GENERALIZABILITY OF NEURAL NETWORKS MINIMIZING EMPIRICAL RISK BASED ON EXPRESSIVE ABILITY

Anonymous authors
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ABSTRACT

The primary objective of learning methods is generalization. Classic generalization bounds, which rely on VC-dimension or Rademacher complexity, are uniformly applicable to all networks in the hypothesis space. On the other hand, algorithm-dependent generalization bounds, like stability bounds, address more practical scenarios and provide generalization conditions for neural networks trained using SGD. However, these bounds often rely on strict assumptions, such as the NTK hypothesis or convexity of the empirical loss, which are typically not met by neural networks. Furthermore, uniform generalization bounds fail to explain the significant attribute that over-parameterized models in deep learning exhibit nice generalizability. In order to establish generalizability under less stringent assumptions, which can also account for the effective generalizability of over-parameterized models, this paper investigates the generalizability of neural networks that minimize empirical risk. A lower bound for accuracy is established based on the expressiveness of these networks, which indicates that with an adequate large number of training samples and network sizes, these networks can generalize effectively. Additionally, we provide a lower bound necessary for generalization, demonstrating that, for certain data distributions, the quantity of training data required to ensure generalization exceeds the network size needed to represent the corresponding data distribution. Finally, we provide theoretical insights into several phenomena in deep learning, including robust generalization, importance of over-parameterization networks, and effects of loss functions.

1 INTRODUCTION

Understanding the mechanisms behind the nice generalization ability of deep neural networks remains a fundamental challenge problem in deep learning theory. By generalization, it means that neural networks trained on finite data give high predict accuracy on unseen data. The generalization bound serves as a critical theoretical framework for evaluating the generalizability of learning algorithms. Let $\mathcal{F}$ be a network, L a loss function, and $\mathcal{D}$ the data distribution. For a hypothesis space $\mathbf{H}$ and any $\mathcal{F} \in \mathbf{H}$, with probability $1-\delta$ of dataset $\mathcal{D}_{tr}$ selected i.i.d. from $\mathcal{D}$, we have the classic generalization bound $|\mathbb{E}_{(x,y)\sim\mathcal{D}}[L(\mathcal{F}(x),y)] - \mathbb{E}_{(x,y)\in\mathcal{D}_{tr}}[L(\mathcal{F}(x),y)]| < \frac{2}{\rho}\text{Rad}_{\mathcal{D}_{tr}}(\mathbf{H}) + \sqrt{\frac{\ln 1/\delta}{2|\mathcal{D}_{tr}|}})$ (Mohri et al., 2018), where $\text{Rad}_{\mathcal{D}_{tr}}(\mathbf{H})$ is the Radermacher Complexity of $\mathbf{H}$ under dataset $\mathcal{D}_{tr}$ and ρ is a constant. If we take $L(\mathcal{F}(x),y) = \mathbb{I}(\widehat{\mathcal{F}}(x) = y)$ where $\widehat{\mathcal{F}}(x)$ is the classification result of $\mathcal{F}(x)$, then it becomes the accuracy of the network $\mathcal{F}$ over $\mathcal{D}$ or population accuracy. There exist similar generalization bounds using VC-dimension (Mohri et al., 2018).

The above-mentioned generalization bounds are satisfied for all networks in the hypothesis space. In practice, the generalizability of the networks trained by gradient descent is desirable. For that purpose, algorithmic-dependent generalization bounds are derived. It is shown that if the data satisfy the NTK condition, the two-layer networks have a small generalization risk after training (Jacot et al., 2018; Ji & Telgarsky, 2019). Stability generalization bounds are also obtained (Hardt et al., 2016; Wang & Ma, 2022) by assuming the convexity and Lipschitz properties of the loss function. Unfortunately, most of these algorithmic-dependent generalization bounds make strong and unrealistic assumptions about the training procedure. For example, the NTK condition is used to reduce the training to a convex optimization (Ji & Telgarsky, 2019) and strong smoothness and convexity of the empirical loss are used to measure the effect in each training epoch (Hardt et al., 2016). Moreover, uniform

generalization bounds fail to explain the significant attribute that over-parameterized models in deep learning exhibit nice generalizability (Belkin et al., 2019; Bartlett et al., 2021), because ****

In order to give generalization conditions under more relaxed assumptions, and to provide more specific conditions for generalization to account for the nice generalizability of over-parameterized models, we will study the generalization of networks that minimize the empirical risk, that is, the network $\mathcal{F} \in \mathbf{M} = \arg\min_{\mathcal{G}\in\mathbf{H}} \sum_{(x,y)\in\mathcal{D}_{tr}} L(\mathcal{G}(x), y)$. The approach is reasonable because most practical training will lead to a very small value of the empirical risk and the trained networks can be considered to minimize the empirical risk. So our main research objective is: **the generalization of the networks that minimize the empirical risk without requiring strong assumptions.**

In this paper, we consider two-layer networks, like many previous works (Ba et al., 2020; Luo & Yang, 2020; Ji & Telgarsky, 2019; Zeng & Lam, 2022). From the perspective of expressive ability, we show that when the number of training data and the size of network are large enough, the network has generalizability. We further found that the sample complexity and the size of the network depend only on the cost required for the network to express such a distribution. As shown below.

Theorem 1.1 (Informal, Section 4)**.** *Let distribution $\mathcal{D}$ satisfy the requirement that a two-layer network with width W can reach accuracy 1 in this distribution, then if $\mathcal{D}_{tr}$ is selected i.i.d. from the distribution and with more than $\Omega(W^2)$ elements, then with high probability, any network in $\mathbf{M}$ and with width more than $\Omega(W)$ has high accuracy.*

From this result, we can determine the exact amount of training data and the size of the network that can ensure generalizability. We also consider the lower bound required to have generalizability. For some data distribution, to ensure the generalizability of network which minimizes the empirical risk, the required number of data must be greater than the size of neural networks required to express such a distribution. As shown below.

Theorem 1.2 (Informal, Section 5)**.** *For some data distribution, if the width required for a two-layer network with* ReLU *activation function to express such distribution is at least W, then for a dataset with fewer than $O(W)$ elements, the network that minimizes the empirical risk for such dataset may have poor generalization.*

Finally, while networks that minimize the empirical risk exhibit good generalization, numerous classical experimental results indicate that these networks encounter several problems such as robust and so on. Therefore we provide some interpretability for these problems based on our theoretical results. Let $\mathcal{D}_{tr}$ be a dataset and $\mathcal{F} \in \mathbf{M}$. Then, three phenomena of deep learning are discussed with our theoretical results.

Robustness Generalization. (Section 6.1) It is shown that robust memorizing a dataset $\mathcal{D}_{tr}$ is more difficult than memorizing $\mathcal{D}_{tr}$ (Park et al., 2021; Li et al., 2022; Yu et al., 2024). We further show that when robust memorizing $\mathcal{D}_{tr}$ is much more difficult than $\mathcal{D}_{tr}$, then the robustness accuracy of $\mathcal{F}$ over $\mathcal{D}$ has an upper bound and may be low.

Importance of over-parameterization. (Section 6.2) It is recognized that over-parameterized networks have nice generalizaility (Belkin et al., 2019; Bartlett et al., 2021). In this regard, we show that when the network is large enough, a small empirical loss leads to high test accuracy. In contrast, when the network $\mathcal{F}$ is not large enough, there exist networks that achieve good generalization but cannot be found by minimizing the empirical risk.

Loss function. (Section 6.3) We show that for some loss function, generalization may not be achieved. If the loss function reached its minimum value or is a strictly decreasing concave function, $\mathcal{F}$ may have poor generalization.

2 Related Work

Generalization bound. Generalization bound is the central issue of learning theory and has been studied extensively (Valle-Pérez & A. Louis, 2022).

The algorithm-independent generalization bounds usually depend on the VC-dimension or the Rademacher complexity (Mohri et al., 2018). In Harvey et al. (2017); Bartlett et al. (2019); Yang et al. (2023), VC-dimension has been accurate calculated in terms of width, depth, and number of parameters, hence use it into generalization bounds. In Wei & Ma (2019); Arora et al. (2018); Li et al.

(2018), some tighter generalization bound of networks was given based on Radermacher complexity. Consider the network structure, the generalization bound can be further precise calculated,Long & Sedghi (2019); Ledent et al. (2021); Li et al. (2018) gave the generalization bound of CNN, Vardi et al. (2022) gave the sample complexity of small networks, Brutzkus & Globerson (2021) studied the generalization bound of maxpooling networks, Trauger & Tewari (2024); Li et al. (2023) gave the generalization bound of transformers; Ma et al. (2018); Luo & Yang (2020); Ba et al. (2020) studied the two-layer network in many situations. Under the assumption of the network, Neyshabur et al. (2017); Barron & Klusowski (2018); Dziugaite & Roy (2017); Bartlett et al. (2017); Valle-Pérez & A. Louis (2022) gave the upper bounds of the generalization error. By using the PAC methods, Alquier et al. (2024); Hellström et al. (2023) gave the generalization bound. Some negative conclusions Nagarajan & Kolter (2019) have been giving for this type of generalization bound.

Algorithm-dependent generalization bounds were established in the algorithmic stability setting, and we can measure generalizability through the stability of algorithms (Bousquet & Elisseeff, 2002; Elisseeff et al., 2005; Shalev-Shwartz et al., 2010). For gradient descent of network, under some assumptions, Hardt et al. (2016); Wang & Ma (2022); Kuzborskij & Lampert (2018); Lei (2023); Bassily et al. (2020) gave the stability bound of networks, for small networks such as two-layer network, under some assumptions, Ji & Telgarsky (2019); Taheri & Thrampoulidis (2024); Li et al. (2020) gave the generalization of networks. For some other training methods, there were also some analysis of stability and generalization, as in Farnia & Ozdaglar (2021); Xing et al. (2021); Xiao et al. (2022); Wang et al. (2024); Allen-Zhu & Li (2022), the adversarial training was considered; in Regatti et al. (2019); Sun et al. (2023), the Asynchronous SGD was considered. However, these previous algorithmic-dependent generalization bounds always impose strong assumptions on the training process or dataset.

Neural Network Interpretability. Interpretability is dedicated to providing reasonable explanations for phenomena that occur in neural networks. As said in Zhang et al. (2021): Interpretability is not always needed, but it is important for some prediction systems that are required to be highly reliable because an error may cause catastrophic results. For adversarial samples, it was shown that for certain data distributions and network, there must be trade off between accuracy and adversarial accuracy Shafahi et al. (2019); Bastounis et al. (2021). In (Yu et al., 2023), it was proven that a small perturbation of the network parameters will lead to low robustness. In (Allen-Zhu & Li, 2022), it was shown that the generation of adversarial samples after training is due to dense mixtures in the hidden weights. In (Yu et al., 2024; Li et al., 2022), it was shown that ensuring generalization requires more parameters. For overfitting, it was shown that long term training can lead to a decrease in generalization (Xiao et al., 2022; Xing et al., 2021). In (Roelofs et al., 2019), comprehensive analysis of overfitting is given. In (Belkin et al., 2019; Bartlett et al., 2021), the importance of over-parameterized interpolation networks are talking, and in Arora et al. (2019); Cao & Gu (2019); Ji & Telgarsky (2020), the training and generalization of DNNs in the over-parameterized regime were studied. In this paper, we explain these phenomena from the perspective of the expressive ability of networks.

3 NOTATION

In this paper, for any $A \in \mathbb{R}$, $O(A)$ means the value not more than cA for some $c > 0$, $\Omega(A)$ means the value not less than cA for some $c > 0$.

3.1 NEURAL NETWORK

In this paper, we consider two-layer neural network $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}$ which can be writen as:

$$F(x) = \sum_{i=1}^{W} a_i \sigma(W_i x + b_i) + c,$$

where σ is the activation function, $W_i \in \mathbb{R}^{1\times n}$ is the transition matrix, $b_i \in \mathbb{R}$ is the bias part, W is the width of the network, and $a_i, c \in \mathbb{R}$. Denote $\mathbf{H}_W^{\sigma}(n)$ to be the set of all two layer neural networks with input dimension n, width W, activation function σ, and all parameters are in $[-1, 1]$. To simplify the notation, we denote $\mathbf{H}_W^{\mathrm{ReLU}}(n)$ by $\mathbf{H}_W(n)$ when using the ReLU activation function.

3.2 DATA DISTRIBUTION

In this paper, we consider binary classification problems. To avoid extreme cases, we focus primarily on the distribution defined below. And in this article, we will mainly focus on this kind of distribution.

Definition 3.1. For $n \in \mathbb{Z}_+$, $\mathcal{D}(n)$ is the set of distributions $\mathcal{D} \in [0,1]^n \times \{-1,1\}$ which have a *positive separation bound:* $\inf_{(x_1,y_1),(x_2,y_2)\sim\mathcal{D} \text{ and } y_1\neq y_2} ||x_1 - x_2||_2 > 0$.

The accuracy of a network $\mathcal{F}$ on a distribution $\mathcal{D}$ is defined as

$$A_{\mathcal{D}}(\mathcal{F}) = \mathbb{P}_{(x,y)\sim\mathcal{D}}(\text{Sgn}(\mathcal{F}(x)) = y),$$

where Sgn is the sign function. We use $\mathcal{D}_{\text{tr}} \sim \mathcal{D}^N$ to mean that $\mathcal{D}_{\text{tr}}$ is a dataset of N samples drawn i.i.d. according to $\mathcal{D}$.

3.3 MINIMUM EMPIRICAL RISK

Consider the loss function $L(\mathcal{F}(x), y) = \ln(1 + e^{-\mathcal{F}(x)y})$, which is the crossentropy of binary classification problem. For a dataset $\mathcal{D}_{tr} \subset [0,1]^n \times \{-1,1\}$ and a hypothesis space $\mathbf{H}_W^\sigma(n)$. To learn the features of the data in D_{tr}, a traditional method is empirical risk minimization (ERM), which minimizes the value of loss function on dataset $\sum_{(x,y)\in\mathcal{D}_{tr}} L(\mathcal{F}(x), y)$ of the network $\mathcal{F}$.

Under this motivation, in this paper, we mainly consider the networks $\mathcal{F} \in \mathbf{H}_W^\sigma(n)$ that can minimize the empirical risk, that is, networks in

$$\mathbf{M}_W^\sigma(\mathcal{D}_{tr}, n) = \underset{\mathcal{G}\in\mathbf{H}_W^\sigma(n)}{\text{argmin}} \sum_{(x,y)\in\mathcal{D}_{tr}} L(\mathcal{G}(x), y). \tag{1}$$

It should be noted that such networks exist in most cases, as shown below.

Proposition 3.2. *Let $\mathcal{D}_{tr} \subset [0,1]^n \times \{-1,1\}$ and σ be a continuous function. Then for any $W \in \mathbb{Z}^+$, there must exist a $\mathcal{F} \in \mathbf{H}_W^\sigma(n)$ such that $\mathcal{F} \in \mathbf{M}_W^\sigma(\mathcal{D}_{tr}, n)$.*

Proof. Consider the empirical risk as a function of network parameters. Let a_i, b_i, c, W_i be the parameters of $\mathcal{F}$. Then empirical risk $\sum_{(x,y)\in\mathcal{D}_{tr}} L(\mathcal{F}(x), y) = \sum_{(x,y)\in\mathcal{D}_{tr}} \ln e^{-y(\sum_{i=1}^W a_i\sigma(W_i x + b_i) + c)}$. Since σ is a continuous function, $\sum_{(x,y)\in\mathcal{D}_{tr}} L(\mathcal{F}(x), y)$ is a continuous function about a_i, b_i, c and W_i, and the domain of definition of parameters is $[-1,1]^{W_g}$, where $W_g = W(n+2)+1$ is the number of parameters of $\mathcal{F}$. The proposition now comes from the fact that continuous functions have reachable upper and lower bounds on closed domain. □

4 GENERALIZABILITY BASED ON NEURAL NETWORK EXPRESSIVE ABILITY

In this section, we demonstrate that based on the expressive ability of neural networks, the generalization of network which minimizes the empirical risk can be estimated. Specifically, in Section 4.1, we establish the relationship between expressive ability and generalizability. In Section 4.2, we extend our conclusion to local minima. In Section 4.3, we compare our generalization bounds with previous algorithm-independent and algorithm-dependent bounds, showcasing the superiority of our bound.

4.1 A LOWER BOUND FOR ACCURACY BASED ON THE EXPRESSIVE ABILITY

We first define the expressive ability of neural networks to classify data distribution.

Definition 4.1. We say that a distribution $\mathcal{D}$ over $[0,1]^n \times \{-1,1\}$ can be **expressed** by $\mathbf{H}_W^\sigma$ with confidence c, if there exists an $\mathcal{F} \in \mathbf{H}_W^\sigma$ such that

$$\mathbb{P}_{(x,y)\sim\mathcal{D}}(y\mathcal{F}(x) \geq c) = 1.$$

For any distribution $\mathcal{D} \in \mathcal{D}(n)$, we can always find some activation function σ, such that $\mathcal{D}$ can be expressed by $\mathbf{H}_W^\sigma(n)$ with confidence c for some W and c. Therefore, this definition is reasonable. For example, if $\sigma = \text{ReLU}$, according to the universal approximation theorem of neural networks (Cybenko, 1989), any $\mathcal{D} \in \mathcal{D}(n)$ can be represented by a network with ReLU as activation function, as shown by the following proposition. The proof is given in Appendix A.

Proposition 4.2. *For any distribution $\mathcal{D} \in \mathcal{D}(n)$, there exist $W \in \mathbb{N}_+$ and $c > 0$ such that $\mathcal{D}$ can be expressed by $\mathbf{H}_W(n)$ with the confidence c.*

With such a definition, we have the following relationship between expressive ability and generalization ability. The proof is given in Appendix B.

Theorem 4.3. *Let σ be a continuous function with Lipschitz constant L_p, $W_0 \geq 2, n \in \mathbb{N}_+, c \in \mathbb{R}_+$. If $\mathcal{D} \in \mathcal{D}(n)$ can be expressed by $\mathbf{H}^{\sigma}_{W_0}$ with the confidence c, then for any $W \geq W_0 + 1$, $N \in \mathbb{N}_+$, $\delta \in (0, 1)$, with probability at least $1 - \delta$ of $\mathcal{D}_{tr} \sim \mathcal{D}^N$, the following bound stands for any $\mathcal{F} \in \mathbf{M}^{\sigma}_W(\mathcal{D}_{tr}, n)$:*

$$A_{\mathcal{D}}(\mathcal{F}) \geq 1 - O(\frac{W_0}{cW} + \frac{nL_p(W_0 + c)\sqrt{\log(4n)}}{c\sqrt{N}} + \sqrt{\frac{\ln(2/\delta)}{N}}).$$

Proof Idea. There are two main steps in the proof. The first step tries to estimate the minimum value of the empirical risk, which mainly uses the assumption: $\mathcal{D}$ can be expressed by $\mathbf{H}^{\sigma}_{W_0}$ with confidence c. The minimum value is based on W_0, c, W. Then, use such the minimum value to estimate the performance of the network on the dataset. In the second step, we can use the result in the first step and the classic generalization bound to estimate the performance of the network across the entire distribution and get the result. The core idea of this step is that the minimum value of empirical risk does not depend on N, but the Radermacher complexity will reduce when increasing N, so when N is large enough, the performance of networks in distribution and datasets is similar.

