Wave packet systems and connections to spectral analysis of limiting operators

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***Abstract*—We discuss the design of "wave packet systems" that admit strong concentration properties in phase space. We make a connection between this problem and topics in signal processing related to the spectral behavior of spatial and frequency-limiting operators. The results have engineering applications in medical imaging, geophysics, and astronomy.**

***Index Terms*—wave packets, limiting operators, eigenvalue distributions**

I. Introduction

We write $\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x)\exp(-ix\cdot\xi)dx$ for the Fourier transform of $f \in L^2(\mathbb{R}^d)$. Given a measurable set $S \subset \mathbb{R}^d$, we denote $\mathcal{PW}(S)$ for the Paley-Wiener space of S, which consists of all $f \in L^2(\mathbb{R}^d)$ such that $\widehat{f}(\xi) = 0$ for a.e. $\xi \in \mathbb{R}^d \setminus S$. Functions in $\mathcal{PW}(S)$ have Fourier transforms supported on S. Consequently, we refer to functions $f \in \mathcal{PW}(S)$ as S-bandlimited.

If S is bounded, functions $f(x)$ in $\mathcal{PW}(S)$ depend analytically on x; hence, they cannot be compactly supported. However, one can ask to what extent a bandlimited function $f \in \mathcal{PW}(S)$ can be concentrated on a given bounded set $F \subset \mathbb{R}^d$. This question can be characterized by the behavior of the eigenvalues of a certain compact operator on $L^2(\mathbb{R}^d)$.

Given bounded measurable sets F and S in $\mathbb{R}^d$ of positive Lebesgue measure, let $P_F : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be the orthogonal projection on the subspace $L^2(F)$ of all L^2 functions supported on F, and let $B_S : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be the orthogonal projection on $\mathcal{PW}(S)$. A *limiting operator* is given by alternating these projection operators as

$$\mathcal{T}_{F,S} = B_S P_F B_S. \tag{I.1}$$

This operator is self-adjoint, positive definite, and compact, and possesses a decreasing sequence of eigenvalues $\{\lambda_k(F,S)\}$ in $(0,1)$ satisfying $\lambda_k(F,S) \to 0$ as $k \to \infty$.

The eigenfunctions of $\mathcal{T}_{F,S}$ form an orthonormal basis $\{\Psi_k = \Psi_k(F,S) : k \geq 1\}$ for $\mathcal{PW}(S)$, and they satisfy a double orthogonality property – specifically, they are orthogonal with respect to the $L^2(\mathbb{R}^d)$ inner product, and also, their restrictions to F are orthogonal with respect to the $L^2(F)$ inner product [18]. This makes them an important tool in many practical problems, e.g., in the development of stable extrapolation methods for bandlimited functions (see [10]).

The eigenfunctions associated with the largest eigenvalues of $\mathcal{T}_{F,S}$, along with any function in the subspace spanned by these eigenfunctions, are bandlimited and exhibit the optimal energy concentration on the domain F. Indeed, by the variational characterization of eigenvalues, the subspace $V_k := \mathrm{Span}\{\Psi_1, \cdots, \Psi_k\} \subset \mathcal{PW}(S)$ and the eigenvalue $\lambda_k = \lambda_k(F,S)$ solve the following optimization problem:

$$\lambda_k = \max_{V \subset \mathcal{PW}(S)} \min_{f \in V \setminus \{0\}} \left\{ \int_F |f(x)|^2 dx / \|f\|^2 : \dim V = k \right\}.$$

The eigenfunctions Ψ_k and associated subspaces V_k play a major role in applications such as imaging, where the shapes of F and S are dictated by various acquisition constraints [17, 13, 14, 18, 19]. For practical reasons, such as in thresholding and denoising, one needs to determine the dimension of such subspaces. Thus, counting the eigenvalues $\lambda_k(F,S)$ near 1 and those far from both 1 and 0 is essential. Generally, the solution to this problem depends on the geometric properties of the space and frequency domains.

