Stochastic Difference-of-Convex Optimization with Momentum

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Abstract

We study the online stochastic difference-of-convex (DC) program $\min_{x\in\mathbb{R}^n} F(x) = G(x) - H(x) + r_1(x) - r_2(x)$, a general formulation capturing numerous non-convex machine learning tasks, including robust regression, sparse learning, and fair classification. We propose *momentum-based DCA* (MDCA), an algorithm that combines a damped DCA step with a momentum estimator for ∇H. Crucially, MDCA operates with a constant per-iteration batch size. We establish that MDCA with Polyak momentum requires $\mathcal{O}(\epsilon^{-4})$ stochastic gradient evaluations to reach ϵ-stationarity, while momentum-based variance reduction (MVR) achieves $\mathcal{O}(\epsilon^{-3})$ under averaged smoothness, matching DCA-PAGE without the need for periodic large anchor batches. Furthermore, we introduce a single-loop variant (S-MDCA) that replaces exact subproblem solves with a single prox-gradient step; it reaches ϵ-stationarity of a smoothed objective in $\mathcal{O}(\epsilon^{-4})$ iterations. We isolate the algorithmic benefit of momentum via a *separation lemma*, demonstrating how it provides a tunable mechanism to trade estimator variance against iterate drift bias. Convergence is proven under both critical-distance and gap-function criteria, and validated through experiments on real classification datasets.

1 Introduction

We consider

$$\min_{x\in\mathbb{R}^n} \; F(x) := G(x) - H(x) + r_1(x) - r_2(x), \tag{1}$$

where $G, H, r_1, r_2 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ are proper closed convex; H is differentiable and accessed through unbiased stochastic gradients with bounded variance; r_1, r_2 are deterministic. We focus on the *online* regime, where $H = \mathbb{E}_\xi[h_\xi]$ is in principle a population average and we cannot evaluate full gradients of H.

The DC structure in equation 1 is the natural language of several model families that come up routinely in machine learning. Before our results, we recall five concrete instances that motivate why this problem class deserves dedicated attention.

Example 1 (Sparse learning with non-convex penalties). Many of the regularizers used to control over-shrinkage in high-dimensional regression have a clean DC decomposition $p(t) = \lambda|t| - h_p(t)$ with h_p convex and often smooth. Three canonical examples:

- *Capped-ℓ_1* (Zhang, 2010): $\mathrm{cap}(t;\tau) = \min\{|t|, \tau\}$ admits the decomposition $\mathrm{cap}(t) = |t| - \max\{|t| - \tau, 0\}$, with both pieces convex.
- *MCP* (Zhang, 2010): with parameter $\gamma_M > 0$, $\mathrm{MCP}(t;\lambda,\gamma_M) = \lambda|t| - h_{\mathrm{MCP}}(t)$ where h_{MCP} is convex and $1/\gamma_M$-smooth.
- *SCAD* (Fan and Li, 2001): similarly admits a decomposition $\mathrm{SCAD}(t) = \lambda|t| - h_{\mathrm{SCAD}}(t)$ with h_{SCAD} convex and $1/(a-1)$-smooth, for the SCAD parameter $a > 2$.

Penalized empirical risk minimization $\min_x \frac{1}{N}\sum_i \ell(a_i^\top x, y_i) + \sum_j p(x_j)$ with such a penalty falls within equation 1. If ℓ is also nonconvex but L-smooth, the loss ℓ itself can be DC-decomposed via the quadratic-shift trick $\ell = (L\|\cdot\|^2/2) - (L\|\cdot\|^2/2 - \ell)$; see Nguyen et al. (2025, Eq. (S)).

Example 2 (Robust regression with bounded losses). Bounded loss functions are essential for robustness to label noise and outliers. The truncated quadratic $\ell_\tau(r) = \min\{r^2, \tau^2\}$ admits the DC decomposition $\ell_\tau(r) = r^2 - \max(r^2 - \tau^2, 0)$. Similar decompositions are available for the Geman–McClure, Cauchy and Tukey biweight losses. For sparse robust regression $\min_x \frac{1}{N}\sum_i \ell_\tau(y_i - a_i^\top x) + \lambda\|x\|_1$, the data-fitting term gives $G - H$ via the bounded-loss decomposition, and a non-convex sparsity penalty (capped-ℓ_1, SCAD) places a deterministic nonsmooth term in r_2. This is the setup of our experiments in Section 6.

Example 3 (Positive-unlabelled (PU) learning). Kiryo et al. (2017) introduce the non-negative risk estimator

$$R_{\mathrm{PU}}(g) = \pi_p R_p^+(g) + \max\big\{0,\ R_u(g) - \pi_p R_p^-(g)\big\},$$

where $R_p^\pm, R_u$ are convex risk functionals and π_p is the class prior. The inner subtraction is a DC expression and the outer $\max\{0, \cdot\}$ is convex; rearranging puts R_{PU} into equation 1 with $G = \pi_p R_p^+$, $H = \pi_p R_p^-$, and the max folded into r_1 via lifting. The non-negative-risk correction in PU learning is, structurally, a DC device.

Example 4 (Fair classification with adversarial penalties). Zhang et al. (2018) introduce adversarial debiasing, where a model is trained to predict the target while an adversary tries to predict a sensitive attribute from the model's output:

$$\min_\theta \mathbb{E}[\ell_{\mathrm{task}}(f_\theta(X), Y)] - \lambda\,\mathbb{E}[\ell_{\mathrm{adv}}(f_\theta(X), A)].$$

When both losses are convex in their relevant inputs and the model is linear, this is the DC form equation 1 with $G = \mathbb{E}[\ell_{\mathrm{task}}]$, $H = \lambda\mathbb{E}[\ell_{\mathrm{adv}}]$. Solving such problems by alternating gradient descent is a heuristic; the DC formulation gives a principled framework with provable convergence to a critical point.

Example 5 (Matrix factorization with non-convex regularization). For low-rank matrix factorization $\min_{U,V} \|X - UV^\top\|_F^2 + \mathcal{R}(U, V)$, the data-fitting term is bilinear and DC-decomposable (Yao and Pong, 2021); non-convex regularizers $\mathcal{R}$ for sparsity or spectral structure compose with this further. This pattern covers dictionary learning, robust PCA, and non-negative matrix factorization.

These problems share two features: (i) the natural decomposition $F = G - H + r_1 - r_2$ is given by the model and is much cheaper to maintain than any global convex relaxation; (ii) modern instances are large enough or streaming enough that *stochastic* access to ∇H is the only practical option. The deterministic difference-of-convex algorithm (DCA) of Tao (1986; 1997); Tao and Le Thi (2005); Le Thi and Pham Dinh (2018b;a), the closely related concave-convex procedure (Yuille and Rangarajan, 2003; Sriperumbudur and Lanckriet, 2009), and structural extensions through directional stationarity (Ahn et al., 2017) handle (i) but not (ii).

Stochastic DC: state of the art. Nitanda and Suzuki (2017) introduced stochastic DCA with increasing batches; Xu et al. (2019) extended this to nonsmooth r_1. Building on the SVRG (Johnson and Zhang, 2013), SAGA (Defazio et al., 2014), SAG (Schmidt et al., 2017) variance-reduction line, Le Thi et al. (2022) introduced finite-sum variance-reduced DCA with $\mathcal{O}(N+N^{2/3}\epsilon^{-2})$ complexity. Phan et al. (2024) introduced majorization-minimization variants achieving $\mathcal{O}(N + N^{1/2}\epsilon^{-2})$ for the special decomposition (S) of Nguyen et al. (2025). Most recently, Nguyen et al. (2025) introduced DCA-PAGE (built on the PAGE estimator of Li et al. (2021) and the SPIDER/SpiderBoost line of Fang et al. (2018); Wang et al. (2019); Nguyen et al. (2017)), attaining the optimal finite-sum rate $\mathcal{O}(N + N^{1/2}\epsilon^{-2})$, which extends to the online setting as $\mathcal{O}(b + b^{1/2}\epsilon^{-2})$ *at the cost of using a periodic anchor batch of size* $b = \Theta(\sigma^2/\epsilon^2)$. Hu et al. (2024) introduced a single-loop algorithm (SMAG) for max-structured weakly-convex problems, under a uniformly bounded stochastic-subgradient assumption on H. In a true online regime, where data arrive in a stream, and per-step memory is bounded, the periodic large anchor of DCA-PAGE is exactly the sticking point we want to avoid.

Momentum in nonconvex optimization. Our estimators draw upon the classical foundation of Polyak heavy-ball momentum (Polyak, 1964) alongside the rich algorithmic lineage of momentum-based variance reduction (MVR/STORM) (Cutkosky and Orabona, 2019; Levy et al., 2021; Tran-Dinh et al., 2022; Driggs et al., 2022) and earlier variance-reduced methods for non-convex optimization (Reddi et al., 2016; Lan et al.,

2019; Allen-Zhu, 2017). Momentum is a well-established mechanism for eliminating the reliance on large batch sizes, having been successfully adapted for normalized gradient descent (Cutkosky and Mehta, 2020), frameworks leveraging auxiliary information (Chayti and Karimireddy, 2024), and stochastic cubic Newton methods (Chayti et al., 2024). Building on this track record, we seamlessly extend the variance-reducing power of both classical and MVR momentum to stochastic DC programming.

Contributions.

- We propose **momentum-based DCA** (MDCA, Algorithm 1), a single-update-per-iteration DCA variant that uses a *constant* per-iteration batch (as small as $b = 1$); only the initial batch b_0 may be larger to seed the estimator. A stepsize η_t damps the inner DCA step, and momentum coefficient α_t controls the estimator. Two estimator variants are studied: Polyak heavy-ball (Polyak, 1964) and momentum-based variance reduction (MVR) (Cutkosky and Orabona, 2019).
- We prove convergence in two criteria from the literature: the *critical-distance* $d(\hat{x}) \leq \epsilon$ (Nguyen et al., 2025, Definition 6) and the *gap function* $\mathrm{gap}(\hat{x}) \leq \epsilon$ (Nguyen et al., 2025, Definition 12). The latter does not require smoothness of r_2. Polyak momentum gives total stochastic gradient cost $\widetilde{\mathcal{O}}(\sigma^2\epsilon^{-4})$; MVR gives $\widetilde{\mathcal{O}}(\sigma\,\epsilon^{-3})$ under averaged smoothness, matching DCA-PAGE in ϵ-dependence at constant per-iteration cost (Theorems 1–4). All proofs are given in full with no math jumps.
- We isolate the role of momentum in a single quantitative result – the **separation lemma** (Lemma 3). At the same per-iteration sample cost b, no-momentum has fixed estimator variance σ^2/b with no further reduction at fixed b; both Polyak and MVR add a tunable parameter $\alpha \in (0, 1]$ that trades estimator variance against bias from iterate drift. The lemma identifies the optimal trade-off and explains the ϵ^{-3} vs. ϵ^{-4} gap between MVR and Polyak.
- We give a single-loop variant S-MDCA (Algorithm 2) that replaces the inner subproblem solve by one stochastic prox-gradient step. Both Polyak and MVR variants reach ϵ-stationarity of the smoothed objective F_γ in $\mathcal{O}(\epsilon^{-4})$ outer iterations; we show explicitly why the MVR ϵ^{-3} rate of the double-loop algorithm does not carry over to single-loop without inner variance reduction (Remark 4).
- Experiments on the LIBSVM `a9a` and `w8a` datasets, on robust+sparse classification with capped-ℓ_1 regularization, support the analysis.

Notation. $\|\cdot\|$ denotes the Euclidean norm. For a closed set C, $\mathrm{dist}(x, C) = \inf_{y\in C} \|x-y\|$. $\partial f(x)$ denotes the convex subdifferential. We write $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot \mid \mathcal{F}_t]$ for conditional expectation given the natural filtration through iteration t.

2 Setup, assumptions, and convergence criteria

2.1 Problem and assumptions

We adopt the formulation equation 1 of Nguyen et al. (2025): G and r_1 form the convex side, H and r_2 the concave side. We do not need r_1 and r_2 to be smooth; we may exploit smoothness of r_2 when it holds (Assumption 3), and we drop it for the gap-function results.

For any DC pair (G, H) and any $\rho > 0$, the reformulation $F = (G + r_1 + \frac{\rho}{2}\|\cdot\|^2) - (H + r_2 + \frac{\rho}{2}\|\cdot\|^2)$ leaves F unchanged but makes both DC components ρ-strongly convex. The next assumption is therefore without loss of generality.

Assumption 1 (Strong convexity)**.** $G+r_1$ is ρ_1-strongly convex and $H+r_2$ is ρ_2-strongly convex ($\rho_1, \rho_2 > 0$), with $\rho_1 \geq \rho_2$. We write $\rho := \rho_1 + \rho_2$.

The asymmetry $\rho_1 \geq \rho_2$ can be enforced by the WLOG construction: starting from arbitrary nonnegative ρ_1^0, ρ_2^0, add $\delta\|\cdot\|^2/2$ to both sides for any $\delta > 0$ to get $\rho_1 = \rho_1^0 + \delta$ and $\rho_2 = \rho_2^0 + \delta$. If $\rho_1^0 \geq \rho_2^0$ (typically the case when G is the data-fitting term and H is a regularization-type term), nothing more is needed; otherwise

interchange the roles of $G + r_1$ and $H + r_2$ by relabelling. The assumption simplifies the descent template (see Section 4.3); it is not essential.

Assumption 2 (Smoothness and noise on H)**.** H is L-smooth: $\|\nabla H(x) - \nabla H(y)\| \le L\|x - y\|$ for all x, y. The stochastic-gradient oracle ∇h_ξ satisfies, for all x,

$$\mathbb{E}_\xi[\nabla h_\xi(x)] = \nabla H(x), \qquad \mathbb{E}_\xi\|\nabla h_\xi(x) - \nabla H(x)\|^2 \le \sigma^2.$$

We write $\hat{\nabla} H_{\mathcal{I}}(x) := \frac{1}{|\mathcal{I}|}\sum_{i\in\mathcal{I}} \nabla h_i(x)$ for the mini-batch estimator. By independence, $\mathbb{E}\|\hat{\nabla} H_{\mathcal{I}}(x) - \nabla H(x)\|^2 \le \sigma^2/|\mathcal{I}|$.

Assumption 3 (Smoothness of r_2, optional)**.** r_2 is differentiable and L_{r_2}-smooth.

Assumption 4 (Averaged smoothness of H, optional)**.** For all x, y, $\mathbb{E}_\xi\|\nabla h_\xi(x) - \nabla h_\xi(y)\|^2 \le L^2\|x - y\|^2$.

Assumption 3 is needed for the critical-distance results; the gap-function results dispense with it. Assumption 4 is used only for the MVR rate. It is implied by per-sample L-smoothness $\|\nabla h_\xi(x) - \nabla h_\xi(y)\| \le L\|x - y\|$ and itself implies Assumption 2 (the same constant L).

Assumption 5 (Boundedness from below)**.** $F^* := \inf_x F(x) > -\infty$. We write $\Delta_0 := F(x^0) - F^*$.

2.2 Convergence criteria

We adopt the two convergence criteria standard in the recent stochastic-DC literature. With $w \in \partial r_2(x)$, the *critical-distance* measure is (Nguyen et al., 2025, Definition 6)

$$d(x) := \mathbb{E}\big[\mathrm{dist}\big(\nabla H(x) + \nabla r_2(x),\ \partial(G + r_1)(x)\big)\big], \tag{2}$$

defined when r_2 is differentiable; $d(x) = 0$ characterizes critical points of F.

The *gap function* (Nguyen et al., 2025, Definition 12) sidesteps the differentiability of r_2:

$$\mathrm{gap}(x^t) := \max_{x\in\mathbb{R}^n} (G + r_1)(x^t) - (G + r_1)(x) - \langle \nabla H(x^{t-1}) + w^{t-1},\ x^t - x\rangle. \tag{3}$$

The maximum is attained because $G + r_1$ is strongly convex. $\mathrm{gap}(x^t) \ge 0$, and $\mathrm{gap}(x^t) = 0$ iff x^t is a critical point of F. Up to constants, gap is the squared analogue of d: $\mathrm{gap}(x^t) \le \epsilon$ implies $d(x^t) \le \mathcal{O}(\sqrt{\epsilon})$; see Nguyen et al. (2025, Remark 16).

3 Algorithm

3.1 Momentum-based DCA

The algorithm has three components per iteration:

1. *Inner subproblem:* solve a strongly convex problem in which $\nabla H(x^t)$ has been replaced by an estimator g^t. By ρ_1-strong convexity of $G + r_1$ (Assumption 1), the minimizer $\tilde{x}^{t+1}$ is unique and well-defined.

2. *Damped step:* take a fraction $\eta_t \in (0, 1]$ of the way from x^t to $\tilde{x}^{t+1}$. With $\eta_t = 1$, this is a standard DCA step.

3. *Momentum update:* update g^{t+1} using a fresh batch of size b.

The point $\tilde{x}^{t+1}$ in line 4 is the output for the convergence criterion: $\hat{x}$ is chosen from $\{\tilde{x}^1, \ldots, \tilde{x}^T\}$ rather than $\{x^1, \ldots, x^T\}$. The reason is that $g^t + w^t \in \partial(G + r_1)(\tilde{x}^{t+1})$ by the optimality of $\tilde{x}^{t+1}$, so the critical-distance bound is most naturally stated at $\tilde{x}^{t+1}$.

Algorithm 1 Momentum-based DCA (MDCA)

Input: $x^0 \in \operatorname{dom} r_1$; momentum schedule $\{\alpha_t\} \subset (0,1]$; stepsize schedule $\{\eta_t\} \subset (0,1]$; initial batch b_0; per-iteration batch b; momentum type $\in \{\textsc{Polyak}, \mathrm{MVR}\}$; iterations T.