This result shows that increasing N and W leads to a better test accuracy. It is reasonable that more data make better generalization and a larger network makes better generalization, which also confirms the observation about the nice generalization ability of over-parameterized networks. This differs from classical algorithm-independent generalization bounds, which lack this advantageous property. Since the values of N and W to ensure generalization are only influenced by the size required for the network to express the data distribution, we can infer the following corollary.

Corollary 4.4. *With probability $1 - \delta$ of $\mathcal{D}_{tr} \sim \mathcal{D}^N$, it holds $A_{\mathcal{D}}(\mathcal{F}) \geq 1 - \epsilon$ for any $\mathcal{F} \in \mathbf{M}^{\sigma}_W(\mathcal{D}_{tr}, n)$, when $W \geq \Omega(W_0/(c\epsilon))$ and $N \geq \Omega(\frac{L_p(W_0+c)n\sqrt{\log(4n)}}{c\epsilon})^2 + \Omega(\frac{\ln(2/\delta)}{\epsilon^2})$.*

The above bounds of N and W depend only on constants about expressive ability W_0, c, Lipschitz constant L_p and ϵ, δ, which shows that as long as there are enough samples and enough large network size based on the expressive ability, the neural network that minimizes the empirical risk will have generalization ability.

Remark 4.5. For deep networks, we can show that if the depth and width of the network and the number of data exceed a distribution-dependent threshold, then with high probability, the network minimizing the empirical risk can ensure generalization, as demonstrated in Appendix K. However, due to the complexity of deep networks, accurately determining the required depth, width, and data volume remains a challenge.

4.2 Generalization For Local Optimal Point

In practice, it is often challenging to accurately find the network parameters that minimize the empirical risk, but instead parameters are find which locally minimize the empirical risk. In this section, we show that for networks with such parameters, if the value of the empirical risk is small, its generalization can also be guaranteed. We define such a set of networks.

Definition 4.6. For any $q \geq 1$ and dataset $\mathcal{D}_{tr}$, we say $\mathcal{F} \in \mathbf{H}^{\sigma}_W(n)$ is a q-approximation of minimize empirical risk if

$$\sum_{(x,y)\in\mathcal{D}_{tr}} L(\mathcal{F}(x), y) \leq q \min_{f\in\mathbf{H}^{\sigma}_W(n)} \sum_{(x,y)\in\mathcal{D}_{tr}} L(f(x), y).$$

For all q-approximation networks, we have the following result. The proof is in Appendix C.

Proposition 4.7. *Let σ be a continuous function with Lipschitz constant L_p, $W_0 \geq 2, n \in \mathbb{N}_+, c \in \mathbb{R}_+$. If $\mathcal{D} \in \mathcal{D}(n)$ can be expressed by $\mathbf{H}^{\sigma}_{W_0}$ with confidence c, then for any $W \geq W_0 + 1$, $N \in \mathbb{N}_+$,*

$q \geq 1$ and $\delta \in (0,1)$, with probability at least $1-\delta$ of $\mathcal{D}_{tr} \sim \mathcal{D}^N$, we have

$$A_{\mathcal{D}}(\mathcal{F}) \geq 1 - O(\frac{qW_0}{cW} + \frac{nL_p(W_0+c)\sqrt{\log(4n)}}{c\sqrt{N}} + \sqrt{\frac{\ln(2/\delta)}{N}}),$$

for any q-approximation $\mathcal{F} \in \mathbf{H}_W^{\sigma}(n)$ to minimize the empirical risk.

The theorem demonstrates that if a local optimal point is a q-approximation to minimize the empirical risk, then we can obtain similar conclusions as Theorem 4.3.

4.3 Comparison with classical conclusions

In this section, we compare our generalization bounds with previous ones. Compared to algorithm-independent generalization bounds, our bound performs better when the data size is not significantly larger than the network size. Compared to algorithm-dependent generalization bounds, our bound does not require overly strong assumptions as prerequisites.

Compare with the algorithm-independent generalization bound. When the scale of the network is bounded, a general generalization bound can be calculated by the VC dimension.

Theorem 4.8 (P.217 of (Mohri et al., 2018), Informal)**.** *Let $\mathcal{D}_{\text{tr}} \sim \mathcal{D}^N$ be the training set. For the hypothesis space $\mathbf{H} = \{\text{Sgn}(\mathcal{F}(x)) \,\|\, \mathcal{F}(x) : \mathbb{R}^n \to \mathbb{R}\}$ and $\delta \in \mathbb{R}_+$, with probability at least $1-\delta$, for any $\text{Sgn}(\mathcal{F}(x)) \in \mathbf{H}$, we have*

$$|A_{\mathcal{D}}(\text{Sgn}(\mathcal{F})) - \mathbb{E}_{(x,y)\in\mathcal{D}_{\text{tr}}}[I(\text{Sgn}(\mathcal{F}(x)) = y)]| \leq O(\sqrt{\frac{\text{VC}(\mathbf{H})+\ln(1/\delta)}{N}}) \tag{2}$$

Theorem 4.3 provides the number of data and the size of the network that is required to ensure generalization. These bounds mainly depend on the distribution, but not on the hypothesis space. Theorem 4.8 demonstrates the relationship between the number of data and the size of the network to ensure generalization, which points out that when the number of data is much more than the VC-dimension of the network hypothesis space, generalization can be ensured. Considering that the VC-dimension can be calculated by the size of the network (Bartlett et al., 2019), Theorem 4.8 means that to ensure generalization, the number of data must be greater than the size of the network, this is contradictory to over-parameterized. So, when the data volume is not significantly larger than the network size, Theorem 4.3 demonstrates superior performance.

Compare with the algorithm-dependent generalization bound. In the study of algorithm-dependent generalization bound, some works derive generalization bounds based on gradient descent and corresponding strong assumptions, which we do not utilize.

Theorem 4.9 (Ji & Telgarsky (2019))**.** *Let $\epsilon \in (0,1)$, $\delta \in (0,1/4)$ and distribution $\mathcal{D}$ over $[0,1]^n$ satisfy the NTK conditions with constant γ, and λ and M defined as*

$$\lambda = \frac{\sqrt{2\ln(4n/\delta)} + \ln(4/\epsilon)}{\gamma/4}, M = \frac{4096\lambda^2}{\gamma^6},$$

If the two-layer network with width $W > M$ and training step $\eta \leq 1$, with probability $1-4\delta$ of $\mathcal{D}_{tr} \sim \mathcal{D}^N$ and training initiation point, after at most $\frac{2\lambda^2}{\eta\epsilon}$ times gradient descent on $\mathcal{D}_{tr}$, for the trained network $\mathcal{F}$, it holds

$$A_{\mathcal{D}}(\mathcal{F}) \geq 1 - 2\epsilon - 16\frac{\sqrt{2\ln(4N/\delta)} + \ln(4/\epsilon)}{\gamma^2\sqrt{N}} - 6\sqrt{\frac{\ln(2/\delta)}{N}}.$$

Theorem 4.3 only requires that the distribution with positive separation can be fitted by the network, and it stands for any distribution $\mathcal{D} \in \mathcal{D}(n)$ as mentioned in Proposition 4.2. But Theorem 4.9 requires NTK conditions for distribution. These conditions can make the training approach to a convex optimization, which is an overly strong condition.

5 Lower bound for sample complexity based on expressive ability

In this section, on the other hand, we consider the lower bound of data complexity necessary for generalization.

5.1 UPPER BOUND FOR ACCURACY WITHOUT ENOUGH DATA

This section illustrates that in the worst-case scenario, the minimum number of data points needed to guarantee accuracy is constrained by the VC-dimension of the smallest hypothesis space necessary to represent a distribution. We give a definition first.

Definition 5.1. For a hypothesis space $\mathbf{H} \subset \mathbb{R}^n \to \mathbb{R}$, $\text{VC}(\mathbf{H})$ is the maximum number of data in $[0,1]^n$ that $\mathbf{H}$ can shatter. Precisely, there exist $\text{VC}(\mathbf{H})$ samples $\{x_i\}_{i=1}^{\text{VC}(\mathbf{H})} \subset [0,1]^n$, such that for any $\{y_i\}_{i=1}^{\text{VC}(\mathbf{H})} \in \{-1,1\}$, there is an $\mathcal{F} \in H$ such that $\text{Sgn}(\mathcal{F}(x_i)) = y_i$ for all $i \in [\text{VC}(\mathbf{H})]$. But there do not exist $\text{VC}(\mathbf{H})+1$ such samples.

We have the following theorem. The proof is in Appendix D.

Theorem 5.2. *For any $n, W, W_0 \in \mathbb{N}_+$ and activation function σ, there is a $\mathcal{D} \in \mathcal{D}(n)$ such that*

(1) There is an $\mathcal{F} \in \mathbf{H}^{\sigma}_{W_0}(n)$ such that $A_{\mathcal{D}}(\mathcal{F}) = 1$;

(2) For any given $\epsilon, \delta \in (0,1)$, if $N \leq \text{VC}(\mathbf{H}^{\sigma}_{W_0}(n))(1-4\epsilon-\delta)$, then with probability $1-\delta$ of $\mathcal{D}_{tr} \sim \mathcal{D}^N$, we have $A_{\mathcal{D}}(\mathcal{F}) < 1-\epsilon$ for some $\mathcal{F} \in \mathbf{M}^{\sigma}_{W}(\mathcal{D}_{tr}, n)$.

This conclusion indicates that for distributions that require networks with width W_0 to express, some of them requires at least $\Omega(\text{VC}(\mathbf{H}^{\sigma}_{W_0}(n)))$ data to ensure generalization. It is worth mentioning that this conclusion is true for any given W in the theorem. It is easy to see that a larger W_0 makes $\text{VC}(\mathbf{H}^{\sigma}_{W_0}(n))$ larger, so as the cost of expression increases, generalization becomes difficult. However, it is difficult to accurately calculate $\text{VC}(\mathbf{H}^{\sigma}_{W_0}(n))$ for some given σ. If we focus on ReLU networks, by the result in (Bartlett et al., 2019), we have

Corollary 5.3. *For any given $n, W, W_0 \in \mathbb{N}_+$, there is a $\mathcal{D} \in \mathcal{D}(n)$ such that:*

(1) There is an $\mathcal{F} \in \mathbf{H}_{W_0}(n)$ such that $A_{\mathcal{D}}(\mathcal{F}) = 1$;

(2) For any given $\epsilon, \delta \in (0,1)$, if $N \leq O(nW_0(1-4\epsilon-\delta))$, then for all $\mathcal{D}_{tr} \sim \mathcal{D}^N$, it holds $A_{\mathcal{D}}(\mathcal{F}) < 1-\epsilon$ for some $\mathcal{F} \in \mathbf{M}^{\sigma}_{W}(\mathcal{D}_{tr}, n)$.

Besides, for any distribution, we can show that if the parameters required to express a distribution tend to infinity, the required number of data to ensure the generalization for such distribution must also tend infinity. As shown in the following theorem. The proof is in Appendix E.

Theorem 5.4. *Suppose $\mathcal{D} \in \mathcal{D}(n)$, $W_0 \geq 2^{n+1}$, and $A_{\mathcal{D}}(\mathcal{F}) \leq 1-\epsilon$ for any ϵ and $\mathcal{F} \in \mathbf{H}_{W_0}(n)$. If $N \leq W_0^{\frac{1}{n+1}}(n+1)/e$, then for any $\mathcal{D}_{tr} \sim \mathcal{D}^N$ and $W \in \mathbb{N}_+$, there is an $\mathcal{F} \in \mathbf{M}_{W}(\mathcal{D}_{tr}, n)$ such that $A_{\mathcal{D}}(\mathcal{F}) \leq 1-\epsilon$.*

However, since the Theorem 5.4 is correct for all distributions and dataset, it can only provide a relatively loose bound. If the distribution is given, we can calculate the relationship between the minimum number of data required and the minimum required number of parameters to fit it, as shown in the following section.

5.2 APPROPRIATE NETWORK MODEL HELPS WITH GENERALIZATION

As mentioned in the above sections, expressive ability and generalization ability are closely related. Section 4.1 demonstrates that simpler expressions facilitate generalization; Section 5 reveals that, in the worst-case scenario, the amount of data required to guarantee generalization approximates the VC-dim of the hypothesis space that can express the distribution.

Therefore, for a given distribution, selecting an appropriate network model which can fit the distribution easily may help to facilitate better expression with fewer data and network size, ultimately leading to improved generalization. In this paper, focusing on two-layer networks, we illustrate that selecting an appropriate activation function for the neural network according to the target distribution enhances generalization.

To better explain this conclusion, let us examine the following distribution.

Definition 5.5. Let $\mathcal{D}_n \in [0,1]^n \times \{-1,1\}$ be a distribution defined as: $\mathcal{D}_n$ is defined on $\{(\frac{i}{n}\mathbf{1}, \mathbb{I}(i))\}_{i=1}^n$ where $\mathbf{1}$ is the all one weight in $\mathbb{R}^n$, $\mathbb{I}(x) = 1$ if x is odd and $\mathbb{I}(x) = -1$ if x is even, annd the probability of each point is the same.

ReLU networks need $\Omega(n)$ widths to express this distribution and require $\Omega(n)$ data to ensure generalization. The proof is given in Appendix F.

Proposition 5.6. *(1) For any n, $A_{\mathcal{D}_n}(\mathcal{F}) < 1$ for any $\mathcal{F} \in \mathbf{H}_W(n)$ when $W < n/2$;*

(2) If $N \leq \delta n$ where $\delta \in (0,1)$, then for all $\mathcal{D}_{tr} \sim \mathcal{D}_n^N$ and $W \in \mathbb{N}_+$, it holds $A_{\mathcal{D}}(\mathcal{F}) \leq 0.5 + 2\delta$ for some $\mathcal{F} \in \mathbf{M}_W(\mathcal{D}_{tr}, n)$.

But if we use the activation function $\sigma(x) = \sin(\pi x)$, the networks only need $O(1)$ widths to express such a distribution and require fewer data to ensure generalization. The proof is given in Appendix G.

Proposition 5.7. *(1) For any n, $\mathcal{D}_n$ can be expressed by $\mathbf{H}_1^\sigma(n)$ with confidence 1;*

(2) For any $W \geq 2, n > 2$, $\delta \in (0,1)$ and $N \geq 4\frac{\ln(\delta/2)}{\ln(0.5+1/n)}$, with probability $1-\delta$ of $\mathcal{D}_{tr} \sim \mathcal{D}_n^N$, it holds $A_{\mathcal{D}}(\mathcal{F}) = 1$ for all $\mathcal{F} \in \mathbf{M}_W^\sigma(\mathcal{D}_{tr}, n)$.

As shown in the above example, using $\sigma(x) = \sin(\pi x)$ as activation function only requires $O(\ln(\delta/2))$ samples and $O(1)$ widths to ensure generalization, but ReLU networks require at least $\Omega(n)$ samples and widths to ensure generalization. This demonstrates the crucial role of selecting the appropriate network model.

Remark 5.8. It is worth mentioning that for some very simple distributions like the Bernoulli distribution, the performance of various activation functions is similar, so we cannot provide a general conclusion for any distribution.

6 Explanation of some Phenomena in Deep Neural Network

Although networks minimizing empirical risk are good for generalization, many classic experimental results have shown that the networks still have problems. In this section, we will provide explanations for some classic experimental results based on our theoretical results.

6.1 Why do general networks lack robustness?

Experiments show that using ERM to train a network can easily lead to low robustness accuracy (Szegedy, 2013). In this section, we provide some explanations for this fact.

The *robustness accuracy* of network $\mathcal{F}$ under distribution $\mathcal{D}$ and robust radius ϵ is defined as

$$\mathrm{Rob}_{\mathcal{D},\epsilon}(\mathcal{F}) = \mathbb{P}_{(x,y)\sim\mathcal{D}}(\mathbb{I}(\widehat{\mathcal{F}}(x') = y), \forall x' \in \mathbb{B}(x,\epsilon) \cap [0,1]^n).$$

The robustness accuracy requires not only correctness on the samples but also correctness within a neighborhood of the sample. We introduce a notation.

Definition 6.1. For a dataset $\mathcal{D} = \{(x_i, y_i)\}_{i=1}^N$ and an $\epsilon > 0$, define

$$R(\mathcal{D}, \epsilon) = \{\mathcal{D}_r \parallel \mathcal{D}_r = \mathcal{D} \cup \{(x_i + \epsilon_i, y_i)\}_{i=1}^N, \text{ for some } ||\epsilon_i|| \leq \epsilon\}\}.$$

It is easy to see that $R(\mathcal{D}, \epsilon)$ contains all the sets formed by adding a perturbation with budget ϵ to $\mathcal{D}$. In the above section, we mainly discussed the network expression ability in distribution. On the other hand, there are also some studies on the network expression ability on dataset such as memorization. Moreover, previous studies (Park et al., 2021; Li et al., 2022; Yu et al., 2024) have shown that robustly memorizing a dataset may be much more difficult than memorizing a dataset. So, for a given hypothesis space $\mathbf{H}$ that can express a normal data set well, it may not be able to express the dataset after disturbance. In this case, in order to minimize the empirical risk, the network will prioritize simple features that are easy to fit, but will ignore the complex robust features, which leads to low robustness. As shown in the following theorem. The proof is given in Appendix H.

Theorem 6.2. *Let $\mathcal{D} \in \mathcal{D}(n)$ and L_p be the Lipschitz constant of activation function σ. If $N_0, W_0 \in \mathbb{N}_+$ and $\epsilon, \delta, c_0, c_1 > 0$ satisfy that with probability $1-\delta$ of $\mathcal{D}_{tr} \sim \mathcal{D}^{N_0}$, we have*

(1) there exists an $\mathcal{F} \in \mathbf{H}_{W_0}^\sigma(n)$ such that $y\mathcal{F}(x) \geq c_0$ for all $(x,y) \in \mathcal{D}_{tr}$;

(2) there exists a $\mathcal{D}_r \in R(\mathcal{D}_{tr}, \epsilon)$, such that $\sum_{(x,y)\in\mathcal{D}_r} \frac{y\mathcal{F}(x)}{|\mathcal{D}_r|} \leq c_1$ for any $\mathcal{F} \in \mathbf{H}_{W_0}^\sigma(n)$.

Then, for any $W \geq W_0 + 1$, with probability $1 - O(\delta)$ of $\mathcal{D}_{tr} \sim \mathcal{D}^{N_0}$ and $\mathcal{F} \in \mathbf{M}_W^{\sigma}(\mathcal{D}_{tr}, n)$, there are $\mathrm{Rob}_{\mathcal{D},\epsilon}(\mathcal{F}) \leq 1 - \Omega(\frac{c_0 - 2c_1}{L_p W_0 n} - \frac{c_1}{L_p W_0 n}(\frac{W_0}{W} + \frac{1}{W_0}) - \sqrt{\frac{\ln(n/\delta)}{N_0}})$.

This theorem states that if the dataset after adding perturbations becomes more difficult to fit, the network may have a low robustness generalization. Please note that although the conclusion is directly unrelated to ϵ, because c_1 is related to ϵ, ϵ also affects the conclusion.

Remark 6.3. Conditions (1) and (2) required in the theorem are reasonable. It is obvious that as ϵ increases, c_1 will decrease, and when ϵ is large enough, we have $c_0 >> c_1 \approx 0$. Hence, in some situation, a small ϵ is also enough to make $c_0 >> c_1$, such as the example given in the proof of Theorem 4.3 in (Li et al., 2022).