In one-dimension, the eigenfunctions of the limiting operator $\mathcal{T}_{I,J}$ for intervals $I, J \subset \mathbb{R}$ are known as prolate spheroidal wave functions, and the eigenvalues $\lambda_k(I,J)$ depend only on the product of the time-frequency bandwidth, $c = |I| \cdot |J|$. As such, we represent these eigenvalues as $\lambda_k(c)$. In [17, 13, 14], it was established that these eigenvalues satisfy a strong clustering property: For any $\epsilon \in (0, 1/2)$, approximately $c/2\pi - O_\epsilon(\log(c))$ eigenvalues lie in $[1-\epsilon, 1)$, fewer than $O_\epsilon(\log(c))$ many eigenvalues lie in the "plunge region" $(\epsilon, 1-\epsilon)$, and the remaining eigenvalues in $(0, \epsilon]$ decay to 0 exponentially fast. Furthermore, it is known that the sequence $\lambda_k(c)$ crosses $\lambda = 1/2$ when k differs from the floor or ceiling of $c/2\pi$ by at most 1 [12]. Therefore, there is a phase transition at $k = c/2\pi$: For $k \lesssim c/2\pi$ the eigenvalues $\lambda_k(c)$ are exponentially close to 1 (for c large), and for $k \gtrsim c/2\pi$, $\lambda_k(c)$ is exponentially close to 0. When $k \sim c/2\pi$, the eigenvalues have intermediate size between 0 and 1. Recent advancements have yielded sharp quantitative bounds, making these statements precise – see [6, 16, 4, 2, 9].

Let $Q = [0,1]^d$ be the unit hypercube, and let $S \subset \mathbb{R}^d$ be a convex domain. We suppose S is *coordinate-wise*

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symmetric – that is, $\tau_j(x) \in S$ for every $x \in S$ and any coordinate reflection $\tau_j : \mathbb{R}^d \to \mathbb{R}^d$, $1 \le j \le d$, $\tau_j(x_1, \cdots, x_j, \cdots, x_d) = (x_1, \cdots, -x_j, \cdots, x_d)$. We write $B(x, r) \subset \mathbb{R}^d$ for the Euclidean ball centered at $x \in \mathbb{R}^d$ of radius $r > 0$. We write $S(r) := \{rx : x \in S\}$ for the r-dilate of S.

The following theorem appeared before in [7].

Theorem I.1. *Let $Q = [0,1]^d$, and let $S \subset B(0,1)$ be a compact coordinate-wise symmetric convex set in $\mathbb{R}^d$.*

There exists a dimensional constant $C_d > 0$ (independent of S) such that, for any $\epsilon \in (0, 1/2)$ and $r \ge 1$,

$$\#\{k : \lambda_k(Q, S(r)) \in (\epsilon, 1-\epsilon)\} \le C_d E_d(\epsilon, r), \qquad \text{(I.2)}$$

where $E_d(\epsilon, r) := \max\{r^{d-1}\log(r/\epsilon)^{5/2}, \log(r/\epsilon)^{5d/2}\}$.

In the following section, we introduce wave packet systems, generalizing the classical Gabor and wavelet systems, and discuss the concentration and packing properties for families of functions in $L^2(\mathbb{R}^d)$. We then use wave packets to construct an orthonormal basis for $L^2([0,1]^d)$ with strong concentration properties. We conclude with the proof of Theorem I.1.

II. Methodology and Main Results

A. Wave Packet Systems

A classical technique in signal analysis is the decomposition of signals into components which are localized in the physical (time) and frequency domains. For the analysis, one is interested to construct orthogonal systems (or more generally, frames) that admit strong concentration properties. Two commonly employed methods for signal analysis are the short-time Fourier transform (STFT) and wavelet transform. Related to these transforms are the associated notion of a Gabor system or wavelet system.

Definition II.1. *A* frame *for a closed subspace $H \subset L^2(\mathbb{R}^d)$ is a family of functions $\{\psi_\nu\}$ in H satisfying the condition:*

$$A\|f\|^2 \le \sum_\nu |\langle f, \psi_\nu\rangle|^2 \le B\|f\|^2 \quad \forall f \in H.$$

We refer to the constants $0 < A \le B < \infty$ as the (lower and upper) frame bounds *for $\{\psi_\nu\}$. A* unit-norm frame *also satisfies $\|\psi_\nu\| = 1$.*

The frame condition asserts that $\{\psi_\nu\}$ is a redundant (overcomplete) spanning set for H.

