Diffusion Models with Group Equivariance

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Abstract

In recent years, diffusion models have risen to prominence as the foremost technique for distribution learning. This paper focuses on structure-preserving diffusion models (SPDM), a subset of diffusion processes tailored to distributions with inherent structures, such as group symmetries. We complement existing sufficient conditions for constructing SPDM by proving complementary necessary ones. Additionally, we propose a new framework that considers the geometric structures affecting the diffusion process. Within this framework, we propose a method of preserving the alignment between endpoint couplings in bridge models to design a novel structure-preserving bridge model. We validate our findings over a variety of equivariant diffusion models by learning symmetric distributions and the transitions among them. Empirical studies on real-world medical images indicate that our models adhere to our theoretical framework, ensuring equivariance without compromising the quality of sampled images. Furthermore, we showcase the practical utility of our framework by achieving reliable equivariant image noise reduction and style transfer, irrespective of prior knowledge of image orientation, by implementing an equivariant denoising diffusion bridge model (DDBM).

1. Introduction

Diffusion models (Song & Ermon, 2019; Ho et al., 2020; Song et al., 2021a;b; Rombach et al., 2022; Karras et al., 2022; Song et al., 2023) have become the leading method in a plethora of generative modelling tasks including image generation (Song & Ermon, 2019; Ho et al., 2020; Song et al., 2021a), audio synthesis (Kong et al., 2021), image segmentation (Baranchuk et al., 2022; Wolleb et al., 2022), image editing, and style transfer (Meng et al., 2022; Zhou et al., 2024).

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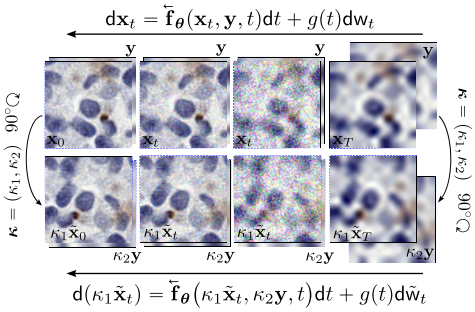


Figure 1: In DDBM (Zhou et al., 2024; Bortoli et al., 2023), $\mathbf{y} = \mathbf{x}_T$ is the starting point of the backward process, a blurry image of cells in the example. SPDM enables a theoretically equivariant inference trajectory: if $(\mathbf{x}_T, \mathbf{y})$ undergos a 90° rotation (via operator $\boldsymbol{\kappa}$), the denoised output precisely mirrors this rotation.

In many generation tasks, the data involved often exhibit inherent "structures" that result in their distributions remaining invariant – or the mappings between them being equivariant – under a set of transformations. In tasks such as image denoising or inpainting, where the orientation of an image is not provided, it is natural to require the denoised or inpainted image to retain the same orientation as the input; i.e., the denoising or inpainting processes should be equivariant (see Fig 1). For example, medical diagnostic images, such as X-ray imaging, are captured from a variety of different orientations (Lafarge et al., 2021; Shao et al., 2023). It such applications is imperative that denoising techniques obey equivariant properties to ensures consistent and reliable disease detection, regardless of the angle at which images are obtained.

This paper investigates structure-preserving diffusion models (SPDM), a family of diffusion processes that preserve the group-invariant properties of distributions. Our framework extends previous research (Yim et al., 2023; Xu et al., 2022; Hoogeboom et al., 2022; Qiang et al., 2023; Martinkus et al., 2023) on drift equivariance by incorporating additional factors that influence diffusion trajectories in both classical and bridge diffusion models, establish an equivalence relation

between the group equivariance of the drifts and the structural preservation of the distributions of induced flows, see Sec 4. Fig 1 illustrates an example of this, for denoising diffusion bridge models (DDBM) (Zhou et al., 2024; Bortoli et al., 2023), where variables $\mathbf{y}$ correspond to starting point $\mathbf{x}_T$ of the backward sampling process. This extension allows us to build a bridge model that captures the equivariant coupling between $\mathbf{x}_T$ and $\mathbf{x}_0$ where a rotation on $\mathbf{x}_T$ results in the same rotation on $\mathbf{x}_0$.

Our developments complement existing discussions around structural diffusion models, which have predominantly focused on identifying sufficient conditions, by providing necessary conditions. We exemplify the utility of our framework by presenting two equivariant score-based models that achieve theoretically guaranteed capabilities for invariant data generation and equivariant data editing: (1) SPDM+WT, using a weight-tied implementation to reduce training and sampling costs at the expense of image quality, and (2) SPDM+FA, employing Framing-Averaging (Puny et al., 2022) to combine outputs from conventionally trained diffusion models, attaining the same theoretical guarantees while achieving a sample quality comparable to that of a standard diffusion model. Notably, unlike other equivariant implementations of diffusion models that incorporate FA into the initial training stage (Martinkus et al., 2023; Duval et al., 2023), our method only applies FA during inference time, significantly decreasing training cost. Additionally, we propose a straightforward method to incorporate a regularization term into existing diffusion models to promote invariant and equivariant properties, see Appx C. We conducted empirical studies on artificial and practical medical image datasets to support our theoretical claims.

2. Related work

The problem of conditioning neural networks to respect group-invariant (or equivariant) distributions (Shawe-Taylor, 1993) has been a longstanding issue within the domains of physical modelling, computer vision, and, more recently, generative modelling. This is underscored by the widespread utilization of diverse forms of data augmentation. However, achieving group invariance (or equivariance) by data augmentation alone necessitates an infeasible many training samples, with models often falling short of being adequately conditioned through data augmentation alone (Elesedy & Zaidi, 2021; Gao et al., 2022).

Equviariant VAEs have been constructed (Huang et al., 2022; Visani et al., 2024) by conditioning the latent space representation to be invariant. This is done by employing techniques similar to those used for conditioning CNNs (e.g., Cohen & Welling, 2016; Ravanbakhsh et al., 2017; Esteves et al., 2018; Kondor & Trivedi, 2018; Knigge et al., 2022; Yarotsky, 2022), such as formulating convolutional kernels and layer pooling operations that preserve the desired equivariance between layers of the network.

Recently, work has appeared on conditioning GANs (Goodfellow et al., 2014; Goodfellow, 2016) to obey generalized group equivariant properties inherent within the data distribution (Dey et al., 2021; Birrell et al., 2022). Initial approaches intuitively replaced all CNN layers within the GAN with group equivariant ones proposed in (Cohen & Welling, 2016). The resulting model was later argued to be incomplete by (Birrell et al., 2022; Duval et al., 2023), who addressed this issue and also proposed a family of invariant divergence measures.

Flow-based frameworks, which are inherently related to diffusion, such as those proposed in (Köhler et al., 2019; Köhler et al., 2020; Rezende et al., 2019; Liu et al., 2019; Biloš & Günnemann, 2021; Satorras et al., 2021; Albergo & Vanden-Eijnden, 2023; Albergo et al., 2023) make use of equivariant normalizing flows to push-forward a group-invariant prior to the target distribution, which ensures that the learned distribution is invariant.

The study of group invariance within diffusion models (Song & Ermon, 2019; Song et al., 2021a; Ho et al., 2020; Karras et al., 2022; Kim et al., 2023; Yim et al., 2023), and now diffusion bridge models (Bortoli et al., 2021; Liu et al., 2023b; Zhou et al., 2024; Lee et al., 2024), has been primarily applied to molecular generation (i.e., molecular conformation, and protein backbone generation) (Shi et al., 2021; Xu et al., 2022; Hoogeboom et al., 2022; Yim et al., 2023; Jing et al., 2022; Corso et al., 2023; Martinkus et al., 2023). Most approaches focus on conditioning the diffusion process on a graph prior and employ a transformation (applied to the inner molecular atomic distances - such as the relative torsion angle coordinates) that elevates the graph into a group-invariant form (or one that is more robust to the selected group transformations). Thereby, resulting in a representation that is sufficient to ensure the diffusion process is equivariant. More generally, the works (De Bortoli et al., 2022; Mathieu et al., 2023), and (Yim et al., 2023), investigate distribution invariance over more general geometries (e.g., Riemannian manifolds generated by Lie groups). The study of distribution invariance comes about naturally as a result of finding a limiting probability distribution over the geometry in these settings, a requirement for the diffusion process to be well-defined. The latter is the most similar to the work presented herein.

3. Preliminary

3.1. The diffusion processes and diffusion bridges

Let $\{\mathbf{x}_t\}_{t=0}^T$ denote a set of time-indexed random variables in $\mathbb{R}^d$ such that $\mathbf{x}_t \sim p_t(\mathbf{x}_t)$, where $p_t(\mathbf{x}_t)$ are the marginal

distributions of an underlying diffusion process defined by a stochastic differential equation (SDE):

$$\mathrm{d}\mathbf{x}_t = \mathbf{u}(\mathbf{x}_t, t)\,\mathrm{d}t + g(t)\,\mathrm{d}\mathsf{w}_t, \quad \mathbf{x}_0 \sim p_0(\mathbf{x}_0). \tag{1}$$

Here, $\mathbf{u} : \mathbb{R}^d \times [0, T] \to \mathbb{R}^d$ is the *drift*, $g : [0, T] \to \mathbb{R}$ is a scalar *diffusion coefficient*, and $\mathsf{w}_t \in \mathbb{R}^d$ denotes a Wiener process. In generative diffusion models, we take $p_0 = p_{\text{data}}$ and $p_T = p_{\text{prior}}$; thereby, the diffusion process constructs a path p_t from p_{data} to p_{prior}.

In practice, $\mathbf{u}$ and g are chosen to accelerate the sampling of $\mathbf{x}_t$ in (1). Table 1 lists two popular choices of $\mathbf{f}$ and g, respectively corresponding to the variance preserving (VP) (Ho et al., 2020; Song et al., 2021a) and variance exploding (VE) SDEs (Song et al., 2021b). For these selections, $p(\mathbf{x}_t|\mathbf{x}_0)$ is an easy-to-sample spherical Gaussian, and the sampling of $\mathbf{x}_t$ is carried out by first picking $\mathbf{x}_0 \sim p_0(\mathbf{x}_0)$ and sampling from $p(\mathbf{x}_t|\mathbf{x}_0)$.

For any SDE, there is a corresponding reverse SDE with the same marginal distribution p_t for all $t \in [0, T]$ (Anderson, 1982). In fact, the forward SDE in (1) has a family of reverse-time SDEs (Zhang & Chen, 2023):

$$\begin{aligned}\mathrm{d}\mathbf{x}_t =& \Big[\mathbf{u}(\mathbf{x}_t, t) - \tfrac{1+\lambda^2}{2} g^2(t) \nabla_{\mathbf{x}_t} \log p_t(\mathbf{x}_t)\Big)\Big]\,\mathrm{d}t \\ &+ \lambda g(t)\,\mathrm{d}\mathsf{w}_t \;\text{ for all } \lambda \geq 0.\end{aligned} \tag{2}$$

Setting $\lambda = 1$ in (2) simplifies the equation to the original reverse SDE as derived in (Anderson, 1982). For $\lambda = 0$, the process transforms into a deterministic ODE process, known as the probability flow ODE (PF-ODE) (Song et al., 2021b). Here, the only unknown term is the score $\nabla_{\mathbf{x}} \log p_t(\mathbf{x})$, which can be estimated using a neural network $\mathbf{s}_{\boldsymbol{\theta}}(\mathbf{x}, t)$ trained through score matching (Song et al., 2021b) or an equivalent denoising loss (Karras et al., 2022). Subsequently, from $\mathbf{x}_T \sim p_T(\mathbf{x}_T)$, one can sample $\mathbf{x}_0 \sim p_{\text{data}}(\mathbf{x}_0)$ by solving the SDE (or ODE if $\lambda = 0$) given in (2).

Instead of building path p_t from a data distribution to a prior, given $q_{\text{data}}(\mathbf{x}, \mathbf{y})$, diffusion bridges (DBs) create path q_t such that $q_0(\mathbf{x}) = q_{\text{data}}(\mathbf{x})$ and $q_T(\mathbf{y}) = q_{\text{data}}(\mathbf{y})$. For $(\mathbf{x}, \mathbf{y}) \sim q_{\text{data}}(\mathbf{x}, \mathbf{y})$, DBs leverage the distribution p_t induced by (1) with $\mathbf{x}_0 = \mathbf{x}$ and $\mathbf{x}_T = \mathbf{y}$ to sample $\mathbf{x}_t$. In this way, $q_t(\mathbf{x}_t) = \mathbb{E}_{(\mathbf{x},\mathbf{y})\sim q_{\text{data}}(\mathbf{x},\mathbf{y})}\big[p_t(\mathbf{x}_t|\mathbf{x}_0 = \mathbf{x}, \mathbf{x}_T = \mathbf{y})\big]$ and admits the forward SDE:

$$\mathrm{d}\mathbf{x}_t = \mathbf{u}(\mathbf{x}_t, t) + g(t)^2 \mathbf{h}(\mathbf{x}, \mathbf{x}_T, t) + g(t)\,\mathrm{d}\mathbf{w}_t, \tag{3}$$

given $\mathbf{x}_T = \mathbf{y}$ and $\mathbf{h}(\mathbf{x}, \mathbf{x}_T, t) = \nabla_{\mathbf{x}_t} \log p(\mathbf{x}_T|\mathbf{x})$ is the gradient of the log transition kernel induced by (1). For $\mathbf{u}$ and g in Table 1, $p_t(\mathbf{x}_t|\mathbf{x}_0, \mathbf{x}_T)$ can be sampled efficiently. Thus, $\mathbf{x}_t \sim q_t(\mathbf{x}_t)$ can be obtained by first sampling $(\mathbf{x}, \mathbf{y})$ and then $\mathbf{x}_t$. DBs can be broadly categorized into those that condition on $\mathbf{x}_T$ and those that do not. Bortoli et al. show that conditioned implementation is necessary for effectively learning the coupling encoded in q_{data}. This is crucial for many practical tasks, such as image denoising, where the denoised image should match the blurry input. For the conditioned DBs (Zhou et al., 2024), the family of the backward SDE, conditioned on $\mathbf{x}_T = \mathbf{y}$, is

$$\begin{aligned}\mathrm{d}\mathbf{x}_t =& \mathbf{u}(\mathbf{x}_t, t) + g(t)^2 \mathbf{h}(\mathbf{x}, \mathbf{x}_T, t) \\ &- \tfrac{1+\tau^2}{2} g(t)^2 \mathbf{s}(\mathbf{x}_t|\mathbf{x}_T, t) + \tau g(t)\,\mathrm{d}\mathbf{w}_t,\end{aligned} \tag{4}$$

for all $\tau \geq 0$, where $\mathbf{s}(\mathbf{x}_t|\mathbf{x}_T, t) = \nabla_{\mathbf{x}_t} \log q_t(\mathbf{x}_t|\mathbf{x}_T)$ is the score of the DB's distribution q_t given that $\mathbf{x}_T = \mathbf{y}$.

Notably, in addition to the noisy sample $\mathbf{x}_t$, the drift terms in (3) and (4) also depend on $\mathbf{y} = \mathbf{x}_T$, motivating us to consider a more general diffusion process

$$\mathrm{d}\mathbf{x}_t = \mathbf{f}(\mathbf{x}_t, \mathbf{y}, t)\,\mathrm{d}t + g(t)\,\mathrm{d}\mathsf{w}_t \qquad \mathbf{x}_0 \sim p(\mathbf{x}_0|\mathbf{y}), \tag{5}$$

where $\mathbf{y}$ represents other factors affecting the process and does not need to have the same shape as $\mathbf{x}_t$.

3.2. Group invariance and equivariance

A set of functions $\mathcal{G} = \{\kappa : \mathcal{X} \to \mathcal{X}\}$ equip with a associative binary operation $\circ : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, composition in this case, is called a *group* if (1) for any $\kappa_1, \kappa_2 \in \mathcal{G}$, $\kappa_1 \circ \kappa_2 \in \mathcal{G}$; (2) $\mathcal{G}$ has an identity operator $\mathbf{e}$ with $\mathbf{e} \circ \kappa = \kappa \circ \mathbf{e} = \kappa$; and (3) for any $\kappa \in \mathcal{G}$, there exists an inverse operator κ^{-1} such that $\kappa^{-1} \circ \kappa = \kappa \circ \kappa^{-1} = \mathbf{e}$. For instance, let f_x be the operator that flips images horizontally. Then $\mathcal{G} = \{\mathsf{f}_x, \mathbf{e}\}$ is a group where $\mathsf{f}_x^{-1} = \mathsf{f}_x$.

In this paper, we are interested in the κ that do not alter the distributions of Wiener processes, which have a standard Gaussian increment. As the density of a standard Gaussian is fully determined by the distance from the sample to the origin, we hereafter assume that $\mathcal{G}$ consists of isometries κ that fix zero; i.e., $\|\kappa\mathbf{x}\|_2 = \|\mathbf{x}\|_2$ and $\|\kappa\mathbf{x} - \kappa\mathbf{y}\|_2 = \|\mathbf{x} - \mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$. Under this assumption, we can show that κ is linear isometry. (Proof in Appx A.1.) Therefore, $\kappa \in \mathcal{G}$ can be written as

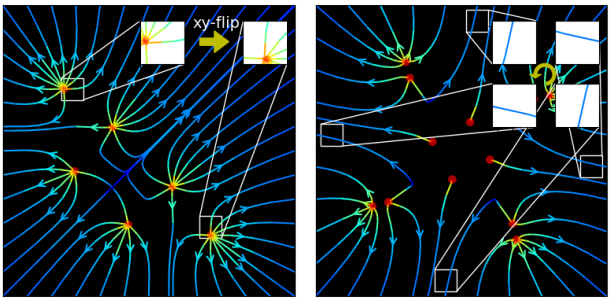
$$\kappa\mathbf{x} = A_\kappa \mathbf{x} \tag{6}$$

for some orthogonal $A_\kappa \in \mathbb{R}^{d\times d}$. To explore the structure-preserving property of the diffusion process in (5), which depends on $\mathbf{x}_t \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^n$, we consider a "stacked" linear isometry group $\mathbf{G} = \{\boldsymbol{\kappa} = (\kappa_1, \kappa_2) | \kappa_1 : \mathbb{R}^m \to \mathbb{R}^m, \kappa_2 : \mathbb{R}^n \to \mathbb{R}^n\}$ defined in $\mathbb{R}^{m+n}$ such that $\boldsymbol{\kappa}(\mathbf{x}, \mathbf{y}) = (\kappa_1\mathbf{x}, \kappa_2\mathbf{y}) = (A_{\kappa_1}\mathbf{x}, A_{\kappa_2}\mathbf{y})$ with orthogonal $A_{\kappa_1} \in \mathbb{R}^{m\times m}$ and $A_{\kappa_2} \in \mathbb{R}^{n\times n}$.