6.2 Importance of over-parameterized networks

In the above section, we mainly consider $\mathcal{F} \in \mathbf{M}_W(\mathcal{D}_{tr}, n)$. But what we really need is $\mathcal{F} \in \underset{\mathcal{G} \in \mathbf{H}_W(n)}{\operatorname{argmax}} A_{\mathcal{D}}(\mathcal{G})$. By Theorem 4.3, it is easy to show that when number of data and size of network are large enough, generalization of $\mathcal{F} \in \mathbf{M}_W(\mathcal{D}_{tr}, n)$ and $\mathcal{F} \in \underset{f \in \mathbf{H}_W(n)}{\operatorname{argmax}} A_{\mathcal{D}}(f)$ are close, as shown below.

Corollary 6.4. *Following Theorem 4.3, for all $\mathcal{F}_1 \in \mathbf{M}_W(\mathcal{D}_{tr}, n)$ and $\mathcal{F}_2 \in \underset{f \in \mathbf{H}_W(n)}{\operatorname{argmax}} A_{\mathcal{D}}(f)$, we have $A_{\mathcal{D}}(\mathcal{F}_2) - A_{\mathcal{D}}(\mathcal{F}_1) \leq O(\frac{W_0}{cW} + \frac{nL_p(W_0+c)\sqrt{\log(4n)}}{c\sqrt{N}} + \sqrt{\frac{\ln(2/\delta)}{N}})$.*

Proof. Since $1 \geq A_{\mathcal{D}}(\mathcal{F}_2) \geq A_{\mathcal{D}}(\mathcal{F}_1)$, we have $A_{\mathcal{D}}(\mathcal{F}_2) - A_{\mathcal{D}}(\mathcal{F}_1) \leq 1 - A_{\mathcal{D}}(\mathcal{F}_1)$, and by Theorem 4.3, we obtain the result. □

The above corollary shows that if the size of the network is large enough, the gap will be small. In the following, we point out that for some distribution $\mathcal{D}$, if the size of network is too small, even with enough data, it may lead to a large gap of $A_{\mathcal{D}}(\mathcal{F}_2) - A_{\mathcal{D}}(\mathcal{F}_1)$. This emphasizes the importance of over-parameterization, as shown in the following. The proof is given in the Appendix I.

Proposition 6.5. *For some distribution $\mathcal{D} \in \mathcal{D}(n)$, there is a $W_0 > 0$, such that*

(1) There exists an $\mathcal{F} \in \mathbf{H}_{W_0}(n)$ such that $A_{\mathcal{D}}(\mathcal{F}) \geq 0.99$;

(2) For any $\delta > 0$, if $N \geq \Omega(n^2 \ln(n/\delta))$, with probability $1 - O(\delta)$ of $\mathcal{D}_{tr} \sim \mathcal{D}^N$, we have $A_{\mathcal{D}}(\mathcal{F}) \leq 0.6$ for all $\mathcal{F} \in \mathbf{M}_{W_0}(\mathcal{D}_{tr}, n)$.

Remark 6.6. 0.99 can be changed to any real number in $(0, 1)$ and 0.6 can be changed to any real number in $(0.5, 1)$, and the result is still correct.

It is obvious that according to Corollary 6.4, a large width does not make (2) in Proposition 6.5 true. So, the above conclusion indicates that for some distributions, when the network structure is not large enough, even if there are some networks that have good generalization, they cannot be found by minimizing the empirical loss. The distribution we are mainly considering here is the distribution with some special outliers. In order to fit these special outliers, the small network may have to reduce some generalization. It is worth mentioning that this is not true for all distributions, so we cannot draw conclusions for all distributions.

6.3 The impact of Loss Function

In order to ensure generalizability of the network after empirical risk minimization, it is necessary to choose an appropriate loss function because minimizing some types of loss function is not good for generalization. In the previous sections, we mainly discussed the crossentropy loss function. In this section, we point out that not all loss functions can reach conclusions similar to Theorem 4.3.

Definition 6.7. We say that the loss function $L_b : \mathbb{R}^2 \to \mathbb{R}$ is bad if (1) or (2) is valid.

(1) There exist $x_{-1}, x_1 \in \mathbb{R}$ such that $L_b(x_{-1}, -1) = \min_{x \in \mathbb{R}} L_b(x, -1)$ and $L_b(x_1, 1) = \min_{x \in \mathbb{R}} L_b(x, 1)$.

(2) $L_b(\mathcal{F}(x), y) = \phi(y\mathcal{F}(x))$, where ϕ is a strictly decreasing concave function.

Condition (1) in the definition means that the loss function can reach its minimum value and condition (2) means that the loss function is a concave function. For example, some common loss functions like the MSE loss function $L_{\text{MSE}}(\mathcal{F}(x), y) = ||\mathcal{F}(x) - y||_2$, or $L_q(\mathcal{F}(x), y) = -y\mathcal{F}(x)$ are all bad loss functions.

For such bad loss functions, we have:

Theorem 6.8. *For any n and bad loss function L_b, there is a distribution $\mathcal{D} \in \mathcal{D}(n)$, such that for any $N \geq 0$, there is a $W_0 \geq 0$, such that if $W \geq W_0$, then with probability 0.99 of $\mathcal{D}_{tr} \sim \mathcal{D}^N$, we have $A_{\mathcal{D}}(\mathcal{F}) \leq 0.5$ for some $\mathcal{F} \in \arg\min_{\mathcal{G} \in \mathbf{H}_W(n)} \sum_{(x,y)\in\mathcal{D}_{tr}} L_b(\mathcal{G}(x), y)$.*

This theorem means that to ensure generalizability, it is important to choose the appropriate loss function. The proof is given in the Appendix J.

7 Conclusion

In this paper, we mainly give a lower bound for the accuracy of the neural networks that minimize the empirical risk, which implies that as long as there exist enough training data and the network is large enough, generalization can be achieved. The data and network sizes required only depend on the size required for the network to represent the target data distribution. Furthermore, we show that if the scale required for the network to represent a data distribution increases, the amount of data required to achieve generalization on that distribution will also inevitably increase. Finally, the results are used to explain some phenomena that occur in deep learning.

Limitation and future work. Although considering 2 layer networks is quite common in theoretical analysis of deep learning, it is still desirable to extend the result to deep neural networks. Preliminary results for deep neural networks are given in Appendix K, which need to be further studied. A more accurate estimate of the cost required to represent a given data distribution is needed to guide data selection.

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A PROOF OF PROPOSITION 4.2

A function σ is sigmoidal if $\text{limit}_{x\to-\infty}\sigma(x) = 0$ and $\text{limit}_{x\to\infty}\sigma(x) = 1$. Then, we have

Theorem A.1 (Theorem 1 in Cybenko (1989))**.** *For any continuous sigmoidal activation function σ, $\epsilon \in (0,1)$ and continuous function $f : [0,1]^n \to \mathbb{R}$, there exist $W \geq 0$ and $F \in \mathbf{H}^{\sigma}_{W}(n)$ such that $|f(x) - F(x)| \leq \epsilon$.*

We prove Proposition 4.2 by using the above Theorem.

Proof. It is easy to see that $\sigma(x) = \text{ReLU}(x+1) - \text{ReLU}(x)$ is a continuous sigmoidal activation function.

Denote $Z^{\sigma}_{W}(n)$ as the set of all two-layer neural networks with input dimension n, width W, and activation function σ. For simplicity, $Z_W(n)$ means $Z^{\text{ReLU}}_{W}(n)$.

Firstly, it is easy to see that $Z^{\sigma}_{W}(n) \subset Z_{2W}(n)$ for any $W \in \mathbb{N}_+$.

Then, because $\mathcal{D}$ has a positive separation distance with different label, there is a continuous function f such that: $f(x) = 1$ if x has label 1 in distribution $\mathcal{D}$; $f(x) = -1$ if x has label -1 in distribution $\mathcal{D}$.

Finally, by Theorem A.1, there exist a W and a $\mathcal{F} \in Z^{\sigma}_{W}(n)$ such that $|\mathcal{F}(x) - f(x)| \leq 0.1$ for all $x \in [0,1]^n$. Thus, $\mathcal{F} \in Z^{\sigma}_{W}(n) \subset Z_{2W}(n)$ and $P_{(x,y)\sim\mathcal{D}}(y\mathcal{F}(x) \geq 0.9) = 1$.

Let the maximum of the absolute value of parameters of $\mathcal{F}$ be A. If $A \leq 1$, then $\mathcal{F}$ is what we want. If $A > 1$, then let $\mathcal{F}_A$ be a network whose parameter is corresponding parameter of $\mathcal{F}$ divided by A, so $\mathcal{F}_A = \mathcal{F}/A^2$. Hence, there are $\mathcal{F}_A \in \mathbf{H}_{2W}(n)$ and $P_{(x,y)\sim\mathcal{D}}(y\mathcal{F}(x) \geq 0.9/A^2) = 1$. The proposition is proved. □

B PROOF OF THEOREM 4.3

B.1 PREPARATORY WORK

We give some definitions of the hypothesis space.

Definition B.1. For a network $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}$ and an $a > 0$, let $\mathcal{F}_{-a,a}(x) = \min\{\max\{-a, \mathcal{F}(x)\}, a\}$, that is, clamp $\mathcal{F}$ in $[-a, a]$. Then for any hypothesis space $\mathbf{H}$, let $\mathbf{H}_{-a,a} = \{\mathcal{F}_{-a,a} \| \mathcal{F} \in H\}$.

We define the Radermacher complexity.

Definition B.2. For a hypothesis space $\mathbf{H}$ and dataset $\mathcal{D}$, the Radermacher complexity of $\mathbf{H}$ under dataset $\mathcal{D}$ is:

$$\text{Rad}_{\mathbf{H}}(\mathcal{D}) = \mathbb{E}_{(q_i)_{i=1}^{|\mathcal{D}|}}\left[\sup_{\mathcal{F}\in\mathbf{H}} \frac{\sum_{x_i\in\mathcal{D}} q_i\mathcal{F}(x_i)}{|\mathcal{D}|}\right]$$

where q_i satisfies that $P(q_i = 1) = P(q_i = -1) = 0.5$ and q_i are i.i.d.

Here are some results about the Radermacher complexity:

Lemma B.3. *For any hypothesis space $\mathbf{H}$, let $\mathbf{H}_{+a} = \{\mathcal{F} + a \| \mathcal{F} \in \mathbf{H}\}$, where $a \in \mathbb{R}$. Then for any hypothesis space $\mathbf{H}$, $a \in \mathbb{R}$ and dataset $\mathcal{D}$, there are $\text{Rad}_{\mathbf{H}}(\mathcal{D}) = \text{Rad}_{\mathbf{H}_{+a}}(\mathcal{D})$.*

Let the $L_{1,\infty}$ norm of a matrix W be the maximum value of the L_1 norm for each row of the matrix W.

Lemma B.4. *Let $\mathcal{F}_{n,d,(L_i),(c_i)} : \mathbb{R}^n \to \mathbb{R}$ be a network with d hidden layers, L_i Lipschitz-continuous activation function for i-th activation function, and the output layer does not contain an activation function. Let w_i be the i-th transition matrix and b_i be the i-th bias vector. Then the $L_{1,\infty}$ norm of w_i plus the $L_{1,\infty}$ norm of b_i is not more than c_i.*

Let $\mathbf{H}_{n,d,(L_i),(c_i)} = \{\mathcal{F}_{n,d,(L_i),(c_i)}\}$. Then when $L_i \geq 1$, $c_i \geq 1$, for any $\{x_i\}_{i=1}^N \subset [0,1]^n$, there are:

$$\text{Rad}_{H_{n,d,(L_i),(c_i)}}(\{x_i\}_{i=1}^N) \leq \frac{\Pi_{i=1}^d L_i \Pi_{i=1}^{d+1} c_i}{\sqrt{N}}(\sqrt{(d+3)\log(4)} + \sqrt{2\log(2n)}).$$

This lemma is an obvious corollary of Theorem 1 in (Wen et al., 2021). By the above two lemmas we can calculate the Radermacher complexity of $\mathbf{H}_W^\sigma(n)_{-a,a}$.

Lemma B.5. *Let σ be a L_p Lipschitz-continuous activation function and $L_p \geq 1$, and let $\mathbf{H} = \{F(x,y) : F(x,y) = y\mathcal{F}(x), \mathcal{F}(x) \in \mathbf{H}_W^\sigma(n)_{-a,a}\}$ where $a > 0$ is given in Definition B.1. Then for any $S = \{(x_i, y_i)\}_{i=1}^N \subset [0,1]^n \times \{-1,1\}$, there are*

$$\text{Rad}_{\mathbf{H}}(S) \leq \frac{2L_p(n+1)(W+1+a)}{\sqrt{N}}(\sqrt{5\log(4)} + \sqrt{2\log(2n)}).$$

Proof. First, there are $\text{Rad}_{\mathbf{H}}(S) = \text{Rad}_{\mathbf{H}}(\{(x_i,y_i)\}_{i=1}^N) = \mathbb{E}_{(q_i)_{i=1}^N}[\sup_{f\in\mathbf{H}_W^\sigma(n)_{-a,a}} \frac{\sum_{i=1}^N q_i y_i f(x_i)}{|\mathcal{D}|}]$. Taking into account the definition of q_i in definition B.2, there are $\text{Rad}_{\mathbf{H}}(\{(x_i,y_i)\}_{i=1}^N) = \mathbb{E}_{(q_i)_{i=1}^N}[\sup_{f\in\mathbf{H}_W^\sigma(n)_{-a,a}} \frac{\sum_{i=1}^N q_i f(x_i)}{|D|}] = \text{Rad}_{\mathbf{H}_W^\sigma(n)_{-a,a}}(\{x_i\}_{i=1}^N)$.

So, we just need to calculate $\text{Rad}_{\mathbf{H}_W^\sigma(n)_{-a,a}}(\{x_i\}_{i=1}^N)$.

First, for any function f and $a > 0$, $k \in N^+$, we have

$$\begin{aligned} & f_{-a,a}(x) \\ = \; & \text{ReLU}(f(x)+a) - \text{ReLU}(f(x)-a) - a \\ = \; & \textstyle\sum_{i=1}^k (\text{ReLU}(f(x)/k + a/k) - \text{ReLU}(f(x)/k - a/k)) - a \end{aligned}$$

On the other hand, let $H_{+a} = \{f + a \| f \in \mathbf{H}_W^\sigma(n)_{-a,a}\}$. Then for any $F \in \mathbf{H}_{+a}$, there are $F = f_{-a,a}(x) + a$ for some $f \in \mathbf{H}_W^\sigma(n)$. Then by the above form of expression, take $k = [W/2]$, F and write it as a network with:

(1): Depth 3. Because f has depth 2, after adding a ReLU activation function, it was depth 3.

(2): The first layer has an L_p Lipschitz-continuous activation function; the second layer has a 1 Lipschitz-continuous activation function, that is, ReLU.

(3): The $L_{1,\infty}$ norm of the three transition matrices plus bias vectors are $n+1$, $\frac{W+1+a}{[W/2]}$ and $2[W/2]$.

So, by Lemmas B.4 and B.3, there are $\text{Rad}_{\mathbf{H}_{+a}}(\{x_i\}_{i=1}^N) = \text{Rad}_{\mathbf{H}_W^\sigma(n)_{-a,a}}(\{x_i\}_{i=1}^N) = \frac{2L_p(n+1)(W+1+a)}{\sqrt{N}}(\sqrt{5\log(4)} + \sqrt{2\log(2n)})$. The theorem is proved. □

Another important Theorem is required.

Theorem B.6 (Theorem in Mohri et al. (2018))**.** *Let $H = \{F : \mathbb{R}^n \to [-a,a]\}$, and $\mathcal{D}$ be a distribution, then with probability $1-\delta$ of $\mathcal{D}_{tr} \sim \mathcal{D}^N$, there are:*

$$|\mathbb{E}_{x\sim\mathcal{D}}[F(x)] - \sum_{x\in\mathcal{D}_{tr}} \frac{F(x)}{N}| \leq 2\text{Rad}_H(\mathcal{D}_{tr}) + 6a\sqrt{\frac{\ln(2/\delta)}{2N}},$$

for any $F \in H$.

We give a simple lemma:

Lemma B.7. *(1): When $0 < x \leq e$, there are $\ln(1+x) \geq x/(e+1)$.*

(2): When $x > 0$, there are $xe^{-x} \leq 1/e$.

Proof. For (1): Consider $f(x) = \ln(1+x) - x/(e+1)$, there are $f'(x) = 1/(1+x) - 1/(1+e) \geq 0$, so $f(x) \geq f(0) = 0$, which means that $\ln(1+x) - x/(e+1) \geq 0$.

For (2): Consider $f(x) = xe^{-x}$, there are $f'(x) = e^{-x}(1-x)$, it is easy to see that $f'(x)$ become positive then negative when x from 0 to ∞, and $f'(1) = 0$, so $f(x) \leq f(1) = 1/e$. □

B.2 PROOF OF THEOREM 4.3

Proof. Let $\mathcal{D}_{tr} \sim \mathcal{D}^N$ and $\mathcal{F}$ be a network in $\mathbf{M}_W^\sigma(\mathcal{D}_{tr}, n)$. We prove Theorem 4.3 in four parts:

Part one: There are $\sum_{(x,y)\in\mathcal{D}_{tr}} L(\mathcal{F}(x), y) \leq N\ln(1 + e^{-c[\frac{W}{W_0+1}]})$.

Because $\mathcal{D}$ can be expressed by $\mathbf{H}_{W_0}^\sigma(n)$ with confidence c, so there is a network $\mathcal{F}_0 = \sum_{i=1}^{W_0} a_i\sigma(W_i x + b_i) + c_1$ such that $y\mathcal{F}(x) \geq c$ for all $(x, y) \sim \mathcal{D}$. Moreover, we can write such network as: $\mathcal{F}_0 = \sum_{i=1}^{W_0+1} a_i\sigma(W_i x + b_i)$, where $a_{W_0+1} = \text{Sgn}(c_1)$, $W_{W_0+1} = 0$, $b_{W_0+1} = |c_1|$.

Now, we consider the following network in $\mathbf{H}_W^\sigma(n)$:

$$\mathcal{F}_W = \sum_{i=1}^{(W_0+1)[\frac{W}{W_0+1}]} a_{i\%(W_0+1)}\sigma(W_{i\%(W_0+1)}x + b_{i\%(W_0+1)}),$$

Here, we stipulate that $i\%(W_0 + 1) = W_0 + 1$ when $W_0 + 1|i$. Then we have $\mathcal{F}_W(x) = [\frac{W}{W_0+1}]\mathcal{F}_0(x)$ and $\mathcal{F}_W(x) \in \mathbf{H}_W^\sigma(n)$. Moreover, there are $y\mathcal{F}_W(x) = y[\frac{W}{W_0+1}]\mathcal{F}_0(x) \geq [\frac{W}{W_0+1}]c$ for all $(x, y) \sim \mathcal{D}$, so $\sum_{(x,y)\in\mathcal{D}_{tr}} L(\mathcal{F}_W(x), y) \leq N\ln(1 + e^{-c[\frac{W}{W_0+1}]})$. So for any $\mathcal{F} \in \arg\min_{f\in\mathbf{H}_W^\sigma(n)} \sum_{(x,y)\in\mathcal{D}_{tr}} L(f(x), y)$, there are $\sum_{(x,y)\in\mathcal{D}_{tr}} L(\mathcal{F}(x), y) \leq \sum_{(x,y)\in\mathcal{D}_{tr}} L(\mathcal{F}_W(x), y) \leq N\ln(1 + e^{-c[\frac{W}{W_0+1}]})$.