Let $g \in L^2(\mathbb{R}^d)$ be a smooth window function, decaying rapidly at ∞, and let $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \subset \mathbb{R}^d \times \mathbb{R}^d$, $\mathcal{S}_1 = \{x_\nu\}_{\nu\in\mathcal{I}}$, $\mathcal{S}_2 = \{\xi_\mu\}_{\mu\in\mathcal{J}}$ be a set of spatial translation and frequency modulation pairs. A *Gabor system* generated by g and $\mathcal{S}$ is given by applying the translations and modulations to the window function,

$$\mathcal{G}(g, \mathcal{S}) := \left\{ g_{\nu\mu}(x) = e^{i(x-x_\nu)\cdot\xi_\mu} g(x - x_\nu) \right\}_{\nu\in\mathcal{I}, \mu\in\mathcal{J}}. \quad \text{(II.1)}$$

In the literature, Gabor systems are frequently studied in the special case where $\mathcal{S}_1$ and $\mathcal{S}_2$ are lattices in $\mathbb{R}^d$ [5].

If $\mathcal{S}$ is structured appropriately (often as a Cartesian product of sufficiently dense lattices), the system $\mathcal{G}(g, \mathcal{S})$ is complete in $L^2(\mathbb{R}^d)$, and even forms a frame. One should regard $g_{\nu\mu}$ as localized to the region $|x - x_\nu| < 1$ in physical domain and to the region $|\xi - \xi_\mu| < 1$ in the frequency domain. Gabor systems are widely used in signal processing to perform localization of signals. Given a signal $f \in L^2(\mathbb{R}^d)$, the inner products $\{\langle f, g_{\nu\mu}\rangle\}$ capture the local spatial and frequency content of the signal near the set of points $\{(x_\nu, \xi_\mu)\}$ in phase space $\mathbb{R}^d \times \mathbb{R}^d$. If the system $\{g_{\nu\mu}\}$ forms a frame in $L^2(\mathbb{R}^d)$, the signal f can then be reconstructed from the inner products $\{\langle f, g_{\nu\mu}\rangle\}$. This recovery problem is solved using the synthesis operator, defined via a suitable dual frame. Therefore, Gabor systems provide a convenient tool for performing analog-to-digital conversion.

Another example of localization arises in the wavelet representation of a function. Starting with a function g, often called a "mother wavelet", one defines a family given by applying a family of translations and dyadic dilations via

$$\mathcal{W}(g) := \left\{ w_{j,\vec{k}}(x) = 2^{-jd/2} g(2^{-j}x - \vec{k}) \right\}_{j\in\mathbb{Z}, \vec{k}\in\mathbb{Z}^d}. \quad \text{(II.2)}$$

Under suitable conditions on g, this system $\mathcal{W}(g)$ can be shown to be an orthonormal basis for a subspace of $L^2(\mathbb{R}^d)$. One should regard the function $w_{j,\vec{k}}$ as localized to a region $2^{-j-1} < |\xi| < 2^{-j}$ in the frequency domain, and to the region $|x - \vec{k}2^j| < 2^j$ in the physical domain. Given a signal f, the inner products $\langle f, w_{j,\vec{k}}\rangle$ measure the content of f within the corresponding region of the phase space $\mathbb{R}^d \times \mathbb{R}^d$.