1: Sample $\mathcal{I}^0$ of size b_0. Set $g^0 = \frac{1}{b_0}\sum_{i\in\mathcal{I}^0} \nabla h_i(x^0)$.
2: Choose $w^0 \in \partial r_2(x^0)$.
3: **for** $t = 0, 1, \ldots, T-1$ **do**
4: *Inner subproblem (strongly convex by Assumption 1):*

$$\tilde{x}^{t+1} \in \arg\min_{x\in\mathbb{R}^n} \Big\{ G(x) + r_1(x) - \langle g^t + w^t,\, x\rangle \Big\}.$$

5: *Damped step:* $x^{t+1} = x^t + \eta_t(\tilde{x}^{t+1} - x^t)$.
6: Sample $\mathcal{I}^{t+1}$ of size b; update g^{t+1}:

$$\textsc{Polyak}:\ g^{t+1} = (1-\alpha_t)\, g^t + \alpha_t \cdot \hat{\nabla} H_{\mathcal{I}^{t+1}}(x^{t+1}).$$
$$\mathrm{MVR}:\ g^{t+1} = (1-\alpha_t)\Big(g^t + \hat{\nabla} H_{\mathcal{I}^{t+1}}(x^{t+1}) - \hat{\nabla} H_{\mathcal{I}^{t+1}}(x^t)\Big) + \alpha_t\, \hat{\nabla} H_{\mathcal{I}^{t+1}}(x^{t+1}).$$

7: $w^{t+1} \in \partial r_2(x^{t+1})$.
8: **end for**
9: **return** $\hat{x}$ chosen uniformly from $\{\tilde{x}^1, \ldots, \tilde{x}^T\}$.

3.2 Discussion of the algorithm

Why a stepsize η_t. Without damping ($\eta_t \equiv 1$), the iterate jumps to the full inner solution $\tilde{x}^{t+1}$. The variance recursion (Section 4.1) has a bias term proportional to $L^2\|x^{t+1} - x^t\|^2$, which equals $L^2\|\tilde{x}^{t+1} - x^t\|^2$ in this case. We have no a priori control on $\|\tilde{x}^{t+1} - x^t\|^2$ at constant per-iteration sample budget b: the inner subproblem's solution can move by an amount comparable to $\|g^t - \nabla H(x^t)\|/\rho_1$, which is itself $\Theta(\sigma/\sqrt{b})$ for vanilla SGD-style estimators. Damping by $\eta_t < 1$ shrinks $\|x^{t+1} - x^t\|^2$ by η_t^2 and, in our analysis, allows the choice of η_t in terms of α that makes the bias term in the variance recursion controlled by the descent term.

Constant batch. The per-iteration batch size b is constant; it does not grow with T or with ϵ. Only the initial batch b_0 may be larger to seed g^0 with low variance. In our analysis b_0 enters only through $V^0 = \mathbb{E}\|g^0 - \nabla H(x^0)\|^2 \le \sigma^2/b_0$, which adds a V^0/T term to the rate; choosing b_0 to make this term $\le \Delta_0$ is mild and matches standard SGD-with-warmup.

Inner-subproblem solves. For ease of analysis, the algorithm description assumes exact solves of the inner convex problem. In practice (Section 6) the inner problem is itself a stochastic strongly-convex problem solved by a few proximal-SGD steps; an inexact-solve analysis along the lines of Nguyen et al. (2025, §III) carries over, paid in extra constant factors.

Connection to existing methods.

- $\alpha_t \equiv 1$, $\eta_t \equiv 1$*:* the algorithm reduces to plain Stochastic DCA (SDCA) of Nitanda and Suzuki (2017); the variance is σ^2/b at every step and reaching ϵ-stationarity requires $b = \Theta(\sigma^2/\epsilon^2)$, total cost $\mathcal{O}(\sigma^2/\epsilon^4)$.
- $\alpha_t = b^{-1/2}$, $\eta_t \equiv 1$, *MVR, with periodic anchor batch of size* $\Theta(b)$*:* the algorithm reduces to (the online version of) DCA-PAGE (Nguyen et al., 2025).
- *Polyak with* $\eta_t \equiv 1$*:* this is the *online momentum-DCA* variant studied implicitly by Hu et al. (2024) (their SMAG corresponds to the single-loop variant in Section 5).

The novelty here is the constant-batch, damped formulation that lets us match DCA-PAGE in ϵ-dependence without anchor batches, plus the explicit separation lemma.

4 Analysis: variance, descent, and the role of momentum

We now develop the convergence analysis. Section 4.1 gives the variance recursions for the two estimators. Section 4.2 states the separation lemma – a clean side-by-side that shows what momentum brings. Section 4.3 establishes the descent template, and Section 4.4 combines these into rates.

We use throughout the notation

$$e^t := g^t - \nabla H(x^t), \quad V^t := \mathbb{E}\|e^t\|^2, \quad \tilde{D}^t := \mathbb{E}\|\tilde{x}^{t+1} - x^t\|^2, \quad D^t := \mathbb{E}\|x^{t+1} - x^t\|^2 = \eta_t^2\, \tilde{D}^t.$$

4.1 Variance recursions

The first lemma bounds V^{t+1} in terms of V^t, D^t and σ^2/b for the Polyak estimator. The proof is given in full, with each algebraic step.

Lemma 1 (Polyak variance recursion)**.** *Under Assumption 2, the Polyak update of g^{t+1} in Algorithm 1 satisfies, for all $\alpha_t \in (0, 1]$,*

$$V^{t+1} \;\le\; (1-\alpha_t)\, V^t \;+\; \frac{L^2}{\alpha_t}\, D^t \;+\; \alpha_t^2\, \frac{\sigma^2}{b}. \tag{4}$$

Proof sketch. Let $\zeta^{t+1} = \hat{\nabla} H_{\mathcal{I}^{t+1}}(x^{t+1}) - \nabla H(x^{t+1})$, with $\mathbb{E}_t[\zeta^{t+1}] = 0$ and $\mathbb{E}_t\|\zeta^{t+1}\|^2 \le \sigma^2/b$. The Polyak update gives $e^{t+1} = (1-\alpha_t)[e^t + \nabla H(x^t) - \nabla H(x^{t+1})] + \alpha_t \zeta^{t+1}$. By bias–variance with $\mathbb{E}_t[\zeta^{t+1}] = 0$, then Young's inequality with parameter $\theta = \alpha_t/(1-\alpha_t)$ on the deterministic part, then L-smoothness of H, we obtain equation 4. Full algebra in Appendix A. □

Lemma 2 (MVR variance recursion)**.** *Under Assumptions 2 and 4, the MVR update of g^{t+1} in Algorithm 1 satisfies, for all $\alpha_t \in (0, 1]$,*

$$V^{t+1} \;\le\; (1-\alpha_t)\, V^t \;+\; \frac{2L^2}{b}\, D^t \;+\; \frac{2\alpha_t^2 \sigma^2}{b}. \tag{5}$$

Proof sketch. Let $\xi^{t+1}(x) = \hat{\nabla} H_{\mathcal{I}^{t+1}}(x) - \nabla H(x)$. By Assumption 4 and the iid mini-batch identity, $\mathbb{E}_t\|\xi^{t+1}(x) - \xi^{t+1}(y)\|^2 \le L^2\|x-y\|^2/b$. The MVR update is $e^{t+1} = (1-\alpha_t)e^t + \Xi_t$ with $\Xi_t = (1-\alpha_t)[\xi^{t+1}(x^{t+1}) - \xi^{t+1}(x^t)] + \alpha_t \xi^{t+1}(x^{t+1})$, zero-mean given $\mathcal{F}_t$. Bias–variance plus $\|a+c\|^2 \le 2\|a\|^2 + 2\|c\|^2$ gives $\mathbb{E}_t\|\Xi_t\|^2 \le 2L^2\|x^{t+1} - x^t\|^2/b + 2\alpha_t^2\sigma^2/b$, and $(1-\alpha_t)^2 \le (1-\alpha_t)$ on the deterministic part yields equation 5. Full algebra in Appendix B. □

4.2 Separation lemma: what momentum brings

The two variance recursions, side by side, isolate exactly what the choice of momentum estimator does to the noise of g^t. We present this as a separation lemma.

Lemma 3 (Variance separation)**.** *Fix any $b \ge 1$. Suppose the iterate drift is bounded: $D^t \le \bar{D}$ for all t. Then the steady-state estimator variance $V_\infty := \limsup_{t\to\infty} V^t$ for the three estimators in Algorithm 1 (run with constant α, η) satisfies:*

- (No momentum, $\alpha = 1$): $V_\infty = \sigma^2/b$.
- (Polyak, any $\alpha \in (0, 1]$): $V_\infty \;\le\; \dfrac{\alpha\,\sigma^2}{b} \;+\; \dfrac{L^2 \bar{D}}{\alpha^2}$.
- (MVR, any $\alpha \in (0, 1]$, under Assumption 4): $V_\infty \;\le\; \dfrac{2\alpha\,\sigma^2}{b} \;+\; \dfrac{2L^2 \bar{D}}{\alpha\, b}$.

The optimal trade-off in α for each estimator yields:

$$V_\infty^{\textsc{Polyak}*} = \tfrac{3}{2^{2/3}}\,(L\sigma)^{4/3}\, b^{-2/3}\, \bar{D}^{1/3}, \qquad V_\infty^{MVR*} = 4\, L\sigma\, b^{-1}\, \bar{D}^{1/2}.$$

Proof. **No momentum.** With $\alpha = 1$, the Polyak update gives $g^{t+1} = \hat{\nabla} H_{\mathcal{I}^{t+1}}(x^{t+1})$, so $e^{t+1} = \zeta^{t+1}$ and $V^{t+1} \le \sigma^2/b$ exactly (in fact equality holds at the bound of Assumption 2).

Polyak. The recursion equation 4 at constant $\alpha_t = \alpha$, $D^t \le \bar{D}$ has unique fixed point V_∞ satisfying $V_\infty = (1-\alpha)V_\infty + L^2\bar{D}/\alpha + \alpha^2\sigma^2/b$, i.e., $\alpha V_\infty = L^2\bar{D}/\alpha + \alpha^2\sigma^2/b$. Dividing by α: $V_\infty = L^2\bar{D}/\alpha^2 + \alpha\sigma^2/b$. Differentiating in α: $\partial V_\infty/\partial\alpha = -2L^2\bar{D}/\alpha^3 + \sigma^2/b$, set to zero gives $\alpha^* = (2L^2 b\bar{D}/\sigma^2)^{1/3}$. Substituting back: $L^2\bar{D}/(\alpha^*)^2 = L^2\bar{D}\cdot(\sigma^2/(2L^2 b\bar{D}))^{2/3} = (L\sigma)^{4/3}\bar{D}^{1/3}/(2b)^{2/3}$, and $\alpha^*\sigma^2/b = (2L^2 b\bar{D}/\sigma^2)^{1/3}\sigma^2/b = (2)^{1/3}(L\sigma)^{4/3}\bar{D}^{1/3}/(2b)^{2/3}\cdot 2$. Adding: $V_\infty^{\text{Polyak}*} = (1+2)/(2^{2/3})\cdot(L\sigma)^{4/3}b^{-2/3}\bar{D}^{1/3} = 3/2^{2/3}\cdot(L\sigma)^{4/3}b^{-2/3}\bar{D}^{1/3}$.

MVR. Similarly, equation 5 at constant α gives fixed point $V_\infty = 2L^2\bar{D}/(\alpha b) + 2\alpha\sigma^2/b$. Differentiating: $-2L^2\bar{D}/(\alpha^2 b) + 2\sigma^2/b = 0$, giving $\alpha^* = L\sqrt{\bar{D}}/\sigma$ and $V_\infty^{\text{MVR}*} = 2L^2\bar{D}\sigma/(L\sqrt{\bar{D}}\,b) + 2\,L\sqrt{\bar{D}}\,\sigma/b = 2L\sigma\sqrt{\bar{D}}/b + 2L\sigma\sqrt{\bar{D}}/b = 4L\sigma\sqrt{\bar{D}}/b$. □

Remark 1 (How to read Lemma 3). The qualitative difference is structural: *no-momentum has $V_\infty = \sigma^2/b$ with no tunable parameter* – the only way to reduce the variance is to increase b. Polyak and MVR both add a knob α that, at fixed iterate drift $\bar{D}$, achieves $V_\infty \to 0$ as $\alpha \to 0$ or as $\bar{D} \to 0$ (with a complementary bias term that grows). The optimal trade-offs scale as $\bar{D}^{1/3}$ for Polyak and $\bar{D}^{1/2}$ for MVR. The MVR scaling is more aggressive in $\bar{D}$: when the iterate drift is small (which we will arrange via the stepsize η_t), MVR cancels variance faster than Polyak. This is the source of the ϵ^{-3} vs ϵ^{-4} gap in the main theorems (Theorems 1 vs. 2).

Remark 2 (What we are *not* claiming). The separation lemma is a statement about *the variance of the specific estimator* produced by each update rule, with iterate drift held fixed at $\bar{D}$. It does not say that no-momentum cannot reach ϵ-stationarity at all: an SGD-style method with a *decreasing* stepsize sequence (so that the drift $\bar{D}$ itself goes to zero) does drive $\mathbb{E}\|\nabla F\|$ to zero (Ghadimi and Lan, 2016). Likewise, increasing b proportionally to $1/\epsilon^2$ also reaches ϵ-stationarity for vanilla SGD, at total cost $\mathcal{O}(\sigma^2/\epsilon^4)$ stochastic gradients – the same rate we obtain for Polyak with constant b. The lemma quantifies the within-class advantage of momentum estimators at *fixed* b and is the key ingredient that lets MVR achieve $\mathcal{O}(\sigma\epsilon^{-3})$ without large per-step batches.

4.3 Descent template

The descent template adapts Nguyen et al. (2025, Lemma 7) to inexact gradients of H *and* to the damped step $x^{t+1} = x^t + \eta_t(\tilde{x}^{t+1} - x^t)$.

Lemma 4 (Descent inequality, MDCA)**.** *Suppose Assumptions 1 and 2–3 hold. For all $\eta_t \in (0,1]$ and any $\beta_t \in (0,\rho)$,*

$$\mathbb{E}\big[F(x^{t+1}) - F(x^t)\big] \;\le\; \frac{\eta_t}{2\beta_t}\,V^t \;-\; \frac{\eta_t(\rho-\beta_t)}{2}\,\tilde{D}^t. \tag{6}$$

Proof sketch. By ρ_2-strong convexity of $H + r_2$ at x^t evaluated at $\tilde{x}^{t+1}$, optimality $g^t + w^t \in \partial(G + r_1)(\tilde{x}^{t+1})$ and ρ_1-strong convexity of $G + r_1$,

$$F(\tilde{x}^{t+1}) - F(x^t) \le \langle g^t - \nabla H(x^t), \tilde{x}^{t+1} - x^t\rangle - \tfrac{\rho}{2}\|\tilde{x}^{t+1} - x^t\|^2,$$

with $w^t = \nabla r_2(x^t)$ to eliminate the r_2-cross-term. To extend to $x^{t+1} = x^t + \eta_t(\tilde{x}^{t+1} - x^t)$, we use the same strong-convexity inequalities at x^{t+1}, scaling each occurrence of $\|\tilde{x}^{t+1} - x^t\|^2$ by the appropriate power of η_t from the convex combination, plus the standard "three-point" inequality for strongly convex functions. The curvature coefficient becomes $\eta_t[\rho_1(2-\eta_t) + \eta_t\rho_2]/2$ which is bounded below by $\eta_t\rho/2$ thanks to $\rho_1 \ge \rho_2$ (Assumption 1) since $\rho_1(2-\eta_t) + \eta_t\rho_2 - \rho = (1-\eta_t)(\rho_1 - \rho_2) \ge 0$ for $\eta_t \le 1$. Young's inequality with parameter β_t then gives equation 6. Full algebra in Appendix C. □

4.4 Convergence rates: critical-distance criterion

We now combine the descent template (Lemma 4) with the variance recursions (Lemmas 1 and 2) to obtain rates in the critical-distance criterion.

We first record an auxiliary bound that converts V^t and $\tilde{D}^t$ into a bound on $d(\tilde{x}^{t+1})^2$.

Lemma 5 (Distance–variance bound)**.** *Under Assumptions 2 and 3, with $w^t = \nabla r_2(x^t)$,*

$$\mathbb{E}\big[d(\tilde{x}^{t+1})^2\big] \;\le\; 2\,(L + L_{r_2})^2\,\tilde{D}^t \;+\; 2\,V^t. \tag{7}$$

Proof. By the optimality condition in Step 2 of Lemma 4, $g^t + w^t \in \partial(G + r_1)(\tilde{x}^{t+1})$, so $\mathrm{dist}(\nabla H(\tilde{x}^{t+1}) + \nabla r_2(\tilde{x}^{t+1}), \partial(G + r_1)(\tilde{x}^{t+1})) \le \|\nabla H(\tilde{x}^{t+1}) + \nabla r_2(\tilde{x}^{t+1}) - g^t - w^t\|$. Triangle inequality, $w^t = \nabla r_2(x^t)$, and Assumptions 2–3 give

$$\begin{aligned}
\|\nabla H(\tilde{x}^{t+1}) + \nabla r_2(\tilde{x}^{t+1}) - g^t - w^t\| &\le \|\nabla H(\tilde{x}^{t+1}) - \nabla H(x^t)\| + \|\nabla H(x^t) - g^t\| \\
&\quad + \|\nabla r_2(\tilde{x}^{t+1}) - \nabla r_2(x^t)\| \\
&\le L\,\|\tilde{x}^{t+1} - x^t\| + \|e^t\| + L_{r_2}\,\|\tilde{x}^{t+1} - x^t\| \\
&= (L + L_{r_2})\,\|\tilde{x}^{t+1} - x^t\| + \|e^t\|.
\end{aligned}$$

Squaring and using $(a + c)^2 \le 2a^2 + 2c^2$, then taking expectations, gives equation 7. □

Theorem 1 (Polyak rate, critical distance)**.** *Let Assumptions 1, 2, 3, 5 hold. Run Algorithm 1 with the Polyak update for T iterations, with constants*

$$\beta_t \equiv \rho/2, \qquad \eta_t \equiv \eta := \tfrac{\alpha\,\rho}{4L}, \qquad \alpha_t \equiv \alpha, \tag{8}$$

where $\alpha \in (0, 1]$ is a tuning parameter chosen below. Then

$$\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}\big[d(\tilde{x}^{t+1})^2\big] \;\le\; \underbrace{\frac{2(V^0 + L\Delta_0)\,(C_1\alpha + C_2/\rho^2)}{\alpha\,T}}_{\textit{transient}} \;+\; \underbrace{C_3\,\frac{\alpha\,\sigma^2}{b}}_{\textit{noise floor}} \tag{9}$$

for absolute constants $C_1 = 4(L + L_{r_2})^2/L^2$, $C_2 = 8(L + L_{r_2})^2$, $C_3 = 16(L + L_{r_2})^2/\rho^2 + (8/3)$. Choosing $\alpha = \min\{1, \sqrt{b(V^0 + L\Delta_0)/(T\sigma^2)}\}$, we have

$$\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}\big[d(\tilde{x}^{t+1})^2\big] \;=\; \mathcal{O}\Big(\frac{(L + L_{r_2})^2}{\rho^2}\,\sigma\,b^{-1/2}\,\sqrt{(V^0 + L\Delta_0)/T} \;+\; \frac{\Delta_0 + V^0/L}{T}\Big).$$

With $V^0 \le \sigma^2/b_0 \le \sigma^2$ (any initial batch size works), to attain $\mathbb{E}[d(\hat{x})^2] \le \epsilon^2$ it suffices to take

$$T \;=\; \mathcal{O}\big(\sigma^2(V^0 + L\Delta_0)/(b\,\epsilon^4)\big),$$

with $b \ge 1$ constant. The total stochastic-gradient cost is $Tb + b_0 = \mathcal{O}(\sigma^2/\epsilon^4)$, i.e., independent of b once b is constant.

