Personalized Federated Learning via Low-Rank Matrix Optimization

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Abstract

Personalized Federated Learning (pFL) has gained significant attention for building a suite of models tailored to different clients. In pFL, the challenge lies in balancing the reliance on local datasets, which may lack representativeness, against the diversity of other clients' models, whose quality and relevance are uncertain. Focusing on the clustered FL scenario, where devices are grouped based on similarities in their data distributions without prior knowledge of cluster memberships, we develop a mathematical model for pFL using low-rank matrix optimization. Building on this formulation, we propose a pFL approach leveraging the Burer-Monteiro factorization technique. We examine the convergence guarantees of the proposed method, and present numerical experiments on training deep neural networks, demonstrating the empirical performance of the proposed method in scenarios where personalization is crucial.

1 Introduction

Federated Learning (FL) is an important paradigm in machine learning, holding a great promise for training machine learning models over a large network with restricted data sharing. It is most suitable when clients require collaboration—often due to the absence of a large, representative dataset available at hand that can capture the diversity and variability of the underlying behavior—but in an environment where sharing datasets with collaborators is prohibited—often driven by concerns and regulations surrounding data sharing and storage. Consequently, FL research has been focused on designing algorithms that can solve optimization and learning problems on a network without sharing essential data; but instead communicating decision variables (referred to as local models in machine learning) or other auxiliary variables like gradients or step-directions, alongside privacy improvement strategies such as encryption or noise injection. However, the restriction of data sharing presents challenges beyond the training process. Primarily, limitations on data sharing hinder effective control over the quality and relevance of the data provided by participating clients—a major concern that led to the rise of personalized federated learning models (pFL).

The primary goal of pFL is finding the right balance between the two conflicting forces: the reliability of local datasets which may lack representativeness, and the diversity of collaborators' models whose quality and relevance are uncertain. As a result, pFL lacks a clear definition and direction without explicit specifications regarding data distribution of the clients. Regrettably, it appears there is no universally accepted measurable quantitative goal to evaluate the success in personalization. Adding to this concern, many existing pFL methods are tested in settings that are inherently unsuited for pFL, where either Federated Averaging (FedAvg) or local training produces the best accuracies. It is evident that when FedAvg yields optimal results, indicating uniform data distributions across clients, there is no room for personalization. On the other hand, if local training performs well, suggesting that local datasets represent the problem sufficiently well, the necessity of FL is called into question.

Motivated by these observations, our first step is to formulate a precise mathematical problem that highlights the role and necessity of a pFL approach. Suppose there are n clients collaborating on an FL system, indexed by $i = 1, \ldots, n$, and assume that the data for each client comes from a specific data distribution, denoted by

$\mathcal{D}_i$. We define the true objective function for each client as follows:

$$f_i^\natural(\mathbf{x}) := \mathbb{E}_{\xi\sim\mathcal{D}_i}\, \ell_i(\mathbf{x}, \xi), \tag{1}$$

where $\ell_i : \mathbb{R}^d \times \mathbb{R}^p \to \mathbb{R}$ is a loss function. When data distributions are known, a solution to this problem can be found by minimizing $f_i^\natural(\mathbf{x}_i)$ locally:

$$\min_{\mathbf{x}_i}\ f_i^\natural(\mathbf{x}_i). \tag{2}$$

However, the true distribution $\mathcal{D}_i$ is unknown in practice. Instead, clients have access to an empirical sample $\mathcal{M}_i$ with the corresponding objective function:

$$f_i(\mathbf{x}) := \frac{1}{|\mathcal{M}_i|} \sum_{\xi\in\mathcal{M}_i} \ell_i(\mathbf{x}, \xi). \tag{3}$$

Throughout, we operate under the assumption that the dataset $\mathcal{M}_i$ is not large enough for clients to accurately approximate a solution to problem (2) locally on their own. Otherwise, FL would not be required.

An effective solution to this problem is possible only if the distributions $\mathcal{D}_i$ exhibit some correlation that we can exploit. At one extreme, when all distributions are the same, the standard FL template can be used, which can be formulated as

$$\min_{\mathbf{x}_1,\ldots,\mathbf{x}_n}\ \frac{1}{n}\sum_{i=1}^{n} f_i(\mathbf{x}_i) \quad \text{s.t.} \quad \mathbf{x}_1 = \cdots = \mathbf{x}_n\,, \tag{4}$$

or equivalently as

$$\min_{\mathbf{x}}\ \frac{1}{n}\sum_{i=1}^{n} f_i(\mathbf{x})\,. \tag{5}$$

A significant portion of existing pFL methods are designed by relaxing the equality constraint (also called consensus constraint); examples include Moreau envelope smoothing and quadratic penalty regularization (T Dinh et al., 2020; Li et al., 2020). However, these approaches that penalize model dissimilarity using a specific norm have limitations, as they rely on the (implicit) assumption that similarity in distributions $\mathcal{D}_i$ translates to the proximity of client models in a given norm. The following simple examples demonstrate these limitations:

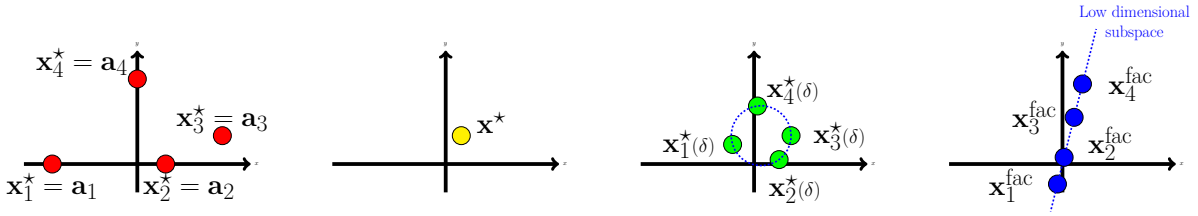
Example 1 (Label noise in classification)**.** *Consider a linear binary classification problem with two groups of clients that differ in their sign conventions. Specifically, these groups label the positive and negative classes in opposite ways due to a misalignment in how they interpret the binary outcomes. Clearly, the data distributions of these two groups are nearly identical, differing only by one bit. However, for this linear classifier, this difference results in the optimal models for the two groups having opposite signs, leading to solutions* $\mathbf{x}^\star_{group1} = -\mathbf{x}^\star_{group2}$*, which are distant in all norms.*

Although presented as a toy example here, mislabeling is a common problem in classification tasks, especially in domains like healthcare where human experts are involved in data collection. In real-world FL systems, where data remains private, detecting or preventing such issues is challenging. Therefore, algorithms must be designed robust against these inconsistencies.

Example 2 (Clustered FL)**.** *Suppose each client draws data from one of r distinct distributions, forming r clusters of clients. We assume that cluster memberships are unknown, and the challenge is to establish effective collaboration without knowing in advance which clients share similar data distributions.*

A specific class of pFL methods for clustered FL problems focuses on identifying clusters using various similarity measures and then performing cluster-aware aggregation. However, in a real-world FL system where data remains private, estimating these clusters is challenging without exposing additional information. This difficulty persists unless we rely on the stringent assumption that similar distributions produce models that are close in a certain norm, which could then be used to estimate clusters during training.

Example 3 (Collaborative filtering)**.** *Consider the classical problem of recommendation systems. Suppose there are n clients and p items. Let $\mathbf{x}_i \in \mathbb{R}^p$ represent the relevance scores of client i for the items. The data*



(a) Local (b) FedAvg (c) Distance based (d) Low-rank (ours)

Figure 1: Solutions to the minimization problem $\frac{1}{n}\sum_{i=1}^{n} \|\mathbf{x}_i - \mathbf{a}_i\|^2$, where $\mathbf{x}_i, \mathbf{a}_i \in \mathbb{R}^2$. The red points represent individual minimizers of the problem which are equivalent to $\mathbf{x}_i^\star = \mathbf{a}_i$s. FedAvg solution is $\mathbf{x}^\star = \frac{1}{n}\sum_{i=1}^{n} \mathbf{a}_i$ which is shown in yellow. Considering $\|\mathbf{x}_i - \mathbf{x}_j\| \leq \delta$ constraint one can find $\mathbf{x}_i^\star(\delta)$ as shown in green. Solving this problem with the low-rank assumption ($r = 1$) gives us solutions $\mathbf{x}_i^{\text{fac}}$, blue points, lying in a low-dimensional subspace.

consists of the actual scores rated by the clients, where each client rates only a subset of the items, denoted by $S_i \subseteq \{1, \ldots, p\}$. We denote these scores by x_{ij}^* for $j \in S_i$. The goal is predicting unknown scores that are not in S_i, based on hidden patterns among clients.[1]

Models based on Euclidean distance regularizations are known to fail in accurately predicting movie preferences. Instead, low-rank matrix factorization is among the most popular and successful approaches to collaborative filtering, as demonstrated by the Netflix competition 2008 progress prize winning team (Koren et al., 2009). The typical explanation for the empirical success of low-rank models in this problem is that movie preferences are well-parameterized by a few meaningful factors, such as genre, cast, language, and year. A more nuanced argument generalizes this by noting that low-rank matrices naturally arise in latent variable models (LVMs). While this is standard for LVMs with linear parameterizations, (Udell & Townsend, 2019) demonstrate that low-rank models are effective for a broad class of (possibly high-dimensional) LVMs parameterized by a piecewise analytic function.