We now introduce the notion of a *wave packet system*, unifying the previous examples. We let $\Theta = \{\theta_\nu : \mathbb{R}^d \to [0, \infty)\}_{\nu\in\mathcal{I}}$ be a family of C^∞ functions satisfying a uniform decay estimate, $|\theta_\nu(x)| \le C_d/(1 + |x|^d)$, with C_d independent of ν, and normalized so that $\|\theta_\nu\| = 1$. Fix a family of affine maps $\mathcal{A} = \{T_\nu : \mathbb{R}^n \to \mathbb{R}^n\}$, with $T_\nu(x) = A_\nu(x - x_\nu)$ for $x_\nu \in \mathbb{R}^n$ and $A_\nu \in GL_n(\mathbb{R})$. Fix a family of frequency modulation vectors $\Xi = \{\xi_\mu \in \mathbb{R}^n\}_{\mu\in\mathcal{J}}$. Define a *wave packet system* associated with these families as

$$\mathcal{P}(\Theta, \mathcal{A}, \Xi) = \{g_{\nu\mu}(x) = c_\nu e^{iT_\nu(x)\cdot\xi_\mu} \theta_\nu(T_\nu(x))\}, \quad \text{(II.3)}$$

with $c_\nu = |\det A_\nu|^{1/2}$ picked so that $\|g_{\nu\mu}\| = 1$. Often, it is convenient to let $\theta_\nu = \theta$ be independent of ν. Notice that Gabor systems (II.1) are a special case of wave packets systems, where the maps T_ν are simply translations, $x \mapsto x - x_\nu$. Also, wavelet systems (II.2) are a special case of wave packet systems in which the affine maps T_ν specify a family of dyadic dilations and translations along dyadic lattice points, and there is no frequency modulation, $\Xi = \{0\}$. In general, a wave packet $g_{\nu\mu}$ should be regarded as localized to an ellipsoidal region $A_\nu^{-1}(B(0,1)) + x_\nu$ in physical space, and localized to the dual region $A_\nu^T(B(0,1)) + \xi_\mu$ in frequency space.

We end the section by stating several problems related to the concentration properties of systems of functions in $\mathbb{R}^d$. For this discussion, we fix domains $F, S \subset \mathbb{R}^d$. We will view $F \times S$ as a domain in the phase space $\mathbb{R}^d \times \mathbb{R}^d$. Phase space will be parametrized by $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$. We are interested in constructing a system of functions $\{\psi_\nu\}_{\nu\in\mathcal{I}}$ in $L^2(\mathbb{R}^d)$ that are highly concentrated in the spatial and frequency domains F

and S (although not necessarily perfectly supported on either). The following definition makes this precise.

Definition II.2 (ϵ-concentrated system)**.** *A system of unit-norm functions $\{\psi_\nu\}$ in $L^2(\mathbb{R}^d)$ is ϵ-concentrated on $F \times S$ provided*

$$\sum_\nu \|\psi_\nu\|^2_{L^2(\mathbb{R}^d \setminus F)} + \sum_\nu \|\widehat{\psi_\nu}\|^2_{L^2(\mathbb{R}^d \setminus S)} \leq \epsilon^2.$$

We note that the previous definition imposes no orthogonality constraints on $\{\psi_\nu\}$.

Definition II.3 (ϵ-packing)**.** *Given compact sets $F, S \subset \mathbb{R}^d$, given $\epsilon > 0$, we say that a finite family of unit-norm functions $\mathcal{F} = \{\psi_\nu\}_{\nu \in \mathcal{I}}$ in $L^2(\mathbb{R}^d)$ is an ϵ-packing for $F \times S$ if*

1) *$\{\psi_\nu\}$ is an ϵ-concentrated on $F \times S$.*
2) *$|\langle \psi_\nu, \psi_{\nu'} \rangle| < \epsilon$ for all ν, ν', $\nu \neq \nu'$.*

Condition 2 in the definition of a packing asserts that the family $\{\psi_\nu\}$ is approximately orthogonal. Of course, every domain $F \times S$ admits the empty packing, $\mathcal{F} = \emptyset$. In the previous definition, we would like to construct packings with large cardinality $\#\mathcal{F}$. We refer to this as *the packing problem*. Ideally, we seek bounds on $\#\mathcal{F}$ in terms of ϵ and the Lebesgue measures of F and S.

The following lemma is essentially proven in [9] – see the discussion in Section 4. It connects the packing problem to lower bounds on the eigenvalues of limiting operators. We provide the proof here to illustrate the variational technique.