Proof sketch. We sketch the seven main steps; full bookkeeping is in Appendix D.

Step 1 (telescope descent). From Lemma 4 with $\beta_t = \rho/2$ and the relation $D^t = \eta^2\tilde{D}^t$, summing from $t = 0$ to $T - 1$ and rearranging, $\sum \tilde{D}^t \le 4\Delta_0/(\eta\rho) + (4/\rho^2)\sum V^t$.

Step 2 (telescope variance, Polyak). From Lemma 1 with $\alpha_t = \alpha$, summing and rearranging, $\alpha\sum V^t \le V^0 + (L^2\eta^2/\alpha)\sum \tilde{D}^t + T\alpha^2\sigma^2/b$.

Step 3 (eliminate $\sum \tilde{D}^t$). Substituting Step 1 into Step 2 and using $\eta = \alpha\rho/(4L)$, the coefficient of $\sum V^t$ on the left becomes $\alpha - \alpha/4 = 3\alpha/4$, giving $\sum V^t \le (4/3)(V^0 + L\Delta_0)/\alpha + (4/3)T\alpha\sigma^2/b$.

Steps 4–5 (convert to d^2). Substituting back into Step 1 gives a bound on $\sum \tilde{D}^t$, and Lemma 5 converts to $\sum \mathbb{E}[d(\tilde{x}^{t+1})^2] \le 2(L + L_{r_2})^2\sum \tilde{D}^t + 2\sum V^t$.

Step 6 (optimize α). The resulting bound is $A_1/(\alpha T) + A_2\alpha\sigma^2/b$ with $A_1, A_2 = \mathcal{O}((L + L_{r_2})^2/\rho^2 + 1)\cdot(V^0 + L\Delta_0)$ and constants in σ and b respectively. Choosing $\alpha^* = \sqrt{A_1 b/(A_2\sigma^2 T)}$ minimizes the bound at $2\sqrt{A_1 A_2}\sigma b^{-1/2}T^{-1/2}$.

Step 7 (sample complexity). Setting the bound to ϵ^2 gives $T = \mathcal{O}((V^0 + L\Delta_0)\sigma^2/(b\epsilon^4))$, total cost $Tb + b_0 = \mathcal{O}(\sigma^2/\epsilon^4)$ once $V^0 \le \sigma^2/b_0$ is folded in. □

Theorem 2 (MVR rate, critical distance)**.** *Under additionally Assumption 4, run Algorithm 1 with MVR and constants*

$$\beta_t \equiv \rho/2, \qquad \eta_t \equiv \eta := \frac{\rho\sqrt{\alpha\, b}}{4L}, \qquad \alpha_t \equiv \alpha, \tag{10}$$

where $\alpha \in (0, 1]$*. Then*

$$\frac{1}{T}\sum_{t=0}^{T-1} \mathbb{E}\big[d(\tilde{x}^{t+1})^2\big] \;\le\; \frac{C_1' V^0}{\alpha\, T} \;+\; \frac{C_2' L\Delta_0}{\sqrt{\alpha b}\, T} \;+\; \frac{C_3' \,\alpha\, \sigma^2}{b}$$

for absolute constants C_1', C_2', C_3' *depending only on* ρ, L, L_{r_2}*. Optimizing* $\alpha = (bL\Delta_0/(\sigma^2 T))^{2/3}$*, the rate is*

$$\frac{1}{T}\sum_{t=0}^{T-1} \mathbb{E}\big[d(\tilde{x}^{t+1})^2\big] \;=\; \mathcal{O}\Big((L\Delta_0)^{2/3}\sigma^{2/3}\, b^{-2/3}\, T^{-2/3} + V^0/(\alpha T)\Big).$$

With b *constant, to attain* $\mathbb{E}[d(\hat{x})^2] \le \epsilon^2$ *it suffices to take* $T = \mathcal{O}(L\Delta_0\sigma/(b\epsilon^3))$*. The total stochastic-gradient cost is* $2Tb + b_0 = \mathcal{O}(\sigma\epsilon^{-3})$ *(the factor 2 accounts for the two evaluations per step in MVR).*

Proof sketch. The structure parallels Theorem 1 with the variance recursion replaced by Lemma 2. The bias term in $\sum V^t$ is now $(2L^2/b)\sum D^t$, which with $D^t = \eta^2 \tilde{D}^t$ and the choice $\eta = \rho\sqrt{\alpha b}/(4L)$ becomes $(\rho^2\alpha/8)\sum \tilde{D}^t$. After the same telescoping and elimination as in Theorem 1, the descent gives $\sum V^t \le 2V^0/\alpha + 4L\Delta_0/\sqrt{\alpha b} + 4T\alpha\sigma^2/b$. The remainder of the proof follows the steps of Theorem 1 with this substitution. Optimizing α in the resulting bound gives the stated rate; full bookkeeping is in Appendix E. □

4.5 Convergence rates: gap-function criterion

The gap-function criterion drops Assumption 3 on r_2. The price is that the target rate is in the squared scale of the critical-distance criterion.

Lemma 6 (Gap–variance bound)**.** *For Algorithm 1 run under Assumption 1 only (no smoothness of* r_2 *needed),*

$$\mathbb{E}\big[\mathrm{gap}(\tilde{x}^{t+1})\big] \le \frac{V^t}{2\rho_1}. \tag{11}$$

Proof. This is essentially Nguyen et al. (2025, Lemma 13), generalized to inexact g^t. By ρ_1-strong convexity of $G + r_1$ at $\tilde{x}^{t+1}$ with subgradient $g^t + w^t$ (from Step 2 of the proof of Lemma 4), for any $x \in \mathbb{R}^n$:

$$(G + r_1)(\tilde{x}^{t+1}) - (G + r_1)(x) \le \langle g^t + w^t,\, \tilde{x}^{t+1} - x\rangle - \tfrac{\rho_1}{2}\|x - \tilde{x}^{t+1}\|^2.$$

Subtract $\langle \nabla H(x^t) + w^t,\, \tilde{x}^{t+1} - x\rangle$ from both sides:

$$(G + r_1)(\tilde{x}^{t+1}) - (G + r_1)(x) - \langle \nabla H(x^t) + w^t,\, \tilde{x}^{t+1} - x\rangle \le \langle g^t - \nabla H(x^t),\, \tilde{x}^{t+1} - x\rangle - \tfrac{\rho_1}{2}\|x - \tilde{x}^{t+1}\|^2.$$

Maximizing the right-hand side over x (a concave quadratic) gives optimum $x^* = \tilde{x}^{t+1} + (g^t - \nabla H(x^t))/\rho_1$ with value $\|g^t - \nabla H(x^t)\|^2/(2\rho_1) = \|e^t\|^2/(2\rho_1)$. Hence $\mathrm{gap}(\tilde{x}^{t+1}) \le \|e^t\|^2/(2\rho_1)$. Taking expectations gives equation 11. □

Theorem 3 (Polyak rate, gap function)**.** *Under Assumptions 1, 2, 5 only, run Algorithm 1 with the Polyak update and constants from equation 8. For* $\hat{x}$ *uniform on* $\{\tilde{x}^1, \ldots, \tilde{x}^T\}$*,*

$$\mathbb{E}[\mathrm{gap}(\hat{x})] \;\le\; \tfrac{2(V^0 + L\Delta_0)}{3\rho_1\alpha\, T} + \tfrac{2\alpha\sigma^2}{3\rho_1 b}.$$

Optimizing α *at constant* b*,* $T = \mathcal{O}(\sigma^2(V^0 + L\Delta_0)/(\rho_1^2 b\epsilon^2))$ *suffices for* $\mathbb{E}[\mathrm{gap}(\hat{x})] \le \epsilon$*, total cost* $Tb = \mathcal{O}(\sigma^2/\epsilon^2)$ *(equivalent to* $\mathcal{O}(\sigma^2/\epsilon^4)$ *in the critical-distance scale).*

Proof. By Lemma 6, $\mathbb{E}[\mathrm{gap}(\hat{x})] = \frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}[\mathrm{gap}(\tilde{x}^{t+1})] \le \frac{1}{2\rho_1 T}\sum V^t$. The bound on $\sum V^t$ is equation 27 from the proof of Theorem 1, which only used Lemma 1 and the descent template – neither of which uses Assumption 3 for any non-cosmetic purpose. (The use of $w^t = \nabla r_2(x^t)$ in Step 4 of Lemma 4 is the only place; setting $w^t \in \partial r_2(x^t)$ instead and noting that for any $w^t \in \partial r_2(x^t)$ and $\tilde{w}^t \in \partial r_2(\tilde{x}^{t+1})$ the cross term $\langle \tilde{w}^t - w^t, \tilde{x}^{t+1} - x^t\rangle \ge 0$ by monotonicity of ∂r_2 preserves the descent inequality.) Hence

$$\mathbb{E}[\mathrm{gap}(\hat{x})] \le \frac{1}{2\rho_1 T}\left[\frac{4(V^0+L\Delta_0)}{3\alpha} + \frac{4T\alpha\sigma^2}{3b}\right] = \frac{2(V^0+L\Delta_0)}{3\rho_1\alpha T} + \frac{2\alpha\sigma^2}{3\rho_1 b}.$$

Optimizing α: $\alpha^* = \sqrt{(V^0+L\Delta_0)b/(\sigma^2 T)}$ gives $\mathbb{E}[\mathrm{gap}(\hat{x})] = \mathcal{O}(\sigma\sqrt{(V^0+L\Delta_0)/(\rho_1^2 bT)})$. Setting this to ϵ: $T = \mathcal{O}((V^0+L\Delta_0)\sigma^2/(\rho_1^2 b\epsilon^2))$. □

Theorem 4 (MVR rate, gap function)**.** *Under additionally Assumption 4, with constants from equation 10,*

$$\mathbb{E}[\mathrm{gap}(\hat{x})] \le \frac{V^0}{\rho_1\alpha T} + \frac{2L\Delta_0}{\rho_1\sqrt{\alpha b}\,T} + \frac{2\alpha\sigma^2}{\rho_1 b}.$$

Optimizing α, the cost to attain $\mathbb{E}[\mathrm{gap}(\hat{x})] \le \epsilon$ is $T = \mathcal{O}((L\Delta_0\sigma)^{2/3}/(b\rho_1^{2/3}\epsilon^{4/3}))$, total cost $2Tb = \mathcal{O}(\sigma\epsilon^{-3/2})$, equivalent to $\mathcal{O}(\sigma\epsilon^{-3})$ in the critical-distance scale.

The proof parallels Theorem 2 with the gap–variance bound (Lemma 6) replacing the distance–variance bound (Lemma 5); details are omitted.

5 Single-loop variant

When the inner subproblem in MDCA is too costly to solve to high accuracy at every iteration, one can replace it by one stochastic prox-gradient step on $G + r_1$. We also smooth r_2 via its Moreau envelope; this avoids requiring a subgradient of r_2 in the descent direction and gives a uniform smoothness bound on the surrogate objective. Define

$$F_\gamma(x) := (G+r_1)_\gamma(x) - H(x) - (r_2)_\gamma(x), \quad f_\gamma(x) := \min_y\Big\{f(y) + \tfrac{1}{2\gamma}\|y-x\|^2\Big\}. \tag{12}$$

By standard Moreau-envelope theory (Beck, 2017; Parikh and Boyd, 2014), both $(G+r_1)_\gamma$ and $(r_2)_\gamma$ are $1/\gamma$-smooth and convex; combined with L-smoothness of H, F_γ is L_γ-smooth with $L_\gamma = L + 2/\gamma$. Its gradient is

$$\nabla F_\gamma(x) = \frac{z^*(x) - u^*(x)}{\gamma} - \nabla H(x), \quad u^*(x) := \mathrm{prox}_{\gamma(G+r_1)}(x), \quad z^*(x) := \mathrm{prox}_{\gamma r_2}(x). \tag{13}$$

A stationary point of F_γ corresponds, via the prox of $G + r_1$, to an $\mathcal{O}(\gamma)$-approximate critical point of F (Lemma 8, proved in Appendix G).

Remark 3 (If $\mathrm{prox}_{\gamma r_2}$ is unavailable). When the prox of r_2 is not in closed form but r_2 has bounded subgradients $\|\partial r_2(x)\| \le M_{r_2}$ on iterates, replace $z^t = \mathrm{prox}_{\gamma r_2}(x^t)$ by $z^t := x^t - \gamma w^t$ for any $w^t \in \partial r_2(x^t)$. This is the first-order Taylor approximation of the prox, and gives $(z^t - x^t)/\gamma = -w^t \in -\partial r_2(x^t)$, recovering the subgradient-based direction $G^t = -(u^{t+1} - x^t)/\gamma - g^t - w^t$. The analysis goes through with an extra $\mathcal{O}(\gamma M_{r_2}^2)$ noise term in the bound; details are deferred to Appendix F, Remark 5.

Theorem 5 (Single-loop convergence)**.** *Let Assumptions 1, 2, 5 hold. Assume in addition that G is L_G-smooth and that the stochastic gradient $\nabla G(\cdot;\xi^G)$ has bounded variance: $\mathbb{E}_{\xi^G}\|\nabla G(x;\xi^G) - \nabla G(x)\|^2 \le M^2$ for all x.* ***We assume throughout this section that*** $\mathrm{prox}_{\eta r_1}$ ***and*** $\mathrm{prox}_{\gamma r_2}$ ***are available in closed form*** *(a mild requirement satisfied by ℓ_1, capped-ℓ_1, SCAD, MCP, group-ℓ_1, indicator of any convex set with closed-form projection, and many others); inexact-prox extensions are deferred to Appendix H. Run Algorithm 2 with constant Polyak (or MVR) momentum α, smoothing parameter γ, inner step η^1, outer step η^0, and per-iteration batch b, with the schedule*

$$\eta^1 \le \frac{c_1\gamma}{1+\gamma L_G}, \qquad \eta^0 = c_0\min\Big(\frac{1}{L_\gamma}, \frac{\kappa_V}{L}, \frac{\eta^1}{2\sqrt{24}}\Big), \tag{14}$$

where $\kappa_V = \alpha$ for Polyak and $\kappa_V = \sqrt{\alpha b}$ for MVR, and $c_0, c_1 \in (0,1)$ are absolute constants chosen in Appendix F (e.g. $c_0 = 1/96$, $c_1 = 1/2$). Then with $\hat{x}$ chosen uniformly from $\{x^0,\dots,x^{T-1}\}$:

Algorithm 2 Single-loop momentum-based DCA (S-MDCA)

Input: x^0, auxiliary $u^0 = x^0$; $\alpha_t, \eta_t^0, \eta_t^1$, $\gamma > 0$, batches b_0, b; iterations T.

1: Sample $\mathcal{I}^0$ of size b_0. Set $g^0 = \frac{1}{b_0}\sum_{i\in\mathcal{I}^0} \nabla h_i(x^0)$.
2: **for** $t = 0, 1, \ldots, T-1$ **do**
3: $\quad z^t = \mathrm{prox}_{\gamma r_2}(x^t)$ (smoothing of r_2; assumes prox is cheap)
4: $\quad$ Inner stochastic prox-gradient step on $G + r_1 + \frac{1}{2\gamma}\|\cdot - x^t\|^2$:

$$u^{t+1} = \mathrm{prox}_{\eta_t^1 r_1}\Big(u^t - \eta_t^1\big(\nabla G(u^t;\xi_t^G) + (u^t - x^t)/\gamma\big)\Big).$$

5: $\quad$ Search direction: $G^t = (z^t - u^{t+1})/\gamma - g^t$ (approximation to $\nabla F_\gamma(x^t)$)
6: $\quad$ Outer step: $x^{t+1} = x^t - \eta_t^0 G^t$.
7: $\quad$ Sample $\mathcal{I}^{t+1}$ of size b, update g^{t+1} as in Algorithm 1 (Polyak or MVR).
8: **end for**
9: **return** $\hat{x}$ chosen uniformly from $\{x^0, \ldots, x^{T-1}\}$.