Inspired by these examples, we explore how to formulate pFL without relying on a specific distance metric. This leads us to investigate low-dimensional subspace formulations, where personalized models are related by their membership to a low-dimensional subspace rather than their proximity in a distance metric. This approach allows us to conceptualize pFL by focusing on the inherent structure of the model relationships rather than their spatial closeness, as illustrated in Figure 1. Drawing parallels to collaborative filtering, we specifically focus on low-rank formulations.

We can now summarize **our main contributions:**

- We introduce a novel formulation for pFL based on low-rank matrix optimization in Section 3. Utilizing a nonconvex matrix factorization method applied to this formulation, we propose a new method called Personalized Federated Learning via Matrix Factorization (pFL^{MF}).
- We investigate the convergence guarantees of the proposed method in Section 4. For the smooth nonconvex minimization problem, we show that the proposed method converges to a first-order stationary point at a rate of $\mathcal{O}(1/T)$; with the stochastic gradients, the rate becomes $\mathcal{O}(1/\sqrt{T})$.
- We present numerical experiments on training various types of neural networks in Section 5. We compare the performance of the proposed method against the baseline in scenarios where personalization is crucial, such as classification in clustered FL with label misalignment or non-homogeneous data distributions.

2 Related Work

Many pFL algorithms impose the closeness of the learning models; the main assumption is that personal models are close with respect to some measures (T Dinh et al., 2020). Learning a mixture of the global

[1]The decision variable in matrix completion reveals the data, limiting the privacy benefits of FL. Nevertheless, the problem highlights the challenge of distributed learning with personalized models.

model and the local models is proposed in (Hanzely & Richtárik, 2020), where these personalized models are encouraged to stay relatively close to their average by incorporating a quadratic penalty. However, research has shown that model similarities among different neural networks, particularly in their classifier layers, are highly correlated with the similarity of the training data distributions (Tan et al., 2023). This implies that assuming closeness between models in Euclidean sense is tantamount to assuming similar data distributions across different clients.

Recently, model decoupling methods have been proposed (Mishchenko et al., 2023; Arivazhagan et al., 2019; Oh et al., 2021) showing a better performance than distance-based pFL methods. The main idea is to decouple each local model into two blocks, a feature extractor block followed by a classifier block. The feature extractor block is communicated and aggregated over clients and the classifier block is trained locally by each client. Arivazhagan et al. (2019) introduced a personalization of some specific layers of the neural network that all user devices share a set of base layers with the same weights and have distinct personalization layers that can potentially adapt to individual data. The base layers are shared with the server while the personalization layers are kept private by each device. In (Oh et al., 2021), the entire network is decomposed into the body (extractor), which is related to universality, and the head (classifier), which is related to personalization. This reduces the update and aggregation parts from the entire model to the body of the model during federated training.

Anelli et al. (2022) investigate federated pair-wise learning for factorization models in a recommendation scenario. Huang et al. (2022) propose an FL framework for solving the POI (Point-of-Interest) recommendation problem. Ammad-Ud-Din et al. (2019) introduces a federated implementation of collaborative filtering that is limited to recommendation systems. Liang et al. (2020) introduce LG-FEDAVG combines local representation learning with global model learning in an end-to-end manner. Each local device learns to extract higher-level representations from raw data before a global model operates on the representations (rather than raw data) from all devices. Tan et al. (2023) propose a decoupling algorithm that also personalizes feature extractors by adjusting aggregation weights based on classifier similarity. Deng et al. (2020) introduce APFL algorithm which. aims to learn a personalized model for each user that is a convex combination of local and global models, and coefficients of these linear combinations are adaptively learned during the training. Hao et al. (2022) assume factorized weights for neural networks and, instead of learning a unique global model, aims at learning a dictionary of rank-1 weight factor matrices. Each client then uses this dictionary to construct a model customized to its unique data distribution. Jeong & Hwang (2022) consider factorization of the model parameters and allows clients to perform a selective aggregation scheme to utilize only the knowledge from the relevant participants for each client.

Perhaps the most relevant work to ours is (Collins et al., 2021), where the server tries to learn the common low-dimensional features of the data, and each client learns local features suited to its requirements. This method, Federated Representation Learning (FedRep), leverages all of the data stored across clients to learn a global low-dimensional representation using gradient-based updates. Further, it enables each client to compute a personalized, low-dimensional classifier, which we term as the clients head, which accounts for the unique labeling of each clients local data. The main difference between this method and $\texttt{pFL}^{\texttt{MF}}$ is that we consider that the concatenation of the parameters of the clients lie on a low dimensional space while this paper assumes that each client has a low-rank parameter. In other words, their feature extractor extracts the features from a single shared global model while $\texttt{pFL}^{\texttt{MF}}$ trains a set of models and each client uses a combination of these models as its personalized model. In another word, it is assumed that $\mathbf{x}_i$ are low rank not $\mathbf{X} := [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n]$. Also, this work focuses on the linear representation setting with quadratic loss. (Zhang et al., 2024) introduces LR-BPFL, a Bayesian personalized federated learning method that learns a global deterministic model along with personalized low-rank Bayesian corrections.