Lemma II.4. *Let $\mathcal{F} = \{\psi_\nu\}$ be an ϵ-packing for $F \times S$, with $n := \#\mathcal{F}$, and $\epsilon < \frac{1}{2n}$. Then $\lambda_n(P_F B_S P_F) > 1 - 5\epsilon n^{1/2}$.*

Proof. We write $\mathcal{F} = \{\psi_1, \cdots, \psi_n\}$, and let $V \subset L^2(\mathbb{R}^d)$ be the subspace spanned by $\mathcal{F}$. We claim that $\mathcal{F}$ is a linearly independent set. Indeed, let G be the $n \times n$ Gram matrix of $\mathcal{F}$, i.e., $G_{\nu\nu'} = \langle \psi_\nu, \psi_{\nu'} \rangle$, $1 \leq \nu, \nu' \leq n$. The diagonal entries of G are 1, and the off-diagonal entries of G have magnitude at most ϵ. Let I denote the $n \times n$ identity matrix. Observe that the Frobenius norm of $I - G$ is at most $\epsilon n < 1/2$. The spectral norm is bounded by the Frobenius norm, and so, $\|I - G\| < 1/2$, hence G is invertible, and so $\mathcal{F}$ is linearly independent. Thus, $\dim V = n$.

By the variational characterization of eigenvalues (the max-min theorem), the bound $\lambda_n(F, S) > 1 - C\epsilon n^{1/2}$ is implied by

$$\frac{\|P_F B_S P_F \psi\|}{\|\psi\|} > 1 - C\epsilon n^{1/2} \quad \forall \psi \in V,\ \psi \neq 0.$$

The claim that $\{\psi_\nu\}$ is ϵ-concentrated on $F \times S$ implies that $\|(Id - P_F)\psi_\nu\| \leq \epsilon$ and $\|(Id - B_S)\psi_\nu\| \leq \epsilon$ for all ν. Observe that $Id - P_F B_S P_F = (Id - P_F) + P_F(Id - B_S) + P_F B_S (Id - P_F)$. Since P_F and B_S are projections, by the triangle inequality we obtain $\|(Id - P_F B_S P_F)\psi_\nu\| \leq 3\epsilon$.

Let $\psi \in V$, and write $\psi = \sum_\nu a_\nu \psi_\nu$. By the triangle inequality and the estimation in the previous paragraph, we have $\|(Id - P_F B_S P_F)\psi\| \leq 3\epsilon \sum_\nu |a_\nu|$, which implies that

$$\|P_F B_S P_F \psi\| \geq \|\psi\| - 3\epsilon \sum_\nu |a_\nu| \geq \|\psi\| - 3\epsilon n^{1/2} \sqrt{\sum_\nu |a_\nu|^2}.$$

By expanding out the inner product, we write

$$\|\psi\|^2 = \sum_\nu |a_\nu|^2 + \sum_{\nu \neq \mu} a_\nu \overline{a_\mu} \langle \psi_\nu, \psi_\mu \rangle \geq (1 - n\epsilon) \sum_\nu |a_\nu|^2,$$

where in the last inequality we used $\left| \sum_{\nu \neq \mu} a_\nu \overline{a_\mu} \right| \leq (\sum |a_\nu|)^2 \leq n \sum_\nu |a_\nu|^2$, as well as the approximate orthogonality of $\mathcal{F}$. Combining the previous estimates,

$$\frac{\|P_F B_S P_F \psi\|}{\|\psi\|} \geq 1 - 3\epsilon n^{1/2} (1 - n\epsilon)^{-1/2} \geq 1 - 5\epsilon n^{1/2}.$$

This estimate holds for any nonzero $\psi \in V$. This concludes the proof of the lemma. □

In dimension $d = 1$, when $I, J \subset \mathbb{R}$ are intervals, [9] gives the construction of packings of $I \times J$ using the time-frequency translates of a family of Hermite polynomials multiplied by a Gaussian window. To state their result, fix $c := |I| \cdot |J|$ to be sufficiently large. Given $\delta < 1$, there exists $\alpha(\delta) < 1$ and an ϵ-packing $\mathcal{F}$ of $I \times J$ such that $\epsilon = \alpha(\delta)^c$ and $\#\mathcal{F} \geq (1 - \delta)c/2\pi$.

