On the Convergence of Low-Precision LoRA Training

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Abstract

Low-rank adaptation (LoRA) has become a standard approach for parameter-efficient fine-tuning, substantially reducing the number of trainable parameters and the optimizer memory required during training. Quantization offers a further route to resource-efficient fine-tuning, but many existing quantized LoRA methods focus on weight-only compression, leaving unclear how low precision affects optimization when it is used throughout training. In this work, we initiate a systematic theoretical study of low-precision LoRA training in which the data, pretrained weights, adapters, gradients, and intermediate matrix multiplications are quantized during optimization. For spectrally initialized LoRA under a low-rank linear model with sub-Gaussian data and relative quantization error, we establish finite-horizon convergence upper bounds that consist of a full-precision linear convergence term plus a quantization-dependent additive error floor. Our results show that low-precision LoRA can retain convergence behavior close to its full-precision counterpart, provided that the mantissa length grows only logarithmically with the inverse of the full-precision contraction rate.

1. Introduction

Low-rank adaptation (LoRA) has emerged as the most prevalent paradigm for parameter-efficient fine-tuning (PEFT) of large-scale pretrained models by learning low-rank updates through a pair of smaller adapter matrices [11]. However, as parameter counts continue to scale, reducing the number of trainable parameters alone does not remove the computational and memory cost of the full training execution graph, including forward activations, backward gradients, and matrix multiplications [2]. Efficient acceleration and memory reduction of LoRA fine-tuning are therefore essential to enable timely, scalable, and cost-effective deployment of large pretrained models under practical computational constraints.

A growing body of empirical research demonstrates that quantizing pretrained weights can significantly reduce the memory footprint of fine-tuning without compromising downstream performance [6, 10, 19, 22, 28, 32]. However, many existing methods are largely static and weight-only: they compress the frozen backbone while keeping the LoRA adapters, gradients, and intermediate matrix multiplication operations in higher precision. This leaves a critical gap in our understanding of training-time low precision: Can the LoRA training execution graph, including forward and backward arithmetic, be quantized while preserving the behavior of full-precision adaptation? It remains unclear when low-precision arithmetic can be used throughout LoRA training without fundamentally changing its optimization behavior.

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Theoretically, beyond analyzing expressive power [35] and optimization landscapes [13, 18, 23], recent works investigate the training dynamics and convergence guarantees of LoRA [24, 26, 30, 33, 38]. Xu et al. [33] studied LoRA under gradient flow for matrix factorization, revealing an alignment phase where the singular vectors of the LoRA weights gradually orient themselves to correct the misalignment between the pre-trained model and the target task. For discrete gradient descent, Mu and Klabjan [26] proves that LoRA gradient descent converges to a stationary point at rate $O(1/\log T)$, assuming only that the original loss function is Lipschitz smooth and lower bounded. Zhang et al. [38] proves linear convergence for spectrally initialized LoRA under sub-Gaussian downstream features and mean-squared loss. However, these theoretical frameworks are confined to the full-precision setting and do not analyze the convergence behavior when the quantities used in the forward and backward passes are quantized. This leaves a fundamental question unanswered: *How does quantization alter the optimization behavior and convergence guarantees of LoRA?*

In this work, we initiate a theoretical study of low-precision LoRA training. We analyze gradient descent (GD) for multi-output linear regression with a frozen pretrained matrix $\mathbf{W}^\natural$ and a rank-r^* target update $\Delta = \mathbf{W}^* - \mathbf{W}^\natural$, learned through LoRA adapters $\mathbf{A} \in \mathbb{R}^{d\times r}$ and $\mathbf{B} \in \mathbb{R}^{r\times k}$. To model training-time low-precision arithmetic, we quantize every tensor entering the matrix multiplications in the forward and backward passes: data features, pretrained weights, adapter factors, output residuals, and intermediate activations. Our goal is to bound the approximation error $R_t = \|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F$ and to identify precision regimes under which the low-precision recursion preserves the linear convergence behavior of its full-precision counterpart.

Our theoretical results. Under sub-Gaussian downstream data, a low-rank target condition $r^* \leq r$, spectral initialization, and coordinate-wise relative quantization error, we prove finite-horizon convergence rates for low-precision LoRA. Informally, writing ϵ for the worst relative quantization error, our results take the form

$$R_T \lesssim \sqrt{r}\left[(1-\rho)^T + D(\epsilon)\right], \quad D(\epsilon) = \begin{cases} \epsilon\sqrt{d}, & r = r^*, \\ \epsilon\sqrt{d} + \left(T\eta\epsilon\sqrt{d}\right)^2, & r > r^*, \end{cases} \tag{1}$$

where ρ is the full-precision contraction factor determined by the stepsize η and the spectrum of Δ. The first term is the same geometric decay that appears in full-precision LoRA. The second term is a quantization-dependent floor. Our analysis shows that exact rank LoRA ($r = r^*$) and over-parameterized LoRA ($r > r^*$) have qualitatively different low-precision behavior: both retain the same linear contraction term and a static $\epsilon\sqrt{d}$ floor, but over-parameterization introduces an additional horizon-dependent term $(T\eta\epsilon\sqrt{d})^2$ caused by quantization drift in non-signal rank directions. The formal statements are given in Theorems 1 and 2.

Our contributions. We summarize the main contributions as follows.

- We formulate a training-time low-precision LoRA recursion that quantizes not only frozen weights, but also data features, adapter factors, residuals, and intermediate products in both forward and backward matrix multiplications. For this recursion, we prove convergence bounds for the approximation error $\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F$ between the learned LoRA update and the target update, showing that quantization affects LoRA convergence through an additive error floor.
- We prove that in the exact rank regime $r = r^*$, low-precision LoRA has a horizon-independent additive floor (Theorem 1), while in the over rank regime $r > r^*$, the floor contains an additional horizon-dependent accumulated term (Theorem 2).

- We derive explicit precision conditions under which the low-precision LoRA convergence rate matches the full-precision rate in both exact rank and over rank cases. Theoretical results show that over rank case needs more bits to recover the full-precision rate compared with the exact rank case.

Notations. For two positive-valued functions $f(x)$ and $g(x)$, we write $f(x) \lesssim g(x)$ (or equivalently, $f(x) \leq O(g(x))$) or $f(x) \gtrsim g(x)$ (or equivalently, $f(x) \geq \Omega(g(x))$) if $f(x) \leq cg(x)$ or $f(x) \geq cg(x)$ holds for some absolute (if not otherwise specified) constant $c > 0$ respectively. We write $f(x) \eqsim g(x)$ (or equivalently, $f(x) = \Theta(g(x))$) if $f(x) \lesssim g(x) \lesssim f(x)$. We use $\|\cdot\|_{op}$ to denote the operator norm and $\|\cdot\|_F$ to denote the Frobenius norm for matrices. For a general matrix $\mathbf{M}$, let $\lambda_i(\mathbf{M})$ denote its i-th largest singular value. For Δ, denote $\lambda_i^* = \lambda_i(\Delta)$ and condition number $\kappa = \lambda_1^*/\lambda_{r^*}^*$.

2. Theoretical Setup

2.1. Low-precision LoRA training

We study LoRA fine-tuning for multi-output linear regression with a frozen pretrained matrix $\mathbf{W}^\natural \in \mathbb{R}^{d\times k}$. Let $\widetilde{\mathbf{X}} = [\widetilde{\mathbf{x}}_1, \ldots, \widetilde{\mathbf{x}}_N]^\top \in \mathbb{R}^{N\times d}$ and $\widetilde{\mathbf{Y}} = [\widetilde{\mathbf{y}}_1, \ldots, \widetilde{\mathbf{y}}_N]^\top \in \mathbb{R}^{N\times k}$ denote the downstream design and label matrices. LoRA learns a rank-r update $\mathbf{AB}$, where $\mathbf{A} \in \mathbb{R}^{d\times r}$ and $\mathbf{B} \in \mathbb{R}^{r\times k}$, by minimizing

$$\widetilde{L}(\mathbf{A}, \mathbf{B}) := \frac{1}{2N} \left\| \widetilde{\mathbf{X}}(\mathbf{W}^\natural + \mathbf{AB}) - \widetilde{\mathbf{Y}} \right\|_F^2. \tag{2}$$

We work in the data-rich regime $N \gg d$ and keep $r = \Theta(1)$, following Zhang et al. [38]. With stepsizes $\eta_1, \eta_2 > 0$, the corresponding full-precision gradient descent step is

$$\begin{aligned} \mathbf{A}_{t+1} &= \mathbf{A}_t - \frac{\eta_1}{N} \widetilde{\mathbf{X}}^\top \left(\widetilde{\mathbf{X}}(\mathbf{W}^\natural + \mathbf{A}_t\mathbf{B}_t) - \widetilde{\mathbf{Y}} \right) \mathbf{B}_t^\top, \\ \mathbf{B}_{t+1} &= \mathbf{B}_t - \frac{\eta_2}{N} \mathbf{A}_t^\top \widetilde{\mathbf{X}}^\top \left(\widetilde{\mathbf{X}}(\mathbf{W}^\natural + \mathbf{A}_t\mathbf{B}_t) - \widetilde{\mathbf{Y}} \right). \end{aligned} \tag{3}$$

To model training-time low-precision arithmetic, we quantize every tensor that enters a matrix multiplication. Define coordinate-wise quantizers $\mathcal{Q}_d$, $\mathcal{Q}_A$, $\mathcal{Q}_B$, $\mathcal{Q}_o$ and $\mathcal{Q}_w$ for data features, model parameters $\mathbf{A}$ and $\mathbf{B}$, output gradients and pretrained weight $\mathbf{W}^\natural$ respectively. Under these quantizers, standard dual-branch LoRA still requires multiplication among quantized parameters, quantized data and quantized output gradients. Hence, we further introduce intermediate quantization operations $\mathcal{Q}_l$ and $\mathcal{Q}_r$. In the forward pass, the intermediate hidden state $\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A(\mathbf{A}_t) \in \mathbb{R}^{N\times r}$ must be quantized by $\mathcal{Q}_l$ before it can be multiplied by $\mathcal{Q}_B(\mathbf{B}_t)$. Similarly, during the backward pass, the intermediate hidden state from the multiplication between quantized output gradient and quantized parameter $\mathcal{Q}_B(\mathbf{B}_t)$ must then be quantized by $\mathcal{Q}_r$ before the final multiplication with $\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top$. Figure 1 summarizes the resulting low-precision execution graph.

Formally, denote the forward activation $\mathbf{Z}_t = \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_w(\mathbf{W}^\natural) + \mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A(\mathbf{A}_t)\right)\mathcal{Q}_B(\mathbf{B}_t)$, then the quantized GD updates are given by:

$$\begin{aligned} \mathbf{A}_{t+1} &= \mathbf{A}_t - \frac{\eta_1}{N} \mathcal{Q}_d(\widetilde{\mathbf{X}})^\top \mathcal{Q}_r \left(\mathcal{Q}_o(\mathbf{Z}_t - \widetilde{\mathbf{Y}}) \mathcal{Q}_B(\mathbf{B}_t)^\top \right), \\ \mathbf{B}_{t+1} &= \mathbf{B}_t - \frac{\eta_2}{N} \mathcal{Q}_l \left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A(\mathbf{A}_t) \right)^\top \mathcal{Q}_o(\mathbf{Z}_t - \widetilde{\mathbf{Y}}). \end{aligned} \tag{quantized GD}$$

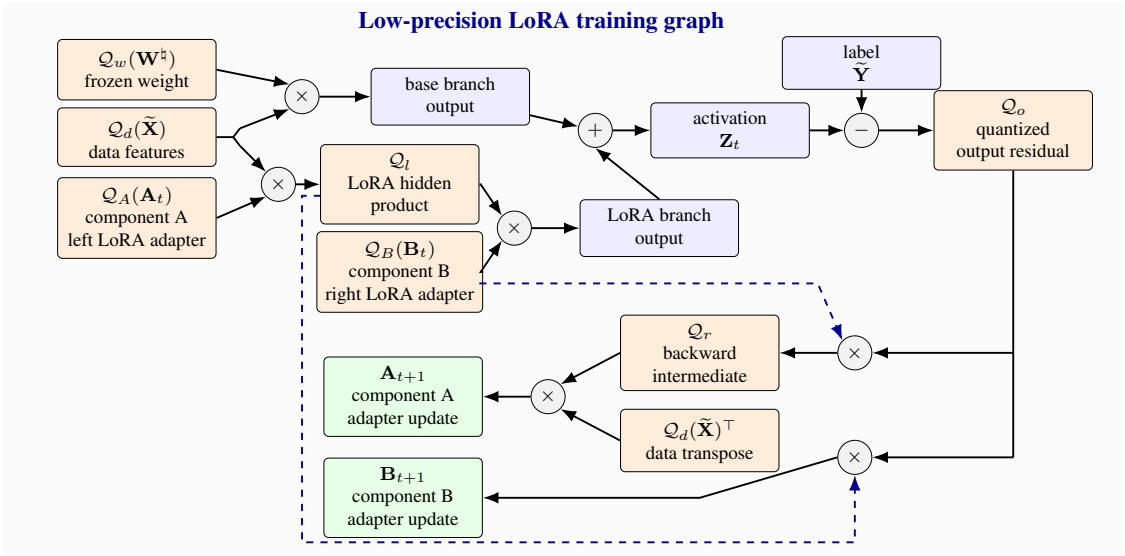

Figure 1: Low-precision LoRA training architecture analyzed in this paper. The forward pass forms the quantized residual; the backward pass proceeds from right to left.

Spectral initialization. We initialize the adapters using the spectral construction of Zhang et al. [38]. Specifically, let the compact singular value decomposition of $\mathbf{G}_f^\natural = -\frac{1}{N}\widetilde{\mathbf{X}}^\top(\widetilde{\mathbf{X}}\mathbf{W}^\natural - \widetilde{\mathbf{Y}})$ be $\mathbf{G}_f^\natural = \mathbf{U}_\mathbf{G}\mathbf{S}_\mathbf{G}\mathbf{V}_\mathbf{G}^\top$, and let $\mathbf{U}_r \in \mathbb{R}^{d\times r}$ and $\mathbf{V}_r \in \mathbb{R}^{k\times r}$ denote the top-r left and right singular matrices, and let $\mathbf{S}_r \in \mathbb{R}^{r\times r}$ denote the top-r sub-matrix of $\mathbf{S}_\mathbf{G}$. Then

$$\mathbf{A}_0 = \mathbf{U}_r\mathbf{S}_r^{1/2}, \quad \mathbf{B}_0 = \mathbf{S}_r^{1/2}\mathbf{V}_r^\top. \qquad \text{(spectral initialization)}$$

2.2. Data model and Quantization

Following Zhang et al. [38], we assume that each downstream label is generated by a ground-truth linear predictor $\mathbf{W}^* \in \mathbb{R}^{d\times k}$: $\widetilde{\mathbf{y}} = \mathbf{W}^{*\top}\widetilde{\mathbf{x}}$. We impose the standard low-rank and concentration assumptions used in the full-precision LoRA and matrix factorization analysis [38].

Assumption 2.1 *Assume that* $\mathrm{rank}(\Delta) = r^* \le r = \Theta(1)$ *and that* $\{\widetilde{\mathbf{x}}_i\}_{i=1}^N$ *are i.i.d. isotropic, centered sub-Gaussian random vectors.*

Denote λ_i^* be the i-th largest singular values of Δ and condition number $\kappa = \lambda_1^*/\lambda_{r^*}^*$. The goal is to derive convergence rate for $\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F^2$. This error also imply generalization performance, because the generalization error for a new data $(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$ satisfies $\mathbb{E}_{\widetilde{\mathbf{x}}}\left[\left\|\widetilde{\mathbf{y}} - (\mathbf{W}^\natural + \mathbf{A}_t\mathbf{B}_t)\widetilde{\mathbf{x}}\right\|_F\right] = \|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F^2$ under the noiseless linear data model and Assumption 2.1.

We model the quantizer $\mathcal{Q}$ and its error properties focusing on widely-adopted floating point quantization. Following Tang et al. [31], we assume the absence of underflow and overflow which is well justified by practical engineering techniques such as per-tensor or per-channel scaling [8, 27]. Under this condition, the relative quantization error decays exponentially with the mantissa bits.

Assumption 2.2 *Assume that for each* $i = d, A, B, o, w, l, r$, *there exists* ϵ_i *such that the coordinate-wise quantization* $\mathcal{Q}_i$ *satisfies*

$$|\mathcal{Q}_i(u) - u| \le \epsilon_i|u|, \quad \textit{for any } u \in \mathbb{R},$$

where $\epsilon_i = \Theta(2^{-M_i})$, and M_i is the mantissa length of the target floating-point format.

3. Main Theory

In this section, we establish theoretical bounds for the approximation error $\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F$. For simplicity, we consider $\eta_1 = \eta_2 =: \eta > 0$ and denote $\epsilon_{\max} = \max_{i=d,A,B,o,w,l,r} \epsilon_i$. We also assume that the leading signal quantities and the LoRA rank under consideration are at constant level.

Assumption 3.1 *Assume that $\lambda_1^*, \lambda_{r^*}^*, \|\mathbf{W}^\natural\|_{op} = \Theta(1)$ and $r = O(1)$.*

Without Assumption 3.1, the same analytic framework yields spectrum-explicit guarantees; see Corollary 12 for the exact rank case and Corollary 16 for the general rank case.

3.1. Exact Rank

Theorem 1 (Convergence rate in exact rank case) *Under Assumptions 2.1, 2.2, and 3.1, consider the exact rank case $r = r^*$ with* (spectral initialization) *and* (quantized GD) *using $\eta_1 = \eta_2 = \eta > 0$. Suppose $0 < \eta \lesssim \frac{\lambda_1^*}{\kappa^3(1+\lambda_1^*)^3}$, $0 \le \epsilon_{\max} \lesssim \frac{1}{T\sqrt{d}}$, for an integer iteration horizon $T > 0$. Then, with probability at least $1 - e^{-\Omega(N)}$,*

$$\|\mathbf{A}_T\mathbf{B}_T - \Delta\|_F \lesssim \sqrt{r^*}\left[\left(1 - \eta\frac{\lambda_{r^*}^*}{64\kappa}\right)^T + \epsilon_{\max}\sqrt{d}\right].$$

The bound in Theorem 1 can be illustrated as follows: the first term represents the geometric convergence characteristic of full-precision LoRA under spectral initialization [38]; the second term establishes an additive quantization floor: once the optimization error diminishes, the convergence is bottlenecked by the perturbation level introduced by low-precision arithmetic.

Crucially, under the conditions of Theorem 1, quantization does not degrade the linear convergence rate. Intuitively, this linear contraction is governed by the preservation of nondegenerate signal within the learned LoRA factors along the true left and right singular spaces of Δ, rather than being directly dictated by coordinate-wise rounding errors. Although quantization injects minor perturbations into the gradient updates, the small-error perturbation ensures that the smallest aligned singular values do not collapse, preserving the geometric decay factor.

Denote $\rho = \eta\lambda_{r^*}^*/(64\kappa)$. A direct implication of Theorem 1 is that if $\epsilon_{\max} \lesssim \frac{(1-\rho)^T}{\sqrt{d}}$, the quantization error is absorbed into the geometric term. In this regime, the convergence of the low-precision iterates asymptotically matches the full-precision rate $\mathcal{O}\left(\sqrt{r^*}(1-\rho)^T\right)$.

For bit level, let $M_{\min} := \min_{i=d,A,B,o,w,l,r} M_i$ denote the smallest mantissa length among the quantized tensors. Since Assumption 2.2 implies $\epsilon_{\max} = \Theta(2^{-M_{\min}})$ up to format-dependent constants, the condition for the quantization floor to be absorbed by the full-precision term is $M_{\min} \gtrsim \frac{1}{2}\log_2 d + T\log_2\frac{1}{1-\rho}$.

3.2. General Rank

Theorem 2 (Convergence rate in general-rank case) *Under Assumptions 2.1, 2.2, and 3.1, consider the general rank case $r \ge r^*$ with* (spectral initialization) *and* (quantized GD) *using $\eta_1 =$*

$\eta_2 = \eta > 0$. *Suppose* $0 < \eta \lesssim \frac{1}{\kappa(1+\lambda_1^*)}$, $0 \le \epsilon_{\max} \lesssim \frac{\lambda_{r^*}^*}{\sqrt{d}\kappa(\lambda_1^*+\kappa^2)(1+T\eta)^{3/2}}$, *for an integer iteration horizon* $T > 0$. *Then, with probability at least* $1 - \exp(-\Omega(\frac{N}{(1+T\eta)^2}))$,

$$\|\mathbf{A}_T\mathbf{B}_T - \Delta\|_F \lesssim \sqrt{r}\left[\left(1 - \eta\frac{\lambda_{r^*}^*}{64\kappa}\right)^T + \epsilon_{\max}\sqrt{d} + \left(T\eta\epsilon_{\max}\sqrt{d}\right)^2\right].$$

Theorem 2 demonstrates that over-parameterization preserves the same linear optimization rate as the exact rank setting, but alters the additive quantization floor. Beyond the iteration-independent contribution $\epsilon_{\max}\sqrt{d}$, the general-rank bound introduces an iteration-dependent term $(T\eta\epsilon_{\max}\sqrt{d})^2$. This additional term stems from the non-signal subspace created by the surplus rank directions ($r > r^*$). In the exact rank case, all learned factors align strictly with the true left and right singular spaces of Δ. In contrast, the over-parameterized factors contain orthogonal components that carry no signal from Δ and, consequently, do not benefit from the geometric contraction governing the signal-space dynamics. As a result, quantization perturbations can accumulate within this unregularized non-signal subspace over the effective horizon $T\eta$, yielding the T-dependent floor. When the effective horizon is short (e.g., $T\eta \lesssim 1$), this accumulation is dominated by the static floor, and Theorem 2 recovers the additive-error structure of Theorem 1, scaling with $\sqrt{r}$ rather than $\sqrt{r^*}$.

Similarly, if $\epsilon_{\max} \lesssim \min\{(1-\rho)^T, \frac{(1-\rho)^{T/2}}{T\eta}\}/\sqrt{d}$, then the low-precision iterates to asymptotically match the full-precision convergence rate. Here, the first argument of the minimum guarantees that the static quantization floor is absorbed into the geometric term, while the second controls the accumulated non-signal-subspace floor. Compared to the exact rank setting, maintaining the optimal rate in the over-parameterized regime imposes a more restrictive upper bound on the quantization error, as the precision must simultaneously suppress both the direct perturbation and its cumulative drift in the orthogonal directions. For bit level, a sufficient condition to match the full-precision linear convergence rate under the floating point quantization error model is $M_{\min} \gtrsim \frac{1}{2}\log_2 d + \max\left\{T\log_2\frac{1}{1-\rho}, \log_2(T\eta) + \frac{T}{2}\log_2\frac{1}{1-\rho}\right\}$.

4. Conclusion and Limitations

We formulate a low-precision training LoRA framework and establish convergence bounds for the approximation error $\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F$ and reveal that quantization affects LoRA convergence through an additive error floor. Theoretical results show a critical divergence in exact rank ($r = r^*$) and over rank ($r > r^*$) cases. Compared with the horizon-independent additive error floor in exact rank case, over rank case contains an additional horizon-dependent accumulated error. Building on this theory, we derive explicit precision conditions under which the low-precision LoRA convergence rate matches the full-precision rate, demonstrating that more bit widths are required in case of over rank.

Future work may address two key limitations of this study: (1) extending the theoretical framework to non-linear models; and (2) analyzing other optimization methods.

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Appendix

We contain related works in Section A, numerical experiments in Section B. Detailed proofs are also contained in the subsequent Appendix. Theorem 1 follows from Corollary 13, and Theorem 2 follows from Corollary 17.

Proof roadmap. The appendix is organized as follows. Section C rewrites the quantized LoRA recursion as full-precision LoRA dynamics plus perturbation terms, decomposes the error into signal and non-signal blocks, and proves the one-step propagation inequalities used by both main results. Section D bounds the quantization perturbations, states the initialization guarantees, and proves the finite-horizon induction arguments. The exact-rank proof is given in Section D.3.1; the general-rank proof is given in Section D.3.2. Section E contains a standalone operator-norm contraction lemma used in the propagation analysis.

Appendix A. Related Work

Empirical quantized variants of LoRA. One prominent line of research focuses on weight-only quantized LoRA, which reduces memory by quantizing the frozen backbone while keeping activations, gradients, adapters, and most training arithmetic in higher precision. QLoRA [6] pioneered this direction through 4-bit NF4 weight quantization and double quantization, while QA-LoRA [32] integrated quantization-aware adaptation with LoRA so that the fine-tuned update can be merged into a quantized model. Follow-up methods improve ultra-low-bit fine-tuning by changing which low-dimensional variables are trained or how adaptation capacity is allocated [17, 19, 29, 34]. Other works reduce quantization mismatch through initialization or error compensation: LoftQ [22] initializes LoRA adapters jointly with quantized base weights, LQ-LoRA [10] decomposes pretrained weights into a quantized component and a high-precision low-rank residual, and subsequent methods preserve information, calibrate initialization, or compensate quantization residuals [5, 28]. Closest in spirit to our setting, RoLoRA [12] studies rotated outlier-free models for weight-activation quantization, and FALQON [3] accelerates LoRA fine-tuning by merging adapters into an FP8-quantized backbone and reformulating forward and backward computation with quantized weights and activations.

Theoretical analysis of LoRA. Theoretical studies of LoRA have analyzed its expressive power [35], optimization landscape, and training dynamics. In the Neural Tangent Kernel (NTK) regime, Jang et al. [13] showed that LoRA has no spurious local minima when the parameterized rank satisfies $r \gtrsim \sqrt{N}$, and Kim et al. [18] extended landscape analysis to nonlinear settings, showing that a local minimizer is either a small-rank global minimizer or a large-magnitude spurious minimizer. Liu et al. [23] compared LoRA and full-rank fine-tuning through the Polyak-Łojasiewicz (PL*) region and found that this benign region can be much smaller for LoRA. Beyond landscapes, recent work studies LoRA trajectories and convergence. RAC-LoRA [24] and Bernoulli-LoRA [30] analyze variants that train one adapter while freezing the other as a fixed projection. For standard LoRA, Zhang et al. [38] proved that gradient descent aligns adapters with the top singular subspaces of the one-step full fine-tuning gradient, motivating spectral initialization and yielding linear convergence for $\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F$. From a continuous-time perspective, Xu et al. [33] studied matrix-factorization gradient flow and identified an alignment phase, while Mu and Klabjan [26] proved an $O(1/\log T)$ stationary-point rate for LoRA gradient descent under smoothness and lower-boundedness assumptions.

Theoretical analysis of low-precision training. A fundamental line of work establishes convergence guarantees for quantized training algorithms. Early analyses studied SGD with quantized gradients, often under unbiased quantizers and fixed quantization schemes [1, 4, 14]; Faghri et al. [7] extended this direction to adaptive quantization. To handle biased quantizers, subsequent work introduced error-feedback mechanisms that compensate quantization bias while preserving convergence [16, 21, 39]. Complementary results analyze quantized model weights [25], vector quantized SGD [9], and quantized adaptive optimizers such as Adam [20] and Muon [15] [31]. Beyond optimization, Zhang et al. [36] established generalization bounds for low-precision SGD in linear regression, and Zhang et al. [37] derived precision scaling laws under Gaussian sketched linear regression. These works provide important tools for understanding low-precision training, but they do not address the coupled bilinear dynamics and training quantization effects that arise in LoRA.

Appendix B. Numerical Experiment

We complement the preceding bounds with a controlled synthetic experiment using a 4090 GPU.

