theta}}{\partial \mathbf{z}}(\mathbf{z}^\star;\mathbf{x}) \;=\; (1-\eta)\mathbf{I} + \eta\mathbf{J},$$

so for every eigenvalue $\lambda_i$ of $\mathbf{J}$, the corresponding eigenvalue of the damped Jacobian is

$$\mu_i(\eta) \;=\; 1 - \eta + \eta\lambda_i \;=\; 1 + \eta(\lambda_i - 1).$$

Local asymptotic stability is implied by $|\mu_i(\eta)| < 1$ for all $i$. Writing $\lambda_i = a_i + \mathrm{i}\, b_i$ with $a_i = \Re(\lambda_i)$,

$$|\mu_i(\eta)|^2 \;=\; 1 + 2\eta(a_i - 1) + \eta^2|\lambda_i - 1|^2.$$

Hence $|\mu_i(\eta)| < 1$ if and only if $\eta|\lambda_i - 1|^2 < 2(1 - a_i)$, i.e.,

$$0 \;<\; \eta \;<\; \frac{2(1 - \Re(\lambda_i))}{|\lambda_i - 1|^2}.$$

By assumption $\Re(\lambda_i) < 1$ for every $i$, so each upper bound is strictly positive. Setting

$$\eta_0 \;=\; \min\left\{ 1,\; \min_i \frac{2(1 - \Re(\lambda_i))}{|\lambda_i - 1|^2} \right\} \;>\; 0,$$

we obtain $|\mu_i(\eta)| < 1$ for every $i$ and every $\eta \in (0, \eta_0)$. Therefore $\mathbf{z}^\star$ is locally asymptotically stable under the damped iteration $\mathbf{z}_{t+1} = g_{\eta,\theta}(\mathbf{z}_t; \mathbf{x})$, and the iterates converge to $\mathbf{z}^\star$ from any sufficiently close initialization. ■

### B.4. Error of truncated-BPTT

**Proof** Since $\|\mathbf{J}\|_2 = \sigma < 1$, the Neumann series is convergent, and we have $(\mathbf{I} - \mathbf{J})^{-1} = \sum_{j=0}^{\infty} \mathbf{J}^j$. Therefore, the error of the $k$-term truncated approximation is

$$\begin{aligned}\left\| (\mathbf{I} - \mathbf{J})^{-1} - \sum_{j=0}^{k-1} \mathbf{J}^j \right\|_F &= \left\| \sum_{j=k}^{\infty} \mathbf{J}^j \right\|_F \\ &\leq \sum_{j=k}^{\infty} \left\| \mathbf{J}^j \right\|_F .\end{aligned}$$

Using the relation $\|\mathbf{A}\|_F \leq \sqrt{D}\|\mathbf{A}\|_2$ for $\mathbf{A} \in \mathbb{R}^{D \times D}$, together with submultiplicativity of the spectral norm, we get

$$\begin{aligned}\left\| \mathbf{J}^j \right\|_F &\leq \sqrt{D} \left\| \mathbf{J}^j \right\|_2 \\ &\leq \sqrt{D} \left\| \mathbf{J} \right\|_2^j \\ &= \sqrt{D} \cdot \sigma^j .\end{aligned}$$

Substituting this into the previous inequality gives

$$\begin{aligned}\left\| (\mathbf{I} - \mathbf{J})^{-1} - \sum_{j=0}^{k-1} \mathbf{J}^j \right\|_F &\leq \sqrt{D} \sum_{j=k}^{\infty} \sigma^j \\ &= \sqrt{D} \cdot \frac{\sigma^k}{1 - \sigma}.\end{aligned}$$

Thus, the approximation error decays as $\mathcal{O}(\sigma^k)$.

To make the corresponding gradient statement explicit, let $\mathbf{P} = \frac{\partial f_\theta}{\partial \theta}(\mathbf{z}^\star; \mathbf{x})$ and $\boldsymbol{\delta} = \frac{\partial \mathcal{L}}{\partial \mathbf{z}^\star}$. The exact implicit gradient is $\nabla_\theta \mathcal{L} = \mathbf{P}^\top (\mathbf{I} - \mathbf{J})^{-\top} \boldsymbol{\delta}$, whereas the $k$-step truncated BPTT gradient is

$\widehat{\nabla}_\theta^{(k)}\mathcal{L} = \mathbf{P}^\top \left(\sum_{j=0}^{k-1} \left(\mathbf{J}^\top\right)^j\right) \boldsymbol{\delta}$. Therefore,

$$\begin{aligned}
\left\|\nabla_\theta \mathcal{L} - \widehat{\nabla}_\theta^{(k)}\mathcal{L}\right\|_2 &\leq \|\mathbf{P}\|_2 \left\|(\mathbf{I}-\mathbf{J})^{-\top} - \sum_{j=0}^{k-1}\left(\mathbf{J}^\top\right)^j\right\|_2 \|\boldsymbol{\delta}\|_2 \\
&\leq \|\mathbf{P}\|_2 \left(\sum_{j=k}^{\infty} \|\mathbf{J}\|_2^j\right) \|\boldsymbol{\delta}\|_2 \\
&= \|\mathbf{P}\|_2 \frac{\sigma^k}{1-\sigma} \|\boldsymbol{\delta}\|_2 .
\end{aligned}$$

Hence, the truncated BPTT gradient error also decays exponentially with the number of backward passes $k$. This completes the proof. ■

## Appendix C. A Toy Failure Mode for Recurrent Post-norm

In this section, we provide a toy example to showcase a failure mode of post-norm in a small setting. We take random $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{n\times d}$ to be the input-output pair of sequences, with sequence length $n = 100$ and hidden-size $d = 2$, and set $\mathbf{z}_0^0 = \mathbf{x}$. Let $f_\theta(\mathbf{z}; \mathbf{x})$ be a neural network with a single sub-layer $f_{\theta^1}^1(\mathbf{z})$ defined as:

$$f_{\theta^1}^1(\mathbf{z}) = \mathbf{z}W(\mathbf{w}),$$

where we define the rank-one map:

$$W(\mathbf{w}) = \mathbf{w}\mathbf{1}^\top \in \mathbb{R}^{2\times 2}, \qquad \mathbf{w} = (w_1, w_2)^\top,$$

with $\theta^1 = \mathbf{w}$.

The **post-norm recurrence** is defined as

$$\mathbf{z}_{i+1}^0 = \mathrm{Norm}_{\mathrm{post}}\left(\mathbf{z}_i^0 + f_{\theta^1}\left(\mathbf{z}_i^0\right)\right),$$

while the **pre-norm recurrence** is defined as

$$\mathbf{z}_{i+1}^0 = (1-\beta)\cdot \mathbf{z}_i^0 + \beta \cdot f_{\theta^1}\left(\mathrm{Norm}_{\mathrm{post}}\left(\mathbf{z}_i^0\right)\right), \qquad \beta = \frac{1}{2}.$$

Figure 4 gives a minimal version of the normalization trade-off discussed in the main text based on this toy model. For both models we sweep a $200 \times 200$ grid over $\mathbf{w} \in \mathcal{G} = [-5, 5]^2$ and plot

$$\Delta\mathcal{L}(\mathbf{w}) = \mathcal{L}(\mathbf{w}) - \min_{\mathbf{v}\in\mathcal{G}} \mathcal{L}(\mathbf{v}),$$

where we define the loss as

$$\mathcal{L}(\mathbf{w}) = \frac{1}{nd}\left\|\mathbf{z}_{20}^0(\mathbf{w}) - \mathbf{y}\right\|_F^2,$$

i.e., at the $20^{th}$ effective layer.