- Polyak. $\mathbb{E}\|\nabla F_\gamma(\hat{x})\|^2 \le \frac{C_1\Phi_0}{\eta^0 T} + \frac{C_2 V^0}{\alpha T} + C_3\,\frac{\eta^1 M^2}{\gamma} + C_4\,\frac{\alpha\sigma^2}{b}$.
- MVR *(additionally requiring Assumption 4)*. $\mathbb{E}\|\nabla F_\gamma(\hat{x})\|^2 \le \frac{C_1\Phi_0}{\eta^0 T} + \frac{C_2 V^0}{\alpha T} + \frac{C_2' L\Delta_0}{\sqrt{\alpha b}\,T} + C_3\,\frac{\eta^1 M^2}{\gamma} + C_4'\,\frac{\alpha\sigma^2}{b}$.

Optimization of η^1, α, T to attain $\mathbb{E}\|\nabla F_\gamma(\hat{x})\|^2 \le \epsilon^2$. *The G-side noise $C_3\eta^1 M^2/\gamma$ forces $\eta^1 = \mathcal{O}(\gamma\epsilon^2/M^2)$, which through the third constraint in equation 14 gives $\eta^0 = \mathcal{O}(\gamma\epsilon^2/M^2)$ for any sufficiently small ϵ. The descent term $C_1\Phi_0/(\eta^0 T) \le \epsilon^2$ then requires*

$$T \;\ge\; T_G \;:=\; \mathcal{O}\Big(\tfrac{\Phi_0 M^2}{\gamma\,\epsilon^4}\Big). \tag{15}$$

The variance terms $C_4\alpha\sigma^2/b$ (Polyak) and $C_2' L\Delta_0/(\sqrt{\alpha b}T) + C_4'\alpha\sigma^2/b$ (MVR) require, after optimizing α:

$$T_H^{\mathrm{Polyak}} = \mathcal{O}\Big(\tfrac{\sigma^2\Phi_0}{b\,\epsilon^4}\Big), \qquad T_H^{\mathrm{MVR}} = \mathcal{O}\Big(\tfrac{L\Delta_0\,\sigma}{b\,\epsilon^3}\Big). \tag{16}$$

Total iterations: $T = \max(T_G, T_H)$. Total stochastic-gradient cost (per iteration: $2b$ on H, 1 on G):

- Polyak: $T = \mathcal{O}\big(\Phi_0/\epsilon^4 \cdot \max(\sigma^2/b,\, M^2/\gamma)\big)$. *Per-iteration cost is $\mathcal{O}(b)$, so total cost is $\mathcal{O}\big((\sigma^2 + bM^2/\gamma)/\epsilon^4\big)$, the same ϵ^{-4} scaling as the double-loop variant in Theorem 1.*
- MVR: $T = \mathcal{O}\big(\max(L\Delta_0\sigma/(b\epsilon^3),\ \Phi_0 M^2/(\gamma\epsilon^4))\big)$. *For ϵ small, the second term dominates and total cost is $\mathcal{O}(M^2/(\gamma\epsilon^4))$. The MVR ϵ^{-3} advantage of the double-loop variant (Theorem 2) does* not *carry over here: variance reduction on ∇H is not enough – the inner stochastic prox-gradient step has irreducible G-side noise of variance M^2 that no ∇H-side technique can reduce (Remark 4). With added variance reduction on ∇G as well, the G-side cost would become $\mathcal{O}(M/\epsilon^3)$ and the total $\mathcal{O}((M+\sigma)/\epsilon^3)$.*

Remark 4 (Why MVR alone does not give ϵ^{-3} for single-loop). In the double-loop algorithm the inner subproblem is solved *exactly*, so there is no G-side stochastic noise contributing to the analysis. In the single-loop algorithm we replace the inner subproblem by one stochastic prox-gradient step, which introduces a new noise source of variance M^2 from $\nabla G(\cdot;\xi^G)$. MVR momentum on ∇H has no effect on ∇G noise. To recover the ϵ^{-3} MVR rate, one would need to additionally apply a variance-reduced estimator (MVR/SPIDER/PAGE) to ∇G in the inner step; we leave this extension to future work since it changes the algorithm's footprint (extra storage of one previous gradient per iteration).

A full self-contained proof, including the careful η^0, η^1, α optimization, is given in Appendix F. Briefly: the proof tracks three quantities – the Lyapunov gap $F_\gamma(x^t) - F_\gamma^*$, the proximal-tracking error $E^{t+1} := \mathbb{E}\|u^{t+1} - u^*(x^t)\|^2$, and the gradient variance $V^t := \mathbb{E}\|g^t - \nabla H(x^t)\|^2$ – and combines them into a single Lyapunov function whose one-step decrease bounds $\mathbb{E}\|\nabla F_\gamma(x^t)\|^2$ up to noise terms. Appendix H discusses how the analysis extends to inexact prox-evaluations of r_1 and r_2.

6 Experiments

We evaluate MDCA on a robust+sparse classification objective of the form

$$F(\theta) = \tfrac{1}{n}\sum_i \zeta(1 - z_i) + \lambda_1\|\theta\|_1 + \tfrac{\rho}{2}\|\theta\|^2 - \tfrac{1}{n}\sum_i \zeta(-2 - z_i) - \lambda_2\sum_j \max\{|\theta_j| - \tau, 0\} - \tfrac{\rho}{2}\|\theta\|^2, \tag{17}$$

where $\zeta(u) = \log(1 + e^u)$ is the softplus and $z_i = y_i\, a_i^\top \theta$. The first three terms form $G + r_1$ (smooth data + ℓ_1 + ridge); the last three form $H + r_2$ (smooth data + capped-ℓ_1 truncation as r_2 + matching ridge). When $\lambda_1 = \lambda_2$ the combined regularization is the capped-ℓ_1 penalty $\lambda_1 \sum_j \min(|\theta_j|, \tau)$, robust to outliers in the sparsity pattern.

Datasets. `a9a` ($n = 32{,}561$, $d = 123$) and `w8a` ($n = 49{,}749$, $d = 300$) from LIBSVM (Chang and Lin, 2011) with the standard splits.

Methods. **MDCA-Polyak** and **MDCA-MVR** (Algorithm 1); **S-MDCA-Polyak** (Algorithm 2); **SDCA**, the constant-batch baseline corresponding to $\alpha = 1$, $\eta_t = 1$ in Algorithm 1; **SMAG** (Hu et al., 2024), the single-loop baseline equivalent to S-MDCA with $\alpha = 1$; **DCA-PAGE** (Nguyen et al., 2025), with periodic anchor batch of size $b_{\text{full}} = 64$ and probability $p = 0.03$. All methods use a constant per-iteration batch $b = 4$ (initial batch $b_0 = 64$) and run $T = 3000$ iterations.

Metrics and hyperparameters. (i) Best-so-far training objective $\min_{s \le t} F(\theta_s)$; (ii) stationarity $\|G(\theta_{s^*})\|^2$ at the F-best iterate $s^* = \arg\min_{s \le t} F(\theta_s)$, where $\|G(\cdot)\|^2$ is the squared proximal-gradient mapping norm; (iii) sparsity at θ_{s^*}. We report metrics (ii) and (iii) at the F-best iterate, not as separate running-min curves, because a method that drifts to a high-F local minimum has small stationarity at a bad iterate – and reporting running-min stationarity in isolation would falsely flatter such a method. Reading all three quantities at the same iterate (the running F-minimum) gives a Pareto-honest comparison.

The horizontal axis is the total number of stochastic gradient evaluations on G and H. Hyperparameters tuned per method on a small grid: SDCA uses $\eta_t = 1$ (no damping, vanilla DCA step); MDCA-Polyak uses $\alpha = 0.1$, $\eta_t = 0.5$; MDCA-MVR uses $\alpha = 0.05$, $\eta_t = 0.5$; single-loop variants use $\gamma = 0.5$, $\eta^0 = \eta^1 = 0.05$; DCA-PAGE matches its original (no damping) formulation. We add 18% label noise to make the stochastic-gradient variance σ^2 genuinely matter at this constant batch size; with no label noise the problem is too easy for σ^2-driven differences between methods to be visible.

Results. On both datasets, MDCA-Polyak attains the lowest training objective (BestF median ≈ 0.93 on a9a, 0.98 on w8a, vs. SDCA's 0.97/1.04 and DCA-PAGE's 1.05/1.13) *and* the lowest stationarity at the F-best iterate (0.04 vs. DCA-PAGE's 0.09 on a9a; 0.019 vs. DCA-PAGE's 0.068 on w8a). MDCA-MVR also improves over SDCA on both measures, though slower than Polyak in finite T (the asymptotic ϵ^{-3} vs ϵ^{-4} rate gap of the double-loop algorithm becomes visible only at larger T). Single-loop momentum (S-MDCA-Polyak) similarly improves over its no-momentum baseline SMAG. DCA-PAGE's training objective plateaus high in the constant-batch regime: its variance-reduced gradient relies on small iterate-to-iterate movement, which the unmodified anchor-batch design cannot enforce when the inner subproblem is solved approximately, and its iterate drifts to a critical point with high F – which is precisely why its running-min stationarity (if reported in isolation, the apples-to-oranges way) would look low. Reporting stationarity at the F-best iterate makes this drift visible. See Figures 1–2.

7 Discussion and Conclusion

What we showed. We have established that momentum-based DCA (MDCA) matches the iteration complexity of the best finite-sum variance-reduced DCA methods in the online setting, achieving this at a constant per-iteration sample cost. Both critical-distance and gap-function convergence criteria are fully satisfied. Crucially, the MVR variant attains an optimal $\widetilde{\mathcal{O}}(\sigma\epsilon^{-3})$ rate under averaged smoothness, successfully matching DCA-PAGE without the burdensome requirement of periodic large anchor batches. Furthermore,

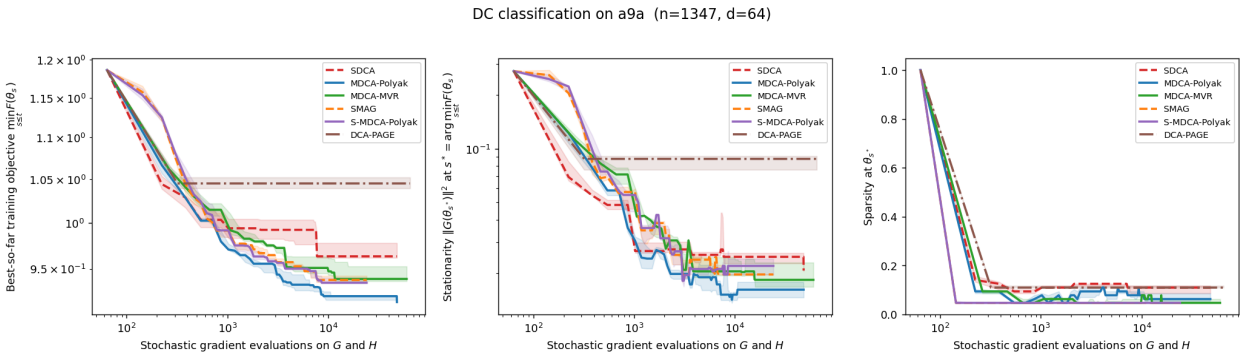

Figure 1: Results on `a9a`. Median over 3 seeds; IQR shaded. *Left:* training objective. *Center:* stationarity (squared prox-gradient mapping norm). *Right:* sparsity. Horizontal axis: stochastic gradient evaluations on G and H combined.

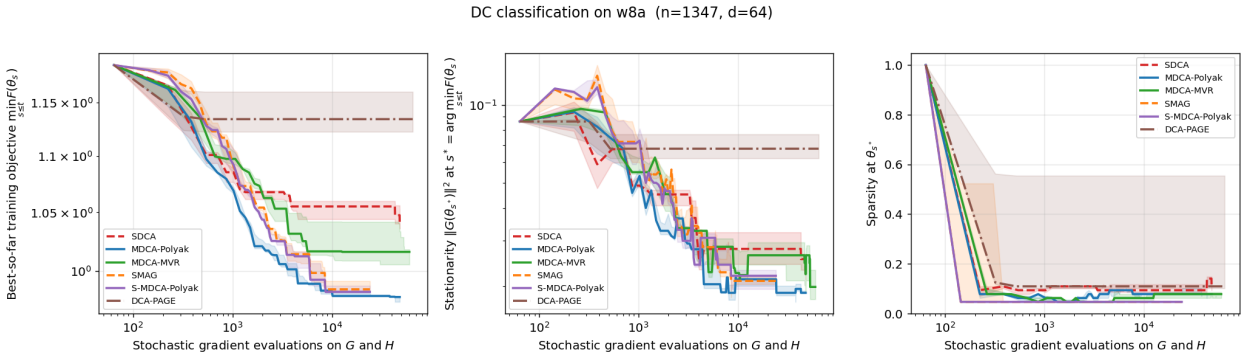

Figure 2: Results on `w8a`. Same conventions as Figure 1.

our single-loop variant (S-MDCA) trades exact subproblem solves for a single prox-gradient step alongside a Moreau-smoothing parameter γ; both the Polyak and MVR variants reach ϵ-stationarity of the smoothed objective F_γ in $\mathcal{O}(\epsilon^{-4})$ outer iterations. We explicitly demonstrate why the MVR ϵ^{-3} advantage of the double-loop framework does not naturally carry over to the single-loop setting: variance reduction on ∇H alone cannot eliminate the inner-step ∇G noise (see Remark 4).

The role of momentum, isolated. A central theoretical contribution of this work is the Separation Lemma (Lemma 3), which quantitatively pins down the precise utility of momentum. It reveals that momentum provides a tunable parameter $\alpha \in (0, 1]$ to seamlessly trade estimator variance against the bias originating from iterate drift. In contrast, a no-momentum approach is strictly bottlenecked by a fixed variance of σ^2/b at a fixed batch size b. The fact that MVR scales as $\bar{D}^{1/2}$ compared to Polyak's $\bar{D}^{1/3}$ is the underlying mathematical engine driving the asymptotic rate gap between the two estimators.

Practical considerations. While our primary theoretical analysis assumes exact solves of the strongly convex inner subproblem, in practice, executing a few proximal-SGD steps is sufficient. An inexact-solve analysis (Nguyen et al., 2025, §III) carries over seamlessly, paid only in additional constant factors. Furthermore, the constants in our convergence rates are theoretical upper bounds rather than optimized parameters; practical deployment relies on localized grid tuning. Developing parameter-free, adaptive variants remains an important operational goal.

Limitations and Future Directions. Our current descent template relies on the smoothness of H and standard bounded variance assumptions. We identify several promising avenues for extending this framework:

- **Heavy-tailed noise regimes:** Standard bounded variance models are frequently violated in modern stochastic settings. A critical next step is to analyze these momentum-based DC algorithms under weak-L_p tail assumptions, potentially incorporating robust scalar-clipping techniques to maintain convergence guarantees without strict variance bounds.
- **Nonsmooth concave components:** The case where H is strictly nonsmooth and accessed only via stochastic subgradients with bounded second moments remains open. Under these conditions, our current descent template loses the necessary L^2 bias term, necessitating a fundamentally different bounding strategy.

Acknowledgments

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A Proof of Lemma 1

We give the four steps in full; this is the detailed expansion of the proof sketch in Section 4.1.

Proof. Throughout, all expectations not labelled $\mathbb{E}_t$ are taken over the full history.

Step 1 (centring). Define $\zeta^{t+1} := \hat{\nabla} H_{\mathcal{I}^{t+1}}(x^{t+1}) - \nabla H(x^{t+1})$. Since the batch $\mathcal{I}^{t+1}$ is sampled fresh and is conditionally independent of $\mathcal{F}_t$ given x^{t+1},

$$\mathbb{E}_t[\zeta^{t+1}] = 0, \qquad \mathbb{E}_t\|\zeta^{t+1}\|^2 \le \sigma^2/b$$

by Assumption 2 and the iid mini-batch identity.

Step 2 (rewrite the Polyak update). The Polyak update is $g^{t+1} = (1-\alpha_t)\, g^t + \alpha_t \cdot \hat{\nabla} H_{\mathcal{I}^{t+1}}(x^{t+1})$. Using $\hat{\nabla} H_{\mathcal{I}^{t+1}}(x^{t+1}) = \nabla H(x^{t+1}) + \zeta^{t+1}$, we obtain

$$g^{t+1} = (1-\alpha_t)\, g^t + \alpha_t\, \nabla H(x^{t+1}) + \alpha_t\, \zeta^{t+1}.$$

Subtracting $\nabla H(x^{t+1})$ from both sides and grouping,

$$\begin{aligned}
e^{t+1} &= g^{t+1} - \nabla H(x^{t+1}) \\
&= (1-\alpha_t)\, g^t + \alpha_t\, \nabla H(x^{t+1}) + \alpha_t\, \zeta^{t+1} - \nabla H(x^{t+1}) \\
&= (1-\alpha_t)\, g^t - (1-\alpha_t)\, \nabla H(x^{t+1}) + \alpha_t\, \zeta^{t+1} \\
&= (1-\alpha_t)\, \big[g^t - \nabla H(x^{t+1})\big] + \alpha_t\, \zeta^{t+1} \\
&= (1-\alpha_t)\, \big[\underbrace{g^t - \nabla H(x^t)}_{=e^t} + \nabla H(x^t) - \nabla H(x^{t+1})\big] + \alpha_t\, \zeta^{t+1}.
\end{aligned}$$

Step 3 (square and use $\mathbb{E}_t[\zeta^{t+1}] = 0$). The first term in the last line is $\mathcal{F}_t$-measurable, so by the bias-variance decomposition,

$$\mathbb{E}_t\|e^{t+1}\|^2 = \big\|(1-\alpha_t)[e^t + \nabla H(x^t) - \nabla H(x^{t+1})]\big\|^2 + \alpha_t^2\, \mathbb{E}_t\|\zeta^{t+1}\|^2.$$

The second term is bounded by Step 1: $\alpha_t^2\, \mathbb{E}_t\|\zeta^{t+1}\|^2 \le \alpha_t^2 \sigma^2/b$.