Experiment setup. We generate noiseless multi-output linear regression data with $N = 4096$ samples and dimensions $d = k = 64$. The covariates are i.i.d. centered isotropic Gaussian, $\widetilde{\mathbf{x}}_i \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_d)$. The target update Δ has rank $r^* = 4$, with random orthonormal singular vectors and singular values geometrically spaced between 1 and 0.6. We compare LoRA ranks $r \in \{4, 8, 16, 32\}$, covering the exact rank case and three over rank cases. The adapters are initialized by the spectral initialization (spectral initialization). We use constant stepsize $\eta = 0.08$ and average over eight random seeds. We record the relative recovery error $E_t = \|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F / \|\Delta\|_F$.

Observed behaviors. Figure 2 summarizes three diagnostics. Panel (a) measures the early log-linear convergence rate by fitting $\log E_t$ over $0 \le t \le 200$ and normalizing by the corresponding full-precision slope. For $\epsilon = 10^{-8}$, the normalized slopes are 0.99997, 0.99997, 0.99997, and 0.99996 for $r = 4, 8, 16, 32$, respectively. Hence sufficiently fine quantization recovers the full-precision linear phase across all ranks, in line with the common contraction factor in Theorems 1 and 2. As quantization error level increases, the trajectory departs from the full-precision log-linear phase earlier in the over rank cases. This aligns with our theory that over rank case requires smaller quantization error level to match the full-precision convergence rate. Panel (b) reports the terminal recovery error after $T = 1200$ iterations. The error grows approximately linearly with ϵ over the displayed smaller precision range, consistent with the additive $\epsilon_{\max}\sqrt{d}$ term in the bounds. This also supports the qualitative prediction that surplus directions in over rank case make the limiting neighborhood larger, even though the geometric contraction term itself has the same rate. Panel (c) isolates the horizon-dependent over rank effect by plotting $E_T(r, \epsilon) - E_T(r^*, \epsilon)$ with $r > r^*$ at $\epsilon = 10^{-4}$. As T increases from 100 to 1200, the excess error grows with T, validating the finite-horizon accumulated term in Theorem 2. Overall, the experiment supports and probes the mechanisms appearing in Theorems 1 and 2.

Appendix C. Algebraic Reductions and One-Step Propagation

This section contains deterministic reductions that are common to both the exact-rank and general-rank proofs. We first separate the quantized update into the full-precision LoRA update plus low-precision residuals. We then decompose the approximation error into blocks aligned with the singular subspaces of the target update Δ, and derive one-step propagation bounds for these blocks.

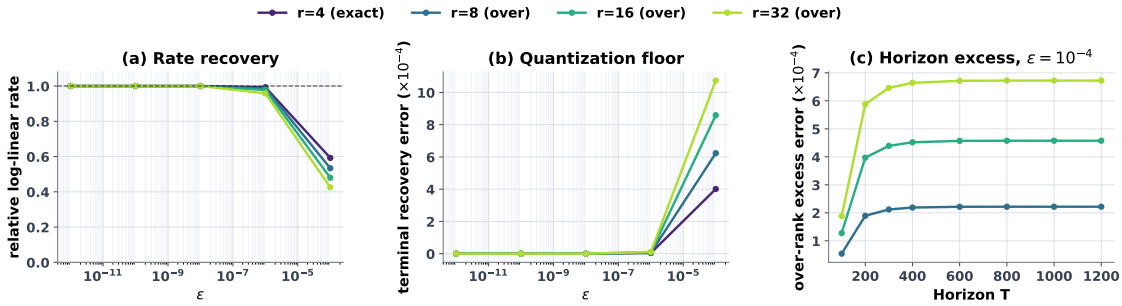

Figure 2: Numerical verification of the theoretical results. Left: sufficiently small ϵ preserves the full-precision log-linear rate, while larger ϵ makes over rank runs leave the full-precision phase earlier. Middle: the terminal recovery error grows with ϵ and with surplus rank. Right: the over rank excess error grows with the horizon.

C.1. Separating the Quantized Update

We first focus on the parameter update rule. Define the first-step negative gradient of full fine-tuning in quantized data space:

$$\mathbf{G}^{\natural} = -\frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top \left[\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{W}^{\natural} - \widetilde{\mathbf{Y}}\right]. \tag{4}$$

Define the quantized data covariance as $\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} = \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top \mathcal{Q}_d(\widetilde{\mathbf{X}})$.

Intermediate quantization errors. To isolate the effects of quantization, we define the following quantization errors.

$$\mathbf{E}_t^{(r)} := \mathcal{Q}_r\left(\mathcal{Q}_o\left(\mathbf{Z}_t - \widetilde{\mathbf{Y}}\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)^\top\right) - \mathcal{Q}_o\left(\mathbf{Z}_t - \widetilde{\mathbf{Y}}\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)^\top,$$
$$\mathbf{E}_t^{(l)} := \mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right) - \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right),$$
$$\mathbf{E}_t^{(o)} := \mathcal{Q}_o\left(\mathbf{Z}_t - \widetilde{\mathbf{Y}}\right) - \left(\mathbf{Z}_t - \widetilde{\mathbf{Y}}\right),$$
$$\mathbf{E}^{(w)} := \mathcal{Q}_w(\mathbf{W}^{\natural}) - \mathbf{W}^{\natural}, \quad \mathbf{E}_t^{(A)} := \mathcal{Q}_A(\mathbf{A}_t) - \mathbf{A}_t, \quad \mathbf{E}_t^{(B)} := \mathcal{Q}_B(\mathbf{B}_t) - \mathbf{B}_t, \quad \mathbf{E}^{(X)} = \mathcal{Q}_d(\widetilde{\mathbf{X}}) - \widetilde{\mathbf{X}}.$$

Then we are ready to simplify the parameter update rule, which is summarized as follows.

Lemma 3 (Parameter update rule) *The quantized gradient descent (quantized GD) can be written as*

$$\mathbf{A}_{t+1} = \mathbf{A}_t + \eta_1\left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\Delta + \mathbf{E}_t^{(T)}\right)\left(\mathbf{B}_t + \mathbf{E}_t^{(B)}\right)^\top - \eta_1\mathbf{H}_{t,\mathbf{A}},$$
$$\mathbf{B}_{t+1} = \mathbf{B}_t + \eta_2\left(\mathbf{A}_t + \mathbf{E}_t^{(A)}\right)^\top\left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\Delta + \mathbf{E}_t^{(T)}\right) - \eta_2\mathbf{H}_{t,\mathbf{B}},$$

where $\mathbf{E}_t^{(T)} = -\frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathbf{E}^{(X)}\mathbf{W}^* - \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\left(\mathbf{E}_t^{(o)} + \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{E}^{(w)}\right)$, *and*

$$\mathbf{H}_{t,\mathbf{A}} = \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)^\top + \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathbf{E}_t^{(r)},$$
$$\mathbf{H}_{t,\mathbf{B}} = \frac{1}{N}\mathcal{Q}_A\left(\mathbf{A}_t\right)^\top\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right)\mathcal{Q}_B\left(\mathbf{B}_t\right) + \frac{1}{N}\mathbf{E}_t^{(l)\top}\left(\mathbf{E}_t^{(o)} + \mathbf{Z}_t - \widetilde{\mathbf{Y}}\right).$$

Proof We first consider the update of parameter $\mathbf{A}_t$. By (quantized GD),

$$\begin{aligned}\mathbf{A}_{t+1} =& \mathbf{A}_t - \frac{\eta_1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top \mathcal{Q}_r\left(\mathcal{Q}_o\left(\mathbf{Z}_t - \widetilde{\mathbf{Y}}\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)^\top\right)\\ =& \mathbf{A}_t - \frac{\eta_1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\left[\mathbf{E}_t^{(r)} + \mathcal{Q}_o\left(\mathbf{Z}_t - \widetilde{\mathbf{Y}}\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)^\top\right]\\ =& \mathbf{A}_t - \frac{\eta_1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\left[\mathbf{E}_t^{(r)} + \left(\mathbf{E}_t^{(o)} + \mathbf{Z}_t - \widetilde{\mathbf{Y}}\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)^\top\right].\end{aligned}$$

Recall that

$$\begin{aligned}\mathbf{Z}_t =& \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_w(\mathbf{W}^\natural) + \mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)\\ =& \mathcal{Q}_d(\widetilde{\mathbf{X}})(\mathbf{W}^\natural + \mathbf{E}^{(w)}) + \mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right)\mathcal{Q}_B\left(\mathbf{B}_t\right),\end{aligned}$$

and utilizing the first-step negative gradient of full fine-tuning (4), we have

$$\begin{aligned}\mathbf{A}_{t+1} =& \mathbf{A}_t - \frac{\eta_1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\left[\mathbf{E}_t^{(r)} + \left(\mathbf{E}_t^{(o)} + \mathbf{Z}_t - \widetilde{\mathbf{Y}}\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)^\top\right]\\ =& \mathbf{A}_t + \eta_1\left(\mathbf{G}^\natural - \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\left(\mathbf{E}_t^{(o)} + \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{E}^{(w)}\right)\right)\left(\mathbf{B}_t + \mathbf{E}_t^{(B)}\right)^\top\\ &-\frac{\eta_1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)^\top - \frac{\eta_1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathbf{E}_t^{(r)}.\end{aligned} \tag{5}$$

Next, we consider the update of parameter $\mathbf{B}_t$. By (quantized GD),

$$\begin{aligned}\mathbf{B}_{t+1} =& \mathbf{B}_t - \frac{\eta_2}{N}\mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A(\mathbf{A}_t)\right)^\top\mathcal{Q}_o\left(\mathbf{Z}_t - \widetilde{\mathbf{Y}}\right)\\ =& \mathbf{B}_t - \frac{\eta_2}{N}\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A(\mathbf{A}_t) + \mathbf{E}_t^{(l)}\right)^\top\left(\mathbf{E}_t^{(o)} + \mathbf{Z}_t - \widetilde{\mathbf{Y}}\right)\\ =& \mathbf{B}_t - \frac{\eta_2}{N}\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})(\mathbf{A}_t + \mathbf{E}_t^{(A)}) + \mathbf{E}_t^{(l)}\right)^\top\left(\mathbf{E}_t^{(o)} + \mathbf{Z}_t - \widetilde{\mathbf{Y}}\right).\end{aligned}$$

Expanding the terms and once again utilizing the first-step negative gradient of full fine-tuning (4), we obtain:

$$\begin{aligned}\mathbf{B}_{t+1} =& \mathbf{B}_t + \eta_2\left(\mathbf{A}_t + \mathbf{E}_t^{(A)}\right)^\top\left(\mathbf{G}^\natural - \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\left(\mathbf{E}_t^{(o)} + \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{E}^{(w)}\right)\right)\\ &-\frac{\eta_2}{N}\mathcal{Q}_A\left(\mathbf{A}_t\right)^\top\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)\\ &-\frac{\eta_2}{N}\mathbf{E}_t^{(l)\top}\left(\mathbf{E}_t^{(o)} + \mathbf{Z}_t - \widetilde{\mathbf{Y}}\right).\end{aligned} \tag{6}$$

We then analyze the dynamics of $\mathbf{A}_t$ (5) and $\mathbf{B}_t$ (6) by dealing with $\mathbf{G}^\natural$. Noticing that

$$\begin{aligned}\mathbf{G}^\natural =& -\frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\left[\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{W}^\natural - \widetilde{\mathbf{Y}}\right]\\ =& -\frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\left[\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{W}^\natural - \widetilde{\mathbf{X}}\mathbf{W}^*\right]\\ =& -\frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\left[\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{W}^\natural - \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{W}^* + (\mathcal{Q}_d(\widetilde{\mathbf{X}}) - \widetilde{\mathbf{X}})\mathbf{W}^*\right]\\ =& -\frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\left[-\mathcal{Q}_d(\widetilde{\mathbf{X}})\Delta + (\mathcal{Q}_d(\widetilde{\mathbf{X}}) - \widetilde{\mathbf{X}})\mathbf{W}^*\right]\\ =& \widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\Delta - \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathbf{E}^{(X)}\mathbf{W}^*,\end{aligned}$$

denoting

$$\mathbf{E}_t^{(T)} = -\frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top \mathbf{E}^{(X)}\mathbf{W}^* - \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top \left(\mathbf{E}_t^{(o)} + \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{E}^{(w)}\right),$$

it follows by (5) and (6) that

$$\begin{aligned}\mathbf{A}_{t+1} = \mathbf{A}_t + \eta_1 \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\Delta + \mathbf{E}_t^{(T)}\right)\left(\mathbf{B}_t + \mathbf{E}_t^{(B)}\right)^\top \\ \underbrace{-\frac{\eta_1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top \mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)^\top - \frac{\eta_1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top \mathbf{E}_t^{(r)}}_{-\eta_1\mathbf{H}_{t,\mathbf{A}}},\end{aligned} \tag{7}$$

$$\begin{aligned}\mathbf{B}_{t+1} = \mathbf{B}_t + \eta_2 \left(\mathbf{A}_t + \mathbf{E}_t^{(A)}\right)^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\Delta + \mathbf{E}_t^{(T)}\right) \\ \underbrace{-\frac{\eta_2}{N}\mathcal{Q}_A\left(\mathbf{A}_t\right)^\top \mathcal{Q}_d(\widetilde{\mathbf{X}})^\top \mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right)\mathcal{Q}_B\left(\mathbf{B}_t\right) - \frac{\eta_2}{N}\mathbf{E}_t^{(l)^\top}\left(\mathbf{E}_t^{(o)} + \mathbf{Z}_t - \widetilde{\mathbf{Y}}\right)}_{-\eta_2\mathbf{H}_{t,\mathbf{B}}}.\end{aligned} \tag{8}$$

■

C.2. Signal and Non-Signal Error Decomposition

To avoid the complicated analysis for $\mathbf{A}_t\mathbf{B}_t\mathbf{B}_t^\top\mathbf{A}_t^\top$, we analyze the operator norm $\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_{op}$ and the Frobenius norm is controlled by

$$\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F \le \sqrt{\mathrm{rank}(\mathbf{A}_t\mathbf{B}_t) + r^*}\,\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_{op}.$$

We then perform error decomposition for $\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_{op}$. To proceed, we first perform SVD for Δ:

$$\Delta = \widetilde{\mathbf{U}}\widetilde{\mathbf{S}}^*\widetilde{\mathbf{V}}^\top = \begin{bmatrix}\mathbf{U} & \mathbf{U}_\perp\end{bmatrix}\begin{bmatrix}\mathbf{S}^* & \mathbf{0}_{r^*\times(k-r^*)} \\ \mathbf{0}_{(d-r^*)\times r^*} & \mathbf{0}_{(d-r^*)\times(k-r^*)}\end{bmatrix}\begin{bmatrix}\mathbf{V}^\top \\ \mathbf{V}_\perp^\top\end{bmatrix}. \tag{9}$$

For simplicity, we also define the following notations:

$$\mathbf{A}_t^{\mathbf{U}} = \mathbf{U}^\top\mathbf{A}_t, \quad \mathbf{B}_t^{\mathbf{V}} = \mathbf{B}_t\mathbf{V}, \quad \mathbf{A}_t^{\perp} = \mathbf{U}_\perp^\top\mathbf{A}_t, \quad \mathbf{B}_t^{\perp} = \mathbf{B}_t\mathbf{V}_\perp.$$

We then decompose the error $\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_{op}$ via (9).

Lemma 4 (Error decomposition)

$$\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_{op} \le \|\mathbf{R}_t^*\|_{op} + \left\|\mathbf{A}_t^{\perp}\mathbf{B}_t^{\mathbf{V}}\right\|_{op} + \left\|\mathbf{A}_t^{\mathbf{U}}\mathbf{B}_t^{\perp}\right\|_{op} + \left\|\mathbf{A}_t^{\perp}\mathbf{B}_t^{\perp}\right\|_{op},$$

where $\mathbf{R}_t^* = \mathbf{A}_t^{\mathbf{U}}\mathbf{B}_t^{\mathbf{V}} - \mathbf{S}^*$.

Proof Noticing that $\mathbf{I}_k = \mathbf{V}\mathbf{V}^\top + \mathbf{V}_\perp\mathbf{V}_\perp^\top$, it holds

$$\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_{op} \le \|(\mathbf{A}_t\mathbf{B}_t - \Delta)\mathbf{V}\|_{op} + \|(\mathbf{A}_t\mathbf{B}_t - \Delta)\mathbf{V}_\perp\|_{op}. \tag{10}$$

For the first term, noticing that $\mathbf{I}_d = \mathbf{U}\mathbf{U}^\top + \mathbf{U}_\perp\mathbf{U}_\perp^\top$, we have

$$(\mathbf{A}_t\mathbf{B}_t - \Delta)\mathbf{V} = \mathbf{U}\underbrace{(\mathbf{A}_t^{\mathbf{U}}\mathbf{B}_t^{\mathbf{V}} - \mathbf{S}^*)}_{\mathbf{R}_t^*} + \mathbf{U}_\perp\mathbf{A}_t^{\perp}\mathbf{B}_t^{\mathbf{V}}.$$

It follows that

$$\|(\mathbf{A}_t\mathbf{B}_t - \Delta)\mathbf{V}\|_{op} \leq \|\mathbf{R}_t^*\|_{op} + \left\|\mathbf{A}_t^{\perp}\mathbf{B}_t^{\mathbf{V}}\right\|_{op}. \tag{11}$$

For the second term, similarly, we have

$$(\mathbf{A}_t\mathbf{B}_t - \Delta)\mathbf{V}_{\perp} = \mathbf{U}\mathbf{A}_t^{\mathbf{U}}\mathbf{B}_t^{\perp} + \mathbf{U}_{\perp}\mathbf{A}_t^{\perp}\mathbf{B}_t^{\perp}.$$

It follows that

$$\|(\mathbf{A}_t\mathbf{B}_t - \Delta)\mathbf{V}_{\perp}\|_{op} \leq \left\|\mathbf{A}_t^{\mathbf{U}}\mathbf{B}_t^{\perp}\right\|_{op} + \left\|\mathbf{A}_t^{\perp}\mathbf{B}_t^{\perp}\right\|_{op}. \tag{12}$$

Together with (10), (11) and (12), we have

$$\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_{op} \leq \|\mathbf{R}_t^*\|_{op} + \left\|\mathbf{A}_t^{\perp}\mathbf{B}_t^{\mathbf{V}}\right\|_{op} + \left\|\mathbf{A}_t^{\mathbf{U}}\mathbf{B}_t^{\perp}\right\|_{op} + \left\|\mathbf{A}_t^{\perp}\mathbf{B}_t^{\perp}\right\|_{op}.$$

■

C.3. One-Step Propagation of the Decomposed Errors

We then derive update rule for reparameterized updates $\mathbf{A}_t^{\mathbf{U}}, \mathbf{A}_t^{\perp}, \mathbf{B}_t^{\mathbf{V}}, \mathbf{B}_t^{\perp}$. Denote

$$\mathbf{R}_t^{(1)} = \mathbf{A}_t^{\perp}\mathbf{B}_t^{\mathbf{V}}, \quad \mathbf{R}_t^{(2)} = \mathbf{A}_t^{\mathbf{U}}\mathbf{B}_t^{\perp}, \quad \mathbf{R}_t^{(3)} = \mathbf{A}_t^{\perp}\mathbf{B}_t^{\perp}.$$

Lemma 5 (Update rule of reparameterized iterates) *The dynamics of the reparameterized components are given by:*

$$\begin{aligned}
\mathbf{A}_{t+1}^{\mathbf{U}} =& \mathbf{A}_t^{\mathbf{U}} - \eta_1\mathbf{R}_t^*(\mathbf{B}_t^{\mathbf{V}})^\top - \eta_1\mathbf{R}_t^{(2)}(\mathbf{B}_t^{\perp})^\top + \eta_1\mathbf{U}^\top\mathbf{E}_{t,\mathbf{A}} \\
&-\eta_1\mathbf{U}^\top\left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right)\left[(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_{\perp}\mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_{\perp}\mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right].
\end{aligned}$$

$$\begin{aligned}
\mathbf{A}_{t+1}^{\perp} =& \mathbf{A}_t^{\perp} - \eta_1\mathbf{R}_t^{(1)}(\mathbf{B}_t^{\mathbf{V}})^\top - \eta_1\mathbf{R}_t^{(3)}(\mathbf{B}_t^{\perp})^\top + \eta_1\mathbf{U}_{\perp}^\top\mathbf{E}_{t,\mathbf{A}} \\
&-\eta_1\mathbf{U}_{\perp}^\top\left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right)\left[(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_{\perp}\mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_{\perp}\mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right].
\end{aligned}$$

$$\begin{aligned}
\mathbf{B}_{t+1}^{\mathbf{V}} =& \mathbf{B}_t^{\mathbf{V}} - \eta_2(\mathbf{A}_t^{\mathbf{U}})^\top\mathbf{R}_t^* - \eta_2(\mathbf{A}_t^{\perp})^\top\mathbf{R}_t^{(1)} + \eta_2\mathbf{E}_{t,\mathbf{B}}\mathbf{V} \\
&-\eta_2\left[(\mathbf{A}_t^{\mathbf{U}})^\top\mathbf{U}^\top + (\mathbf{A}_t^{\perp})^\top\mathbf{U}_{\perp}^\top\right]\left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right)\left(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_{\perp}\mathbf{R}_t^{(1)}\right).
\end{aligned}$$

$$\begin{aligned}
\mathbf{B}_{t+1}^{\perp} =& \mathbf{B}_t^{\perp} - \eta_2(\mathbf{A}_t^{\mathbf{U}})^\top\mathbf{R}_t^{(2)} - \eta_2(\mathbf{A}_t^{\perp})^\top\mathbf{R}_t^{(3)} + \eta_2\mathbf{E}_{t,\mathbf{B}}\mathbf{V}_{\perp} \\
&-\eta_2\left[(\mathbf{A}_t^{\mathbf{U}})^\top\mathbf{U}^\top + (\mathbf{A}_t^{\perp})^\top\mathbf{U}_{\perp}^\top\right]\left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right)\left(\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_{\perp}\mathbf{R}_t^{(3)}\right).
\end{aligned}$$

Here, with $\mathbf{E}_t^{(T)}, \mathbf{H}_{t,\mathbf{A}}$ and $\mathbf{H}_{t,\mathbf{B}}$ defined in Lemma 3, $\mathbf{E}_{t,\mathbf{A}}$ and $\mathbf{E}_{t,\mathbf{B}}$ satisfy

$$\begin{aligned}
\mathbf{E}_{t,\mathbf{A}} &= \widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\Delta\mathbf{E}_t^{(B)^\top} + \mathbf{E}_t^{(T)}\left(\mathbf{B}_t + \mathbf{E}_t^{(B)}\right)^\top + \widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\mathbf{A}_t\mathbf{B}_t\mathbf{B}_t^\top - \mathbf{H}_{t,\mathbf{A}}, \\
\mathbf{E}_{t,\mathbf{B}} &= (\mathbf{E}_t^{(A)})^\top\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\Delta + (\mathbf{A}_t + \mathbf{E}_t^{(A)})^\top\mathbf{E}_t^{(T)} + \mathbf{A}_t^\top\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\mathbf{A}_t\mathbf{B}_t - \mathbf{H}_{t,\mathbf{B}}.
\end{aligned}$$

Proof By Lemma 3, the update rule of parameter $\mathbf{A}_t$ satisfies

$$\mathbf{A}_{t+1} = \mathbf{A}_t + \eta_1 \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\Delta + \mathbf{E}_t^{(T)}\right)\left(\mathbf{B}_t + \mathbf{E}_t^{(B)}\right)^\top - \eta_1 \mathbf{H}_{t,\mathbf{A}}.$$

Denote the low-precision counterpart of $\mathbf{H}_{t,\mathbf{A}}$ as $\mathbf{E}_{t,\mathbf{A}}^{(H)} = \mathbf{H}_{t,\mathbf{A}} - \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{A}_t\mathbf{B}_t\mathbf{B}_t^\top$. Then

$$\begin{aligned}
\mathbf{A}_{t+1} =& \mathbf{A}_t + \eta_1 \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\Delta + \mathbf{E}_t^{(T)}\right)\left(\mathbf{B}_t + \mathbf{E}_t^{(B)}\right)^\top - \frac{\eta_1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{A}_t\mathbf{B}_t\mathbf{B}_t^\top - \eta_1 \mathbf{E}_{t,\mathbf{A}}^{(H)} \\
=& \mathbf{A}_t + \eta_1 \widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\Delta\mathbf{B}_t^\top - \eta_1 \widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\mathbf{A}_t\mathbf{B}_t\mathbf{B}_t^\top + \eta_1 \underbrace{\left[\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\Delta\mathbf{E}_t^{(B)\top} + \mathbf{E}_t^{(T)}\left(\mathbf{B}_t + \mathbf{E}_t^{(B)}\right)^\top - \mathbf{E}_{t,\mathbf{A}}^{(H)}\right]}_{\mathbf{E}_{t,\mathbf{A}},\ \text{low-precision part of } \mathbf{A}_{t+1}} \\
=& \mathbf{A}_t - \eta_1(\mathbf{A}_t\mathbf{B}_t - \Delta)\mathbf{B}_t^\top - \eta_1\left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right)(\mathbf{A}_t\mathbf{B}_t - \Delta)\mathbf{B}_t^\top + \eta_1\mathbf{E}_{t,\mathbf{A}}.
\end{aligned} \tag{13}$$

Next, we analyze $\mathbf{A}_t^{\mathbf{U}} = \mathbf{U}^\top\mathbf{A}_t$ and $\mathbf{A}_t^{\perp} = \mathbf{U}_\perp^\top\mathbf{A}_t$ respectively. By (13), we have

$$\begin{aligned}
\mathbf{U}^\top\mathbf{A}_{t+1} =& \mathbf{U}^\top\mathbf{A}_t - \eta_1\mathbf{U}^\top(\mathbf{A}_t\mathbf{B}_t - \Delta)(\mathbf{V}\mathbf{V}^\top + \mathbf{V}_\perp\mathbf{V}_\perp^\top)\mathbf{B}_t^\top \\
&-\eta_1\mathbf{U}^\top\left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right)(\mathbf{A}_t\mathbf{B}_t - \Delta)(\mathbf{V}\mathbf{V}^\top + \mathbf{V}_\perp\mathbf{V}_\perp^\top)\mathbf{B}_t^\top + \eta_1\mathbf{U}^\top\mathbf{E}_{t,\mathbf{A}} \\
=& \mathbf{U}^\top\mathbf{A}_t - \eta_1(\mathbf{U}^\top\mathbf{A}_t\mathbf{B}_t\mathbf{V} - \mathbf{U}^\top\Delta\mathbf{V})(\mathbf{B}_t\mathbf{V})^\top - \eta_1(\mathbf{U}^\top\mathbf{A}_t\mathbf{B}_t\mathbf{V}_\perp - \mathbf{U}^\top\Delta\mathbf{V}_\perp)(\mathbf{B}_t\mathbf{V}_\perp)^\top + \eta_1\mathbf{U}^\top\mathbf{E}_{t,\mathbf{A}} \\
&-\eta_1\mathbf{U}^\top\left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right)(\mathbf{A}_t\mathbf{B}_t\mathbf{V} - \Delta\mathbf{V})(\mathbf{B}_t\mathbf{V})^\top - \eta_1\mathbf{U}^\top\left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right)(\mathbf{A}_t\mathbf{B}_t\mathbf{V}_\perp - \Delta\mathbf{V}_\perp)(\mathbf{B}_t\mathbf{V}_\perp)^\top.
\end{aligned}$$

Substituting the definition of $\mathbf{R}_t^*,\ \mathbf{R}_t^{(i)},\ i = 1, 2, 3$, it holds

$$\begin{aligned}
\mathbf{U}^\top\mathbf{A}_{t+1} =& \mathbf{U}^\top\mathbf{A}_t - \eta_1\mathbf{R}_t^*(\mathbf{B}_t^{\mathbf{V}})^\top - \eta_1\mathbf{R}_t^{(2)}(\mathbf{B}_t^{\perp})^\top + \eta_1\mathbf{U}^\top\mathbf{E}_{t,\mathbf{A}} \\
&-\eta_1\mathbf{U}^\top\left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right)\left[(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_\perp\mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_\perp\mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right].
\end{aligned}$$

Regarding $\mathbf{U}_\perp^\top\mathbf{A}_t$, we can similarly deduce that

$$\begin{aligned}
\mathbf{A}_{t+1}^{\perp} =& \mathbf{A}_t^{\perp} - \eta_1\mathbf{R}_t^{(1)}(\mathbf{B}_t^{\mathbf{V}})^\top - \eta_1\mathbf{R}_t^{(3)}(\mathbf{B}_t^{\perp})^\top + \eta_1\mathbf{U}_\perp^\top\mathbf{E}_{t,\mathbf{A}} \\
&-\eta_1\mathbf{U}_\perp^\top\left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right)\left[(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_\perp\mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_\perp\mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right].
\end{aligned}$$

Regarding the update rule of parameter $\mathbf{B}_t$, by Lemma 3,

$$\mathbf{B}_{t+1} = \mathbf{B}_t + \eta_2\left(\mathbf{A}_t + \mathbf{E}_t^{(A)}\right)^\top\left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\Delta + \mathbf{E}_t^{(T)}\right) - \eta_2\mathbf{H}_{t,\mathbf{B}}.$$

Similarly, we extract the primary term from $\mathbf{H}_{t,\mathbf{B}}$ and define its low-precision residual as $\mathbf{E}_{t,\mathbf{B}}^{(H)} = \mathbf{H}_{t,\mathbf{B}} - \mathbf{A}_t^\top\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\mathbf{A}_t\mathbf{B}_t$, yielding:

$$\begin{aligned}
\mathbf{B}_{t+1} =& \mathbf{B}_t + \eta_2\left(\mathbf{A}_t + \mathbf{E}_t^{(A)}\right)^\top\left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\Delta + \mathbf{E}_t^{(T)}\right) - \eta_2\mathbf{A}_t^\top\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\mathbf{A}_t\mathbf{B}_t - \eta_2\mathbf{E}_{t,\mathbf{B}}^{(H)} \\
=& \mathbf{B}_t - \eta_2\mathbf{A}_t^\top(\mathbf{A}_t\mathbf{B}_t - \Delta) - \eta_2\mathbf{A}_t^\top\left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right)(\mathbf{A}_t\mathbf{B}_t - \Delta) + \eta_2\mathbf{E}_{t,\mathbf{B}},
\end{aligned} \tag{14}$$

where $\mathbf{E}_{t,\mathbf{B}} = (\mathbf{E}_t^{(A)})^\top \widehat{\mathbf{\Sigma}}_{\mathcal{Q}} \Delta + (\mathbf{A}_t + \mathbf{E}_t^{(A)})^\top \mathbf{E}_t^{(T)} - \mathbf{E}_{t,\mathbf{B}}^{(H)}$.