Another line of research is partitioning the variables, Mishchenko et al. (2023); Pillutla et al. (2022) partition the model parameters into two groups: the shared parameters and the personal parameters. Clients do simultaneous or alternating updates and only share shared parameters.

It is worth mentioning that our work is fundamentally different from the following set of works. (1) Low-rank structure for the network assumption, such as (Pinto et al., 2023), proposed algorithm projects the private dataset onto a low-dimensional space spanned by the top principal components estimated with the public

unlabeled dataset and then applies gradient-based private algorithms (e.g., Noisy-SGD) to learn a linear classifier on top of the projected features, (Zhao et al., 2016; Cai et al., 2014), one of the layers in the neural net is assumed to be low rank, (Liu et al., 2024) assumes homogeneous pre-factorized low-rank layers in the model for clients, (Tran et al., 2025) factorizes the local prompt into two lower rank components with an additional residual term. And (2) Low-rank structure for the gradient assumption such as (Kasiviswanathan, 2021; Yu et al., 2021; Gooneratne et al., 2020). Yao et al. (2021) introduced FEDHM, that low-rank factorized neural networks with a specified size are trained, and the server translates this to the full rank global model using model shape alignment method.

3 Algorithm

We propose a novel formulation for pFL based on low-rank matrix optimization:

$$\min_{\mathbf{X}\in\mathbb{R}^{d\times n}} \; F(\mathbf{X}) := \frac{1}{n}\sum_{i=1}^{n} \; f_i(\mathbf{x}_i) \quad \text{s.t.} \quad \text{rank}(\mathbf{X}) \leq r. \tag{6}$$

Here, $\mathbf{X} \in \mathbb{R}^{d\times n}$ denotes the system-level decision variable obtained by concatenating clients' decision variables as $\mathbf{X} := [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n]$, and r is problem specific tuning parameter. Note that this formulation suits well for the examples we discussed in the introduction.

There exists a rich literature on rank-constrained matrix optimization problems, approaches including hard thresholding algorithms Jain et al. (2010); Goldfarb & Ma (2011); Kyrillidis & Cevher (2014), convex relaxation methods Candes & Recht (2012); Recht et al. (2010), and nonconvex matrix factorization techniques Burer & Monteiro (2003); Sun & Luo (2016); Bhojanapalli et al. (2016); Park et al. (2017). The first two class of algorithms require expensive spectral decomposition steps, hence they are not suitable for federated implementation unless the server possesses sufficient computational power to perform such decompositions. Consequently, we adopt the nonconvex matrix factorization technique, also known as the Burer-Monteiro (BM) factorization.

BM factorization strategy replaces the system-level decision variable $\mathbf{X} \in \mathbb{R}^{d\times n}$ with a factorized form of $\mathbf{X} = \mathbf{U}\mathbf{V}^\top$. This transformation leads to the following optimization problem:

$$\begin{aligned} &\min_{\mathbf{U}\in\mathbb{R}^{d\times r},\, \mathbf{V}\in\mathbb{R}^{n\times r}} \psi(\mathbf{U},\mathbf{V}), \\ &\psi(\mathbf{U},\mathbf{V}) := F(\mathbf{U}\mathbf{V}^\top) = \frac{1}{n}\sum_{i=1}^{n} \; f_i(\mathbf{U}\mathbf{v}_i). \end{aligned} \tag{7}$$

We denote by $\mathbf{V}^\top := [\mathbf{v}_1, \cdots, \mathbf{v}_n] \in \mathbb{R}^{r\times n}$. In this notation, personalized model parameters can be computed as $\mathbf{x}_i = \mathbf{U}\mathbf{v}_i \in \mathbb{R}^d$. One can interpret $\mathbf{U} \in \mathbb{R}^{d\times r}$ as a shared feature representation in the FL problem, computed by the server, and $\mathbf{v}_i \in \mathbb{R}^r$ as the feature extractor specific to client i.

While various optimization techniques can address problem (7), we develop our algorithm based on the simple block-coordinate gradient updates. We can compute the gradient of ψ with respect to $\mathbf{U}$ and $\mathbf{v}_i$ as follows:

$$\begin{aligned} \nabla_{\mathbf{U}}\psi(\mathbf{U},\mathbf{V}) &= \frac{1}{n}\sum_{i=1}^{n} \; \nabla f_i(\mathbf{U}\mathbf{v}_i)\; \mathbf{v}_i^\top \\ \nabla_{\mathbf{v}_i}\psi(\mathbf{U},\mathbf{V}) &= \frac{1}{n}\mathbf{U}^\top \nabla f_i(\mathbf{U}\mathbf{v}_i). \end{aligned} \tag{8}$$