The figure illustrates that **boundedness does not imply trainability**. Post-norm keeps the recurrent state bounded by construction: after every step, each row is projected back onto the unit

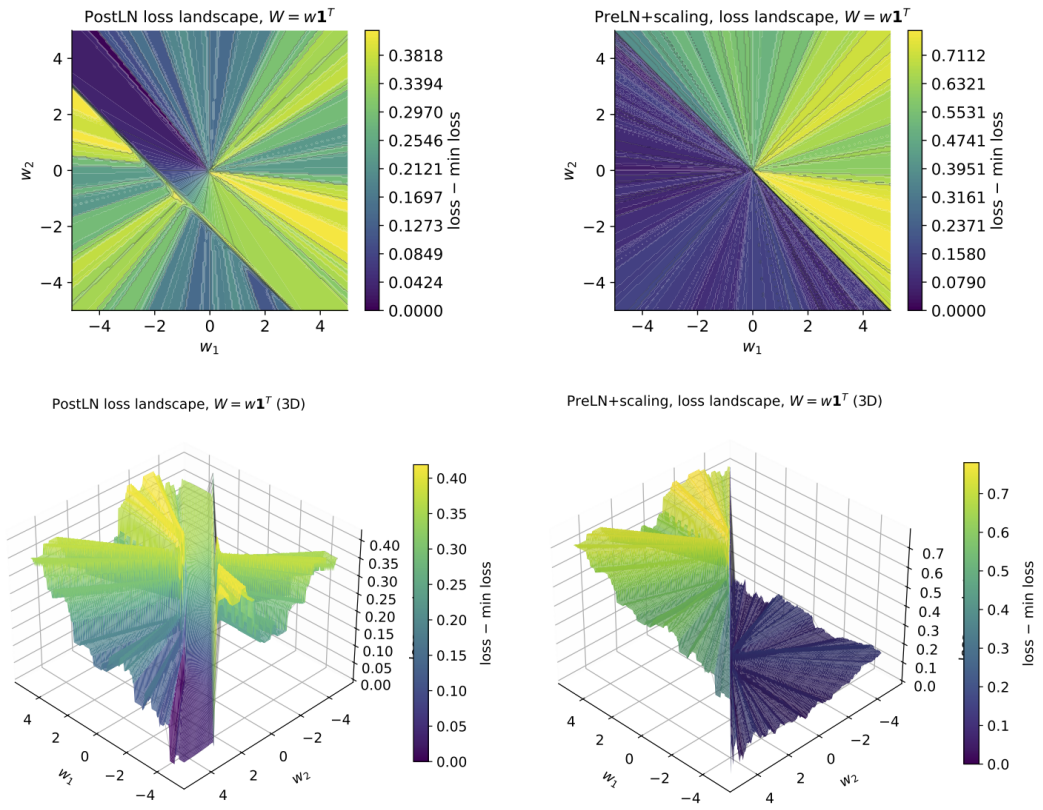



*Figure 4:* Landscape visualization for the setup proposed in Section C.

sphere. However, the same projection also removes radial information at every iteration. In this two-parameter slice, the resulting loss is organized into thin angular sectors with sharp ridges and narrow low-loss regions. Thus a random initialization of $\mathbf{w}$ is likely to start in a bounded but poorly conditioned part of the landscape, where the gradient does not point into a useful basin. This is the toy analogue of the optimization difficulty of recurrent post-norm layers.

The right panel is not bare pre-norm; it is pre-norm with residual scaling. This matters because naive pre-norm removes the projection that controls the recurrent state and can lead to activation growth (Figure 1(b)). With the scaled update, each row satisfies

$$\|\mathbf{z}_{i+1}^0\|_2 \leq (1-\beta)\|\mathbf{z}_i^0\|_2 + \beta.$$

So the toy dynamics remain bounded while preserving a live residual stream. The broader low-loss region in Figure 4 is therefore consistent with the architecture we use in Theorem 1: pre-normalization improves signal propagation, while residual scaling replaces the boundedness mechanism that post-normalization provided.

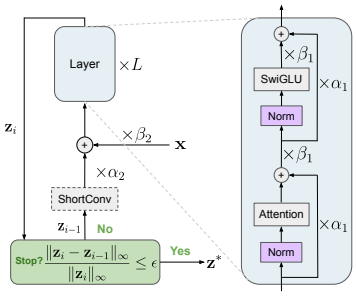



*Figure 5:* **FPRM architecture.** Our fixed-point Looped Transformer uses pre-norm and residual scaling for improved signal propagation.

## Appendix D. Further Details About the Architecture

### D.1. Signal propagation in Transformer block

We start by introducing pre-norm and post-norm in a Transformer layer. A Transformer layer consists of two sub-layers (with $f^{\ell}_{\theta^\ell}(.)$ denoting the $\ell^{th} \in \{1, \ldots, 2L\}$ sub-layer): multi-head attention and a feed-forward network. We denote the Looped Transformer model as defined in Equation (1), consisting of multiple Transformer layers, by $f_\theta(.)$. In the Transformer layer, the two sub-layers are interleaved with layer normalization (Norm). Two canonical placements of the normalization define two variants of the layer. The original *post-norm* formulation ($\mathrm{Norm}_{\mathrm{post}}$) [64] applies normalization after the residual addition with the residual stream ($\mathbf{z}^{\ell-1}$), while the *pre-norm* variant ($\mathrm{Norm}_{\mathrm{pre}}$) [67] applies normalization to the input of each sub-layer without modifying the residual stream:

$$\mathbf{z}^\ell = \mathrm{Norm}_{\mathrm{post}}\left(\mathbf{z}^{\ell-1} + f^{\ell}_{\theta^\ell}\left(\mathrm{Norm}_{\mathrm{pre}}\left(\mathbf{z}^{\ell-1}\right)\right)\right), \qquad \ell = 1, \ldots, 2L,$$

where $L$ is the number of layers (Transformer blocks). In fixed-depth models, both normalization placements have been linked to training issues at large depth: post-norm bounds activation magnitudes but induces a signal propagation problem [43, 51], while pre-norm improves signal propagation but causes exponential growth in residual magnitude [43, 67]. Therefore, while using pre-norm in a deep neural network is desirable from the signal propagation standpoint, it can introduce instability due to unbounded activations.

### D.2. Residual scaling and fixed-point convergence

However, boundedness still does not guarantee that the looping to converge to a fixed-point, which we propose to utilize for adaptivity. In the following, we show that there exists a choice of $\alpha_2$ that satisfies this requirement. Consequently, as empirically shown by Bansal et al. [8], the model may become locally contractive during training.

**Theorem 2 (Small $\alpha_2$ implies convergence)** *Let $\lambda_f$ be the Lipschitz constant of the $L$-layer model defined by Equation (2), i.e. the map $\mathbf{z}^0 \mapsto \mathbf{z}^{2L}$. Then the looped step $f_\theta(\,\cdot\,;\mathbf{x})$ of Equations 2–3 is Lipschitz in its first argument with constant $\alpha_2\lambda_f$. In particular, if $\alpha_2\lambda_f < 1$, then $f_\theta(\,\cdot\,;\mathbf{x})$ is a contraction and the iteration $\mathbf{z}_{i+1} = f_\theta(\mathbf{z}_i;\mathbf{x})$ converges to a unique fixed-point $\mathbf{z}^\star = f_\theta(\mathbf{z}^*;\mathbf{x})$ at a linear rate:*

$$\|f_\theta(\mathbf{z}_i;\mathbf{x}) - \mathbf{z}_i\| \leq (\alpha_2\lambda_f)^i \, \|f_\theta(\mathbf{z}_0;\mathbf{x}) - \mathbf{z}_0\| \,.$$

The proof is in Section B.2.