Next, the analysis for $\mathbf{B}_t^{\mathbf{V}}$ and $\mathbf{B}_t^{\perp}$ can also be deduced. Specifically, it holds

$$\begin{aligned}
\mathbf{B}_{t+1}^{\mathbf{V}} =& \mathbf{B}_t^{\mathbf{V}} - \eta_2 \mathbf{A}_t^\top (\mathbf{U}\mathbf{U}^\top + \mathbf{U}_\perp \mathbf{U}_\perp^\top)(\mathbf{A}_t \mathbf{B}_t - \Delta)\mathbf{V} \\
& - \eta_2 \mathbf{A}_t^\top (\mathbf{U}\mathbf{U}^\top + \mathbf{U}_\perp \mathbf{U}_\perp^\top) \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d \right) (\mathbf{A}_t \mathbf{B}_t - \Delta)\mathbf{V} + \eta_2 \mathbf{E}_{t,\mathbf{B}} \mathbf{V} \\
=& \mathbf{B}_t^{\mathbf{V}} - \eta_2 (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^* - \eta_2 (\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(1)} + \eta_2 \mathbf{E}_{t,\mathbf{B}} \mathbf{V} \\
& - \eta_2 \left[(\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{U}^\top + (\mathbf{A}_t^{\perp})^\top \mathbf{U}_\perp^\top \right] \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d \right) \left(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_\perp \mathbf{R}_t^{(1)} \right),
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{B}_{t+1}^{\perp} =& \mathbf{B}_t^{\perp} - \eta_2 (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^{(2)} - \eta_2 (\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(3)} + \eta_2 \mathbf{E}_{t,\mathbf{B}} \mathbf{V}_\perp \\
& - \eta_2 \left[(\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{U}^\top + (\mathbf{A}_t^{\perp})^\top \mathbf{U}_\perp^\top \right] \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d \right) \left(\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_\perp \mathbf{R}_t^{(3)} \right).
\end{aligned}$$

■

Lemma 6 (Error norm propagation) *For a general matrix $\mathbf{M}$, let $\lambda_i(\mathbf{M})$ denote its i-th largest singular value; for a symmetric positive semi-definite matrix $\mathbf{P}$, let $\lambda_{\min}(\mathbf{P})$ denote its smallest eigenvalue. Denote $\mathcal{M}_t = \max \left\{ \|\mathbf{R}_t^*\|_{op}, \left\|\mathbf{R}_t^{(1)}\right\|_{op}, \left\|\mathbf{R}_t^{(2)}\right\|_{op} \right\}$, $\delta = \left\| \widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d \right\|_{op}$ and λ_i as the i-th singular value. Assume additionally that*

$$\eta_1 \left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}^2 + \|\mathbf{B}_t^{\perp}\|_{op}^2 \right) + \eta_2 \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}^2 + \|\mathbf{A}_t^{\perp}\|_{op}^2 \right) \leq 1.$$

It holds

$$\begin{aligned}
& \|\mathbf{R}_{t+1}^*\|_{op} \leq \left(1 - \eta_1 \lambda_{r^*}^2(\mathbf{B}_t^{\mathbf{V}}) - \eta_2 \lambda_{r^*}^2(\mathbf{A}_t^{\mathbf{U}})\right) \mathcal{M}_t \\
& + \eta_1 \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \mathcal{M}_t + \eta_2 \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{A}_t^{\perp}\|_{op} \mathcal{M}_t \\
& + \eta_1 \eta_2 \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} \right) \mathcal{M}_t \\
& + \eta_1 \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \eta_2 \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} + \eta_1 \eta_2 \|\mathbf{E}_{t,\mathbf{A}}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
& + \eta_1 \eta_2 \left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{A}_t^{\perp}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{B}_t^{\perp}\|_{op} \right) \mathcal{M}_t^2 \\
& + \eta_1 \delta \left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op} \right) \left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \eta_2 \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \mathcal{M}_t + \eta_2 \|\mathbf{A}_t^{\perp}\|_{op} \mathcal{M}_t + \eta_2 \|\mathbf{E}_{t,\mathbf{B}}\|_{op} \right) \mathcal{M}_t \\
& + \eta_1 \delta \|\mathbf{B}_t^{\perp}\|_{op} \left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \eta_2 \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \mathcal{M}_t + \eta_2 \|\mathbf{A}_t^{\perp}\|_{op} \mathcal{M}_t + \eta_2 \|\mathbf{E}_{t,\mathbf{B}}\|_{op} \right) \|\mathbf{R}_t^{(3)}\|_{op} \\
& + 2\eta_2 \delta \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op} \right) \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \eta_1 \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \mathcal{M}_t + \eta_1 \|\mathbf{B}_t^{\perp}\|_{op} \mathcal{M}_t + \eta_1 \|\mathbf{E}_{t,\mathbf{A}}\|_{op} \right) \mathcal{M}_t \\
& + 2\eta_1 \eta_2 \delta^2 \left[\left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op} \right) \mathcal{M}_t + \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(3)}\|_{op} \right] \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op} \right) \mathcal{M}_t.
\end{aligned}$$

$$
\begin{aligned}
&\|\mathbf{R}_{t+1}^{(1)}\|_{op} \leq \left(1 - \eta_1 \lambda_{r^*}^2(\mathbf{B}_t^{\mathbf{V}}) - \eta_2 \lambda_{\min}(\mathbf{A}_t^{\perp} {\mathbf{A}_t^{\perp}}^{\top})\right) \mathcal{M}_t \\
&+\eta_1 \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{R}_t^{(3)}\|_{op} + \eta_2 \|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \mathcal{M}_t \\
&+\eta_1 \eta_2 \left[\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right) \mathcal{M}_t + \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} \|\mathbf{R}_t^{(3)}\|_{op}\right] \\
&+\eta_1 \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \eta_2 \|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} + \eta_1 \eta_2 \|\mathbf{E}_{t,\mathbf{A}}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
&+\eta_1 \eta_2 \left[\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{A}_t^{\perp}\|_{op}\right) \mathcal{M}_t + \left(\|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{A}_t^{\perp}\|_{op}\right) \|\mathbf{R}_t^{(3)}\|_{op}\right] \mathcal{M}_t \\
&+\eta_1 \delta \left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op}\right) \left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \eta_2 \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \mathcal{M}_t + \eta_2 \|\mathbf{A}_t^{\perp}\|_{op} \mathcal{M}_t + \eta_2 \|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right) \mathcal{M}_t \\
&+\eta_1 \delta \|\mathbf{B}_t^{\perp}\|_{op} \left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \eta_2 \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \mathcal{M}_t + \eta_2 \|\mathbf{A}_t^{\perp}\|_{op} \mathcal{M}_t + \eta_2 \|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right) \|\mathbf{R}_t^{(3)}\|_{op} \\
&+2\eta_2 \delta \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right) \left(\|\mathbf{A}_t^{\perp}\|_{op} + \eta_1 \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \mathcal{M}_t + \eta_1 \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(3)}\|_{op} + \eta_1 \|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right) \mathcal{M}_t \\
&+2\eta_1 \eta_2 \delta^2 \left[\left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op}\right) \mathcal{M}_t + \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(3)}\|_{op}\right] \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right) \mathcal{M}_t.
\end{aligned}
$$

$$
\begin{aligned}
&\|\mathbf{R}_{t+1}^{(2)}\|_{op} \leq \left(1 - \eta_1 \lambda_{\min}({\mathbf{B}_t^{\perp}}^{\top} \mathbf{B}_t^{\perp}) - \eta_2 \lambda_{r^*}^2(\mathbf{A}_t^{\mathbf{U}})\right) \mathcal{M}_t \\
&+\eta_1 \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \mathcal{M}_t + \eta_2 \|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{R}_t^{(3)}\|_{op} \\
&+\eta_1 \eta_2 \left[\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right) \mathcal{M}_t + \|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{E}_{t,\mathbf{A}}\|_{op} \|\mathbf{R}_t^{(3)}\|_{op}\right] \\
&+\eta_1 \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \eta_2 \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} + \eta_1 \eta_2 \|\mathbf{E}_{t,\mathbf{A}}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
&+\eta_1 \eta_2 \left[\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{A}_t^{\mathbf{U}}\|_{op}\right) \mathcal{M}_t + \left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{A}_t^{\perp}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{A}_t^{\perp}\|_{op}\right) \|\mathbf{R}_t^{(3)}\|_{op}\right] \mathcal{M}_t \\
&+\eta_1 \delta \left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op}\right) \left(\|\mathbf{B}_t^{\perp}\|_{op} + \eta_2 \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \mathcal{M}_t + \eta_2 \|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(3)}\|_{op} + \eta_2 \|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right) \mathcal{M}_t \\
&+\eta_1 \delta \|\mathbf{B}_t^{\perp}\|_{op} \left(\|\mathbf{B}_t^{\perp}\|_{op} + \eta_2 \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \mathcal{M}_t + \eta_2 \|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(3)}\|_{op} + \eta_2 \|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right) \|\mathbf{R}_t^{(3)}\|_{op} \\
&+\eta_2 \delta \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right) \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \eta_1 \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \mathcal{M}_t + \eta_1 \|\mathbf{B}_t^{\perp}\|_{op} \mathcal{M}_t + \eta_1 \|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right) \left(\mathcal{M}_t + \|\mathbf{R}_t^{(3)}\|_{op}\right) \\
&+\eta_1 \eta_2 \delta^2 \left[\left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op}\right) \mathcal{M}_t + \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(3)}\|_{op}\right] \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right) \left(\mathcal{M}_t + \|\mathbf{R}_t^{(3)}\|_{op}\right).
\end{aligned}
$$

$$
\begin{aligned}
&\|\mathbf{R}_{t+1}^{(3)}\|_{op} \leq \left(1-\eta_1\lambda_{\min}(\mathbf{B}_t^{\perp\top}\mathbf{B}_t^{\perp})-\eta_2\lambda_{\min}(\mathbf{A}_t^{\perp}\mathbf{A}_t^{\perp\top})\right)\|\mathbf{R}_t^{(3)}\|_{op}\\
&+\eta_1\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\mathcal{M}_t+\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\mathcal{M}_t\\
&+\eta_1\eta_2\left[\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\mathcal{M}_t+\left(\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op}\right]\\
&+\eta_1\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}+\eta_1\eta_2\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\\
&+\eta_1\eta_2\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\mathcal{M}_t^2+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}^2\right)\\
&+\eta_1\eta_2\left(\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}\right)\mathcal{M}_t\|\mathbf{R}_t^{(3)}\|_{op}\\
&+\eta_1\delta\left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\right)\left(\|\mathbf{B}_t^{\perp}\|_{op}+\eta_2\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\mathcal{M}_t+\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}+\eta_2\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\mathcal{M}_t\\
&+\eta_1\delta\|\mathbf{B}_t^{\perp}\|_{op}\left(\|\mathbf{B}_t^{\perp}\|_{op}+\eta_2\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\mathcal{M}_t+\eta_2\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}+\eta_2\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op}\\
&+\eta_2\delta\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\|\mathbf{A}_t^{\perp}\|_{op}+\eta_1\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\mathcal{M}_t+\eta_1\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}+\eta_1\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right)\mathcal{M}_t\\
&+\eta_2\delta\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\|\mathbf{A}_t^{\perp}\|_{op}+\eta_1\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\mathcal{M}_t+\eta_1\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}+\eta_1\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op}\\
&+\eta_1\eta_2\delta^2\left[\left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\right)\mathcal{M}_t+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\right]\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)(\mathcal{M}_t+\|\mathbf{R}_t^{(3)}\|_{op}).
\end{aligned}
$$

Proof We prove by Lemma 5 and Lemma 18. In the sequel, all quantization/covariance perturbation terms (including mixed terms such as $-\eta_1\eta_2\mathbf{U}^\top\mathbf{E}_{t,\mathbf{A}}(\mathbf{A}_t^{\mathbf{U}})^\top\mathbf{R}_t^*$) are upper bounded additively by sub-multiplicativity of $\|\cdot\|_{op}$. We first write the update rule for $\mathbf{R}_t^*$.

$$
\begin{aligned}
\mathbf{R}_{t+1}^* =&\mathbf{A}_{t+1}^{\mathbf{U}}\mathbf{B}_{t+1}^{\mathbf{V}}-\mathbf{S}^*\\
=&\mathbf{R}_t^*-\eta_1\mathbf{R}_t^*(\mathbf{B}_t^{\mathbf{V}})^\top\mathbf{B}_t^{\mathbf{V}}-\eta_1\mathbf{R}_t^{(2)}(\mathbf{B}_t^{\perp})^\top\mathbf{B}_t^{\mathbf{V}}+\eta_1\mathbf{U}^\top\mathbf{E}_{t,\mathbf{A}}\mathbf{B}_t^{\mathbf{V}}\\
&-\eta_1\mathbf{U}^\top\left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}-\mathbf{I}_d\right)\left[(\mathbf{U}\mathbf{R}_t^*+\mathbf{U}_\perp\mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top+(\mathbf{U}\mathbf{R}_t^{(2)}+\mathbf{U}_\perp\mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right]\mathbf{B}_t^{\mathbf{V}}\\
&-\eta_2\mathbf{A}_t^{\mathbf{U}}(\mathbf{A}_t^{\mathbf{U}})^\top\mathbf{R}_t^*+\eta_1\eta_2\mathbf{R}_t^*(\mathbf{B}_t^{\mathbf{V}})^\top(\mathbf{A}_t^{\mathbf{U}})^\top\mathbf{R}_t^*\\
&+\eta_1\eta_2\mathbf{R}_t^{(2)}(\mathbf{B}_t^{\perp})^\top(\mathbf{A}_t^{\mathbf{U}})^\top\mathbf{R}_t^*-\eta_1\eta_2\mathbf{U}^\top\mathbf{E}_{t,\mathbf{A}}(\mathbf{A}_t^{\mathbf{U}})^\top\mathbf{R}_t^*\\
&-\eta_1\mathbf{U}^\top\left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}-\mathbf{I}_d\right)\left[(\mathbf{U}\mathbf{R}_t^*+\mathbf{U}_\perp\mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top+(\mathbf{U}\mathbf{R}_t^{(2)}+\mathbf{U}_\perp\mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right](-\eta_2(\mathbf{A}_t^{\mathbf{U}})^\top\mathbf{R}_t^*)\\
&-\eta_2\mathbf{A}_t^{\mathbf{U}}(\mathbf{A}_t^{\perp})^\top\mathbf{R}_t^{(1)}+\eta_1\eta_2\mathbf{R}_t^*(\mathbf{B}_t^{\mathbf{V}})^\top(\mathbf{A}_t^{\perp})^\top\mathbf{R}_t^{(1)}\\
&+\eta_1\eta_2\mathbf{R}_t^{(2)}(\mathbf{B}_t^{\perp})^\top(\mathbf{A}_t^{\perp})^\top\mathbf{R}_t^{(1)}-\eta_1\eta_2\mathbf{U}^\top\mathbf{E}_{t,\mathbf{A}}(\mathbf{A}_t^{\perp})^\top\mathbf{R}_t^{(1)}\\
&-\eta_1\mathbf{U}^\top\left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}-\mathbf{I}_d\right)\left[(\mathbf{U}\mathbf{R}_t^*+\mathbf{U}_\perp\mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top+(\mathbf{U}\mathbf{R}_t^{(2)}+\mathbf{U}_\perp\mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right](-\eta_2(\mathbf{A}_t^{\perp})^\top\mathbf{R}_t^{(1)})\\
&+\eta_2\mathbf{A}_t^{\mathbf{U}}\mathbf{E}_{t,\mathbf{B}}\mathbf{V}-\eta_1\eta_2\mathbf{R}_t^*(\mathbf{B}_t^{\mathbf{V}})^\top\mathbf{E}_{t,\mathbf{B}}\mathbf{V}-\eta_1\eta_2\mathbf{R}_t^{(2)}(\mathbf{B}_t^{\perp})^\top\mathbf{E}_{t,\mathbf{B}}\mathbf{V}+\eta_1\eta_2\mathbf{U}^\top\mathbf{E}_{t,\mathbf{A}}\mathbf{E}_{t,\mathbf{B}}\mathbf{V}\\
&-\eta_1\mathbf{U}^\top\left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}-\mathbf{I}_d\right)\left[(\mathbf{U}\mathbf{R}_t^*+\mathbf{U}_\perp\mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top+(\mathbf{U}\mathbf{R}_t^{(2)}+\mathbf{U}_\perp\mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right]\eta_2\mathbf{E}_{t,\mathbf{B}}\mathbf{V}\\
&-\eta_2\mathbf{A}_{t+1}^{\mathbf{U}}\left[(\mathbf{A}_t^{\mathbf{U}})^\top\mathbf{U}^\top+(\mathbf{A}_t^{\perp})^\top\mathbf{U}_\perp^\top\right]\left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}-\mathbf{I}_d\right)\left(\mathbf{U}\mathbf{R}_t^*+\mathbf{U}_\perp\mathbf{R}_t^{(1)}\right).
\end{aligned}
$$

Regarding $\mathbf{R}_t^{(1)}$,

$$
\begin{aligned}
\mathbf{R}_{t+1}^{(1)} =& \mathbf{A}_{t+1}^{\perp}\mathbf{B}_{t+1}^{\mathbf{V}} \\
=& \mathbf{R}_t^{(1)} - \eta_1 \mathbf{R}_t^{(1)}(\mathbf{B}_t^{\mathbf{V}})^\top \mathbf{B}_t^{\mathbf{V}} - \eta_1 \mathbf{R}_t^{(3)}(\mathbf{B}_t^{\perp})^\top \mathbf{B}_t^{\mathbf{V}} + \eta_1 \mathbf{U}_\perp^\top \mathbf{E}_{t,\mathbf{A}} \mathbf{B}_t^{\mathbf{V}} \\
& - \eta_1 \mathbf{U}_\perp^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left[(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_\perp \mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_\perp \mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right] \mathbf{B}_t^{\mathbf{V}} \\
& - \eta_2 \mathbf{A}_t^{\perp}(\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^* + \eta_1\eta_2 \mathbf{R}_t^{(1)}(\mathbf{B}_t^{\mathbf{V}})^\top (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^* \\
& + \eta_1\eta_2 \mathbf{R}_t^{(3)}(\mathbf{B}_t^{\perp})^\top (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^* - \eta_1\eta_2 \mathbf{U}_\perp^\top \mathbf{E}_{t,\mathbf{A}} (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^* \\
& + \eta_1\eta_2 \mathbf{U}_\perp^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left[(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_\perp \mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_\perp \mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right] (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^* \\
& - \eta_2 \mathbf{A}_t^{\perp}(\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(1)} + \eta_1\eta_2 \mathbf{R}_t^{(1)}(\mathbf{B}_t^{\mathbf{V}})^\top (\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(1)} \\
& + \eta_1\eta_2 \mathbf{R}_t^{(3)}(\mathbf{B}_t^{\perp})^\top (\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(1)} - \eta_1\eta_2 \mathbf{U}_\perp^\top \mathbf{E}_{t,\mathbf{A}} (\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(1)} \\
& + \eta_1\eta_2 \mathbf{U}_\perp^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left[(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_\perp \mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_\perp \mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right] (\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(1)} \\
& + \eta_2 \mathbf{A}_t^{\perp} \mathbf{E}_{t,\mathbf{B}} \mathbf{V} - \eta_1\eta_2 \mathbf{R}_t^{(1)}(\mathbf{B}_t^{\mathbf{V}})^\top \mathbf{E}_{t,\mathbf{B}} \mathbf{V} - \eta_1\eta_2 \mathbf{R}_t^{(3)}(\mathbf{B}_t^{\perp})^\top \mathbf{E}_{t,\mathbf{B}} \mathbf{V} + \eta_1\eta_2 \mathbf{U}_\perp^\top \mathbf{E}_{t,\mathbf{A}} \mathbf{E}_{t,\mathbf{B}} \mathbf{V} \\
& - \eta_1\eta_2 \mathbf{U}_\perp^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left[(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_\perp \mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_\perp \mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right] \mathbf{E}_{t,\mathbf{B}} \mathbf{V} \\
& - \eta_2 \mathbf{A}_{t+1}^{\perp} \left[(\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{U}^\top + (\mathbf{A}_t^{\perp})^\top \mathbf{U}_\perp^\top\right] \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_\perp \mathbf{R}_t^{(1)}\right).
\end{aligned}
$$

Regarding $\mathbf{R}_t^{(2)}$,

$$
\begin{aligned}
\mathbf{R}_{t+1}^{(2)} =& \mathbf{A}_{t+1}^{\mathbf{U}}\mathbf{B}_{t+1}^{\perp} \\
=& \mathbf{R}_t^{(2)} - \eta_1 \mathbf{R}_t^*(\mathbf{B}_t^{\mathbf{V}})^\top \mathbf{B}_t^{\perp} - \eta_1 \mathbf{R}_t^{(2)}(\mathbf{B}_t^{\perp})^\top \mathbf{B}_t^{\perp} + \eta_1 \mathbf{U}^\top \mathbf{E}_{t,\mathbf{A}} \mathbf{B}_t^{\perp} \\
& - \eta_1 \mathbf{U}^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left[(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_\perp \mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_\perp \mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right] \mathbf{B}_t^{\perp} \\
& - \eta_2 \mathbf{A}_t^{\mathbf{U}}(\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^{(2)} + \eta_1\eta_2 \mathbf{R}_t^*(\mathbf{B}_t^{\mathbf{V}})^\top (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^{(2)} \\
& + \eta_1\eta_2 \mathbf{R}_t^{(2)}(\mathbf{B}_t^{\perp})^\top (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^{(2)} - \eta_1\eta_2 \mathbf{U}^\top \mathbf{E}_{t,\mathbf{A}} (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^{(2)} \\
& + \eta_1\eta_2 \mathbf{U}^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left[(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_\perp \mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_\perp \mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right] (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^{(2)} \\
& - \eta_2 \mathbf{A}_t^{\mathbf{U}}(\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(3)} + \eta_1\eta_2 \mathbf{R}_t^*(\mathbf{B}_t^{\mathbf{V}})^\top (\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(3)} \\
& + \eta_1\eta_2 \mathbf{R}_t^{(2)}(\mathbf{B}_t^{\perp})^\top (\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(3)} - \eta_1\eta_2 \mathbf{U}^\top \mathbf{E}_{t,\mathbf{A}} (\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(3)} \\
& + \eta_1\eta_2 \mathbf{U}^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left[(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_\perp \mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_\perp \mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right] (\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(3)} \\
& + \eta_2 \mathbf{A}_t^{\mathbf{U}} \mathbf{E}_{t,\mathbf{B}} \mathbf{V}_\perp - \eta_1\eta_2 \mathbf{R}_t^*(\mathbf{B}_t^{\mathbf{V}})^\top \mathbf{E}_{t,\mathbf{B}} \mathbf{V}_\perp - \eta_1\eta_2 \mathbf{R}_t^{(2)}(\mathbf{B}_t^{\perp})^\top \mathbf{E}_{t,\mathbf{B}} \mathbf{V}_\perp + \eta_1\eta_2 \mathbf{U}^\top \mathbf{E}_{t,\mathbf{A}} \mathbf{E}_{t,\mathbf{B}} \mathbf{V}_\perp \\
& - \eta_1\eta_2 \mathbf{U}^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left[(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_\perp \mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_\perp \mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right] \mathbf{E}_{t,\mathbf{B}} \mathbf{V}_\perp \\
& - \eta_2 \mathbf{A}_{t+1}^{\mathbf{U}} \left[(\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{U}^\top + (\mathbf{A}_t^{\perp})^\top \mathbf{U}_\perp^\top\right] \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left(\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_\perp \mathbf{R}_t^{(3)}\right).
\end{aligned}
$$