It is crucial that ψ is separable with respect to $\mathbf{v}_i$, enabling clients to compute $\nabla_{\mathbf{v}_i}\psi(\mathbf{U},\mathbf{V})$ in parallel without requiring access to data or model parameters from other clients, given the features $\mathbf{U}$. Consequently, for a given step-size $\eta_i > 0$, local training steps can be independently formulated and performed by each participating client as:

$$\mathbf{v}_i^{t+1} = \mathbf{v}_i^t - \eta_i \frac{1}{n}\mathbf{U}^{t\,\top} \; \nabla f_i(\mathbf{U}^t\mathbf{v}_i^t). \tag{9}$$

Algorithm 1 Personalized Federated Learning via Matrix Factorization (pFL^{MF})

set $\mathbf{U}^0 \in \mathbb{R}^{m\times r}, \mathbf{v}_i^0 \in \mathbb{R}^r \ \forall i \in [n]$.
for round $t = 0, 1, \ldots, T-1$ **do**
  — **Client**-level local training ————————-
  for client $i \in \mathcal{S}_t$ **do**
    set $\mathbf{v}_i^{t,1} = \mathbf{v}_i^t$.
    for $k = 0, \ldots, K-1$ **do**
      $\mathbf{v}_i^{t,k+1} = \mathbf{v}_i^{t,k} - \eta_i \frac{1}{n} \mathbf{U}^{t\top} \ \nabla f_i(\mathbf{U}^t \mathbf{v}_i^{t,k})$
    end for
    $\mathbf{v}_i^{t+1} = \mathbf{v}_i^{t,K}$
    $\mathbf{G}_i^t = \big(\nabla f_i(\mathbf{U}^t \mathbf{v}_i^t)\big) \ \mathbf{v}_i^{t\top}$
    Client communicates $\mathbf{G}_i^t$ to the server.
  end for
  — **Server**-level aggregation ————————————
  $\mathbf{U}^{t+1} = \mathbf{U}^t - \eta_i \frac{1}{|\mathcal{S}_t|} \sum_{i\in\mathcal{S}_t} \ \mathbf{G}_i^t$
  Server communicates $\mathbf{U}^{t+1}$ to the clients.
end for

On the other hand, ψ is not separable with respect to the rows or columns of $\mathbf{U}$, necessitating collaboration among clients for computing $\nabla_{\mathbf{U}}\psi(\mathbf{U}, \mathbf{V})$. Consequently, the gradient step in $\mathbf{U}$ requires communication and will be performed at the server, forming our aggregation step:

$$\mathbf{U}^{t+1} = \mathbf{U}^t - \frac{1}{n} \sum_{i=1}^{n} \ \eta_i \big(\nabla f_i(\mathbf{U}^t \mathbf{v}_i^t)\big) \ \mathbf{v}_i^{t\top}. \tag{10}$$

Algorithm 1 depicts the pseudo-code of our algorithm. Here, K is the number of local passes each client performs, and the output of the algorithm is a set of personalized parameters $\mathbf{x}_i = \mathbf{U}\mathbf{v}_i$ that each client can compute locally using its feature extractors $\mathbf{v}_i$ and the shared feature representation $\mathbf{U}$.

4 Convergence Guarantees

Several works have studied the convergence for the problem 7 under different assumptions; we refer to (Chi et al., 2019) and references therein. (Bhojanapalli et al., 2016; Park et al., 2018) proved linear/sub-linear rates for smooth functions and smooth and strongly convex functions, respectively. Due to the nonconvex nature of BM factorization, even in cases where $f(.)$ is convex in $\mathbf{X}$, it is not possible to prove a convergence theorem to the global minimum. For more specialized cases (e.g., matrix sensing problems under some technical assumptions called restricted isometry property, convergence to a global solution can be characterized with careful initialization procedures Park et al. (2018); Jain et al. (2013); Zheng & Lafferty (2016); Park et al. (2016). Since our focus is primarily on neural network applications, where objectives are already nonconvex in $\mathbf{X}$, we derive convergence guarantees to a stationary point.

We begin our presentation of the main convergence guarantee by first listing our assumptions. We assume $f_i(\mathbf{U}\mathbf{v}_i)$ are directionally smooth, and that we have access to unbiased stochastic gradients $\tilde{\nabla}_{\mathbf{V}} F(\mathbf{U}\mathbf{V}^\top)$ and $\tilde{\nabla}_{\mathbf{U}} F(\mathbf{U}\mathbf{V}^\top)$ with bounded variance.