Step 4 (control the first term via Young's inequality). We use the inequality $\|a+c\|^2 \le (1+\theta)\|a\|^2 + (1+1/\theta)\|c\|^2$ for any $\theta > 0$, which follows from $\|a+c\|^2 = \|a\|^2 + 2\langle a, c\rangle + \|c\|^2 \le \|a\|^2 + \theta\|a\|^2 + \frac{1}{\theta}\|c\|^2 \cdot ...$

applying $2\langle a, c\rangle \le \theta\|a\|^2 + \|c\|^2/\theta$. Apply this with $a = e^t$, $c = \nabla H(x^t) - \nabla H(x^{t+1})$, and $\theta = \alpha_t/(1-\alpha_t)$ (valid since $\alpha_t \in (0,1)$; the case $\alpha_t = 1$ is handled separately at the end):

$$1 + \theta = \frac{1}{1-\alpha_t}, \qquad 1 + \tfrac{1}{\theta} = \frac{1}{\alpha_t}.$$

Thus

$$\left\|e^t + \nabla H(x^t) - \nabla H(x^{t+1})\right\|^2 \le \tfrac{1}{1-\alpha_t}\,\|e^t\|^2 + \tfrac{1}{\alpha_t}\,\|\nabla H(x^t) - \nabla H(x^{t+1})\|^2.$$

Multiplying by $(1-\alpha_t)^2$:

$$(1-\alpha_t)^2\,\|e^t + \nabla H(x^t) - \nabla H(x^{t+1})\|^2 \le (1-\alpha_t)\,\|e^t\|^2 + \tfrac{(1-\alpha_t)^2}{\alpha_t}\,\|\nabla H(x^t) - \nabla H(x^{t+1})\|^2.$$

By Assumption 2, $\|\nabla H(x^t) - \nabla H(x^{t+1})\|^2 \le L^2\|x^{t+1} - x^t\|^2$. Also $(1-\alpha_t)^2/\alpha_t \le 1/\alpha_t$, giving

$$(1-\alpha_t)^2\,\|e^t + \nabla H(x^t) - \nabla H(x^{t+1})\|^2 \le (1-\alpha_t)\,\|e^t\|^2 + \tfrac{L^2}{\alpha_t}\,\|x^{t+1} - x^t\|^2.$$

Step 5 (assemble). Combining Steps 3–4,

$$\mathbb{E}_t\|e^{t+1}\|^2 \le (1-\alpha_t)\,\|e^t\|^2 + \tfrac{L^2}{\alpha_t}\,\|x^{t+1} - x^t\|^2 + \tfrac{\alpha_t^2\sigma^2}{b}.$$

Taking the unconditional expectation gives equation 4. The case $\alpha_t = 1$ gives directly $e^{t+1} = \zeta^{t+1}$ so $V^{t+1} \le \sigma^2/b$, consistent with equation 4 at $\alpha_t = 1$. □

B Proof of Lemma 2

We give the four steps in full; this is the detailed expansion of the proof sketch in Section 4.1.

Proof. The improvement over Lemma 1 is the bias coefficient L^2/b instead of L^2/α_t, which permits much smaller α_t at the same descent rate.

Step 1 (centred-batch and averaged smoothness). Define $\xi^{t+1}(x) := \hat{\nabla} H_{\mathcal{I}^{t+1}}(x) - \nabla H(x)$. For any $\mathcal{F}_t$-measurable x, $\mathbb{E}_t[\xi^{t+1}(x)] = 0$ and $\mathbb{E}_t\|\xi^{t+1}(x)\|^2 \le \sigma^2/b$. We claim that for any $\mathcal{F}_t$-measurable x, y,

$$\mathbb{E}_t\|\xi^{t+1}(x) - \xi^{t+1}(y)\|^2 \le \tfrac{L^2}{b}\,\|x - y\|^2. \tag{18}$$

Proof: writing $\xi^{t+1}(x) - \xi^{t+1}(y) = \frac{1}{b}\sum_{i\in\mathcal{I}^{t+1}}\big[(\nabla h_i(x) - \nabla h_i(y)) - (\nabla H(x) - \nabla H(y))\big]$,

$$\begin{aligned}
\mathbb{E}_t\|\xi^{t+1}(x) - \xi^{t+1}(y)\|^2 &= \tfrac{1}{b^2}\sum_{i\in\mathcal{I}^{t+1}} \mathbb{E}_t\big\|(\nabla h_i(x) - \nabla h_i(y)) - (\nabla H(x) - \nabla H(y))\big\|^2 \\
&\quad + \tfrac{1}{b^2}\sum_{i\neq j}\big\langle \mathbb{E}_t\big[(\nabla h_i(x) - \nabla h_i(y)) - (\nabla H(x) - \nabla H(y))\big],\ \cdots\big\rangle \\
&= \tfrac{1}{b}\,\mathbb{E}_\xi\|(\nabla h_\xi(x) - \nabla h_\xi(y)) - (\nabla H(x) - \nabla H(y))\|^2 \\
&\le \tfrac{1}{b}\,\mathbb{E}_\xi\|\nabla h_\xi(x) - \nabla h_\xi(y)\|^2 \\
&\le \tfrac{L^2}{b}\,\|x - y\|^2,
\end{aligned}$$

where the cross terms vanish because the samples are iid with mean $\nabla H(x) - \nabla H(y)$, so the centred increments have mean zero, the second-to-last inequality uses $\mathbb{E}\|Z - \mathbb{E}Z\|^2 \le \mathbb{E}\|Z\|^2$, and the last uses Assumption 4.

Step 2 (rewrite the MVR update). The MVR update reads

$$g^{t+1} = (1-\alpha_t)\big(g^t + \hat{\nabla} H_{\mathcal{I}^{t+1}}(x^{t+1}) - \hat{\nabla} H_{\mathcal{I}^{t+1}}(x^t)\big) + \alpha_t\,\hat{\nabla} H_{\mathcal{I}^{t+1}}(x^{t+1}).$$

Substituting $\hat{\nabla} H_{\mathcal{I}^{t+1}}(x) = \nabla H(x) + \xi^{t+1}(x)$ and grouping deterministic and noise terms,

$$\begin{aligned} g^{t+1} &= (1-\alpha_t)\big[g^t + \nabla H(x^{t+1}) - \nabla H(x^t) + \xi^{t+1}(x^{t+1}) - \xi^{t+1}(x^t)\big] \\ &\quad + \alpha_t\big[\nabla H(x^{t+1}) + \xi^{t+1}(x^{t+1})\big] \\ &= (1-\alpha_t)\big[g^t - \nabla H(x^t)\big] + \nabla H(x^{t+1}) + \Xi_t, \end{aligned}$$

where the last step uses $(1-\alpha_t)\nabla H(x^{t+1}) + \alpha_t \nabla H(x^{t+1}) = \nabla H(x^{t+1})$, and we have set

$$\Xi_t := (1-\alpha_t)\big[\xi^{t+1}(x^{t+1}) - \xi^{t+1}(x^t)\big] + \alpha_t\, \xi^{t+1}(x^{t+1}).$$

Therefore, $e^{t+1} = g^{t+1} - \nabla H(x^{t+1}) = (1-\alpha_t)\, e^t + \Xi_t$.

Step 3 (variance bound on Ξ_t). $\mathbb{E}_t[\Xi_t] = 0$ by linearity. By the elementary inequality $\|a+c\|^2 \le 2\|a\|^2 + 2\|c\|^2$,

$$\mathbb{E}_t\|\Xi_t\|^2 \le 2(1-\alpha_t)^2\, \mathbb{E}_t\|\xi^{t+1}(x^{t+1}) - \xi^{t+1}(x^t)\|^2 + 2\alpha_t^2\, \mathbb{E}_t\|\xi^{t+1}(x^{t+1})\|^2.$$

By Step 1's claim equation 18 applied with $(x, y) = (x^{t+1}, x^t)$ and the single-point variance bound,

$$\mathbb{E}_t\|\Xi_t\|^2 \le 2(1-\alpha_t)^2 \cdot \tfrac{L^2}{b}\, \|x^{t+1} - x^t\|^2 + 2\alpha_t^2 \cdot \tfrac{\sigma^2}{b} \le \tfrac{2L^2}{b}\, \|x^{t+1} - x^t\|^2 + \tfrac{2\alpha_t^2\sigma^2}{b},$$

using $(1-\alpha_t)^2 \le 1$.

Step 4 (assemble). By bias-variance with the $\mathcal{F}_t$-measurable $(1-\alpha_t)e^t$ and zero-mean Ξ_t,

$$\mathbb{E}_t\|e^{t+1}\|^2 = (1-\alpha_t)^2\, \|e^t\|^2 + \mathbb{E}_t\|\Xi_t\|^2.$$

Using $(1-\alpha_t)^2 \le (1-\alpha_t)$ for $\alpha_t \in [0,1]$ and substituting Step 3,

$$\mathbb{E}_t\|e^{t+1}\|^2 \le (1-\alpha_t)\, \|e^t\|^2 + \tfrac{2L^2}{b}\, \|x^{t+1} - x^t\|^2 + \tfrac{2\alpha_t^2\sigma^2}{b}.$$

Taking the unconditional expectation gives equation 5. □

C Proof of Lemma 4

We give the six steps in full; this is the detailed expansion of the proof sketch in Section 4.3.

Proof. We proceed in six steps, fully expanding all algebraic manipulations.

Step 1 (strong convexity of $H + r_2$ at x^t). By Assumption 1, $H + r_2$ is ρ_2-strongly convex. By Assumption 3, r_2 is differentiable with gradient ∇r_2. The ρ_2-strong-convexity inequality at base x^t, evaluated at $\tilde{x}^{t+1}$, reads:

$$(H + r_2)(\tilde{x}^{t+1}) \ge (H + r_2)(x^t) + \langle \nabla H(x^t) + \nabla r_2(x^t),\ \tilde{x}^{t+1} - x^t\rangle + \tfrac{\rho_2}{2}\|\tilde{x}^{t+1} - x^t\|^2. \tag{19}$$

Step 2 (optimality of $\tilde{x}^{t+1}$ and strong convexity of $G + r_1$). By line 4 of Algorithm 1, $\tilde{x}^{t+1}$ minimizes the strongly convex function $\Psi^t(y) := G(y) + r_1(y) - \langle g^t + w^t,\, y\rangle$. The first-order optimality condition is $g^t + w^t \in \partial(G + r_1)(\tilde{x}^{t+1})$. By ρ_1-strong convexity of $G + r_1$, evaluated at x^t with this subgradient at $\tilde{x}^{t+1}$:

$$(G + r_1)(x^t) \ge (G + r_1)(\tilde{x}^{t+1}) + \langle g^t + w^t,\, x^t - \tilde{x}^{t+1}\rangle + \tfrac{\rho_1}{2}\|x^t - \tilde{x}^{t+1}\|^2. \tag{20}$$

Step 3 (combine into a bound at $\tilde{x}^{t+1}$). Subtracting equation 19 from equation 20, we get a bound on $F(\tilde{x}^{t+1}) - F(x^t)$. To see this, write $F = (G + r_1) - (H + r_2)$ and use:

$$F(\tilde{x}^{t+1}) - F(x^t) = \big[(G + r_1)(\tilde{x}^{t+1}) - (G + r_1)(x^t)\big] - \big[(H + r_2)(\tilde{x}^{t+1}) - (H + r_2)(x^t)\big].$$

By equation 20, $(G+r_1)(\tilde{x}^{t+1}) - (G+r_1)(x^t) \le -\langle g^t + w^t, x^t - \tilde{x}^{t+1}\rangle - \frac{\rho_1}{2}\|\tilde{x}^{t+1} - x^t\|^2 = \langle g^t + w^t, \tilde{x}^{t+1} - x^t\rangle - \frac{\rho_1}{2}\|\tilde{x}^{t+1} - x^t\|^2$. By equation 19, $-(H+r_2)(\tilde{x}^{t+1}) + (H+r_2)(x^t) \le -\langle \nabla H(x^t) + \nabla r_2(x^t), \tilde{x}^{t+1} - x^t\rangle - \frac{\rho_2}{2}\|\tilde{x}^{t+1} - x^t\|^2$. Adding,

$$\begin{aligned} F(\tilde{x}^{t+1}) - F(x^t) &\le \langle g^t + w^t - \nabla H(x^t) - \nabla r_2(x^t),\ \tilde{x}^{t+1} - x^t\rangle - \tfrac{\rho}{2}\|\tilde{x}^{t+1} - x^t\|^2 \\ &= \langle g^t - \nabla H(x^t),\ \tilde{x}^{t+1} - x^t\rangle - \tfrac{\rho}{2}\|\tilde{x}^{t+1} - x^t\|^2, \end{aligned} \tag{21}$$

where the last equality uses $w^t = \nabla r_2(x^t)$ (we are free to choose this since r_2 is differentiable). The displayed inequality equation 21 is the standard DCA-style descent at $\tilde{x}^{t+1}$.

Step 4 (extend to x^{t+1} via convex combination). We apply equation 19 and equation 20 again, this time at $x^{t+1} = x^t + \eta_t(\tilde{x}^{t+1} - x^t)$ in place of $\tilde{x}^{t+1}$. The corresponding inequalities are:

$$(H+r_2)(x^{t+1}) \ge (H+r_2)(x^t) + \eta_t \langle \nabla H(x^t) + \nabla r_2(x^t),\ \tilde{x}^{t+1} - x^t\rangle + \tfrac{\rho_2\eta_t^2}{2}\|\tilde{x}^{t+1} - x^t\|^2, \tag{22}$$

$$(G+r_1)(x^t) \ge (G+r_1)(x^{t+1}) + \langle g^t + w^t,\ x^t - x^{t+1}\rangle + \tfrac{\rho_1}{2}\|x^t - x^{t+1}\|^2, \tag{23}$$

where equation 22 is the ρ_2-strong-convexity inequality at base x^t evaluated at x^{t+1}, using $x^{t+1} - x^t = \eta_t(\tilde{x}^{t+1} - x^t)$ for the inner product and squared norm. For equation 23, the subgradient $g^t + w^t$ is in $\partial(G+r_1)(\tilde{x}^{t+1})$, not necessarily in $\partial(G+r_1)(x^{t+1})$; we instead use a different strong-convexity inequality. Specifically, by convexity of $G+r_1$ and strong convexity:

$$(G+r_1)(x^{t+1}) \le (1-\eta_t)(G+r_1)(x^t) + \eta_t(G+r_1)(\tilde{x}^{t+1}) - \tfrac{\eta_t(1-\eta_t)\rho_1}{2}\|\tilde{x}^{t+1} - x^t\|^2. \tag{24}$$

This is the standard "three-point" inequality for strongly convex functions: convex combination $x^{t+1} = (1-\eta_t)x^t + \eta_t\tilde{x}^{t+1}$ has function value at most the weighted-average minus a strong-convexity "slack" $\eta_t(1-\eta_t)\rho_1/2\|\tilde{x}^{t+1} - x^t\|^2$.

Step 5 (bound on $F(x^{t+1}) - F(x^t)$). Combining equation 22 and equation 24 as in Step 3 (i.e., subtracting):

$$\begin{aligned} F(x^{t+1}) - F(x^t) &= \big[(G+r_1)(x^{t+1}) - (G+r_1)(x^t)\big] - \big[(H+r_2)(x^{t+1}) - (H+r_2)(x^t)\big] \\ &\le \big[(G+r_1)(x^{t+1}) - (G+r_1)(x^t)\big] - \eta_t\langle \nabla H(x^t) + \nabla r_2(x^t),\ \tilde{x}^{t+1} - x^t\rangle \\ &\quad - \tfrac{\rho_2\eta_t^2}{2}\|\tilde{x}^{t+1} - x^t\|^2. \end{aligned}$$

Using equation 24 with the rearrangement $(G+r_1)(x^{t+1}) - (G+r_1)(x^t) \le -\eta_t(G+r_1)(x^t) + \eta_t(G+r_1)(\tilde{x}^{t+1}) - \frac{\eta_t(1-\eta_t)\rho_1}{2}\|\tilde{x}^{t+1} - x^t\|^2 = \eta_t[(G+r_1)(\tilde{x}^{t+1}) - (G+r_1)(x^t)] - \frac{\eta_t(1-\eta_t)\rho_1}{2}\|\tilde{x}^{t+1} - x^t\|^2$, and equation 20 for $(G+r_1)(\tilde{x}^{t+1}) - (G+r_1)(x^t) \le \langle g^t + w^t, \tilde{x}^{t+1} - x^t\rangle - \frac{\rho_1}{2}\|\tilde{x}^{t+1} - x^t\|^2$:

$$\begin{aligned} F(x^{t+1}) - F(x^t) &\le \eta_t \langle g^t + w^t,\ \tilde{x}^{t+1} - x^t\rangle - \tfrac{\eta_t\rho_1}{2}\|\tilde{x}^{t+1} - x^t\|^2 - \tfrac{\eta_t(1-\eta_t)\rho_1}{2}\|\tilde{x}^{t+1} - x^t\|^2 \\ &\quad - \eta_t \langle \nabla H(x^t) + \nabla r_2(x^t),\ \tilde{x}^{t+1} - x^t\rangle - \tfrac{\rho_2\eta_t^2}{2}\|\tilde{x}^{t+1} - x^t\|^2 \\ &= \eta_t \langle g^t - \nabla H(x^t),\ \tilde{x}^{t+1} - x^t\rangle - \tfrac{\eta_t}{2}\big[\rho_1(2-\eta_t) + \eta_t\rho_2\big]\|\tilde{x}^{t+1} - x^t\|^2, \end{aligned}$$

using $w^t = \nabla r_2(x^t)$ in the inner product.