Regarding $\mathbf{R}_t^{(3)}$,

$$
\begin{aligned}
\mathbf{R}_{t+1}^{(3)} =& \mathbf{A}_{t+1}^{\perp}\mathbf{B}_{t+1}^{\perp} \\
=& \mathbf{R}_t^{(3)} - \eta_1 \mathbf{R}_t^{(1)}(\mathbf{B}_t^{\mathbf{V}})^\top \mathbf{B}_t^{\perp} - \eta_1 \mathbf{R}_t^{(3)}(\mathbf{B}_t^{\perp})^\top \mathbf{B}_t^{\perp} + \eta_1 \mathbf{U}_{\perp}^\top \mathbf{E}_{t,\mathbf{A}} \mathbf{B}_t^{\perp} \\
& - \eta_1 \mathbf{U}_{\perp}^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left[(\mathbf{U}\mathbf{R}_t^{*} + \mathbf{U}_{\perp}\mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_{\perp}\mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right] \mathbf{B}_t^{\perp} \\
& - \eta_2 \mathbf{A}_t^{\perp}(\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^{(2)} + \eta_1\eta_2 \mathbf{R}_t^{(1)}(\mathbf{B}_t^{\mathbf{V}})^\top (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^{(2)} \\
& + \eta_1\eta_2 \mathbf{R}_t^{(3)}(\mathbf{B}_t^{\perp})^\top (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^{(2)} - \eta_1\eta_2 \mathbf{U}_{\perp}^\top \mathbf{E}_{t,\mathbf{A}} (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^{(2)} \\
& + \eta_1\eta_2 \mathbf{U}_{\perp}^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left[(\mathbf{U}\mathbf{R}_t^{*} + \mathbf{U}_{\perp}\mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_{\perp}\mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right] (\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{R}_t^{(2)} \\
& - \eta_2 \mathbf{A}_t^{\perp}(\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(3)} + \eta_1\eta_2 \mathbf{R}_t^{(1)}(\mathbf{B}_t^{\mathbf{V}})^\top (\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(3)} \\
& + \eta_1\eta_2 \mathbf{R}_t^{(3)}(\mathbf{B}_t^{\perp})^\top (\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(3)} - \eta_1\eta_2 \mathbf{U}_{\perp}^\top \mathbf{E}_{t,\mathbf{A}} (\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(3)} \\
& + \eta_1\eta_2 \mathbf{U}_{\perp}^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left[(\mathbf{U}\mathbf{R}_t^{*} + \mathbf{U}_{\perp}\mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_{\perp}\mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right] (\mathbf{A}_t^{\perp})^\top \mathbf{R}_t^{(3)} \\
& + \eta_2 \mathbf{A}_t^{\perp} \mathbf{E}_{t,\mathbf{B}} \mathbf{V}_{\perp} - \eta_1\eta_2 \mathbf{R}_t^{(1)}(\mathbf{B}_t^{\mathbf{V}})^\top \mathbf{E}_{t,\mathbf{B}} \mathbf{V}_{\perp} - \eta_1\eta_2 \mathbf{R}_t^{(3)}(\mathbf{B}_t^{\perp})^\top \mathbf{E}_{t,\mathbf{B}} \mathbf{V}_{\perp} + \eta_1\eta_2 \mathbf{U}_{\perp}^\top \mathbf{E}_{t,\mathbf{A}} \mathbf{E}_{t,\mathbf{B}} \mathbf{V}_{\perp} \\
& - \eta_1\eta_2 \mathbf{U}_{\perp}^\top \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left[(\mathbf{U}\mathbf{R}_t^{*} + \mathbf{U}_{\perp}\mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_{\perp}\mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right] \mathbf{E}_{t,\mathbf{B}} \mathbf{V}_{\perp} \\
& - \eta_2 \mathbf{A}_{t+1}^{\perp} \left[(\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{U}^\top + (\mathbf{A}_t^{\perp})^\top \mathbf{U}_{\perp}^\top\right] \left(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right) \left(\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_{\perp}\mathbf{R}_t^{(3)}\right).
\end{aligned}
$$

Denote $\delta = \left\|\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right\|_{op}$, we then upper bound $\left\|\mathbf{R}_{t+1}^{*}\right\|_{op}, \left\|\mathbf{R}_{t+1}^{(1)}\right\|_{op}, \left\|\mathbf{R}_{t+1}^{(2)}\right\|_{op}, \left\|\mathbf{R}_{t+1}^{(3)}\right\|_{op}$ using Lemma 18. To simplify bounds, we also denote $\mathcal{M}_t = \max\left\{\|\mathbf{R}_t^{*}\|_{op}, \left\|\mathbf{R}_t^{(1)}\right\|_{op}, \left\|\mathbf{R}_t^{(2)}\right\|_{op}\right\}$.

Regarding $\mathbf{R}_t^*$,

$$
\begin{aligned}
&\|\mathbf{R}_{t+1}^*\|_{op} \leq \left(1 - \eta_1 \lambda_{r^*}^2(\mathbf{B}_t^{\mathbf{V}}) - \eta_2 \lambda_{r^*}^2(\mathbf{A}_t^{\mathbf{U}})\right) \|\mathbf{R}_t^*\|_{op} \\
&+\eta_1 \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{R}_t^{(2)}\|_{op} + \eta_2 \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(1)}\|_{op} \\
&+\eta_1\eta_2 \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{R}_t^*\|_{op} \|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \eta_1\eta_2 \|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(1)}\|_{op} \|\mathbf{E}_{t,\mathbf{A}}\|_{op} \\
&+\eta_1\eta_2 \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{R}_t^*\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} + \eta_1\eta_2 \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(2)}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
&+\eta_1 \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \eta_2 \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} + \eta_1\eta_2 \|\mathbf{E}_{t,\mathbf{A}}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
&+\eta_1\eta_2 \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{R}_t^*\|_{op}^2 + \eta_1\eta_2 \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{R}_t^{(2)}\|_{op} \|\mathbf{R}_t^*\|_{op} \\
&+\eta_1\eta_2 \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{R}_t^*\|_{op} \|\mathbf{R}_t^{(1)}\|_{op} + \eta_1\eta_2 \|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(2)}\|_{op} \|\mathbf{R}_t^{(1)}\|_{op} \\
&+\eta_1\delta \left[\|\mathbf{B}_t^{\mathbf{V}}\|_{op}(\|\mathbf{R}_t^*\|_{op} + \|\mathbf{R}_t^{(1)}\|_{op}) + \|\mathbf{B}_t^{\perp}\|_{op}(\|\mathbf{R}_t^{(2)}\|_{op} + \|\mathbf{R}_t^{(3)}\|_{op})\right] \left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \eta_2 \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{R}_t^*\|_{op}\right) \\
&+\eta_1\eta_2\delta \left[\|\mathbf{B}_t^{\mathbf{V}}\|_{op}(\|\mathbf{R}_t^*\|_{op} + \|\mathbf{R}_t^{(1)}\|_{op}) + \|\mathbf{B}_t^{\perp}\|_{op}(\|\mathbf{R}_t^{(2)}\|_{op} + \|\mathbf{R}_t^{(3)}\|_{op})\right] \left(\|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(1)}\|_{op} + \|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right) \\
&+\eta_2\delta \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right) \left(\|\mathbf{R}_t^*\|_{op} + \|\mathbf{R}_t^{(1)}\|_{op}\right) \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \eta_1 \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{R}_t^*\|_{op}\right) \\
&+\eta_1\eta_2\delta \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right) \left(\|\mathbf{R}_t^*\|_{op} + \|\mathbf{R}_t^{(1)}\|_{op}\right) \left(\|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(2)}\|_{op} + \|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right) \\
&+\eta_1\eta_2\delta^2 \left[\|\mathbf{B}_t^{\mathbf{V}}\|_{op}(\|\mathbf{R}_t^*\|_{op} + \|\mathbf{R}_t^{(1)}\|_{op}) + \|\mathbf{B}_t^{\perp}\|_{op}(\|\mathbf{R}_t^{(2)}\|_{op} + \|\mathbf{R}_t^{(3)}\|_{op})\right] \\
&\cdot \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right) \left(\|\mathbf{R}_t^*\|_{op} + \|\mathbf{R}_t^{(1)}\|_{op}\right).
\end{aligned}
$$

Using the definition of $\mathcal{M}_t$, we have

$$
\begin{aligned}
&\|\mathbf{R}_{t+1}^*\|_{op} \leq \left(1 - \eta_1 \lambda_{r^*}^2(\mathbf{B}_t^{\mathbf{V}}) - \eta_2 \lambda_{r^*}^2(\mathbf{A}_t^{\mathbf{U}})\right) \mathcal{M}_t \\
&+\eta_1 \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \mathcal{M}_t + \eta_2 \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{A}_t^{\perp}\|_{op} \mathcal{M}_t \\
&+\eta_1\eta_2 \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right) \mathcal{M}_t \\
&+\eta_1 \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \eta_2 \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} + \eta_1\eta_2 \|\mathbf{E}_{t,\mathbf{A}}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
&+\eta_1\eta_2 \left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{A}_t^{\perp}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{B}_t^{\perp}\|_{op}\right) \mathcal{M}_t^2 \\
&+\eta_1\delta \left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op}\right) \left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \eta_2 \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \mathcal{M}_t + \eta_2 \|\mathbf{A}_t^{\perp}\|_{op} \mathcal{M}_t + \eta_2 \|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right) \mathcal{M}_t \\
&+\eta_1\delta \|\mathbf{B}_t^{\perp}\|_{op} \left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \eta_2 \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \mathcal{M}_t + \eta_2 \|\mathbf{A}_t^{\perp}\|_{op} \mathcal{M}_t + \eta_2 \|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right) \|\mathbf{R}_t^{(3)}\|_{op} \\
&+2\eta_2\delta \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right) \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \eta_1 \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \mathcal{M}_t + \eta_1 \|\mathbf{B}_t^{\perp}\|_{op} \mathcal{M}_t + \eta_1 \|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right) \mathcal{M}_t \\
&+2\eta_1\eta_2\delta^2 \left[\left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op}\right) \mathcal{M}_t + \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(3)}\|_{op}\right] \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right) \mathcal{M}_t.
\end{aligned}
$$

Regarding $\mathbf{R}_t^{(1)}$,

$$
\begin{aligned}
&\|\mathbf{R}_{t+1}^{(1)}\|_{op} \leq \left(1 - \eta_1 \lambda_{r^*}^2(\mathbf{B}_t^{\mathbf{V}}) - \eta_2 \lambda_{\min}(\mathbf{A}_t^{\perp} {\mathbf{A}_t^{\perp}}^{\top})\right) \|\mathbf{R}_t^{(1)}\|_{op} \\
&+\eta_1 \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{R}_t^{(3)}\|_{op} + \eta_2 \|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{R}_t^{*}\|_{op} \\
&+\eta_1\eta_2 \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{R}_t^{*}\|_{op} \|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \eta_1\eta_2 \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{R}_t^{(1)}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
&+\eta_1\eta_2 \|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(1)}\|_{op} \|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \eta_1\eta_2 \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(3)}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
&+\eta_1 \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \eta_2 \|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} + \eta_1\eta_2 \|\mathbf{E}_{t,\mathbf{A}}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
&+\eta_1\eta_2 \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{R}_t^{(1)}\|_{op} \|\mathbf{R}_t^{*}\|_{op} + \eta_1\eta_2 \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{R}_t^{(3)}\|_{op} \|\mathbf{R}_t^{*}\|_{op} \\
&+\eta_1\eta_2 \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(1)}\|_{op} \|\mathbf{R}_t^{(1)}\|_{op} + \eta_1\eta_2 \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(3)}\|_{op} \|\mathbf{R}_t^{(1)}\|_{op} \\
&+\eta_1 \delta \left[\|\mathbf{B}_t^{\mathbf{V}}\|_{op}(\|\mathbf{R}_t^{*}\|_{op} + \|\mathbf{R}_t^{(1)}\|_{op}) + \|\mathbf{B}_t^{\perp}\|_{op}(\|\mathbf{R}_t^{(2)}\|_{op} + \|\mathbf{R}_t^{(3)}\|_{op})\right] \left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \eta_2 \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{R}_t^{*}\|_{op}\right) \\
&+\eta_1\eta_2 \delta \left[\|\mathbf{B}_t^{\mathbf{V}}\|_{op}(\|\mathbf{R}_t^{*}\|_{op} + \|\mathbf{R}_t^{(1)}\|_{op}) + \|\mathbf{B}_t^{\perp}\|_{op}(\|\mathbf{R}_t^{(2)}\|_{op} + \|\mathbf{R}_t^{(3)}\|_{op})\right] \left(\|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(1)}\|_{op} + \|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right) \\
&+\eta_2 \delta \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right) \left(\|\mathbf{R}_t^{*}\|_{op} + \|\mathbf{R}_t^{(1)}\|_{op}\right) \left(\|\mathbf{A}_t^{\perp}\|_{op} + \eta_1 \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{R}_t^{(1)}\|_{op}\right) \\
&+\eta_1\eta_2 \delta \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right) \left(\|\mathbf{R}_t^{*}\|_{op} + \|\mathbf{R}_t^{(1)}\|_{op}\right) \left(\|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(3)}\|_{op} + \|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right) \\
&+\eta_1\eta_2 \delta^2 \left[\|\mathbf{B}_t^{\mathbf{V}}\|_{op}(\|\mathbf{R}_t^{*}\|_{op} + \|\mathbf{R}_t^{(1)}\|_{op}) + \|\mathbf{B}_t^{\perp}\|_{op}(\|\mathbf{R}_t^{(2)}\|_{op} + \|\mathbf{R}_t^{(3)}\|_{op})\right] \\
&\cdot \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right) \left(\|\mathbf{R}_t^{*}\|_{op} + \|\mathbf{R}_t^{(1)}\|_{op}\right).
\end{aligned}
$$

Using the definition of $\mathcal{M}_t$, we have

$$
\begin{aligned}
&\|\mathbf{R}_{t+1}^{(1)}\|_{op} \leq \left(1 - \eta_1 \lambda_{r^*}^2(\mathbf{B}_t^{\mathbf{V}}) - \eta_2 \lambda_{\min}(\mathbf{A}_t^{\perp} {\mathbf{A}_t^{\perp}}^{\top})\right) \mathcal{M}_t \\
&+\eta_1 \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{R}_t^{(3)}\|_{op} + \eta_2 \|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \mathcal{M}_t \\
&+\eta_1\eta_2 \left[\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right) \mathcal{M}_t + \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} \|\mathbf{R}_t^{(3)}\|_{op}\right] \\
&+\eta_1 \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \eta_2 \|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} + \eta_1\eta_2 \|\mathbf{E}_{t,\mathbf{A}}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
&+\eta_1\eta_2 \left[\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{A}_t^{\perp}\|_{op}\right) \mathcal{M}_t + \left(\|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{A}_t^{\perp}\|_{op}\right) \|\mathbf{R}_t^{(3)}\|_{op}\right] \mathcal{M}_t \\
&+\eta_1 \delta \left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op}\right) \left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \eta_2 \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \mathcal{M}_t + \eta_2 \|\mathbf{A}_t^{\perp}\|_{op} \mathcal{M}_t + \eta_2 \|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right) \mathcal{M}_t \\
&+\eta_1 \delta \|\mathbf{B}_t^{\perp}\|_{op} \left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \eta_2 \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \mathcal{M}_t + \eta_2 \|\mathbf{A}_t^{\perp}\|_{op} \mathcal{M}_t + \eta_2 \|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right) \|\mathbf{R}_t^{(3)}\|_{op} \\
&+2\eta_2 \delta \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right) \left(\|\mathbf{A}_t^{\perp}\|_{op} + \eta_1 \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \mathcal{M}_t + \eta_1 \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(3)}\|_{op} + \eta_1 \|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right) \mathcal{M}_t \\
&+2\eta_1\eta_2 \delta^2 \left[\left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op}\right) \mathcal{M}_t + \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(3)}\|_{op}\right] \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right) \mathcal{M}_t.
\end{aligned}
$$

Regarding $\mathbf{R}_t^{(2)}$,

$$
\begin{aligned}
&\|\mathbf{R}_{t+1}^{(2)}\|_{op} \leq \left(1 - \eta_1 \lambda_{\min}({\mathbf{B}_t^{\perp}}^\top \mathbf{B}_t^{\perp}) - \eta_2 \lambda_{r^*}^2(\mathbf{A}_t^{\mathbf{U}})\right) \|\mathbf{R}_t^{(2)}\|_{op} \\
&+\eta_1 \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{R}_t^{*}\|_{op} + \eta_2 \|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{R}_t^{(3)}\|_{op} \\
&+\eta_1\eta_2 \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{R}_t^{(2)}\|_{op} \|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \eta_1\eta_2 \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{R}_t^{*}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
&+\eta_1\eta_2 \|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(3)}\|_{op} \|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \eta_1\eta_2 \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(2)}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
&+\eta_1 \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \eta_2 \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} + \eta_1\eta_2 \|\mathbf{E}_{t,\mathbf{A}}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
&+\eta_1\eta_2 \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{R}_t^{(2)}\|_{op} \|\mathbf{R}_t^{*}\|_{op} + \eta_1\eta_2 \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{R}_t^{(2)}\|_{op} \|\mathbf{R}_t^{(2)}\|_{op} \\
&+\eta_1\eta_2 \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(3)}\|_{op} \|\mathbf{R}_t^{*}\|_{op} + \eta_1\eta_2 \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(3)}\|_{op} \|\mathbf{R}_t^{(2)}\|_{op} \\
&+\eta_1 \delta \left[\|\mathbf{B}_t^{\mathbf{V}}\|_{op}(\|\mathbf{R}_t^{*}\|_{op} + \|\mathbf{R}_t^{(1)}\|_{op}) + \|\mathbf{B}_t^{\perp}\|_{op}(\|\mathbf{R}_t^{(2)}\|_{op} + \|\mathbf{R}_t^{(3)}\|_{op})\right] \left(\|\mathbf{B}_t^{\perp}\|_{op} + \eta_2 \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{R}_t^{(2)}\|_{op}\right) \\
&+\eta_1\eta_2 \delta \left[\|\mathbf{B}_t^{\mathbf{V}}\|_{op}(\|\mathbf{R}_t^{*}\|_{op} + \|\mathbf{R}_t^{(1)}\|_{op}) + \|\mathbf{B}_t^{\perp}\|_{op}(\|\mathbf{R}_t^{(2)}\|_{op} + \|\mathbf{R}_t^{(3)}\|_{op})\right] \left(\|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(3)}\|_{op} + \|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right) \\
&+\eta_2 \delta \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right) \left(\|\mathbf{R}_t^{(2)}\|_{op} + \|\mathbf{R}_t^{(3)}\|_{op}\right) \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \eta_1 \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{R}_t^{*}\|_{op}\right) \\
&+\eta_1\eta_2 \delta \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right) \left(\|\mathbf{R}_t^{(2)}\|_{op} + \|\mathbf{R}_t^{(3)}\|_{op}\right) \left(\|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(2)}\|_{op} + \|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right) \\
&+\eta_1\eta_2 \delta^2 \left[\|\mathbf{B}_t^{\mathbf{V}}\|_{op}(\|\mathbf{R}_t^{*}\|_{op} + \|\mathbf{R}_t^{(1)}\|_{op}) + \|\mathbf{B}_t^{\perp}\|_{op}(\|\mathbf{R}_t^{(2)}\|_{op} + \|\mathbf{R}_t^{(3)}\|_{op})\right] \\
&\cdot \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right) \left(\|\mathbf{R}_t^{(2)}\|_{op} + \|\mathbf{R}_t^{(3)}\|_{op}\right).
\end{aligned}
$$

Using the definition of $\mathcal{M}_t$, we have

$$
\begin{aligned}
&\|\mathbf{R}_{t+1}^{(2)}\|_{op} \leq \left(1 - \eta_1 \lambda_{\min}({\mathbf{B}_t^{\perp}}^\top \mathbf{B}_t^{\perp}) - \eta_2 \lambda_{r^*}^2(\mathbf{A}_t^{\mathbf{U}})\right) \mathcal{M}_t \\
&+\eta_1 \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \mathcal{M}_t + \eta_2 \|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{R}_t^{(3)}\|_{op} \\
&+\eta_1\eta_2 \left[\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right) \mathcal{M}_t + \|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{E}_{t,\mathbf{A}}\|_{op} \|\mathbf{R}_t^{(3)}\|_{op}\right] \\
&+\eta_1 \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \eta_2 \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} + \eta_1\eta_2 \|\mathbf{E}_{t,\mathbf{A}}\|_{op} \|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
&+\eta_1\eta_2 \left[\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{A}_t^{\mathbf{U}}\|_{op}\right) \mathcal{M}_t + \left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op} \|\mathbf{A}_t^{\perp}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{A}_t^{\perp}\|_{op}\right) \|\mathbf{R}_t^{(3)}\|_{op}\right] \mathcal{M}_t \\
&+\eta_1 \delta \left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op}\right) \left(\|\mathbf{B}_t^{\perp}\|_{op} + \eta_2 \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \mathcal{M}_t + \eta_2 \|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(3)}\|_{op} + \eta_2 \|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right) \mathcal{M}_t \\
&+\eta_1 \delta \|\mathbf{B}_t^{\perp}\|_{op} \left(\|\mathbf{B}_t^{\perp}\|_{op} + \eta_2 \|\mathbf{A}_t^{\mathbf{U}}\|_{op} \mathcal{M}_t + \eta_2 \|\mathbf{A}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(3)}\|_{op} + \eta_2 \|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right) \|\mathbf{R}_t^{(3)}\|_{op} \\
&+\eta_2 \delta \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right) \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \eta_1 \|\mathbf{B}_t^{\mathbf{V}}\|_{op} \mathcal{M}_t + \eta_1 \|\mathbf{B}_t^{\perp}\|_{op} \mathcal{M}_t + \eta_1 \|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right) \left(\mathcal{M}_t + \|\mathbf{R}_t^{(3)}\|_{op}\right) \\
&+\eta_1\eta_2 \delta^2 \left[\left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op}\right) \mathcal{M}_t + \|\mathbf{B}_t^{\perp}\|_{op} \|\mathbf{R}_t^{(3)}\|_{op}\right] \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right) \left(\mathcal{M}_t + \|\mathbf{R}_t^{(3)}\|_{op}\right).
\end{aligned}
$$

Regarding $\mathbf{R}_t^{(3)}$,

$$
\begin{aligned}
&\|\mathbf{R}_{t+1}^{(3)}\|_{op} \leq \left(1 - \eta_1 \lambda_{\min}(\mathbf{B}_t^{\perp\top}\mathbf{B}_t^{\perp}) - \eta_2 \lambda_{\min}(\mathbf{A}_t^{\perp}\mathbf{A}_t^{\perp\top})\right) \|\mathbf{R}_t^{(3)}\|_{op} \\
&+\eta_1 \|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{R}_t^{(1)}\|_{op} + \eta_2 \|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{R}_t^{(2)}\|_{op} \\
&+\eta_1\eta_2 \|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{R}_t^{(2)}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \eta_1\eta_2 \|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{R}_t^{(1)}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
&+\eta_1\eta_2 \|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \eta_1\eta_2 \|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
&+\eta_1 \|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \eta_2 \|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} + \eta_1\eta_2 \|\mathbf{E}_{t,\mathbf{A}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
&+\eta_1\eta_2 \|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{R}_t^{(2)}\|_{op}\|\mathbf{R}_t^{(1)}\|_{op} + \eta_1\eta_2 \|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\|\mathbf{R}_t^{(2)}\|_{op} \\
&+\eta_1\eta_2 \|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\|\mathbf{R}_t^{(1)}\|_{op} + \eta_1\eta_2 \|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op} \\
&+\eta_1\delta \left[\|\mathbf{B}_t^{\mathbf{V}}\|_{op}(\|\mathbf{R}_t^{*}\|_{op} + \|\mathbf{R}_t^{(1)}\|_{op}) + \|\mathbf{B}_t^{\perp}\|_{op}(\|\mathbf{R}_t^{(2)}\|_{op} + \|\mathbf{R}_t^{(3)}\|_{op})\right] \left(\|\mathbf{B}_t^{\perp}\|_{op} + \eta_2\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{R}_t^{(2)}\|_{op}\right) \\
&+\eta_1\eta_2\delta \left[\|\mathbf{B}_t^{\mathbf{V}}\|_{op}(\|\mathbf{R}_t^{*}\|_{op} + \|\mathbf{R}_t^{(1)}\|_{op}) + \|\mathbf{B}_t^{\perp}\|_{op}(\|\mathbf{R}_t^{(2)}\|_{op} + \|\mathbf{R}_t^{(3)}\|_{op})\right] \left(\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op} + \|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right) \\
&+\eta_2\delta \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right) \left(\|\mathbf{R}_t^{(2)}\|_{op} + \|\mathbf{R}_t^{(3)}\|_{op}\right) \left(\|\mathbf{A}_t^{\perp}\|_{op} + \eta_1\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{R}_t^{(1)}\|_{op}\right) \\
&+\eta_1\eta_2\delta \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right) \left(\|\mathbf{R}_t^{(2)}\|_{op} + \|\mathbf{R}_t^{(3)}\|_{op}\right) \left(\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op} + \|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right) \\
&+\eta_1\eta_2\delta^2 \left[\|\mathbf{B}_t^{\mathbf{V}}\|_{op}(\|\mathbf{R}_t^{*}\|_{op} + \|\mathbf{R}_t^{(1)}\|_{op}) + \|\mathbf{B}_t^{\perp}\|_{op}(\|\mathbf{R}_t^{(2)}\|_{op} + \|\mathbf{R}_t^{(3)}\|_{op})\right] \\
&\cdot \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right) \left(\|\mathbf{R}_t^{(2)}\|_{op} + \|\mathbf{R}_t^{(3)}\|_{op}\right).
\end{aligned}
$$

Using the definition of $\mathcal{M}_t$, we have

$$
\begin{aligned}
&\|\mathbf{R}_{t+1}^{(3)}\|_{op} \leq \left(1 - \eta_1 \lambda_{\min}(\mathbf{B}_t^{\perp\top}\mathbf{B}_t^{\perp}) - \eta_2 \lambda_{\min}(\mathbf{A}_t^{\perp}\mathbf{A}_t^{\perp\top})\right) \|\mathbf{R}_t^{(3)}\|_{op} \\
&+\eta_1 \|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\mathcal{M}_t + \eta_2 \|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\mathcal{M}_t \\
&+\eta_1\eta_2 \left[\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\mathcal{M}_t + \left(\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op}\right] \\
&+\eta_1 \|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \eta_2 \|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} + \eta_1\eta_2 \|\mathbf{E}_{t,\mathbf{A}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \\
&+\eta_1\eta_2 \left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\mathcal{M}_t^2 + \|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}^2\right) \\
&+\eta_1\eta_2 \left(\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}\right)\mathcal{M}_t\|\mathbf{R}_t^{(3)}\|_{op} \\
&+\eta_1\delta \left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op}\right) \left(\|\mathbf{B}_t^{\perp}\|_{op} + \eta_2\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\mathcal{M}_t + \eta_2\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op} + \eta_2\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\mathcal{M}_t \\
&+\eta_1\delta\|\mathbf{B}_t^{\perp}\|_{op} \left(\|\mathbf{B}_t^{\perp}\|_{op} + \eta_2\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\mathcal{M}_t + \eta_2\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op} + \eta_2\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op} \\
&+\eta_2\delta \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right) \left(\|\mathbf{A}_t^{\perp}\|_{op} + \eta_1\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\mathcal{M}_t + \eta_1\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op} + \eta_1\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right)\mathcal{M}_t \\
&+\eta_2\delta \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right) \left(\|\mathbf{A}_t^{\perp}\|_{op} + \eta_1\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\mathcal{M}_t + \eta_1\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op} + \eta_1\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op} \\
&+\eta_1\eta_2\delta^2 \left[\left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op}\right)\mathcal{M}_t + \|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\right] \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right)(\mathcal{M}_t + \|\mathbf{R}_t^{(3)}\|_{op}).
\end{aligned}
$$

■

Appendix D. Convergence Analysis

This section turns the deterministic propagation bounds from Section C into the finite-horizon guarantees used in the main text. The proof has three layers. First, Lemma 7 bounds the low-precision residuals $\mathbf{E}_{t,\mathbf{A}}$ and $\mathbf{E}_{t,\mathbf{B}}$ in terms of the quantization levels and iterate norms. Second, Lemmas 8 and 9 provide the high-probability initialization and covariance control needed to start the induction. Third, Sections D.3.1 and D.3.2 prove separate finite-horizon inductions for $r = r^*$ and $r \geq r^*$, respectively, and convert the resulting operator-norm block bounds into Frobenius-norm error bounds.