While a contractive map is needed for convergence, an excessively contractive one severely limits expressivity [2, 5]. We avoid this by making $\alpha_1$ and $\alpha_2$ learnable. In practice, we find that initializing the network to be more contractive by setting $\alpha_2$ to be small yields better performance (Table 3). In Figure 13, we observe that after training, the distribution of $\alpha$ values widens, with the median very close to the initial point. Intriguingly, these observations are also in line with the common solutions to signal propagation and rank-collapse problems in deep neural networks [51, 63], suggesting a connection between rank-collapse and the signal propagation issue in looped architectures.

### D.3. Oscillation around the fixed-point

So far, we have been able to establish that, given a small enough $\alpha_2$, the model introduced in Equation (3) becomes locally contractive and converges to a fixed-point. However, in practice the contraction factor of Theorem 2 is not itself guaranteed. For some inputs, we observe that the model often descends into an oscillatory behavior, causing the iteration to stay in a small region of latent space without converging. This non-convergent behavior is not in tension with Theorem 2, as the theorem gives a sufficient condition for convergence, not a complete characterization of the iteration's behavior. In fact, oscillation around the fixed-point can happen when the Jacobian satisfies certain conditions.

Linearizing the iteration near a fixed-point $\mathbf{z}^\star$ gives $\mathbf{z}_{i+1} - \mathbf{z}^\star \approx \mathbf{J}\,(\mathbf{z}_i - \mathbf{z}^\star)$, where $\mathbf{J} = \partial f_\theta / \partial \mathbf{z}\,|_{\mathbf{z}^\star}$. Oscillation around $\mathbf{z}^\star$ arises when $\mathbf{J}$ has an eigenvalue with $\Re(\lambda_i) < 1$ but $|\lambda_i| \geq 1$, in which case the iteration spirals around $\mathbf{z}^\star$ rather than contracting toward it. The half-plane condition $\Re(\lambda_i) < 1$ is exactly what licenses a runtime fix that does not require modifying $f_\theta$:

**Theorem 3 (Damping stabilizes oscillatory fixed-point dynamics)** *Suppose $f_\theta(\,\cdot\,;\mathbf{x})$ is continuously differentiable in a neighborhood of a fixed-point $\mathbf{z}^\star$, and that every eigenvalue $\lambda_i$ of the Jacobian $\mathbf{J}$ at $\mathbf{z}^\star$ satisfies $\Re(\lambda_i) < 1$. Define the damped iteration map*

$$g_{\eta,\theta}(\mathbf{z};\mathbf{x}) \;:=\; \eta\, f_\theta(\mathbf{z};\mathbf{x}) + (1-\eta)\,\mathbf{z}.$$

*Then there exists $\eta_0 \in (0,1)$ such that, for every $\eta \in (0,\eta_0)$, the iteration $\mathbf{z}_{i+1} = g_{\eta,\theta}(\mathbf{z}_i;\mathbf{x})$ converges locally to $\mathbf{z}^\star$. Moreover, $g_{\eta,\theta}(\,\cdot\,;\mathbf{x})$ and $f_\theta(\,\cdot\,;\mathbf{x})$ have the same fixed-points.*

The proof is in Section B.3.

Theorem 3 shows that a suitable damping factor $\eta$ eliminates the oscillations while preserving the fixed-points of $f_\theta(\mathbf{z};\mathbf{x})$. We measure convergence to a fixed-point at iteration $i$ as

$$r_i \;=\; \frac{\|\mathbf{z}_i - f_\theta(\mathbf{z}_i;\mathbf{x})\|_\infty}{\|f_\theta(\mathbf{z}_i;\mathbf{x})\|_\infty + \epsilon},$$

which serves as the halting signal: the iteration stops once $r_i$ falls below a tolerance $\tau$. To choose $\eta$ adaptively at inference time, we use a patience mechanism that decreases $\eta$ whenever this residual stops improving. We track the smallest residual observed so far, $r^\star = \min_{j\leq i} r_j$. A geometric decay $\eta \leftarrow \gamma\,\eta$ with $\gamma \in (0,1)$ is applied to the step-size after $P$ consecutive iterations with no

improvement in the residuals. The full procedure is given in Algorithm 1, which is based on the implementation provided by Movahedi et al. [50].

### D.4. Optimization of fixed-point models

One advantage of contractive fixed-point models is that they can be trained using truncated back-propagation through time (BPTT) [29]. Let $\mathbf{J} = \frac{\partial f_\theta}{\partial \mathbf{z}}(\mathbf{z}^\star; \mathbf{x})$ and $\mathbf{P} = \frac{\partial f_\theta}{\partial \theta}(\mathbf{z}^\star; \mathbf{x})$ denote the Jacobians of $f_\theta$ at the fixed-point with respect to the state and the parameters, respectively. Following the implicit function theorem we can write the gradient w.r.t. the parameters of the model as [5]:

$$\frac{d\mathbf{z}^\star}{d\theta} \;=\; (\mathbf{I} - \mathbf{J})^{-1}\,\mathbf{P}. \tag{4}$$

The Neumann series $(\mathbf{I} - \mathbf{J})^{-1} = \sum_{j \geq 0} \mathbf{J}^j$ converges, assuming $f_\theta(\mathbf{z}; \mathbf{x})$ is contractive. Therefore, truncating the series at depth $k$ yields the estimate:

$$\frac{d\mathbf{z}^\star}{d\theta} \;\approx\; \textstyle\sum_{j=0}^{k-1} \mathbf{J}^j\,\mathbf{P}, \tag{5}$$

which is closely related to Jacobian-free backpropagation, where the full implicit linear solve is replaced by cheaper approximate gradients [24]. The truncation depth $k$ trades off computation against accuracy. The following proposition bounds the resulting error under contractivity.

**Proposition 4 (Exponential decay of truncated-BPTT error)** *Let $\mathbf{z}^\star = f_\theta(\mathbf{z}^\star; \mathbf{x})$ and $\mathbf{J} = \frac{\partial f_\theta}{\partial \mathbf{z}}(\mathbf{z}^\star; \mathbf{x}) \in \mathbb{R}^{D \times D}$. If $\mathbf{J}$ is contractive in spectral norm, $\|\mathbf{J}\|_2 = \sigma < 1$, then for every $k \geq 0$,*

$$\left\| (\mathbf{I} - \mathbf{J})^{-1} - \textstyle\sum_{j=0}^{k-1} \mathbf{J}^j \right\|_F \;\leq\; \sqrt{D}\,\frac{\sigma^k}{1 - \sigma}.$$

The proof is in Section B.4. An essentially equivalent result appears in the proof of Theorem 2 of Geng et al. [29].

Theorem 4 allows for a fixed memory footprint during training, essentially decoupling the number of loops from the memory complexity of the model. In the same spirit, HRM [65] and TRM [39] approximate the gradient with a small number of backward passes at the fixed-point, although TRM argues that the fixed-point condition is unnecessary in practice.

**Fixed-point solver.** Let $\mathbf{z}_i \in \mathbb{R}^{B \times T \times d}$ denote the $i^{th}$ latent representation (with $B$ denoting the batch index, $T$ the sequence index, and $d$ the hidden size), and $\mathbf{z}_{i+1}$ the next latent representation. We index the $b^{th}$ batch dimension as $\mathbf{z}_i[b]$. Convergence is measured per sample $\mathbf{z}_i[b]$ by the relative $L_\infty$ norm of the residual,

$$\mathbf{r}_i[b] = \frac{\|\mathbf{z}_{i+1}[b] - \mathbf{z}_i[b]\|_\infty}{\|\mathbf{z}_{i+1}[b]\|_\infty + \epsilon} \in \mathbb{R}.$$

A sample is declared converged when $r_i[b] < \tau$. In practice, we set $\tau$ to 0.1. But we observe that for a reasonably small choice of $\tau$, the model is not sensitive to this value. Two safeguards bound the loop: **(1)** a hard cap on the number of iterations, and **(2)** early termination if the adaptive step size collapses below a minimum.