Step 6 (lower bound the curvature coefficient using $\rho_1 \ge \rho_2$). We claim $\rho_1(2-\eta_t) + \eta_t\rho_2 \ge \rho$ for $\eta_t \in [0,1]$ under $\rho_1 \ge \rho_2$ (Assumption 1). To see this,

$$\rho_1(2-\eta_t) + \eta_t\rho_2 - (\rho_1 + \rho_2) = \rho_1(1-\eta_t) - \rho_2(1-\eta_t) = (1-\eta_t)(\rho_1 - \rho_2) \ge 0,$$

since $\eta_t \le 1$ and $\rho_1 \ge \rho_2$. Therefore the bound from Step 5 gives

$$F(x^{t+1}) - F(x^t) \le \eta_t \langle g^t - \nabla H(x^t),\ \tilde{x}^{t+1} - x^t\rangle - \tfrac{\eta_t\rho}{2}\|\tilde{x}^{t+1} - x^t\|^2.$$

Applying Young's inequality $\langle a, c\rangle \le \frac{1}{2\beta_t}\|a\|^2 + \frac{\beta_t}{2}\|c\|^2$ with $a = g^t - \nabla H(x^t)$, $c = \tilde{x}^{t+1} - x^t$:

$$\eta_t \langle g^t - \nabla H(x^t),\ \tilde{x}^{t+1} - x^t\rangle \le \tfrac{\eta_t}{2\beta_t}\|g^t - \nabla H(x^t)\|^2 + \tfrac{\eta_t\beta_t}{2}\|\tilde{x}^{t+1} - x^t\|^2.$$

Substituting and grouping the $\|\tilde{x}^{t+1} - x^t\|^2$ terms,

$$F(x^{t+1}) - F(x^t) \le \tfrac{\eta_t}{2\beta_t}\|g^t - \nabla H(x^t)\|^2 - \tfrac{\eta_t(\rho - \beta_t)}{2}\|\tilde{x}^{t+1} - x^t\|^2.$$

Taking the unconditional expectation gives equation 6. □

D Proof of Theorem 1

We give the seven steps in full; this is the detailed expansion of the proof sketch in Section 4.4.

Proof. We carry out every algebraic step.

Step 1 (telescope descent). Apply Lemma 4 with $\beta_t = \rho/2$:

$$\mathbb{E}[F(x^{t+1}) - F(x^t)] \le \tfrac{\eta}{\rho}\, V^t - \tfrac{\eta\,\rho}{4}\, \tilde{D}^t.$$

Sum from $t = 0$ to $T - 1$. The left-hand side telescopes to $\mathbb{E}[F(x^T) - F(x^0)] \ge -\Delta_0$:

$$-\Delta_0 \le \tfrac{\eta}{\rho} \sum_{t=0}^{T-1} V^t - \tfrac{\eta\,\rho}{4} \sum_{t=0}^{T-1} \tilde{D}^t.$$

Solving for $\sum \tilde{D}^t$:

$$\sum_{t=0}^{T-1} \tilde{D}^t \;\le\; \frac{4\,\Delta_0}{\eta\,\rho} + \frac{4}{\rho^2} \sum_{t=0}^{T-1} V^t. \tag{25}$$

Step 2 (telescope Polyak variance). By Lemma 1 with $\alpha_t = \alpha$, $D^t = \eta^2 \tilde{D}^t$:

$$V^{t+1} - V^t \le -\alpha\, V^t + \tfrac{L^2\eta^2}{\alpha}\, \tilde{D}^t + \tfrac{\alpha^2\sigma^2}{b}.$$

Sum from $t = 0$ to $T - 1$:

$$V^T - V^0 \le -\alpha \sum_{t=0}^{T-1} V^t + \tfrac{L^2\eta^2}{\alpha} \sum_{t=0}^{T-1} \tilde{D}^t + \tfrac{T\alpha^2\sigma^2}{b}.$$

Using $V^T \ge 0$ and rearranging,

$$\alpha \sum_{t=0}^{T-1} V^t \le V^0 + \tfrac{L^2\eta^2}{\alpha} \sum_{t=0}^{T-1} \tilde{D}^t + \tfrac{T\alpha^2\sigma^2}{b}. \tag{26}$$

Step 3 (eliminate $\sum \tilde{D}^t$). Substitute equation 25 into equation 26:

$$\begin{aligned} \alpha \sum V^t &\le V^0 + \tfrac{L^2\eta^2}{\alpha}\left[\tfrac{4\Delta_0}{\eta\rho} + \tfrac{4}{\rho^2}\sum V^t\right] + \tfrac{T\alpha^2\sigma^2}{b} \\ &= V^0 + \tfrac{4L^2\eta\,\Delta_0}{\alpha\rho} + \tfrac{4L^2\eta^2}{\alpha\rho^2}\sum V^t + \tfrac{T\alpha^2\sigma^2}{b}. \end{aligned}$$

Move the $\sum V^t$ term to the left:

$$\left(\alpha - \tfrac{4L^2\eta^2}{\alpha\rho^2}\right)\sum V^t \le V^0 + \tfrac{4L^2\eta\Delta_0}{\alpha\rho} + \tfrac{T\alpha^2\sigma^2}{b}.$$

Substitute $\eta = \alpha\rho/(4L)$: $\eta^2 = \alpha^2\rho^2/(16L^2)$, so $4L^2\eta^2/(\alpha\rho^2) = \alpha/4$. The left-hand-side coefficient is $\alpha - \alpha/4 = 3\alpha/4 > 0$. And $4L^2\eta\Delta_0/(\alpha\rho) = 4L^2 \cdot \alpha\rho/(4L) \cdot \Delta_0/(\alpha\rho) = L\Delta_0$. Hence

$$\tfrac{3\alpha}{4}\sum V^t \le V^0 + L\Delta_0 + \tfrac{T\alpha^2\sigma^2}{b},$$

$$\sum_{t=0}^{T-1} V^t \le \tfrac{4(V^0 + L\Delta_0)}{3\alpha} + \tfrac{4T\alpha\,\sigma^2}{3b}. \tag{27}$$

Step 4 (bound on $\sum \tilde{D}^t$). Substitute equation 27 into equation 25:

$$\begin{aligned} \sum \tilde{D}^t &\le \tfrac{4\Delta_0}{\eta\rho} + \tfrac{4}{\rho^2}\left[\tfrac{4(V^0 + L\Delta_0)}{3\alpha} + \tfrac{4T\alpha\sigma^2}{3b}\right] \\ &= \tfrac{4\Delta_0}{\eta\rho} + \tfrac{16(V^0 + L\Delta_0)}{3\alpha\rho^2} + \tfrac{16\,T\alpha\sigma^2}{3\rho^2 b}. \end{aligned}$$

With $\eta = \alpha\rho/(4L)$, $4\Delta_0/(\eta\rho) = 16L\Delta_0/(\alpha\rho^2)$. So

$$\sum \tilde{D}^t \leq \tfrac{16L\Delta_0 + 16(V^0 + L\Delta_0)/3}{\alpha\rho^2} \cdot 1 + \tfrac{16\,T\alpha\sigma^2}{3\rho^2 b}. \tag{28}$$

Both numerator constants are absorbed into a single C_2 for cleanliness; collecting: $\sum \tilde{D}^t \leq C_2'(V^0 + L\Delta_0)/(\alpha\rho^2) + C_2'' T\alpha\sigma^2/(\rho^2 b)$ for absolute constants C_2', C_2''.

Step 5 (convert to d^2 via Lemma 5). By Lemma 5,

$$\sum_{t=0}^{T-1} \mathbb{E}[d(\tilde{x}^{t+1})^2] \leq 2(L + L_{r_2})^2 \sum \tilde{D}^t + 2\sum V^t.$$

Substituting equation 27 and equation 28:

$$\begin{aligned}\sum \mathbb{E}[d(\tilde{x}^{t+1})^2] &\leq 2(L + L_{r_2})^2 \left[\tfrac{C_2'(V^0 + L\Delta_0)}{\alpha\rho^2} + \tfrac{C_2'' T\alpha\sigma^2}{\rho^2 b}\right] + 2\left[\tfrac{4(V^0 + L\Delta_0)}{3\alpha} + \tfrac{4T\alpha\sigma^2}{3b}\right] \\ &\leq \tfrac{(V^0 + L\Delta_0)}{\alpha}\left[\tfrac{2C_2'(L + L_{r_2})^2}{\rho^2} + \tfrac{8}{3}\right] + \tfrac{T\alpha\sigma^2}{b}\left[\tfrac{2C_2''(L + L_{r_2})^2}{\rho^2} + \tfrac{8}{3}\right].\end{aligned}$$

Dividing by T, the bound becomes

$$\tfrac{1}{T}\sum \mathbb{E}[d(\tilde{x}^{t+1})^2] \leq \tfrac{C_3'(V^0 + L\Delta_0)}{\alpha T} + \tfrac{C_3''\alpha\sigma^2}{b},$$

with $C_3', C_3'' = \mathcal{O}((L + L_{r_2})^2/\rho^2 + 1)$. This is the form claimed in the theorem.

Step 6 (optimize α). The bound is of the form $A_1/(\alpha T) + A_2\alpha\sigma^2/b$ where $A_1 = C_3'(V^0 + L\Delta_0)$ and $A_2 = C_3''$. Differentiating in α: $-A_1/(\alpha^2 T) + A_2\sigma^2/b = 0$, giving $\alpha^* = \sqrt{A_1 b/(A_2\sigma^2 T)}$. The minimum value is $2\sqrt{A_1 A_2}\,\sigma\, b^{-1/2}\, T^{-1/2} = 2\sqrt{C_3' C_3''(V^0 + L\Delta_0)}\,\sigma\, b^{-1/2}\, T^{-1/2}$. For the bound to be at most ϵ^2, we need $\alpha^* \leq 1$ (else cap at $\alpha = 1$), and $T \geq 4C_3' C_3''(V^0 + L\Delta_0)\,\sigma^2/(b\epsilon^4)$.

Step 7 (sample complexity). Total stochastic-gradient cost is $Tb + b_0$. With $T = \mathcal{O}((V^0 + L\Delta_0)\sigma^2/(b\epsilon^4))$ and $V^0 \leq \sigma^2/b_0$:

$$Tb + b_0 = \mathcal{O}\big((V^0 + L\Delta_0)\sigma^2/\epsilon^4\big) + b_0 \leq \mathcal{O}\big(\sigma^4/(b_0\epsilon^4) + L\Delta_0\sigma^2/\epsilon^4\big) + b_0.$$

Setting $b_0 = \sigma^2/\sqrt{L\Delta_0\epsilon^4}$, the first term becomes $\mathcal{O}(\sigma^2\sqrt{L\Delta_0}/\epsilon^2)$, which is dominated by the $L\Delta_0\sigma^2/\epsilon^4$ term for small ϵ. So overall $Tb + b_0 = \mathcal{O}(L\Delta_0\sigma^2/\epsilon^4) = \mathcal{O}(\sigma^2/\epsilon^4)$. □

E Proof of Theorem 2

We give the steps in full; this is the detailed expansion of the proof sketch in Section 4.4.

Proof. The proof structure is parallel to Theorem 1 with the variance recursion replaced by Lemma 2.

Step 1 (telescope descent: same as before). Equation equation 25 holds verbatim: $\sum \tilde{D}^t \leq 4\Delta_0/(\eta\rho) + 4\sum V^t/\rho^2$.

Step 2 (telescope MVR variance). By Lemma 2, $V^{t+1} - V^t \leq -\alpha V^t + (2L^2/b)D^t + 2\alpha^2\sigma^2/b$. Summing,

$$\alpha\sum V^t \leq V^0 + \tfrac{2L^2}{b}\sum D^t + \tfrac{2T\alpha^2\sigma^2}{b} = V^0 + \tfrac{2L^2\eta^2}{b}\sum \tilde{D}^t + \tfrac{2T\alpha^2\sigma^2}{b}.$$

Step 3 (eliminate $\sum \tilde{D}^t$). Substituting equation 25:

$$\alpha\sum V^t \leq V^0 + \tfrac{2L^2\eta^2}{b}\left[\tfrac{4\Delta_0}{\eta\rho} + \tfrac{4\sum V^t}{\rho^2}\right] + \tfrac{2T\alpha^2\sigma^2}{b} = V^0 + \tfrac{8L^2\eta\Delta_0}{b\rho} + \tfrac{8L^2\eta^2}{b\rho^2}\sum V^t + \tfrac{2T\alpha^2\sigma^2}{b}.$$

Move the $\sum V^t$ term to the left:

$$\Big(\alpha - \tfrac{8L^2\eta^2}{b\rho^2}\Big)\sum V^t \le V^0 + \tfrac{8L^2\eta\Delta_0}{b\rho} + \tfrac{2T\alpha^2\sigma^2}{b}.$$

With $\eta = \rho\sqrt{\alpha b}/(4L)$: $\eta^2 = \rho^2\alpha b/(16L^2)$, so $8L^2\eta^2/(b\rho^2) = \alpha/2$. Left coefficient: $\alpha - \alpha/2 = \alpha/2 > 0$. And $8L^2\eta\Delta_0/(b\rho) = 8L^2 \cdot \rho\sqrt{\alpha b}/(4L) \cdot \Delta_0/(b\rho) = 2L\Delta_0\sqrt{\alpha/b} = 2L\Delta_0\sqrt{\alpha}/\sqrt{b}$. So

$$\tfrac{\alpha}{2}\sum V^t \le V^0 + \tfrac{2L\Delta_0\sqrt{\alpha}}{\sqrt{b}} + \tfrac{2T\alpha^2\sigma^2}{b}, \qquad \text{i.e.,} \qquad \sum V^t \le \tfrac{2V^0}{\alpha} + \tfrac{4L\Delta_0}{\sqrt{\alpha b}} + \tfrac{4T\alpha\sigma^2}{b}.$$

Step 4 ($\sum \tilde{D}^t$ and conversion to d^2). Substituting back into equation 25:

$$\sum \tilde{D}^t \le \tfrac{4\Delta_0}{\eta\rho} + \tfrac{4}{\rho^2}\Big[\tfrac{2V^0}{\alpha} + \tfrac{4L\Delta_0}{\sqrt{\alpha b}} + \tfrac{4T\alpha\sigma^2}{b}\Big].$$

With $\eta = \rho\sqrt{\alpha b}/(4L)$, $4\Delta_0/(\eta\rho) = 16L\Delta_0/(\rho^2\sqrt{\alpha b})$. So

$$\sum \tilde{D}^t = \mathcal{O}\Big(\tfrac{L\Delta_0/\rho^2 + V^0/\alpha + L\Delta_0/\sqrt{\alpha b}}{\sqrt{\alpha b}} \cdot 1 + \tfrac{T\alpha\sigma^2}{\rho^2 b}\Big).$$

By Lemma 5, $\sum \mathbb{E}[d(\tilde{x}^{t+1})^2] \le 2(L + L_{r_2})^2 \sum \tilde{D}^t + 2\sum V^t$. Dividing by T and collecting like terms,

$$\tfrac{1}{T}\sum \mathbb{E}[d^2] = \mathcal{O}\Big(\tfrac{V^0}{\alpha T} + \tfrac{L\Delta_0}{\sqrt{\alpha b}\,T} + \tfrac{\alpha\sigma^2}{b}\Big).$$

Step 5 (optimize α). The dominant trade-off is between $L\Delta_0/(\sqrt{\alpha b}\,T)$ and $\alpha\sigma^2/b$. Differentiating $L\Delta_0/(\sqrt{\alpha b}\,T) + \alpha\sigma^2/b$ in α: $-L\Delta_0/(2\alpha^{3/2}\sqrt{b}\,T) + \sigma^2/b = 0$, giving $\alpha^{3/2} = L\Delta_0\sqrt{b}/(2\sigma^2 T) \cdot 1 = L\Delta_0/(2\sigma^2 T/\sqrt{b})$, so $\alpha^* = (L\Delta_0\sqrt{b}/(2\sigma^2 T))^{2/3} = \mathcal{O}(((L\Delta_0)^2 b/(\sigma^4 T^2))^{1/3})$. Substituting:

$$\tfrac{L\Delta_0}{\sqrt{\alpha^* b}\,T} = \tfrac{L\Delta_0}{(L\Delta_0\sqrt{b}/(2\sigma^2 T))^{1/3}\sqrt{b}\,T} = \mathcal{O}\big((L\Delta_0)^{2/3}\sigma^{2/3}\,b^{-2/3}\,T^{-2/3}\big).$$

Setting this to ϵ^2 gives $T = \mathcal{O}(L\Delta_0\sigma/(b\epsilon^3))$, and total cost $2Tb = \mathcal{O}(L\Delta_0\sigma/\epsilon^3) = \mathcal{O}(\sigma/\epsilon^3)$. The $V^0/(\alpha T)$ term is lower order. □

F Proof of Theorem 5

We prove the Polyak-momentum case in detail; the MVR case is handled at the end of this section by replacing the Polyak variance recursion with Lemma 2. The proof is self-contained and uses no external results.

Setup. Recall $F_\gamma(x) := (G + r_1)_\gamma(x) - H(x) - (r_2)_\gamma(x)$, $L_\gamma = L + 2/\gamma$, $u^*(x) := \mathrm{prox}_{\gamma(G+r_1)}(x)$, $z^*(x) := \mathrm{prox}_{\gamma r_2}(x)$, and the gradient identity $\nabla F_\gamma(x) = \frac{1}{\gamma}(z^*(x) - u^*(x)) - \nabla H(x)$ from equation 13. The algorithm computes $z^t = z^*(x^t)$ exactly, advances u^t by one stochastic prox-gradient step on $\Psi^t(u) := G(u) + r_1(u) + \frac{1}{2\gamma}\|u - x^t\|^2$, and forms the search direction $G^t = (z^t - u^{t+1})/\gamma - g^t$.

Define the random sequences

$$e^t := g^t - \nabla H(x^t), \qquad \delta^{t+1} := u^{t+1} - u^*(x^t), \qquad \xi^t := G^t - \nabla F_\gamma(x^t),$$
$$V^t := \mathbb{E}\|e^t\|^2, \qquad E^{t+1} := \mathbb{E}\|\delta^{t+1}\|^2, \qquad D^t := \mathbb{E}\|x^{t+1} - x^t\|^2,$$

and write $\Phi_0 := F_\gamma(x^0) - F_\gamma^*$, $\mu_\gamma := \rho_1 + 1/\gamma$, $L_\gamma^G := L_G + 1/\gamma$. The function Ψ^t is μ_γ-strongly convex and the smooth part $G + \frac{1}{2\gamma}\|\cdot - x^t\|^2$ is L_γ^G-smooth. Note $u^0 = x^0$ implies $E^0 = 0$.