For a distribution with density p defined on $\mathbb{R}^d$, we say p is $\mathcal{G}$*-invariant* if $p(\mathbf{x}) = p(\kappa\mathbf{x})$ for all $\kappa \in \mathcal{G}$. Likewise, for a conditional distribution $p(\mathbf{x}|\mathbf{y})$ with $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^n$,

Table 1: Choices of $\mathbf{u}(\mathbf{x}, t)$ and $g(\mathbf{x})$ where $\eta_t = \frac{\alpha_t^2}{\sigma_t^2}$ (Zhou et al., 2024). VE is a special case of VP that $\alpha_t = 1$

SDE	$\mathbf{u}(\mathbf{x}, t)$	$g(t)^2$	$p(\mathbf{x}_t \vert \mathbf{x}_0)$	$\nabla_{\mathbf{x}_t} \log p(\mathbf{x}_T \vert \mathbf{x}_t)$	$p_t(\mathbf{x}_t \vert \mathbf{x}_0, \mathbf{x}_T)$
VP	$\frac{\mathrm{d}\log\alpha_t}{\mathrm{d}t}\mathbf{x}$	$\frac{\mathrm{d}\sigma_t^2}{\mathrm{d}t} - \frac{\mathrm{d}\log\alpha_t^2}{\mathrm{d}t}\sigma_t^2$	$\mathcal{N}(\alpha_t\mathbf{x}_0, \sigma_t^2\mathbf{I})$	$\frac{(\alpha_t/\alpha_T)\mathbf{x}_T - \mathbf{x}_t}{\sigma_t^2(\eta_t/\eta_T - 1)}$	$\mathcal{N}\Big(\frac{\eta_T}{\eta_t}\frac{\alpha_t}{\alpha_T}\mathbf{x}_T + \alpha_t\mathbf{x}_0(1 - \frac{\eta_T}{\eta_t}), \sigma_t^2(1 - \frac{\eta_T}{\eta_t})\Big)$
VE	$\mathbf{0}$	$\frac{\mathrm{d}\sigma_t^2}{dt}$	$\mathcal{N}(\mathbf{x}_0, \sigma_t^2\mathbf{I})$	$\frac{\mathbf{x}_T - \mathbf{x}_t}{\sigma_T^2 - \sigma_t^2}$	$\mathcal{N}\Big(\frac{\sigma_t^2}{\sigma_T^2}\mathbf{x}_T + (1 - \frac{\sigma_t^2}{\sigma_T^2})\mathbf{x}_0, \sigma_t^2(1 - \frac{\sigma_t^2}{\sigma_T^2})\Big)$



(a) (b)

Figure 2: The vector fields of score functions that are equivariant under xy-flip (left) and $(90 \cdot k)^\circ$ rotation for $k = 0, 1, 2, 3$ (right).

we say it is $\mathbf{G}$-*invariant* if $p(\kappa_1\mathbf{x}|\kappa_2\mathbf{y}) = p(\mathbf{x}|\mathbf{y})$ for all $(\kappa_1, \kappa_2) \in \mathbf{G}$, $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^n$. The following lemma shows if a distribution is $\mathcal{G}$-invariant, the learned score has to be $\mathcal{G}$-*equivariant*. (All the proofs can be found in Appx A.)

Lemma 1. *$p(\mathbf{x}|\mathbf{y})$ is $\mathbf{G}$-invariant if and only if $\mathbf{s}(\kappa_1\mathbf{x}|\kappa_2\mathbf{y}) = \kappa_1 \circ \mathbf{s}(\mathbf{x}|\mathbf{y})$ for all $(\kappa_1, \kappa_2) \in \mathbf{G}$, $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^n$. Likewise, $p(\mathbf{x})$ is $\mathcal{G}$-invariant if and only if $\mathbf{s}(\kappa\mathbf{x}) = \kappa \circ \mathbf{s}(\mathbf{x})$ for all $\kappa \in \mathcal{G}$.*

To see how Lem 1 works, in Fig 2, we illustrate this commutative relationship for unconditional distributions when subject to xy-flips and rotations. In Fig 2a, it is evident that when we select symmetric points relative to the diagonal $x = y$, the score at these two points also exhibits symmetry. This implies that we can obtain an identical score vector by either flipping a point and then computing its score or by evaluating its score first and subsequently applying the flip operation. Similarly, in Fig 2b, we can derive the score vector by rotating a point about the origin and then evaluating its score or by evaluating its score first and then applying the rotation operation.

4. Structure preserving diffusion processes

In this section, we discuss explore the sufficient and necessary configurations of diffusion processes that preserve a distributions invariance throughout the entire diffusion trajectory. These insights will serve as a foundation for the design of the corresponding equivariant neural networks in Sec 5. We present our main theoretical results in Prop 1:

Proposition 1. *Given a diffusion process in (5) with $\mathbf{G}$-invariant $p_0(\mathbf{x}_0|\mathbf{y})$, let $[\mathbf{0}]_{p_t}$ be the set of ODE drifts preserving the distribution p_t. Then $p_t(\mathbf{x}_t|\mathbf{y})$ is $\mathbf{G}$-invariant for all $t \geq 0$ if and only if*

$$\kappa_1^{-1} \circ \mathbf{f}(\kappa_1\mathbf{x}, \kappa_2\mathbf{y}, t) - \mathbf{f}(\mathbf{x}, \mathbf{y}, t) \in [\mathbf{0}]_{p_t} \tag{7}$$

for all $t > 0$, $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^n$ and $\boldsymbol{\kappa} \in \mathbf{G}$.

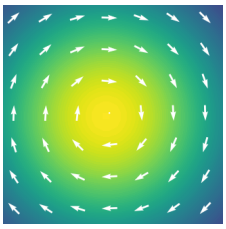


Figure 3: A distribution preserving drift of $\mathcal{N}(\mathbf{0}, \mathbf{I})$ in $\mathbb{R}^2$.

This proposition tells us that a diffusion process retains the $\mathbf{G}$-invariance property of the data distribution p_0 throughout the trajectory if and only if $\kappa_1^{-1} \circ \mathbf{f}(\kappa_1\mathbf{x}, \kappa_2\mathbf{y}, t) - \mathbf{f}(\mathbf{x}, \mathbf{y}, t) = \mathbf{0}$ up to some terms $\boldsymbol{\delta}(\mathbf{x}, \mathbf{y}, t)$ satisfying Louisville equation (Oksendal, 2003; Ehrendorfer, 2006):

$$0 = \frac{\partial}{\partial t}p_t(\mathbf{x}|\mathbf{y}) = -\nabla_{\mathbf{x}} \cdot \big(p_t(\mathbf{x}|\mathbf{y})\boldsymbol{\delta}(\mathbf{x}, \mathbf{y}, t)\big). \tag{8}$$

Note, $\boldsymbol{\delta}$ is not necessarily zero; .e.g., Fig 3 presents a non-zero drift that does not change $\mathcal{N}(\mathbf{0}, \mathbf{I})$ in $\mathbb{R}^2$. (More discussions are given in Appx A.3)

Moreover, while Prop 1 is presented based on the conditional distribution $p_t(\mathbf{x}_t|\mathbf{y})$, by setting $n = 0$ we recover the unconditional case, as $p_t(\mathbf{x}_t|\mathbf{y})$ reduces to $p_t(\mathbf{x}_t)$, $\mathbf{G}$ simplifies to $\mathcal{G}$ consisting of κ_1, and (7) becomes $\kappa_1^{-1} \circ \mathbf{f}(\kappa_1\mathbf{x}, t) - \mathbf{f}(\mathbf{x}, t) \in [\mathbf{0}]_{p_t}$. Existing structure-preserving diffusion models (Yim et al., 2023; Xu et al., 2022; Hoogeboom et al., 2022; Qiang et al., 2023; Martinkus et al., 2023) are based on the special case that $\kappa_1^{-1} \circ \mathbf{f}(\kappa_1\mathbf{x}, t) - \mathbf{f}(\mathbf{x}, t) = \mathbf{0}$.

To gain an intuitive understanding of why (7) results in the $\mathbf{G}$-invariance of $p_t(\mathbf{x}|\mathbf{y})$, we illustrate how the proposition works in DB models where the stacked group $\mathbf{G} = \{\boldsymbol{\kappa} = (\kappa, \kappa) | \kappa \in \mathcal{G}\}$. For simplicity, we assume $\mathbf{f}(\kappa\mathbf{x}, \kappa\mathbf{y}, t) - \kappa \circ \mathbf{f}(\mathbf{x}, \mathbf{y}, t) = \mathbf{0}$, which implies $\mathbf{f}$ is equivariant. When $\mathbf{f}$ is equivariant, it essentially says for an infinitesimal step δ, the transition probability induced by the SDE in (5) satisfies

$$p(\mathbf{x}_{t-\delta}|\mathbf{x}_t, \mathbf{x}_T) = p(\kappa\mathbf{x}_{t-\delta}|\kappa\mathbf{x}_t, \kappa\mathbf{x}_T). \tag{9}$$

As $p_T(\mathbf{x}_T|\mathbf{x}_T) = p_T(\kappa\mathbf{x}_T|\kappa\mathbf{x}_T)$, which is $\mathbf{G}$-invariant, applying this relationship recursively from $t = T$ to 0 implies $p(\mathbf{x}_t|\mathbf{x}_T) = p(\kappa\mathbf{x}_t|\kappa\mathbf{x}_T)$ for $t \in [0, T]$. (Since the SDE is solved reversely, the base case becomes the invariance of p_T instead of p_0.) In Fig 4 (**Left**), we visualize the evolution of conditional p_t when the conditioned end point $\mathbf{x}_T$ is flipped w.r.t. $x = 0$. As we can see when $\mathbf{x}_T$ is flipped, the trajectory from $\mathbf{x}_T$ to $\mathbf{x}_t$ is also flipped, which corresponds to the invariance of $p(\mathbf{x}_t|\mathbf{x}_T)$. In Fig 4 (**Right**), we also present an example of unconditional p_t in Prop 1 by setting $m = 0$. In this case, to ensure that p_t is invariant, it suffices that $\mathbf{f}(\kappa_1\mathbf{x}, t) - \kappa_1 \circ \mathbf{f}(\mathbf{x}, t) = \mathbf{0}$. Here, we visualize the evolution of p_t driven by two different diffusion processes with p_0 invariant to flipping with respect to $x = 0$ (that is, $\kappa_1(x) = -x$). Specifically, for the upper plot, we have drift $\mathbf{f}(x, t) = \frac{1-x}{1-t}$ that pushes x to 1 and is not flip-equivariant for $t \geq 0$. As we observe, for all $t > 0$, p_t is no longer flip-invariant, which corroborates Prop 1. For the second plot, we select the VP-SDE (see Table 1) with $\alpha_t = 1-t$ for $t \in (0, 1)$. Then its drift $\mathbf{f}(x, t) = -\frac{x}{2(1-t)}$ is equivariant to flipping as $\mathbf{f}(-x, t) = -\mathbf{f}(x, t)$. As suggested in the plot, p_t has a symmetric density for all $t \geq 0$, which is also aligned with Prop 1.

In summary, Prop 1 shows for conditional p_t, (7) ensures the coupling relationship between condition $\mathbf{y}$ and the noise sample $\mathbf{x}_t$ is $\mathbf{G}$-invariant. In contrast, for unconditional p_t, it ensures the sample $\mathbf{x} \sim p_t$ follows the same distribution when transformed by $\kappa \in \mathcal{G}$. This result can be generalized to all possible groups consisting of linear isometries and linear drifts where $\mathbf{u}(\mathbf{x}, t) = \mathbf{x}u(t)$:

Proposition 2. *Assume $\mathbf{u}(\mathbf{x}, t) = u(t)\mathbf{x}$ for some scalar function $u : \mathbb{R} \to \mathbb{R}$. Given any group $\mathcal{G}$ (or $\mathbf{G}$) composed of linear isometries, if the unconditional p_t induced by (1) is G-invariant at $t = 0$, then it is $\mathcal{G}$-invariant for all $t \geq 0$. Likewise, if the conditional $q_t(\mathbf{x}_t|\mathbf{x}_T)$ induced by (3) is $\mathbf{G}$-invariant at $t = 0$ then it is $\mathbf{G}$-invariant for all $t \geq 0$.*

Since the drift terms of both VP and VE-SDE in Table 1 take the form $u(t)\mathbf{x}$, Prop 2 indicates that the induced diffusion process and the corresponding diffusion bridges are structure-preserving for any group composed of linear isometries.

5. Structure preserving diffusion models

In this section, we apply the insights from Sec 4 to ensure the data generated by SPDM adheres to a $\mathbf{G}$-invariant distribution. As mentioned in Sec 3, sampling a diffusion model involves solving the SDE in (2) or (4) by estimating the score using a neural network $\mathbf{s}_{\boldsymbol{\theta}}$. We will discuss several effective methods to design and train $\mathbf{s}_{\boldsymbol{\theta}}$ so that it meets the properties outlined in Prop 1, achieving theoretically guaranteed structure-preserving sampling.

5.1. Structure preserving sampling

Unconditioned distribution sampling. By Prop 1, when a diffusion process is structure preserving, p_t is $\mathcal{G}$-invariant for all $t \geq 0$. So given the prior distribution p_T is $\mathcal{G}$-invariant and by Lem 1, for all $t \geq 0$, the score $\nabla_{\mathbf{x}} \log p_t(\mathbf{x})$ is $\mathcal{G}$-equivariant. As a result, as long as the score estimator $\mathbf{s}_{\boldsymbol{\theta}}(\mathbf{x}, t)$ perfectly learns the $\mathcal{G}$-equivariant property and satisfies (7), the drift of reverse-time SDE (2):

$$\overleftarrow{\mathbf{f}}_{\boldsymbol{\theta},\lambda}(\mathbf{x}_t, t) = \mathbf{f}(\mathbf{x}_t, t) - \tfrac{1}{2}(1 + \lambda^2)g^2(t)\, \mathbf{s}_{\boldsymbol{\theta}}(\mathbf{x}, t) \quad (10)$$

also satisfies (7) as $[\mathbf{0}]_{p_t}$ is closed under addition (see Appx A.3). Applying Prop 1 with reversed t, we can then conclude that the generated samples must follow a $\mathcal{G}$-invariant distribution.

Equivariant style-transfer through diffusion bridges conditioned on $\mathbf{x}_T$. When the drift $\mathbf{u}(\mathbf{x}, t)$ of original SDE in (1) satisfies (7), given stacked group $\mathbf{G} = \{(\kappa, \kappa)|\kappa \in \mathcal{G}\}$, we can show that the drift $\mathbf{u}(\mathbf{x}_t, t) + g(t)^2\mathbf{h}(\mathbf{x}_t, \mathbf{x}_T, t)$ in (3) also satisfies (7) (see Lem 13 in Appx A.5 for the proof). As a result, by Prop 1, $p_t(\mathbf{x}_t|\mathbf{x}_T)$ is $\mathbf{G}$-invariant for all $t \in [0, T]$ and thus by Lem 1, its score is equivariant and thus satisfies (7). In this way, if the score estimator $\mathbf{s}_{\boldsymbol{\theta}}(\mathbf{x}_t, \mathbf{x}_T, t)$ perfectly learns the equivariant property such that $\mathbf{s}_{\boldsymbol{\theta}}(\kappa\mathbf{x}_t, \kappa\mathbf{x}_T, t) = \kappa\mathbf{s}_{\boldsymbol{\theta}}(\mathbf{x}_t, \mathbf{x}_T, t)$, the drift of reverse-time SDE (4):

$$\begin{aligned} \mathrm{d}\mathbf{x}_t =& \mathbf{u}(\mathbf{x}_t, t) + g(t)^2\mathbf{h}(\mathbf{x}_t, \mathbf{x}_T, t) \\ & - \tfrac{1+\tau^2}{2} g(t)^2 \mathbf{s}_{\boldsymbol{\theta}}(\mathbf{x}_t, \mathbf{x}_T, t) \end{aligned} \quad (11)$$

satisfies (7); therefore, the invariant coupling between $\mathbf{y} = \mathbf{x}_T$ and $\mathbf{x}_t$ is preserved for all $t \in [0, T]$ during the sampling process.

Due to this observation, to ensure the generated data preserve the necessary geometric structure, it is sufficient to train a group equivariant score estimator $\mathbf{s}_{\boldsymbol{\theta}}$. Here, we present two theoretically guaranteed $\mathcal{G}$-equivariant implementations of $\mathbf{s}_{\boldsymbol{\theta}}$, SPDM+WT and SPDM+FA.

Weight Tying (SPDM+WT). Currently, almost all existing diffusion models are based on the U-Net backbone (Salimans et al., 2017; Ronneberger et al., 2015). As the only components that are not equivariant are CNNs, we replace them with group-equivariant CNNs (Cohen & Welling, 2016; Ravanbakhsh et al., 2017; Esteves et al., 2018; Kondor & Trivedi, 2018; Knigge et al., 2022; Yarotsky, 2022) to make the entire network equivariant.

In particular, as we only consider linear isometry groups, we can make CNNs equivariant by tying the weights of the convolution kernels k, which will also reduce the total number of parameters and improve the computation efficiency (Ravanbakhsh et al., 2017). (For more general groups, refer to (Cohen & Welling, 2016; Knigge et al., 2022) for methods to make CNNs $\mathcal{G}$-equivariant.) We provide more details

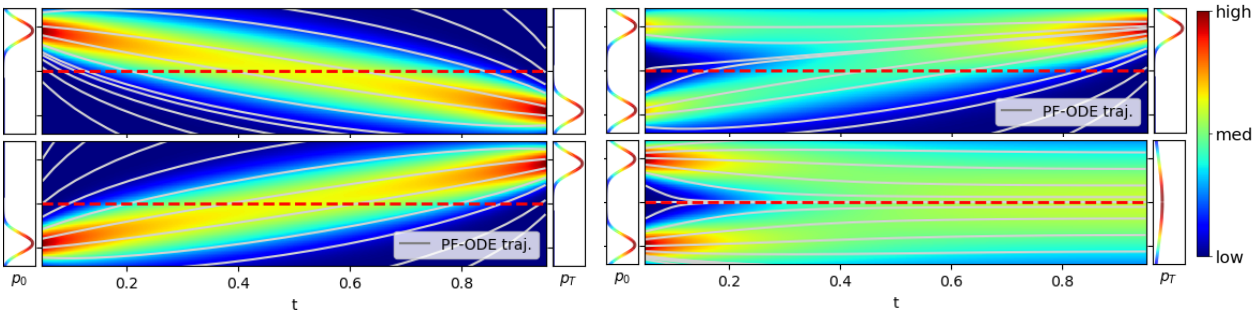


Figure 4: **Left:** The evolution of p_t driven by DB processes induced by the VE-SDE in Table 1 conditioned on the end point $\mathbf{x}_T = -1$ (upper) and $\mathbf{x}_T = 1$ (lower). **Right:** The upper plot has $f(x,t) = \frac{1-x}{1-t}$ and $g(t) = 1$. The lower is the VP-SDE in Table 1 with $\alpha_t = 1 - t$.

on our selections of weight-tied kernels for empirical study in Appx B.