**Deep supervision.** We adopt a similar deep supervision mechanism as HRM [65] and TRM [39]. Let $T_{\text{sup}}$ denote the deep supervision interval. After every $T_{\text{sup}}$ iterations, the intermediate activations of the model are decoded through the output head, the loss is computed, and truncated-BPTT is performed through the $k$ latest iterations. Then, the computation graph is detached from the previous step. For each sequence, this process is continued until the fixed-point of the input is reached. The number of backward passes per forward pass is therefore $\lceil k/T_{\text{sup}} \rceil$. In our model, we set $T_{\text{sup}} = k$, while in TRM and HRM, $T_{\text{sup}}$ is usually set to a larger number. However, we observe that in practice, FPRM performs a smaller number of forward and backward passes during training compared to TRM, lowering the training cost.

**Depth-wise convolutions.** Depth-wise convolutions have proven effective in improving the performance of looped models, at a small time and parameter complexity [25, 59]. Therefore, in FPRM we apply depth-wise convolutions on the latent representations at the beginning of each loop, which we find to be most effective. An overview of FPRM is available in Figure 5. We consider both 1D and 2D convolutions, and we find the 2D variant to be more effective at 2-dimensional tasks such as Sudoku and ARC, while the 1D variant is essential in state-tracking. However, as observed in Table 2, depth-wise convolutions seem to have a detrimental impact on the performance of TRM.

## Appendix E. Description of Figure 1

In this figure, we categorize the puzzles into three groups: easy, medium, and hard. The grouping is based on difficulty, which is measured by the number of empty cells in the puzzle. The sample sizes for each difficulty level are balanced and set at around 1000 samples. For FPRM, we mark the halting decision for the entire group based on the residual of the group: if the mean residual is smaller than a pre-determined threshold (set to 0.1), then the model makes the halting decision. For TRM, the halting decision is marked when the ACT module signals halting for more than half of the samples. For the sake of exposition, we exclude the hardest puzzles from the groups, since they fail to halt at the current max. set budget of 10000 effective layers.

## Appendix F. Empirical illustration of fixed-point residuals and halting

We provide the test accuracy and the fixed-point residuals achieved by FPRM as a function of effective layer in Figure 7. The residuals for more difficult problems decay at a much slower rate, indicating that they demand more compute. Furthermore, accuracy stops improving at roughly the same effective layer where the residual plateaus, supporting the use of fixed-points as a halting criterion.

## Appendix G. Additional Experimental Details

### G.1. Dataset description

In the following, we provide a brief description for each dataset used in this paper. We refer the reader to the cited literature for more information.

**Sudoku-Extreme.** The task consists of exceptionally challenging, partially filled $9 \times 9$ Sudoku puzzles with unique solutions, introduced by Wang et al. [65]. Each sample is flattened into a sequence of length 81. The train data consists of 1000 unique samples, each augmented 1000 times,

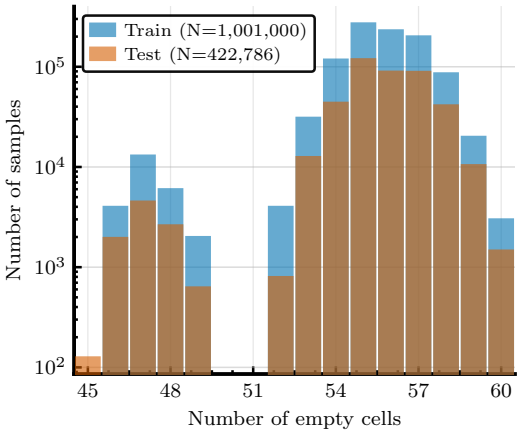



*Figure 6:* **Sudoku-Extreme dataset is imbalanced.** The number of samples per difficulty level (number of empty cells).

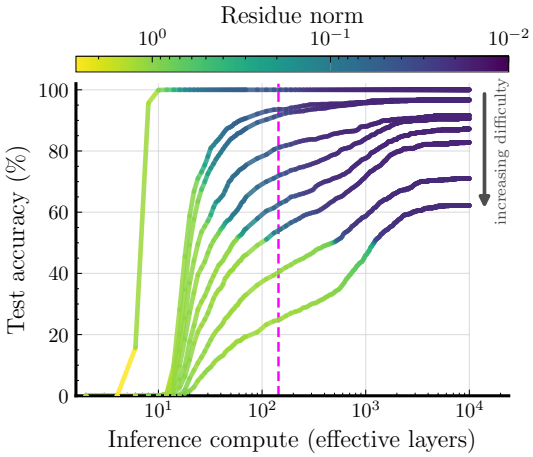



*Figure 7:* **FPRM allocates more compute to harder problems.** Harder inputs need more iterations before halting and peak beyond the training compute limit (dashed line); color shows residual norm.

giving a total of ˜1M train samples. The test data contains 422,786 samples. The evaluation metric is exact (sequence) accuracy.

**Maze-Hard.** The task consists of difficult $30 \times 30$ shortest-path maze puzzles with unique solutions, introduced by Wang et al. [65]. The train data and the test data each consist of 1000 unique samples. Following Wang et al. [65], we do not use augmentation for this problem. The evaluation metric is exact (sequence) accuracy.

**ARC-1 and ARC-2.** The Abstraction and Reasoning Corpus-1 (ARC-1) and -2 (ARC-2), introduced by Chollet et al. [14], aim to assess the ability of the model to solve novel problems from minimal examples. ARC-1 consists of 2-3 2D grid-based input-output demonstration pairs with variable size (up to $30\times30$), through which the model is supposed to learn an underlying transformation rule and apply it to a held-out sample. The train data and the test data each contain ˜400 samples. ARC-2 has similar characteristics to ARC-1, but with much more challenging problems involving several complex transformations, which are less susceptible to brute-force solutions. The benchmark contains ˜1000 training samples and ˜360 test samples. The evaluation metric is exact (sequence) pass@2 (top-2 predictions) accuracy. Importantly, pretrained reasoning LLMs with CoT often struggle with these tasks. For example, DeepSeek-R1 (671B model) achieves 15.8% on ARC-1 and 1.3% on ARC-2, Claude 3.7 Sonnet 16K achieves 28.6% on ARC-1 and 0.7% on ARC-2.

**State tracking.** The $A_5$ and $S_5$ state-tracking tasks, introduced by Merrill et al. [49], are algorithmic benchmarks based on permutation composition. Each sample consists of an initial state and a sequence of $k$ update permutations; the model must apply the updates in order and predict the resulting final state. $A_5$ uses the alternating group on five elements, i.e., the subgroup of even permutations, while $S_5$ uses the full symmetric group on five elements. These tasks are useful prox-

*Table 2:* Effect of adding FPRM's architectural modifications to TRM: pre-norm and residual scaling ($\alpha_2$ only, or both $\alpha_1$ and $\alpha_2$), individually and in combination, evaluated with and without the conv2d layer in the TRM core. Each column reports the change in test sequence accuracy on Sudoku-Extreme (%) relative to its own post-norm, no-scale baseline measured in this sweep, with the absolute accuracy shown alongside.

| | w/ conv | | w/o conv | |
|---|---|---|---|---|
| Configuration | $\Delta$ (%) | Acc. (%) | $\Delta$ (%) | Acc. (%) |
| Original TRM (post-norm, no residual scaling) | — | 63.98 | — | 72.60 |
| + residual scaling ($\alpha_2$ only) | $-6.71$ | 57.27 | $-4.44$ | 68.16 |
| + residual scaling ($\alpha_1$, $\alpha_2$) | $-4.47$ | 59.51 | $-12.24$ | 60.36 |
| − post-norm + pre-norm + residual scaling ($\alpha_2$ only) | $-49.00$ | 14.98 | $-58.87$ | 13.73 |
| − post-norm + pre-norm + residual scaling ($\alpha_1$, $\alpha_2$) | $-24.52$ | 39.46 | $\mathbf{-52.83}$ | 19.77 |

ies for stateful reasoning problems such as entity tracking, code execution, and game-state tracking, since solving them requires learning a composable update rule rather than memorizing computations at fixed lengths. We train on sequences containing up to 32 updates and evaluate out-of-distribution length generalization on sequences containing up to 128 updates. The evaluation metric is exact final-state accuracy.