Part Two: Let $k = [\frac{W}{W_0+1}]$, by the assumption in Theorem, there is $k \geq 1$. We show that there are $|\{(x, y) : (x, y) \in \mathcal{D}_{tr}, y\mathcal{F}(x) \leq kc/2\}| \leq Ne^{-kc/2+2}$.

Let $S = \{(x, y) : (x, y) \in \mathcal{D}_{tr}, y\mathcal{F}(x) \leq kc/2\}$, then according to part one, there are: $|S|\ln(1 + e^{-kc/2}) \leq \sum_{(x,y)\in S} L(\mathcal{F}(x), y) \leq \sum_{(x,y)\in\mathcal{D}_{tr}} L(\mathcal{F}(x), y) \leq N\ln 1 + e^{-kc} \leq Ne^{-kc}$. So, there are $|S|\ln 1 + e^{-kc/2} \leq Ne^{-kc}$.

By Lemma B.7, there are $|S|e^{-kc/2}/(e + 1) \leq |S|\ln 1 + e^{-kc/2} \leq Ne^{-kc}$, so $|S| \leq Ne^{-kc/2}(e + 1) < Ne^{-kc/2+2}$.

Part Three: By Definition B.1, let network $g = \mathcal{F}_{-kc/2,kc/2}$, we show that, with high probability, $\mathbb{E}_{(x,y)\sim\mathcal{D}} yg(x)$ has a lower bound.

Firstly, by part two, there are $\sum_{(x,y)\in\mathcal{D}_{tr}} yg(x) \geq N(kc(1 - e^{-kc/2+2})/2 - kce^{-kc/2+2}/2) = Nkc(1 - 2e^{-kc/2+2})/2$.

Then, let $H = \{y\mathcal{F}(x) : \mathcal{F}(x) \in \mathbf{H}_W^\sigma(n)_{-kc/2,kc/2}\}$, by Lemma B.5, there are $Rad_H(\mathcal{D}_{tr}) \leq \frac{2(n+1)(W+1+kc/2)L_p}{\sqrt{N}}(\sqrt{5\log(4)} + \sqrt{2\log(2n)})$, $Rad_H(\mathcal{D}_{tr})$ is defined in definition B.2.

So, considering that $yg(x) \in H$ and by Theorem B.6, with probability $1 - \delta$ of $\mathcal{D}_{tr}$, there are

$$\begin{aligned}
& \mathbb{E}_{(x,y)\sim\mathcal{D}} yg(x) \\
\geq\ & \tfrac{1}{N}\textstyle\sum_{(x,y)\in\mathcal{D}_{tr}} yg(x) - 2\text{Rad}([\mathbf{H}_W^\sigma(n)]_{-kc/2,kc}) - 3kc\sqrt{\tfrac{\ln(2/\delta)}{2N}} \\
\geq\ & kc(1 - 2e^{-kc/2+2})/2 - \tfrac{2(n+1)L_p(W+1+kc/2)}{\sqrt{N}}(\sqrt{5\log(4)} + \sqrt{2\log(2n)}) - 3kc\sqrt{\tfrac{\ln(2/\delta)}{2N}}.
\end{aligned}$$

Part Four: Now, we prove Theorem 4.3.

Firstly, there are $A_{\mathcal{D}}(g) = \mathbb{P}_{(x,y)\sim\mathcal{D}}(yg(x) > 0) \geq \mathbb{E}_{(x,y)\sim\mathcal{D}}[yg(x)]/(kc/2)$, we use $|g(x)| \leq kc/2$ in here. So, by part three, with probability of $\mathcal{D}_{tr}$, there are

$$A_{\mathcal{D}}(g) \geq 1 - 2e^{-kc/2+2} - \frac{4(n + 1)L_p(W + 1 + kc/2)}{\sqrt{N}kc}(\sqrt{5\log(4)} + \sqrt{2\log(2n)}) - 6\sqrt{\frac{\ln(2/\delta)}{2N}}.$$

By Lemma B.7 and $k = [W/(W_0+1)] \geq \frac{W}{2W_0}$ which is because $[W/(W_0+1)] = k \geq 1$ and $W_0 \geq 2$, there are $2e^{-kc/2+2} \leq \frac{4e}{kc} = \frac{4e}{c[\frac{W}{W_0+1}]} \leq \frac{8eW_0}{Wc}$; and it is easy to see that $\frac{4(n+1)L_p(W+1+kc/2)}{\sqrt{N}kc} \leq$

$\frac{4(n+1)WL_p(2+kc/2W)}{\sqrt{N}kc} \leq \frac{8nWL_p(2+c/2W_0)}{\sqrt{N}[W/(W_0+1)]c} \leq \frac{8nL_p(4W_0+c)}{\sqrt{N}c}$, the last inequality uses $[W/(W_0+1)] \geq \frac{W}{2W_0}$.

The last step uses $k = [W/(W_0+1)] \geq \frac{W}{2W_0}$. And $\sqrt{5\log(4)}+\sqrt{2\log(2n)} \leq (\sqrt{5}+\sqrt{2})\sqrt{\log(4n)}$. So there are:

$$A_{\mathcal{D}}(g) \geq 1 - \frac{8eW_0}{Wc} - \frac{8nL_p(1+4\frac{W_0}{c})}{\sqrt{N}}(\sqrt{5}+\sqrt{2})\sqrt{\log(4n)} - 6\sqrt{\frac{\ln(2/\delta)}{2N}}.$$

Lastly, because $A_{\mathcal{D}}(g) = A_{\mathcal{D}}(\mathcal{F})$, there are $A_{\mathcal{D}}(\mathcal{F}) \geq 1 - O(\frac{W_0}{Wc} + \frac{nL_p(W_0+c)\sqrt{\log(4n)}}{\sqrt{N}c} + \sqrt{\frac{\ln(2/\delta)}{N}})$.

The theorem is proved. □

C Proof of Proposition 4.7

The proof is similar to the proof of Theorem 4.3, so we just follow the proof of Theorem 4.3.

Proof. Let $\mathcal{D}_{tr} \sim \mathcal{D}^N$, $\mathcal{F}$ be a network in $\mathbf{M}_W^{\sigma}(\mathcal{D}_{tr}, n)$, and $\mathcal{F}_q$ be a network which is a q-approximation of minimization empirical risk.

We prove the Theorem 4.7 in four parts:

Part one: There are $\sum_{(x,y)\in\mathcal{D}_{tr}} L(\mathcal{F}(x), y) \leq N\ln(1+e^{-c[\frac{W}{W_0+1}]})$. This is as same as in Part one in the proof of Theorem 4.3

Part Two: Let $k = [\frac{W}{W_0+1}] \geq 1$, there are $|\{(x,y) : (x,y) \in \mathcal{D}_{tr}, y\mathcal{F}_q(x) \leq kc/2\}| \leq qNe^{-kc/2+2}$.

Let $S = \{(x,y) : (x,y) \in \mathcal{D}_{tr}, y\mathcal{F}_q(x) \leq kc/2\}$, then according to part one, there are: $|S|\ln(1+e^{-kc/2}) \leq \sum_{(x,y)\in S} L(\mathcal{F}_q(x), y) \leq q\sum_{(x,y)\in\mathcal{D}_{tr}} L(\mathcal{F}(x), y) \leq qN\ln 1 + e^{-kc} \leq qNe^{-kc}$. So, there are $|S|\ln 1 + e^{-kc/2} \leq qNe^{-kc}$.

By Lemma B.7, there are $|S|e^{-kc/2}/(e+1) \leq |S|\ln 1 + e^{-kc/2} \leq qNe^{-kc}$, so $|S| \leq qNe^{-kc/2}(e+1) < qNe^{-kc/2+2}$.

Part Three: By definition B.1, let network $g = (\mathcal{F}_q)_{-kc/2,kc/2}$, we show that, with high probability, $\mathbb{E}_{(x,y)\sim\mathcal{D}} yg(x)$ has a lower bound.

Firstly, by part two, there are $\sum_{(x,y)\in\mathcal{D}_{tr}} yg(x) \geq N(kc(1-qe^{-kc/2+2})/2 - qkce^{-kc/2+2}/2) = Nkc(1-2qe^{-kc/2+2})/2$.

So, with probability $1-\delta$ of $\mathcal{D}_{tr}$, there are

$$\begin{aligned}
& \mathbb{E}_{(x,y)\sim\mathcal{D}} yg(x) \\
\geq\ & \tfrac{1}{N}\textstyle\sum_{(x,y)\in\mathcal{D}_{tr}} yg(x) - 2\mathrm{Rad}([\mathbf{H}_W^{\sigma}(n)]_{-kc/2,kc}) - 3kc\sqrt{\tfrac{\ln(2/\delta)}{2N}} \\
\geq\ & kc(1-2qe^{-kc/2+2})/2 - \tfrac{2(n+1)L_p(W+1+kc/2)}{\sqrt{N}}(\sqrt{5\log(4)}+\sqrt{2\log(2n)}) - 3kc\sqrt{\tfrac{\ln(2/\delta)}{2N}}
\end{aligned}$$

Part Four: Now, we prove Proposition 4.7.

Firstly, there are $A_{\mathcal{D}}(g) = \mathbb{P}_{(x,y)\sim\mathcal{D}}(yg(x) > 0) \geq \mathbb{E}_{(x,y)\sim\mathcal{D}}[yg(x)]/(kc/2)$. So, by part three, with $1-\delta$ probability of $\mathcal{D}_{tr}$, there are

$$A_{\mathcal{D}}(g) \geq 1 - 2qe^{-kc/2+2} - \frac{4(n+1)L_p(W+1+kc/2)}{\sqrt{N}kc}(\sqrt{5\log(4)}+\sqrt{2\log(2n)}) - 6\sqrt{\frac{\ln(2/\delta)}{2N}}.$$

Then, similar as part four in proof of Theorem 4.3, there are

$$A_{\mathcal{D}}(g) \geq 1 - \frac{8qeW_0}{Wc} - \frac{8nL_p(1+4\frac{W_0}{c})}{\sqrt{N}}(\sqrt{5}+\sqrt{2})\sqrt{\log(4n)} - 6\sqrt{\frac{\ln(2/\delta)}{2N}},$$

which is what we want. □

D Proof of Theorem 5.2

Proof. Assume that Theorem 5.2 is wrong, then there exist n, W and W_0 such that

For a given $\epsilon, \delta \in (0,1)$, if $\mathcal{D} \in \mathcal{D}(n)$ and $N \geq \text{VC}(\mathbf{H}^{\sigma}_{W_0}(n))(1-4\epsilon-\delta)$, with probability $1-\delta$ of $\mathcal{D}_{tr}$, we have $A_{\mathcal{D}}(\mathcal{F}) \geq 1-\epsilon$ for all $\mathcal{F} \in \arg\min_{f \in \mathbf{H}_W(n)} \sum_{(x,y)\in\mathcal{D}_{tr}} L(f(x), y)$.

We will derive contradictions on the basis of this conclusion.

Part 1: Find some points and values.

For a simple expression, let $k = \text{VC}(\mathbf{H}^{\sigma}_{W_0}(n))$, and $\{u_i\}_{i=1}^k$ be k points that can be shattered by $\text{VC}(\mathbf{H}^{\sigma}_{W_0}(n))$. Let $q = \text{VC}(\mathbf{H}^{\sigma}_{W_0}(n))(1-4\epsilon-\delta)$.

Now, we consider the following types of distribution $\mathcal{D}$:

(c1): $\mathcal{D}$ is a distribution in $\mathcal{D}(n)$ and $\mathbb{P}_{(x,y)\sim\mathcal{D}}(x \in \{u_i\}_{i=1}^k) = 1$.

(c2): $\mathbb{P}_{(x,y)\sim\mathcal{D}}(x = u_i) = \mathbb{P}_{(x,y)\sim\mathcal{D}}(x = u_j) = 1/k$ for any $i, j \in [k]$.

Let S be the set that contains all such distributions, and it is easy to see that for any $\mathcal{D} \in S$, it can be expressed by $\mathbf{H}^{\sigma}_{W_0}(n)$.

Part 2: Some definition.

Moreover, for $\mathcal{D} \in S$, we define $S(\mathcal{D})$ as the following set:

$Z \in S(\mathcal{D})$ if and only if $Z \in [k]^q$ is a vector satisfying: Define $D(Z)$ as $D(Z) = \{(u_{z_i}, y_{z_i})\}_{i=1}^q$, then $A_{\mathcal{D}}(\mathcal{F}) \geq 1-\epsilon$ for all $\mathcal{F} \in \arg\min_{f\in\mathbf{H}_W(n)} \sum_{(x,y)\in D_Z} L(f(x), y)$, where z_i is the i-th weight of Z and y_{z_i} is the label of u_{z_i} in distribution $\mathcal{D}$.

It is easy to see that if we i.i.d. select q samples in distribution $\mathcal{D}$ to form a dataset $\mathcal{D}_{\text{tr}}$, then by $c2$, with probability 1, $\mathcal{D}_{\text{tr}}$ only contain the samples (u_j, y_j) where $j \in [k]$.

Now for any $\mathcal{D}_{tr}$ selected from $\mathcal{D}$, we construct a vector in $[k]^q$ as follows: the index of i-th selected samples as the i-th component of the vector. Then each selection situation corresponds to a vector in $[k]^q$ which is constructed as before. Then by the definition of $S(\mathcal{D})$, we have $A_{\mathcal{D}}(\mathcal{F}) \geq 1-\epsilon$ for all $\mathcal{F} \in \arg\min_{f\in\mathbf{H}_W(n)} \sum_{(x,y)\in\mathcal{D}_{tr}} L(f(x), y)$ if and only if the corresponding vector of $\mathcal{D}_{\text{tr}}$ is in $S(\mathcal{D})$.

By the above result and by the assumption at the beginning of the proof, for any $\mathcal{D} \in S$ we have $\frac{|S(\mathcal{D})|}{q^k} \geq 1-\delta$.

Part 3: Prove the Theorem.

Let S_s be a subset of S, and $S_s = \{\mathcal{D}_{i_1,i_2,\ldots,i_k}\}_{i_j\in\{-1,1\}, j\in[k]} \subset S$, where the distribution $\mathcal{D}_{i_1,i_2,\ldots,i_k}$ satisfies the label of u_j is i_j, where $j \in [k]$.

We will show that there exists at least one $\mathcal{D} \subset S_s$, such that $|S(\mathcal{D})| < (1-\delta)q^k$, which is contrary to the inequality $\frac{|S(\mathcal{D})|}{q^k} \geq 1-\delta$ as shown in the above. To prove that, we only need to prove that $\sum_{\mathcal{D}\in S_s} |S(\mathcal{D})| < (1-\delta)2^k q^k$, use $|S_s| = 2^k$ here.

To prove that, for any vector $Z \in [k]^q$, we estimate how many $\mathcal{D} \in S_s$ make Z included in $S(\mathcal{D})$.

Part 3.1, situation of a given vector Z and a given distribution $\mathcal{D}$.

For a $Z = (z_i)_{i=1}^q$ and $\mathcal{D}$ such that $Z \in S(\mathcal{D})$, let $\text{len}(Z) = \{c \in [k] : \exists i, c = z_i\}$. We consider the distributions in S_s that satisfy the following condition: for $i \in \text{len}(Z)$, the label of u_i is equal to the label of u_i in $\mathcal{D}$. Obviously, we have $2^{k-|\text{len}(Z)|}$ distributions that can satisfy the above condition in

S_s. Let such distributions make up a set $S_{ss}(\mathcal{D}, Z)$. Now, we estimate how many distributions $\mathcal{D}_s$ in $S_{ss}(\mathcal{D}, Z)$ satisfy $Z \in S(\mathcal{D}_s)$.

It is easy to see that if $\mathcal{D}_s \in S_{ss}(\mathcal{D}, Z)$ such that there are more than $[2k\epsilon]$ of $i \in [k]$, $\mathcal{D}_s$ and D have different labels of u_i, then $\min\{A_{\mathcal{D}}(\mathcal{F}), A_{\mathcal{D}_s}(\mathcal{F})\} < 1 - \epsilon$ for any $\mathcal{F}$. So considering $A_{\mathcal{D}}(\mathcal{F}) \geq 1 - \epsilon$ for all $\mathcal{F} \in \arg\min_{f \in \mathbf{H}_W(n)} \sum_{(x,y) \in D_Z} L(f(x), y)$, by the above result, such kind of $\mathcal{D}_s$ is at most $\sum_{i=0}^{[2k\epsilon]} C^i_{k-|\text{len}(Z)|}$. So, we have that: There are at most $\sum_{i=0}^{[2k\epsilon]} C^i_{k-|\text{len}(Z)|}$ numbers of distributions $\mathcal{D}_s$ in $S_{ss}(\mathcal{D}, Z)$ satisfy $Z \in S(\mathcal{D}_s)$.

Part 3.2, for any vector Z and distribution $\mathcal{D}$.

For any distribution $\mathcal{D} \in S_s$, let $y(\mathcal{D})_i$ be the label of u_i in distribution $\mathcal{D}$.

Firstly, for a given Z, we have at most $2^{|len(Z)|}$ different $\mathcal{S}_{ss}(\mathcal{D}, Z)$ for $\mathcal{D} \in \mathcal{D}_S$. Because when $\mathcal{D}_1$ and $\mathcal{D}_2$ satisfy $y(\mathcal{D}_1)_i = y(\mathcal{D}_2)_i$ for any $i \in \text{len}(Z)$, we have $\mathcal{D}_{ss}(\mathcal{D}_1, Z) = \mathcal{D}_{ss}(\mathcal{D}_2, Z)$, and $2^{|\text{len}(Z)|}$ situations of label of u_i where $i \in \text{len}(Z)$, so there exist at most $2^{|\text{len}(Z)|}$ different $\mathcal{S}_{ss}(\mathcal{D}, Z)$.

Then, by part 3.1, for an $\mathcal{S}_{ss}(\mathcal{D}, Z)$, at most $\sum_{i=0}^{[2k\epsilon]} C^i_{k-|\text{len}(Z)|}$ of $\mathcal{D}_s \in S_{ss}(\mathcal{D}, Z)$ satisfies $Z \in S(\mathcal{D}_s)$. So by the above result and consider that $\mathcal{D}_s = \cup_{\mathcal{D} \in \mathcal{D}_s} \mathcal{S}_{ss}(\mathcal{D}, Z)$, at most $2^{|\text{len}(Z)|} \sum_{i=0}^{[2k\epsilon]} C^i_{k-|\text{len}(Z)|}$ number of $\mathcal{D}_s \in S_s$ such that $Z \in S(\mathcal{D}_s)$.

And there exist q^k different Z, so $\sum_{\mathcal{D} \in S_s} |S(\mathcal{D})| = \sum_Z \sum_{\mathcal{D} \in S_s} I(Z \in S(\mathcal{D})) \leq \sum_Z 2^{|\text{len}(Z)|} \sum_{i=0}^{[2k\epsilon]} C^i_{k-|\text{len}(Z)|} \leq \sum_Z 2^k(1-\delta) = q^k 2^k(1-\delta)$. For the last inequality, we use $\sum_{i=0}^{[2k\epsilon]} C^i_{k-|\text{len}(Z)|} < 2^{k-|\text{len}(Z)|}(1-\delta)$, which can be shown by $|\text{len}(Z)| \leq q \leq k(1-4\epsilon-\delta)$ and Lemma D.1.

This is what we want. we proved the Theorem. □

A required lemma is given.