Output Combining (SPDM+FA). When $\mathcal{G}$ contains finite elements, we can achieve $\mathcal{G}$-equivariance through frame averaging (FA) (Puny et al., 2022), leveraging the following fact: for any function $\mathbf{r} : \mathbb{R}^d \to \mathbb{R}^d$,

$$\tilde{\mathbf{r}}(\mathbf{x}) = \frac{1}{|\mathcal{G}|} \sum\nolimits_{\kappa \in \mathcal{G}} \kappa^{-1}\, \mathbf{r}(\kappa \mathbf{x}, \kappa \mathbf{y}) \tag{12}$$

is $\mathcal{G}$-equivariant, where $|\mathcal{G}|$ denotes the number of elements in $\mathcal{G}$ and the second argument of $\mathbf{r}$ can be discarded for the approximation of the score not conditioned on $\mathbf{y}$. Based on this fact, we can obtain an equivariant estimator $\tilde{\mathbf{s}}_{\boldsymbol{\theta}}(\cdot, t)$ of the score by setting $\mathbf{r}(\cdot) = \mathbf{s}_{\boldsymbol{\theta}}(\cdot, t)$. Note that unlike other FA-based diffusion models (Martinkus et al., 2023; Duval et al., 2023), our method trains $\mathbf{s}_{\boldsymbol{\theta}}(\cdot, t)$ using regular score-matching and only adopts FA to build the theoretically guaranteed equivariant score estimator during the inference time. This design significantly saves training costs. To see why FA is not necessary during training, we note that our previous discussion has shown that the ground truth score function is equivariant; therefore, when $\mathbf{s}_{\boldsymbol{\theta}}$ is well trained, estimator $\kappa^{-1}\mathbf{s}_{\boldsymbol{\theta}}(\kappa\, \cdot, t)$ for $\kappa \in \mathcal{G}$ will produce very similar output, estimating the value of the score function at $\mathbf{x}_t$ (given $\mathbf{y}$). Thus, their average is also a valid estimator. To boost the model's performance, additional regularizers can be used to encourage $\kappa^{-1}\mathbf{s}_{\boldsymbol{\theta}}(\kappa\, \cdot, t)$ for $\kappa \in \mathcal{G}$ to have the same output; we discuss this technique with extra details in Appx C.

5.2. $\mathcal{G}$-equivariant trajectory

The sampling trajectory of the SPDM is governed by the SDE in (2) or (4), which can be collectively written as

$$\mathrm{d}\mathbf{x}_t = \overleftarrow{\mathbf{f}}_{\boldsymbol{\theta},\lambda}(\mathbf{x}_t, \mathbf{y}, t)\, \mathrm{d}t + \lambda g(t)\, \mathrm{d}\mathsf{w}_t \tag{13}$$

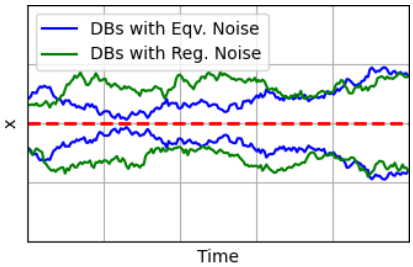


Figure 5: Equivariant DB trajectories with/without equivariant noises.

where $\mathbf{y}$ can be optionally discarded. In practice, the sampling process solves (13) through:

$$\mathbf{x}_{i-1} \leftarrow \overleftarrow{\mathbf{f}}_{\boldsymbol{\theta},\lambda}(\mathbf{x}_i, \mathbf{y}, t_i)(t_{i-1} - t_i) + \lambda g(t)\sqrt{t_i - t_{i-1}}\, \boldsymbol{\epsilon}_i \tag{14}$$

with preset time steps $\{t_i\}_{i=1}^n$ and $\boldsymbol{\epsilon}_i \sim \mathcal{N}(\mathbf{0}, I)$. While the techniques discussed in Sec 5.1 ensure the equivariance of $\overleftarrow{\mathbf{f}}_{\boldsymbol{\theta},\lambda}$, they do not guarantee the equivariance of the sampled noise sequence $\{\boldsymbol{\epsilon}_i\}_{i=1}^n$. As a result, the sample trajectory $\mathbf{x}_t$ may not be equivariant. This is visualized in Fig 5 with green curves, where the drifts of DBs are equivariant to the flip about $x = 0$ but the trajectory is not.

Due to this asymmetry, the output of the SPDM will not be theoretically equivariant. One option to address this problem is to adopt ODE sampling by setting $\lambda = 0$. However, this method is not always preferred as SDE sampling can significantly improve image quality (Karras et al., 2022; Song et al., 2021b; Zhou et al., 2024). For $\lambda > 0$, we also need $\{\boldsymbol{\epsilon}_i\}_{i=1}^n$ to be "equivariant" such that for $\kappa \in \mathcal{G}$, if the starting point $\mathbf{x}_n$ is updated to $\kappa\mathbf{x}_n$, then $\{\boldsymbol{\epsilon}_i\}_{i=1}^n$ is also updated to $\{\tilde{\boldsymbol{\epsilon}}_i\}_{i=1}^n$ with $\tilde{\boldsymbol{\epsilon}}_i = \kappa\boldsymbol{\epsilon}_i$. In this way, the trajectory becomes equivariant, as shown by the blue curves in Fig 5. In Appx D, we present a simple technique to achieve this by fixing the random seed and matching some artificial features between $\mathbf{x}_n$ and $\boldsymbol{\epsilon}_n$. In our empirical study, we use this

method to inject noise into $\mathbf{x}_t$ as shown in Fig 1 so that the rotation of the input results in a precise output rotation.

6. Empirical study

In this section, we present experiments demonstrating the effectiveness of the methods from Sec 5. The results support our theoretical work in Sec 4 and offer additional insights.

Datasets. Our study adopts, rotated MNIST (Larochelle et al., 2007), LYSTO (Jiao et al., 2023), and ANHIR (Borovec et al., 2020a) that have been used in the past to evaluate the equivaraince of generative models. We also validate our models effectiveness on denoinsing LYSTO images, and style transfer from CT scan images to PET scan images of the same patients from the CT-PET dataset (Gatidis et al., 2022).

Rotated MNIST dataset (Larochelle et al., 2007) contains random 90° rotations of MNIST images (Deng, 2012), resulting in a C_4-invariant distribution. This dataset is commonly used to evaluate group-invariant CNN models, as seen in (Dey et al., 2021; Birrell et al., 2022), with experiments on 1% (600), 5% (3000), and 10% (6000) of the dataset. The LYSTO dataset (Jiao et al., 2023) includes 20,000 image patches from breast, colon, and prostate cancer samples stained with CD3 or CD8 dyes, exhibiting D_4 invariance due to natural rotational and mirror invariance. Following (Birrell et al., 2022), models are trained on randomly selected 64x64x3 crops from the scaled-down (128x128x3) LYSTO dataset. ANHIR dataset (Borovec et al., 2020b) provides 15kx15k images of lesions, lung-lobes, and mammary-glands, from which we extract random 64x64x3 patches from lung-lobes according to the method used in (Dey et al., 2021).

For the denoising and style-transfer tasks, LYSTO 64x64x3 patches are downscaled to $1/4$ resolution and then upscaled back to form training pairs. The CT-PET dataset (Gatidis et al., 2022) includes 1014 annotated whole-body paired FDG-PET/CT scans of patients with malignant lymphoma, melanoma, and non-small cell lung cancer. This dataset is flipping invariant. The style-transfer task transforms a patient's CT scan into a PET scan.

6.1. Models

We implement regular diffusion models and bridge models (DDBM) (Zhou et al., 2024) based on VP-SDEs (Ho et al., 2020; Song et al., 2021a), which are structure-preserving with respect to C_4, D_4, and flipping, as per Prop 2. Except for SPDM-WT, all models are trained with data augmentation using randomly selected operators from their respective groups. For generation tasks, we present the performance of the standard diffusion model, VP-SDE, as a baseline, along with SP-GAN (Birrell et al., 2022), the only GAN-based model with theoretical group invariance guarantees. We also report the mean performance of GE-GAN (Dey et al., 2021). The tested models and their invariance and equivariance properties are summarized in Table 5. To boost performance, we apply non-leaky augmentation as in EDM (Karras et al., 2022) and self-conditioning (Chen et al., 2023) to improve sample quality.

For style-transfer tasks, in addition to the original DDBM implementation (Zhou et al., 2024), we report the performance of the popular style-transfer method Pix2Pix (Isola et al., 2017) and the unconditional diffusion bridge model $\mathrm{I}^2\mathrm{SB}$ (Liu et al., 2023a). For the denoising task on LYSTO, all models use pixel-level implementation. For the CT-PET dataset, all models except Pix2Pix are trained in the latent space, with images first encoded by a fine-tuned pretrained VAE from Stable Diffusion (Rombach et al., 2022). FA was applied during fine-tuning and inference to ensure equivariance. Detailed configurations are provided in Appx F.

6.2. Image generation

We report the FID score (Heusel et al., 2017) of each model in Table 2 and Table 3. To ensure consistency, we reproduced the results of SP-GAN and GE-GAN and computed the FID using the standard InceptionV3 model. We note that the reproduced FID of GE-GAN is significantly higher than reported by Dey et al., as their score is based on a customized InceptionV3 finetuned on LYSTO and ANHIR. While included in the table for reference, these scores are not comparable with other FIDs (details on our FID calculation are provided in Appx G). All FIDs are based on 50,000 randomly generated images. Sample images from each model are presented in Appx H. Table 2 and Table 3 shows that diffusion-based models consistently outperform SP-GAN across all datasets. Notably, SPDM+WT and SPDM+FA provide the same invariance guarantee. In addition, we propose two metrics to measure model invariance conditioning.

Invariant sampling distribution. To measure the degree of $\mathcal{G}$-invariance of the sampling distribution, we propose a metric called *Inv-FID*. Given a set of generated images $\mathcal{D}_s$, Inv-FID computes the maximum FID between $\kappa_1(\mathcal{D}_s)$ and $\kappa_2(\mathcal{D}_s)$ for $\kappa_1, \kappa_2 \in \mathcal{G}$. If $\mathcal{D}_s$ is perfectly $\mathcal{G}$-invariant, $\kappa \in \mathcal{G}$ does not change the distribution of $\mathcal{D}_s$, and Inv-FID is zero; otherwise, the score is strictly greater than zero. As shown in Table 2 and Table 3, diffusion models with theoretical guarantees have lower scores, and the equivariance regularizer further improves invariance. Notably, the differences in Inv-FID scores among diffusion models are relatively small, suggesting that diffusion models inherently learn invariant properties. Therefore, in scenarios where sampling distribution invariance is not critical, regular diffusion models may suffice.

Table 2: Model Comparison on Rotated MNIST.

Model	FID↓ 1%	FID↓ 5%	FID↓ 10%	FID↓ 100%	Inv-FID↓ 100%	$\Delta\hat{\mathbf{x}}_0$ ↓ 100%
VP-SDE	5.97	**3.05**	3.47	2.81	2.21	0.2997
SPDM+WT	5.80	3.34	3.57	3.50	2.20	0.0004
SPDM+FA	**5.42**	3.09	**2.83**	**2.64**	**2.07**	**0.0002**
SP-GAN	149	99	88	81	–	–
SP-GAN (Reprod.)	16.59	11.28	9.02	10.95	19.92	–
GE-GAN	–	–	4.25	2.90	–	–
GE-GAN (Reprod.)	15.82	7.44	5.92	4.17	58.61	–

Table 3: Model Comparison on LYSTO and ANHIR.

Model	LYSTO FID↓	LYSTO Inv-FID↓	LYSTO $\Delta\hat{\mathbf{x}}_0$ ↓	ANHIR FID↓	ANHIR Inv-FID↓	ANHIR $\Delta\hat{\mathbf{x}}_0$ ↓
VP-SDE	7.88	0.66	0.0845	8.03	0.57	0.3123
SPDM+WT	12.75	**0.59**	0.0002	11.73	0.43	0.0002
SPDM+FA	**5.31**	0.6	**0.0001**	**7.57**	**0.31**	**0.0001**
SP-GAN	192	–	–	90	–	–
SP-GAN (Reprod.)	16.29	0.66	–	17.12	0.28	–
GE-GAN	3.90	–	–	5.19	–	–
GE-GAN (Reprod.)	23.20	27.84	–	14.16	6.87	–

Validating equivariant sampling trajectory. To empirically validate our methods' theoretical guarantees for equivariant sampling trajectories, we implemented an image-denoising task using SDEdit (Meng et al., 2022), as shown in Fig 1. Given a low-resolution (or corrupted) image $\tilde{\mathbf{x}}_0$, we add equivariant noise to obtain $\mathbf{x}_t$ using the technique from Sec 5.2. This technique is also applied when solving backward SDEs to obtain the denoised image $\mathrm{dn}(\tilde{\mathbf{x}}_0)$, where dn represents the denoising process. As discussed in Sec 5.2, if a diffusion model is $\mathcal{G}$-equivariant, we should have $\mathrm{dn}(\kappa\,\tilde{\mathbf{x}}_0) - \kappa\,\mathrm{dn}(\tilde{\mathbf{x}}_0) \approx \mathbf{0}$ for all $\kappa \in \mathcal{G}$. In Table 2 and Table 3, we report the average maximum pixel-wise distance $\Delta\mathbf{x}_0$ between $\mathrm{dn}(\kappa\,\tilde{\mathbf{x}}_0)$ and $\kappa\,\mathrm{dn}(\tilde{\mathbf{x}}_0)$ over 16 randomly sampled corrupted $\tilde{\mathbf{x}}_0$ images. κ is randomly picked for each $\tilde{\mathbf{x}}_0$. The results show that theoretically equivariant models consistently have nearly zero $\Delta\mathbf{x}_0$, while models without theoretical guarantees produce significantly different outputs, which could be problematic in applications like medical image analysis. Likewise, in Table 4, for a model $\mathbf{m}_\theta$, we adopt a similar idea to measure its equivariance $\Delta\hat{\mathbf{x}}_0$ by reporting the average maximum pixel-wise distance between $\mathbf{m}_\theta(\kappa\,\tilde{\mathbf{y}})$ and $\kappa\,\mathbf{m}_\theta(\tilde{\mathbf{y}})$, where $\mathbf{y}$ is the input.

Sampling quality of SPDM Among models with theoretically guaranteed structure-preserving properties, SPDM+WT struggles to achieve FID scores comparable to FA methods on complex datasets like LYSTO. This is likely due to the weight-tying technique limiting the model's expressiveness and optimization. In contrast, SPDM+FA maintains sample quality and achieves the best performance on most datasets. This result corroborates our discussion made in Sec 5.1: it is sufficient to train a score-based model using regular score-matching and combine the score-based model's outputs during the inference time to ensure equivariance without compromising the model's performance.

Table 4: Model Comparison on LYSTO denoising and CT-PET style transfer datasets.

Model	LYSTO FID↓	LYSTO L_1 ↓	LYSTO SSIM↑	LYSTO $\Delta\hat{\mathbf{x}}_0$ ↓	CT-PET FID↓	CT-PET L_1 ↓	CT-PET SSIM↑	CT-PET $\Delta\hat{\mathbf{x}}_0$ ↓
DDBM	17.28	0.076	0.696	0.8884	18.13	**0.041**	0.861	0.9233
SPDM+FA	**16.21**	**0.071**	0.721	**0.0001**	**17.74**	0.042	0.860	**0.0000**
Pix2Pix	78.43	0.087	0.654	0.8629	20.26	0.043	**0.862**	1.3196
I^2SB	20.45	0.073	**0.722**	0.8683	27.51	0.051	0.832	1.2123

6.3. Equivariant image style transfer

We compare the performance of Pix2Pix (Isola et al., 2017), I^2SB (Liu et al., 2023a), and our SPDM+FA models in two style-transfer tasks. SPDM+FA is based on DDBM, with its score-based model combined through FA to ensure theoretically guaranteed group equivariance. In addition to FID for measuring sample quality and $\Delta\hat{\mathbf{x}}_0$ for measuring the model's equivariance, we report the L_1 loss between the output and ground truth for local structure similarity and SSIM (Wang et al., 2004) for global feature alignment. SPDM+FA achieves nearly perfect group equivariance for both tasks, the best scores in most measures, and close-to-the-best scores in the rest. These results corroborate the effectiveness of the techniques presented in Sec 5 and suggest that our framework covers the setting of diffusion bridges, guiding the construction of equivariant bridge models.

7. Discussion

In this paper, we investigated structure-preserving diffusion models (SPDM), an extended diffusion framework that accounts for the geometric structure of additional factors influencing the diffusion process. This extension allows us to effectively characterize the structure-preserving properties of a broader range of diffusion processes, including diffusion bridges used by DDBM (Zhou et al., 2024). Building upon this more general framework, we presented an equivalent condition on the drift term necessary to achieve a structure-preserving process, complementing existing work that primarily focuses on sufficient conditions. Based on the developed theoretical insights, we discussed several effective techniques to ensure the invariant distributions of samples and the equivariant properties of diffusion bridges. Empirical results on image generation and style-transfer tasks support our theoretical claims and demonstrate the effectiveness of the proposed methods in achieving structure-preserving sampling while maintaining high image quality.

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A. Derivation details of the theoretical results

In this section, we provide detailed derivations of our theoretical results. For conciseness, we complete most of the proofs in measure theory notation and show their equivalence to those presented in the main text.

In Appx A.1, we demonstrate that the isometry assumption on group operators results in their linearity. In Appx A.2, we briefly review the Liouville equations, which play a crucial role in characterizing the distribution evolution of particles driven by an ODE drift. In Appx A.3, we discuss a special family of ODE drifts that preserve distributions, characterizing the equivalence of various drifts by inducing the same evolution of p_t. This result helps derive the equivalent conditions on drifts to achieve structure-preserving ODE and SDE processes. Appx A.4 discusses the structure-preserving conditions for ODE processes, and we extend the results to SDE processes in Appx A.5.