### G.2. Training details

**Optimization of FPRM.** During training, the model performs looping in windows of $k$ iterations, with $k$ being a hyperparameter, which determines the truncated-BPTT value. During inference, we set $k = 1$. At each forward pass, the fixed-point optimizer introduced in Algorithm 1 is called to dampen the fixed-point iteration steps. Then, we get a prediction from the current state $\mathbf{z}$ of the model and perform a deep supervision step, following Jolicoeur-Martineau [39], Wang et al. [65]. To truncate the computation graph between deep supervision steps during training, we detach the state $\mathbf{z}$ from the graph and stop when the optimizer detects fixed-points. This happens when the residual falls below the tolerance, or the step-size becomes too small.

**Weight initialization.** It seems that initializing the weights using a truncated normal distribution (LeCun initialization) is common practice in looped architectures. In our experiments, it accelerates the convergence but there is very little material difference in sequence accuracy after convergence.

**Grokking.** There is some evidence for grokking in looped architectures, but on the maze task we observe convergence on the training data. And training the models for a longer period (up to 7 days) did not yield better performance.

**Hyperparameters, device specification.** In Tables 4, 5, 7 we provide the values for some of the most important hyperparameters in the paper, per each model and dataset.

## Appendix H. How to effectively spend loops in TRM?

The default proposed TRM uses 16 deep-supervision steps (outer loops) and variable L- and H-steps (inner loops). The L-steps outnumber the H-steps, typically by about $2\times$. However, other configurations for the number of loops spent for deep supervision vs. inner loops are possible. We

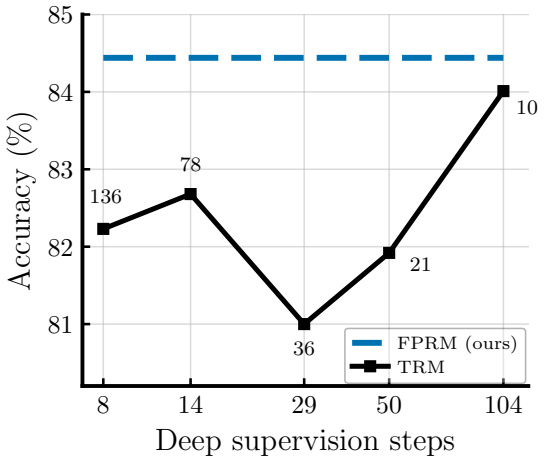


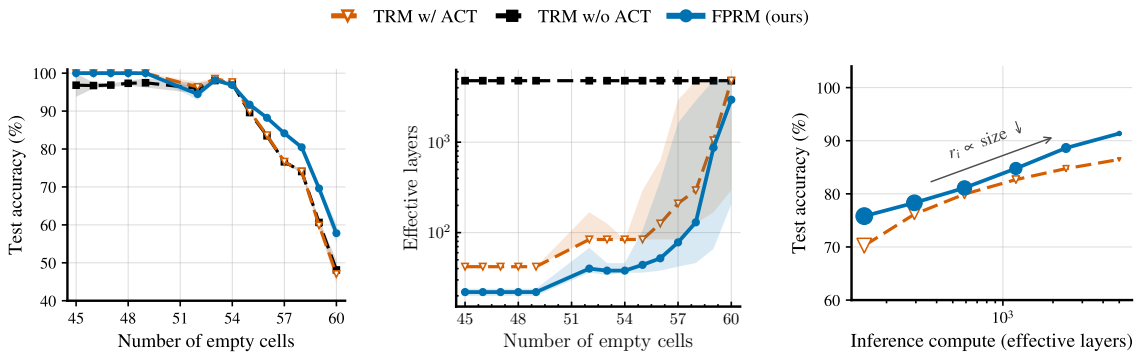

*Figure 8:* **The optimal way to spend the fixed looping compute is to maximize deep supervision steps.** The numbers next to markers are inner recurrence depths per each deep supervision step. The total depth of effective layers is approximately the same across all configurations of TRM and FPRM on the Sudoku-Extreme task.

test the performance of other configurations with experiments shown in Figure 8. We fix the L-to-H ratio at 2 and vary deep-supervision steps (segments) against per-segment recurrence depth (inner loops, shown with numbers next to the black markers in Figure 8). We measure test accuracy as a function of the number of deep-supervision steps, with fixed inference budget at approximately 1040 steps. This budget also matches the max. number of effective layers reached by the baseline FPRM on this task. This isolates how a fixed inference budget is best allocated: toward more outer refinement steps or deeper inner recurrence. Figure 8 shows that the budget is best spent on outer, deep-supervision steps. This matches the finding that, in TRM's post-norm Transformer, the gains from added effective layer depth get smaller compared to FPRM (Figure 10). Fewer effective layers per segment is therefore the better strategy at a fixed compute budget. We adopt it for all experiments where we scale TRM compute (effective layers).

## Appendix I. Additional Empirical Analyses

### I.1. Depth-induced signal propagation issues

Here, we aim to investigate the role of pre-norm with residual scaling in mitigating the depth-induced signal propagation issues in looped models. For a comprehensive investigation, we approach the problem from three angles: **(1)** *trainability*, where we show that pre-norm with residual scaling keeps activations bounded and enables stable training at large depth (Section I.1.1); **(2)** *depth utilization*, where we show that boundedness alone is not sufficient, and that pre-norm with residual scaling additionally improves depth utilization in FPRM [63] (Section I.1.2); and **(3)** *residual scaling*, where we analyze the training dynamics of the scaling parameters (Section I.3.1).




**(a)** Accuracy vs. difficulty **(b)** Inference compute vs. difficulty

*Figure 9:* **FPRM achieves (a) better accuracy, while (b) adapting more efficiently to the task difficulty.** Difficulty is measured by the number of empty cells in the Sudoku grid. The max. compute budget is matched across models (4788 effective layers). From (b): effective layers are reported as medians with 25th–75th percentiles bands. The default behavior of TRM is without ACT at inference time (in black), which exhausts the max. budget for all sample difficulties.

*Figure 10:* **Test-time scaling lowers residuals while improving accuracy.** Both models run until the matched compute budget is exhausted. FPRM achieves higher accuracy across the entire range. Marker size denotes residual.

### I.1.1. BOUNDEDNESS OF ACTIVATION NORMS AND TRAINABILITY

As noted in Section 2.1, the normalization scheme governs a trade-off between activation stability and signal propagation, which sharpens with depth. Post-norm keeps activations bounded but suffers from signal propagation issues. Pre-norm enables better signal propagation but lets activations grow exponentially. To isolate this trade-off, we adopt the Looped Transformer framework [15, 40, 58] with a fixed number of effective layers. This controls for dynamic halting, which can reduce the effective layers, as introduced in Section 2.