Lemma D.1. *If $\epsilon, \delta \in (0,1)$ and $k, x \in \mathbb{Z}_+$ satisfy that: $x \leq k(1-2\epsilon-\delta)$, then $2^x(\sum_{j=0}^{[k\epsilon]} C^j_{k-x}) < 2^k(1-\delta)$.*

Proof. We have

$$2^x(\textstyle\sum_{j=0}^{[k\epsilon]} C^j_{k-x}) \leq 2^x 2^{k-x} \frac{[k\epsilon]}{k-x} \leq 2^k \frac{k\epsilon}{k-x} < 2^k(1-\delta).$$

The first inequality sign uses $\sum_{j=0}^{m} C^m_n \leq m2^n/n$ where $m \leq n/2$, and by $x \leq k(1-2\epsilon-\delta)$, so $[k\epsilon] \leq (k-x)/2$. The third inequality sign uses the fact $x \leq k(1-2\epsilon-\delta)$. □

E Proof of Theorem 5.4

We give the proof of Theorem 5.4.

Proof. Let $\mathcal{D}_{tr} \sim \mathcal{D}^N$ and $\mathcal{D}_{tr} = \{(x_i, y_i)\}_{i=1}^N$.

For any given W, let $\mathcal{F}$ be a network in $\mathbf{M}^{\sigma}_W(\mathcal{D}_{tr}, n)$, and $\mathcal{F} = \sum_{i=1}^{W} a_i \text{ReLU}(W_i x + b_i) + c$.

Then, we consider another network F_f which is constructed in the following way:

(1): For a $v \in \{-1, 1\}^N$, we say $i \in S_v$ if: $\text{ReLU}(W_i x_j + b_i) \geq 0$ for all j such that $v_j = 1$; $\text{ReLU}(W_i x_j + b_i) < 0$ for all j such that $v_j = -1$.

(2): For any $v \in \{-1, 1\}^N$, if $S_v \neq \phi$, let $\mathbb{P}_v = \sum_{i \in S_v} a_i W_i / |S_v|$ and $Q_v = \sum_{i \in S_v} a_i b_i / |S_v|$.

(3): Define F_f as: $F_f(x) = \sum_{v \in \{-1,1\}^N, S_v \neq \phi} \sum_{i=1}^{|S_v|} \text{ReLU}(\mathbb{P}_v x + Q_v) + c$.

Then we have the following result:

(r1): $F_f \in \arg\min_{f \in \mathbf{H}_W^\sigma(n)} \sum_{(x,y)\in\mathcal{D}_{tr}} L(f(x), y)$.

Firstly, it is easy to see that each parameter of F_f is in $[-1, 1]$, because for any v, $||\mathbb{P}_v||_\infty = ||\sum_{i\in S_v} \frac{a_i W_i}{|S_v|}||_\infty \leq \sum_{i\in S_v} \frac{||a_i W_i||_\infty}{|S_v|} \leq |S_v| \frac{1}{|S_v|} = 1$, and $||Q_v||_\infty = ||\sum_{i\in S_v} \frac{a_i b_i}{|S_v|}||_\infty \leq \sum_{i\in S_v} \frac{||a_i b_i||_\infty}{|S_v|} \leq |S_v| \frac{1}{|S_v|} = 1$.

Then, F_f has width W, because for each i, there is only one v such that $i \in S_v$, so $\sum_{v\in\{-1,1\}^N, S_v\neq\phi} \sum_{i=1}^{|S_v|} 1 = W$, which implies that F_f has width W.

Finally, there are $\mathcal{F}_f(x_i) = \mathcal{F}(x_i)$ for all $(x_i, y_i) \in \mathcal{D}_{tr}$. We just need to show that for x_1, others are similar.

There are $\mathcal{F}(x_1) = \sum_{i=1}^W a_i \mathrm{ReLU}(W_i x_1 + b_i) + c = \sum_{i\in[W], W_i x_1 + b_i \geq 0} a_i(W_i x_1 + b_i) + c$. Hence, letting $V1 = \{v : v \in \{-1, 1\}^N, v_1 = 1\}$, then there is $F_f(x_1) = \sum_{v\in\{-1,1\}^N, S_v\neq\phi} \sum_{i=1}^{|S_v|} \mathrm{ReLU}(\mathbb{P}_v x_1 + Q_v) + c = \sum_{v\in V1, S_v\neq\phi} \sum_{i=1}^{|S_v|} (\mathbb{P}_v x_1 + Q_v) + c$.

Consider that $\{i \in [W], W_i x_1 + b_i \geq 0\} = \{i : i \in S_v, v \in V1\}$, so:

$$\begin{array}{rl} & \mathcal{F}(x_1) \\ = & \sum_{i\in[W], W_i x_1 + b_i > 0} a_i(W_i x_1 + b_i) + c \\ = & \sum_{i: i\in S_v, v\in V1} a_i(W_i x_1 + b_i) + c \\ = & \sum_{v\in V1, S_v\neq\phi} \sum_{i\in S_v} a_i(W_i x_1 + b_i) + c \\ = & \sum_{v\in V1, S_v\neq\phi} |S_v|(\mathbb{P}_v x_1 + b_v) + c \\ = & \mathcal{F}_f(x_1). \end{array}$$

By such three points and considering $\mathcal{F} \in \arg\min_{f\in\mathbf{H}_W^\sigma(n)} \sum_{(x,y)\in\mathcal{D}_{tr}} L(f(x), y)$, so there are $F_f \in \arg\min_{f\in\mathbf{H}_W^\sigma(n)} \sum_{(x,y)\in\mathcal{D}_{tr}} L(f(x), y)$.

(r2): $A_{\mathcal{D}}(F_f) \leq 1 - \delta$ when $N \leq W_0^{\frac{1}{n+1}}(n+1)/e$, where W_0 is defined in Theorem. This is what we want.

Firstly, we show that $|\{v : S_v \neq \phi\}| \leq max\{2^{n+1}, \frac{eN}{n+1}^{n+1}\}$, just by Lemma E.1.

Secondly, consider the network $F_{f1} = \sum_{v\in\{-1,1\}^N, S_v\neq\phi} \mathrm{ReLU}(|S_v|\mathbb{P}_v x_1 + |S_v| Q_v) + c$. By the assumption of $\mathcal{D}$ and $|\{v : S_v \neq \phi\}| \leq max\{2^{n+1}, \frac{eN}{n+1}^{n+1}\}$, then we know that, when $N \leq W_0^{\frac{1}{n+1}}(n+1)/e$, there are $A_{\mathcal{D}}(F_{f1}) \leq 1 - \delta$.

Moreover, there are $F_{f1}(x) = \sum_{v\in\{-1,1\}^N, S_v\neq\phi} \mathrm{ReLU}(|S_v|\mathbb{P}_v x + |S_v| Q_v) + c = \sum_{v\in\{1,1\}^N, S_v\neq\phi} \sum_{i=1}^{|S_v|} \mathrm{ReLU}(\mathbb{P}_v x + Q_v) + c = F_f(x)$, so $A_{\mathcal{D}}(\mathcal{F}_f) = A_{\mathcal{D}}(F_{f1}) \leq 1 - \delta$, this is what we want. □

A required lemma is given:

Lemma E.1. *For any* $S = \{x_i\}_{i=1}^N \subset \mathbb{R}^n$, *let* $\Pi(S) = \{(\mathrm{Sgn}(W x_i + b))_{i=1}^n : W \in \mathbb{R}^n, b \in \mathbb{R}\}$. *Then* $|\Pi(S)| \leq max\{2^{n+1}, \frac{eN}{n+1}^{n+1}\}$.

Proof. It is easy to see that $|\Pi(S)| \leq 2^N$ because $\mathrm{Sgn}(W x_i + b) \in \{-1, 1\}$. So, when $N \leq n+1$, it is obviously correct.

When $N > n+1$. Consider that the VC-dim of the linear space is $n+1$, and $\Pi(S) = \{(\mathrm{Sgn}(W x_i + b))_{i=1}^n : W \in \mathbb{R}^n, b \in \mathbb{R}\}$ is the growth function of linear space under dataset S. So by Theorem 1 of (Sauer, 1972), there are $|\Pi(S)| \leq \sum_{i=0}^{n+1} C_N^i$.

Moreover, there are $\sum_{i=0}^{n+1} C_N^i \leq \frac{eN}{n+1}^{n+1}$ as shown in (Sauer, 1972), this is what we want. □

F Proof of Proposition 5.6

We give the proof of Proposition 5.6.

Proof. Firstly, it is easy to show that $\mathcal{D}_n$ cannot be expressed by $\mathbf{H}_W(n)$ when $W < n/2$ by Lemma F.1, so we have proved (1) of Proposition 5.6.

Let $\mathcal{D}_{tr} \sim \mathcal{D}_n^N$ and $N \leq n\delta$, for any given W, let $\mathcal{F}$ be a network in $\mathbf{M}_W^\sigma(\mathcal{D}_{tr}, n)$, and $\mathcal{F} = \sum_{i=1}^W a_i \mathrm{ReLU}(W_i x + b_i) + c$.

Now we prove (2) of Proposition 5.6. Let $\mathcal{D}_{tr} = \{(\frac{x_i}{n}\mathbf{1}, \mathbb{I}(x_i))\}_{i=1}^N$ where $x_i \in [n]$ be selected from the distribution, without loss of generality, let $x_i < x_{i+1}$ for any $i \in [N]$.

We will divide $[W]$ into several subsets based on the intersection of the plane $W_j x + b$ and the line $-\infty\mathbf{1} \to \infty\mathbf{1}$, let $[W] = \cup_{i=1}^{2N} s_i$, and:

1. For any $i \in [N-1]$: if $j \in [W]$ such that $\frac{x_i}{n} W_j \mathbf{1} + b_i < 0$ and $\frac{x_{i+1}}{n} W_j \mathbf{1} + b_j \geq 0$, then $j \in s_i$;

2. If $j \in [W]$ such that $\frac{x_i}{n} W_j \mathbf{1} + b_i < 0$ for any $i \in [N]$, then $j \in s_N$;

3. For any $i \in \{N+1, N+2, \ldots, 2N-1\}$: if $j \in [W]$ such that $\frac{x_{i-N}}{n} W_j \mathbf{1} + b_i \geq 0$ and $\frac{x_{i-N+1}}{n} W_j \mathbf{1} + b_j < 0$, then $j \in s_i$;

4. If $j \in [W]$ such that $\frac{x_i}{n} W_j \mathbf{1} + b_i \geq 0$ for any $i \in [N]$, then $j \in s_{2N}$.

Now, by such $2N$ subset, we consider another network $\mathcal{F}_f$ that is defined as:

For any $i \in [2N]$, if $S_i \neq \phi$, define $P_i = \sum_{j \in S_i} a_i W_i / |S_i|$ and $Q_i = \sum_{j \in S_i} a_i b_i / |S_i|$. Then $F_f = \sum_{i \in [2N], S_i \neq \phi} \sum_{j=1}^{|S_i|} \mathrm{ReLU}(P_i x + Q_i) + c = \sum_{i \in [2N], S_i \neq \phi} |S_i| \mathrm{ReLU}(P_i x + Q_i) + c$.

Because there is only one intersection point between a straight line and a plane, each $j \in [W]$ is only in one subset s_i. So, $\mathcal{F}_f \in \mathbf{H}_W^\sigma(n)$. Moreover, we show that $\mathcal{F}_f(x) = \mathcal{F}(x)$ for any $(x, y) \in \mathcal{D}_{tr}$, which implies $\mathcal{F}_f \in \arg\min_{f \in \mathbf{H}_W^\sigma(n)} \sum_{(x,y) \in \mathcal{D}_{tr}} L(f(x), y)$.

For any $j \in [N]$, by the definition of s_i, we know that $\frac{x_j}{n} W_i \mathbf{1} + b_i \geq 0$ if and only if $i \in \{1, 2, \ldots, j-1\} \cup \{N+j, N+j+1, \ldots, 2N\}$, so:

$$
\begin{aligned}
& \mathcal{F}_f(x_j) \\
= \; & \textstyle\sum_{i \in [2N], S_i \neq \phi} \sum_{j=1}^{|S_i|} \mathrm{ReLU}(P_i x_j + Q_i) + c \\
= \; & \textstyle\sum_{i \in \{1,2,\ldots,j-1\} \cup \{N+j, N+j+1, \ldots, 2N\}, S_i \neq \phi} \sum_{j=1}^{|S_i|} (P_i x_j + Q_i) + c \\
= \; & \textstyle\sum_{k \in \bigcup_{i \in \{1,2,\ldots,j-1, N+j, N+j+1, \ldots, 2N\}} s_i} (W_k x_j + b_k) + c \\
= \; & \textstyle\sum_{k \in [W]} \mathrm{ReLU}(W_k x_j + b_k) + c \\
= \; & \mathcal{F}(x_j)
\end{aligned}
$$

This is what we want. At last, by $\mathcal{F}_f = \sum_{i \in [2N], S_i \neq \phi} |S_i| \mathrm{ReLU}(P_i x + Q_i) + c$ has width at most $2N$ and Lemma F.1, and consider that $N \leq n\delta$, we have that: $A_{\mathcal{D}}(\mathcal{F}_f) \leq 0.5 + 2\delta$, this is what we want.

□

A required lemma is given:

Lemma F.1. *If $x_1 < x_2 < x_3 < \cdots < x_N$, and x_i has label $y_i = 1$ when i is odd, or x_i has label $y_i = -1$. We consider dataset $S = \{(x_i \mathbf{1}(n), y_i)\}$, where $\mathbf{1}$ is all-one vector in $\mathbb{R}^n$. Then: For any two-layer network width M, this network can correctly classify at most to $M + \frac{N}{2}$ samples in S.*

Proof. Let $\mathcal{F} = \sum_{i=1}^M a_i \mathrm{ReLU}(W_i x + b_i) + c$. Let $W_i x + b_i$ and the line $-\infty\mathbf{1}(n) \to \infty\mathbf{1}(n)$ intersect at one point $P_i \mathbf{1}(n)$. Let $P_i \leq P_j$ if $i \leq j$. Let $P_{M+1} = \infty$.

Then it is easy to see that in line segment $P_i \mathbf{1}(n) \to P_{i+1} \mathbf{1}(n)$, $\mathcal{F}(x)$ is a linear function. So, there is $P_{i+0.5} \in (P_i, \mathbb{P}_{i+1})$ such that $\mathcal{F}$ maintains positive and negative polarity unchanged in $P_i \mathbf{1}(n) \to P_{i+0.5} \mathbf{1}(n)$ and $P_{i+0.5} \mathbf{1}(n) \to P_{i+1} \mathbf{1}(n)$.

So if $P_i \leq x_u < x_{u+1} < \cdots < x_{u+k} < P_{i+0.5}$, F gives the same label to $x_u \mathbf{1}(n), x_{u+1} \mathbf{1}(n), \ldots, x_{u+k} \mathbf{1}(n)$, which means $\mathcal{F}$ can classify at most $[\frac{(k+1)+1}{2}]$ samples in them. Similar to when $P_{i+0.5} \leq x_u < x_{u+1} < \cdots < x_{u+k} < P_{i+1}$.

Let $q_i = |\{j : P_{i/2} \leq x_j < P_{i/2+0.5}\}|$ where $i \in [2M]$. Consider that each sample in S is appeared in a $P_i\mathbf{1}(n) \to P_{i+0.5}\mathbf{1}(n)$ or $P_{i+0.5}\mathbf{1}(n) \to P_{i+1}\mathbf{1}(n)$, so $\sum_{i=1}^{2M} q_i = N$.

So, the whole network can classify at most $\sum_{i=1}^{2M}[\frac{1+q_i}{2}] \leq \sum_{i=1}^{2M} \frac{1+q_i}{2} = M + \frac{N}{2}$. □

G Proof of Proposition 5.7

Proof. Proof of (1): Let $\mathbf{1}$ be the all one vector, $\sum x = \sum_{i=1}^{n} x_i$ where x_i is the i-th weight of x. We show that $\mathcal{F} = \sigma(\mathbf{1}x - 0.5) \in \mathbf{H}_1^{\sigma}(n)$ is what we want. Because if $\sum x$ is odd, then $\sigma(\mathbf{1}x - 0.5) = \sigma(\sum x - 0.5) = sin(\pi(\sum x - 0.5)) = 1$; if $\sum x$ is even, then $\sigma(\mathbf{1}x - 0.5) = \sigma(\sum x - 0.5) = sin(\pi(\sum x - 0.5)) = -1$.

Proof of (2): we will prove it into three parts:

Part one: For any W and $\mathcal{D}_{tr} \sim \mathcal{D}_n^N$, let $\mathcal{F} \in \mathbf{M}_W^{\sigma}(\mathcal{D}_{tr}, n)$ and $\mathcal{F} = \sum_{i=1}^{W} \sigma(W_i x + b_i) + c$. Then there are: for any $(x, y) \in \mathcal{D}_{tr}$, there are $y\sigma(W_i x + b_i) = 1$ for any $i \in [W]$.

If not, without loss of generality, assume that $y\sigma(W_1 x + b_1) < 1$ for some $(x, y) \in \mathcal{D}_{tr}$. According to the proof of (1), there are W_0 and b_0 such that $y\sigma(W_0 x + b_0) = 1$ for any $(x, y) \in \mathcal{D}_{tr}$. Now we consider the network $\mathcal{F}_c(x) = \sum_{i=2}^{W} \sigma(W_i x + b_i) + \sigma(W_0 x + b_0) + c$, then we have that:

Firstly, it is easy to see that $\mathcal{F}_c \in \mathbf{H}_W^{\sigma}(n)$.

Secondly, we show that $y\mathcal{F}(x) \leq y\mathcal{F}_c(x)$ for any $(x, y) \in \mathcal{D}_{tr}$ and $y\mathcal{F}(x) < y\mathcal{F}_c(x)$ for some $(x, y) \in \mathcal{D}_{tr}$.

By the definition of $\mathcal{F}$ and $\mathcal{F}_c$, for any $(x, y) \in \mathcal{D}_{tr}$, there are $y\mathcal{F}_c(x) - y\mathcal{F}(x) = y(\sigma(W_0 x + b_0) - \sigma(W_1 x + b_1)) = 1 - y\sigma(W_1 x + b_1) \geq 0$, and by the assumption, there is a $(x, y) \in \mathcal{D}_{tr}$ such that $1 > y\sigma(W_1 x + b_1))$, then $y\mathcal{F}_c(x) - y\mathcal{F}(x) > 0$ for such $(x, y) \in \mathcal{D}_{tr}$, this is what we want.

By the above two results, and considering that $L(\mathcal{F}(x), y)$ is a strictly decreasing function about $y\mathcal{F}(x)$, there are $\sum_{(x,y)\in\mathcal{D}_{tr}} L(\mathcal{F}(x), y) > \sum_{(x,y)\in\mathcal{D}_{tr}} L(\mathcal{F}_c(x), y)$, which is contradictory to $\mathcal{F} \in \underset{f\in\mathbf{H}_W^{\sigma}(n)}{\operatorname{argmin}} \sum_{(x,y)\in\mathcal{D}_{tr}} L(f(x), y)$. So we prove part one.