D.1. Quantization Perturbation Bounds

Lemma 7 (Quantization-error norm bounds under operator norm) *Suppose for any $i \in \{d, A, B, o, w, l, r\}$, there exists a constant ϵ_i such that quantization $\mathcal{Q}_i$ satisfies Assumption 2.2 with ϵ_i, then we have*

$$\|\mathbf{E}_{t,\mathbf{A}}\|_{op} \leq \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N} \cdot (\|\mathbf{B}_t\|_{op} + \epsilon_B\|\mathbf{B}_t\|_F) \cdot \left(\epsilon_1\|\widetilde{\mathbf{Y}}\|_F + \epsilon_d\|\widetilde{\mathbf{X}}\|_F\|\mathbf{W}^*\|_{op} + \epsilon_2\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathbf{W}^\natural\|_F\right)$$
$$+\frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N} \cdot \left[\epsilon_3(\|\mathbf{B}_t\|_{op} + \epsilon_B\|\mathbf{B}_t\|_F)^2 \cdot \|\mathbf{A}_t\|_F + \epsilon_B\|\mathbf{B}_t\|_F\left(\|\Delta\|_{op} + \|\mathbf{A}_t\|_{op}\left(2\|\mathbf{B}_t\|_{op} + \epsilon_B\|\mathbf{B}_t\|_F\right)\right)\right],$$

and

$$\begin{aligned}
\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \leq & \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N}\|\Delta\|_{op}\|\mathbf{A}_t\|_F \cdot \epsilon_A \\
& + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N}\left[\epsilon_o(\|\mathbf{A}_t\|_{op} + \epsilon_A\|\mathbf{A}_t\|_F) + \epsilon_l(1+\epsilon_A)(1+\epsilon_o)\|\mathbf{A}_t\|_F\right]\|\widetilde{\mathbf{Y}}\|_F \\
& + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N}(\|\mathbf{A}_t\|_{op} + \epsilon_A\|\mathbf{A}_t\|_F)\epsilon_d\|\widetilde{\mathbf{X}}\|_F\|\mathbf{W}^*\|_{op} \\
& + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N}(\|\mathbf{A}_t\|_{op} + \epsilon_A\|\mathbf{A}_t\|_F)[\epsilon_w + (1+\epsilon_w)\epsilon_o]\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathbf{W}^\natural\|_F \\
& + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N}\|\mathbf{A}_t\|_F\epsilon_l(1+\epsilon_w)(1+\epsilon_o)(1+\epsilon_A)\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathbf{W}^\natural\|_F \\
& + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N}\|\mathbf{A}_t\|_F(\|\mathbf{A}_t\|_{op} + \epsilon_A\|\mathbf{A}_t\|_F)(\|\mathbf{B}_t\|_{op} + \epsilon_B\|\mathbf{B}_t\|_F) \cdot [\epsilon_l(1+\epsilon_A) + \epsilon_A + \epsilon_o(1+\epsilon_l)(1+\epsilon_A)] \\
& + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N}\|\mathbf{A}_t\|_{op}\|\mathbf{A}_t\|_F(\|\mathbf{B}_t\|_{op} + \epsilon_B\|\mathbf{B}_t\|_F) \cdot \epsilon_A \\
& + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N}\|\mathbf{A}_t\|_{op}^2\|\mathbf{B}_t\|_F \cdot \epsilon_B \\
& + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N}\|\mathbf{A}_t\|_F^2(\|\mathbf{B}_t\|_{op} + \epsilon_B\|\mathbf{B}_t\|_F) \cdot \epsilon_l(1+\epsilon_o)(1+\epsilon_A)^2(1+\epsilon_l).
\end{aligned}$$

where $\mathbf{E}_{t,\mathbf{A}}, \mathbf{E}_{t,\mathbf{B}}$ are defined in Lemma 5 and

$$\begin{aligned}
\epsilon_1 &= \epsilon_r(1+\epsilon_o) + \epsilon_o, \\
\epsilon_2 &= (1+\epsilon_w)[\epsilon_o + \epsilon_r(1+\epsilon_o)] + \epsilon_w, \\
\epsilon_3 &= [\epsilon_o + \epsilon_r(1+\epsilon_o)](1+\epsilon_l)(1+\epsilon_A) + \epsilon_l(1+\epsilon_A) + \epsilon_A.
\end{aligned}$$

Proof Recall that by (13) and (14),

$$\mathbf{E}_{t,\mathbf{A}} = \widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\Delta(\mathbf{E}_t^{(B)})^\top + \mathbf{E}_t^{(T)}(\mathbf{B}_t + \mathbf{E}_t^{(B)})^\top - \mathbf{E}_{t,\mathbf{A}}^{(H)},$$

$$\mathbf{E}_{t,\mathbf{B}} = (\mathbf{E}_t^{(A)})^\top\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\Delta + (\mathbf{A}_t + \mathbf{E}_t^{(A)})^\top\mathbf{E}_t^{(T)} - \mathbf{E}_{t,\mathbf{B}}^{(H)},$$

where $\mathbf{E}_t^{(T)} = -\frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathbf{E}^{(X)}\mathbf{W}^* - \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\left(\mathbf{E}_t^{(o)} + \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{E}^{(w)}\right)$, and

$$\begin{aligned}
\mathbf{E}_{t,\mathbf{A}}^{(H)} =& \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)^\top + \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathbf{E}_t^{(r)} \\
&-\frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{A}_t\mathbf{B}_t\mathbf{B}_t^\top, \\
\mathbf{E}_{t,\mathbf{B}}^{(H)} =& \frac{1}{N}\mathcal{Q}_A\left(\mathbf{A}_t\right)^\top\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top\mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right)\mathcal{Q}_B\left(\mathbf{B}_t\right) + \frac{1}{N}\mathbf{E}_t^{(l)^\top}\left(\mathbf{E}_t^{(o)} + \mathbf{Z}_t - \widetilde{\mathbf{Y}}\right) \\
&-\mathbf{A}_t^\top\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\mathbf{A}_t\mathbf{B}_t.
\end{aligned}$$

We then prove by the fact under Assumption 2.2: for any matrix $\mathbf{M}$ and quantizer $\mathcal{Q}_i$,

$$\|\mathcal{Q}_i(\mathbf{M}) - \mathbf{M}\|_{op} \le \epsilon_i\|\mathbf{M}\|_F, \quad \|\mathcal{Q}_i(\mathbf{M})\|_{op} \le \|\mathbf{M}\|_{op} + \epsilon_i\|\mathbf{M}\|_F. \tag{15}$$

Specifically, it holds

$$\begin{aligned}
\|\mathbf{E}_{t,\mathbf{A}}\|_{op} \le& \|\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\|_{op}\|\Delta\|_{op}\|\mathbf{E}_t^{(B)}\|_{op} + \|\mathbf{E}_t^{(T)}\|_{op}\|\mathbf{B}_t + \mathbf{E}_t^{(B)}\|_{op} + \|\mathbf{E}_{t,\mathbf{A}}^{(H)}\|_{op} \\
\le& \|\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\|_{op}\|\Delta\|_{op}\epsilon_B\|\mathbf{B}_t\|_F + \|\mathbf{E}_t^{(T)}\|_{op}(\|\mathbf{B}_t\|_{op} + \epsilon_B\|\mathbf{B}_t\|_F) + \|\mathbf{E}_{t,\mathbf{A}}^{(H)}\|_{op},
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \le& \|\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\|_{op}\|\Delta\|_{op}\|\mathbf{E}_t^{(A)}\|_{op} + \|\mathbf{E}_t^{(T)}\|_{op}\|\mathbf{A}_t + \mathbf{E}_t^{(A)}\|_{op} + \|\mathbf{E}_{t,\mathbf{B}}^{(H)}\|_{op} \\
\le& \|\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\|_{op}\|\Delta\|_{op}\epsilon_A\|\mathbf{A}_t\|_F + \|\mathbf{E}_t^{(T)}\|_{op}(\|\mathbf{A}_t\|_{op} + \epsilon_A\|\mathbf{A}_t\|_F) + \|\mathbf{E}_{t,\mathbf{B}}^{(H)}\|_{op}.
\end{aligned} \tag{17}$$

It remains to bound the operator norm of $\mathbf{E}_t^{(T)}$, $\mathbf{E}_{t,\mathbf{A}}^{(H)}$ and $\mathbf{E}_{t,\mathbf{B}}^{(H)}$. By definition,

$$\begin{aligned}
\|\mathbf{E}_t^{(T)}\|_{op} \le& \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N}\left(\|\mathbf{E}^{(X)}\|_{op}\|\mathbf{W}^*\|_{op} + \|\mathbf{E}_t^{(o)}\|_{op} + \|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathbf{E}^{(w)}\|_{op}\right) \\
\le& \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N}\left(\epsilon_d\|\widetilde{\mathbf{X}}\|_F\|\mathbf{W}^*\|_{op} + \|\mathbf{E}_t^{(o)}\|_{op} + \epsilon_w\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathbf{W}^\natural\|_F\right) \\
\le& \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N}\left(\epsilon_d\|\widetilde{\mathbf{X}}\|_F\|\mathbf{W}^*\|_{op} + \epsilon_o\left(\|\mathbf{Z}_t\|_F + \|\widetilde{\mathbf{Y}}\|_F\right) + \epsilon_w\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathbf{W}^\natural\|_F\right).
\end{aligned} \tag{18}$$

Moreover, it holds

$$\begin{aligned}
\|\mathbf{Z}_t\|_F =& \left\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_w(\mathbf{W}^\natural) + \mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)\right\|_F \\
\le& \left[\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathcal{Q}_w(\mathbf{W}^\natural)\|_F + \left\|\mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right)\right\|_F\|\mathcal{Q}_B\left(\mathbf{B}_t\right)\|_{op}\right] \\
\le& \|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\left[(1+\epsilon_w)\|\mathbf{W}^\natural\|_F + (1+\epsilon_l)(1+\epsilon_A)\|\mathbf{A}_t\|_F(\|\mathbf{B}_t\|_{op} + \epsilon_B\|\mathbf{B}_t\|_F)\right].
\end{aligned} \tag{19}$$

Regarding

$$\begin{aligned}\mathbf{E}_{t,\mathbf{A}}^{(H)} =& \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top \mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)^\top + \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top \mathbf{E}_t^{(r)} \\ & -\frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{A}_t\mathbf{B}_t\mathbf{B}_t^\top \\ =& \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top \mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)^\top - \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{A}_t\mathcal{Q}_B\left(\mathbf{B}_t\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)^\top \\ & +\frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{A}_t\mathcal{Q}_B\left(\mathbf{B}_t\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)^\top - \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{A}_t\mathbf{B}_t\mathbf{B}_t^\top + \frac{1}{N}\mathcal{Q}_d(\widetilde{\mathbf{X}})^\top \mathbf{E}_t^{(r)},\end{aligned}$$

we have

$$\begin{aligned}\|\mathbf{E}_{t,\mathbf{A}}^{(H)}\|_{op} \leq& \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N}\left\|\mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right) - \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{A}_t\right\|_{op}\|\mathcal{Q}_B\left(\mathbf{B}_t\right)\|_{op}^2 \\ & +\frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N}\left(\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathbf{A}_t\|_{op}\|\mathcal{Q}_B\left(\mathbf{B}_t\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)^\top - \mathbf{B}_t\mathbf{B}_t^\top\|_{op} + \|\mathbf{E}_t^{(r)}\|_{op}\right).\end{aligned} \tag{20}$$

Moreover, it holds

$$\begin{aligned}& \left\|\mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right) - \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{A}_t\right\|_{op} \\ \leq& \left\|\mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right) - \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right) + \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right) - \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{A}_t\right\|_{op} \\ \leq& \left\|\mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right) - \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right\|_{op} + \left\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right) - \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{A}_t\right\|_{op} \\ \leq& \epsilon_l\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathcal{Q}_A\left(\mathbf{A}_t\right)\|_F + \epsilon_A\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathbf{A}_t\|_F \\ \leq& \left(\epsilon_l(1+\epsilon_A)+\epsilon_A\right)\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathbf{A}_t\|_F,\end{aligned} \tag{21}$$

$$\begin{aligned}\|\mathcal{Q}_B\left(\mathbf{B}_t\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)^\top - \mathbf{B}_t\mathbf{B}_t^\top\|_{op} \leq& \|\mathcal{Q}_B\left(\mathbf{B}_t\right)\mathcal{Q}_B\left(\mathbf{B}_t\right)^\top - \mathcal{Q}_B\left(\mathbf{B}_t\right)\mathbf{B}_t^\top\|_{op} \\ & +\|\mathcal{Q}_B\left(\mathbf{B}_t\right)\mathbf{B}_t^\top - \mathbf{B}_t\mathbf{B}_t^\top\|_{op} \\ \leq& \epsilon_B\|\mathcal{Q}_B\left(\mathbf{B}_t\right)\|_{op}\|\mathbf{B}_t\|_F + \epsilon_B\|\mathbf{B}_t\|_{op}\|\mathbf{B}_t\|_F \\ \leq& \epsilon_B\|\mathbf{B}_t\|_F\left(2\|\mathbf{B}_t\|_{op} + \epsilon_B\|\mathbf{B}_t\|_F\right),\end{aligned} \tag{22}$$

and

$$\|\mathbf{E}_t^{(r)}\|_{op} \leq \epsilon_r\|\mathcal{Q}_o(\mathbf{Z}_t - \widetilde{\mathbf{Y}})\mathcal{Q}_B(\mathbf{B}_t)^\top\|_F \leq \epsilon_r(1+\epsilon_o)\left(\|\mathbf{Z}_t\|_F + \|\widetilde{\mathbf{Y}}\|_F\right)(\|\mathbf{B}_t\|_{op} + \epsilon_B\|\mathbf{B}_t\|_F). \tag{23}$$

Therefore, together with (16), (18), (19), (20), (21), (22) and (23), we have

$$\begin{aligned}\|\mathbf{E}_{t,\mathbf{A}}\|_{op} \leq& \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N}\cdot\left(\|\mathbf{B}_t\|_{op} + \epsilon_B\|\mathbf{B}_t\|_F\right)\cdot\left(\epsilon_1\|\widetilde{\mathbf{Y}}\|_F + \epsilon_d\|\widetilde{\mathbf{X}}\|_F\|\mathbf{W}^*\|_{op} + \epsilon_2\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathbf{W}^\natural\|_F\right) \\ &+\frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N}\cdot\left[\epsilon_3(\|\mathbf{B}_t\|_{op} + \epsilon_B\|\mathbf{B}_t\|_F)^2\cdot\|\mathbf{A}_t\|_F + \epsilon_B\|\mathbf{B}_t\|_F\left(\|\Delta\|_{op} + \|\mathbf{A}_t\|_{op}\left(2\|\mathbf{B}_t\|_{op} + \epsilon_B\|\mathbf{B}_t\|_F\right)\right)\right],\end{aligned}$$

where

$$\begin{aligned}\epsilon_1 &= \epsilon_r(1+\epsilon_o) + \epsilon_o, \\ \epsilon_2 &= (1+\epsilon_w)[\epsilon_o + \epsilon_r(1+\epsilon_o)] + \epsilon_w, \\ \epsilon_3 &= [\epsilon_o + \epsilon_r(1+\epsilon_o)](1+\epsilon_l)(1+\epsilon_A) + \epsilon_l(1+\epsilon_A) + \epsilon_A.\end{aligned}$$

Regarding $\mathbf{E}_{t,\mathbf{B}}^{(H)}$, similarly, write

$$\begin{aligned}\mathbf{E}_{t,\mathbf{B}}^{(H)} =& \frac{1}{N}\mathcal{Q}_A\left(\mathbf{A}_t\right)^\top \mathcal{Q}_d(\widetilde{\mathbf{X}})^\top \mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right)\mathcal{Q}_B\left(\mathbf{B}_t\right) - \mathcal{Q}_A\left(\mathbf{A}_t\right)^\top \widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\mathbf{A}_t\mathcal{Q}_B\left(\mathbf{B}_t\right)\\ &+\mathcal{Q}_A\left(\mathbf{A}_t\right)^\top \widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\mathbf{A}_t\mathcal{Q}_B\left(\mathbf{B}_t\right) - \mathbf{A}_t^\top\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\mathbf{A}_t\mathcal{Q}_B\left(\mathbf{B}_t\right)\\ &+\mathbf{A}_t^\top\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\mathbf{A}_t\mathcal{Q}_B\left(\mathbf{B}_t\right) - \mathbf{A}_t^\top\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\mathbf{A}_t\mathbf{B}_t\\ &+\frac{1}{N}\mathbf{E}_t^{(l)\top}\left(\mathbf{E}_t^{(o)} + \mathbf{Z}_t - \widetilde{\mathbf{Y}}\right),\end{aligned}$$

then we have

$$\begin{aligned}\|\mathbf{E}_{t,\mathbf{B}}^{(H)}\|_{op} \leq& \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N}\|\mathcal{Q}_A(\mathbf{A}_t)\|_{op}\|\mathcal{Q}_B(\mathbf{B}_t)\|_{op}\left\|\mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right) - \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{A}_t\right\|_{op}\\ &+\|\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\|_{op}\|\mathbf{A}_t\|_{op}\|\mathcal{Q}_B\left(\mathbf{B}_t\right)\|_{op}\|\mathcal{Q}_A\left(\mathbf{A}_t\right) - \mathbf{A}_t\|_{op}\\ &+\|\mathbf{A}_t\|_{op}^2\|\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}\|_{op}\|\mathcal{Q}_B\left(\mathbf{B}_t\right) - \mathbf{B}_t\|_{op}\\ &+\frac{\|\mathbf{E}_t^{(l)}\|_{op}}{N}(1+\epsilon_o)\left(\|\mathbf{Z}_t\|_F + \|\widetilde{\mathbf{Y}}\|_F\right).\end{aligned} \tag{24}$$

Moreover, it holds

$$\begin{aligned}\left\|\mathcal{Q}_l\left(\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right) - \mathcal{Q}_d(\widetilde{\mathbf{X}})\mathbf{A}_t\right\|_{op} \leq& \epsilon_l\left\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\mathcal{Q}_A\left(\mathbf{A}_t\right)\right\|_F + \epsilon_A\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathbf{A}_t\|_F\\ \leq& [\epsilon_l(1+\epsilon_A) + \epsilon_A]\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathbf{A}_t\|_F,\end{aligned} \tag{25}$$

and

$$\|\mathbf{E}_t^{(l)}\|_{op} \leq \epsilon_l(1+\epsilon_A)\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathbf{A}_t\|_F. \tag{26}$$

Together with (17), (18), (19), (24), (25) and (26), we have

$$\begin{aligned}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \leq& \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N}\|\Delta\|_{op}\|\mathbf{A}_t\|_F\cdot\epsilon_A\\ &+\frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N}(\|\mathbf{A}_t\|_{op} + \epsilon_A\|\mathbf{A}_t\|_F)\left(\epsilon_d\|\widetilde{\mathbf{X}}\|_F\|\mathbf{W}^*\|_{op} + \epsilon_o\left(\|\mathbf{Z}_t\|_F + \|\widetilde{\mathbf{Y}}\|_F\right) + \epsilon_w\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathbf{W}^\natural\|_F\right)\\ &+\frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N}\|\mathbf{A}_t\|_F(\|\mathbf{A}_t\|_{op} + \epsilon_A\|\mathbf{A}_t\|_F)(\|\mathbf{B}_t\|_{op} + \epsilon_B\|\mathbf{B}_t\|_F)[\epsilon_l(1+\epsilon_A) + \epsilon_A]\\ &+\frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N}\|\mathbf{A}_t\|_{op}\|\mathbf{A}_t\|_F(\|\mathbf{B}_t\|_{op} + \epsilon_B\|\mathbf{B}_t\|_F)\cdot\epsilon_A\\ &+\frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N}\|\mathbf{A}_t\|_{op}^2\|\mathbf{B}_t\|_F\cdot\epsilon_B\\ &+\frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N}\|\mathbf{A}_t\|_F(1+\epsilon_o)\left(\|\mathbf{Z}_t\|_F + \|\widetilde{\mathbf{Y}}\|_F\right)\cdot\epsilon_l(1+\epsilon_A).\end{aligned}$$

Merging terms and using

$$\|\mathbf{Z}_t\|_F \leq \|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\left[(1+\epsilon_w)\|\mathbf{W}^\natural\|_F + (1+\epsilon_l)(1+\epsilon_A)\|\mathbf{A}_t\|_F(\|\mathbf{B}_t\|_{op} + \epsilon_B\|\mathbf{B}_t\|_F)\right],$$

we have

$$
\begin{aligned}
\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \leq & \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N}\|\Delta\|_{op}\|\mathbf{A}_t\|_F \cdot \epsilon_A \\
& + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N}\left[\epsilon_o(\|\mathbf{A}_t\|_{op} + \epsilon_A\|\mathbf{A}_t\|_F) + \epsilon_l(1+\epsilon_A)(1+\epsilon_o)\|\mathbf{A}_t\|_F\right]\|\widetilde{\mathbf{Y}}\|_F \\
& + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N}(\|\mathbf{A}_t\|_{op} + \epsilon_A\|\mathbf{A}_t\|_F)\epsilon_d\|\widetilde{\mathbf{X}}\|_F\|\mathbf{W}^*\|_{op} \\
& + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N}(\|\mathbf{A}_t\|_{op} + \epsilon_A\|\mathbf{A}_t\|_F)[\epsilon_w + (1+\epsilon_w)\epsilon_o]\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathbf{W}^\natural\|_F \\
& + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N}\|\mathbf{A}_t\|_F\epsilon_l(1+\epsilon_w)(1+\epsilon_o)(1+\epsilon_A)\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathbf{W}^\natural\|_F \\
& + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N}\|\mathbf{A}_t\|_F(\|\mathbf{A}_t\|_{op} + \epsilon_A\|\mathbf{A}_t\|_F)(\|\mathbf{B}_t\|_{op} + \epsilon_B\|\mathbf{B}_t\|_F) \cdot [\epsilon_l(1+\epsilon_A) + \epsilon_A + \epsilon_o(1+\epsilon_l)(1+\epsilon_A)] \\
& + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N}\|\mathbf{A}_t\|_{op}\|\mathbf{A}_t\|_F(\|\mathbf{B}_t\|_{op} + \epsilon_B\|\mathbf{B}_t\|_F) \cdot \epsilon_A \\
& + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N}\|\mathbf{A}_t\|_{op}^2\|\mathbf{B}_t\|_F \cdot \epsilon_B \\
& + \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N}\|\mathbf{A}_t\|_F^2(\|\mathbf{B}_t\|_{op} + \epsilon_B\|\mathbf{B}_t\|_F) \cdot \epsilon_l(1+\epsilon_o)(1+\epsilon_A)^2(1+\epsilon_l).
\end{aligned}
$$

■

D.2. Initialization and Covariance Control

Lemma 8 (Lemma C.11 in [38]) *Consider* (spectral initialization)*, under Assumption 2.1, let* $\kappa = \frac{\lambda_1(\Delta)}{\lambda_{r^*}(\Delta)} := \frac{\lambda_1^*}{\lambda_{r^*}^*}$*, then with probability at least* $1 - 2C\exp(-\epsilon^2 N)$ *for a universal constant* $C > 0$*, we have* $\left\|\frac{1}{N}\widetilde{\mathbf{X}}^\top\widetilde{\mathbf{X}} - \mathbf{I}\right\|_{op} \leq \epsilon$ *and then*

$$
\|\mathbf{A}_0\mathbf{B}_0 - \Delta\|_{op} \leq \left\|\frac{1}{N}\widetilde{\mathbf{X}}^\top\widetilde{\mathbf{X}} - \mathbf{I}\right\|_{op}\|\Delta\|_{op} \leq \epsilon\|\Delta\|_{op}.
$$

Moreover, if Assumption 2.2 holds for $\mathcal{Q}_d$ *with parameter* ϵ_d*, then on the same event,*

$$
\delta = \left\|\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right\|_{op} \leq \epsilon + \frac{2\epsilon_d\|\widetilde{\mathbf{X}}\|_{op}\|\widetilde{\mathbf{X}}\|_F + \epsilon_d^2\|\widetilde{\mathbf{X}}\|_F^2}{N} \leq \epsilon + (1+\epsilon)\left(2\epsilon_d\sqrt{d} + \epsilon_d^2 d\right) := \delta_q.
$$

Proof The first claim is Lemma C.11 in [38]. For the quantized covariance bound, write

$$
\mathcal{Q}_d(\widetilde{\mathbf{X}}) = \widetilde{\mathbf{X}} + \mathbf{E}^{(X)}.
$$

By Assumption 2.2,

$$
\|\mathbf{E}^{(X)}\|_{op} \leq \|\mathbf{E}^{(X)}\|_F \leq \epsilon_d\|\widetilde{\mathbf{X}}\|_F.
$$

Expanding $\widehat{\mathbf{\Sigma}}_{\mathcal{Q}}$ gives

$$\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d = \left(\frac{1}{N}\widetilde{\mathbf{X}}^\top\widetilde{\mathbf{X}} - \mathbf{I}_d\right) + \frac{1}{N}\left(\widetilde{\mathbf{X}}^\top\mathbf{E}^{(X)} + \mathbf{E}^{(X)\top}\widetilde{\mathbf{X}} + \mathbf{E}^{(X)\top}\mathbf{E}^{(X)}\right).$$

Therefore, on the event $\left\|\frac{1}{N}\widetilde{\mathbf{X}}^\top\widetilde{\mathbf{X}} - \mathbf{I}_d\right\|_{op} \le \epsilon$,

$$\delta \le \epsilon + \frac{2\|\widetilde{\mathbf{X}}\|_{op}\|\mathbf{E}^{(X)}\|_{op} + \|\mathbf{E}^{(X)}\|_{op}^2}{N} \le \epsilon + \frac{2\epsilon_d\|\widetilde{\mathbf{X}}\|_{op}\|\widetilde{\mathbf{X}}\|_F + \epsilon_d^2\|\widetilde{\mathbf{X}}\|_F^2}{N}.$$

Since $\|\widetilde{\mathbf{X}}\|_{op}^2/N \le 1+\epsilon$ and $\|\widetilde{\mathbf{X}}\|_F \le \sqrt{d}\|\widetilde{\mathbf{X}}\|_{op}$ on the same event, the simplified bound follows. ■

D.3. Finite-Horizon Induction Arguments

For simplicity, we consider $\eta_1 = \eta_2 = \eta$. The finite-horizon analysis below follows the same three-step pattern in both rank regimes. First, an induction-ready propagation corollary packages Lemma 6 into a compact recursion with explicit envelopes. Second, a finite-horizon induction theorem proves that the iterates stay in a bounded tube, the signal singular values remain bounded away from zero, and the decomposed errors contract up to quantization-dependent envelopes. Third, a Frobenius-error corollary converts the operator-norm block bounds into the approximation error $\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F$, followed by a simplified corollary that yields the corresponding main-text theorem under the constant-level assumptions.