In Figure 1(b) we show the norm of the final activations of the residual branch of Looped Transformer at initialization. From the figure, we conclude that the magnitude of the activations grows exponentially in the case of pre-norm Looped Transformer. In contrast, when using post-norm or pre-norm with residual scaling, the architecture benefits from bounded activations. In Figure 1(a) we give evidence that boundedness is a prerequisite for reaching deeper effective layers. Pre-norm architecture with its unbounded activations diverges and can't reach deep effective layers, while bounded architectures ensure trainability. Therefore, a Looped Transformer with pre-norm and without residual scaling cannot achieve deep effective layers, highlighting one aspect of the importance of residual scaling in pre-norm.

### I.1.2. PRE-NORM WITH RESIDUAL SCALING ENABLES HIGHER DEPTH UTILIZATION

An important measure for evaluating signal propagation issues in deep neural networks is depth utilization [63]. Depth utilization measures whether all layers contribute meaningfully to the model. It is commonly probed by removing layers from a trained model and measuring the effect on performance at test-time. In models with signal propagation problems, removing deeper layers (closer to the output) usually does not impact the performance significantly, highlighting that these models

fail to utilize the compute due to signal propagation issues. However, in a looped model there is no fixed stack of layers to remove, since the same block is applied to the previous latent representations. Consequently, we cannot perform the same experiment here. We therefore probe the same property from the opposite direction: instead of *removing computation* and measuring degradation, we *add computation* and measure improvement. We do so in two complementary regimes. In the first, using the state-tracking task, we ask whether the model can be trained to use the depth that a harder task requires. In the second, using the Sudoku-Extreme task, we ask whether a trained model can convert depth beyond its training regime into further gains at test time.

**State-tracking.** In Section I.1.1, we discussed the necessity of using normalization layers that ensure the boundedness of the activations, which is required for the stable training. However, bounded activations are not a sufficient condition to ensure effective utilization of large depth, once it is reached. We demonstrate this in Figure 1(a) using a state-tracking task, where increasing difficulty, corresponding to longer sequences, requires training at greater depths [50]. We train a Looped Transformer model with its number of loops (effective layer) tied to the train sequence length. We choose the sequence length from the set $\{8, 16, 32, 64, 128, 256, 512\}$. We plot the maximum sequence length solved with $> 90\%$ accuracy against effective layer. Because we also at test-time match effective layer to the training sequence length, a model with no signal-propagation bottleneck should solve exactly the length its depth permits, tracing the identity line $y = x$ (Figure 3 in Movahedi et al. [50]). We observe that this behavior is only present in the model equipped with pre-norm and residual scaling. We interpret this observation as strong evidence for improved trainability in Looped Transformers with pre-norm and residual scaling.

**Sudoku-Extreme.** Compared to the previous experiment on the state-tracking task, here we run each model far beyond its trained depth, trying to detect the point where more compute no longer translates into improvements at test-time. We expect the performance of a model with fewer signal propagation issues to saturate later and with more effective layers, indicating that the model is capable of reaching deeper effective layer. On the other hand, the performance of a model bottlenecked by signal propagation problems is expected to saturate early, indicating that it cannot convert the extra compute into better predictions. For this experiment, we focus on two variants of the FPRM model, one with pre-norm and residual scaling, and the other with post-norm, both trained on the Sudoku-Extreme task.

In Figure 10(a), we demonstrate that scaling test-time compute improves performance for both types of normalization. Furthermore, considering Figure 10(b), we also observe that the majority of the improvement comes before at least half of the samples halt. However, there are clear differences between the effective layer of the two normalization methods in Figure 10(a), as the pre-norm model's performance saturates at almost twice as much compute, indicating improved signal propagation through depth.

The advantage of pre-norm with residual scaling also extends to the cross-model comparison in Figure 10, between FPRM (pre-norm with residual scaling) and TRM (post-norm). In this inference-time scaling experiment, all samples are run to a varied maximum looping compute budget. For TRM, compute is scaled through deep supervision steps, which we find optimal relative to scaling L- and H-steps (see Section H). FPRM outperforms TRM across a range of effective-layer depths reached, with the gap widening at higher compute budgets, consistent with FPRM making better use of its depth.

#### I.1.3. ARCHITECTURE ABLATIONS IN FPRM

Given that FPRM performs on par with or better than the hierarchical baselines that use post-norm on the tasks from Table 1, we hypothesize that the hierarchy might be alleviating signal propagation issues in these models. However, if this hypothesis holds, hierarchy's benefit appears limited: in Figure 10, both models improve as effective layer grows, but the performance gap widens in favor of FPRM, and TRM's performance stays below FPRM at matched effective layers.

Moreover, in Table 2, we attempt to investigate the effect of introducing our architectural modifications to the Transformer layers used in TRM on its performance on Sudoku-Extreme. We find the proposed modifications to have a detrimental impact on the performance of the models, which we attribute to, among other things, hyperparameter optimization and a careful redesigning of the looping.

### I.2. Further experiments showing adaptivity

**Sudoku-Extreme.** We compare the performance and efficiency of the halting mechanisms in FPRM and TRM on Sudoku puzzles of varying difficulty, measured by the number of empty cells (Figure 9). The sample counts per difficulty level are shown in Figure 6. We observe that halting mechanisms in both TRM and FPRM adapt the number of loops, i.e., inference compute, to the sample difficulty. In contrast, the default behavior of TRM during inference (deactivated ACT) does not adapt the compute to the task difficulty. Even when we enable ACT with TRM, FPRM proves to be more efficient and accurate.

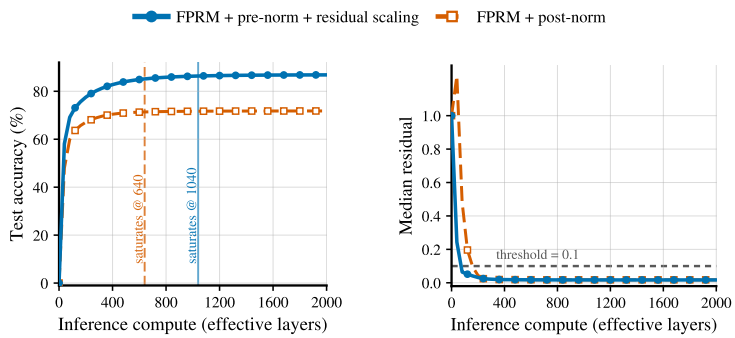
### I.3. Fixed-point optimizer hyperparameters

We additionally study how the fixed-point optimizer controls the compute–accuracy trade-off at inference time. Figure 12 shows test accuracy and effective depth as a function of the decay rate $\gamma$ for two values of the maximum patience $P$. Larger decay rates improve accuracy but require more effective layers before halting, whereas the choice of patience has a comparatively small effect.

#### I.3.1. THE TRAINING DYNAMICS OF RESIDUAL SCALES

While our original motivation for residual scaling was to prevent unbounded activations in pre-norm, we note a parallel relationship between our solution and common solutions to signal propagation problems [51, 63]. Specifically, it has been known that scaling down the output of the sub-layers when introducing them to the residual stream is beneficial to increasing effective layer, which is equivalent to increasing $\alpha_1$ in Equation (2). Moreover, in Theorem 2, we show that the looping will become more stable for smaller $\alpha_2$ in Equation (3). In order to investigate the impact of these two parameters, we perform a coarse-grained ablation on the initial value of $\alpha_1, \alpha_2$ on the Sudoku-Extreme task.