Part Two. For any $j \in Z$, let $x_j = \frac{j}{n}\mathbf{1}$ and $y_j = \mathbb{I}(j)$, where $\mathbb{I}(x)$ is defined in the definition of distribution $\mathcal{D}_n$. If $i_j \in \mathbb{Z}$ where $j \in [4]$ such that $i_1 - i_2$ and $i_3 - i_4$ are co-prime, then there are: if $W_0 \in [-1, 1]^n$ and $b_0 \in [-1, 1]$ such that $y_{i_j}\sigma(W_0 x_{i_j} + b_0) = 1$ for any $j \in [4]$, then $y_p\sigma(W_0 x_p + b_0) = 1$ for all $p \in Z$.

When there is $y_{i_j}\sigma(W_0 x_{i_j} + b_0) = y_{i_j} sin(\pi(W_0 x_{i_j} + b_0)) = y_{i_j} sin(\pi(< W_0, \mathbf{1} > i_j/n + b_0)) = 1$, consider that $y_{i_j} \in \{-1, 1\}$, then there is $< W_0, \mathbf{1} > i_j/n + b_0 = m_{i_j} - 0.5$ for $m_{i_j} \in Z$, moreover, m_{i_j} and i_j are same parity.

Now consider $(W_0 x_{i_1} + b_0) - (W_0 x_{i_2} + b_0)$ and $(W_0 x_{i_3} + b_0) - (W_0 x_{i_4} + b_0)$, there are $< W_0, \mathbf{1} > (i_1 - i_2)/n = m_{i_1} - m_{i_2}$ and $< W_0, \mathbf{1} > (i_3 - i_4)/n = m_{i_3} - m_{i_4}$. So, there are $\frac{i_1 - i_2}{i_3 - i_4} = \frac{m_{i_1} - m_{i_2}}{m_{i_3} - m_{i_4}}$.

By $i_1 - i_2$ and $i_3 - i_4$ are co-prime, and $|m_{i_1} - m_{i_2}| = | < W_0, \mathbf{1} > (i_1 - i_2)/n| \leq |i_1 - i_2|$, $|m_{i_3} - m_{i_5}| = | < W_0, \mathbf{1} > (i_3 - i_4)/n| \leq |i_3 - i_4|$, there are $< W_0, \mathbf{1} > /n = 1$ or $< W_0, \mathbf{1} > /n = -1$.

Hence, by $m_{i_j} - i_j =< W_0, \mathbf{1} > i_j/n + b_0 + 0.5 - i_j$ and $< W_0, \mathbf{1} > /n = 1$ or $< W_0, \mathbf{1} > /n = -1$, consider that m_{i_j} and i_j are the same parity, so $b = -0.5$.

So for any $p \in \mathbb{Z}$, there are $y_p\sigma(W_0 x_p + b_0) = y_p sin(\pi(< W_0, \mathbf{1} > p/n + b_0)) = y_p sin(\pi(p - 0.5)) = 1$, this is what we want.

Part Three, if $\mathcal{D}_{tr} \sim \mathcal{D}_n^N$ and $N \geq 4\frac{\ln(\delta/2)}{\ln(0.5+1/n)}$, with probability $1 - \delta$, there are four samples (x_i, y_i) where $i \in [4]$ in $\mathcal{D}_{tr}$, such that $x_i = \frac{m_i}{n}\mathbf{1}$, $m_1 - m_2$ and $m_3 - m_4$ are co-prime.

By the definition of $\mathcal{D}_n$, it is equivalent to: repeatable randomly select $N \geq 4\frac{\ln(\delta/2)}{\ln(0.5+1/n)}$ points from $[n]$, with probability $1 - \delta$, there are four samples m_i such that $m_1 - m_2$ and $m_3 - m_4$ are co-prime.

By Lemma G.1, when $N \geq 4\frac{\ln(\delta/2)}{\ln(0.5+1/n)}$, with probability at least $1-(0.5+1/n)^{\frac{\ln(\delta/2)}{\ln(0.5+1/n)}}/(0.5+1/n) = 1-\delta/(1+2/n) \geq 1-\delta$. This is what we want.

Part Four, we prove the result.

Let $\mathcal{D}_{tr} \sim \mathcal{D}_n^N$. For any W, let $\mathcal{F} \in \arg\min_{f\in\mathbf{H}_W^\sigma(n)} \sum_{(x,y)\in\mathcal{D}_{tr}} L(f(x),y)$ and $\mathcal{F} = \sum_{i=1}^W \sigma(W_i x + b_i) + c$.

Firstly, with probability $1-\delta$, there are four samples in $\mathcal{D}_{tr}$ satisfying part three. Then, according to part one, there are $y\sigma(W_i x + b_i) = 1$ for such four samples. Finally, in part two, there are $y\sigma(W_i x + b_i) = y\sigma(\sum x) = 1$ for any $(x,y) \sim \mathcal{D}_n$. So, $y\mathcal{F}(x) \geq W-1 > 0$ for any $(x,y)\sim\mathcal{D}_n$, we prove the result. □

A required lemma is given.

Lemma G.1. *Randomly select N points from $[n]$, where $n \geq 3$ and $N \geq 4$. With probability $1-(0.5+1/n)^{N/4-1}$, there are four samples m_i such that m_1-m_2 and m_3-m_4 are co-prime.*

Proof. Firstly, we consider the situation that $N=4$, let $\{m_i\}_{i=1}^4$ are the selected number. Then we have

$$\begin{array}{ll} & P((m_1-m_2, m_3-m_4)=1) \\ = & P(m_1-m_2\neq 0, m_3-m_4\neq 0) - P((m_1-m_2,m_3-m_4)\neq 1, m_1-m_2\neq 0, m_3-m_4\neq 0) \\ = & (1-1/n)^2(1-P((|m_1-m_2|,|m_3-m_4|)\neq 1 | m_1-m_2\neq 0, m_3-m_4\neq 0)) \\ \geq & (1-1/n)^2(1-\sum_{q\in Prime} P(q|(|m_1-m_2|,|m_3-m_4|)|m_1-m_2\neq 0, m_3-m_4\neq 0)) \\ \geq & (1-1/n)^2(1-\sum_{q\in Prime}\frac{1}{q^2}) \\ \geq & 0.5(1-1/n)^2 \geq 0.5-1/n \end{array}$$

where $Prime$ is the set of all primes. For the second inequality sign, we use

$$\begin{array}{ll} & P(q|m_1-m_2\,|m_1-m_2\neq 0) \\ = & \sum_{i=1}^{n-1} P(q|i, i=|m_1-m_2|\,|m_1-m_2\neq 0) \\ = & [(n-1)/q]*\frac{1}{n-1} \\ \leq & 1/q. \end{array}$$

Similar for m_3-m_4. For the last inequality sign, we use $P(2)=\sum_{i\in Prime}\frac{1}{i^2} < 0.5$, where P is Riemann function.

So, when we select N samples, it contains $[N/4] > N/4-1$ pairs of four independent samples randomly selected. So, with probability $1-(0.5+1/n)^{N/4-1}$, there are four samples m_i such that m_1-m_2 and m_3-m_4 are co-prime. □

H Proof of Theorem 6.2

Now, we prove Theorem 6.2.

Proof. we prove the proposition into three parts.

Part One, with probability $1-2\delta$ of $\mathcal{D}_{tr} \sim D^{N_0}$, there are $\mathbb{E}_{x\sim\mathcal{D}}[y\mathcal{F}(x)] \geq c_0 N_0[\frac{W}{W_0+1}] - 2\frac{L_p(W+1)(n+1)(\sqrt{4\log(4)}+\sqrt{2\log(2n)})}{\sqrt{N_0}} - 6\mathcal{F}_{\max}\sqrt{\frac{\ln(2/\delta)}{2N_0}}$ for all $\mathcal{F}\in\mathbf{M}_W^\sigma(\mathcal{D}_{tr},n)$, where $\mathcal{F}_{\max} = \max_{x+\delta\in[0,1]^n}|\mathcal{F}(x+\delta)|$.

Firstly, we show that there are $\sum_{(x,y)\in\mathcal{D}_{tr}} y\mathcal{F}(x) \geq N_0[\frac{W}{W_0+1}]c_0$ for all $\mathcal{F}\in\mathbf{M}_W^\sigma(\mathcal{D}_{tr},n)$ when $\mathcal{D}_{tr}\sim\mathcal{D}^{N_0}$ satisfies the conditions of the proposition.

Because $\mathcal{D}_{tr}$ can be expressed in the network space $\mathbf{H}_{W_0}^\sigma(n)$ with confidence c_0, there is a network $\mathcal{F}_0 = \sum_{i=1}^{W_0} a_i\sigma(W_i x + b_i) + c$ such that $y\mathcal{F}(x) \geq c_0$ for all $(x,y)\in\mathcal{D}_{tr}$. Moreover, we can write such networks as: $\mathcal{F}_0 = \sum_{i=1}^{W_0+1} a_i\sigma(W_i x + b_i)$, where $a_{W_0+1} = \text{Sgn}(c)$, $W_{W_0+1}=0$, $b_{W_0+1}=|c|$.

Now, we consider the following network in $\mathbf{H}_W^\sigma(n)$:

$$\mathcal{F}_W = \sum_{i=1}^{(W_0+1)[\frac{W}{W_0+1}]} a_{i\%(W_0+1)}\sigma(W_{i\%(W_0+1)}x + b_{i\%(W_0+1)}),$$

Here, we stipulate that $i\%(W_0+1) = W_0+1$ when $W_0+1|i$. Then we have $\mathcal{F}_W(x) = [\frac{W}{W_0+1}]\mathcal{F}_0(x)$ and $\mathcal{F}_W(x) \in \mathbf{H}_W^\sigma(n)$. Moreover, there are $y\mathcal{F}_W(x) = y[\frac{W}{W_0+1}]\mathcal{F}_0(x) \geq [\frac{W}{W_0+1}]c_0$ for all $(x,y) \in \mathcal{D}_{tr}$, so $\sum_{(x,y)\in\mathcal{D}_{tr}} L(\mathcal{F}_W(x), y) \leq N_0 \ln(1 + e^{-c_0[\frac{W}{W_0+1}]})$.

Then, because $\ln 1 + e^x$ is a convex function, so that:

$$\begin{array}{rl} & N_0 \ln 1 + e^{-\frac{\sum_{(x,y)\in\mathcal{D}_{tr}} y\mathcal{F}(x)}{N}} \\ \leq & \sum_{(x,y)\in\mathcal{D}_{tr}} \ln 1 + e^{-y\mathcal{F}(x)} \\ = & \sum_{(x,y)\in\mathcal{D}_{tr}} L(\mathcal{F}(x), y) \\ \leq & \sum_{(x,y)\in\mathcal{D}_{tr}} L(\mathcal{F}_W(x), y) \\ \leq & N_0 \ln(1 + e^{-c_0[\frac{W}{W_0+1}]}) \end{array}$$

So $\sum_{(x,y)\in\mathcal{D}_{tr}} y\mathcal{F}(x) \geq c_0 N[\frac{W}{W_0+1}]$.

Hence, by Lemma B.4 and Theorem B.6, with probability $1-\delta$ of $\mathcal{D}_{tr}$, there are:

$$|\mathbb{E}_{x\sim\mathcal{D}}[\mathcal{F}(x)] - \sum_{x\in\mathcal{D}_{tr}} \frac{\mathcal{F}(x)}{N_0}| \leq 2\frac{L_p(W+1)(n+1)(\sqrt{4\log(4)} + \sqrt{2\log(2n)})}{\sqrt{N_0}} + 6\mathcal{F}_{\max}\sqrt{\frac{\ln(2/\delta)}{2N_0}},$$

for all $\mathcal{F} \in \mathbf{H}_W(n)$.

Finally, combining the above two results, with probability $1-2\delta$, there is $\mathbb{E}_{x\sim\mathcal{D}}[\mathcal{F}(x)] \geq c_0 N_0[\frac{W}{W_0+1}] - 2\frac{L_p(W+1)(n+1)(\sqrt{4\log(4)}+\sqrt{2\log(2n)})}{\sqrt{N_0}} - 6\mathcal{F}_{\max}\sqrt{\frac{\ln(2/\delta)}{2N_0}}$.

Part Two, there is an upper bound of $\mathbb{E}_{(x,y)\sim\mathcal{D}}[\min_{||\delta||\leq\epsilon} y\mathcal{F}(x+\delta)]$, if $\mathcal{D}_{tr}$ satisfies Part One.

For any $\mathcal{F} \in \mathbf{H}_W(n)$, we can write $\mathcal{F} = \sum_{i=0}^{\lceil\frac{W}{W_0}\rceil-1}\sum_{j=1}^{W_0} \mathrm{ReLU}(W_{iW_0+j}x + b_{iW_0+j}) + c$, which is a representation of the sum of $\lceil\frac{W}{W_0}\rceil$ small networks with width of W_0. So by part one and by the assumption in the theorem, with probability $1-\delta$ of $\mathcal{D}_{tr} \sim \mathcal{D}^N$, there is a $\mathcal{D}_r \in R(\mathcal{D}_{tr}, \epsilon)$ such that $\sum_{(x,y)\in\mathcal{D}_r} y\mathcal{F}_1(x) \leq 2N_0c_1$ for all $\mathcal{F}_1 \in \mathbf{H}_{W_0}(n)$. Then we have $\sum_{(x,y)\in\mathcal{D}_r} y\mathcal{F}(x) \leq 2N_0c_1\lceil\frac{W}{W_0}\rceil$, by the definition of $\mathcal{D}_r$, which implies that $\sum_{(x,y)\in\mathcal{D}_{tr}} \min_{||\delta||\leq\epsilon} y\mathcal{F}(x+\delta) + y\mathcal{F}(x) \leq 2N_0c_1\lceil\frac{W}{W_0}\rceil$.

And then, by McDiarmid inequality, with probability $1-\delta$ of $\mathcal{D}_{tr} \sim \mathcal{D}^{N_0}$, there are $|\mathbb{E}_{(x,y)\sim\mathcal{D}}[\min_{||\delta||\leq\epsilon} y\mathcal{F}(x+\delta) + y\mathcal{F}(x)] - \frac{1}{N_0}\sum_{(x,y)\in\mathcal{D}_{tr}} \min_{||\delta||\leq\epsilon} y\mathcal{F}(x+\delta) + y\mathcal{F}(x)| \leq 2\mathcal{F}_{\max}\sqrt{\frac{\ln 1/\delta}{2N_0}}$. So if there are $\mathbb{E}_{(x,y)\sim\mathcal{D}}[\min_{||\delta||\leq\epsilon} y\mathcal{F}(x+\delta) + y\mathcal{F}(x)] > 2c_1\lceil\frac{W}{W_0}\rceil + 2\mathcal{F}_{\max}\sqrt{\frac{\ln 1/\delta}{2N_0}}$, according to McDiarmid inequality, with probability $1-\delta$ of $\mathcal{D}_{tr} \sim \mathcal{D}^{N_0}$, $\sum_{(x,y)\in\mathcal{D}_{tr}} \min_{||\delta||\leq\epsilon} y\mathcal{F}(x+\delta) + y\mathcal{F}(x) > 2N_0c_1\lceil\frac{W}{W_0}\rceil$ stand, which is a contradiction with the above result.

So there must be $\mathbb{E}_{(x,y)\sim\mathcal{D}}[\min_{||\delta||\leq\epsilon} y\mathcal{F}(x+\delta) + y\mathcal{F}(x)] \leq 2c_1\lceil\frac{W}{W_0}\rceil + 2\mathcal{F}_{\max}\sqrt{\frac{\ln 1/\delta}{2N_0}}$. Finally, considering the result in Part one, we have that:

$$\begin{array}{rl} & \mathbb{E}_{(x,y)\sim\mathcal{D}}[\min_{||\delta||\leq\epsilon} y\mathcal{F}(x+\delta)] \\ \leq & 2c_1\lceil\frac{W}{W_0}\rceil - c_0[\frac{W}{W_0+1}] + 2\frac{L_p(W+1)(n+1)(\sqrt{4\log(4)}+\sqrt{2\log(2n)})}{\sqrt{N_0}} + 8\mathcal{F}_{\max}\sqrt{\frac{\ln(2/\delta)}{2N_0}} \end{array}$$

Part Three, Now we can get the result.

By Lemma H.1 and part two, there are $Rob_{\mathcal{D},\epsilon}(\mathcal{F}) \leq 1 - \frac{c_0[\frac{W}{W_0+1}]-2c_1\lceil\frac{W}{W_0}\rceil}{\mathcal{F}_{\max}} + 8\sqrt{\frac{\ln 2/\delta}{2N_0}} + 2\frac{L_p(W+1)(n+1)(\sqrt{4\log(4)}+\sqrt{2\log(2n)})}{\sqrt{N_0}\mathcal{F}_{\max}}$, and we consider that each parameter of $\mathcal{F}$ is not greater than 1 and Lipschitz constant of σ is not more than L_p, so $\mathcal{F}_{\max} = \max_{x+\delta\in[0,1]^n} |\mathcal{F}(x+\delta)| = \max_{x\in[0,1]^n} |\mathcal{F}(x)| \leq L_pW(n+1)+1$.

Let $T = [\frac{W}{W_0+1}]$, by $L_p, n, W_0 \geq 1$ and $W \geq W_0+1$, there are:

$$\begin{aligned}
& \frac{c_0[\frac{W}{W_0+1}]-2c_1\lceil\frac{W}{W_0}\rceil}{L_pW(n+1)+1} \\
\geq\ & \frac{c_0T-2c_1(\frac{(T+1)(W_0+1)}{W_0}+1)}{L_p(T+1)(W_0+1)(n+1)+1} \\
=\ & \frac{c_0T-2c_1T}{L_p(T+1)(W_0+1)(n+1)+1} - \frac{4c_1}{L_p(T+1)(W_0+1)(n+1)+1} - \frac{2c_1}{L_pW_0(W_0+1)(n+1)+1} \\
\geq\ & \frac{c_0-2c_1}{8L_pW_0n} - \frac{4c_1}{L_pW_0n}(\frac{1}{W/W_0}+\frac{1}{W_0})
\end{aligned}$$

and

$$\begin{aligned}
& \frac{L_p(W+1)(n+1)(\sqrt{4\log(4)}+\sqrt{2\log(2n)})}{\sqrt{N_0}\mathcal{F}_{\max}} \\
\geq\ & \frac{L_p(W+1)(n+1)(\sqrt{4\log(4)}+\sqrt{2\log(2n)})}{2\sqrt{N_0}L_p(W+1)(n+1)} \\
=\ & 2\frac{\sqrt{4\log(4)}+\sqrt{2\log(2n)}}{\sqrt{N_0}}
\end{aligned}$$

So, there are $\mathrm{Rob}_{\mathcal{D},\epsilon}(\mathcal{F}) \leq 1 - \frac{c_0-2c_1}{8L_pW_0n} + \frac{4c_1}{L_pW_0n}(\frac{1}{W/W_0}+\frac{1}{W_0}) + 2\sqrt{\frac{\ln 2/\delta}{2N_0}} + 4\frac{\sqrt{4\log(4)}+\sqrt{2\log(2n)}}{\sqrt{N_0}}$. Merge some items and ignore constants, this is what we want. □

A required lemma is given:

Lemma H.1. *If $\mathcal{F}: \mathbb{R}^n \to \mathbb{R}$ and distribution $\mathcal{D} \in [0,1]^n \times \{-1,1\}$ satisfy $\mathbb{E}_{(x,y)\sim\mathcal{D}}[y\mathcal{F}(x)] \leq A$ and $\max_{x\in[0,1]^n} |\mathcal{F}(x)| \leq B$, then $A_{\mathcal{D}}(\mathcal{F}) \leq 1 + \frac{A}{B}$.*

Proof. There are $\mathbb{E}_{(x,y)\sim\mathcal{D}}[y\mathcal{F}(x)] \geq -(\max_{x\in[0,1]^n} |\mathcal{F}(x)|)\mathbb{P}_{(x,y)\sim\mathcal{D}}(y \neq \mathrm{Sgn}(\mathcal{F}(x))) = -B(1-A_{\mathcal{D}}(\mathcal{F}))$, so $A \geq -B + BA_{\mathcal{D}}(\mathcal{F})$, that is, $A_{\mathcal{D}}(\mathcal{F}) \leq 1 - \frac{A}{B}$. □

I PROOF OF PROPOSITION 6.5

Proof. We take a $c > 0$ such that $\ln(1+e^{-c}) \geq \ln 2 - \ln 2/800$, $1-(1/e)^{4c} < 0.1$. Then take an n such that $\ln(1+e^{-n/2+2c}) < \ln 2/2$. Let N satisfy $(\frac{4(n+1)(\sqrt{5\log(4)}+\sqrt{2log2n})}{\sqrt{98N/200}} + 6(n+2)\sqrt{\frac{\ln(2/\delta)}{2N}}) < \ln 2/800$.