Assumption 1 (Directional smoothness)**.** *We assume that $F(\mathbf{U}\mathbf{V}^\top)$ is smooth with respect to $\mathbf{U}$ and $\mathbf{V}$, i.e., there exist constants $L_U, L_V \geq 0$ such that for all $\mathbf{U}_1, \mathbf{U}_2 \in \mathbb{R}^{d\times r}$ and $\mathbf{V}_1, \mathbf{V}_2 \in \mathbb{R}^{n\times r}$:*

$$\big\|\nabla_{\mathbf{U}} F(\mathbf{U}_1\mathbf{V}_1^\top) - \nabla_{\mathbf{U}} F(\mathbf{U}_2\mathbf{V}_2^\top)\big\|_F \leq L_U \Big(\|\mathbf{U}_1 - \mathbf{U}_2\|_F + \|\mathbf{V}_1 - \mathbf{V}_2\|_F \Big)$$
$$\big\|\nabla_{\mathbf{V}} F(\mathbf{U}_1\mathbf{V}_1^\top) - \nabla_{\mathbf{V}} F(\mathbf{U}_2\mathbf{V}_2^\top)\big\|_F \leq L_V \Big(\|\mathbf{U}_1 - \mathbf{U}_2\|_F + \|\mathbf{V}_1 - \mathbf{V}_2\|_F \Big)$$

Assumption 2 (Stochastic gradients)**.** *We assume access to an unbiased stochastic gradient estimator with bounded variance, i.e.,, there exists $\sigma < +\infty$ such that for all $\mathbf{U} \in \mathbb{R}^{d\times r}$ and $\mathbf{V} \in \mathbb{R}^{n\times r}$:*

$$\mathbb{E}\left[\tilde{\nabla} F(\mathbf{U}\mathbf{V}^\top)\right] = \nabla F(\mathbf{U}\mathbf{V}^\top) \qquad \text{and} \qquad \begin{aligned} &\mathbb{E}\left[\left\|\tilde{\nabla}_{\mathbf{U}} F(\mathbf{U}\mathbf{V}^\top) - \nabla_{\mathbf{U}} F(\mathbf{U}\mathbf{V}^\top)\right\|^2\right] \leq \sigma^2 \\ &\mathbb{E}\left[\left\|\tilde{\nabla}_{\mathbf{V}} F(\mathbf{U}\mathbf{V}^\top) - \nabla_{\mathbf{V}} F(\mathbf{U}\mathbf{V}^\top)\right\|^2\right] \leq \sigma^2. \end{aligned}$$

Assumption 3 (Stochastic gradient estimator)**.** *We assume access to an unbiased stochastic gradient estimator with bounded variance, i.e., there exists $\sigma < +\infty$ such that for all $\mathbf{U} \in \mathbb{R}^{d\times r}$ and $\mathbf{V} \in \mathbb{R}^{n\times r}$:*

$$\begin{aligned} \mathbb{E}\left[\tilde{\nabla}_{\mathbf{U}} F(\mathbf{U}\mathbf{V}^\top)\right] &= \nabla_{\mathbf{U}} F(\mathbf{U}\mathbf{V}^\top) \\ \mathbb{E}\left[\tilde{\nabla}_{\mathbf{V}} F(\mathbf{U}\mathbf{V}^\top)\right] &= \nabla_{\mathbf{V}} F(\mathbf{U}\mathbf{V}^\top), \end{aligned} \tag{11}$$

$$\begin{aligned} \mathbb{E}\left[\left\|\tilde{\nabla}_{\mathbf{U}} F(\mathbf{U}\mathbf{V}^\top) - \nabla_{\mathbf{U}} F(\mathbf{U}\mathbf{V}^\top)\right\|_F^2\right] &\leq \sigma^2 \\ \mathbb{E}\left[\left\|\tilde{\nabla}_{\mathbf{V}} F(\mathbf{U}\mathbf{V}^\top) - \nabla_{\mathbf{V}} F(\mathbf{U}\mathbf{V}^\top)\right\|_F^2\right] &\leq \sigma^2. \end{aligned} \tag{12}$$

Theorem 1. *Consider problem (7) with smooth loss functions $f_i(.)$ in the sense that assumption 1 holds. Assume access to a stochastic gradient estimator such that assumption 2 holds. Furthermore, assume that every client participates in each round with probability p and performs K local steps per iteration. Then, the sequence $\mathbf{U}^t, \mathbf{V}^t$ generated by* pFL^{MF} *with step-sizes $\eta_v = \frac{p\eta_u}{K}$ and $\eta_u < \frac{1}{2L}$, where $L := \max\{L_U, L_V\}$, satisfies the following bound:*

$$\frac{1}{T}\sum_{t=0}^{T-1}\left(\mathbb{E}\left[\left\|\nabla_{\mathbf{U}} F(\mathbf{U}^t{\mathbf{V}^t}^\top)\right\|^2\right] + \mathbb{E}\left[\frac{1}{K}\sum_{k=0}^{K-1}\left\|\nabla_{\mathbf{V}} F(\mathbf{U}^t{\mathbf{V}_k^t}^\top)\right\|^2\right]\right) \leq \frac{2\left(F(\mathbf{U}^0{\mathbf{V}^0}^\top) - F^\star\right)}{\eta T\left(1 - 2\eta L\right)} + \frac{2\eta L\sigma^2}{1 - 2\eta L}.$$