Lemma 9 (Induction base bounds for full-precision spectral initialization) *Consider spectral initialization* (spectral initialization)*. Let* $\widehat{\mathbf{\Sigma}} = \frac{1}{N}\widetilde{\mathbf{X}}^\top\widetilde{\mathbf{X}}$ *and write* $\lambda_i^* = \lambda_i(\Delta)$*. Under Assumption 2.1, for* $\epsilon \in (0,1)$*, with probability at least* $1 - 2C\exp(-\epsilon^2 N)$ *for a universal constant* $C > 0$*, the following bounds hold:*

$$\begin{aligned}
\|\mathbf{A}_0\mathbf{B}_0 - \Delta\|_{op} &\le \epsilon\lambda_1^*,\\
\|\mathbf{A}_0\|_{op} = \|\mathbf{B}_0\|_{op} &\le \sqrt{(1+\epsilon)\lambda_1^*},\\
\|\mathbf{A}_0^\perp\|_{op} &\le \frac{\epsilon\lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}}, \qquad \|\mathbf{B}_0^\perp\|_{op} = 0,\\
\|\mathbf{A}_0^{\mathbf{U}}\|_{op},\ \|\mathbf{B}_0^{\mathbf{V}}\|_{op} &\le \sqrt{(1+\epsilon)\lambda_1^*},\\
\lambda_{r^*}(\mathbf{A}_0^{\mathbf{U}}) &\ge \sqrt{(1-\epsilon)\lambda_{r^*}^*} - \frac{\epsilon\lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}},\\
\lambda_{r^*}(\mathbf{B}_0^{\mathbf{V}}) &\ge \sqrt{(1-\epsilon)\lambda_{r^*}^*},\\
\|\mathbf{R}_0^*\|_{op} &\le \epsilon\lambda_1^*, \qquad \|\mathbf{R}_0^{(1)}\|_{op} \le \epsilon\lambda_1^*,\\
\|\mathbf{R}_0^{(2)}\|_{op} &= 0, \qquad \|\mathbf{R}_0^{(3)}\|_{op} = 0.
\end{aligned}$$

Proof Since $\widetilde{\mathbf{Y}} = \widetilde{\mathbf{X}}\mathbf{W}^*$, the full-precision first gradient satisfies $\mathbf{G}_f^\natural = \widehat{\mathbf{\Sigma}}\Delta$. On the event $\|\widehat{\mathbf{\Sigma}} - \mathbf{I}_d\|_{op} \le \epsilon$, all eigenvalues of $\widehat{\mathbf{\Sigma}}$ lie in $[1-\epsilon, 1+\epsilon]$. Hence, by singular-value inequalities applied to $\widehat{\mathbf{\Sigma}}\Delta$,

$$(1-\epsilon)\lambda_i^* \le \lambda_i(\mathbf{G}_f^\natural) \le (1+\epsilon)\lambda_i^*, \qquad i \le r^*.$$

Since $\mathrm{rank}(\mathbf{G}_f^\natural) \le r^* \le r$, the top-$r$ spectral factorization reconstructs $\mathbf{G}_f^\natural$ exactly, so $\mathbf{A}_0\mathbf{B}_0 = \mathbf{G}_f^\natural$. This gives

$$\|\mathbf{A}_0\mathbf{B}_0 - \Delta\|_{op} = \|(\widehat{\mathbf{\Sigma}} - \mathbf{I}_d)\Delta\|_{op} \le \epsilon\lambda_1^*,$$

and $\|\mathbf{A}_0\|_{op} = \|\mathbf{B}_0\|_{op} = \sqrt{\lambda_1(\mathbf{G}_f^\natural)} \le \sqrt{(1+\epsilon)\lambda_1^*}$.

Next, $\mathbf{G}_f^\natural\mathbf{V}_\perp = \widehat{\mathbf{\Sigma}}\Delta\mathbf{V}_\perp = \mathbf{0}$. Therefore the nonzero right singular vectors of $\mathbf{G}_f^\natural$ lie in the span of $\mathbf{V}$, and the zero singular directions do not contribute to $\mathbf{B}_0$; hence $\mathbf{B}_0^\perp = \mathbf{B}_0\mathbf{V}_\perp = \mathbf{0}$. If $\mathbf{V}_{r^*}$ and $\mathbf{S}_{r^*}$ denote the nonzero right singular vectors and singular values of $\mathbf{G}_f^\natural$, then

$$\mathbf{A}_0^\perp = \mathbf{U}_\perp^\top\mathbf{G}_f^\natural\mathbf{V}_{r^*}\mathbf{S}_{r^*}^{-1/2} = \mathbf{U}_\perp^\top(\widehat{\mathbf{\Sigma}} - \mathbf{I}_d)\Delta\mathbf{V}_{r^*}\mathbf{S}_{r^*}^{-1/2},$$

where we used $\mathbf{U}_\perp^\top\Delta = \mathbf{0}$. Thus

$$\|\mathbf{A}_0^\perp\|_{op} \le \frac{\epsilon\lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}}.$$

The upper bounds for $\mathbf{A}_0^\mathbf{U}$ and $\mathbf{B}_0^\mathbf{V}$ follow from the bounds on $\|\mathbf{A}_0\|_{op}$ and $\|\mathbf{B}_0\|_{op}$. Since $\mathbf{B}_0^\perp = \mathbf{0}$, the nonzero singular values of $\mathbf{B}_0^\mathbf{V}$ equal those of $\mathbf{B}_0$, and therefore $\lambda_{r^*}(\mathbf{B}_0^\mathbf{V}) \ge \sqrt{(1-\epsilon)\lambda_{r^*}^*}$. For $\mathbf{A}_0^\mathbf{U}$, Weyl's singular-value perturbation inequality gives

$$\lambda_{r^*}(\mathbf{A}_0^\mathbf{U}) \ge \lambda_{r^*}(\mathbf{A}_0) - \|\mathbf{A}_0^\perp\|_{op} \ge \sqrt{(1-\epsilon)\lambda_{r^*}^*} - \frac{\epsilon\lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}}.$$

Finally, because $\mathbf{A}_0\mathbf{B}_0 = \mathbf{G}_f^\natural$,

$$\mathbf{R}_0^* = \mathbf{U}^\top(\mathbf{G}_f^\natural - \Delta)\mathbf{V}, \qquad \mathbf{R}_0^{(1)} = \mathbf{U}_\perp^\top(\mathbf{G}_f^\natural - \Delta)\mathbf{V},$$
$$\mathbf{R}_0^{(2)} = \mathbf{U}^\top\mathbf{G}_f^\natural\mathbf{V}_\perp, \qquad \mathbf{R}_0^{(3)} = \mathbf{U}_\perp^\top\mathbf{G}_f^\natural\mathbf{V}_\perp.$$

The first two are bounded by $\epsilon\lambda_1^*$, and the last two are zero by $\mathbf{G}_f^\natural\mathbf{V}_\perp = \mathbf{0}$. The final high-probability statement is exactly the covariance concentration event from Lemma 8. ■

D.3.1. EXACT RANK

The exact-rank proof uses the fact that $\mathbf{B}_t^\mathbf{V}$ is square and remains well-conditioned. This allows $\mathbf{A}_t^\perp$ to be controlled through $\mathbf{R}_t^{(1)} = \mathbf{A}_t^\perp\mathbf{B}_t^\mathbf{V}$, so the non-signal drift remains at the static quantization scale. The proof proceeds through an induction-ready propagation corollary, a finite-horizon induction theorem, and two corollaries converting the induction to the simplified main-text bound.

Corollary 10 (Exact-rank induction-ready propagation with separated envelopes) *Consider $r = r^*$. Assume the hypotheses of Lemma 6 with $\eta_1 = \eta_2 = \eta$, and define*

$$P_t = \max\{1, \|\mathbf{A}_t\|_{op}, \|\mathbf{B}_t\|_{op}\}, \quad q_t = \max\{\|\mathbf{E}_{t,\mathbf{A}}\|_{op}, \|\mathbf{E}_{t,\mathbf{B}}\|_{op}\},$$

$$e_t = \mathcal{M}_t, \quad s_t = \min\left\{\lambda_{r^*}^2(\mathbf{A}_t^\mathbf{U}), \lambda_{r^*}^2(\mathbf{B}_t^\mathbf{V})\right\}, \quad \delta = \left\|\widehat{\mathbf{\Sigma}}_\mathcal{Q} - \mathbf{I}_d\right\|_{op}.$$

If $r = r^*$, and if nonnegative numbers $\alpha_t, \beta_t, \zeta_t$ satisfy

$$\|\mathbf{A}_t^{\perp}\|_{op} \le \alpha_t, \qquad \|\mathbf{B}_t^{\perp}\|_{op} \le \beta_t, \qquad \|\mathbf{R}_t^{(3)}\|_{op} \le \zeta_t,$$

then, with

$$K_t^{\mathrm{sep}} = e_t + \alpha_t + \beta_t + \zeta_t,$$

define

$$\begin{aligned}\widetilde{\Phi}_t = {} & C\eta P_t(\alpha_t + \beta_t)(e_t + \zeta_t) + C\eta^2 P_t q_t K_t^{\mathrm{sep}} + C\eta P_t q_t + C\eta^2 q_t^2 \\ & + C\eta^2 P_t^2 (K_t^{\mathrm{sep}})^2 + C\eta\delta P_t \left(P_t + \eta P_t K_t^{\mathrm{sep}} + \eta q_t\right) K_t^{\mathrm{sep}} + C\eta^2\delta^2 P_t^2 (K_t^{\mathrm{sep}})^2,\end{aligned}$$

for a universal constant $C > 0$, it holds

$$\begin{aligned}\mathcal{M}_{t+1} &\le (1 - \eta s_t) e_t + \widetilde{\Phi}_t, \\ \|\mathbf{A}_{t+1}^{\perp}\|_{op} &\le (1 - \eta s_t)\alpha_t + \eta q_t + C\eta\delta \left[P_t(e_t + P_t\alpha_t) + (P_t\beta_t + \zeta_t)\beta_t\right], \\ \|\mathbf{B}_{t+1}^{\perp}\|_{op} &\le (1 - \eta s_t)\beta_t + \eta q_t + C\eta\delta P_t(P_t\beta_t + \zeta_t).\end{aligned}$$

Moreover,

$$\|\mathbf{R}_{t+1}^{(3)}\|_{op} = \|\mathbf{A}_{t+1}^{\perp}\mathbf{B}_{t+1}^{\perp}\|_{op} \le \|\mathbf{A}_{t+1}^{\perp}\|_{op}\,\|\mathbf{B}_{t+1}^{\perp}\|_{op}.$$

Proof Apply Lemma 6 with $\eta_1 = \eta_2 = \eta$. Throughout the proof we use the envelope facts

$$\|\mathbf{A}_t^{\mathbf{U}}\|_{op}, \|\mathbf{A}_t^{\perp}\|_{op}, \|\mathbf{B}_t^{\mathbf{V}}\|_{op}, \|\mathbf{B}_t^{\perp}\|_{op} \le P_t,$$

$$\|\mathbf{R}_t^{*}\|_{op}, \|\mathbf{R}_t^{(1)}\|_{op}, \|\mathbf{R}_t^{(2)}\|_{op} \le e_t, \quad \|\mathbf{A}_t^{\perp}\|_{op} \le \alpha_t, \quad \|\mathbf{B}_t^{\perp}\|_{op} \le \beta_t, \quad \|\mathbf{R}_t^{(3)}\|_{op} \le \zeta_t,$$

and $\|\mathbf{E}_{t,\mathbf{A}}\|_{op}, \|\mathbf{E}_{t,\mathbf{B}}\|_{op} \le q_t$. The leading coefficients in the first three residual recursions are

$$\begin{aligned}&1 - \eta\lambda_{r^*}^2(\mathbf{B}_t^{\mathbf{V}}) - \eta\lambda_{r^*}^2(\mathbf{A}_t^{\mathbf{U}}), \\ &1 - \eta\lambda_{r^*}^2(\mathbf{B}_t^{\mathbf{V}}) - \eta\lambda_{\min}(\mathbf{A}_t^{\perp}{\mathbf{A}_t^{\perp}}^{\top}), \\ &1 - \eta\lambda_{\min}({\mathbf{B}_t^{\perp}}^{\top}\mathbf{B}_t^{\perp}) - \eta\lambda_{r^*}^2(\mathbf{A}_t^{\mathbf{U}}).\end{aligned}$$

By the step-size hypothesis these coefficients are nonnegative, and each is at most $1 - \eta s_t$ after dropping the nonnegative term not included in s_t.

The deterministic first-order non-signal groups from Lemma 6 are

$$\begin{aligned}&\eta\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{B}_t^{\mathbf{V}}\|_{op}e_t + \eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}e_t, \\ &\eta\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op} + \eta\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}e_t, \\ &\eta\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{B}_t^{\mathbf{V}}\|_{op}e_t + \eta\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}.\end{aligned}$$

They are bounded by

$$C\eta P_t(\alpha_t + \beta_t)(e_t + \zeta_t).$$

The one-quantization and pure-quantization groups are

$$\begin{aligned}
&\eta^2\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)e_t\\
&\quad+\eta\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}+\eta^2\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op},\\
&\eta^2\left[\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right)e_t+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\right]\\
&\quad+\eta\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\eta\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}+\eta^2\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op},\\
&\eta^2\left[\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)e_t+\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\right]\\
&\quad+\eta\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}+\eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}+\eta^2\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}.
\end{aligned}$$

Using $\|\mathbf{A}_t^{\perp}\|_{op}, \|\mathbf{B}_t^{\perp}\|_{op} \le P_t$ in the direct ηq_t terms, they are bounded by

$$C\eta^2 P_t q_t K_t^{\mathrm{sep}} + C\eta P_t q_t + C\eta^2 q_t^2,$$

since $e_t + \zeta_t \le K_t^{\mathrm{sep}}$. The quadratic residual groups are

$$\begin{aligned}
&\eta^2\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{B}_t^{\perp}\|_{op}\right)e_t^2,\\
&\eta^2\left[\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}\right)e_t+\left(\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op}\right]e_t,\\
&\eta^2\left[\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\right)e_t+\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op}\right]e_t.
\end{aligned}$$

Each displayed product has two residual or separated non-signal factors and at most two parameter factors, so they are bounded by

$$C\eta^2 P_t^2 (K_t^{\mathrm{sep}})^2,$$

where we used $e_t + \zeta_t \le K_t^{\mathrm{sep}}$. The covariance-linear groups in the first three recursions are

$$\begin{aligned}
&\eta\delta\left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\right)\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op}e_t+\eta\|\mathbf{A}_t^{\perp}\|_{op}e_t+\eta\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)e_t\\
&\quad+\eta\delta\|\mathbf{B}_t^{\perp}\|_{op}\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op}e_t+\eta\|\mathbf{A}_t^{\perp}\|_{op}e_t+\eta\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op}\\
&\quad+2\eta\delta\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\eta\|\mathbf{B}_t^{\mathbf{V}}\|_{op}e_t+\eta\|\mathbf{B}_t^{\perp}\|_{op}e_t+\eta\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right)e_t,\\
&\eta\delta\left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\right)\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op}e_t+\eta\|\mathbf{A}_t^{\perp}\|_{op}e_t+\eta\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)e_t\\
&\quad+\eta\delta\|\mathbf{B}_t^{\perp}\|_{op}\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op}e_t+\eta\|\mathbf{A}_t^{\perp}\|_{op}e_t+\eta\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op}\\
&\quad+2\eta\delta\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\|\mathbf{A}_t^{\perp}\|_{op}+\eta\|\mathbf{B}_t^{\mathbf{V}}\|_{op}e_t+\eta\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}+\eta\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right)e_t,\\
&\eta\delta\left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\right)\left(\|\mathbf{B}_t^{\perp}\|_{op}+\eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op}e_t+\eta\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}+\eta\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)e_t\\
&\quad+\eta\delta\|\mathbf{B}_t^{\perp}\|_{op}\left(\|\mathbf{B}_t^{\perp}\|_{op}+\eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op}e_t+\eta\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}+\eta\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)\|\mathbf{R}_t^{(3)}\|_{op}\\
&\quad+\eta\delta\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\eta\|\mathbf{B}_t^{\mathbf{V}}\|_{op}e_t+\eta\|\mathbf{B}_t^{\perp}\|_{op}e_t+\eta\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right)\left(e_t+\|\mathbf{R}_t^{(3)}\|_{op}\right).
\end{aligned}$$

They are bounded by

$$C\eta\delta P_t\left(P_t+\eta P_t K_t^{\mathrm{sep}}+\eta q_t\right)K_t^{\mathrm{sep}}.$$

The covariance-quadratic groups have the common form

$$\eta^2\delta^2\left[\left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\|\mathbf{B}_t^{\perp}\|_{op}\right)e_t+\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\right]\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}+\|\mathbf{A}_t^{\perp}\|_{op}\right)\left(e_t+\|\mathbf{R}_t^{(3)}\|_{op}\right),$$

with the last factor equal to e_t in the first two recursions. Hence, they are bounded by

$$C\eta^2\delta^2P_t^2K_t^{\mathrm{sep}}(e_t+\zeta_t)\le C\eta^2\delta^2P_t^2(K_t^{\mathrm{sep}})^2.$$

Combining the leading contraction and the five perturbation groups gives

$$\begin{aligned}\mathcal{M}_{t+1}\le{}&(1-\eta s_t)e_t+C\eta P_t(\alpha_t+\beta_t)(e_t+\zeta_t)+C\eta^2P_tq_tK_t^{\mathrm{sep}}+C\eta P_tq_t+C\eta^2q_t^2\\&+C\eta^2P_t^2(K_t^{\mathrm{sep}})^2+C\eta\delta P_t\left(P_t+\eta P_tK_t^{\mathrm{sep}}+\eta q_t\right)K_t^{\mathrm{sep}}+C\eta^2\delta^2P_t^2(K_t^{\mathrm{sep}})^2\\={}&(1-\eta s_t)e_t+\widetilde{\Phi}_t.\end{aligned}$$

It remains to prove the sharpened perpendicular recursions. For $\mathbf{A}_{t+1}^{\perp}$, the deterministic full-precision part of the $\mathbf{A}_{t+1}^{\perp}$ update equals

$$\mathbf{A}_t^{\perp}\left[\mathbf{I}_r-\eta\mathbf{B}_t^{\mathbf{V}}(\mathbf{B}_t^{\mathbf{V}})^\top-\eta\mathbf{B}_t^{\perp}(\mathbf{B}_t^{\perp})^\top\right].$$

Since $r=r^*$, $\mathbf{B}_t^{\mathbf{V}}$ is square. The step-size condition in Lemma 6 implies

$$0\preceq\mathbf{I}_r-\eta\mathbf{B}_t^{\mathbf{V}}(\mathbf{B}_t^{\mathbf{V}})^\top-\eta\mathbf{B}_t^{\perp}(\mathbf{B}_t^{\perp})^\top\preceq\mathbf{I}_r.$$

Therefore,

$$\begin{aligned}&\left\|\mathbf{I}_r-\eta\mathbf{B}_t^{\mathbf{V}}(\mathbf{B}_t^{\mathbf{V}})^\top-\eta\mathbf{B}_t^{\perp}(\mathbf{B}_t^{\perp})^\top\right\|_{op}\\=&1-\eta\lambda_{\min}\left(\mathbf{B}_t^{\mathbf{V}}(\mathbf{B}_t^{\mathbf{V}})^\top+\mathbf{B}_t^{\perp}(\mathbf{B}_t^{\perp})^\top\right)\le1-\eta\lambda_{r^*}^2(\mathbf{B}_t^{\mathbf{V}}).\end{aligned}$$

The remaining terms in the $\mathbf{A}_{t+1}^{\perp}$ update are the quantization term and the covariance term:

$$\|\mathbf{U}_\perp^\top\mathbf{E}_{t,\mathbf{A}}\|_{op}\le q_t$$

and

$$\begin{aligned}&\left\|(\mathbf{U}\mathbf{R}_t^*+\mathbf{U}_\perp\mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top+(\mathbf{U}\mathbf{R}_t^{(2)}+\mathbf{U}_\perp\mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right\|_{op}\\\le&\left(\|\mathbf{R}_t^*\|_{op}+\|\mathbf{R}_t^{(1)}\|_{op}\right)\|\mathbf{B}_t^{\mathbf{V}}\|_{op}+\left(\|\mathbf{R}_t^{(2)}\|_{op}+\|\mathbf{R}_t^{(3)}\|_{op}\right)\|\mathbf{B}_t^{\perp}\|_{op}\\\le&C\left[P_t(e_t+P_t\alpha_t)+(P_t\beta_t+\zeta_t)\beta_t\right],\end{aligned}$$

where the last line uses $\mathbf{R}_t^{(1)}=\mathbf{A}_t^{\perp}\mathbf{B}_t^{\mathbf{V}}$ and $\mathbf{R}_t^{(2)}=\mathbf{A}_t^{\mathbf{U}}\mathbf{B}_t^{\perp}$. Combining these bounds gives

$$\|\mathbf{A}_{t+1}^{\perp}\|_{op}\le\left(1-\eta\lambda_{r^*}^2(\mathbf{B}_t^{\mathbf{V}})\right)\|\mathbf{A}_t^{\perp}\|_{op}+\eta q_t+C\eta\delta\left[P_t(e_t+P_t\alpha_t)+(P_t\beta_t+\zeta_t)\beta_t\right].$$

Since $s_t\le\lambda_{r^*}^2(\mathbf{B}_t^{\mathbf{V}})$, this implies the displayed bound with $1-\eta s_t$.

For $\mathbf{B}_{t+1}^{\perp}$, the deterministic full-precision part is

$$\left[\mathbf{I}_r-\eta(\mathbf{A}_t^{\mathbf{U}})^\top\mathbf{A}_t^{\mathbf{U}}-\eta(\mathbf{A}_t^{\perp})^\top\mathbf{A}_t^{\perp}\right]\mathbf{B}_t^{\perp}.$$

Since $r = r^*$, $\mathbf{A}_t^{\mathbf{U}}$ is square, and the same step-size condition gives

$$\left\|\mathbf{I}_r - \eta(\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{A}_t^{\mathbf{U}} - \eta(\mathbf{A}_t^{\perp})^\top \mathbf{A}_t^{\perp}\right\|_{op} \le 1 - \eta\lambda_{r^*}^2(\mathbf{A}_t^{\mathbf{U}}).$$

The quantization term is bounded by q_t. Its covariance term only involves the right-perpendicular residual blocks:

$$\begin{aligned}
&\left\|\left[(\mathbf{A}_t^{\mathbf{U}})^\top \mathbf{U}^\top + (\mathbf{A}_t^{\perp})^\top \mathbf{U}_\perp^\top\right](\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d)(\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_\perp \mathbf{R}_t^{(3)})\right\|_{op} \\
\le& \delta\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\|\mathbf{R}_t^{(2)}\|_{op} + \|\mathbf{R}_t^{(3)}\|_{op}\right) \\
\le& C\delta P_t(P_t\beta_t + \zeta_t),
\end{aligned}$$

where $\|\mathbf{R}_t^{(2)}\|_{op} \le \|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{B}_t^{\perp}\|_{op} \le P_t\beta_t$. Hence

$$\|\mathbf{B}_{t+1}^{\perp}\|_{op} \le \left(1 - \eta\lambda_{r^*}^2(\mathbf{A}_t^{\mathbf{U}})\right)\|\mathbf{B}_t^{\perp}\|_{op} + \eta q_t + C\eta\delta P_t(P_t\beta_t + \zeta_t).$$

Since $s_t \le \lambda_{r^*}^2(\mathbf{A}_t^{\mathbf{U}})$, this implies the displayed bound with $1 - \eta s_t$.

Finally, $\mathbf{R}_{t+1}^{(3)} = \mathbf{A}_{t+1}^{\perp}\mathbf{B}_{t+1}^{\perp}$ by definition, so the product bound follows from submultiplicativity. ■

Next, we are ready to state the main induction lemma. For simplicity, denote $\epsilon_{\max} = \max\{\epsilon_d, \epsilon_A, \epsilon_B, \epsilon_o, \epsilon_w, \epsilon_l, \epsilon_r\}$.

Theorem 11 (Exact-rank finite-horizon induction) *Under Assumptions 2.1 and 2.2, consider the spectral initialization* (spectral initialization) *and the quantized GD iteration* (quantized GD) *with* $\eta_1 = \eta_2 = \eta > 0$. *Assume* $r = r^*$ *and write* $\lambda_i^* = \lambda_i(\Delta)$ *and* $\kappa = \lambda_1^*/\lambda_{r^*}^*$. *There are universal constants* $c_0 \in (0,1)$ *and* $C > 1$ *such that the following holds. For any* $0 < \epsilon < 1$ *and* $\tau \in \mathbb{N}_0$, *define*

$$C_\epsilon = 1 + (1 + 2\sqrt{\lambda_1^*})^2 + \lambda_1^* + \frac{\epsilon\lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}}, \qquad \delta_q = \epsilon + (1+\epsilon)\left(2\epsilon_{\max}\sqrt{d} + \epsilon_{\max}^2 d\right).$$

Let

$$\begin{aligned}
\mathfrak{e}_q =& C\epsilon_{\max}(1 + \epsilon_{\max}\sqrt{r^*})^3(1+\epsilon)(1+\epsilon_{\max}\sqrt{d})^2(1 + 2\sqrt{\lambda_1^*}) \\
&\cdot\left[(1+\sqrt{r^*})\|\mathbf{W}^*\|_F + \sqrt{d}\|\mathbf{W}^*\|_{op} + (1+\sqrt{r^*})\|\mathbf{W}^\natural\|_F + \sqrt{r^*}\lambda_1^* + r^*(1+2\sqrt{\lambda_1^*})^2\right],
\end{aligned} \tag{27}$$

let

$$\Phi_q = C\left[\frac{(1+2\sqrt{\lambda_1^*})(1+\eta)}{\lambda_{r^*}^*/(64\kappa)} + \frac{(1+2\sqrt{\lambda_1^*})(1+\eta(1+2\sqrt{\lambda_1^*}) + \delta_q)}{(\lambda_{r^*}^*/(64\kappa))^2}\right]C_\epsilon^2\mathfrak{e}_q + C\mathfrak{e}_q + \frac{C\eta}{\lambda_{r^*}^*/(64\kappa)}\mathfrak{e}_q^2, \tag{28}$$

and let

$$\Phi_A(\tau) = C\eta\delta_q(1+2\sqrt{\lambda_1^*})\sum_{j=0}^{\tau-1}\left(1 - \eta\frac{\lambda_{r^*}^*}{64\kappa}\right)^{\tau-1-j}\left[\left(1 - \eta\frac{\lambda_{r^*}^*}{64\kappa}\right)^j \epsilon\lambda_1^* + \Phi_q\right] + \Phi_q.$$

Set

$$P_\star = 1 + 2\sqrt{\lambda_1^*}, \qquad \gamma = \frac{\lambda_{r^*}^*}{64\kappa}, \qquad L_\perp = \frac{4\sqrt{\lambda_1^*}}{\lambda_{r^*}^*},$$

and

$$E_\star = \epsilon\lambda_1^* + \Phi_q, \qquad \Xi_\star = 2 + L_\perp(1 + \Phi_q).$$

Suppose

$$\delta_q \le c_0 \min\left\{\frac{1}{\kappa}, \frac{\lambda_{r^*}^*}{\kappa(\lambda_1^* + \kappa^2)}, \frac{\gamma}{(1 + 2\sqrt{\lambda_1^*})C_\epsilon}\right\}, \quad 0 \le \epsilon_{\max} \le 1, \tag{29}$$

$$\eta \le \min\left\{\frac{c_0}{\kappa\lambda_1^*}, \frac{c_0}{\lambda_1^*}, \frac{c_0\gamma}{(1 + 2\sqrt{\lambda_1^*})^2 C_\epsilon^2}, \frac{1}{4(1 + 2\sqrt{\lambda_1^*})^2}\right\},$$

$$\frac{\epsilon\lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}} \le c_0 \frac{\lambda_{r^*}^*}{\sqrt{\lambda_1^*}}, \quad 0 < \epsilon \le \min\left\{\frac{1}{2}, c_0 \frac{\gamma\lambda_{r^*}^*}{\lambda_1^*\sqrt{\lambda_1^*}P_\star}\right\},$$

$$\max\left\{\frac{\Phi_q}{\lambda_{r^*}^*}, \frac{\sqrt{\lambda_1^*}\Phi_q}{\lambda_{r^*}^*}, \frac{\tau\eta(1+\delta_q)(1+2\sqrt{\lambda_1^*})\sqrt{\lambda_1^*}\Phi_q}{\lambda_{r^*}^*}\right\} \le c_0, \quad C_\epsilon\Phi_q \le c_0 P_\star,$$

$$\max\left\{P_\star\Xi_\star^2 E_\star,\, \eta P_\star\Xi_\star E_\star,\, \eta P_\star^2\Xi_\star^2 E_\star,\, \delta_q P_\star(P_\star + \eta P_\star\Xi_\star E_\star + \eta\mathfrak{e}_q)\Xi_\star,\, \eta\delta_q^2 P_\star^2\Xi_\star^2 E_\star\right\} \le c_0\gamma.$$

Then with probability at least $1 - 2C'\exp(-\epsilon^2 N)$ for a universal constant $C' > 0$, on the same high-probability event, for every integer $0 \le t \le \tau$, the following bounds hold.