We consider the following distribution $\mathcal{D}$:

(c1): Let $s_1 = \{(x,1) : x \in [0,1], \sum x = n/2+c, ||x||_{-\infty} \geq 2c/n\}$, $||x||_{-\infty}$ mean the minimum of the weight of $|x|$; $s_2 = \{(x,-1) : x \in [0,1], \sum x = n/2-c, ||x||_{\infty} \leq 1-2c/n\}$; $s_3 = \{(x,-1) : x \in [0,1], \sum x = n-c\}$;

(c2): $\mathbb{P}_{(x,y)\sim\mathcal{D}}(\sum x = n-c) = 1/100$, and $\mathcal{D}$ is a uniform distribution in s_3;

(c3): $\mathbb{P}_{(x,y)\sim\mathcal{D}}(\sum x = n/2+c) = \mathbb{P}_{(x,y)\sim\mathcal{D}}(\sum x = n/2-c) = 99/200$, and $\mathcal{D}$ is a uniform distribution in $s_1 \cup s_2$.

Let $W_0 = 1$, then we show this distribution and W_0 are what we want.

(1) in Theorem: Let $\mathcal{F}_1 = Relu(\mathbf{1}x) - c/2 \in \mathbf{H}_1(n)$. Then $\mathcal{F}_1(x) > 0$ for all x such that $\sum x = c$, and $\mathcal{F}_1(x) < 0$ for all x such that $\sum x = -c$, so $A_{\mathcal{D}}(\mathcal{F}_1) \geq 0.99$.

(2) in Theorem: We use the following parts to show the (2) in the Theorem.

Part One. With probability at least $1-3e^{-2N/200^2}$ of $\mathcal{D}_{tr}\sim\mathcal{D}^N$, there are at least $N/200$ points in $\mathcal{D}_{tr}\cap s_3$, and at least $98/200N$ points with label 1 in $\mathcal{D}_{tr}$, at least $98/200N$ points with label -1 in $\mathcal{D}_{tr}$.

Using the Hoeffding inequality and $\mathbb{P}_{(x,y)\sim\mathcal{D}}(\sum x=n-c)=1/100$, we know that with probability at least $1-e^{-2N/200^2}$ of $\mathcal{D}_{tr}$, there are at least $N/200$ points in s_3. Using also the Hoeffding inequality and $\mathbb{P}_{(x,y)\sim\mathcal{D}}(y=1)=99/200$, we know that with probability at least $1-e^{-2N(99/200-98/200)^2}$ of $\mathcal{D}_{tr}$, there are at least $98/200N$ points with label 1 in $\mathcal{D}_{tr}$; similar, with probability at least $1-e^{-2N(101/200-98/200)^2}$ of $\mathcal{D}_{tr}$, there are at least $98/200N$ points with label -1 in $\mathcal{D}_{tr}$. Adding them, we get the result.

Part Two. For a $\mathcal{D}_{tr}$ that satisfies Part One, if $\mathcal{F}\in\underset{f\in\mathbf{H}_W(n)}{\operatorname{argmin}}\sum_{(x,y)\in\mathcal{D}_{tr}}L(f(x),y)$, then there is $\sum_{(x,y)\in\mathcal{D}_{tr}}L(\mathcal{F}(x),y)\leq\frac{199\ln 2+\ln(1+e^{-n/2+2c})}{200}N$.

We just consider the following network $\mathcal{F}_1\in\mathbf{H}_1(n)$: $\mathcal{F}_1=-\mathrm{ReLU}(\mathbf{1}x-(n/2+c))$, then $\sum_{(x,y)\in\mathcal{D}_{tr}}L(\mathcal{F}_1(x),y)=\ln 2|\mathcal{D}_{tr}/s_3|+\ln 1+e^{-n/2+2c}|\mathcal{D}_{tr}\cap s_3|\leq\frac{199\ln 2+\ln(1+e^{-n/2+2c})}{200}$. Hence, for any $\mathcal{F}\in\underset{f\in\mathbf{H}_W(n)}{\operatorname{argmin}}\sum_{(x,y)\in\mathcal{D}_{tr}}L(f(x),y)$, there must be $\sum_{(x,y)\in\mathcal{D}_{tr}}L(\mathcal{F}(x),y)\leq\sum_{(x,y)\in\mathcal{D}_{tr}}L(\mathcal{F}_1(x),y)\leq\frac{199\ln 2+\ln(1+e^{-n/2+2c})}{200}N$, which is what we want.

Part Three. If $\mathcal{F}\in\mathbf{H}_1(n)$ such that $\mathcal{F}(x)\geq 0$ for all $(x,-1)\in s_3$. Then $\mathbb{E}_{(x,y)\sim\mathcal{D}}[L(\mathcal{F}(x),y)]\geq 99/100\ln 1+e^{-c}+1/100\ln 2$.

Consider that for any $(x_1,1)\in s_1$, there must be $(x_1-2c\mathbf{1}/n,-1)\in s_2$; on the other hand, if $(x_2,-1)\in s_2$, there must be $(x_2+2c\mathbf{1}/n,1)\in s_1$. So we can match the points in s_1 and s_2 one by one by adding or subtracting a vector $2c\mathbf{1}/n$.

Moreover, for any $x\in[0,1]$ and $x\in\mathbf{H}_1(n)$, there are $|\mathcal{F}(x)-\mathcal{F}(x-2c\mathbf{1}/n)|\leq 2c$, which implies $L(\mathcal{F}(x),1)+L(\mathcal{F}(x-2c\mathbf{1}/n),-1)=\ln(1+e^{-\mathcal{F}(x)})+\ln(1+e^{\mathcal{F}(x-2c\mathbf{1}/n)})\geq 2\ln 1+e^{-c}$. So for a $(x_1,1)\in s_1$ and $(x_2,-1)\in s_2$ where $x_2=x_1-2c\mathbf{1}/n$, there must be $L(\mathcal{F}(x_1),1)+L(\mathcal{F}(x_2),-1)\geq 2\ln 1+e^{-c}$.

Hence, by $\mathcal{F}(x)>0$ for all $(x,-1)\in s_3$, $\mathbb{E}_{(x,y)\sim\mathcal{D}}[L(\mathcal{F}(x),y)]\geq 99/200\ln(1+e^{-c})+\ln 2/100$.

Part Four. For any network $\mathcal{F}\in\mathbf{H}_1(n)$ such that $\mathcal{F}(x)<0$ for a $x\in s_3$, then $A_{\mathcal{D}}(\mathcal{F})<60\%$.

Firstly, we show that if z_1,z_2,z_3 are collinear, without loss of generality, assuming z_2 is between z_1 and z_3, then $\mathcal{F}(z_1)\geq\mathcal{F}(z_2)\geq\mathcal{F}(z_3)$ or $\mathcal{F}(z_1)\leq\mathcal{F}(z_2)\leq\mathcal{F}(z_3)$. Consider that z_1,z_2,z_3 are collinear, so $z_2=\lambda z_1+(1-\lambda)z_3$ for some $\lambda\in(0,1)$. So let $f(k)=\mathrm{ReLU}(k(Wz_1+b)+(1-k)(Wz_3+b))$, there are $f(0)=\mathrm{ReLU}(Wz_3+b)$, $f(1)=\mathrm{ReLU}(Wz_1+b)$ and $f(\lambda)=\mathrm{ReLU}(\lambda(Wz_1+b)+(1-\lambda)(Wz_3+b))=\mathrm{ReLU}(Wz_2+b)$. Consider that $\mathrm{ReLU}(\cdot)$ is a monotonic function, so that $f(k)$ is also an monotonic function about $k\in\mathbb{R}$, so we get the result.

Secondly, for any $(z,-1)\in s_2$, let x_z satisfy: $(x_z,1)\in s_1$ and x,x_z,z are collinear. Then we have that:

(1): For any $(z,-1)\in s_2$, $\mathcal{F}$ must give the wrong label to x_z or z. If not, there are $\mathcal{F}(x)<0$, $\mathcal{F}(x_z)>0$ and $\mathcal{F}(z)<0$. By the above result, it is not possible.

(2): Let $S=\{x_z:(z,1)\in s_2\}\subset s_1$, then $\mathbb{P}_{(x,y)\sim\mathcal{D}}(x\in S|x\in s_1)\geq(1-4c/n)^{n-1}$. Because for any $(z,1)\in s_2$, $\frac{||x-x_z||_2}{||x-z||_2}=\frac{\sum(x-x_z)}{\sum(x-z)}=\frac{n/2-2c}{n/2}$, which is a constant value, where $\sum x$ means the sum of the weights of x, so S is a proportional scaling of s_1 with the ratio $\frac{n-4c}{n}$, we get the result.

So, there are: $A_{\mathcal{D}}(\mathcal{F})\leq max\{\mathbb{P}_{(x,y)\sim\mathcal{D}}((x,y)\in s_2),\mathbb{P}_{(x,y)\sim\mathcal{D}}((x,y)\in S)\}+\mathbb{P}_{(x,y)\sim\mathcal{D}}((x,y)\in s_3)+\mathbb{P}_{(x,y)\sim\mathcal{D}}(s_1/S)\leq\frac{101+99(1-(1-4c/n)^{n-1})}{200}\leq 101/200+99/200*(1-(1/e)^{4c})\leq 0.6$, use the definition of c.

Part Five. Prove the Theorem.

We show that with probability $1 - 3e^{-2N/200^2} - \delta$ of $\mathcal{D}_{tr}$, for any $\mathcal{F} \in \underset{f\in\mathbf{H}_W(n)}{\operatorname{argmin}} \sum_{(x,y)\in\mathcal{D}_{tr}} L(f(x), y)$, $\mathcal{F}$ must give the correct label to some points in s_3. Then by part four, we can get the result.

By part one, with probability at least $1 - 3e^{-2N/200^2}$ of $\mathcal{D}_{tr}$, there are at least $N/200$ points in $\mathcal{D}_{tr} \cap s_3$, and at least $98N/200(98N/200)$ points has label 1(-1). Hence, by Lemma I.1 and Theorem 4.8, we know that, with probability $1 - \delta$ of $\mathcal{D}_{tr}$, there are $|\sum_{(x,y)\in\mathcal{D}_{tr}} L(\mathcal{F}(x), y)/N - \mathbb{E}_{(x,y)\sim\mathcal{D}}[L(\mathcal{F}(x), y)]| \geq \frac{4(n+1)(\sqrt{5\log(4)}+\sqrt{2log2n})}{\sqrt{98N/200}} + 6(n+2)\sqrt{\frac{\ln(2/\delta)}{2N}})$. So, with probability $1 - 3e^{-2N/200^2} - \delta$, $\mathcal{D}_{tr}$ satisfies the above two conditions.

For such a $\mathcal{D}_{tr}$, assume that $\mathcal{F} \in \underset{f\in\mathbf{H}_W(n)}{\operatorname{argmin}} \sum_{(x,y)\in\mathcal{D}_{tr}} L(f(x), y)$, and $\mathcal{F}$ must give the correct label to some points in s_3.

If not, by part two, we know that $\sum_{(x,y)\in\mathcal{D}_{tr}} L(\mathcal{F}(x), y) \leq \frac{199\ln 2+\ln(1+e^{-n/2+2c})}{200} N$.

Then, by part three, $\mathbb{E}_{(x,y)\sim\mathcal{D}} L(\mathcal{F}(x), y) \geq 99/100 \ln 1 + e^{-c} + 1/100 \ln 2$. Hence, by the definition of $\mathcal{D}_{tr}$, there are $\sum_{(x,y)\in\mathcal{D}_{tr}} L(\mathcal{F}(x), y) \geq N(99/100 \ln 1 + e^{-c} + 1/100 \ln 2) - N(\frac{4(n+1)(\sqrt{5\log(4)}+\sqrt{2log2n})}{\sqrt{98N/200}} - 6(n+2)\sqrt{\frac{\ln(2/\delta)}{2N}})$.

By the definition of c, n and N, there are $\sum_{(x,y)\in\mathcal{D}_{tr}} L(\mathcal{F}(x), y)/N \geq (99/100 \ln 1 + e^{-c} + 1/100 \ln 2) - (\frac{4(n+1)(\sqrt{5\log(4)}+\sqrt{2log2n})}{\sqrt{98N/200}} + 6(n+2)\sqrt{\frac{\ln(2/\delta)}{2N}}) \geq \frac{199.5\ln 2}{200} > \frac{199\ln 2+\ln(1+e^{-n/2+2c})}{200} \geq \sum_{(x,y)\in\mathcal{D}_{tr}} L(\mathcal{F}(x), y)/N$, which leads to contradiction. And we prove the result. □

A required lemma is given:

Lemma I.1. *For any given $D = \{(x_i, y_i)\}_{i=1}^N$, if there are at least K samples have label 1 in it and there are at least K samples have label -1 in it, then there are:*

$$\mathbb{E}_{\sigma_i}[\max_{\mathcal{F}\in\mathbf{H}_1(n)} \frac{1}{N}\sum_{i=1}^N \sigma_i L(\mathcal{F}(x_i), y_i)] \leq \frac{4(n+1)(\sqrt{5\log(4)} + \sqrt{2\log(2n)})}{\sqrt{K}},$$

where σ_i are i.i.d and $P(\sigma_i = 1) = P(\sigma_i = -1) = 0.5$.

Proof. We have

$$\begin{aligned} & \mathbb{E}_{\sigma_i}[\max_{\mathcal{F}\in\mathbf{H}_1(n)} \tfrac{1}{N}\textstyle\sum_{i=1}^N \sigma_i L(\mathcal{F}(x_i), y_i)] \\ = \; & \mathbb{E}_{\sigma_i}[\max_{\mathcal{F}\in\mathbf{H}_1(n)} \tfrac{1}{N}\textstyle\sum_{i=1}^N \sigma_i \ln 1 + e^{y_i\mathcal{F}(x_i)}] \\ \leq \; & \mathbb{E}_{\sigma_i}[\max_{\mathcal{F}\in\mathbf{H}_1(n)} \tfrac{1}{|D_1|}\textstyle\sum_{x\in D_1} \sigma_i \ln 1 + e^{\mathcal{F}(x)}] + \mathbb{E}_{\sigma_i}[\max_{\mathcal{F}\in\mathbf{H}_1(n)} \tfrac{1}{|D_2|}\textstyle\sum_{x\in D_2} \sigma_i \ln 1 + e^{\mathcal{F}(x)}] \end{aligned}$$

Hence, see $2\ln(1 + e^x)$ as an activation of the second layer, and the output layer is $\mathcal{F}_2(x) = x/2$. By Lemma B.4, we have $\mathbb{E}_{\sigma_i}[\max_{\mathcal{F}\in\mathbf{H}_1(n)} \frac{1}{|D_1|}\sum_{x\in D_1} \sigma_i \ln 1 + e^{\mathcal{F}(x)}] \leq \frac{2(n+1)(\sqrt{5\log(4)}+\sqrt{2\log(2n)})}{\sqrt{|D_1|}}$. Similar for an other part, so we get the result. □

J Proof of Theorem 6.8

We give the proof of when loss function L_b satisfies (1) in definition 6.7 at first.

Proof. We first define some symbols.

Let the loss function L_b be a bad loss function that satisfies (1) in Definition 6.7, let $L_b(z_1, 1) = \min_{x\in\mathbb{R}} L_b(x, 1)$ and $L_b(z_{-1}, -1) = \min_{x\in\mathbb{R}} L_b(x, -1)$, assume $|z_1| + |z_{-1}| = z$. For any

given $x \in \mathbb{R}^n$, let $x_t = (x_2, x_3, \ldots, x_n) \in \mathbb{R}^{n-1}$, where x_i is the i-the weight of x; let $x^t = (0, x_1, x_2, x_3, \ldots, x_n) \in \mathbb{R}^{n+1}$.

Then we prove the Theorem in three parts:

Part One: We construct the following distribution $\mathcal{D}_b \in [0,1]^n \times \{-1, 1\}$:

(1): $\mathcal{D}_b$ is defined on $\{x : x \in [0,1]^n, 0.6 \leq x_1 \; or \; x_1 \leq 0.4\} \times \{-1, 1\}$, where x_1 is the first weight of x.

(2): x has label 1 if and only if $x_1 \geq 0.6$, or x has label -1.

(3): The marginal distribution about x of $\mathcal{D}_b$ is an uniform distribution.

Part Two: For any $\mathcal{D}_{tr} \sim \mathcal{D}_b^N$, we consider the following network $\mathcal{F}_{\mathcal{D}_{tr}}$.

Let $\mathcal{D}_{tr-t} = \{(x_t, y) \| (x, y) \in \mathcal{D}_{tr}\}$. By Lemma J.2, with probability 0.99, there is a $\mathcal{F}_t$ with width W not greater than $O(zN^5n^2)$ such that: if $(x_t, 1) \in \mathcal{D}_{tr-t}$, there are $\mathcal{F}_t(x_t) = z_1$; if $(x_t, -1) \in \mathcal{D}_{tr-t}$, there are $\mathcal{F}_t(x_t) = z_{-1}$. Let $\mathcal{F}_t(x) = \sum_{i=1}^{W} a_i \mathrm{ReLU}(W_i x + b_i) + c$.

Then, we construct $\mathcal{F}_{\mathcal{D}_{tr}} : \mathbb{R}^n \to \mathbb{R}$ as $\mathcal{F} = \sum_{i=1}^{W} a_i \mathrm{ReLU}(W_i^t x + b_i) + c$.

Part Three: We prove the Theorem.

For any $\mathcal{D}_{tr} \sim \mathcal{D}_b^N$, we consider the network $\mathcal{F}_{\mathcal{D}_{tr}}$ mentioned in part two. Firstly, we show that $\mathcal{F}_{\mathcal{D}_{tr}}(x) \in \arg\min_{\mathcal{F} \in \mathcal{H}_W(n)} \sum_{(x,y) \in \mathcal{D}_{tr}} L(\mathcal{F}(x), y)$. Because $\mathcal{F}_{\mathcal{D}_{tr}}(x) = \mathcal{F}_t(x_t) = z_1$ when $(x, 1) \in \mathcal{D}_{tr}$ and $\mathcal{F}_{\mathcal{D}_{tr}}(x) = \mathcal{F}_t(x_t) = z_{-1}$ when $(x, -1) \in \mathcal{D}_{tr}$. So $L(\mathcal{F}_{\mathcal{D}_{tr}}(x), y)$ reaches the minimum value for any $(x, y) \in \mathcal{D}_{tr}$, which implies $\mathcal{F}_{\mathcal{D}_{tr}} \in \arg\min_{\mathcal{F} \in \mathcal{H}_W(n)}$.