As noted in [9], these results and Lemma II.4 imply that for any $\delta > 0$, there exists $\alpha = \alpha(\delta) < 1$, so that, for $c = |I| \cdot |J|$,

$$\lambda_n(\mathcal{T}_{I,J}) = \lambda_n(P_I B_J P_I) \geq 1 - 5\alpha^c c^{1/2} \quad \forall\, n \leq (1 - \delta)c/2\pi.$$

This gives the sharpest known lower bounds on the eigenvalues of time-frequency limiting operators in one-dimension, before the "plunge region" (i.e., for $n < c/2\pi$).

Open Problem 1: Let $I, J \subset \mathbb{R}$ be intervals with $c := |I| \cdot |J|$ sufficiently large. Given $\delta < 1$, does there exist a wave packet system $\mathcal{F}$ in $L^2(\mathbb{R})$, given as a family of time-frequency translates of a *single* smooth window function, that is an ϵ-packing for $I \times J$ with $\#\mathcal{F} \geq (1 - \delta)c/2\pi$ and $\epsilon = \alpha^c$ for $\alpha = \alpha(\delta) < 1$? Further, can this result be extended to higher dimensions, for the packing of a Cartesian product of Euclidean balls?

The next lemma is the key tool used in the proof of Theorem I.1. It connects the existence of a concentrated system with a bound on the set of the eigenvalues of a limiting operator.

Lemma II.5. *Suppose there exists a unit-norm frame $\mathcal{F} = \{\psi_\nu\}$ for $L^2(F)$ with frame constants $0 < A \leq B < \infty$, that admits a partition into three subfamilies, $\mathcal{F} = \mathcal{F}_{low} \cup \mathcal{F}_{hi} \cup \mathcal{F}_{res}$, such that $\mathcal{F}_{low}$ is $\sqrt{A/4} \cdot \epsilon$-concentrated on $F \times S$, $\mathcal{F}_{hi}$ is $\sqrt{A/4} \cdot \epsilon$-concentrated on $F \times (\mathbb{R}^d \setminus S)$, and the family $\mathcal{F}_{res}$ is finite. Then the set of eigenvalues of the operator $P_F B_S P_F$ satisfies the following spectral bound:*

$$\#\{\lambda_k(P_F B_S P_F) \in (\epsilon, 1 - \epsilon)\} \leq (2/A)\#\mathcal{F}_{res}.$$

This lemma is proven using the variational principle, and variants of this lemma appeared previously in the literature – see Lemma 1 of [6] for a related result when $\mathcal{F}$ is an orthonormal system. This result, for general frames, is implied by Lemma 3.1 in [15] (we present the reduction below).

Proof of Lemma II.5. Since $\mathcal{F} \subset L^2(F)$ we have $P_F \psi = \psi$ for all $\psi \in \mathcal{F}$. Since P_F is a projection, it follows that

$$\|(Id - P_F B_S P_F)\psi\| = \|P_F(Id - B_S)\psi\| \leq \|(Id - B_S)\psi\|$$

and $\|P_F B_S P_F \psi\| \leq \|B_S \psi\|$. Therefore, by Plancherel's theorem, and the concentration properties of $\mathcal{F}_{low}$ and $\mathcal{F}_{hi}$,

$$\sum_{\psi \in \mathcal{F}_{low}} \|(Id - P_F B_S P_F)\psi\|^2 + \sum_{\psi \in \mathcal{F}_{hi}} \|P_F B_S P_F \psi\|^2$$
$$\leq \sum_{\psi \in \mathcal{F}_{low}} \|\widehat{\psi}\|^2_{L^2(\mathbb{R}^d \setminus S)} + \sum_{\psi \in \mathcal{F}_{hi}} \|\widehat{\psi}\|^2_{L^2(S)} \leq \frac{A}{2}\epsilon^2.$$

The result follows then by Lemma 3.1 of [15]. □

B. Concentrated wave packet systems on the unit cube.

1) Coifman-Meyer local sine basis: We introduce an orthonormal basis for $L^2([0,1])$, consisting of compactly supported functions whose Fourier transforms have rapidly decaying tails. Here, we will use a variant of the *local sine basis* of Coifman-Meyer [3].