- ***(H1) Error:***

$$\begin{aligned}\mathcal{M}_t &\le \left(1 - \eta\frac{\lambda_{r^*}^*}{64\kappa}\right)^t \epsilon\lambda_1^* + \Phi_q,\\ \|\mathbf{R}_t^{(3)}\|_{op} &\le \Phi_q\left[\left(1 - \eta\frac{\lambda_{r^*}^*}{64\kappa}\right)^t \frac{\epsilon\lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}} + \Phi_A(t)\right].\end{aligned} \tag{30}$$

- ***(H2) Parameter upper bounds:***

$$\|\mathbf{A}_t\|_{op} \le 1 + 2\sqrt{\lambda_1^*}, \qquad \|\mathbf{B}_t\|_{op} \le 1 + 2\sqrt{\lambda_1^*}. \tag{31}$$

- ***(H3) Signal lower bounds:***

$$\lambda_{r^*}(\mathbf{A}_t^{\mathbf{U}}) \ge \frac{\lambda_{r^*}^*}{4\sqrt{\lambda_1^*}}, \qquad \lambda_{r^*}(\mathbf{B}_t^{\mathbf{V}}) \ge \frac{\lambda_{r^*}^*}{4\sqrt{\lambda_1^*}}. \tag{32}$$

- ***(H4) Approximate alignment:***

$$\begin{aligned}\|\mathbf{A}_t^\perp\|_{op} &\le \left(1 - \eta\frac{\lambda_{r^*}^*}{64\kappa}\right)^t \frac{\epsilon\lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}} + \Phi_A(t),\\ \|\mathbf{B}_t^\perp\|_{op} &\le \Phi_q.\end{aligned} \tag{33}$$

Proof By Lemma 8, with probability at least $1 - 2C' \exp(-\epsilon^2 N)$,

$$\delta = \left\| \widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d \right\|_{op} \leq \delta_q.$$

The same concentration bound gives

$$\|\widetilde{\mathbf{X}}\|_{op} \leq \sqrt{N(1+\epsilon)}, \qquad \|\widetilde{\mathbf{X}}\|_F \leq \sqrt{Nd(1+\epsilon)}.$$

By Assumption 2.2,

$$\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op} \leq \|\widetilde{\mathbf{X}}\|_{op} + \epsilon_{\max}\|\widetilde{\mathbf{X}}\|_F \leq \sqrt{N(1+\epsilon)}(1+\epsilon_{\max}\sqrt{d}).$$

For simplicity, we use the theorem notation

$$\rho = \eta\gamma.$$

We first prove the base case at $t = 0$. By Lemma 9, we have

$$\mathcal{M}_0 \leq \epsilon\lambda_1^*, \qquad \|\mathbf{A}_0^{\perp}\|_{op} \leq \frac{\epsilon\lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}}, \qquad \mathbf{B}_0^{\perp} = \mathbf{0}, \qquad \mathbf{R}_0^{(3)} = \mathbf{0},$$

The bound $0 < \epsilon \leq 1/2$ gives

$$\|\mathbf{A}_0\|_{op},\ \|\mathbf{B}_0\|_{op} \leq \sqrt{(1+\epsilon)\lambda_1^*} \leq 1 + 2\sqrt{\lambda_1^*}.$$

The initial-perpendicular smallness condition in (29) gives

$$\frac{\epsilon\lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}} \leq c_0 \frac{\lambda_{r^*}^*}{\sqrt{\lambda_1^*}}.$$

Since $\epsilon \leq 1/2$ and $\lambda_{r^*}^* \leq \lambda_1^*$,

$$\sqrt{(1-\epsilon)\lambda_{r^*}^*} \geq \frac{1}{\sqrt{2}}\sqrt{\lambda_{r^*}^*} \geq \frac{1}{\sqrt{2}}\frac{\lambda_{r^*}^*}{\sqrt{\lambda_1^*}}.$$

Choose the universal constant $c_0 \leq 1/\sqrt{2} - 1/2$. Lemma 9 then gives the stronger base signal bound

$$\lambda_{r^*}(\mathbf{A}_0^{\mathbf{U}}),\ \lambda_{r^*}(\mathbf{B}_0^{\mathbf{V}}) \geq \frac{\lambda_{r^*}^*}{2\sqrt{\lambda_1^*}},$$

and hence (H2) and (H3) hold at $t = 0$.

Assume the induction claims hold for every time j with $0 \leq j \leq t < \tau$. Use the notation of Corollary 10, we have

$$\begin{aligned} P_t &= \max\{1, \|\mathbf{A}_t\|_{op}, \|\mathbf{B}_t\|_{op}\} \leq 1 + 2\sqrt{\lambda_1^*}, \\ e_t &= \mathcal{M}_t \leq \left(1 - \eta\frac{\lambda_{r^*}^*}{64\kappa}\right)^t \epsilon\lambda_1^* + \Phi_q, \\ s_t &= \min\left\{\lambda_{r^*}^2(\mathbf{A}_t^{\mathbf{U}}), \lambda_{r^*}^2(\mathbf{B}_t^{\mathbf{V}})\right\} \geq \frac{(\lambda_{r^*}^*)^2}{16\lambda_1^*} = 4\frac{\lambda_{r^*}^*}{64\kappa}. \end{aligned} \tag{34}$$

Define

$$E_t = (1-\rho)^t \epsilon \lambda_1^* + \Phi_q, \qquad a_t = (1-\rho)^t \frac{\epsilon \lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}} + \Phi_A(t), \qquad b_t = \Phi_q, \qquad z_t = a_t b_t.$$

By (H4) and the exact-rank identity $\mathbf{R}_t^{(3)} = \mathbf{A}_t^{\perp}\mathbf{B}_t^{\perp}$,

$$\|\mathbf{A}_t^{\perp}\|_{op} \le a_t, \qquad \|\mathbf{B}_t^{\perp}\|_{op} \le b_t, \qquad \|\mathbf{R}_t^{(3)}\|_{op} \le z_t.$$

Since $r = r^*$, (H2) gives

$$\|\mathbf{A}_t\|_F,\ \|\mathbf{B}_t\|_F \le \sqrt{r^*}P_\star, \qquad \|\mathbf{A}_t\|_{op},\ \|\mathbf{B}_t\|_{op} \le P_\star.$$

Moreover

$$\|\Delta\|_{op} = \lambda_1^*, \qquad \|\widetilde{\mathbf{Y}}\|_F = \|\widetilde{\mathbf{X}}\mathbf{W}^*\|_F \le \sqrt{N(1+\epsilon)}\|\mathbf{W}^*\|_F,$$

and $\epsilon_1, \epsilon_2, \epsilon_3 \le C\epsilon_{\max}(1+\epsilon_{\max})^3$. Substituting these estimates, $\epsilon_i \le \epsilon_{\max}$, and the displayed data bounds into Lemma 7 yields

$$\begin{aligned}\|\mathbf{E}_{t,\mathbf{A}}\|_{op} \le & C\epsilon_{\max}(1+\epsilon_{\max}\sqrt{r^*})^3(1+\epsilon)(1+\epsilon_{\max}\sqrt{d})^2 P_\star \\ & \cdot\Big[(1+\sqrt{r^*})\|\mathbf{W}^*\|_F + \sqrt{d}\|\mathbf{W}^*\|_{op} + (1+\sqrt{r^*})\|\mathbf{W}^\natural\|_F + \sqrt{r^*}\lambda_1^* + r^* P_\star^2\Big].\end{aligned}$$

The displayed bound for $\mathbf{E}_{t,\mathbf{B}}$ in Lemma 7 gives, with the same substitutions,

$$\begin{aligned}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \le & C\epsilon_{\max}(1+\epsilon_{\max}\sqrt{r^*})^3(1+\epsilon)(1+\epsilon_{\max}\sqrt{d})^2 P_\star \\ & \cdot\Big[(1+\sqrt{r^*})\|\mathbf{W}^*\|_F + \sqrt{d}\|\mathbf{W}^*\|_{op} + (1+\sqrt{r^*})\|\mathbf{W}^\natural\|_F + \sqrt{r^*}\lambda_1^* + r^* P_\star^2\Big].\end{aligned}$$

Thus

$$q_t = \max\{\|\mathbf{E}_{t,\mathbf{A}}\|_{op}, \|\mathbf{E}_{t,\mathbf{B}}\|_{op}\} \le \mathfrak{e}_q.$$

We record the consequences of the envelopes and smallness hypotheses that will be used below:

$$P_\star \le C_\epsilon, \qquad P_\star^2 \le C_\epsilon, \qquad \delta_q P_\star C_\epsilon \le c_0\gamma, \qquad C_\epsilon \Phi_q \le c_0 P_\star, \qquad q_t \le \gamma\Phi_q, \qquad \eta q_t \le \rho\Phi_q. \tag{35}$$

The first two inequalities follow from the definition of C_ϵ, and the next two are explicit assumptions in (29). Also, by choosing the universal constant in (28) larger than the constants in the quantization estimate, the bound $q_t \le \mathfrak{e}_q$ gives

$$\Phi_q \ge C\left(\frac{P_\star}{\gamma}q_t + q_t + \frac{\eta}{\gamma}q_t^2\right), \tag{36}$$

which implies the last two inequalities in (35).

We now turn to the case at $t+1$. The step-size hypothesis needed for Corollary 10 is satisfied because

$$\eta\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}^2 + \|\mathbf{B}_t^{\perp}\|_{op}^2\right) + \eta\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}^2 + \|\mathbf{A}_t^{\perp}\|_{op}^2\right) \le 4\eta(1+2\sqrt{\lambda_1^*})^2 \le 1.$$

Since $r = r^*$, $\mathbf{B}_t^{\mathbf{V}}$ is square. By (H3),

$$\|(\mathbf{B}_t^{\mathbf{V}})^{-1}\|_{op} \le L_\perp.$$

Therefore the identity $\mathbf{R}_t^{(1)} = \mathbf{A}_t^{\perp}\mathbf{B}_t^{\mathbf{V}}$ gives the sharper exact-rank envelope

$$\|\mathbf{A}_t^{\perp}\|_{op} \le L_\perp E_t, \qquad \|\mathbf{B}_t^{\perp}\|_{op} \le \Phi_q, \qquad \|\mathbf{R}_t^{(3)}\|_{op} \le L_\perp E_t \Phi_q.$$

(H1) for $\mathcal{M}_{t+1}$. Apply Corollary 10 with the sharper envelopes

$$\alpha_t = L_\perp E_t, \qquad \beta_t = \Phi_q, \qquad \zeta_t = L_\perp E_t \Phi_q.$$

Then

$$\mathcal{M}_{t+1} \le (1-4\rho)e_t + \widetilde{\Phi}_t,$$

where [1]

$$\begin{aligned}\widetilde{\Phi}_t \le & C\eta P_t(L_\perp E_t + \Phi_q)(E_t + L_\perp E_t \Phi_q) + C\eta P_t q_t + C\eta^2 P_t q_t \Xi_\star E_t + C\eta^2 q_t^2 \\ & + C\eta^2 P_t^2 \Xi_\star^2 E_t^2 + C\eta\delta_q P_t \left(P_t + \eta P_t \Xi_\star E_t + \eta q_t\right) \Xi_\star E_t + C\eta^2 \delta_q^2 P_t^2 \Xi_\star^2 E_t^2,\end{aligned}$$

where we used $E_t \le E_\star$, $\Phi_q \le E_t$, and hence

$$e_t + L_\perp E_t + \Phi_q + L_\perp E_t \Phi_q \le \Xi_\star E_t.$$

The quantization-only terms are absorbed as follows. Choose the universal constant in (28) so that the first two terms below are each at most $\rho\Phi_q/4$. Then (36), $P_t \le P_\star$, and $E_t \le E_\star$ give

$$\begin{aligned} C\eta P_t q_t &\le \frac{\rho\Phi_q}{4}, \\ C\eta^2 q_t^2 &\le \frac{\rho\Phi_q}{4}, \\ C\eta^2 P_t q_t \Xi_\star E_t &\le C\frac{\eta P_\star \Xi_\star E_\star}{\gamma}\rho\Phi_q \le Cc_0\rho\Phi_q, \end{aligned}$$

where the last inequality uses $\eta P_\star \Xi_\star E_\star \le c_0\gamma$ from (29). Choose c_0 so that $Cc_0 \le 1/2$. Then

$$C\eta P_t q_t + C\eta^2 q_t^2 + C\eta^2 P_t q_t \Xi_\star E_t \le \rho\Phi_q.$$

The last smallness condition in (29) gives the term-by-term bounds

$$\begin{aligned} C\eta P_t(L_\perp E_t + \Phi_q)(E_t + L_\perp E_t \Phi_q) &\le Cc_0\rho E_t, \\ C\eta^2 P_t^2 \Xi_\star^2 E_t^2 &\le Cc_0\rho E_t, \\ C\eta\delta_q P_t(P_t + \eta P_t \Xi_\star E_t + \eta q_t)\Xi_\star E_t &\le Cc_0\rho E_t, \\ C\eta^2\delta_q^2 P_t^2 \Xi_\star^2 E_t^2 &\le Cc_0\rho E_t. \end{aligned}$$

Choose c_0 so that the four $Cc_0\rho E_t$ terms sum to at most $3\rho E_t$. Then

$$\widetilde{\Phi}_t \le 3\rho E_t + \rho\Phi_q \le 3\rho(1-\rho)^t \epsilon\lambda_1^* + 4\rho\Phi_q.$$

Combining this with $e_t \le (1-\rho)^t\epsilon\lambda_1^* + \Phi_q$ in (34) gives

$$\mathcal{M}_{t+1} \le (1-4\rho)\left[(1-\rho)^t\epsilon\lambda_1^* + \Phi_q\right] + 3\rho(1-\rho)^t\epsilon\lambda_1^* + 4\rho\Phi_q = (1-\rho)^{t+1}\epsilon\lambda_1^* + \Phi_q.$$

1. Here we slightly abuse the notation of C by selecting the maximum of the constant C in Theorem 11 and constant C in Corollary 10.

(H4) for $\mathbf{A}_{t+1}^{\perp}$ and $\mathbf{B}_{t+1}^{\perp}$. For the separated perpendicular induction, we return to the envelopes

$$\alpha_t = a_t, \qquad \beta_t = b_t = \Phi_q, \qquad \zeta_t = z_t = a_t\Phi_q.$$

The definition of $\Phi_A(t)$ and the geometric sum bound $\sum_{j=0}^{t-1}(1-\rho)^{t-1-j}\eta \le 1/\gamma$ give

$$\begin{aligned} a_t &\le \frac{\epsilon\lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}} + \frac{C\delta_q P_\star}{\gamma}(\epsilon\lambda_1^* + \Phi_q) + \Phi_q \\ &\le CC_\epsilon. \end{aligned}$$

The last inequality follows from (29), $0 < \epsilon \le 1/2$, and the definition of C_ϵ. The perpendicular bounds in Corollary 10 imply

$$\|\mathbf{A}_{t+1}^{\perp}\|_{op} \le (1-4\rho)a_t + \eta q_t + C\eta\delta_q\left[P_t(e_t + P_t a_t) + (P_t b_t + z_t)b_t\right].$$

Since $P_t \le P_\star$, $a_t \le CC_\epsilon$, and (35) holds, the first covariance term is absorbed into the contraction:

$$C\eta\delta_q P_t^2 a_t \le C\eta\delta_q P_\star C_\epsilon a_t \le Cc_0\eta\gamma a_t \le \rho a_t,$$

where c_0 is fixed so that the displayed universal factor satisfies $Cc_0 \le 1$. For the remaining covariance contribution, $E_t \ge \Phi_q$ and

$$(P_\star\Phi_q + a_t\Phi_q)\Phi_q \le CC_\epsilon\Phi_q^2 \le Cc_0 P_\star\Phi_q \le P_\star E_t.$$

Hence

$$C\eta\delta_q\left[P_t e_t + (P_t b_t + z_t)b_t\right] \le C\eta\delta_q P_\star E_t.$$

Hence

$$\|\mathbf{A}_{t+1}^{\perp}\|_{op} \le (1-\rho)a_t + C\eta\delta_q(1 + 2\sqrt{\lambda_1^*})\left[(1-\rho)^t\epsilon\lambda_1^* + \Phi_q\right] + \eta q_t.$$

By (35), $\eta q_t \le \rho\Phi_q$. The definition of $\Phi_A(t+1)$ yields

$$\|\mathbf{A}_{t+1}^{\perp}\|_{op} \le (1-\rho)^{t+1}\frac{\epsilon\lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}} + \Phi_A(t+1).$$

The same corollary gives

$$\|\mathbf{B}_{t+1}^{\perp}\|_{op} \le (1-\eta s_t)\|\mathbf{B}_t^{\perp}\|_{op} + \eta q_t + C\eta\delta_q P_t(P_t b_t + z_t).$$

Since $b_t = \Phi_q$, $z_t = a_t\Phi_q$, $a_t \le CC_\epsilon$, and $P_t \le P_\star$, (35) gives

$$C\eta\delta_q P_t(P_t b_t + z_t) \le C\eta\delta_q P_\star C_\epsilon\Phi_q \le Cc_0\eta\gamma\Phi_q \le \rho\Phi_q$$

with the same choice of c_0. By $s_t \ge 4\gamma$ and $\eta q_t \le \rho\Phi_q$, the last two terms are at most $2\rho\Phi_q$. Thus

$$\|\mathbf{B}_{t+1}^{\perp}\|_{op} \le (1-4\rho)\Phi_q + 2\rho\Phi_q \le \Phi_q.$$

(H1) for $\mathbf{R}_{t+1}^{(3)}$. We use the identity

$$\mathbf{R}_{t+1}^{(3)} = \mathbf{A}_{t+1}^{\perp}\mathbf{B}_{t+1}^{\perp}.$$

The two (H4) bounds just proved imply

$$\|\mathbf{R}_{t+1}^{(3)}\|_{op} \le \Phi_q \left[(1-\rho)^{t+1}\frac{\epsilon\lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}} + \Phi_A(t+1)\right],$$

which is the second (H1) claim at time $t+1$.

(H2) and (H3) at $t+1$. It remains to keep the aligned factors in the same bounded and nonsingular tube. For every $0 \le j \le t$, (H1) and (H3) at time j imply that $\mathbf{B}_j^{\mathbf{V}}$ is invertible and

$$\|\mathbf{A}_j^{\perp}\|_{op} = \|\mathbf{R}_j^{(1)}(\mathbf{B}_j^{\mathbf{V}})^{-1}\|_{op} \le L_{\perp}\left[(1-\rho)^j\epsilon\lambda_1^* + \Phi_q\right].$$

Together with the (H4) bound $\|\mathbf{B}_j^{\perp}\|_{op} \le \Phi_q$, this gives

$$\|\mathbf{R}_j^{(3)}\|_{op} = \|\mathbf{A}_j^{\perp}\mathbf{B}_j^{\perp}\|_{op} \le L_{\perp}\left[(1-\rho)^j\epsilon\lambda_1^* + \Phi_q\right]\Phi_q \le C\left[(1-\rho)^j\epsilon\lambda_1^* + \Phi_q\right],$$

and hence

$$\|\mathbf{A}_j\mathbf{B}_j - \Delta\|_{op} \le C\left[(1-\rho)^j\epsilon\lambda_1^* + \Phi_q\right].$$

Projecting the component updates in Lemma 5 back to the ambient coordinates gives

$$\begin{aligned}
\mathbf{A}_{j+1} - \mathbf{A}_j &= -\eta(\mathbf{A}_j\mathbf{B}_j - \Delta)\mathbf{B}_j^{\top} - \eta(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d)(\mathbf{A}_j\mathbf{B}_j - \Delta)\mathbf{B}_j^{\top} + \eta\mathbf{E}_{j,\mathbf{A}},\\
\mathbf{B}_{j+1} - \mathbf{B}_j &= -\eta\mathbf{A}_j^{\top}(\mathbf{A}_j\mathbf{B}_j - \Delta) - \eta\mathbf{A}_j^{\top}(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d)(\mathbf{A}_j\mathbf{B}_j - \Delta) + \eta\mathbf{E}_{j,\mathbf{B}}.
\end{aligned}$$

Taking operator norms, using $\delta \le \delta_q$, the parameter bounds, and the preceding error estimate gives

$$\|\mathbf{A}_{j+1} - \mathbf{A}_j\|_{op} + \|\mathbf{B}_{j+1} - \mathbf{B}_j\|_{op} \le C\eta(1+\delta_q)(1+2\sqrt{\lambda_1^*})\left[(1-\rho)^j\epsilon\lambda_1^* + \Phi_q\right] + C\eta q_j.$$

The preceding quantization estimate applies at each $j \le t$, so $q_j \le \mathfrak{e}_q$. Summing this display from $j = 0$ to t gives

$$\begin{aligned}
&\|\mathbf{A}_{t+1} - \mathbf{A}_0\|_{op} + \|\mathbf{B}_{t+1} - \mathbf{B}_0\|_{op}\\
&\quad\le C(1+\delta_q)P_\star\left[\frac{\epsilon\lambda_1^*}{\gamma} + \tau\eta\Phi_q\right] + C\tau\eta\mathfrak{e}_q.
\end{aligned}$$

Here $\sum_{j=0}^{t}\eta(1-\rho)^j \le 1/\gamma$ was used. The $C\mathfrak{e}_q$ term in (28) gives $\mathfrak{e}_q \le C\Phi_q$. Also $\delta_q \le 1$ and $P_\star \ge 1$. Hence the two smallness conditions

$$\frac{P_\star\epsilon\lambda_1^*}{\gamma} \le c_0\frac{\lambda_{r^*}^*}{\sqrt{\lambda_1^*}}, \qquad \frac{\tau\eta(1+\delta_q)P_\star\sqrt{\lambda_1^*}\Phi_q}{\lambda_{r^*}^*} \le c_0$$

imply, provided c_0 is chosen so that the preceding universal constant times c_0 is at most 1, that

$$\|\mathbf{A}_{t+1} - \mathbf{A}_0\|_{op} + \|\mathbf{B}_{t+1} - \mathbf{B}_0\|_{op} \le c_0\min\left\{\sqrt{\lambda_1^*}, \frac{\lambda_{r^*}^*}{\sqrt{\lambda_1^*}}\right\}.$$

Therefore, by Lemma 9, the stronger base signal bound proved above, and Weyl's singular-value inequality,

$$\|\mathbf{A}_{t+1}\|_{op},\ \|\mathbf{B}_{t+1}\|_{op} \le \sqrt{(1+\epsilon)\lambda_1^*} + c_0\sqrt{\lambda_1^*} \le 1 + 2\sqrt{\lambda_1^*},$$

and

$$\lambda_{r^*}(\mathbf{A}_{t+1}^{\mathbf{U}}),\ \lambda_{r^*}(\mathbf{B}_{t+1}^{\mathbf{V}}) \ge \frac{\lambda_{r^*}^*}{2\sqrt{\lambda_1^*}} - c_0\frac{\lambda_{r^*}^*}{\sqrt{\lambda_1^*}} \ge \frac{\lambda_{r^*}^*}{4\sqrt{\lambda_1^*}},$$

because $c_0 \le 1/4$. This closes (H2) and (H3) at time $t+1$, and hence the induction. ■

Corollary 12 (Whole Frobenius error from the exact-rank induction) *For every fixed $t \ge 0$, under the conditions of Theorem 11 with $\tau = t$, on the same high-probability event,*

$$\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F \le \sqrt{r^*}\left(3\mathcal{E}_t^2 + \Phi_q^2\mathcal{A}_t^2\right)^{1/2} \le \sqrt{r^*}\left(\sqrt{3}\,\mathcal{E}_t + \Phi_q\mathcal{A}_t\right),$$

where, with $\gamma = \lambda_{r^*}^*/(64\kappa)$,

$$\mathcal{E}_t = (1-\eta\gamma)^t\epsilon\lambda_1^* + \Phi_q \le (1-\eta\gamma)^t\lambda_{r^*}^* + \Phi_q, \qquad \mathcal{A}_t = (1-\eta\gamma)^t\frac{\epsilon\lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}} + \Phi_A(t).$$

Proof The orthogonal decompositions induced by $\mathbf{U}, \mathbf{U}_\perp, \mathbf{V}, \mathbf{V}_\perp$ give

$$\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F^2 = \|\mathbf{R}_t^*\|_F^2 + \|\mathbf{R}_t^{(1)}\|_F^2 + \|\mathbf{R}_t^{(2)}\|_F^2 + \|\mathbf{R}_t^{(3)}\|_F^2.$$

Because $r = r^*$ in Theorem 11, each block on the right-hand side has rank at most r^*. Hence $\|\mathbf{X}\|_F \le \sqrt{r^*}\|\mathbf{X}\|_{op}$ for each of these four blocks. The induction proved in Theorem 11 gives, on the same high-probability event,

$$\|\mathbf{R}_t^*\|_{op},\ \|\mathbf{R}_t^{(1)}\|_{op},\ \|\mathbf{R}_t^{(2)}\|_{op} \le \mathcal{E}_t, \qquad \|\mathbf{R}_t^{(3)}\|_{op} \le \Phi_q\mathcal{A}_t.$$

Substituting these estimates into the orthogonal block identity yields the first displayed inequality. The second follows from $\sqrt{x+y} \le \sqrt{x} + \sqrt{y}$. $(1-\eta\gamma)^t\epsilon\lambda_1^* + \Phi_q \le (1-\eta\gamma)^t\lambda_{r^*}^* + \Phi_q$ holds by $\epsilon \le \frac{1}{\kappa}$. ■

Corollary 13 (Simplified whole Frobenius error in exact-rank case) *Under Assumptions 2.1 and 2.2, consider the spectral initialization* (spectral initialization) *and the quantized GD iteration* (quantized GD) *with $\eta_1 = \eta_2 = \eta > 0$. Assume $r = r^*$, write $\lambda_i^* = \lambda_i(\Delta)$, $\kappa = \lambda_1^*/\lambda_{r^*}^*$, and*

$$\gamma = \frac{\lambda_{r^*}^*}{64\kappa}.$$

Suppose the signal quantities are at constant level: λ_1^*, $1/\lambda_{r^*}^*$, κ, r^*, and $\|\mathbf{W}^\natural\|_{op}$ are bounded by absolute constants. Define

$$P_\star = 1 + 2\sqrt{\lambda_1^*}, \qquad C_0 = 1 + P_\star^2 + \lambda_1^* + \frac{\lambda_1^*}{\sqrt{\lambda_{r^*}^*/2}}.$$

There exist constants $c \in (0,1)$ and $C > 1$, depending only on the preceding constant-level bounds and the universal constants in Theorem 11, such that the following holds. Choose ϵ, η satisfying

$$0 < \epsilon \le c \min\left\{1,\ \frac{1}{\kappa},\ \frac{\lambda^*_{r^*}}{\kappa(\lambda^*_1 + \kappa^2)},\ \frac{\gamma}{P_\star C_0},\ \frac{\gamma \lambda^*_{r^*}}{P_\star \lambda^*_1 \sqrt{\lambda^*_1}},\ \frac{(\lambda^*_{r^*})^{3/2}}{(\lambda^*_1)^{3/2}}\right\},$$

and

$$0 < \eta \le c \min\left\{1,\ \frac{1}{\kappa \lambda^*_1},\ \frac{1}{\lambda^*_1},\ \frac{1}{P_\star^2},\ \frac{\gamma}{P_\star^2 C_0^2}\right\}.$$

For any $t > 0$, if the low-precision errors obey

$$t \epsilon_{\max} \sqrt{d} \le c,$$

then with probability at least $1 - C\exp(-c\epsilon^2 N)$,

$$\|\mathbf{A}_t \mathbf{B}_t - \Delta\|_F \le C\sqrt{r^*}\left[(1-\eta\gamma)^t + \epsilon_{\max}\sqrt{d}\right].$$

In particular, if additionally

$$\epsilon_{\max}\sqrt{d} \le c(1-\eta\gamma)^t,$$

then the low-precision iterates match the full-precision linear rate:

$$\|\mathbf{A}_t \mathbf{B}_t - \Delta\|_F \le C\sqrt{r^*}(1-\eta\gamma)^t.$$

Proof Let c_0 denote the small universal constant in Theorem 11, and write

$$z = \epsilon_{\max}\sqrt{d}.$$

Throughout the proof, constants denoted by C depend only on the constant-level bounds in the statement and on the universal constants in Theorem 11. By taking the statement constant c small enough, the displayed assumptions imply $\epsilon \le 1/2$, $\eta \le 1$, $z \le 1$, and $\delta_q \le 1$.