Corollary 2. *Choosing $\eta = \frac{1}{2L\sqrt{T}}$ in Theorem 1 yields a rate of $\mathcal{O}(1/\sqrt{T})$ in the stochastic setting. If full gradients are available ($\sigma = 0$), then $\eta = \frac{1}{4L}$ results in a convergence rate of $\mathcal{O}(1/T)$.*

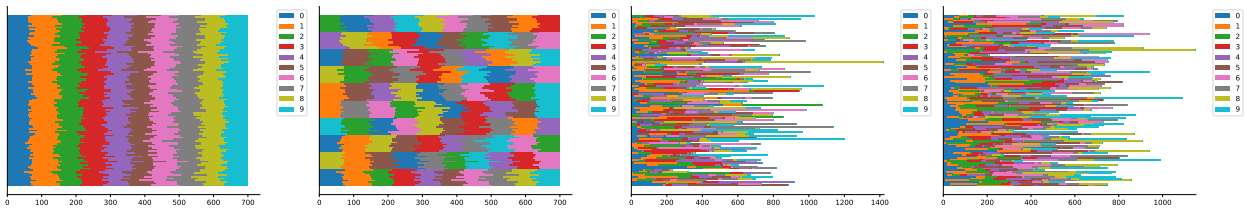
5 Numerical Experiments

In this section, we evaluate the performance of pFL^{MF}. We compare the performance of pFL^{MF} against several baselines, including Local training, FedAvg (McMahan et al., 2017), FedPer Arivazhagan et al. (2019), FedRep (Collins et al., 2021), APFL (Deng et al., 2020), and CFL (Sattler et al., 2020) by implementing pFL^{MF} in the *FL-Bench* benchmark (Tan et al., 2023). We conducted experiments in four different setups:

Setup (1) For the MNIST, CIFAR10, and CIFAR100 datasets, we split the data according to the Dirichlet distribution Dir(0.5) and Dir(1) across 100 clients. The labels' distribution is shown in figs. 2c and 2d. Performance of the algorithms is shown in table 1.

Setup (2) For the CIFAR-100 dataset, we partitioned the 100 classes into 20 groups, each containing 5 distinct labels. Data was then distributed among 500 clients, with each client exclusively assigned data from a single group, resulting in highly heterogeneous data. Results are shown in table 2.

Setup (3) For the MNIST, we follow the experimental setup in (Sattler et al., 2019) and consider 1000 clients divided into 10 groups, and labels in each group are re-mapped (permuted) according to a random permutation map. In other words, clients in group one would have the same numbers $\{0, \cdots, 9\}$ but labeled differently; group one may consider 0 with label 0, and group two may consider 0 with label 8. figs. 2a and 2b show the distribution of the labels before and after re-labeling, respectively. Results are shown in table 2.



(a) **Uniform** (b) **Permuted** (c) **Dir(0.5)** (d) **Dir(1)**

Figure 2: Distribution of the labels for MNIST dataset and 100 clients. The vertical and horizontal axes show clients and the size of each client's data, respectively.

	MNIST		CIFAR10		CIFAR100	
	Dir(0.5)	**Dir(1)**	**Dir(0.5)**	**Dir(1)**	**Dir(0.5)**	**Dir(1)**
Local	92.12% (±0.59)	89.15% (±1.12)	59.14% (±3.74)	48.53% (±2.53)	16.09% (±1.52)	10.66% (±0.98)
FedAvg	96.92% (±0.65)	97.01% (±0.54)	65.21% (±2.11)	65.44% (±1.68)	28.30% (±1.82)	28.36% (±1.32)
FedPer	96.30% (±0.26)	95.16% (±0.58)	66.86% (±3.19)	58.25% (±2.22)	19.98% (±1.6)	14.22% (±1.01)
FedRep	95.04% (±0.40)	93.33% (±0.94)	65.16% (±3.44)	55.4% (±2.06)	17.49% (±1.12)	12.14% (±1.05)
APFL	97.93% (±0.51)	97.64% (±0.39)	65.99% (±2.06)	65.14% (±1.54)	27.07% (±1.57)	27.07% (±1.36)
CFL	96.92% (±0.72)	97.04% (±0.5)	64.97% (±2.68)	65.98% (±1.70)	27.02% (±1.48)	24.84% (±0.91)
pFL^{MF}						
$r = 1$	96.75% (±0.61)	96.53% (±0.59)	43.89% (±3.49)	64.03% (±1.66)	34.32% (±1.96)	35.24% (±1.77)
$r = 5$	96.78% (±0.51)	96.55% (±0.60)	60.73% (±2.86)	65.89% (±1.88)	35.64% (±2.09)	35.75% (±1.24)
$r = 10$	96.98% (±0.70)	96.84% (±0.56)	65.10% (±2.30)	**67.68**% (±1.56)	35.28% (±1.73)	**36.84**% (±1.50)
$r = 15$	**98.24**% (±0.26)	**97.93**% (±0.22)	**68.13**% (±2.43)	65.88% (±1.62)	**35.70**% (±1.77)	36.12% (±1.46)

Table 1: Performance of the algorithms for **Setup (1)**. The best accuracy is shown in boldface, and the second best is underlined.