A.1. Isometries

The groups $\mathcal{G}$ involved in our discussions from Sec 3.2 are assumed to consist of isometries κ satisfying $\|\kappa\mathbf{x}\| = \|\mathbf{x}\|$. Then, for $\kappa \in \mathcal{G}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, we have $\|\kappa\mathbf{x} - \kappa\mathbf{y}\|_2 = \|\mathbf{x} - \mathbf{y}\|_2$. Since $\kappa \in \mathcal{G}$ is bijective, by the Mazur–Ulam theorem (Nica, 2012), κ is affine and thus can be written as

$$\kappa(\mathbf{x}) = A\mathbf{x} + \mathbf{b} \tag{15}$$

for some $A_\kappa \in \mathbb{R}^{d\times d}$ and $\mathbf{b}_\kappa \in \mathbb{R}^d$. Besides, A_κ is orthogonal:

Lemma 2. *If $\kappa(\mathbf{x}) = A_\kappa\mathbf{x} + \mathbf{b}_\kappa$ is an isometry, then $A_\kappa^\top A_\kappa = \mathbf{I}$.*

Proof. As κ is an isometry, then for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$,

$$\|A_\kappa\mathbf{x} - A_\kappa\mathbf{y}\| = \|\kappa\mathbf{x} - \kappa\mathbf{y}\| = \|\mathbf{x} - \mathbf{y}\|. \tag{16}$$

In addition,

$$\langle A_\kappa\mathbf{x}, A_\kappa\mathbf{y}\rangle = \frac{1}{4}\left[\|A_\kappa\mathbf{x} - A_\kappa(-\mathbf{y})\|^2 - \|A_\kappa\mathbf{x} - A_\kappa\mathbf{y}\|^2\right] = \langle\mathbf{x}, \mathbf{y}\rangle \tag{17}$$

That is $\left\langle A_\kappa^\top A_\kappa\mathbf{x}, \mathbf{y}\right\rangle = \langle\mathbf{x}, \mathbf{y}\rangle$, which implies $A_\kappa^\top A_\kappa = \mathbf{I}$. □

Remark 1. *Lem 2 suggests that $D\kappa(\mathbf{x}) = A_\kappa$ for all $\mathbf{x} \in \mathbb{R}^d$.*

In addition, since $\|\kappa\mathbf{x}\|_2 = \|\mathbf{x}\|_2$, we have $\|A_\kappa\mathbf{x} + \mathbf{b}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$. Setting $A_\kappa\mathbf{x} = -\mathbf{b}$ yields $\|\mathbf{b}\| = 0$, or equivalently, $\mathbf{b} = \mathbf{0}$.

Therefore, for all the group operators κ appearing in our discussion, we can write:

$$\kappa\mathbf{x} = A_\kappa\mathbf{x} \tag{18}$$

for some orthogonal $A_\kappa \in \mathbb{R}^{d\times d}$.

The following lemma can significantly simplify the discussion on the geometric properties of diffusion processes in Appx A.2 and Appx A.5:

Lemma 3. *Let $C_c^\infty(\mathbb{R}^d)$ be the set of compactly supported functions. Then for $\kappa \in \mathcal{G}$,*

$$\{\phi \circ \kappa | \phi \in C_c^\infty(\mathbb{R}^d)\} = C_c^\infty(\mathbb{R}^d) \tag{19}$$

Proof. If $\phi \in C_c^\infty(\mathbb{R}^d)$ has a compact support C, then $\phi \circ \kappa$ has a support $\kappa^{-1}C$, which is also compact because $\kappa^{-1} \in \mathcal{G}$ is also affine (thus continuous) and a continuous image of a compact set is compact. Moreover, since ϕ is infinitely differentiable, so is $\phi \circ \kappa$. Thus, $\{\phi \circ \kappa | \phi \in C_c^\infty(\mathbb{R}^d)\} \subseteq C_c^\infty(\mathbb{R}^d)$. In addition, for $\psi \in C_c^\infty(\mathbb{R}^d)$, we have $\phi = \psi \circ \kappa^{-1} \in C_c^\infty(\mathbb{R}^d)$ such that $\phi \circ \kappa = \psi$. Hence, $C_c^\infty(\mathbb{R}^d) \subseteq \{\phi \circ \kappa | \phi \in C_c^\infty(\mathbb{R}^d)\}$. □

A.2. Liouville equation

Our proof relies on the Liouville equation in measure theory notation. We provide an intuitive and easy-to-follow proof here and show its equivalence to the popular version in the probability density notation in Remark 2.

Consider N non-interacting particles moving according to a deterministic ODE in $\mathbb{R}^d$:

$$\mathrm{d}\mathbf{x}_t = \mathbf{u}(\mathbf{x}_t, t)\,\mathrm{d}t. \tag{20}$$

Then their distribution is characterized by a measure $\mu_t^{(N)}$ such that for any compactly supported function $\phi \in C_c^\infty(\mathbb{R}^d)$ we have

$$\int \phi(\mathbf{x})\,\mathrm{d}\mu_t^N(\mathbf{x}) = \frac{1}{N}\sum_{i=1}^N \phi(\mathbf{x}_t^i). \tag{21}$$

Then

$$\frac{\partial}{\partial t}\int \phi(\mathbf{x})\,\mathrm{d}\mu_t^N(\mathbf{x}) = \frac{1}{N}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^N \phi(\mathbf{x}_t^i) = \frac{1}{N}\sum_{i=1}^N \nabla\phi(\mathbf{x}_t^i)\cdot\mathbf{u}(\mathbf{x}_t^i, t) \tag{22}$$

$$= \int \nabla\phi(\mathbf{x})\cdot\mathbf{u}(\mathbf{x}, t)\,\mathrm{d}\mu_t^N(\mathbf{x}). \tag{23}$$

Then if we suppose the initial distribution

$$\mu_0^N(\mathbf{x}, 0) \to^\star \mu_0(\mathbf{x}) \text{ as } N \to \infty \tag{24}$$

in a sense that $\int \phi(\mathbf{x})\,\mathrm{d}\mu_0(\mathbf{x}) \to \int \phi(\mathbf{x})\,\mathrm{d}\mu_0(\mathbf{x})$ for any $\phi \in C_c^\infty(\mathbb{R}^d)$. Then we can establish the limit $\mu_t^N(\mathbf{x}) \to^\star \mu_t(\mathbf{x})$ and μ_t satisfies

$$\frac{\partial}{\partial t}\int \phi(\mathbf{x})\,\mathrm{d}\mu_t(\mathbf{x}) = \int \nabla\phi(\mathbf{x})\cdot\mathbf{u}(\mathbf{x}, t)\,\mathrm{d}\mu_t(\mathbf{x}). \tag{25}$$

Notably, the setting we consider in the main text assume that the drift could optionally depend on some additional (fixed) term $\mathbf{y}$ such that

$$\mathrm{d}\mathbf{x}_t = \mathbf{f}(\mathbf{x}_t, \mathbf{y}, t)\,\mathrm{d}t, \tag{26}$$

where $\mathbf{x}_t \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^n$ with $m > 0$ and $n \geq 0$.[1] In this case, the process can be rewritten as

$$\mathrm{d}\begin{bmatrix}\mathbf{x}_t\\ \mathbf{y}\end{bmatrix} = \begin{bmatrix}\mathbf{f}(\mathbf{x}_t, \mathbf{y}, t)\\ \mathbf{0}\end{bmatrix}\mathrm{d}t = \mathbf{u}\big([\mathbf{x}_t, \mathbf{y}]^\top, t\big)\,\mathrm{d}t. \tag{27}$$

Applying (25), we obtain

$$\boxed{\frac{\partial}{\partial t}\int \phi(\mathbf{x}, \mathbf{y})\,\mathrm{d}\mu_t(\mathbf{x}|\mathbf{y}) = \int \nabla_1\phi(\mathbf{x}, \mathbf{y})\cdot\mathbf{f}(\mathbf{x}_t, \mathbf{y}, t)\,\mathrm{d}\mu_t(\mathbf{x}|\mathbf{y}) \qquad \textit{(Liouville eq. meas.)}} \tag{28}$$

where $\nabla_1\psi(\mathbf{x}, \mathbf{y}, \ldots) := \frac{\partial\psi(\mathbf{x}, \mathbf{y}, \ldots)}{\partial\mathbf{x}}$ denote the gradient with respect to the first argument.

Remark 2. *Let λ denote the Lebesgue measure. When the probability measure $\mu_t(\mathbf{x}|\mathbf{y})$ has density $p_t(\mathbf{x}|\mathbf{y}) \in C^1(\mathbb{R}^m \times \mathbb{R}^n \times [0, T])$ with respect to $\mathbf{x}$, we have*

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_c \phi(\mathbf{x}, \mathbf{y})p_t(\mathbf{x}|\mathbf{y})\,\mathrm{d}\lambda(\mathbf{x}) = \frac{\partial}{\partial t}\int_c \phi(\mathbf{x}, \mathbf{y})\,\mathrm{d}\mu_t(\mathbf{x}|\mathbf{y})\\
&= \int_c \nabla_1\phi(\mathbf{x}, \mathbf{y})\cdot\mathbf{f}(\mathbf{x}, \mathbf{y}, t)\,\mathrm{d}\mu_t(\mathbf{x}|\mathbf{y}) = \int_c \nabla_1\phi(\mathbf{x}, \mathbf{y})\cdot\mathbf{f}(\mathbf{x}, \mathbf{y}, t)p_t(\mathbf{x}|\mathbf{y})\,\mathrm{d}\lambda(\mathbf{x})\\
&= [p_t(\mathbf{x}|\mathbf{y})\mathbf{f}(\mathbf{x}, \mathbf{y}, t)\cdot\phi(\mathbf{x}, \mathbf{y})]_{\partial c} - \int_c \phi(\mathbf{x}, \mathbf{y})\,\nabla_{\mathbf{x}}\cdot\big(p_t(\mathbf{x}|\mathbf{y})\mathbf{f}(\mathbf{x}, \mathbf{y}, t)\big)\,\mathrm{d}\lambda(\mathbf{x})\\
&= -\int_c \phi(\mathbf{x}, \mathbf{y})\,\nabla_{\mathbf{x}}\cdot\big(p_t(\mathbf{x}|\mathbf{y})\mathbf{f}(\mathbf{x}, \mathbf{y}, t)\big)\,\mathrm{d}\lambda(\mathbf{x}).
\end{aligned}$$

[1] We use $n = 0$ to indicate the case that $\mathbf{f}$ does not depend on $\mathbf{y}$. Unless otherwise stated, we will continue to use this convention.

As this holds for all $\phi \in C_c^\infty(\mathbb{R}^{m+n})$, we obtain the regular Liouville equation (Oksendal, 2003; Ehrendorfer, 2006):

$$\boxed{\frac{\partial}{\partial t} p_t(\mathbf{x}|\mathbf{y}) = -\nabla_{\mathbf{x}} \cdot \big(p_t(\mathbf{x}|\mathbf{y})\mathbf{f}(\mathbf{x},\mathbf{y},t)\big). \qquad \text{(Liouville eq. density)}} \tag{29}$$

A.3. Distribution-preserving Drifts.

While a zero drift implies $p_t = p_0$ for all $t > 0$, the converse is not necessarily true:

Example 1. *Let p_0 be the density of a spherical Gaussian $\mathcal{N}(\mathbf{0}, \mathbf{I})$ in $\mathbb{R}^2$. For $\mathbf{f}(\mathbf{x}, t) = [y, -x]^\top$, by the Liouville equation, at $t = 0$*

$$\frac{\partial}{\partial t} p_t(x,y) = -\nabla \cdot \left[\frac{1}{Z}\exp(-\frac{x^2+y^2}{2})\,[y,-x]^\top\right] \tag{30}$$

$$= \frac{\partial}{\partial x}\left[\frac{1}{Z}\exp(-\frac{x^2+y^2}{2})y\right] - \frac{\partial}{\partial y}\left[\frac{1}{Z}\exp(-\frac{x^2+y^2}{2})x\right] \tag{31}$$

$$= \frac{1}{Z}\left[-\exp(-\frac{x^2+y^2}{2})xy + \exp(-\frac{x^2+y^2}{2})xy\right] = 0. \tag{32}$$

As a result, $\mathbf{f}$ does not change p_0, although is not zero.

In general,

Lemma 4. *Given a measure μ, drift $\mathbf{f}$ does not change the distribution if for all $\phi \in C_c^\infty(\mathbb{R}^d)$*

$$0 = \int \nabla\phi(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}, t)\,\mathrm{d}\mu(\mathbf{x}) \tag{33}$$

for all $\mathbf{x}$ and t. We use $[\mathbf{0}]_\mu(\mathbf{x})$ to denote the set of drifts that do not alter distribution μ. That is, if $\mathbf{f}$ satisfies (33), we have $\mathbf{f} \in [\mathbf{0}]_\mu$.

Proof. This is an immedate result of (28) by setting the left-hand side zero. □

Remark 3. *For any μ, $\mathbf{0} \in [\mathbf{0}]_\mu$.*

Remark 4. *If $\mathbf{f}, \mathbf{g} \in [\mathbf{0}]_\mu$, then $\alpha\mathbf{f} + \beta\mathbf{g} \in [\mathbf{0}]_\mu$, for $\alpha, \beta \in \mathbb{R}$.*

Remark 5. *In the main text, we use the notation $[\mathbf{0}]_p$ instead of $[\mathbf{0}]_\mu$ to represent distribution-preserving drifts that maintain a distribution with density p, which corresponds to the distribution measure μ.*

A.4. Structural preserving ODE processes

In this section, we discuss the sufficient and necessary condition of structurally preserved ODE processes. Here, we consider ODE process:

$$\mathrm{d}\mathbf{x}_t = \mathbf{f}(\mathbf{x}_t, \mathbf{y}, t)\,\mathrm{d}t \tag{34}$$

with $\mathbf{x}_t \in \mathbb{R}^m$, and $\mathbf{y} \in \mathbb{R}^n$ denote additional conditions of the process. Here, we assume $m > 0$ and $n \geq 0$, where $n = 0$ denote the case when $\mathbf{f}$ does not depend on $\mathbf{y}$. We note that for a similar setting with discrete time step and drift $\mathbf{f}$ that does not depend on $\mathbf{y}$, a sufficient condition on $\mathcal{G}$-invariance of μ_t for $t \geq 0$ has been discussed by (Papamakarios et al., 2021) and (Köhler et al., 2020).

Let $\mu_t(\mathbf{x}_t|\mathbf{y})$ be the probability measure of $\mathbf{x}_t$ induced by the ODE process (34) conditioned on $\mathbf{y}$. Let $\mathbf{G} = \{\boldsymbol{\kappa} = (\kappa_1, \kappa_2) | \kappa_1 : \mathbb{R}^m \to \mathbb{R}^m, \kappa_2 : \mathbb{R}^n \to \mathbb{R}^n\}$ be a group of isometries defined in $\mathbb{R}^{m+n}$ such that $\boldsymbol{\kappa}(\mathbf{x}, \mathbf{y}) = (\kappa_1\mathbf{x}, \kappa_2\mathbf{y})$. It is easy to see that the sets of κ_1 and κ_2 are also groups of isometries. We will respectively denote them as $\mathcal{G}_1$ and $\mathcal{G}_2$. In addition, by Lem 2, we have

$$\boldsymbol{\kappa}(\mathbf{x}, \mathbf{y}) = \begin{bmatrix} A_{\kappa_1} & \mathbf{0} \\ \mathbf{0} & A_{\kappa_2} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} + \begin{bmatrix} \mathbf{b}_{\kappa_1} \\ \mathbf{b}_{\kappa_2} \end{bmatrix}, \tag{35}$$

where $A_{\kappa_1} \in \mathbb{R}^{m\times m}$ and $A_{\kappa_2} \in \mathbb{R}^{n\times n}$ are orthogonal.

Furthermore, by Remark 1, we have

$$D\boldsymbol{\kappa} = A_{\boldsymbol{\kappa}} = \begin{bmatrix} A_{\kappa_1} & \mathbf{0} \\ \mathbf{0} & A_{\kappa_2} \end{bmatrix}. \tag{36}$$

We say $\mu_t(\mathbf{x}_t|\mathbf{y})$ is $\mathbf{G}$-*invariant* if for all $\boldsymbol{\kappa} \in \mathbf{G}$, $\mu_t(\kappa_1\mathbf{x}_t|\kappa_2\mathbf{y}) = \mu_t(\mathbf{x}_t|\mathbf{y})$. Lem 5 shows that this definition is equivalent to the one given in Sec 3.2.

Lemma 5. *Assume $\mu(\cdot|\cdot)$ has density $p(\mathbf{x}|\mathbf{y})$. Then $\mu(\cdot|\cdot)$ is $\mathbf{G}$-invariant if and only if the density $p(\mathbf{x}|\mathbf{y}) = p(\kappa_1\mathbf{x}|\kappa_2\mathbf{y})$ for all $\boldsymbol{\kappa} \in \mathbf{G}$, $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^n$.*

Proof. $\mu(\cdot|\cdot)$ is $\mathbf{G}$-invariant if and only if for all $\boldsymbol{\kappa} \in \mathbf{G}$, $\phi \in C_c^\infty(\mathbb{R}^{m+n})$,

$$\int \phi(\mathbf{x},\mathbf{y})\,\mathrm{d}\boldsymbol{\mu}(\mathbf{x}|\mathbf{y}) = \int \phi(\boldsymbol{\kappa}^{-1}(\mathbf{x},\mathbf{y}))\,\mathrm{d}\boldsymbol{\mu}(\mathbf{x}|\mathbf{y}). \tag{37}$$

That is,

$$\begin{aligned}
&\int \phi(\mathbf{x},\mathbf{y})p(\mathbf{x}|\mathbf{y})\,\mathrm{d}\lambda(\mathbf{x}) = \int \phi(\boldsymbol{\kappa}^{-1}(\mathbf{x},\mathbf{y}))\,\mathrm{d}\boldsymbol{\mu}(\mathbf{x}|\mathbf{y}) = \int \phi(\mathbf{x},\mathbf{y})\,\mathrm{d}\mu(\kappa_1\mathbf{x}|\kappa_2\mathbf{y}) \\
&= \int \phi(\mathbf{x},\mathbf{y})p(\kappa_1\mathbf{x}|\kappa_2\mathbf{y})\,\mathrm{d}\lambda(\kappa_1\mathbf{x}) \overset{(\text{Lem } 8)}{=} \int \phi(\mathbf{x},\mathbf{y})p(\kappa_1\mathbf{x}|\kappa_2\mathbf{y})\,\mathrm{d}\lambda(\mathbf{x}).
\end{aligned}$$

Therefore, $p(\mathbf{x}|\mathbf{y}) = p(\kappa_1\mathbf{x}|\kappa_2\mathbf{y})$. Since every step is reversible, the proof is completed. □

Then we give the equivalent conditions on the drift terms to ensure the structure-preserving property of ODE flows.