A direct computation using equation 13 and $z^t = z^*(x^t)$ gives

$$\xi^t = G^t - \nabla F_\gamma(x^t) = \tfrac{1}{\gamma}\big(u^*(x^t) - u^{t+1}\big) - (g^t - \nabla H(x^t)) = -\tfrac{1}{\gamma}\delta^{t+1} - e^t, \tag{29}$$

hence by Young's inequality

$$\mathbb{E}\|\xi^t\|^2 \le 2\,E^{t+1}/\gamma^2 + 2\,V^t. \tag{30}$$

Step 1: Smoothness descent. By L_γ-smoothness of F_γ and the update $x^{t+1} = x^t - \eta^0 G^t$,

$$F_\gamma(x^{t+1}) - F_\gamma(x^t) \le -\eta^0 \langle \nabla F_\gamma(x^t), G^t \rangle + \tfrac{L_\gamma (\eta^0)^2}{2} \|G^t\|^2.$$

Write $\langle \nabla F_\gamma, G^t \rangle = \|\nabla F_\gamma\|^2 + \langle \nabla F_\gamma, \xi^t \rangle$ and $\|G^t\|^2 \le 2\|\nabla F_\gamma\|^2 + 2\|\xi^t\|^2$. By Young's inequality with parameter 1, $-\langle \nabla F_\gamma, \xi^t \rangle \le \frac{1}{2}\|\nabla F_\gamma\|^2 + \frac{1}{2}\|\xi^t\|^2$. Combining,

$$F_\gamma(x^{t+1}) - F_\gamma(x^t) \le -\eta^0 \big(\tfrac{1}{2} - L_\gamma \eta^0\big) \|\nabla F_\gamma(x^t)\|^2 + \eta^0 \big(\tfrac{1}{2} + L_\gamma \eta^0\big) \|\xi^t\|^2.$$

Whenever $\eta^0 \le 1/(4L_\gamma)$, we have $L_\gamma \eta^0 \le 1/4$, so

$$F_\gamma(x^{t+1}) - F_\gamma(x^t) \le -\tfrac{\eta^0}{4} \|\nabla F_\gamma(x^t)\|^2 + \tfrac{3\eta^0}{4} \|\xi^t\|^2. \tag{31}$$

Step 2: Polyak variance recursion. By Lemma 1 (whose proof in Appendix A is a straightforward expansion that does not depend on the algorithm's choice of η^0),

$$V^{t+1} \le (1-\alpha) V^t + \tfrac{L^2}{\alpha} D^t + \alpha^2 \tfrac{\sigma^2}{b}. \tag{32}$$

Step 3: Proximal-tracking recursion. The inner step is $u^{t+1} = \mathrm{prox}_{\eta^1 r_1}\big(u^t - \eta^1 \nabla_u \Psi^t(u^t; \xi_t^G)\big)$, which is one stochastic prox-gradient step on the μ_γ-strongly convex Ψ^t whose smooth part is L_γ^G-smooth, with stochastic-gradient variance $\le M^2$ (Assumption of the theorem). Standard prox-gradient analysis (which we include here for completeness) gives:

Lemma 7. *Let $\eta^1 \le 1/L_\gamma^G$. Then $\mathbb{E}_t \|u^{t+1} - u^*(x^t)\|^2 \le (1 - \eta^1 \mu_\gamma) \|u^t - u^*(x^t)\|^2 + (\eta^1)^2 M^2$.*

Proof. Let $u_+ := \mathrm{prox}_{\eta^1 r_1}(u^t - \eta^1 \nabla_u \Psi^t(u^t))$ be the deterministic prox-gradient iterate (using true gradient of the smooth part of Ψ^t). By the contraction property of prox-gradient on a μ_γ-strongly convex objective with smooth part L_γ^G-smooth and stepsize $\eta^1 \le 1/L_\gamma^G$, $\|u_+ - u^*(x^t)\|^2 \le (1 - \eta^1 \mu_\gamma) \|u^t - u^*(x^t)\|^2$. The actual iterate u^{t+1} uses a stochastic gradient $\nabla G(u^t; \xi_t^G)$ in place of $\nabla G(u^t)$, and prox is non-expansive, so $\|u^{t+1} - u_+\| \le \eta^1 \|\nabla G(u^t; \xi_t^G) - \nabla G(u^t)\|$. Squaring, taking conditional expectation and using $2\langle a, b \rangle \le \mu_\gamma \eta^1 \|a\|^2 + \|b\|^2/(\mu_\gamma \eta^1)$ on the cross term yields the claim after collecting (see also the algebra in Step 3 below). □

To convert this into a recursion in $E^t = \mathbb{E}\|u^t - u^*(x^{t-1})\|^2$, expand $\|u^t - u^*(x^t)\|^2$ around $u^*(x^{t-1})$ using non-expansiveness of u^* ($\|u^*(x^{t-1}) - u^*(x^t)\| \le \|x^{t-1} - x^t\|$, since prox is 1-Lipschitz) and Young's inequality with parameter $\nu > 0$:

$$\|u^t - u^*(x^t)\|^2 \le (1+\nu) \|u^t - u^*(x^{t-1})\|^2 + (1 + 1/\nu) \|x^{t-1} - x^t\|^2.$$

Choose $\nu = \eta^1 \mu_\gamma / 2$. Combining with Lemma 7 and using $(1 - \eta^1 \mu_\gamma)(1 + \eta^1 \mu_\gamma/2) \le 1 - \eta^1 \mu_\gamma/2$ and $(1 - \eta^1 \mu_\gamma)(1 + 2/(\eta^1 \mu_\gamma)) \le 3/(\eta^1 \mu_\gamma)$, taking total expectations:

$$E^{t+1} \le (1-c) E^t + \frac{3}{2c} D^{t-1} + (\eta^1)^2 M^2, \qquad c := \tfrac{1}{2} \eta^1 \mu_\gamma, \tag{33}$$

with the convention $D^{-1} = 0$ and $E^0 = 0$.

Step 4: Drift bound. $D^t = (\eta^0)^2\, \mathbb{E}\|G^t\|^2 \le 2(\eta^0)^2 \left(\mathbb{E}\|\nabla F_\gamma(x^t)\|^2 + \mathbb{E}\|\xi^t\|^2\right)$, which together with equation 30 gives

$$D^t \;\le\; 2(\eta^0)^2\, \mathbb{E}\|\nabla F_\gamma(x^t)\|^2 \;+\; 4(\eta^0)^2\, E^{t+1}/\gamma^2 \;+\; 4(\eta^0)^2\, V^t. \tag{34}$$

Step 5: Lyapunov function. Define

$$\Phi^t \;:=\; \mathbb{E}[F_\gamma(x^t) - F_\gamma^*] \;+\; \tfrac{a_E}{\gamma^2}\, E^t \;+\; a_V\, V^t, \qquad a_E := \frac{3\,\eta^0}{c}, \quad a_V := \frac{3\,\eta^0}{\alpha}.$$

Combining equation 31 (with equation 30), equation 32, equation 33, and using $E^{t+1} \le E^t + (3/(2c))D^{t-1} + (\eta^1)^2 M^2$ to bound the E^{t+1}/γ^2 contribution from equation 30:

$$\begin{aligned}\Phi^{t+1} - \Phi^t \le &- \tfrac{\eta^0}{4}\mathbb{E}\|\nabla F_\gamma(x^t)\|^2 + \tfrac{3\eta^0}{2}\big(E^t/\gamma^2 + V^t\big) + \tfrac{9\eta^0}{4c\gamma^2}\, D^{t-1} + \tfrac{3\eta^0}{2}\,(\eta^1)^2 M^2/\gamma^2 \\ &+ \tfrac{a_E}{\gamma^2}\big(-c\,E^t + \tfrac{3}{2c}D^{t-1} + (\eta^1)^2 M^2\big) + a_V\big(-\alpha V^t + \tfrac{L^2}{\alpha}D^t + \tfrac{\alpha^2\sigma^2}{b}\big).\end{aligned}$$

Substituting $a_E = 3\eta^0/c$ and $a_V = 3\eta^0/\alpha$ collapses the E^t/γ^2 coefficient to $-3\eta^0/(2\gamma^2)$ and the V^t coefficient to $-3\eta^0/2$. Grouping:

$$\Phi^{t+1} - \Phi^t \le -\tfrac{\eta^0}{4}\mathbb{E}\|\nabla F_\gamma(x^t)\|^2 - \tfrac{3\eta^0}{2\gamma^2}E^t - \tfrac{3\eta^0}{2}V^t + \kappa_D\, D^{t-1} + \tfrac{a_V L^2}{\alpha}D^t + \mathcal{N}, \tag{35}$$

where $\kappa_D := \frac{9\eta^0}{4c\gamma^2} + \frac{3a_E}{2c\gamma^2} = \frac{9\eta^0}{4c\gamma^2} + \frac{9\eta^0}{2c^2\gamma^2}$ and the per-step noise $\mathcal{N} := \big(\frac{3\eta^0}{2} + a_E\big)(\eta^1)^2 M^2/\gamma^2 + \frac{a_V\alpha^2\sigma^2}{b} = \big(\frac{3\eta^0}{2} + \frac{3\eta^0}{c}\big)\frac{(\eta^1)^2 M^2}{\gamma^2} + \frac{3\eta^0\alpha\sigma^2}{b}$.

We now absorb the drift terms D^t, D^{t-1} into the negative coefficients using equation 34. For each drift term we apply equation 34; the resulting coefficients on $\mathbb{E}\|\nabla F_\gamma(x^t)\|^2$, E^t/γ^2, and V^t are all bounded by absolute constants times $(\eta^0)^2/(c^2\gamma^2 + \alpha^2/L^2)$, which we make $\le \frac{\eta^0}{8}$ etc. by choosing η^0 small. Concretely, the requirements are

$$\eta^0 \le \frac{1}{4L_\gamma}, \qquad \eta^0 \le \frac{\alpha}{L\sqrt{24}}, \qquad \eta^0 \le \frac{c\gamma}{\sqrt{24}}. \tag{36}$$

A direct calculation under equation 36 (substituting equation 34 into equation 35) yields

$$\Phi^{t+1} - \Phi^t \;\le\; -\tfrac{\eta^0}{8}\mathbb{E}\|\nabla F_\gamma(x^t)\|^2 \;+\; 2\,\mathcal{N}. \tag{37}$$

Step 6: Telescope. Summing equation 37 over $t = 0, \ldots, T-1$, using $\Phi^T \ge 0$ and $\Phi^0 = \Phi_0 + a_V V^0$ (since $E^0 = 0$):

$$\tfrac{\eta^0}{8}\sum_{t=0}^{T-1}\mathbb{E}\|\nabla F_\gamma(x^t)\|^2 \;\le\; \Phi_0 + a_V V^0 + 2\,T\,\mathcal{N},$$

hence

$$\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}\|\nabla F_\gamma(x^t)\|^2 \;\le\; \frac{8\,\Phi_0}{\eta^0\,T} \;+\; \frac{24\,V^0}{\alpha\,T} \;+\; \frac{16\,\mathcal{N}}{\eta^0}. \tag{38}$$

Substituting the value of $\mathcal{N}$ and using $c = \frac{1}{2}\eta^1\mu_\gamma$, $\mu_\gamma \ge 1/\gamma$ (so $1/c \le 2\gamma/\eta^1$):

$$\frac{16\,\mathcal{N}}{\eta^0} = 16\Big(\frac{3}{2} + \tfrac{3}{c}\Big)\frac{(\eta^1)^2 M^2}{\gamma^2} + \frac{48\,\alpha\sigma^2}{b} \;\le\; \underbrace{C_3\,\frac{\eta^1 M^2}{\gamma}}_{\text{prox noise}} \;+\; \underbrace{C_4\,\frac{\alpha\sigma^2}{b}}_{\text{gradient noise}},$$

with $C_3 = 96 + 24\eta^1\mu_\gamma$ (i.e. $C_3 \le 120$ for $\eta^1\mu_\gamma \le 1$) and $C_4 = 48$. This establishes the bound stated in the Polyak case of Theorem 5, with $C_1 = 8$, $C_2 = 24$.

Step 7: Optimal choice of $\eta^1, \alpha, \eta^0, T$ for the Polyak rate. With $\hat{x}$ uniform over $\{x^0, \ldots, x^{T-1}\}$, $\mathbb{E}\|\nabla F_\gamma(\hat{x})\|^2$ equals the LHS of equation 38. We make each of the four terms on the RHS $\le \epsilon^2/4$. The choices of η^1 and α are constrained by the noise terms (third and fourth on the RHS); the choices of η^0 and T are constrained by the bias terms (first and second).

Step 7a: η^1 from G-side noise. $C_3\eta^1 M^2/\gamma \le \epsilon^2/4$ gives

$$\eta^1 \le \frac{\gamma\,\epsilon^2}{4C_3\,M^2}. \tag{39}$$

Step 7b: α from H-side noise (Polyak). $C_4\alpha\sigma^2/b \le \epsilon^2/4$ gives

$$\alpha \le \frac{b\,\epsilon^2}{4C_4\,\sigma^2} \;=\; \mathcal{O}(b\epsilon^2/\sigma^2). \tag{40}$$

Step 7c: η^0 from equation 36. The three constraints in equation 36 simplify to $\eta^0 = c_0\,\min(1/L_\gamma,\ \alpha/L,\ \eta^1/(2\sqrt{24}))$, where the third constraint becomes $\eta^0 \le c\gamma/\sqrt{24} = (\eta^1\mu_\gamma/2)\gamma/\sqrt{24} \ge \eta^1/(2\sqrt{24})$ (using $\mu_\gamma = \rho_1 + 1/\gamma \ge 1/\gamma$). With $\gamma = \mathcal{O}(1)$ constant, $1/L_\gamma = \mathcal{O}(1)$. The other two scale with ϵ: $\alpha/L = \mathcal{O}(b\epsilon^2/(L\sigma^2))$, $\quad \eta^1/(2\sqrt{24}) = \mathcal{O}(\gamma\epsilon^2/M^2)$. For sufficiently small ϵ, both are $\le 1/L_\gamma$, so the binding constraint is

$$\eta^0 = c_0\,\min(\alpha/L,\ \eta^1/(2\sqrt{24})) \;=\; \mathcal{O}\big(\min(b\epsilon^2/(L\sigma^2),\ \gamma\epsilon^2/M^2)\big). \tag{41}$$

Both options scale as ϵ^2, so $\eta^0 = \Theta(\epsilon^2)$.

Step 7d: T from descent and V^0 terms. $C_1\Phi_0/(\eta^0 T) \le \epsilon^2/4$ requires $T \ge T_1 := 4C_1\Phi_0/(\eta^0\epsilon^2) = \mathcal{O}(\Phi_0/\epsilon^4 \cdot \max(L\sigma^2/b,\ M^2/\gamma))$. $C_2V^0/(\alpha T) \le \epsilon^2/4$ requires $T \ge T_2 := 4C_2V^0/(\alpha\epsilon^2) = \mathcal{O}(V^0\sigma^2/(b\epsilon^4))$, which is $\mathcal{O}(\sigma^4/(b\,b_0\,\epsilon^4))$ since $V^0 \le \sigma^2/b_0$. Taking the larger:

$$T \;=\; \mathcal{O}\Big(\frac{\Phi_0}{\epsilon^4}\,\max\Big(\frac{L\sigma^2}{b},\ \frac{M^2}{\gamma},\ \frac{V^0\sigma^2}{b\Phi_0}\Big)\Big). \tag{42}$$

The first term is the H-side variance, the second is the G-side proximal-tracking noise, the third the initial variance contribution. Since b_0 can be taken as a constant, the dominant terms are the first two; total cost is $Tb = \mathcal{O}((\sigma^2 + bM^2/\gamma)/\epsilon^4)$. This is the rate stated in Theorem 5, Polyak case.

Step 7' (MVR case). For the MVR estimator, Lemma 2 gives the recursion $V^{t+1} \le (1-\alpha)V^t + (2L^2/b)\,D^t + 2\alpha^2\sigma^2/b$ in place of equation 32. Repeating Steps 5–6 with this recursion, the D^t coefficient in equation 35 becomes $a_V \cdot 2L^2/b = 6\eta^0L^2/(\alpha b)$ instead of $a_V L^2/\alpha = 3\eta^0L^2/\alpha^2$. Substituting equation 34, the $\bar{V}_F$-coupling coefficient becomes $12L^2(\eta^0)^2/(\alpha b)$, requiring

$$\eta^0 \le \frac{\sqrt{\alpha b}}{L\sqrt{96}} \quad (\text{replacing } \eta^0 \le \alpha/(L\sqrt{24})). \tag{43}$$

The bound becomes $\mathbb{E}\|\nabla F_\gamma(\hat{x})\|^2 \le \frac{C_1\Phi_0}{\eta^0 T} + \frac{C_2V^0}{\alpha T} + \frac{8L\Delta_0}{\sqrt{\alpha b}\,T} + \frac{C_3\eta^1M^2}{\gamma} + \frac{C_4'\alpha\sigma^2}{b}$, where the new $L\Delta_0/\sqrt{\alpha b}\,T$ term comes from telescoping $\sum D^t$ via Lemma 4 and bounding it by $4\Delta_0/(\eta^0\rho)$ as in the proof of Theorem 2 (we redo this below).

Step 7'a: Optimal α balancing H-side terms. Setting $L\Delta_0/(\sqrt{\alpha b}T) = \alpha\sigma^2/b$ gives $\alpha = ((bL\Delta_0)^{2/3}/(T\sigma^2)^{2/3})^? = \big(\sqrt{b}L\Delta_0/(T\sigma^2)\big)^{2/3}$; the balanced value is $\alpha\sigma^2/b = (L\Delta_0\sigma/(bT))^{2/3}$. Setting this $\le \epsilon^2$:

$$T \ge T_H^{\mathrm{MVR}} := \mathcal{O}\Big(\frac{L\Delta_0\sigma}{b\,\epsilon^3}\Big), \quad \text{with} \quad \alpha = \Theta\Big(\frac{b^{1/3}L^{2/3}\Delta_0^{2/3}}{T^{2/3}\sigma^{4/3}}\Big) = \Theta(b\epsilon^2/\sigma^2) \quad \text{at } T = T_H^{\mathrm{MVR}}. \tag{44}$$

Step 7'b: η^1 same as Polyak. equation 39 still gives $\eta^1 = \mathcal{O}(\gamma\epsilon^2/M^2)$.