Secondly, there are $A_{\mathcal{D}}(\mathcal{F}_{\mathcal{D}_{tr}}(x)) = 0.5$. If $A_{\mathcal{D}}(\mathcal{F}_{\mathcal{D}_{tr}}(x)) > 0.5$, then there must be a pair of $(x_1, 1)$ and $(x_2, -1)$ in distribution $\mathcal{D}_b$ such that $(x_1)_t = (x_2)_t$ and $\mathcal{F}_{\mathcal{D}_{tr}}(x)$ give the correct label to x_1 and x_2. But it is easy to see that $\mathcal{F}_{\mathcal{D}_{tr}}(x) = \mathcal{F}_t(x_t)$ where $\mathcal{F}_t$ is mentioned in part two, so, $\mathcal{F}_{\mathcal{D}_{tr}}(x_1) = \mathcal{F}_t(x)((x_1)_t) = \mathcal{F}_t(x)((x_2)_t) = \mathcal{F}_{\mathcal{D}_{tr}}(x_2)$, which is in contradiction to $\mathcal{F}_{\mathcal{D}_{tr}}(x)$ gives the correct label to x_1 and x_2. This is what we want.

□

Some required lemmas are given.

Lemma J.1. *For any $v \in \mathbb{R}^n$ and $T \geq 1$, let $u \in \mathbb{R}^n$ be uniformly randomly sampled from the hypersphere S^{n-1}. Then we have $\mathbb{P}(|\langle u, v \rangle| < \frac{||v||_2}{T}\sqrt{\frac{8}{n\pi}}) < \frac{2}{T}$.*

This is Lemma 13 in (Park et al., 2021).

Lemma J.2. *For any N points $\{x_i\}_{i=1}^N$ randomly selected in $[0,1]^n$, and any N given point $\{y_i\}_{i=1}^N$ in $[-a, a]$. With probability 0.99 of $\{x_i\}_{i=1}^N$, there is a network $\mathcal{F}$ with width not more than $O(aN^5n^2)$ and $\mathcal{F}(x_i) = y_i$.*

Proof. **Part One:** First, we show that with probability 0.99, there is $||x_i - x_j||_2 \geq \frac{0.01}{2N^2\sqrt{n}}$ for all pairs i, j.

For any $i, j \in \mathbb{N}$ and $\epsilon > 0$, there are:

$$\begin{array}{rl} & P(||x_i - x_j||_2 \geq \epsilon) \\ = & P(\sum_{k=1}^{n}((x_i)_k - (x_j)_k)^2 \geq \epsilon^2) \\ \geq & \Pi_{k=1}^{n} P(((x_i)_k - (x_j)_k)^2 \geq \epsilon^2/n) \\ \geq & \Pi_{k=1}^{n}(1 - \frac{2\epsilon}{\sqrt{n}}) \\ \geq & 1 - 2\epsilon\sqrt{n} \end{array}$$

So $\mathbb{P}(||x_i - x_j||_2 \geq \epsilon, \forall(i,j)) \geq 1 - \sum_{i \neq j} P(||x_i - x_j||_2 < \epsilon) \geq 1 - 2\epsilon\sqrt{n}N^2$. Take $\epsilon = \frac{0.01}{2\sqrt{n}N^2}$, we get the result.

Part Two: There is a $w \in \mathbb{R}^n$ such that $||w||_2 = 1$ and $|w(x_i - x_j)| \geq \frac{0.01}{4N^4n}\sqrt{\frac{8}{\pi}}$

By Lemma J.1, for any pair i, j, $\mathbb{P}_u(|u(x_i - x_j)| < \frac{||x_i - x_j||_2}{2N^2}\sqrt{\frac{8}{n\pi}}) < \frac{1}{N^2}$. So, $\mathbb{P}_u(|u(x_i - x_j)| \geq \frac{||x_i - x_j||_2}{2N^2}\sqrt{\frac{8}{n\pi}}, \forall(i,j)) \geq 1 - \sum_{i \neq j} \mathbb{P}_u(|u(x_i - x_j)| < \frac{||x_i - x_j||_2}{2N^2}\sqrt{\frac{8}{n\pi}}) > 1 - 1 = 0$, which implies that there is a w such that $||w||_2 = 1$ and for any pair (i, j), there are $|w(x_i - x_j)| \geq \frac{||x_i - x_j||_2}{2N^2}\sqrt{\frac{8}{n\pi}} \geq \frac{0.01}{4N^4 n}\sqrt{\frac{8}{\pi}}$, use the result of part one.

Part Three: Prove the result.

Let w be the vector mentioned in part two, and $wx_i < wx_j$ when $i \neq j$. Let $\delta = \frac{0.01}{4N^4 n}\sqrt{\frac{8}{\pi}}$. Now, we consider the following network:

$$\mathcal{F}(x) = \sum_{i=1}^{N} \frac{y_i}{\delta}(\mathrm{ReLU}(wx - (wx_i + \delta)) + \mathrm{ReLU}(wx - (wx_i - \delta)) - 2\mathrm{ReLU}(wx - wx_i)).$$

Easy to verify $\mathcal{F}(x_i) = y_i$. Consider $|wx_i| \leq n$ and $|\frac{y_i}{\delta}| < 400aN^4 n$, so $\mathcal{F} \in \mathbf{H}_{O(aN^5n^2)}(n)$. This is what we want. □

We now give the proof of when the loss function L_b satisfies (2) in definition 6.7.

Proof. In this proof, we only consider a very simple distribution $\mathcal{D}$: $\mathbb{P}_{(x,y)\sim\mathcal{D}}((x, y) = (0, -1)) = \mathbb{P}_{(x,y)\sim\mathcal{D}}((x, y) = (\mathbf{1}, 1)) = 0.5$, where $\mathbf{1}$ is a all one vector.

We show that for any $\mathcal{D}_{tr}$ and W, let $\mathcal{F} \in \arg\min_{f \in \mathbf{H}_W(n)} \sum_{(x,y)\in\mathcal{D}_{tr}} L_b(f(x), y)$, there are $A_{\mathcal{D}}(\mathcal{F}) = 0.5$.

Part one: When $\mathcal{D}_{tr}$ contains only $(0, -1)$, then there must be $\mathcal{F} = \sum_{i=1}^{W} -\mathrm{ReLU}(w_i x + 1) - 1$ for some x_i, which implies $\mathcal{F}(\mathbf{1}) < 0$, so $A_{\mathcal{D}}(\mathcal{F}) = 0.5$.

Part two: When $\mathcal{D}_{tr}$ contains only $(\mathbf{1}, 1)$, then there must be $\mathcal{F} = \sum_{i=1}^{W} \mathrm{ReLU}(\mathbf{1}x + 1) + 1$, which implies $\mathcal{F}(0) > 0$, so $A_{\mathcal{D}}(\mathcal{F}) = 0.5$.

Part Three: When $\mathcal{D}_{tr}$ contains $(\mathbf{1}, 1)$ and $(0, -1)$, we will show that $\mathcal{F} = \sum_{i=1}^{W} \mathrm{ReLU}(\mathbf{1}x + 1) + 1 \in \arg\min_{f \in \mathbf{H}_W(n)} L_b(f(0), -1) + L_b(f(\mathbf{1}), 1)$. Consider that $A_{\mathcal{D}}(\mathcal{F}) = 0.5$ for such $\mathcal{F}$, we can prove the Theorem.

If $\mathcal{F} = \sum_{i=1}^{W} \mathrm{ReLU}(\mathbf{1}x + 1) + 1 \notin \arg\min_{f \in \mathbf{H}_W(n)} L_b(f(0), -1) + L_b(f(\mathbf{1}), 1)$. Let $\mathcal{F}_0(x) = \sum_{i=1}^{W} a_i \mathrm{ReLU}(W_i x + b_i) + c \in \arg\min_{f \in \mathbf{H}_W(n)} L_b(f(0), -1) + L_b(f(\mathbf{1}), 1)$. Then, let $\mathcal{F}_0(0) = b$ and $\mathcal{F}_0(\mathbf{1}) = a$.

By $\phi(a) + \phi(-b) = L_b(\mathcal{F}_0(0), -1) + L_b(\mathcal{F}_0(\mathbf{1}), 1) < L_b(\mathcal{F}(0), -1) + L_b(\mathcal{F}(\mathbf{1}), 1) = \phi(W(n + 1) + 1) + \phi(-W - 1)$, and ϕ is a decreasing concave function, then there must be $W(n+1) + 1 - a < -b + W + 1$, which implies $|a - b| > Wn$.

Consider $|a - b| = |\sum_{i=1}^{W} a_i \mathrm{ReLU}(b_i) - \sum_{i=1}^{W} a_i \mathrm{ReLU}(W_i \mathbf{1} + b_i)| \leq |\sum_{i=1}^{W} a_i \mathbf{1} W_i| \leq Wn$. This is a contradiction to $|a - b| > Wn$ which was shown above. So, assumption is wrong, so $\mathcal{F} = \sum_{i=1}^{W} \mathrm{ReLU}(\mathbf{1}x + 1) + 1 \in \arg\min_{f \in \mathbf{H}_W(n)} L_b(f(0), -1) + L_b(f(\mathbf{1}), 1)$, this is what we want.

□

K For the general network

For multi-layer neural networks, we can show that if there is enough data and the network is large enough, then generalization can also be ensured for the network which can minimum the empirical risk. Unfortunately, due to the complexity of depth networks, we are unable to provide a good generalization bound of such network.

Denote $\mathbf{H}_{W,D}(n)$ to be the set of all neural networks of layers D with input dimension n, width W for each hidden layer, activation function ReLU, and all parameters of the transition matrix are in $[-1, 1]$. Then, there are:

Theorem K.1. *For any given $n \in \mathbb{N}_+$, if $\mathcal{D} \in \mathcal{D}(n)$ satisfies: there is a network $\mathcal{F} \in \mathbf{H}_{W_0,D_0}(n)$ such that $\mathbb{P}_{(x,y)\sim\mathcal{D}}(y\mathcal{F}(x) > c) = 1$ for a $W_0, D_0 \in \mathbb{N}_+, c > 0$, then we have that:*

For any $W \geq \Omega(W_0)$, $D \geq \Omega(D_0)$ and $\delta > 0$, with probability at least $1-\delta$ of $\mathcal{D}_{tr} \sim \mathcal{D}^N$, there are: $A_{\mathcal{D}}(\mathcal{F}) \geq 1 - O(e^{-W^D/K} + K^n\sqrt{\frac{\ln(K/\delta)}{N}})$ for all $\mathcal{F} \in \mathbf{M}_{W,D}(\mathcal{D}_{tr}, n)$, where $K = (\frac{c}{2^{D_0+2}W_0^{D_0-1}n})^{-1}$.

However, this bound is relatively loose, how to achieve a bound that is polynomial in W_0, D_0, c is an important question.

Proof. **Part One**. For any given $\mathcal{D}_{tr} \sim \mathcal{D}^N$, we show that there is a network $\mathcal{F} \in \mathbf{H}_{W,D}$ such that $y\mathcal{F}(x) \geq [\frac{W}{W_0}]^{D_0-1}\frac{cW^{D-D_0}}{2}$ for any $(x,y) \in \mathcal{D}_{tr}$.

By the assumption of $\mathcal{D}$ in the theorem, let $\mathcal{F}_1 \in \mathbf{H}_{W_0,D_0}(n)$ satisfy $\mathbb{P}_{(x,y)\sim\mathcal{D}}(y\mathcal{F}_1(x) \geq c) = 1$. And W_i is the i-th transition matrix of $\mathcal{F}_1$, b_i is the i-th bias vector of $\mathcal{F}_1$.

We will construct $\mathcal{F}$ as $\mathcal{F} = \mathcal{F}_{p2} \circ \mathcal{F}_{p1}$, and we construct the two networks $\mathcal{F}_{p1}$ and $\mathcal{F}_{p2}$ as following:

$\mathcal{F}_{p1} : \mathbb{R}^n \to \mathbb{R}^W$ which has width W and depth D_0, and the output layer of $\mathcal{F}_{p1}$ also uses the ReLU activation function.

Let W be a matrix in $\mathbb{R}^{a,b}$ where $a, b \in \mathbb{N}^+$, and $T(W, a_1, b_1)$ is a matrix in $\mathbb{R}^{a_1,b_1}$ defined as: for any $i \in [a], j \in [b], k_1, k_2 \in \mathbb{Z}$, there are $T(W, a_1, b_1)_{k_1[\frac{a_1}{a}]+i, k_2[\frac{b_1}{b}]+j} = W_{i,j}$; other weights of $T(W, a_1, b_1)$ are 0. Then $\mathcal{F}_{p1}$ is defined as:

(1): The first transition matrix is $T(W_1, W, n)$, and the first bias vector is $T(b_1, W, 1)$;

(2): When $i > 2$, the i-th transition matrix is $T(W_i, W, W)$, and the i-th bias vector is $[\frac{W}{W_0}]^{i-1}T(b_i, W, 1)$.

Then, we have $\mathcal{F}_{p1}(x) = [\frac{W}{W_0}]^{D_0-1}\mathrm{ReLU}(\mathcal{F}_1(x))$.

For $\mathcal{F}_{p2} : \mathbb{R}^W \to \mathbb{R}$, which has width W and depth $D - D_0$, we define it as:

(1): When $i < D - D_0$, the i-th transition matrix is $\mathbb{I}_{W,W}$, and the i-th bias vector is 0, where $\mathbb{I}$ means all one matrix;

(2): The last transition matrix is $\mathbb{I}(1, W)$, and the last bias vector is $-[\frac{W}{W_0}]^{D_0-1}\frac{cW^{D-D_0}}{2}$.

Then, $\mathcal{F} = \mathcal{F}_2 \circ \mathcal{F}_1$ is what we want.

Part two. Similar to the proof of 4.3, there are at most $Ne^{-[\frac{W}{W_0}]^{D_0-1}\frac{cW^{D-D_0}}{4}+2}$ points in $\mathcal{D}_{tr}$ such that $y\mathcal{F}(x) \leq [\frac{W}{W_0}]^{D_0-1}\frac{cW^{D-D_0}}{4}$.

Part three. If $y\mathcal{F}(x) \geq [\frac{W}{W_0}]^{D_0-1}\frac{cW^{D-D_0}}{4}$, then $y\mathcal{F}(x') > 0$ for all $||x' - x||_\infty \leq \frac{c}{2^{D_0+1}W_0^{D_0-1}n}$.

As shown in Lemma K.2, there are $y\mathcal{F}(x') \geq [\frac{W}{W_0}]^{D_0-1}\frac{cW^{D-D_0}}{4} - W^{L-1}n||x - x'||_\infty$. So when $||x - x'||_\infty \leq \frac{c[\frac{W}{W_0}]^{L_0-1}}{4nW^{L_0-1}} \leq \frac{c}{2^{D_0+1}W_0^{D_0-1}n}$, there are $y\mathcal{F}(x') > 0$.

Part four. Let $r = \frac{c}{2^{D_0+1}W_0^{D_0-1}n}$. we can divide $[0,1]^n$ into $\frac{1}{(r/2)^n}$ disjoint cubes that have side length $r/2$. Then by part three, we know that in a cube, $\mathcal{F}$ gives the same label to every point in such a cube when $|\mathcal{F}(x)| \geq [\frac{W}{W_0}]^{D_0-1}\frac{cW^{D-D_0}}{2}$ for at least one x in such cube.

Part Five. Prove the result.

By part four, name such m cubes as $c_1, c_2, \cdots, c_m$, and let $\mathbb{P}_i = \mathbb{P}_{(x,y)\sim\mathcal{D}}(x \in c_i)$ and $\mathbb{P}_i \geq \mathbb{P}_j$ when $i \geq j$.

As shown in part four, let $S = \{i \in [N], \exists (x,y) \in \mathcal{D}_{tr} \cap c_i, y\mathcal{F}(x) \geq [\frac{W}{W_0}]^{D_0-1}\frac{cW^{D-D_0}}{4}\}$, then we have $A_{\mathcal{D}}(\mathcal{F}) \geq \sum_{i\in S}\mathbb{P}_i$.

For any i, by Hoeffding inequality, with probability $1 - e^{-N\mathbb{P}_i^2/2}$, there are at least $N\mathbb{P}_i/2$ points in cube c_i. So for any given $\epsilon_0 > 0$, let $\mathbb{P}_{k_0} \geq \epsilon_0$, then, with probability at least $1 - \sum_{i=k_0}^{m} e^{-N\epsilon_0^2/2}$ of $\mathcal{D}_{tr}$, there are at least $N\mathbb{P}_i/2$ points in C_i for any $i \geq k_0$.

As shown in part two, there are at most $Ne^{-[\frac{W}{W_0}]^{D_0-1}\frac{cW^{D-D_0}}{4}+2}$ points in $\mathcal{D}_{tr}$ such that $y\mathcal{F}(x) \leq [\frac{W}{W_0}]^{D_0-1}\frac{cW^{D-D_0}}{4}$. So, by the above result, let $T = \{k_0, k_0+1, \ldots, N\}/S$ and $N(C_i)$ is the number of points in C_i, with probability at least $1 - \sum_{i=k_0}^{m} e^{-N\epsilon_0^2/2}$ of $\mathcal{D}_{tr}$, there are $\sum_{i\in T} N\mathbb{P}_i/2 \leq \sum_{i\in T} N(C_i) \leq Ne^{-[\frac{W}{W_0}]^{D_0-1}\frac{cW^{D-D_0}}{4}+2}$.

Hence, there are:

$$\begin{array}{rl} & \mathbb{P}_{\mathcal{D}}(\mathcal{F}) \\ \geq & \sum_{i\in S}\mathbb{P}_i \\ \geq & 1 - \sum_{i\in[k_0]}\mathbb{P}_i - \sum_{i\in T}\mathbb{P}_i \\ \geq & 1 - m\epsilon_0 - 2e^{-[\frac{W}{W_0}]^{D_0-1}\frac{cW^{D-D_0}}{4}+2}. \end{array}$$

Now, we take $\epsilon_0 = \sqrt{\frac{2\ln(m/\delta)}{N}}$, then we get the result. □

A required lemma is given.

Lemma K.2. *If a network with depth L and width W, the L_∞ norm of each transition matrix does not exceed 1. Then $|\mathcal{F}(x) - \mathcal{F}(z)| \leq nW^{L-1}||x - z||_\infty$.*

Proof. It is easy to see that $||Relu(Wx+b) - \mathrm{ReLU}(Wz+b)||_\infty \leq ||W(x-z)||_\infty \leq ||W||_{1,\infty}||x-z||_\infty$. Let $\mathcal{F}_i$ is the output of i-th layer of $\mathcal{F}$, then

$$\begin{array}{rl} & |\mathcal{F}(x) - \mathcal{F}(z)| \\ \leq & W||\mathcal{F}_{D-1}(x) - \mathcal{F}_{D-1}(z)||_\infty \\ \leq & W^2||\mathcal{F}_{D-2}(x) - \mathcal{F}_{D-2}(z)||_\infty \\ \cdots & \\ \leq & W^{D-1}||\mathcal{F}_1(x) - \mathcal{F}_1(z)||_\infty \\ \leq & nW^{D-1}||x - z||_\infty \end{array}$$

which proves the lemma. □