Lemma 6. *Consider the ODE process in (34) with $\mathbf{G}$-invariant $\mu_0(\cdot|\cdot)$. Then, μ_t is $\mathbf{G}$-invariant for all $t \geq 0$ if and only if*

$$A_{\kappa_1}^\top \mathbf{f}(\kappa_1\mathbf{x},\kappa_2\mathbf{y},t) - \mathbf{f}(\mathbf{x},\mathbf{y},t) \in [\mathbf{0}]_{\mu_t}. \tag{38}$$

for all $t \geq 0$, $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^n$ and $\boldsymbol{\kappa} = (\kappa_1,\kappa_2) \in \mathbf{G}$.

Proof. ($\Rightarrow$) Assume that μ_t is $\mathbf{G}$-invariant for all $t \geq 0$. For all $\phi \in C_c^\infty(\mathbb{R}^{m+n})$ and $\boldsymbol{\kappa} \in \mathbf{G}$, let $\psi = \phi \circ \boldsymbol{\kappa}$. We note that by Lem 3, $\psi \in C_c^\infty(\mathbb{R}^{m+n})$. Then, for $t \geq 0$, we have

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\int \phi(\mathbf{x},\mathbf{y})\,\mathrm{d}\mu_t(\mathbf{x}|\mathbf{y}) - \frac{\mathrm{d}}{\mathrm{d}t}\int \phi(\mathbf{x},\mathbf{y})\,\mathrm{d}\mu_t(\kappa_1^{-1}\mathbf{x}|\kappa_2^{-1}\mathbf{y}) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int \phi(\mathbf{x},\mathbf{y})\,\mathrm{d}\mu_t(\mathbf{x}|\mathbf{y}) - \frac{\mathrm{d}}{\mathrm{d}t}\int \phi(\kappa_1\mathbf{x},\kappa_2\mathbf{y})\,\mathrm{d}\mu_t(\mathbf{x}|\mathbf{y}) \\
&\overset{(28)}{=} \int \nabla_1\phi(\mathbf{x},\mathbf{y})^\top \mathbf{f}(\mathbf{x},\mathbf{y},t)\,\mathrm{d}\mu_t(\mathbf{x}|\mathbf{y}) - \int \nabla_1\psi(\mathbf{x},\mathbf{y})^\top \mathbf{f}(\mathbf{x},\mathbf{y},t)\,\mathrm{d}\mu_t(\mathbf{x}|\mathbf{y}) \\
&\overset{(\mathbf{G}-\text{inv})}{=} \int \nabla_1\phi(\kappa_1\mathbf{x},\kappa_2\mathbf{y})^\top \mathbf{f}(\kappa_1\mathbf{x},\kappa_2\mathbf{y},t)\,\mathrm{d}\mu_t(\mathbf{x}|\mathbf{y}) - \int \nabla_1\psi(\mathbf{x},\mathbf{y})^\top \mathbf{f}(\mathbf{x},\mathbf{y},t)\,\mathrm{d}\mu_t(\mathbf{x}|\mathbf{y}) \\
&= \int \nabla_1\psi(\kappa_1\mathbf{x},\kappa_2\mathbf{y})^\top D\kappa_1(\mathbf{x})^\top \mathbf{f}(\kappa_1\mathbf{x},\kappa_2\mathbf{y},t)\,\mathrm{d}\mu_t(\mathbf{x}|\mathbf{y}) - \int \nabla_1\psi(\mathbf{x},\mathbf{y})^\top \mathbf{f}(\mathbf{x},\mathbf{y},t)\,\mathrm{d}\mu_t(\mathbf{x}|\mathbf{y}) \\
&= \int \nabla_1\psi(\kappa_1\mathbf{x},\kappa_2\mathbf{y})^\top \Big(A_{\kappa_1}^\top \mathbf{f}(\kappa_1\mathbf{x},\kappa_2\mathbf{y},t) - \mathbf{f}(\mathbf{x},\mathbf{y},t)\Big)\,\mathrm{d}\mu_t(\mathbf{x}|\mathbf{y}).
\end{aligned}$$

By Lem 3, ψ can be any functions in $C_c^\infty(\mathbb{R}^{m+n})$. Thus, (38) follows.

($\Leftarrow$) Assume (38) holds and $\boldsymbol{\mu}_0$ is $\mathbf{G}$-invariant. For all $\phi \in C_c^\infty(\mathbb{R}^{m+n})$ and $\boldsymbol{\kappa} \in \mathbf{G}$, let $\psi = \phi \circ \boldsymbol{\kappa}$. Then we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \phi(\mathbf{x},\mathbf{y})\,\mathrm{d}\boldsymbol{\mu}_t(\kappa_1^{-1}\mathbf{x}|\kappa_2^{-1}\mathbf{y}) = \frac{\mathrm{d}}{\mathrm{d}t}\int \phi(\kappa_1\mathbf{x},\kappa_2\mathbf{y})\,\mathrm{d}\boldsymbol{\mu}_t(\mathbf{x}|\mathbf{y}) = \frac{\mathrm{d}}{\mathrm{d}t}\int \psi(\mathbf{x},\mathbf{y})\,\mathrm{d}\boldsymbol{\mu}_t(\mathbf{x}|\mathbf{y}) \\
&\overset{(28)}{=} \int (\nabla_1\psi)(\mathbf{x},\mathbf{y})^\top \mathbf{f}(\mathbf{x},\mathbf{y},t)\,\mathrm{d}\mu_t(\mathbf{x}|\mathbf{y}) \overset{(\mathbf{A})}{=} \int (\nabla_1\phi)\,(\kappa_1\mathbf{x},\kappa_2\mathbf{y})^\top \mathbf{f}(\kappa_1\mathbf{x},\kappa_2\mathbf{y},t)\,\mathrm{d}\boldsymbol{\mu}_t(\mathbf{x}|\mathbf{y}) \\
&= \int (\nabla_1\phi)\,(\mathbf{x},\mathbf{y})^\top\ \mathbf{f}(\mathbf{x},\mathbf{y},t)\,\mathrm{d}\boldsymbol{\mu}_t(\kappa_1^{-1}\mathbf{x}|\kappa_2^{-1}\mathbf{y}),
\end{aligned} \tag{39}$$

where ($\mathbf{A}$) is due to:

$$\begin{aligned}
0 &\overset{(38)}{=} \int \nabla_1\psi(\kappa_1\mathbf{x},\kappa_2\mathbf{y})^\top \Big(A_{\kappa_1}^\top \mathbf{f}(\kappa_1\mathbf{x},\kappa_2\mathbf{y},t) - \mathbf{f}(\mathbf{x},\mathbf{y},t)\Big)\,\mathrm{d}\mu_t(\mathbf{x}|\mathbf{y}) \\
&= \int \nabla_1\psi(\kappa_1\mathbf{x},\kappa_2\mathbf{y})^\top D\kappa_1(\mathbf{x})^\top \mathbf{f}(\kappa_1\mathbf{x},\kappa_2\mathbf{y},t)\,\mathrm{d}\mu_t(\mathbf{x}|\mathbf{y}) - \int \nabla_1\psi(\mathbf{x},\mathbf{y})^\top \mathbf{f}(\mathbf{x},\mathbf{y},t)\,\mathrm{d}\mu_t(\mathbf{x}|\mathbf{y}) \\
&= \int \nabla_1\phi(\kappa_1\mathbf{x},\kappa_2\mathbf{y})^\top \mathbf{f}(\kappa_1\mathbf{x},\kappa_2\mathbf{y},t)\,\mathrm{d}\mu_t(\mathbf{x}|\mathbf{y}) - \int \nabla_1\psi(\mathbf{x},\mathbf{y})^\top \mathbf{f}(\mathbf{x},\mathbf{y},t)\,\mathrm{d}\mu_t(\mathbf{x}|\mathbf{y})
\end{aligned}$$

Besides, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \phi_1(\mathbf{x},\mathbf{y})\,\mathrm{d}\mu_t(\mathbf{x}|\mathbf{y}) \overset{(28)}{=} \int \nabla\phi_1(\mathbf{x},\mathbf{y})^\top \mathbf{f}(\mathbf{x},\mathbf{y},t)\,\mathrm{d}\mu_t(\mathbf{x}|\mathbf{y}). \tag{40}$$

As $\mu_0(\mathbf{x}|\mathbf{y}) = \mu_0(\kappa_1^{-1}\mathbf{x}|\kappa_2^{-1}\mathbf{x})$, (39) and (40) together suggest that $\mu_t(\mathbf{x}|\mathbf{y})$ and $\mu_t(\kappa_1^{-1}\mathbf{x}|\kappa_2^{-1}\mathbf{y})$ share the same Liouville's equation. Therefore, $\mu_t(\mathbf{x}|\mathbf{y}) = \mu_t(\kappa_1^{-1}\mathbf{x}|\kappa_2^{-1}\mathbf{y})$ for all $t \geq 0$. □

A.5. Structural preserving SDE processes

In this section, we assume all the measures involved have densities. We first show that Lebesgue measure is $\mathcal{G}$-invariant, where $\mathcal{G}$ is a group of isometries.

Lemma 7. *For all $\kappa \in \mathcal{G}$, $\det D\kappa(\mathbf{x}) = \det A_\kappa = 1$ or -1 for all $\mathbf{x} \in \mathbb{R}^d$.*

Proof. For $\kappa \in \mathcal{G}$, by Lem 2 and Remark 1, we have $D\kappa(\mathbf{x})^\top D\kappa(\mathbf{x}) = A_\kappa^\top A_\kappa = I$. Then $\det(D\kappa(\mathbf{x}))^2 = (\det A_\kappa)^2 = 1$, which implies $\det D\kappa(\mathbf{x}) = \det A_\kappa = \pm 1$ □

Lemma 8. *The Lebesgue measure λ is $\mathcal{G}$-invariant.*

Proof. For all $\phi \in C_c^\infty(\mathbb{R}^d)$ and $\kappa \in \mathcal{G}$, we have

$$\begin{aligned}
\int \phi(\mathbf{x})\,\mathrm{d}\lambda(\mathbf{x}) &= \int \phi(\kappa\mathbf{x})\,\mathrm{d}\lambda(\kappa\mathbf{x}) \overset{(\text{Lem } 8)}{=} \int \phi(\kappa\mathbf{x})\,|\det D\kappa(\mathbf{x})|\,\mathrm{d}\lambda(\mathbf{x}) \\
&= \int \phi(\kappa\mathbf{x})\,\mathrm{d}\lambda(\mathbf{x}) = \int \phi(\mathbf{x})\,\mathrm{d}\lambda(\kappa^{-1}\mathbf{x})
\end{aligned}$$

Therefore, $\lambda = \kappa^\# \lambda$. □

To deal with the invariance property associated with the diffusion term, we prove a lemma similar to Lem F.4 of (Yim et al., 2023). The lemma basically says Laplacian is invariant with respect to isometries:

Lemma 9. *For $\boldsymbol{\kappa} \in \mathbf{G}$ and $v : \mathbb{R}^{m+n} \to \mathbb{R}$, we have*

$$\Delta_1(v \circ \boldsymbol{\kappa})(\mathbf{x},\mathbf{y}) = (\Delta_1 v) \circ \boldsymbol{\kappa}(\mathbf{x},\mathbf{y}), \tag{41}$$

where

$$(\Delta_1 u)\,(\mathbf{x},\mathbf{y}) = \sum_{k=1}^{m} \frac{\partial^2}{\partial x_k^2} u(\mathbf{x},\mathbf{y}) = (\nabla_1 \cdot \nabla_1 u)(\mathbf{x},\mathbf{y}), \tag{42}$$

$$(\nabla_1 u)\,(\mathbf{x},\mathbf{y}) = \frac{\partial u}{\partial \mathbf{x}}(\mathbf{x},\mathbf{y}). \tag{43}$$

Proof. Let

$$M = \begin{bmatrix} \mathbf{I}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{44}$$

where $\mathbf{0}$ denotes a zero matrix of a proper size. Then it is easy to see

$$\Delta_1 v(\mathbf{x}, \mathbf{y}) = \nabla \cdot \Big(M \, \nabla v(\mathbf{x}, \mathbf{y}) \Big). \tag{45}$$

As a result, for all $\phi \in C_c^\infty(\mathbb{R}^{m+n})$, we have

$$\begin{aligned}
&\int \phi(\mathbf{x}, \mathbf{y})(\Delta_1 v) \circ \boldsymbol{\kappa}(\mathbf{x}, \mathbf{y}) \, \mathrm{d}\lambda(\mathbf{x}, \mathbf{y}) = \int \phi(\kappa_1^{-1}\mathbf{x}, \kappa_2^{-1}\mathbf{y}) \, (\Delta_1 v)(\mathbf{x}, \mathbf{y}) \, \mathrm{d}\lambda(\mathbf{x}, \mathbf{y}) \\
&= \int \phi(\kappa_1^{-1}\mathbf{x}, \kappa_2^{-1}\mathbf{y}) \, \nabla \cdot \Big(M \, \nabla v(\mathbf{x}, \mathbf{y}) \Big) \, \mathrm{d}\lambda(\mathbf{x}, \mathbf{y}) \\
&= \big[\phi(\kappa_1^{-1}\mathbf{x}, \kappa_2^{-1}\mathbf{y}) \, M \nabla v(\kappa_1^{-1}\mathbf{x}, \kappa_2^{-1}\mathbf{y}) \big]_{\partial c} - \int M \nabla v(\mathbf{x}, \mathbf{y}) \cdot \nabla(\phi \circ \boldsymbol{\kappa}^{-1})(\mathbf{x}, \mathbf{y}) \, \mathrm{d}\lambda(\mathbf{x}, \mathbf{y}) \\
&= - \int M \nabla v(\mathbf{x}, \mathbf{y}) \cdot \Big((\nabla \phi)(\boldsymbol{\kappa}^{-1}(\mathbf{x}, \mathbf{y}))^\top D\boldsymbol{\kappa}^{-1}(\mathbf{x}, \mathbf{y}) \Big) \, \mathrm{d}\lambda(\mathbf{x}, \mathbf{y}) \\
&= - \int M \nabla v(\mathbf{x}, \mathbf{y}) \cdot \Big(\nabla \phi(\boldsymbol{\kappa}^{-1}(\mathbf{x}, \mathbf{y}))^\top A_{\boldsymbol{\kappa}}^\top \Big) \, \mathrm{d}\lambda(\mathbf{x}, \mathbf{y}) \\
&= - \int M \nabla v(\kappa_1 \mathbf{x}, \kappa_2 \mathbf{y}) \cdot \Bigg(\nabla \phi(\mathbf{x}, \mathbf{y})^\top A_{\boldsymbol{\kappa}}^\top \Bigg) \, \mathrm{d}\lambda(\mathbf{x}, \mathbf{y}) \\
&\overset{\text{(Lem 8)}}{=} - \int \nabla v(\kappa_1 \mathbf{x}, \kappa_2 \mathbf{y})^\top M^\top A_{\boldsymbol{\kappa}} \nabla \phi(\mathbf{x}, \mathbf{y}) \, \mathrm{d}\lambda(\mathbf{x}, \mathbf{y}) \\
&= - \int \nabla v(\kappa_1 \mathbf{x}, \kappa_2 \mathbf{y})^\top \begin{bmatrix} A_{\kappa_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \nabla \phi(\mathbf{x}, \mathbf{y}) \, \mathrm{d}\lambda(\mathbf{x}, \mathbf{y}) \\
&= - \int \big[\nabla_1 (v \circ \boldsymbol{\kappa})(\mathbf{x}, \mathbf{y})^\top \;\; \mathbf{0} \big] \, \nabla \phi(\mathbf{x}, \mathbf{y}) \, \mathrm{d}\lambda(\mathbf{x}, \mathbf{y}) \\
&= - \int \big[\nabla_1 (v \circ \kappa)(\mathbf{x}, \mathbf{y})^\top \;\; \mathbf{0} \big] \, \nabla \phi(\mathbf{x}, \mathbf{y}) \, \mathrm{d}\lambda(\mathbf{x}, \mathbf{y}) + \Big[\phi(\mathbf{x}, \mathbf{y}) \, \big[\nabla_1 (v \circ \boldsymbol{\kappa})(\mathbf{x}, \mathbf{y})^\top \;\; \mathbf{0} \big] \Big]_{\partial c} \\
&= \int \phi(\mathbf{x}, \mathbf{y}) \;\; \nabla \cdot \big[\nabla_1 (v \circ \boldsymbol{\kappa})(\mathbf{x}, \mathbf{y})^\top \;\; \mathbf{0} \big] \, \mathrm{d}\lambda(\mathbf{x}, \mathbf{y}) = \int \phi(\mathbf{x}, \mathbf{y}) \, \Delta_1 (v \circ \boldsymbol{\kappa})(\mathbf{x}, \mathbf{y}) \, \mathrm{d}\lambda(\mathbf{x}, \mathbf{y}).
\end{aligned}$$

Thus, $(\Delta_1 v) \circ \boldsymbol{\kappa}(\mathbf{x}, \mathbf{y}) = \Delta_1 (v \circ \boldsymbol{\kappa})(\mathbf{x}, \mathbf{y})$. □

Lemma 10. *$\mu(\cdot|\cdot)$ is $\mathbf{G}$-invariant if and only if*

$$\mathbf{s}(\kappa_1 \mathbf{x} | \kappa_2 \mathbf{y}) = A_{\kappa_1} \, \mathbf{s}(\mathbf{x}|\mathbf{y}) \tag{46}$$

for all $\boldsymbol{\kappa} \in \mathbf{G}$, $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^n$, where $\mathbf{s}(\mathbf{x})$ denotes the score function $\nabla_{\mathbf{x}} \log p(\mathbf{x}|\mathbf{y})$.