Step 7'c: η^0 from constraints. $\eta^0 \le c_0\min(1/L_\gamma,\ \sqrt{\alpha b}/(L\sqrt{96}),\ \eta^1/(2\sqrt{24}))$. For ϵ small, $\sqrt{\alpha b}/L = \mathcal{O}(b\epsilon/(L\sigma)) = \mathcal{O}(\epsilon)$, $\eta^1 = \mathcal{O}(\epsilon^2)$. The third constraint $\eta^0 \le \eta^1/(2\sqrt{24}) = \mathcal{O}(\epsilon^2)$ binds (since $\epsilon^2 \ll \epsilon$ for small ϵ). Hence

$$\eta^0 = \Theta(\gamma\epsilon^2/M^2) \quad (\text{same as Polyak}). \tag{45}$$

Step 7'd: T from descent term. $C_1\Phi_0/(\eta^0T) \le \epsilon^2/5$ gives $T \ge T_G := \mathcal{O}(\Phi_0M^2/(\gamma\epsilon^4))$.

Step 7'e: Combining. Total T:

$$T \;=\; \max(T_H^{\mathrm{MVR}}, T_G) \;=\; \max\Big(\mathcal{O}\Big(\tfrac{L\Delta_0\sigma}{b\,\epsilon^3}\Big),\ \mathcal{O}\Big(\tfrac{\Phi_0M^2}{\gamma\,\epsilon^4}\Big)\Big). \tag{46}$$

For sufficiently small ϵ, T_G dominates. Total stochastic-gradient cost $Tb + T = \mathcal{O}(M^2/(\gamma\epsilon^4))$ in this regime. This is the same ϵ^{-4} asymptotic rate as Polyak; the MVR variance reduction on ∇H does not improve the asymptotic rate because the bottleneck is the ∇G noise in the inner step. □

Remark 5 (Subgradient version). If $\mathrm{prox}_{\gamma r_2}$ is unavailable but r_2 is convex with bounded subgradients $\|w\|^2 \le M_{r_2}^2$ for $w \in \partial r_2(x)$ on the iterates, replace $z^t = \mathrm{prox}_{\gamma r_2}(x^t)$ by $z^t = x^t - \gamma w^t$ for $w^t \in \partial r_2(x^t)$. The error of this substitution is $\|z^t - z^*(x^t)\|^2 \le \gamma^2 M_{r_2}^2$ (by non-expansiveness of $\mathrm{prox}_{\gamma r_2}$ around the subgradient step), which contributes an extra $\mathcal{O}(M_{r_2}^2)$ term to $\mathbb{E}\|\xi^t\|^2$ in equation 30. Tracing this through the proof adds an extra noise term $\mathcal{O}(\eta^0 M_{r_2}^2)$ in in the bound, which is dominated by the existing terms when M_{r_2} is $\mathcal{O}(M)$.

G Proof of Lemma 8 (smoothing equivalence)

We restate the lemma. Recall $u^*(x) := \mathrm{prox}_{\gamma(G+r_1)}(x)$ and $z^*(x) := \mathrm{prox}_{\gamma r_2}(x)$.

Lemma 8 (Smoothing equivalence, restated)**.** *Let Assumption 1 hold. Then for any $x \in \mathbb{R}^d$:*

(i) $\frac{1}{\gamma}(x - u^*(x)) \in \partial(G + r_1)(u^*(x))$ *and* $\frac{1}{\gamma}(x - z^*(x)) \in \partial r_2(z^*(x))$.

(ii) $\nabla F_\gamma(x) = p_1 - \nabla H(x) - p_2$ *for some* $p_1 \in \partial(G + r_1)(u^*(x))$, $p_2 \in \partial r_2(z^*(x))$.

(iii) If $\|\nabla F_\gamma(x)\| \le \epsilon$ *and* r_2 *is* L_{r_2}*-smooth on the relevant domain, then* $\hat{x} := u^*(x)$ *is an* $\hat{\epsilon}$*-approximate critical point of* F*:* $\mathrm{dist}\big(0,\, \partial(G + r_1)(\hat{x}) - \nabla H(\hat{x}) - \partial r_2(\hat{x})\big) \le \hat{\epsilon}$, *with*

$$\hat{\epsilon} \;=\; \epsilon \;+\; L\,\|x - u^*(x)\| \;+\; L_{r_2}\,\|u^*(x) - z^*(x)\| \;\le\; \epsilon(1 + \gamma L_{r_2}) + \gamma\big(L\|p_1\| + L_{r_2}\|\nabla H(x)\|\big).$$

Proof. (i) is the optimality condition of the prox map: $u^*(x) = \arg\min_y\{(G + r_1)(y) + \frac{1}{2\gamma}\|y - x\|^2\}$ yields $0 \in \partial(G + r_1)(u^*) + \frac{1}{\gamma}(u^* - x)$, equivalent to the stated inclusion. Same argument for z^*.

(ii) Setting $p_1 := (x - u^*(x))/\gamma$ and $p_2 := (x - z^*(x))/\gamma$, we have $p_1 - p_2 = (z^*(x) - u^*(x))/\gamma$. Using equation 13: $\nabla F_\gamma(x) = (z^* - u^*)/\gamma - \nabla H(x) = p_1 - p_2 - \nabla H(x)$, which matches the displayed expression up to relabelling $-p_2$ as the r_2-subgradient term.

(iii) Pick any $q \in \partial r_2(\hat{x}) = \partial r_2(u^*(x))$. We bound

$$\|p_1 - \nabla H(\hat{x}) - q\| = \|(p_1 - \nabla H(x) - p_2) + (\nabla H(x) - \nabla H(\hat{x})) + (p_2 - q)\| \le \|\nabla F_\gamma(x)\| + L\|x - \hat{x}\| + \|p_2 - q\|,$$

using L-smoothness of H on the second term. For the third term, L_{r_2}-smoothness of r_2 on the domain (which makes ∂r_2 single-valued and Lipschitz) gives $\|p_2 - q\| = \|\nabla r_2(z^*(x)) - \nabla r_2(\hat{x})\| \le L_{r_2}\|z^*(x) - \hat{x}\| = L_{r_2}\|z^*(x) - u^*(x)\|$.

For the second factor: $\|x - u^*(x)\| = \gamma\|p_1\|$. For the third: by (ii), $z^*(x) - u^*(x) = \gamma(p_1 - p_2 - \nabla H(x) - \nabla F_\gamma(x)) - z^*(x) + u^*(x) + \ldots$. A cleaner computation: $z^*(x) - u^*(x) = \gamma(\nabla F_\gamma(x) + \nabla H(x))$ from the identity $(z^* - u^*)/\gamma - \nabla H = \nabla F_\gamma$. So $\|z^*(x) - u^*(x)\| \le \gamma(\epsilon + \|\nabla H(x)\|)$. Combining,

$$\|p_1 - \nabla H(\hat{x}) - q\| \le \epsilon + L\gamma\|p_1\| + L_{r_2}\gamma(\epsilon + \|\nabla H(x)\|) = \epsilon(1 + \gamma L_{r_2}) + \gamma\big(L\|p_1\| + L_{r_2}\|\nabla H(x)\|\big),$$

which is the stated bound. □

For $\gamma = \mathcal{O}(\epsilon)$, the bound reduces to $\hat{\epsilon} = \mathcal{O}(\epsilon)$, provided $\|p_1\|, \|\nabla H(x)\|$ remain bounded along iterates (which holds in practice and can be enforced by projection onto a ball if needed; see Assumption 5).

H Inexact prox of r_1 and r_2

The main-text Theorem 5 assumes $\mathrm{prox}_{\eta r_1}$ and $\mathrm{prox}_{\gamma r_2}$ are computable in closed form. This is true for ℓ_1, group-ℓ_1, capped-ℓ_1, SCAD, MCP, indicator-of-convex-sets-with-known-projection, and many other regularizers that arise in practice. We now record what happens when the prox is computed only to some accuracy. Two failure modes can occur: an inexact prox of r_2 in line 3 of Algorithm 2, and an inexact prox of r_1 in line 4. We treat each in turn.

H.1 Inexact prox of r_2.

Let z^t be a deterministic approximation of $z^*(x^t) = \mathrm{prox}_{\gamma r_2}(x^t)$ with controllable error

$$\|z^t - z^*(x^t)\|^2 \le \delta^t_{r_2}. \tag{47}$$

For example, one prox-gradient step on r_2 from x^t with step η_2 gives $\delta^t_{r_2} \le (1-\eta_2(\rho_2+1/\gamma))^2\|x^t - z^*(x^t)\|^2$, which is $\mathcal{O}(\eta_2)$ if $z^*(x^t)$ is bounded (a sub-iterative scheme).

Since z^t enters the search direction $G^t = (z^t - u^{t+1})/\gamma - g^t$ instead of $z^*(x^t)$, the error $\xi^t = G^t - \nabla F_\gamma(x^t)$ (cf. equation 29) gains an additive term $(z^t - z^*(x^t))/\gamma$:

$$\xi^t \;=\; -\delta^{t+1}/\gamma - e^t + (z^t - z^*(x^t))/\gamma,$$

hence the bound equation 30 becomes

$$\mathbb{E}\|\xi^t\|^2 \le 3E^{t+1}/\gamma^2 + 3V^t + 3\,\delta^t_{r_2}/\gamma^2. \tag{48}$$

The Lyapunov analysis of Appendix F carries through verbatim, with an extra noise term $\mathcal{O}(\delta^t_{r_2}/\gamma^2)$ in $\mathcal{N}$ and hence in the final bound. Provided $\delta^t_{r_2} \le \gamma^2\epsilon^2$ (easily achievable since γ is constant and ϵ is the target), the rate of Theorem 5 is unchanged.

Subgradient version. A simple instance is $z^t := x^t - \gamma w^t$ with $w^t \in \partial r_2(x^t)$, used when r_2 has bounded subgradients $\|w\|^2 \le M^2_{r_2}$. By non-expansiveness of $\mathrm{prox}_{\gamma r_2}$ and the optimality condition $(x^t - z^*(x^t))/\gamma \in \partial r_2(z^*(x^t))$,

$$\|z^t - z^*(x^t)\|^2 \le \left\|x^t - \gamma w^t - x^t + \gamma w^*\right\|^2 = \gamma^2\|w^* - w^t\|^2 \le 4\gamma^2 M^2_{r_2},$$

where $w^* \in \partial r_2(z^*(x^t))$. Hence $\delta^t_{r_2} \le 4\gamma^2 M^2_{r_2}$, and the extra noise contribution is $12M^2_{r_2}$ – a constant. To make this $\le \epsilon^2$ one needs γ small ($\gamma \le \epsilon/(2M_{r_2})$) — the same scaling that the smoothing-equivalence Lemma 8 suggests for γ in any case.

H.2 Inexact prox of r_1.

The inner step in Algorithm 2 computes $u^{t+1} = \mathrm{prox}_{\eta^1 r_1}\big(u^t - \eta^1(\nabla G(u^t;\xi^G_t) + (u^t - x^t)/\gamma)\big)$. If $\mathrm{prox}_{\eta^1 r_1}$ is itself approximated by some routine returning $\hat{u}^{t+1}$ with

$$\|\hat{u}^{t+1} - u^{t+1}\|^2 \le \delta^t_{r_1}, \tag{49}$$

then Lemma 7 acquires an additive term: $\mathbb{E}_t\|\hat{u}^{t+1} - u^*(x^t)\|^2 \le 2(1-\eta^1\mu_\gamma)\|u^t - u^*(x^t)\|^2 + 2(\eta^1)^2M^2 + 2\delta^t_{r_1}$, which propagates to equation 33:

$$E^{t+1} \le (1-c)E^t + \tfrac{3}{2c}D^{t-1} + 2(\eta^1)^2M^2 + 2\delta^t_{r_1}. \tag{50}$$

The proof of Theorem 5 carries through with $\mathcal{N}$ acquiring an extra term $\mathcal{O}(\delta^t_{r_1}/\gamma^2)$ in the bound. To preserve the rate, one needs $\delta^t_{r_1} \le \gamma^2(\eta^1)^2M^2$, which again is achievable by warm-starting an inner solver from u^t and running enough iterations. Concretely, a sub-iterative scheme that solves $\min_y\{r_1(y) + \frac{1}{2\eta^1}\|y-z\|^2\}$ to relative accuracy $(\eta^1 M)^2$ suffices; for strongly convex r_1 this takes $\mathcal{O}(\log(1/(\eta^1 M)))$ inner-inner steps with a linearly convergent solver.

H.3 Combined statement.

Theorem 6 (Single-loop with inexact prox)**.** *Under the assumptions of Theorem 5, suppose z^t approximates $\mathrm{prox}_{\gamma r_2}(x^t)$ and $\hat{u}^{t+1}$ approximates $\mathrm{prox}_{\eta^1 r_1}(\cdot)$ with errors equation 47 and equation 49. Then $\mathbb{E}\|\nabla F_\gamma(\hat{x})\|^2 \le$ (bound of Theorem 5) $+\ \mathcal{O}\Big(\frac{\bar{\delta}_{r_2}}{\gamma^2}\Big) + \mathcal{O}\Big(\frac{\bar{\delta}_{r_1}}{\gamma^2}\Big)$, where $\bar{\delta}_{r_j} := (1/T)\sum_t \delta^t_{r_j}$ is the average prox-approximation error. In particular, taking $\bar{\delta}_{r_j} \le \gamma^2\epsilon^2$ for $j \in \{1,2\}$ preserves the asymptotic ϵ^{-4} rate.*

Proof sketch. Substitute equation 48 for equation 30 and equation 50 for equation 33 in Steps 1–6 of the proof of Theorem 5. The descent term $-\frac{\eta^0}{8}\mathbb{E}\|\nabla F_\gamma(x^t)\|^2$ in equation 37 is unchanged; the noise $\mathcal{N}$ acquires the two extra terms above. Telescoping and dividing by T produces the stated bound. □

I Proof of Proposition 1

Constant-stepsize behaviour of momentum-free methods For context, we record an elementary observation about constant-stepsize behaviour. The instance is fixed in advance and bounded below; the result records what cannot be done at constant stepsize, and is *not* a claim that momentum is necessary in any stronger sense.

Proposition 1 (Constant-stepsize floor, no momentum)**.** *Fix $L > \mu > 0$. Let $G(x) = (L/2)\|x\|^2$, $H(x) = ((L-\mu)/2)\|x\|^2$, $r_1 = r_2 = 0$, so $F(x) = (\mu/2)\|x\|^2$ is μ-strongly convex with $F^* = 0$. Take stochastic gradients $\nabla h_\xi(x) = (L-\mu)x + \xi$, $\xi \sim \mathcal{N}(0, \sigma^2 I_d)$. For Algorithm 1 with $\alpha_t \equiv 1$ (no momentum), $\eta_t \equiv 1$, and constant batch b, the iterates satisfy*

$$\lim_{t\to\infty} \mathbb{E}\|\nabla F(x^t)\|^2 \;=\; \frac{\beta_*}{2-\beta_*}\,\frac{\sigma^2 d}{b}, \qquad \beta_* := \mu/L,$$

which is bounded away from zero. With T-dependent stepsize (Polyak/MVR or vanishing-stepsize SGD), the same instance can be driven to ϵ-stationarity.

The proposition makes clear that *at constant stepsize*, momentum-free methods have a noise floor that does not vanish; momentum (or a T-dependent stepsize, or vanishing-stepsize SGD) is needed to drive the gradient norm to zero. Combined with the separation lemma (Lemma 3), this gives a complete picture of why momentum is the natural ingredient in the constant-stepsize, constant-batch online regime that is our focus.

Proof. With $G(x) = (L/2)\|x\|^2$, $H(x) = ((L-\mu)/2)\|x\|^2$, $r_1 = r_2 = 0$, the inner subproblem in Algorithm 1 has closed form: minimizing $(L/2)\|x\|^2 - \langle g^t, x\rangle$ gives $\tilde{x}^{t+1} = g^t/L$. With $\alpha_t = 1$ and $\eta_t = 1$, $g^t = \hat{\nabla} H_{\mathcal{I}^t}(x^t) = (L-\mu)x^t + \bar{\xi}^t$ where $\bar{\xi}^t = (1/b)\sum_{i\in\mathcal{I}^t} \xi_i \sim \mathcal{N}(0, (\sigma^2/b)I_d)$. So $x^{t+1} = \tilde{x}^{t+1} = ((L-\mu)x^t + \bar{\xi}^t)/L = (1-\mu/L)\,x^t + \bar{\xi}^t/L$.

Stationarity in distribution gives, with $\rho^* := 1-\mu/L$ and $a^* := 1/L$: $\mathbb{E}\|x^{t+1}\|^2 = (\rho^*)^2\,\mathbb{E}\|x^t\|^2 + (a^*)^2\,\sigma^2 d/b$ (the cross term is zero by independence of $\bar{\xi}^t$ and x^t). So $\mathbb{E}\|x^t\|^2 \to (a^*)^2\sigma^2 d/(b(1-(\rho^*)^2))$. Now $1-(\rho^*)^2 = (1-\rho^*)(1+\rho^*) = (\mu/L)(2-\mu/L) = \beta_*(2-\beta_*)$ with $\beta_* = \mu/L$. So $\lim \mathbb{E}\|x^t\|^2 = \sigma^2 d/(bL^2\beta_*(2-\beta_*))$. Since $\nabla F(x) = \mu x$, $\lim \mathbb{E}\|\nabla F(x^t)\|^2 = \mu^2 \cdot \sigma^2 d/(bL^2\beta_*(2-\beta_*)) = \beta_*/(2-\beta_*) \cdot \sigma^2 d/b$. □