Setup (4) We sampled a subset of clients, 30% of the total clients, from FEMNIST dataset without changing the underlying data distribution, then we removed clients with less than 10 data points. The remaining set has 1091 clients. We ran the experiments for 1 and 5 numbers of local epochs. Results are shown in table 2.

Model. We used a three-layer neural network, consisting of three linear layers, on the MNIST and FEMNIST datasets and a four-layer convolutional neural network, consisting of two convolutional layers followed by two linear layers, on the CIFAR10 and CIFAR100 datasets. For FedPer and FedRep, we treated the last layer as the classifier, while in pFL^{MF}, we factorized the entire model.

Hyper-parameters. We consider partial participation with probability equal to 0.1. We set the batch size equal to 256 for all algorithms. We tried parameter r values of (7) is set to $r \in \{1, 5, 10, 15\}$. All experiments have 75% train and 25% test data splits on each client's data. We chose the best step size for each algorithm from the set $\{10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}\}$.

Observations. In the heterogeneous experiments, Setup (1), pFL^{MF} outperforms other pFL methods in most cases, though the performance across algorithms is comparable on the MNIST dataset. Remarkably, pFL^{MF} achieves substantial improvements in average test accuracy when different client groups have similar intra-group distributions but differ significantly across groups (see table 2). It is important to note that the low test accuracy observed in the CIFAR-100 experiment, Setup (2), is attributed to the simplicity of the neural network model rather than the algorithms themselves. Additionally, the results indicate that multiple local updates in pFL^{MF} contribute positively to performance. Overall, the findings highlight the practical significance of the proposed method.

	MNIST (permuted labels)	CIFAR100 (super groups)	FEMNIST	
	1000 clients	500 clients	1091 clients	
	1 local epoch	1 local epoch	1 local epoch	5 local epochs
Local	25.36% (±0.013)	10.49% (±0.95)	50.77% (±0.053)	65.69%(0.012)
FedAvg	12.02% (±0.022)	36.40% (±1.31)	65.40% (±0.017)	**77.19**%(0.013)
FedPer	19.86% (±0.141)	14.80% (±0.81)	66.05% (±0.010)	67.72%(0.009)
FedRep	21.30% (±0.148)	12.18% (±0.80)	66.10% (±0.013)	66.29%(±0.010)
pFL^{MF}				
$r = 1$	14.70% (±0.083)	42.94% (±1.22)	67.82% (±0.134)	71.42%(0.044)
$r = 5$	23.75% (±0.027)	44.70% (±1.91)	69.99% (±0.123)	72.09%(±0.208)
$r = 10$	<u>34.23</u>% (±0.090)	**45.57**% (±1.97)	<u>72.56</u>% (±0.023)	72.47%(±0.010)
$r = 15$	**39.31**% (±0.042)	<u>45.43</u>% (±1.23)	**73.59**% (0.092)	<u>76.41</u>%(±0.006)

Table 2: Performance of the algorithms for **Setup (2)**, **Setup (3)**, and **Setup (4)**. The best accuracy is shown in boldface, and the second best is underlined.

6 Conclusion

We introduced a new pFL formulation based on low-rank matrix optimization and developed a novel pFL algorithm utilizing Burer-Monteiro factorization. We further established convergence guarantees for the proposed method: for minimizing a smooth non-convex objective, the algorithm converges to a stationary point at a rate of $\mathcal{O}(1/T)$ with full gradients; and $\mathcal{O}(1/\sqrt{T})$ for the stochastic setting. Evaluations across four experimental setups highlight the practical significance of the proposed method, especially in scenarios where personalization is essential, and standard approaches are unable to adequately capture the complexity of the underlying data distributions.

We conclude by listing some limitations and future directions. Our numerical experiments demonstrate improved performance of pFL^{MF} with multiple local steps; however, this enhancement is not reflected in our theoretical convergence guarantees. Establishing stronger guarantees that reflect this behavior is a valuable direction for future research. Another notable limitation is that our formulation currently factorizes the entire model (decision variable), which can be computationally intensive in some cases, particularly in large-scale neural network applications. A more efficient approach might be to apply the BM factorization selectively, targeting only a subset of the parameters, which could reduce overhead while maintaining its benefits. Exploring such partial factorizations is a promising direction for future research

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