Proof. ($\Rightarrow$) By Lem 5, if μ is $\mathbf{G}$-invariant, its density $p(\mathbf{x}|\mathbf{y}) = p(\kappa_1 \mathbf{x} | \kappa_2 \mathbf{y})$. Taking $\log$ on both sides, followed by taking the derivative with respect to $\mathbf{x}$ yields

$$A_{\kappa_1}^\top \, \mathbf{s}(\kappa_1 \mathbf{x} | \kappa_2 \mathbf{y}) = \mathbf{s}(\mathbf{x}|\mathbf{y}), \tag{47}$$

as $D\kappa_1(\mathbf{x}) = A_{\kappa_1}$.

($\Leftarrow$) Conversely, (46) yields

$$p(\mathbf{x}|\mathbf{y}) = p(\kappa_1 \mathbf{x} | \kappa_2 \mathbf{y}) + C, \tag{48}$$

where C must be zero so that $p(\mathbf{x}|\mathbf{y})$ and $p(\kappa_1 \mathbf{x} | \kappa_2 \mathbf{y})$ are valid densities. □

Then we prove the following Lemma presented in the Sec 3.2.

Lemma 1. *$p(\mathbf{x}|\mathbf{y})$ is $\mathbf{G}$-invariant if and only if $\mathbf{s}(\kappa_1\mathbf{x}|\kappa_2\mathbf{y}) = \kappa_1 \circ \mathbf{s}(\mathbf{x}|\mathbf{y})$ for all $(\kappa_1, \kappa_2) \in \mathbf{G}$, $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^n$. Likewise, $p(\mathbf{x})$ is $\mathcal{G}$-invariant if and only if $\mathbf{s}(\kappa\mathbf{x}) = \kappa \circ \mathbf{s}(\mathbf{x})$ for all $\kappa \in \mathcal{G}$.*

Proof. The conditional density case is immediate given Lem 5 and Lem 10 while the unconditional one is the special case that $n = 0$. □

Lemma 11 ((Song et al., 2021b))**.** *Let p_t be the marginal distribution of $\mathbf{x}_t$ that satisfies SDE:*

$$\mathrm{d}\mathbf{x}_t = \mathbf{f}(\mathbf{x}_t, \mathbf{y}, t)\,\mathrm{d}t + g(t)\,\mathrm{d}\mathsf{w}_t, \quad \mathbf{x}_0 \sim p_0(\mathbf{x}_0|\mathbf{y}). \tag{49}$$

Besides, let $\mathbf{s}_t(\cdot|\mathbf{y}) = \nabla \log p_t(\cdot|\mathbf{y})$. Then, the ODE

$$\mathrm{d}\mathbf{x} = \tilde{\mathbf{f}}(\mathbf{x}, \mathbf{y}, t)\,\mathrm{d}t \tag{50}$$

with

$$\tilde{\mathbf{f}}(\mathbf{x}, \mathbf{y}, t) = \mathbf{f}(\mathbf{x}, \mathbf{y}, t) - \frac{1}{2}g(t)^2\,\mathbf{s}_t(\mathbf{x}|\mathbf{y}) \tag{51}$$

also has the same marginal distribution p_t for all $t \geq 0$.

Proof. The marginal distribution $p_t(\mathbf{x}|\mathbf{y})$ evolution is characterized by the Fokker-Planck equation (Oksendal, 2003):

$$\begin{aligned}
\frac{\partial p_t(\mathbf{x}|\mathbf{y})}{\partial t} &= -\nabla \cdot \big(\mathbf{f}(\mathbf{x}, \mathbf{y}, t)p_t(\mathbf{x}|\mathbf{y})\big) + \frac{1}{2}\nabla \cdot \nabla\big(g(t)^2 p_t(\mathbf{x}|\mathbf{y})\big) && (52)\\
&= -\sum_{i=1}^{d} \frac{\partial}{\partial x_i}[f_i(\mathbf{x}, \mathbf{y}, t)p_t(\mathbf{x}|\mathbf{y})] + \frac{1}{2}\sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}[g(t)^2\, p_t(\mathbf{x}|\mathbf{y})] && (53)\\
&= -\sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left\{[f_i(\mathbf{x}, \mathbf{y}, t)p_t(\mathbf{x}|\mathbf{y})] - \frac{g(t)^2}{2}\big[p_t(\mathbf{x}|\mathbf{y})\frac{\partial}{\partial x_i}\log p_t(\mathbf{x}|\mathbf{y})\big]\right\} && (54)\\
&= -\sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left[f_i(\mathbf{x}, \mathbf{y}, t) - \frac{g(t)^2}{2}\frac{\partial}{\partial x_i}\log p_t(\mathbf{x}|\mathbf{y})\right] p_t(\mathbf{x}|\mathbf{y}), && (55)
\end{aligned}$$

where the last line is the Fokker-Planck equation of

$$\mathrm{d}\mathbf{x} = \tilde{\mathbf{f}}(\mathbf{x}, \mathbf{y}, t)\,\mathrm{d}t \tag{56}$$

with $\tilde{\mathbf{f}}(\mathbf{x}, \mathbf{y}, t)$ given in (51). □

Now we are ready to give the if and only if statement on the structurally preserving property of the distributions induced by

$$\mathrm{d}\mathbf{x}_t = \mathbf{f}(\mathbf{x}_t, \mathbf{y}, t)\,\mathrm{d}t + g(t)\,\mathrm{d}\mathsf{w}_t \tag{57}$$

Notably, a sufficient condition given by (58) with the left-hand side equal to zero is firstly discussed by (Yim et al., 2023).

Then we give the equivalent conditions on the drift terms to ensure the structure-preserving property of SDE flows and its equivalence to the Prop 1 presented in the main text.

Proposition 3. *Given a diffusion process in (57) with $\mathbf{G}$-invariant $\mu_0(\cdot|\cdot)$, $\mu_t(\cdot|\cdot)$ is $\mathbf{G}$-invariant for all $t \geq 0$ if and only if*

$$A_{\kappa_1}^{\top}\mathbf{f}(\kappa_1\mathbf{x}, \kappa_2\mathbf{y}, t) - \mathbf{f}(\mathbf{x}, \mathbf{y}, t) \in [\mathbf{0}]_{\mu_t}. \tag{58}$$

for all $t > 0$, $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^n$ and $\boldsymbol{\kappa} \in \mathbf{G}$.

Proof. ($\Rightarrow$) Let $\tilde{\mathbf{f}}$ denote the corresponding ODE drift shown in (51). Assume $\mu_t(\cdot|\cdot)$ is $\mathbf{G}$-invariant for all $t \geq 0$. Then by Lem 6, for all $\boldsymbol{\kappa} \in \mathbf{G}$, the ODE drift $\tilde{\mathbf{f}}$ satisfies

$$A_{\kappa_1}^\top \mathbf{f}(\kappa_1\mathbf{x}, \kappa_2\mathbf{y}, t) - \mathbf{f}(\mathbf{x}, \mathbf{y}, t) \in [\mathbf{0}]_{\mu_t}. \tag{59}$$

That is,

$$A_{\kappa_1}^\top \Big(\mathbf{f}(\kappa_1\mathbf{x}, \kappa_2\mathbf{y}, t) - \frac{1}{2}g(t)^2\mathbf{s}_t(\kappa_1\mathbf{x}|\kappa_2\mathbf{y})\Big) - \Big(\mathbf{f}(\mathbf{x}, \mathbf{y}, t) - \frac{1}{2}g(t)^2\mathbf{s}_t(\mathbf{x}|\mathbf{y})\Big) \in [\mathbf{0}]_{\mu_t}. \tag{60}$$

By Lem 10, $\mathbf{G}$-invariance of μ_t implies that $A_{\kappa_1}^\top \mathbf{s}_t(\kappa_1\mathbf{x}|\kappa_2\mathbf{y}) = \mathbf{s}_t(\mathbf{x}|\mathbf{y})$. Thus, (58) follows.

($\Leftarrow$) Assume (58) holds. For $\phi \in C_c^\infty(\mathbb{R}^{m+n})$ and $\boldsymbol{\kappa} \in \mathbf{G}$, let $\psi = \phi \circ \boldsymbol{\kappa}$, and then we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \phi(\mathbf{x}, \mathbf{y})\,\mathrm{d}\boldsymbol{\mu}_t(\kappa_1^{-1}\mathbf{x}|\kappa_2^{-1}\mathbf{y}) = \frac{\mathrm{d}}{\mathrm{d}t}\int \psi(\mathbf{x}, \mathbf{y})\,\mathrm{d}\boldsymbol{\mu}_t(\mathbf{x}, \mathbf{y}) \\
&\overset{(28)}{=} \int \nabla_1\psi(\mathbf{x}, \mathbf{y})^\top \tilde{\mathbf{f}}(\mathbf{x}, \mathbf{y}, t)\,\mathrm{d}\mu_t(\mathbf{x}|\mathbf{y}) = \int \nabla_1\psi(\mathbf{x}, \mathbf{y})^\top \Big(\mathbf{f}(\mathbf{x}, \mathbf{y}, t) - \frac{1}{2}g^2(t)\mathbf{s}_t(\mathbf{x}|\mathbf{y})\Big)\,\mathrm{d}\mu_t(\mathbf{x}|\mathbf{y}) \\
&= \underbrace{\int \nabla_1\psi(\mathbf{x}, \mathbf{y})^\top \mathbf{f}(\mathbf{x}, \mathbf{y}, t)\,\mathrm{d}\mu_t(\mathbf{x}|\mathbf{y})}_{\mathbf{I}} - \frac{1}{2}g^2(t)\underbrace{\int \nabla_1\psi(\mathbf{x}, \mathbf{y})^\top \mathbf{s}_t(\mathbf{x}|\mathbf{y})\,\mathrm{d}\mu_t(\mathbf{x}|\mathbf{y})}_{\mathbf{II}}.
\end{aligned} \tag{61}$$

By (58) and applying the same argument to derive (**A**) in the proof of Lem 6. We have

$$\int \nabla_1\phi(\kappa_1\mathbf{x}, \kappa_2\mathbf{y})^\top \mathbf{f}(\kappa_1\mathbf{x}, \kappa_2\mathbf{y}, t)\,\mathrm{d}\mu_t(\mathbf{x}|\mathbf{y}) = \int \nabla_1\psi(\mathbf{x}, \mathbf{y})^\top \mathbf{f}(\mathbf{x}, \mathbf{y}, t)\,\mathrm{d}\mu_t(\mathbf{x}|\mathbf{y}) = \mathbf{I}$$

Then,

$$\begin{aligned}
\mathbf{I} &= \int \nabla_1\phi(\mathbf{x}, \mathbf{y})^\top \mathbf{f}(\mathbf{x}, \mathbf{y}, t)\,\mathrm{d}\mu_t(\kappa_1^{-1}\mathbf{x}|\kappa_2^{-1}\mathbf{y}) \\
&= -\int \phi(\mathbf{x}, \mathbf{y})\,\nabla_{\mathbf{x}} \cdot \big(p_t(\kappa_1^{-1}\mathbf{x}|\kappa_2^{-1}\mathbf{y})\mathbf{f}(\mathbf{x}, \mathbf{y}, t)\big)\,\mathrm{d}\lambda(\mathbf{x}).
\end{aligned}$$

In addition,

$$\begin{aligned}
\mathbf{II} &= \int \nabla_1\psi(\mathbf{x}, \mathbf{y})^\top p_t(\mathbf{x}|\mathbf{y})\,\mathrm{d}\lambda(\mathbf{x}) = -\int \psi(\mathbf{x}, \mathbf{y})\Delta_1 p_t(\mathbf{x}|\mathbf{y})\,\mathrm{d}\lambda(\mathbf{x}) \\
&= -\int \phi(\mathbf{x}, \mathbf{y})\Delta_1 p_t(\kappa_1^{-1}\mathbf{x}|\kappa_2^{-1}\mathbf{y})\,\mathrm{d}\lambda(\mathbf{x}) \overset{(\text{Lem } 9)}{=} -\int \phi(\mathbf{x}, \mathbf{y})\,\Delta_1(p_t \circ \boldsymbol{\kappa}^{-1})(\mathbf{x}|\mathbf{y})\,\mathrm{d}\lambda(\mathbf{x})
\end{aligned}$$

As a result, by Eq (61), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_c \phi(\mathbf{x}, \mathbf{y})p_t(\kappa_1^{-1}\mathbf{x}|\kappa_2^{-1}\mathbf{y})\,\mathrm{d}\lambda(\mathbf{x}) = \frac{\mathrm{d}}{\mathrm{d}t}\int \phi(\mathbf{x}, \mathbf{y})\,\mathrm{d}\boldsymbol{\mu}_t(\kappa_1^{-1}\mathbf{x}, \kappa_2^{-1}\mathbf{y}) \\
&= -\int_c \phi(\mathbf{x}, \mathbf{y})\left[\nabla_{\mathbf{x}} \cdot \big(p_t(\kappa_1^{-1}\mathbf{x}|\kappa_2^{-1}\mathbf{y})\mathbf{f}(\mathbf{x}, \mathbf{y}, t)\big) - \frac{1}{2}g^2(t)\Delta_1(p_t \circ \boldsymbol{\kappa}^{-1})(\mathbf{x}, \mathbf{y})\right]\,\mathrm{d}\lambda(\mathbf{x})
\end{aligned}$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}p_t(\kappa_1^{-1}\mathbf{x}|\kappa_2^{-1}\mathbf{y}) = -\nabla_{\mathbf{x}} \cdot \big(p_t(\kappa_1^{-1}\mathbf{x}|\kappa_2^{-1}\mathbf{y})\mathbf{f}(\mathbf{x}, \mathbf{y}, t)\big) - \frac{1}{2}g^2(t)\Delta_1(p_t \circ \boldsymbol{\kappa}^{-1})(\mathbf{x}, \mathbf{y}). \tag{62}$$

By the Fokker-Planck equation, we also have

$$\frac{\mathrm{d}}{\mathrm{d}t}p_t(\mathbf{x}|\mathbf{y}) = -\nabla_{\mathbf{x}} \cdot (p_t(\mathbf{x}|\mathbf{y})\mathbf{f}(\mathbf{x}, \mathbf{y}, t)) + \frac{1}{2}g^2(t)\,(\Delta_1 p_t)(\mathbf{x}|\mathbf{y}). \tag{63}$$

Therefore, $p_t = p_t \circ \boldsymbol{\kappa}^{-1}$, which, by Lem 5, implies $\boldsymbol{\mu}_t$ is $\mathbf{G}$-invariant. □

Proposition 1. *Given a diffusion process in (5) with $\mathbf{G}$-invariant $p_0(\mathbf{x}_0|\mathbf{y})$, let $[\mathbf{0}]_{p_t}$ be the set of ODE drifts preserving the distribution p_t. Then $p_t(\mathbf{x}_t|\mathbf{y})$ is $\mathbf{G}$-invariant for all $t \geq 0$ if and only if*

$$\kappa_1^{-1} \circ \mathbf{f}(\kappa_1 \mathbf{x}, \kappa_2 \mathbf{y}, t) - \mathbf{f}(\mathbf{x}, \mathbf{y}, t) \in [\mathbf{0}]_{p_t} \tag{7}$$

for all $t > 0$, $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^n$ and $\boldsymbol{\kappa} \in \mathbf{G}$.

Proof. Prop 1 is equivalent to Prop 3 but in probability density notations by Lem 5. □

Finally, we show how our theoretical results can be applied to characterize the structure-preserving properties of the diffusion bridges. The main results are given in Lem 13 and are collectively presented with the counterparts for the regular diffusion processes in Prop 2.

Lemma 12. *Let p_t denote the distribution of $\mathbf{x}_t$ generated by SDE:*

$$\mathrm{d}\mathbf{x}_t = \mathbf{u}(\mathbf{x}_t, t)\,\mathrm{d}t + g(t)\,\mathrm{d}\mathsf{w}_t. \tag{64}$$

Then $p(\kappa \mathbf{x}_T|\kappa \mathbf{x}_t) = p(\mathbf{x}_T|\mathbf{x}_t)$ for all $\kappa \in \mathcal{G}_r$ and $T \geq t$ if

$$A_\kappa^\top \mathbf{u}(\kappa \mathbf{x}, t) - \mathbf{u}(\mathbf{x}, t) \in [\mathbf{0}]_{\mu_t}, \tag{65}$$

for all $\kappa \in \mathcal{G}$.

Proof. Without loss of generality, it is sufficient to show

$$p(\kappa \mathbf{x}_t|\kappa \mathbf{x}_0) = p(\mathbf{x}_t|\mathbf{x}_0) \tag{66}$$

for all $\kappa \in \mathcal{G}$ and $t \geq 0$. Then let $\mathbf{y} = \mathbf{x}_0$, $\mathbf{f}(\mathbf{x}_t, \mathbf{y}, t) = \mathbf{u}(\mathbf{x}, t)$ and $\mathbf{G} = \{(\kappa, \kappa)|\kappa \in \mathcal{G}_r\}$. As (65) implies $A_\kappa^\top \mathbf{u}(\kappa \mathbf{x}, t) - \mathbf{u}(\mathbf{x}, t) \in [\mathbf{0}]_{\mu_t}$. Then, combined with (65), Prop 3 shows $\mu_t(\mathbf{x}_t|\mathbf{y})$ is $\mathbf{G}$-invariant. By Lem 5, we have $p(\kappa \mathbf{x}_t|\kappa \mathbf{x}_0) = p(\mathbf{x}_t|\mathbf{x}_0)$, which completes the proof. □

Lemma 13. *Assume the two ends $(\mathbf{x}_0, \mathbf{x}_T) \in \mathbb{R}^d \times \mathbb{R}^d$ of the diffusion bridges follow a $\mathbf{G}$-invariant conditional distribution $\boldsymbol{\mu}_{0|T}(\mathbf{x}_0|\mathbf{x}_T)$, where $\mathbf{G} = \{(\kappa, \kappa)|\kappa \in \mathcal{G}\}$. Let $\boldsymbol{\mu}_{t|T}$ denote the measure of $(\mathbf{x}_t, \mathbf{x}_T)$ induced by diffusion bridge:*

$$\mathrm{d}\mathbf{x}_t = \big(\mathbf{u}(\mathbf{x}_t, t) + g(t)^2 \mathbf{h}(\mathbf{x}_t, t, \mathbf{x}_T, T)\big)\,\mathrm{d}t + g(t)\,\mathrm{d}\mathsf{w}_t, \tag{67}$$

where $\mathbf{h}(\mathbf{x}_t, t, \mathbf{x}_T, T) = \nabla_{\mathbf{x}_t} \log p(\mathbf{x}_T|\mathbf{x}_t)$ is the gradient of the log transition kernel from t to T generated by the original SDE:

$$\mathrm{d}\mathbf{x}_t = \mathbf{u}(\mathbf{x}_t, t)\,\mathrm{d}t + g(t)\,\mathrm{d}\mathsf{w}_t. \tag{68}$$

If

$$A_\kappa^\top \mathbf{u}(\kappa \mathbf{x}, t) - \mathbf{u}(\mathbf{x}, t) = \mathbf{0} \tag{69}$$

for all $\kappa \in \mathcal{G}$, then $\mathbf{u}(\mathbf{x}, t) + g(t)^2 \mathbf{h}(\mathbf{x}, t, \mathbf{y}, T)$ satisfies (58) and $\mu_{t|T}$ is $\mathbf{G}$-invariant for all $t \in [0, T]$.