**Compute–accuracy trade-off in fixed-point halting.** The previous experiments show that FPRM adapts its compute to task difficulty. We now examine how this adaptation can be controlled. In Figure 12, we demonstrate the effect of the step-size decay rate ($\gamma$) and maximum patience ($P$) from Algorithm 1 on the halting time and performance at test-time. Max. inference-time budget was set to be the same (70k effective layers) for all experiments. We observe that larger decay rates improve the performance. Moreover, we observe that maximum patience has minimal impact on the performance, with its impact completely disappearing for larger decay rates. On the other hand, the




**(a)** Test accuracy **(b)** Fixed-point residuals

*Figure 11:* **Loop-utilization of FPRM on Sudoku. (a)** test accuracy of FPRM with pre-norm with residual scales vs. post-norm. **(b)** median residual. The pre-norm model is better at loop utilization, while both have similar residuals. This indicates similar latent-space convergence, with more meaningful updates in the pre-norm variant, resulting in improved performance.

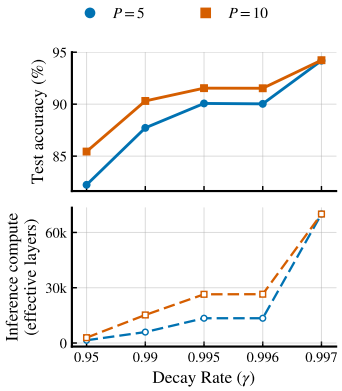



*Figure 12:* **Decay rate and patience.** Test accuracy and effective layer of FPRM with fixed-point halting as a function of decay rate $\gamma$, for maximum-patience $P \in \{5, 10\}$.

increased performance comes at the cost of reaching much deeper effective layers before halting. The largest decay rate that achieves the best performance also exhausts the max. inference budget.

These trends follow from how $\gamma$ and $P$ control the step size $\eta$ from Algorithm 1. A decay rate closer to 1 reduces $\eta$ only slightly once the residual stops improving. The state therefore keeps evolving, halting is deferred, and the model reaches deeper effective layers. The accuracy gain follows directly from these additional iterations. Patience $P$ sets only when a decay occurs, not its magnitude, so its effect is minor; as $\gamma$ approaches 1 each decay becomes negligible and the two patience curves coincide. The decay rate thus controls the compute-accuracy trade-off directly: a larger $\gamma$ spends more compute and yields higher accuracy, while $P$ offers only secondary control that vanishes as compute saturates.

Table 3 demonstrates that the best initialization places $\alpha_1$ at a high value and $\alpha_2$ at a low value. Interestingly, the $\alpha_1$ preference matches the common solutions to signal propagation problems: keeping the residual stream dominant. Moreover, the ablation also highlights the importance of having a more contractive mapping at initialization, as our convergence analysis (Theorem 2) requires a sufficiently small $\alpha_2$ for the loop to reach a fixed-point. On the other hand, comparing the row with the smallest $\alpha_2$ choice ($\alpha_2 = 0.25$) with the column with the largest $\alpha_1$ ($\alpha_1 = 0.75$), we observe that increasing $\alpha_1$ has a slightly higher positive impact on the accuracy than a decreasing $\alpha_2$. We hypothesize that this is because it is easier for the model to recover from a bad choice of $\alpha_2$

*Table 3:* Sensitivity of FPRM to residual scaling initialization on Sudoku-Extreme dataset. Each cell reports best test sequence accuracy (%) for a given pair of initial values.

| | $\alpha_2$ init | | |
|---|---|---|---|
| $\alpha_1$ init | 0.25 | 0.50 | 0.75 |
| 0.25 | 83.44 | 78.10 | 83.24 |
| 0.50 | 84.49 | 89.05 | 86.29 |
| 0.75 | **94.23** | 91.41 | 85.70 |

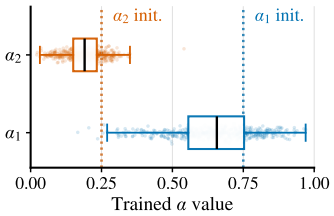



*Figure 13:* The distribution of the residual scales in FPRM after training on the Sudoku-Extreme dataset.

than $\alpha_1$, as the gradients for $\alpha_1$ come from two different sources (the MHA and MLP sub-layers), and thus can be noisier.

In Figure 13, we provide the distribution statistics of the residual scales after training. Interestingly, we observe that while the median of the $\alpha_1, \alpha_2$ values over channels does not deviate significantly from the initial point, the spread of the distributions widens significantly. In the case of $\alpha_1$, the widening happens at a much larger scale, covering both very small and very large values. On the other hand, the $\alpha_2$ distribution remains more concentrated, with the majority of the values actually becoming smaller than the initial point. This can be interpreted as the model learning to become more contractive during training, which is in line with the observations in [8]. Furthermore, this observation also supports our hypothesis that it might be easier for the model to learn the optimal $\alpha_2$ values than the $\alpha_1$ values.

## Appendix J. Discussion

Our experiments support three broader observations about FPRM and looped reasoning models in general, which we discuss in turn.

**Looped fixed-point models are adaptive.** FPRM adapts to the difficulty of the problem more effectively compared to TRM (Figures 3, 9, and Section 3.2), using fewer effective layers (compute), while achieving better performance. This is a consequence of FPRM halting closer to the saturation point of accuracy (Figure 7). In contrast, TRM with its ACT halting mechanism either halts too early, resulting in lower performance, or too late, using excessive compute.

**Enabling ACT at inference time.** The original proposed TRM does not use its trained ACT head at inference time, leading to the non-adaptive behavior. However, we find this to be an engineering challenge rather than a fundamental limitation. Therefore, in Figures 1, 3, 9, we record the number of effective layers reached at the moment when the probability of halting exceeds $0.5$. Similarly, in the case of FPRM, we do so when the residual drops below the set threshold ($0.1$ in this case). However, note that the halted samples remain in the batch until the last sample in the batch halts. We leave the efficient implementation for future work.

**The role of hierarchy in HRM and TRM.** While originally hierarchical reasoning was biologically motivated [65], later explanations involved likening the lower level of the hierarchy to a scratch pad, the latent representation of which is used by the higher level for prediction [39]. However, the role of hierarchy as the driving force behind the success of hierarchical reasoning models has been brought into question recently, with similar architectures without the hierarchy performing as well as hierarchical models [3, 26]. In Section I.1, we observe that a Transformer model with post-norm, which is the building block of TRM and HRM, suffers from a signal propagation issue. On the other hand, in Section 3.1 we were able to show that by improving signal propagation, FPRM improves upon these models without requiring the hierarchy. In Figure 8, we observe that reallocating the compute from the H- and L-steps to the additional deep supervision steps improves TRM's performance. Since more H- and L-steps increase the effective layer of TRM within each supervision step, the signal propagation issue induced by post-norm is amplified. The improvement is therefore consistent with TRM being limited by the same signal-propagation issue we identify in Section I.1. In light of these results, we hypothesize that there might be a simpler explanation for the success of hierarchical models: *the hierarchy improves signal propagation*. We identify the theoretical ex-

planation of the role of hierarchy through the lens of optimization and signal propagation as an interesting direction for future work.

**Scaling behavior of FPRM.** The results of Section 3 combine into a coherent picture of how FPRM scales its computation. First, with better signal propagation FPRM is able to utilize compute more efficiently (Figure 10). Second, as more difficult problems require more compute [49, 50], better test-time scaling of FPRM is mostly visible in harder tasks (Figures 3, 8(a)). Finally, because in FPRM halting is governed by the fixed-point optimizer rather than a learned module, a natural controlling mechanism for the compute-performance trade-off appears in the form of the decay rate $\gamma$ and maximum patience $P$, allowing practitioners to select a desired point on the Pareto front. However, the optimality of the algorithms learned by looped models is not guaranteed, with great variation not only possible but likely. For example, for solving $A_5$, CoT would require a super-logarithmic number of iterations [48], while an optimal algorithm could solve it in logarithmic time. This suboptimal scaling has also been observed in recurrent-in-depth state-space models [50], a behavior that we also observe in Figure 3. Therefore, we propose it as an open challenge to find a latent reasoning architecture that achieves a solution with logarithmic complexity to state-tracking while remaining Turing-complete [15].