Define $\mathcal{W}$ to be a realization of the *Whitney decomposition* of $(0,1)$. That is, the family $\mathcal{W}$ consists of intervals, forming a partition of $(0,1)$, given by

$$\mathcal{W} = \{[2^{-j-1}, 2^{-j})\}_{j \geq 1} \cup \{[1 - 2^{-j}, 1 - 2^{-j-1})\}_{j \geq 1}.$$

Let $\delta_L \in (0,1)$ denote the length of an interval $L \in \mathcal{W}$.

For $a > 0$, define the "envelope function" $\Psi_a(\xi) := \exp(-a \cdot |\xi|^{2/3})$. The main result of the section follows:

Proposition II.6. *There exists an orthonormal basis $\mathcal{B}^1 = \{\phi_{L,k}\}_{(L,k) \in \mathcal{W} \times \mathbb{N}}$ for $L^2([0,1])$, with each $\phi_{L,k}$ a C^∞ function supported on a neighborhood of L, satisfying the Fourier decay condition: for all $\xi \in \mathbb{R}$, and some constants $C, a > 0$,*

$$|\widehat{\phi_{L,k}}(\xi)| \leq C\sqrt{\delta_L} \sum_{\sigma \in \{\pm 1\}} \Psi_a\left(\delta_L \xi - \sigma\pi(k + 1/2)\right). \quad \text{(II.4)}$$

We sketch the proof of Proposition II.6 – for details, see [7]. The system $\{\phi_{L,k}\}$ is a variant of the local sine basis for $L^2([0,1])$ (see [3, 1]). We select cutoff functions θ_L, with each θ_L supported on a neighborhood of the interval $L \in \mathcal{W}$. We define a family, indexed by $L \in \mathcal{W}$ and $k \in \mathbb{N}$,

$$\phi_{L,k}(x) = c_L \theta_L(x) \sin(\pi(k + 1/2)(x - x_L)/\delta_L),$$

with x_L the left endpoint of L, and c_L a normalization constant picked so that $\|\phi_{L,k}\| = 1$. Coifman and Meyer showed that the resulting family is an orthonormal basis for $L^2([0,1])$. The support condition in Proposition II.6 follows by the support condition for the cutoffs θ_L. If the cutoff functions are suitably regular – specifically, if they belong to the Gevrey class $G^{3/2}$ – then the decay condition (II.4) will be satisfied. This concludes our sketch of the proof of Proposition II.6.

Observe that the $\phi_{L,k}$ do not technically form a wave packet system, but rather a weighted combination of two wave packets with modulation frequencies of opposite signs. This follows from the identity $\sin(t) = (-i/2)(e^{it} - e^{-it})$.

2) Tensor systems: We construct an orthonormal basis for $L^2([0,1]^d)$. Let $\mathcal{W}^{\otimes d} = \{\mathbf{L} = L_1 \times \cdots \times L_d : L_1, \cdots, L_d \in \mathcal{W}\}$ be the set of all *d-fold Cartesian products* of intervals in $\mathcal{W}$. Elements $\mathbf{L}$ of $\mathcal{W}^{\otimes d}$ are rectangular boxes in $[0,1]^d$. See Figure 1 for a depiction of the family $\mathcal{W}^{\otimes d}$.

For $\mathbf{L} = L_1 \times \cdots \times L_d \in \mathcal{W}^{\otimes d}$ and $\mathbf{k} = (k_1, \cdots, k_d) \in \mathbb{N}^d$, we define $\psi_{(\mathbf{L},\mathbf{k})}$ in $L^2(\mathbb{R}^d)$ by

$$\psi_{(\mathbf{L},\mathbf{k})}(x_1, \cdots, x_d) = \phi_{L_1,k_1}(x_1) \cdots \phi_{L_d,k_d}(x_d),$$

where $\mathcal{B}^1 = \{\phi_{L,k}\}_{(L,k) \in \mathcal{W} \times \mathbb{N}}$ is as in Proposition II.6.