Proof. By Lem 12, we have $p(\kappa \mathbf{x}_T|\kappa \mathbf{x}_t) = p(\mathbf{x}_T|\mathbf{x}_t)$ for all $\kappa \in \mathcal{G}$. As a result,

$$A_\kappa \nabla_{\mathbf{x}_t} \log p(\mathbf{x}_T|\mathbf{x}_t) = \nabla_{\kappa \mathbf{x}_t} \log p(\kappa \mathbf{x}_T|\kappa \mathbf{x}_t). \tag{70}$$

Or equivalently,

$$\mathbf{h}(\mathbf{x}_t, t, \mathbf{x}_T, T) = A_\kappa^\top \mathbf{h}(\kappa \mathbf{x}_t, t, \kappa \mathbf{x}_T, T). \tag{71}$$

As a result,

$$\mathbf{f}(\mathbf{x}, \mathbf{y}, t) = \mathbf{u}(\mathbf{x}, t) + g(t)^2 \mathbf{h}(\mathbf{x}, t, \mathbf{y}, T). \tag{72}$$

satisfies (58), and thus Prop 3 implies $\boldsymbol{\mu}_{t|T}$ is $\mathbf{G}$-invariant. □

Proposition 2. *Assume $\mathbf{u}(\mathbf{x}, t) = u(t)\mathbf{x}$ for some scalar function $u : \mathbb{R} \to \mathbb{R}$. Given any group $\mathcal{G}$ (or $\mathbf{G}$) composed of linear isometries, if the unconditional p_t induced by (1) is G-invariant at $t = 0$, then it is $\mathcal{G}$-invariant for all $t \geq 0$. Likewise, if the conditional $q_t(\mathbf{x}_t|\mathbf{x}_T)$ induced by (3) is $\mathbf{G}$-invariant at $t = 0$ then it is $\mathbf{G}$-invariant for all $t \geq 0$.*

Proof. The the part of the proposition regarding the unconditional p_t follows Prop 3 while the second part basically restates Lem 13 in density notation where their equivalence can be seen by Lem 5. □

B. Group invariant weight tied convolutional kernels

As stated in Sec 5, we currently limit our attention to linear groups $\mathcal{G}_{\mathcal{L}}$. In this setting, we can directly impose $\mathcal{G}_{\mathcal{L}}$-equivariance into the diffusion model by constructing specific CNN kernels.

In particular, for a given linear group $\mathcal{G}_{\mathcal{L}}$ we can construct a group equivariant convolutional kernel $\mathsf{k} \in \mathbb{R}^{d\times d}$, of the form

$$\mathsf{k} = \begin{array}{|c|c|c|c|} \hline k_{1,1} & k_{1,2} & \cdots & k_{1,d} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & & \vdots \\ \hline k_{d-1,1} & k_{d-1,2} & \cdots & k_{d-1,d} \\ \hline k_{d,1} & k_{d,2} & \cdots & k_{d,d} \\ \hline \end{array}, \tag{73}$$

such that

$$\mathsf{h}(\mathsf{k} * \mathbf{x}) = \mathsf{k} * \mathsf{h}(\mathbf{x})$$

for any $h \in \mathcal{G}_{\mathcal{L}}$ and $\mathbf{x} \sim p_{data}$ by constraining the individual kernel values to obey a system of equalities set by the group invariance condition

$$\mathsf{h}(\mathsf{k}) = \mathsf{k}. \tag{74}$$

Example: Vertical flipping. A concrete example, which was discussed Sec 3.2, is to consider the group $\mathcal{G} = \{\mathsf{f}_x, \mathbf{e}\}$ where f_x is a vertical flipping operation with $\mathsf{f}_x^{-1} = \mathsf{f}_x$. A convolutional kernel $\mathsf{k} \in R^{3\times 3}$ constrained to be equivariant to actions from this group would take the form:

$$\mathsf{k} = \begin{array}{|c|c|c|} \hline d & a & d \\ \hline e & b & e \\ \hline f & c & f \\ \hline \end{array} \tag{75}$$

It should be clear given the form of k that

$$\mathsf{f}_x(\mathsf{k} * \mathbf{x}) = \mathsf{k} * \mathsf{f}_x(\mathbf{x})$$

and consequently also for f_x^{-1}, as desired. Likewise, we also present the weight-tied kernels for C4 and D4.

Example: The C_4 cyclic and D_4 dihedral group. Recall that the C_4 cyclic group is composed of planar $90\,\mathrm{deg}$ rotations about the origin, and can be denoted as $C_4 = \{\mathsf{e}, \mathsf{r}_1, \mathsf{r}_2, \mathsf{r}_3\}$ where r_i represents a rotation of $i \times 90\,\mathrm{deg}$. Taking a convolutional kernel $\mathsf{k} \in \mathbb{R}^{5\times 5}$ and constraining it to be C_4-equivariant results in k being of the form:

$$\mathsf{k} = \begin{array}{|c|c|c|c|c|} \hline a & b & c & d & a \\ \hline d & e & f & e & b \\ \hline c & f & g & f & c \\ \hline b & e & f & e & d \\ \hline a & d & c & b & a \\ \hline \end{array}. \tag{76}$$

The D_4 dihedral group can then be "constructed" from C_4 by adding the vertial flipping operation from the past example; that is, $D_4 = \{\mathsf{e}, \mathsf{r}_1, \mathsf{r}_2, \mathsf{r}_3, \mathsf{f}_x, \mathsf{f}_x \circ \mathsf{r}_1, \mathsf{f}_x \circ \mathsf{r}_2, \mathsf{f}_x \circ \mathsf{r}_3\}$. This requires further constraints to k so that

$$\mathsf{k} = \begin{array}{|c|c|c|c|c|} \hline a & b & c & b & a \\ \hline b & e & f & e & b \\ \hline c & f & g & f & c \\ \hline b & e & f & e & b \\ \hline a & b & c & b & a \\ \hline \end{array}. \tag{77}$$

Naturally, constraining convolutional kernels in this fashion has the advantage of reducing the number of model parameters – with a possible loss in expressiveness when the kernel size is relatively small in comparison to the size of the group and structure of the data. For a more general discussion on $\mathcal{G}$-equivariant convolutional kernels in the context of CNNs we refer the reader to Cohen & Welling (2016) and Knigge et al. (2022).

C. Equivariance regularization

Instead of achieving $\mathcal{G}$-equivalence by adopting specific model architectures, as described in Sec 5, or by frame averaging (Puny et al., 2022), we can also directly add a regularizer to the score-matching loss to inject this preference. Specifically, according to Lem 1, the estimated score $\mathbf{s}_{\boldsymbol{\theta}}(\cdot, t)$ is equivariant if

$$\mathbf{s}_{\boldsymbol{\theta}}(\kappa\mathbf{x}, \kappa\mathbf{y}, t) = \kappa\mathbf{s}_{\boldsymbol{\theta}}(\mathbf{x}, \mathbf{y}, t), \tag{78}$$

for all $\kappa \in \mathcal{G}$. (For the unconditional distribution, similar techniques can be applied by omitting the second argument of $\mathbf{s}_{\boldsymbol{\theta}}$.) Thus, we propose the following regularizer to encourage the two terms to match for all $\mathbf{x}$ and t:

$$\mathcal{R}(\boldsymbol{\theta}, \bar{\boldsymbol{\theta}}) = \mathbb{E}\left[\frac{1}{|\mathcal{G}|}\sum_{\kappa\in\mathcal{G}}\left\|\mathbf{s}_{\boldsymbol{\theta}}(\kappa\mathbf{x}, \kappa\mathbf{y}, t) - \kappa\mathbf{s}_{\bar{\boldsymbol{\theta}}}(\mathbf{x}, \mathbf{y}, t)\right\|^2\right] \tag{79}$$

where the expectation is taken over the same variables in the regular score-matching loss and $\bar{\boldsymbol{\theta}}$ denotes the exponential moving average (EMA) of the model weights

$$\bar{\boldsymbol{\theta}} \leftarrow \text{stopgrad}(\mu\bar{\boldsymbol{\theta}} + (1-\mu)\boldsymbol{\theta}) \;\text{ with }\; \mu \in [0, 1), \tag{80}$$

which helps improve training stability. In practice, iterating over all elements in $\mathcal{G}$ may be intractable. Thus, for each optimization step, $\mathcal{R}(\boldsymbol{\theta}, \bar{\boldsymbol{\theta}})$ is one-sample approximated by:

$$\mathcal{R}(\boldsymbol{\theta}, \bar{\boldsymbol{\theta}}) \approx \mathbb{E}\left[\left\|\mathbf{s}_{\boldsymbol{\theta}}(\kappa\mathbf{x}, \kappa\mathbf{y}, t) - \kappa\mathbf{s}_{\bar{\boldsymbol{\theta}}}(\mathbf{x}, \mathbf{y}, t)\right\|^2\right], \tag{81}$$

with randomly picked $\kappa \in \mathcal{G}$.

D. Constructions of equivariant noisy sequence

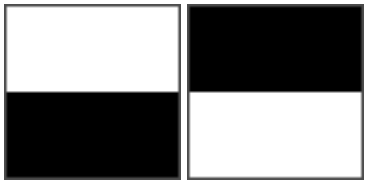
In this section, we present a method to construct an equivariant noisy sequence $\{\boldsymbol{\epsilon}_i\}_{i=1}^n$ with respect to some $\mathbf{x}_n \sim q(\mathbf{x})$ without knowing the "true" orientation of $\mathbf{x}_n$.

Let q denote the distribution of $\mathbf{x}_n$. Construct a function $\phi : \mathbb{R}^d \to \mathbb{R}^d$ such that: (1) for all $\kappa \in \mathcal{G}$, $\mathbf{x} \sim q$ or $\mathbf{x} \sim \mathcal{N}(\mathbf{0}, I)$, $\phi(\kappa\mathbf{x}) = \kappa\phi(\mathbf{x})$ almost surely; (2) for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, there exists a unique $\kappa \in \mathcal{G}$ such that $\phi(\mathbf{x}) = \kappa\phi(\mathbf{y})$. For example, for $\mathbb{R}^2$ with $\mathcal{G}$ consisting of element-swapping operators, ϕ can be the function that outputs one-hot vector indicating the max element of the input. We will present some selections of ϕ for common $\mathcal{G}$ below.

Given starting point $\mathbf{x}_n$ and a noise sequence $\{\boldsymbol{\epsilon}_i\}_{i=1}^n$, choose $\kappa \in \mathcal{G}$ such that $\phi(\mathbf{x}_n) = \kappa\phi(\boldsymbol{\epsilon}_n)$. Then we use the noise sequence $\tilde{\boldsymbol{\epsilon}}_i = \kappa\boldsymbol{\epsilon}_i$ for the evaluation of (14). To see why this approach works, assume that $\mathbf{x}_n$ is updated to $\mathsf{r}\mathbf{x}_n$ for some $\mathsf{r} \in \mathcal{G}$. Then, $\phi(\mathsf{r}\mathbf{x}_n) = \mathsf{r}\phi(\mathbf{x}_n) = (\mathsf{r} \circ \kappa)\phi(\boldsymbol{\epsilon}_n)$, and thus the sequence becomes $\{\mathsf{r} \circ \kappa\, \boldsymbol{\epsilon}_i\}_{i=1}^n = \{\mathsf{r}\tilde{\boldsymbol{\epsilon}}_i\}_{i=1}^n$. Note that this is a general method to create an equivariant noise sequence with respect to any input.

Below, we present some choices of ϕ for some common linear operator groups for 2D images.

Example: Vertical flipping. The function ϕ_v can be chosen to output either of two images:

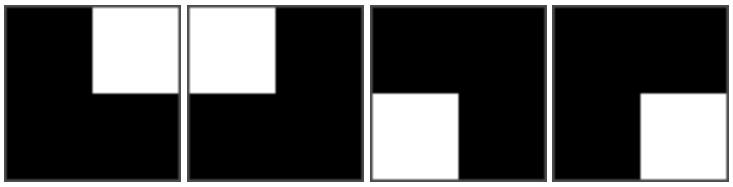


Specifically, if the input image $\mathbf{x}$ has the max value on the upper half of the image, ϕ returns the left plot; otherwise, the right one. (Here, we assume that it is almost surely that the max value cannot appear in both halves.)

It is obvious that if the input is flipped vertically, the output will be flipped in the same way. Therefore, the first condition is satisfied. For the second, if $\phi_v(\mathbf{x})$ and $\phi_v(\mathbf{y})$ have the same output, κ is the identity operator; otherwise, κ is the vertical flipping. For multichannel input, ϕ can be applied independently to each channel.

Applying the same idea, we can derive the corresponding ϕ_h for horizontal flipping.

Example: C_4 cyclic group. We can use a similar idea to derive ϕ_{C4} for C_4 cyclic that is composed of planar $90\,\mathrm{deg}$ rotations about the origin. In this case, ϕ_{C4} has four possible outputs



such that ϕ_{C4} assigns the quadrant white if the input has the max value in that quadrant. (Here, we assume it is almost surely that the max value cannot appear in multiple quadrants.) Then, it is straightforward to see that ϕ_{C4} satisfies the two conditions of ϕ.

Example: D_4 dihedral group. As we have mentioned in Appx B, the D_4 dihedral group can be "constructed" from C_4 by adding the vertial flipping operation. As a result, we can combine ϕ_v and ϕ_{C4} to construct the corresponding ϕ_{D4} for D_4. Assume that ϕ_v assigns one to the elements corresponding to the white pixels and zero to the ones associated with the black. Likewise, let ϕ_{C4} assign two to the elements corresponding to the white pixels and zero to the rest. Then we define $\phi_{D4} = \phi_v + \phi_{C4}$. It is easy to check that both ϕ_v and ϕ_{D4} satisfy the first condition of ϕ for all $\kappa \in D_4$ (i.e., vertical flipping, rotation, and their composition). We also note that the range ϕ_{D4} contains eight distinct elements. Starting from one element, we get all the elements by applying one of the eight operators in $D_4 = \{\mathsf{e}, \mathsf{r}_1, \mathsf{r}_2, \mathsf{r}_3, \mathsf{f}_x, \mathsf{f}_x \circ \mathsf{r}_1, \mathsf{f}_x \circ \mathsf{r}_2, \mathsf{f}_x \circ \mathsf{r}_3\}$, which suggests ϕ_{D4} satisfies the second condition.

E. Dataset details

This section contains detailed discussion on the contents and preprocessing of each dataset mentioned in Sec 6.

E.1. Rotated MNIST

Rotated MNIST dataset (Larochelle et al., 2007) contains random 90° rotations of MNIST images (Deng, 2012), resulting in a C_4-invariant distribution. This was dataset was generated following the description in (Knigge et al., 2022), but modified to only perform random 90° rotations opposed to the arbitrary angles used in the forgoing reference.

E.2. LYSTO

The LYSTO dataset (Jiao et al., 2023) consists of 20,000 labeled image patches at a resolution of 299x299x3 extracted at 40X magnification from breast, colon, and prostate cancer samples stained with CD3 or CD8 dyes. This data was preprocessed by first scaling all the images to 128x128x3 before randomly sampling 64x64x3 image patches for the data. The data was

first scaled to increase the feature density within each randomly sampled patch. The resulting data exhibits D_4 invariance due to natural rotational and mirror invariance.

E.3. LYSTO denosing

To construct the LYSTO denosing dataset, we take the LYSTO 64x64x3 patches, generated as described above, and downscale them to $1/4$ the resolution and then upscaled back using LANCZOS interpolation to form conditional training pairs.

E.4. ANHIR

The ANHIR dataset (Borovec et al., 2020b) provides whole-slide images of lesions, lung-lobes, and mammary-glands at a variety of different resolutions, from 15kx15k to 50kx50k. We only make use of the lung images. This data was processed following the method outlined in (Dey et al., 2021) from which random 64x64x3 image patches are extracted.

E.5. CT-PET

The CT-PET dataset (Gatidis et al., 2022) includes 1014 (501 positives and 512 controls) annotated whole-body paired FDG-PET/CT scans, comprised of 3D voxels, of patients with malignant lymphoma, melanoma, and non-small cell lung cancer. This data is restricted under TCIA restricted lience. Formal access must be filed for and granted before the data can be made available to practitioners from the Caner Imaging Archive (CIA). Scripts for processing this data, such as into slices, are provided by CIA on the dataset page.

The style-transfer dataset was constructed by first slicing the 3D voxel volumes of the patients in to 2D images of the middle of the patients. These images were then cropped to just contain the torso and head and then scaled to 256x256x3. A patient's CT scan slice was then paired with the matching PET scan image slice to form the final dataset.

F. Model details

Table 5: Model summary. VP-SDE denotes the regular diffusion model with variance preservation configuration in Table 1. ✓: Theo. guaranteed ✗: Not theo. guaranteed –: Not Applicable

Model	Arch.	$\mathcal{G}_{\mathcal{L}}$-inv Smpl	Eqv Traj
VP-SDE	U-Net	✗	✗
SPDM+WT	U-Net (WT)	✓	✓
SPDM+FA	U-Net	✓	✓
DDBM	U-NET	✗	✗
SPDM+FA (Bridge)	U-NET	✓	✓
SP-GAN	CNN	✓	–
GE-GAN	CNN	✗	–
Pix2Pix	U-NET & CNN	✗	–
I^2SB	U-NET	✗	✗

The implementation details and hyperparamters used while training the models presented in Sec 6 over the listed datasets are given below. Table 5 provides a summary of all the models discussed within the paper and their theoretical invariance (equivariance) guarantees.

We refer the reader to the reader to Appx.G. of Birrell et al. (2022) for a more in-depth discussion of the implementation details and training parameters used to produce the results of SP-GAN reported in Sec 6. We will only summarize the training parameters we changed from the defaults given in the forgoing reference.