**Limitations.** In a similar spirit to previous literature on end-to-end reasoning [18, 20, 22, 39, 40, 65], we test our model only on algorithmic tasks and not on natural language. It is an open challenge to demonstrate that the compositional reasoning behavior that latent models exhibit on algorithmic tasks translates to other domains. In addition, even though the base architecture of FPRM could adopt any model (e.g. CNN, MLP, state-space models), we limit our experiments to Transformers.

*Table 4:* Hyperparameters for Sudoku-Extreme experiments (Table 1 of the paper). Shared across all models: 1×A100-40GB, batch 768, 60 000 epochs, constant LR after 2 000-step warm-up, EMA enabled (rate 0.999), puzzle-embedding length 16. All models are trained with `AdamW` [47].

| | TRM | FPRM |
|---|---|---|
| *Looping structure* | | |
| $H$-cycles | 3 | – |
| $L$-cycles | 6 | – |
| $H$-layers | 0 | 0 |
| $L$-layers | 2 | 2 |
| $n_{\text{back}}$ | $= L$-cycles $+ 1$ | 6 |
| *Halting* | | |
| mechanism | ACT | fixed-point |
| `halt_max_steps` | 16 | – |
| `max_iter` (train) | – | 12 |
| `max_iter` (eval) | – | 35 000 |
| stepsize-decay / patience (eval) | – | 0.997 / 10 |
| `fp_thresh` | – | 0.1 |
| *Block / signal-prop modifications* | | |
| norm type | post-norm | pre-norm |
| residual scaling | ✗ | ✓ |
| $\alpha_1, \alpha_2$ init | – | 0.75, 0.25 |
| conv branch | – | 2D-conv ($3 \times 3$ kernel) |
| *Optimizer* | | |
| learning rate | $10^{-4}$ | $10^{-3}$ |
| weight decay | 1.0 | $10^{-3}$ |
| puzzle-emb LR | $10^{-4}$ | $10^{-3}$ |
| puzzle-emb WD | 1.0 | $10^{-3}$ |

*Table 5:* Hyperparameters for Maze-Hard experiments (Table 1 of the paper). Shared: trained on `maze-30x30-hard-1k` *without augmentation*, 4×A100-80GB, constant LR after a 2 000-step warm-up, EMA enabled (rate 0.999), puzzle-embedding length 16. FPRM trains for 60 000 epochs (TRM 50 000). FPRM is trained using `Adam-Atan2` [21]; TRM is trained using `AdamW`.

| | TRM | FPRM |
|---|---|---|
| *Looping structure* | | |
| $H$-cycles | 3 | – |
| $L$-cycles | 4 | – |
| $H$-layers | 0 | 0 |
| $L$-layers | 2 | 2 |
| $n_{\text{back}}$ | $= L$-cycles $+ 1$ | 6 |
| *Halting* | | |
| mechanism | ACT | fixed-point |
| `halt_max_steps` | 16 | – |
| `max_iter` (train) | – | 24 |
| `max_iter` (eval) | – | 35 000 |
| stepsize-decay / patience (eval) | – | 0.996 / 10 |
| `fp_thresh` | – | 0.1 |
| *Block / signal-prop modifications* | | |
| norm type | post-norm | pre-norm |
| residual scaling | ✗ | ✓ |
| $\alpha_1, \alpha_2$ init | – | 0.75, 0.25 |
| conv branch | – | 1D-conv ($1 \times 4$ kernel) |
| *Optimizer* | | |
| learning rate | $10^{-4}$ | $10^{-4}$ |
| weight decay | 1.0 | 1.0 |
| puzzle-emb LR | $10^{-4}$ | $10^{-2}$ |
| puzzle-emb WD | 1.0 | 1.0 |

*Table 6:* Hyperparameters for state-tracking experiments on $A_5$ and $S_5$ (Figure 3 of the paper) and for the Looped Transformer signal-propagation analysis (Figure 2). Shared: 1×A100-80GB, global batch 1024, `Adam-Atan2`, no LR warm-up, EMA disabled, no puzzle embedding (`puzzle_emb_len`=0). Trained at $k_{\text{train}}$=32, evaluated for $k \in [2, 128]$. TRM and FPRM train for 50 epochs; the Looped Transformer analysis (Fig. 2) for 30.

| | TRM | FPRM | Looped Transformer (Fig. 2) |
|---|---|---|---|
| *Looping structure* | | | |
| $H$-cycles | 2 | – | – |
| $L$-cycles | 4 | – | – |
| $H$-layers | 0 | 0 | 0 |
| $L$-layers | 4 | 2 | 2 |
| $n_{\text{back}}$ | $= L$-cycles $+ 1$ | 4 | 4 |
| *Halting* | | | |
| mechanism | fixed iters or ACT (inference) | fixed-point | fixed iters |
| `halt_max_steps` | 16 | – | – |
| `max_iter` | – | 128 | $= k_{\text{train}}$ |
| iter. distribution | – | deterministic | deterministic |
| *Block / signal-prop modifications* | | | |
| norm type | post-norm | pre-norm | sweep$^\dagger$ |
| norm placement | – | none | none |
| residual scaling | ✗ | ✓ | sweep$^\dagger$ |
| $\alpha_1, \alpha_2$ init | – | 0.5, 0.5 | sweep$^\dagger$ |
| conv branch | – | 1D-conv ($1 \times 4$ kernel) | – |
| *Optimizer* | | | |
| learning rate | $10^{-4}$ | $10^{-3}$ | $10^{-4}$ |
| weight decay | $10^{-2}$ | $10^{-2}$ | $10^{-2}$ |

$^\dagger$ The Looped Transformer row sweeps the {post-norm, pre-norm, pre-norm + residual-scaling} variants from Figure 1(a); the residual-scaling variant uses $\alpha_1$=0.75, $\alpha_2$=0.5 and $k_{\text{train}} \in \{8, 16, 32, 64\}$.

*Table 7:* Hyperparameters for the FPRM ARC-AGI experiments. The two runs share an identical configuration and differ only in the training corpus (ARC1-Concept vs. ARC2-Concept, both with 1000 augmentations per sample). Shared across both: 4×A100-80GB, batch 768, 100 000 epochs, constant LR with no warm-up, EMA enabled (rate 0.999), puzzle-embedding length 16, hidden size 512, 8 heads, MLP expansion 4, RoPE position encodings. Both models are trained with `Adam-Atan2` ($\beta_1$=0.9, $\beta_2$=0.95).

| | FPRM |
|---|---|
| *Looping structure* | |
| $H$-cycles | – |
| $L$-cycles | – |
| $H$-layers | 0 |
| $L$-layers | 2 |
| $n_{\text{back}}$ | 6 |
| *Halting* | |
| mechanism | fixed-point |
| `halt_max_steps` | – |
| `max_iter` (train) | 8 |
| `max_iter` (eval) | 1000 |
| stepsize / decay / patience (eval) | 1.0 / 0.9 / 5 |
| `fp_thresh` | 0.1 |
| *Block / signal-prop modifications* | |
| norm type | pre-norm |
| residual scaling | ✓ |
| $\alpha_1, \alpha_2$ init | 0.75, 0.25 |
| conv branch | 1D-conv (kernel 4) |
| *Optimizer* | |
| learning rate | $10^{-3}$ |
| weight decay | $10^{-2}$ |
| puzzle-emb LR | $10^{-2}$ |
| puzzle-emb WD | 1.0 |