Set $\mathcal{B}^d = \{\psi_{(\mathbf{L},\mathbf{k})}\}$. Then $\mathcal{B}^d = (\mathcal{B}^1)^{\otimes d}$ is the d-fold tensor product of the basis $\mathcal{B}^1$ for $L^2([0,1])$. We identify the d-fold tensor power of $L^2([0,1])$ with $L^2([0,1]^d)$.

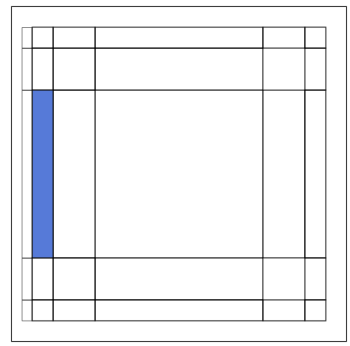

Fig. 1. A decomposition of the square into rectangles $\mathbf{L}$ in dimension $d = 2$.

By properties of tensor products of Hilbert spaces, $\mathcal{B}^d$ is an orthonormal basis for $L^2([0,1]^d)$. Each $\psi_{(\mathbf{L},\mathbf{k})}$ is supported on a neighborhood of the box $\mathbf{L}$, and $\psi_{(\mathbf{L},\mathbf{k})}$ is the sum of 2^d many wave packets. Using the decay condition (II.4), one can then establish the following:

Proposition II.7. *Let $Q = [0,1]^d$. Let $S \subset B(0,1)$ be convex and coordinate-wise symmetric. Let $\epsilon \in (0, 1/2)$ and $r \geq 1$ be given. The system $\mathcal{B}^d = \{\psi_{(\mathbf{L},\mathbf{k})}\}$ is an orthonormal basis for $L^2(Q)$, and admits a partition into three parts, $\mathcal{B}^d = \mathcal{B}^d_{low} \cup \mathcal{B}^d_{res} \cup \mathcal{B}^d_{hi}$, determined by S, r, ϵ, satisfying:*

$$\sum_{\psi \in \mathcal{B}^d_{hi}} \|\widehat{\psi}\|^2_{L^2(S(r))} + \sum_{\psi \in \mathcal{B}^d_{low}} \|\widehat{\psi}\|^2_{L^2(\mathbb{R}^d \setminus S(r))} \leq \epsilon^2/4 \quad \text{(II.5)}$$

with $\#\mathcal{B}^d_{res} \leq C_d \max\{r^{d-1}\log(r/\epsilon)^{5/2}, \log(r/\epsilon)^{(5/2)d})\}$. Here, C_d is a dimensional constant determined solely by d.

For the proof of Proposition II.7, we apply Proposition 5.3 in [7] with $\delta = c_d \epsilon^2 / r^d$. For the proof of (II.5), see the estimates in Lemmas 5.4 and 5.6 of [7].

Proof of Theorem I.1. The operators $\mathcal{T}_{F,S} = B_S P_F B_S$ and $P_F B_S P_F$ have identical eigenvalues (see the proof of Lemma 1 in [11]). The bound follows from Lemma II.5 and Proposition II.7. □

Open Problem 2: Moving beyond tensor systems, generalize the previous construction to prove an analogue of Proposition II.7 in the setting when Q and S are Euclidean balls, i.e., $Q = B(0,1)$, $S(r) = B(0,r)$. That is, we ask to construct a system $\mathcal{B}^d$ that is a frame for $L^2(B(0,1))$ and such that for each $\epsilon > 0$, $\mathcal{B}^d$ admits a decomposition $\mathcal{B}^d = \mathcal{B}^d_{low} \cup \mathcal{B}^d_{res} \cup \mathcal{B}^d_{hi}$, satisfying (II.5) with $\#\mathcal{B}^d_{res} \leq C_d r^{d-1} \log(r/\epsilon)^J$ for some exponent J.

By Lemma 3.1 in [15], this bound would recover the known spectral estimates on the localization operators $\mathcal{T}_{Q,S(r)}$ [15, 8]. Furthermore, we expect that the associated frame will have further use in signal processing tasks, such as signal recovery and denoising.

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