All Diffusion models (VP-SDE, SPDM+WT, SPDM+FA, SPDM+FA(Bridge)) are trained using the Adam optimizer (Kingma & Ba, 2015) with learning rate $\eta = 0.0001\, 0.0002$, and weight decay rate of $\gamma = 0.0$; separately, we make use of the exponential moving average (EMA) of the model weights (80) with $\mu = 0.999,\ 0.9999,\ 0.9999432189950708$, which are the values commonly used when training this style of diffusion model.

F.1. Rotated MNIST

For the Rotated MNIST datasets, the diffusion models are configured using the following model parameters: dropout rate $\mathsf{d} = 0.1$.

Model	Loss	Batch	Cond.	Aug	Attn. res.	Num. Ch.	Num. Heads	Ch. Scal.	Scale Shift
SPDM	L_2	32	True	True	8	128	16	1,2,2	True
SPDM+WT	L_2	32	True	False	8	128	16	1,2,2	True
SPDM+FA	$L_2 + FA$	32	True	True	8	128	16	1,2,2	True

The GAN based methods were trained using the scripts provided by (Dey et al., 2021) and (Birrell et al., 2022) with the following settings:

Model	Loss	Batch	Cond.	Aug	latent dim	gp weight	lr	alpha
SP-GAN	D_2^L	64	True	True	64	10.0	$1e{-}4$	2
GE-GAN	RA	64	True	True	64	10.0	$1e{-}4$	–

F.2. LYSTO

For the LYSTO dataset, the diffusion models are configured using the following model parameters: dropout rate $\mathsf{d} = 0.1$.

Model	Loss	Batch	Cond.	Aug	Attn. res.	Num. Ch.	Num. Heads	Ch. Scal.	Scale Shift
VP-SDE	L_2	32	True	True	32,16,8	128	64	1,2,2,2	True
SPDM+WT	L_2	32	True	False	32,16,8	128	64	1,2,2,2	True
SPDM+FA	$L_2 + FA$	32	True	True	32,16,8	128	64	1,2,2,2	True

The GAN based methods were trained using the scripts provided by (Dey et al., 2021) and (Birrell et al., 2022) with the following settings:

Model	Loss	Batch	Cond.	Aug	latent dim	gp weight	lr	alpha
SP-GAN	D_2^L	32	True	True	128	10.0	$1e{-}4$	2
GE-GAN	RA	32	True	True	128	10.0	$1e{-}4$	–

F.3. ANHIR

For the ANHIR dataset, the diffusion models are configured using the following model parameters: dropout rate $\mathsf{d} = 0.1$.

Model	Loss	Batch	Cond.	Aug	Attn. res.	Num. Ch.	Num. Heads	Ch. Scal.	Scale Shift
VP-SDE	L_2	32	True	True	32,16,8	128	64	1,2,2,2	True
SPDM+WT	L_2	32	True	False	32,16,8	128	64	1,2,2,2	True
SPDM+FA	$L_2 + FA$	32	True	True	32,16,8	128	64	1,2,2,2	True

The GAN based methods were trained using the scripts provided by (Dey et al., 2021) and (Birrell et al., 2022) with the following settings:

F.4. LYSTO denoising task

For the LYSTO denoising dataset, the diffusion models are configured using the following model parameters:

The entries in the above table are left empty for both Pix2Pix and I^2SB as their architectures differ from the other models. In particular, Pix2Pix being a GAN makes use of a U-NET for the generator and custom discriminator architecture. The

Model	Loss	Batch	Cond.	Aug	latent dim	gp weight	lr	alpha
SP-GAN	D_2^L	32	True	True	128	10.0	$1e{-4}$	2
GE-GAN	RA	32	True	True	128	10.0	$1e{-4}$	–

Model	Loss	Batch	Cond.	Aug	Attn. res.	Num. Ch.	Num. Heads	Ch. Scal.	Scale Shift
VP-SDE	L_2	32	True	True	32,16,8	128	64	1,2,2,2	True
SPDM+FA	$L_2 + FA$	32	True	True	32,16,8	128	64	1,2,2,2	True
Pix2Pix	GAN	32	True	True	–	–	–	–	–
I^2SB	L_2	64	True	True	–	–	–	–	–

model comes with several predefined U-NET architecture configurations that can be selected; however, it is lacking a configuration for 64x64 images. We defined a suitable configuration by modifying that provided for 128x128 by reducing the number of down-sampling layers from 7 to 4. All other settings are left as default. The I^2SB model by default it make use of a preconfigured U-NET architecture which it downloads from openai. For this task we simply make use of the default configuration settings without any modification.

F.5. CT-PET style transfer task

For the CT-PET dataset, all models except Pix2Pix are trained in latent space, where the original images are first encoded by a fine-tuned pretrained VAE from stable diffusion (Rombach et al., 2022). FA was applied during the fine-tuning and inference to ensure equivariance. The VAE takes the 256x256x3 images and encodes them into a 32x32x4 latent space representation.

Models are configured using the following model parameters:

Model	Loss	Batch	Cond.	Aug	Attn. res.	Num. Ch.	Num. Heads	Ch. Scal.	Scale Shift
DDBM	L_2	32	True	True	32,16,8	128	64	1,2,2,2	True
SPDM+FA	$L_2 + FA$	32	True	True	32,16,8	128	64	1,2,2,2	True
Pix2Pix	GAN	32	True	True	–	–	–	–	–
I^2SB	L_2	64	True	True	32,16,8	128	64	1,2,2,2	True

The entries in the above table are left empty for both Pix2Pix as the architecture differs from the other models. Pix2Pix makes use of a U-NET for the generator and custom discriminator architecture. The model comes with a predefined U-NET architecture for 256x256x3 images which we use. All other settings are left as default. As discussed above the I^2SB model by default it make use of a preconfigured U-NET architecture. We can't make use of the default architecture for this task as it is incompatible with the latent space embedding produced by the VAE. Instead we configure the U-NET architecture in an identical fashion to SPBM.

F.6. Computational resources

All the model results reported within Sec 6 were trained using 4xNVIDIA A40 (or equivalent). All experiments required on the order of 5-20 days of training time.

G. FID Computation

In Sec 6.2 we report the Fréchet intercept distance (FID) (Heusel et al., 2017) score of the various models on the datasets described in Sec 6 and Appx E, respectively under C_4 and D_4 groups. In order to make the FID score robust to changes in image orientation, meaning the features the underlying InceptionV3 model extracts from the reference dataset can be compared to those extracted from the generated samples, we average the reference statistics over all actions within the group considered. In particular, suppose $\mathcal{D}_{ref}$ is a reference dataset (e.g., all the images within the rotated MNIST dataset) and $\mathcal{D}_s$ is a collection of samples generated from the model being evaluated with respect to group $\mathcal{G}$. Let $\mathrm{T}(\cdot)$ denote the operation

that returns the mean and covariance statistics of the features extracted from a dataset; i.e., $T(\mathcal{D}_f) = (\mu_s, \Sigma_s)$. Moreover, recall that FID is computed using the expression:

$$\mathrm{FID} = d^2(\mathrm{T}(\mathcal{D}_{ref}, \mathrm{T}(\mathcal{D}_s)) = \|\mu_{ref} - \mu_s\|_2^2 + \mathrm{Tr}(\Sigma_{ref} + \Sigma_s - 2(\Sigma_{ref}\Sigma_s)^{1/2}). \tag{82}$$

Then the calculation of the FID with respect to the group $\mathcal{G}$ is done by first computing

$$\mathrm{T}_{\mathcal{G}}(\mathcal{D}) = \frac{1}{|\mathcal{D}|} \sum_{\mathsf{h} \in \mathcal{G}} T(A_{\mathsf{h}}\mathcal{D}) = (\hat{\mu}, \hat{\Sigma}), \tag{83}$$

where $A_{\mathsf{h}}\mathcal{D} = \{A_{\mathsf{h}}\mathsf{x} \mid \mathsf{x} \in \mathcal{D}\}$. Then we compute the final FID score as

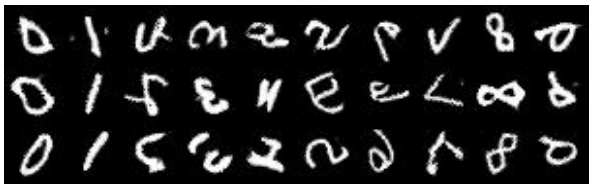
$$\mathrm{FID}_{\mathcal{G}} = d^2(\mathrm{T}_{\mathcal{G}}(\mathcal{D}_{ref}), \mathrm{T}(\mathcal{D}_s)). \tag{84}$$

This formulation ensures that the reference statistics used in computing the FID score of a model conditioned to be equivalent are not biased. All FID values reported in Table 2 and Table 4, potentially excluding those reported by other authors, were calculated in this fashion.

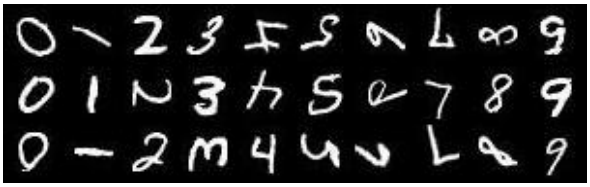
H. Sample images

Here, we include a collection of generated image samples from the models discussed within the paper across the various datasets in Section 6.

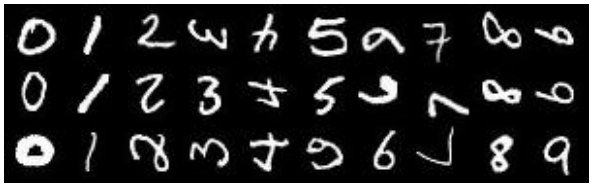
(a) Reference C_4 rotated MNIST images.



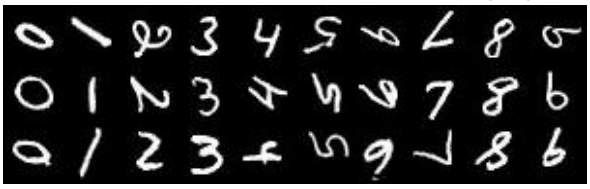
(b) Images generated from SP-GAN.



(c) Images generated from VP-SDE.

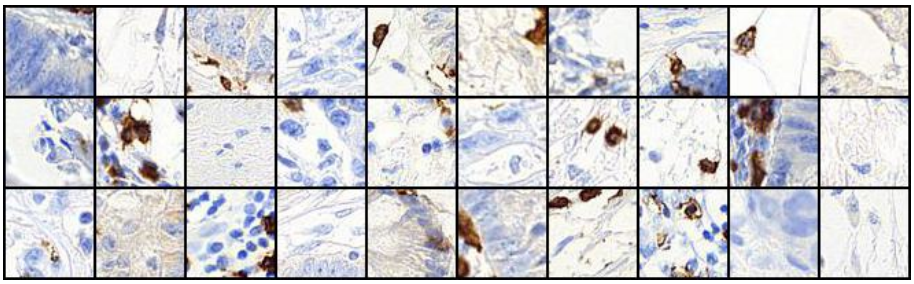


(d) Images generated from SPDM+WT

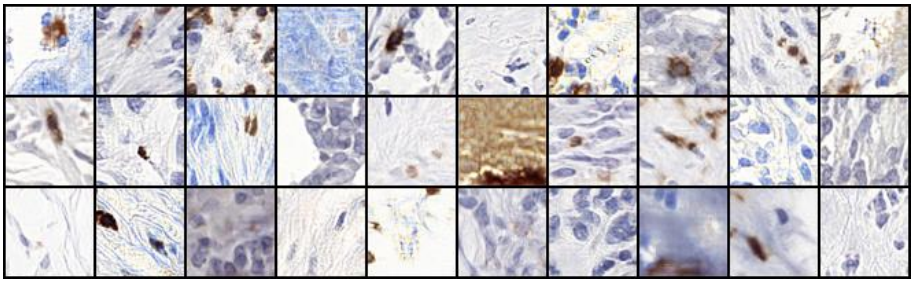


(e) Images generated from SPDM+FA

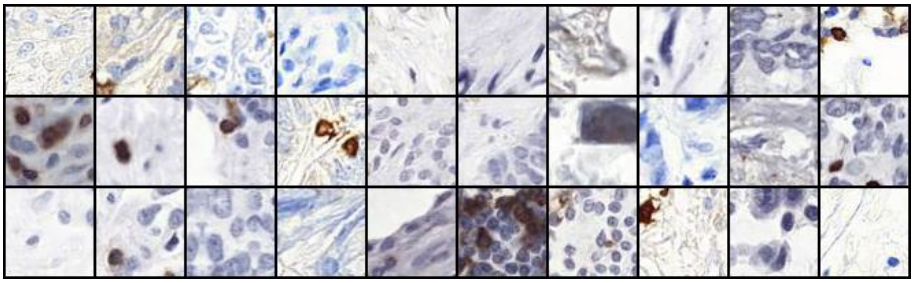
Figure 6: Sample comparison between models trained on the Rotated MNIST 28x28x1 dataset as described in Sec 6.



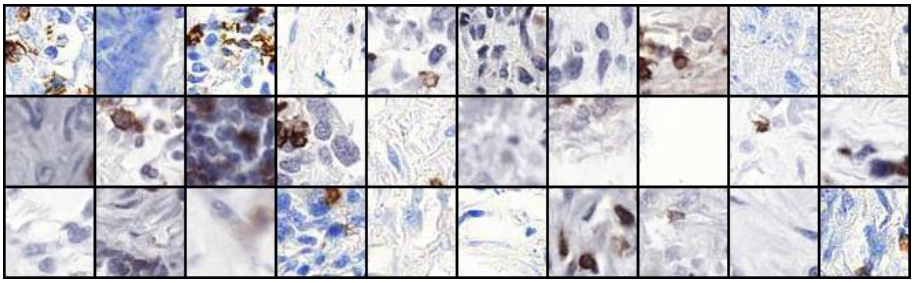
(a) Reference images.



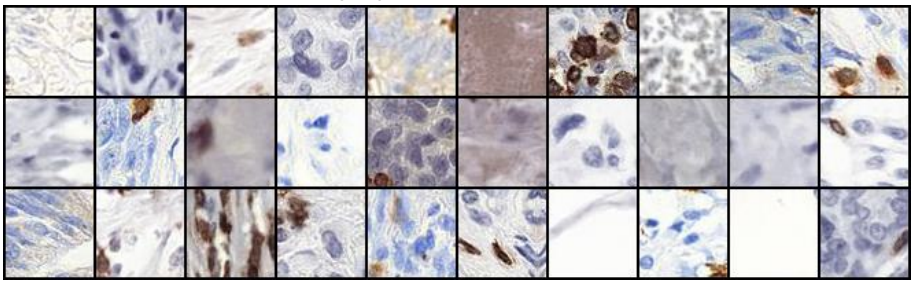
(b) Images generated from SP-GAN.



(c) Images generated from VP-SDE

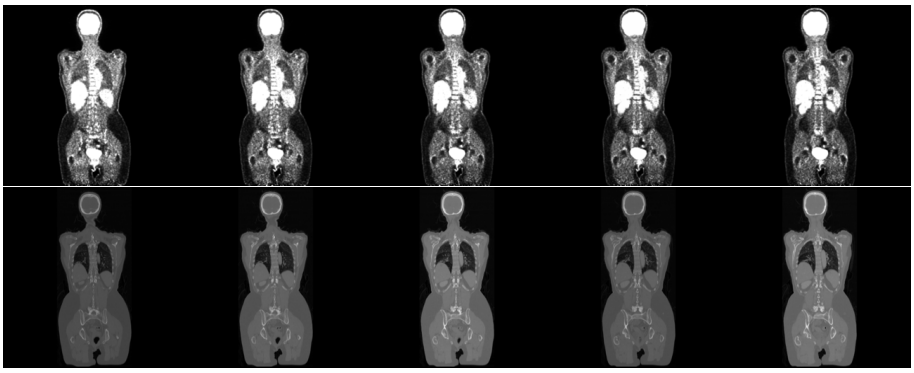


(d) Images generated from SPDM+WT.

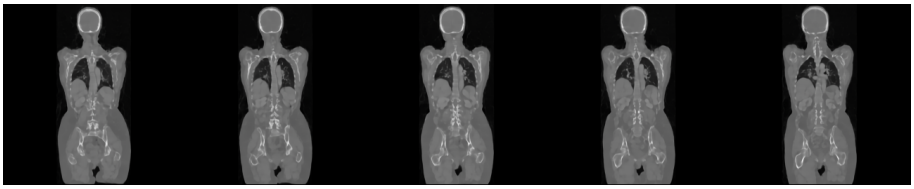


(e) Images generated from SPDM+FA

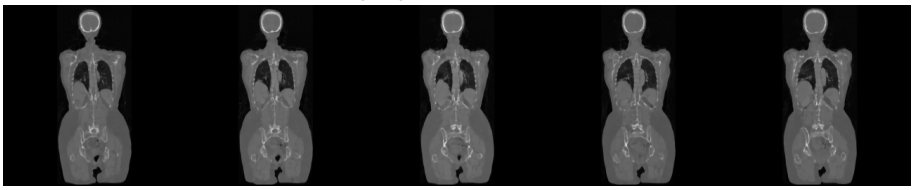
Figure 7: Sample comparison between models trained on the LYSTO 64x64x3 dataset from Sec 6.



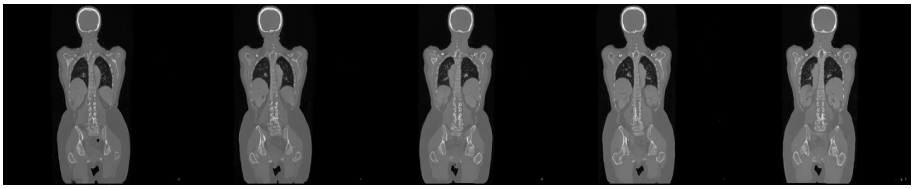
(a) Reference images of CT and PET images.



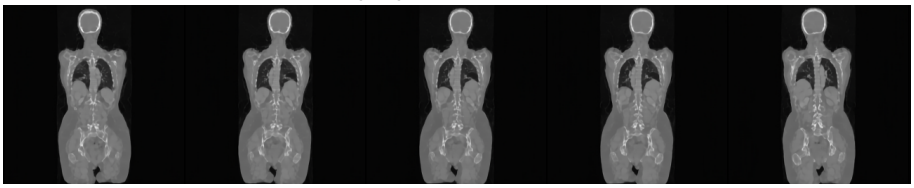
(b) Images generated from DDBM.



(c) Images generated from SPDM+FA



(d) Images generated from Pix2Pix.



(e) Images generated from I^2SB

Figure 8: Sample comparison between models trained on the CT-PET 256x256x3 dataset from Sec 6 and Appx E.