We first record the simplified envelopes entering Theorem 11. Since $\epsilon \le 1/2$, the definition of C_ϵ gives

$$C_\epsilon \le C_0.$$

Also,

$$\delta_q = \epsilon + (1+\epsilon)(2z + z^2) \le \epsilon + 5z.$$

The constant-level assumptions imply

$$\|\Delta\|_F \le \sqrt{r^*}\lambda^*_1 = \mathcal{O}(1), \qquad \|\mathbf{W}^*\|_{op} \le \|\mathbf{W}^\natural\|_{op} + \lambda^*_1 = \mathcal{O}(1),$$

and, because $\mathbf{W}^\natural \in \mathbb{R}^{d\times k}$,

$$\|\mathbf{W}^\natural\|_F \le \sqrt{d}\,\|\mathbf{W}^\natural\|_{op} = \mathcal{O}(\sqrt{d}), \qquad \|\mathbf{W}^*\|_F \le \|\mathbf{W}^\natural\|_F + \|\Delta\|_F = \mathcal{O}(\sqrt{d}).$$

Substituting these bounds and $z \le 1$ into (27) gives

$$\mathfrak{e}_q \le Cz.$$

Then (28), $\eta \le 1$, and $\delta_q \le 1$ give

$$\Phi_q \le Cz.$$

Consequently

$$E_\star = \epsilon\lambda_1^* + \Phi_q \le C(\epsilon + z), \qquad \Xi_\star = 2 + L_\perp(1 + \Phi_q) \le C.$$

We now verify (29). The first covariance-smallness condition follows from the displayed bound on ϵ, the low-precision bound $z \le c$, and the smallness of c:

$$\delta_q \le \epsilon + 5z \le c_0 \min\left\{\frac{1}{\kappa}, \frac{\lambda_{r^*}^*}{\kappa(\lambda_1^* + \kappa^2)}\right\}.$$

The assumptions also give $0 < \epsilon \le 1/2$ and $0 \le \epsilon_{\max} \le z \le 1$. The displayed step-size condition gives

$$\eta \le \min\left\{\frac{c_0}{\kappa\lambda_1^*}, \frac{c_0}{\lambda_1^*}, \frac{1}{4P_\star^2}\right\}$$

after decreasing c if necessary. Moreover,

$$\delta_q P_\star C_\epsilon \le C(\epsilon + z)P_\star C_0 \le c_0\gamma, \qquad \eta P_\star^2 C_\epsilon^2 \le cP_\star^2 C_0^2 \frac{\gamma}{P_\star^2 C_0^2} \le c_0\gamma.$$

The two initialization smallness requirements follow directly from the ϵ condition:

$$\frac{P_\star \epsilon\lambda_1^*}{\gamma} \le c\frac{\lambda_{r^*}^*}{\sqrt{\lambda_1^*}} \le c_0\frac{\lambda_{r^*}^*}{\sqrt{\lambda_1^*}},$$

and

$$\frac{\epsilon\lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}} \le \sqrt{2}\,\epsilon\frac{\lambda_1^*}{\sqrt{\lambda_{r^*}^*}} \le c\frac{\lambda_{r^*}^*}{\sqrt{\lambda_1^*}} \le c_0\frac{\lambda_{r^*}^*}{\sqrt{\lambda_1^*}}.$$

The Φ_q conditions are consequences of $\Phi_q \le Cz$, $z \le c$, and $tz \le c$:

$$C_\epsilon\Phi_q \le CC_0 z \le c_0 P_\star,$$

$$\frac{\Phi_q}{\lambda_{r^*}^*} \le Cz \le c_0, \qquad \frac{\sqrt{\lambda_1^*}\Phi_q}{\lambda_{r^*}^*} \le Cz \le c_0,$$

and

$$\frac{t\eta(1+\delta_q)P_\star\sqrt{\lambda_1^*}\Phi_q}{\lambda_{r^*}^*} \le Ctz \le c_0.$$

Finally, using $E_\star \le C(\epsilon + z)$, $\Xi_\star \le C$, $\delta_q \le C(\epsilon + z)$, and $\mathfrak{e}_q \le Cz$, the last maximum in (29) satisfies

$$P_\star\Xi_\star^2 E_\star, \quad \eta P_\star\Xi_\star E_\star, \quad \eta P_\star^2\Xi_\star^2 E_\star \le C(\epsilon + z) \le c_0\gamma,$$
$$\delta_q P_\star(P_\star + \eta P_\star\Xi_\star E_\star + \eta\mathfrak{e}_q)\Xi_\star \le C(\epsilon + z) \le c_0\gamma,$$

and

$$\eta\delta_q^2 P_\star^2\Xi_\star^2 E_\star \le C(\epsilon + z) \le c_0\gamma.$$

Thus every condition in Theorem 11 holds, so Corollary 12 applies.

For $t = 0$, the summation defining $\Phi_A(0)$ is empty, so $\Phi_A(0) = \Phi_q \le Cz$. The exact definition of $\Phi_A(t)$ and $\gamma^{-1} = \mathcal{O}(1)$ give

$$\Phi_A(t) \le C\left[\eta t(1-\eta\gamma)^{t-1} + \epsilon_{\max}\sqrt{d}\right].$$

The first envelope satisfies, for every $0 \le t$,

$$\mathcal{E}_t \le C\left[(1-\eta\gamma)^t + \epsilon_{\max}\sqrt{d}\right],$$

For the second envelope, the case $t = 0$ gives

$$\Phi_q \mathcal{A}_0 \le C\epsilon_{\max}\sqrt{d}.$$

For $1 \le t$,

$$\Phi_q \mathcal{A}_t \le C\epsilon_{\max}\sqrt{d}\left[(1-\eta\gamma)^t + \eta t(1-\eta\gamma)^{t-1} + \epsilon_{\max}\sqrt{d}\right] \le C\epsilon_{\max}\sqrt{d},$$

because $\eta t(1-\eta\gamma)^{t-1} \le \gamma^{-1} = \mathcal{O}(1)$ and $\epsilon_{\max}\sqrt{d} \le c$. Substituting these estimates into Corollary 12 proves the claim.

If $\epsilon_{\max}\sqrt{d} \le c(1-\eta\gamma)^t$, then for every $0 \le t$,

$$\epsilon_{\max}\sqrt{d} \le c(1-\eta\gamma)^t.$$

Absorbing this term into the geometric term in the preceding display gives the final full-precision-rate statement. ■

D.3.2. GENERAL RANK

The general-rank proof follows the same template as the exact-rank proof but cannot invert the signal block to eliminate all surplus directions. Instead, it tracks explicit envelopes for $\mathbf{A}_t^{\perp}$, $\mathbf{B}_t^{\perp}$, and $\mathbf{R}_t^{(3)} = \mathbf{A}_t^{\perp}\mathbf{B}_t^{\perp}$. This produces the additional horizon-dependent term in the final Frobenius bound.

Corollary 14 (Induction-ready propagation with separated envelopes) *Assume the hypotheses of Lemma 6 with $\eta_1 = \eta_2 = \eta$, and define*

$$P_t = \max\{1, \|\mathbf{A}_t\|_{op}, \|\mathbf{B}_t\|_{op}\}, \quad q_t = \max\{\|\mathbf{E}_{t,\mathbf{A}}\|_{op}, \|\mathbf{E}_{t,\mathbf{B}}\|_{op}\},$$

$$e_t = \mathcal{M}_t, \quad s_t = \min\left\{\lambda_{r^*}^2(\mathbf{A}_t^{\mathbf{U}}), \lambda_{r^*}^2(\mathbf{B}_t^{\mathbf{V}})\right\}, \quad \delta = \left\|\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right\|_{op}.$$

If nonnegative numbers $\alpha_t, \beta_t, \zeta_t$ satisfy

$$\|\mathbf{A}_t^{\perp}\|_{op} \le \alpha_t, \qquad \|\mathbf{B}_t^{\perp}\|_{op} \le \beta_t, \qquad \|\mathbf{R}_t^{(3)}\|_{op} \le \zeta_t,$$

then, with

$$K_t^{\text{sep}} = e_t + \alpha_t + \beta_t + \zeta_t, \qquad H_t = e_t + \zeta_t,$$

define

$$\begin{aligned}\widetilde{\Phi}_t &= C\eta P_t(\alpha_t+\beta_t)H_t + C\eta^2 P_t q_t K_t^{\mathrm{sep}} + C\eta P_t q_t + C\eta^2 q_t^2 \\ &\quad + C\eta^2 P_t^2 H_t\left(K_t^{\mathrm{sep}} + (K_t^{\mathrm{sep}})^2\right) + C\eta\delta P_t\left(P_t + K_t^{\mathrm{sep}} + \eta P_t K_t^{\mathrm{sep}} + \eta q_t\right)H_t \\ &\quad + C\eta^2\delta^2 P_t^2 H_t\left(K_t^{\mathrm{sep}} + (K_t^{\mathrm{sep}})^2\right),\end{aligned}$$

for a universal constant $C > 0$, it holds

$$\begin{aligned}\mathcal{M}_{t+1} &\le (1-\eta s_t)e_t + \widetilde{\Phi}_t, \\ \|\mathbf{A}_{t+1}^{\perp}\|_{op} &\le \alpha_t + \eta q_t + C\eta\delta\left[P_t(e_t + P_t\alpha_t) + (P_t\beta_t + \zeta_t)\beta_t\right], \\ \|\mathbf{B}_{t+1}^{\perp}\|_{op} &\le \beta_t + \eta q_t + C\eta\delta P_t(P_t\beta_t + \zeta_t).\end{aligned}$$

Moreover,

$$\|\mathbf{R}_{t+1}^{(3)}\|_{op} = \|\mathbf{A}_{t+1}^{\perp}\mathbf{B}_{t+1}^{\perp}\|_{op} \le \|\mathbf{A}_{t+1}^{\perp}\|_{op}\,\|\mathbf{B}_{t+1}^{\perp}\|_{op}.$$

In particular, if $q_t = 0$ and $\beta_t = \zeta_t = 0$, then the displayed bounds give $\|\mathbf{B}_{t+1}^{\perp}\|_{op} = 0$ and $\|\mathbf{R}_{t+1}^{(3)}\|_{op} = 0$.

Proof Apply Lemma 6 with $\eta_1 = \eta_2 = \eta$. Throughout the proof we use

$$\|\mathbf{A}_t^{\mathbf{U}}\|_{op}, \|\mathbf{A}_t^{\perp}\|_{op}, \|\mathbf{B}_t^{\mathbf{V}}\|_{op}, \|\mathbf{B}_t^{\perp}\|_{op} \le P_t,$$

$$\|\mathbf{R}_t^{*}\|_{op}, \|\mathbf{R}_t^{(1)}\|_{op}, \|\mathbf{R}_t^{(2)}\|_{op} \le e_t, \quad \|\mathbf{A}_t^{\perp}\|_{op} \le \alpha_t, \quad \|\mathbf{B}_t^{\perp}\|_{op} \le \beta_t, \quad \|\mathbf{R}_t^{(3)}\|_{op} \le \zeta_t,$$

and $\|\mathbf{E}_{t,\mathbf{A}}\|_{op}, \|\mathbf{E}_{t,\mathbf{B}}\|_{op} \le q_t$. The leading coefficients in the first three residual recursions are

$$\begin{aligned}&1 - \eta\lambda_{r^*}^2(\mathbf{B}_t^{\mathbf{V}}) - \eta\lambda_{r^*}^2(\mathbf{A}_t^{\mathbf{U}}), \\ &1 - \eta\lambda_{r^*}^2(\mathbf{B}_t^{\mathbf{V}}) - \eta\lambda_{\min}(\mathbf{A}_t^{\perp}\mathbf{A}_t^{\perp\top}), \\ &1 - \eta\lambda_{\min}(\mathbf{B}_t^{\perp\top}\mathbf{B}_t^{\perp}) - \eta\lambda_{r^*}^2(\mathbf{A}_t^{\mathbf{U}}).\end{aligned}$$

By the step-size hypothesis these coefficients are nonnegative, and each is at most $1 - \eta s_t$ after dropping the nonnegative term not included in s_t.

The deterministic first-order non-signal groups from Lemma 6 are

$$\begin{aligned}&\eta\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{B}_t^{\mathbf{V}}\|_{op}e_t + \eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op}e_t, \\ &\eta\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op} + \eta\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}e_t, \\ &\eta\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{B}_t^{\mathbf{V}}\|_{op}e_t + \eta\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}.\end{aligned}$$

Therefore, they are bounded by

$$C\eta P_t(\alpha_t+\beta_t)H_t,$$

where $H_t = e_t + \zeta_t$. The one-quantization and pure-quantization groups are

$$\begin{aligned}&\eta^2\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)e_t \\ &\quad + \eta\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} + \eta^2\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}, \\ &\eta^2\left[\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\right)e_t + \|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\right] \\ &\quad + \eta\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \eta\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} + \eta^2\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}, \\ &\eta^2\left[\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}\right)e_t + \|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op}\right] \\ &\quad + \eta\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{E}_{t,\mathbf{A}}\|_{op} + \eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op} + \eta^2\|\mathbf{E}_{t,\mathbf{A}}\|_{op}\|\mathbf{E}_{t,\mathbf{B}}\|_{op}.\end{aligned}$$

They are bounded by

$$C\eta^2 P_t q_t K_t^{\mathrm{sep}} + C\eta P_t q_t + C\eta^2 q_t^2.$$

The quadratic residual groups are

$$\eta^2 \left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{B}_t^{\perp}\|_{op} \right) e_t^2,$$
$$\eta^2 \left[\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op} \right) e_t + \left(\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op} \right) \|\mathbf{R}_t^{(3)}\|_{op} \right] e_t,$$
$$\eta^2 \left[\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\mathbf{U}}\|_{op} \right) e_t + \left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{A}_t^{\perp}\|_{op} \right) \|\mathbf{R}_t^{(3)}\|_{op} \right] e_t.$$

They are bounded by

$$C\eta^2 P_t^2 H_t \left(K_t^{\mathrm{sep}} + (K_t^{\mathrm{sep}})^2 \right).$$

The covariance-linear groups are

$$\begin{aligned}
&\eta\delta \left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op} \right) \left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op} e_t + \eta\|\mathbf{A}_t^{\perp}\|_{op} e_t + \eta\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \right) e_t \\
&\quad + \eta\delta\|\mathbf{B}_t^{\perp}\|_{op} \left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op} e_t + \eta\|\mathbf{A}_t^{\perp}\|_{op} e_t + \eta\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \right) \|\mathbf{R}_t^{(3)}\|_{op} \\
&\quad + 2\eta\delta \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op} \right) \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \eta\|\mathbf{B}_t^{\mathbf{V}}\|_{op} e_t + \eta\|\mathbf{B}_t^{\perp}\|_{op} e_t + \eta\|\mathbf{E}_{t,\mathbf{A}}\|_{op} \right) e_t, \\
&\eta\delta \left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op} \right) \left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op} e_t + \eta\|\mathbf{A}_t^{\perp}\|_{op} e_t + \eta\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \right) e_t \\
&\quad + \eta\delta\|\mathbf{B}_t^{\perp}\|_{op} \left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op} e_t + \eta\|\mathbf{A}_t^{\perp}\|_{op} e_t + \eta\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \right) \|\mathbf{R}_t^{(3)}\|_{op} \\
&\quad + 2\eta\delta \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op} \right) \left(\|\mathbf{A}_t^{\perp}\|_{op} + \eta\|\mathbf{B}_t^{\mathbf{V}}\|_{op} e_t + \eta\|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op} + \eta\|\mathbf{E}_{t,\mathbf{A}}\|_{op} \right) e_t, \\
&\eta\delta \left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op} \right) \left(\|\mathbf{B}_t^{\perp}\|_{op} + \eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op} e_t + \eta\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op} + \eta\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \right) e_t \\
&\quad + \eta\delta\|\mathbf{B}_t^{\perp}\|_{op} \left(\|\mathbf{B}_t^{\perp}\|_{op} + \eta\|\mathbf{A}_t^{\mathbf{U}}\|_{op} e_t + \eta\|\mathbf{A}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op} + \eta\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \right) \|\mathbf{R}_t^{(3)}\|_{op} \\
&\quad + \eta\delta \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op} \right) \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \eta\|\mathbf{B}_t^{\mathbf{V}}\|_{op} e_t + \eta\|\mathbf{B}_t^{\perp}\|_{op} e_t + \eta\|\mathbf{E}_{t,\mathbf{A}}\|_{op} \right) \left(e_t + \|\mathbf{R}_t^{(3)}\|_{op} \right).
\end{aligned}$$

In the third residual recursion the leading $\|\mathbf{B}_t^{\perp}\|_{op}$ term is kept as a separated non-signal factor, and all other displayed factors are bounded by the common envelopes. Thus they are bounded by

$$C\eta\delta P_t \left(P_t + K_t^{\mathrm{sep}} + \eta P_t K_t^{\mathrm{sep}} + \eta q_t \right) H_t$$

The covariance-quadratic groups have the common form

$$\eta^2\delta^2 \left[\left(2\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \|\mathbf{B}_t^{\perp}\|_{op} \right) e_t + \|\mathbf{B}_t^{\perp}\|_{op}\|\mathbf{R}_t^{(3)}\|_{op} \right] \left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op} \right) \left(e_t + \|\mathbf{R}_t^{(3)}\|_{op} \right),$$

with the last factor equal to e_t in the first two recursions. Since $\beta_t\zeta_t \le K_t^{\mathrm{sep}} H_t$, the envelope reduction is bounded by

$$C\eta^2\delta^2 P_t^2 H_t \left(K_t^{\mathrm{sep}} + (K_t^{\mathrm{sep}})^2 \right).$$

Combining these estimates proves

$$\mathcal{M}_{t+1} \le (1 - \eta s_t) e_t + \widetilde{\Phi}_t.$$

It remains to prove the separated perpendicular recursions. For $\mathbf{A}_{t+1}^{\perp}$, the deterministic full-precision part of the $\mathbf{A}_{t+1}^{\perp}$ update equals

$$\mathbf{A}_t^{\perp}\left[\mathbf{I}_r - \eta\mathbf{B}_t^{\mathbf{V}}(\mathbf{B}_t^{\mathbf{V}})^\top - \eta\mathbf{B}_t^{\perp}(\mathbf{B}_t^{\perp})^\top\right].$$

The step-size hypothesis implies that the bracketed matrix lies between $\mathbf{0}$ and $\mathbf{I}_r$ in Loewner order, so its operator norm is at most one. Thus this part contributes at most α_t. The quantization term is bounded by q_t. The covariance factor satisfies

$$\begin{aligned}
&\left\|(\mathbf{U}\mathbf{R}_t^* + \mathbf{U}_\perp\mathbf{R}_t^{(1)})(\mathbf{B}_t^{\mathbf{V}})^\top + (\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_\perp\mathbf{R}_t^{(3)})(\mathbf{B}_t^{\perp})^\top\right\|_{op}\\
\le&\left(\|\mathbf{R}_t^*\|_{op} + \|\mathbf{R}_t^{(1)}\|_{op}\right)\|\mathbf{B}_t^{\mathbf{V}}\|_{op} + \left(\|\mathbf{R}_t^{(2)}\|_{op} + \|\mathbf{R}_t^{(3)}\|_{op}\right)\|\mathbf{B}_t^{\perp}\|_{op}\\
\le& C\left[P_t(e_t + P_t\alpha_t) + (P_t\beta_t + \zeta_t)\beta_t\right],
\end{aligned}$$

where we used $\mathbf{R}_t^{(1)} = \mathbf{A}_t^{\perp}\mathbf{B}_t^{\mathbf{V}}$ and $\mathbf{R}_t^{(2)} = \mathbf{A}_t^{\mathbf{U}}\mathbf{B}_t^{\perp}$. Combining these bounds gives

$$\|\mathbf{A}_{t+1}^{\perp}\|_{op} \le \alpha_t + \eta q_t + C\eta\delta\left[P_t(e_t + P_t\alpha_t) + (P_t\beta_t + \zeta_t)\beta_t\right].$$

For $\mathbf{B}_{t+1}^{\perp}$, the deterministic full-precision part is

$$\left[\mathbf{I}_r - \eta(\mathbf{A}_t^{\mathbf{U}})^\top\mathbf{A}_t^{\mathbf{U}} - \eta(\mathbf{A}_t^{\perp})^\top\mathbf{A}_t^{\perp}\right]\mathbf{B}_t^{\perp},$$

whose contribution is at most β_t by the same step-size argument. The quantization term is bounded by q_t, and the covariance factor satisfies

$$\begin{aligned}
&\left\|\left[(\mathbf{A}_t^{\mathbf{U}})^\top\mathbf{U}^\top + (\mathbf{A}_t^{\perp})^\top\mathbf{U}_\perp^\top\right](\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d)(\mathbf{U}\mathbf{R}_t^{(2)} + \mathbf{U}_\perp\mathbf{R}_t^{(3)})\right\|_{op}\\
\le&\delta\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op} + \|\mathbf{A}_t^{\perp}\|_{op}\right)\left(\|\mathbf{R}_t^{(2)}\|_{op} + \|\mathbf{R}_t^{(3)}\|_{op}\right)\\
\le& C\delta P_t(P_t\beta_t + \zeta_t),
\end{aligned}$$

where $\|\mathbf{R}_t^{(2)}\|_{op} \le \|\mathbf{A}_t^{\mathbf{U}}\|_{op}\|\mathbf{B}_t^{\perp}\|_{op} \le P_t\beta_t$. Hence

$$\|\mathbf{B}_{t+1}^{\perp}\|_{op} \le \beta_t + \eta q_t + C\eta\delta P_t(P_t\beta_t + \zeta_t).$$

Finally, $\mathbf{R}_{t+1}^{(3)} = \mathbf{A}_{t+1}^{\perp}\mathbf{B}_{t+1}^{\perp}$ by definition, so the product bound follows from submultiplicativity. ■

Theorem 15 (General-rank finite-horizon induction) *Under Assumptions 2.1 and 2.2, consider the spectral initialization* (spectral initialization) *and the quantized GD iteration* (quantized GD) *with $\eta_1 = \eta_2 = \eta > 0$. Assume $r \ge r^*$, write $\lambda_i^* = \lambda_i(\Delta)$ and $\kappa = \lambda_1^*/\lambda_{r^*}^*$, and set $\epsilon_{\max} = \max\{\epsilon_d, \epsilon_A, \epsilon_B, \epsilon_o, \epsilon_w, \epsilon_l, \epsilon_r\}$. There are universal constants $c_0 \in (0,1)$ and $C > 1$ such that the following holds. For any $0 < \epsilon < 1$ and $\tau \in \mathbb{N}_0$, define*

$$\delta_q = \epsilon + (1+\epsilon)\left(2\epsilon_{\max}\sqrt{d} + \epsilon_{\max}^2 d\right), \qquad P_\star = 1 + 2\sqrt{\lambda_1^*}, \qquad \gamma = \frac{\lambda_{r^*}^*}{64\kappa}, \qquad \rho = \eta\gamma.$$

Let

$$\begin{aligned}\mathfrak{e}_{q,r} =& C\epsilon_{\max}(1+\epsilon_{\max}\sqrt{r})^3(1+\epsilon)(1+\epsilon_{\max}\sqrt{d})^2 P_\star \\ &\cdot\Big[(1+\sqrt{r})\|\mathbf{W}^*\|_F + \sqrt{d}\|\mathbf{W}^*\|_{op} + (1+\sqrt{r})\|\mathbf{W}^\natural\|_F + \sqrt{r}\lambda_1^* + rP_\star^2\Big]. \end{aligned} \tag{37}$$

Define deterministic envelopes $\{\mathcal{E}_t, \mathcal{A}_t, \mathcal{B}_t, \mathcal{Z}_t, \Omega_t\}$ *recursively by*

$$\mathcal{E}_0 = \epsilon\lambda_1^*, \qquad \mathcal{A}_0 = \frac{\epsilon\lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}}, \qquad \mathcal{B}_0 = 0, \qquad \mathcal{Z}_0 = 0.$$

For $0 \le t \le \tau$*, set*

$$\mathcal{K}_t = \mathcal{E}_t + \mathcal{A}_t + \mathcal{B}_t + \mathcal{Z}_t, \qquad \mathcal{H}_t = \mathcal{E}_t + \mathcal{Z}_t.$$

For $0 \le t < \tau$*, define*

$$\begin{aligned}\Omega_t =& C\eta P_\star(\mathcal{A}_t+\mathcal{B}_t)\mathcal{H}_t + C\eta^2 P_\star \mathfrak{e}_{q,r}\mathcal{K}_t + C\eta P_\star\mathfrak{e}_{q,r} + C\eta^2\mathfrak{e}_{q,r}^2 \\ &+ C\eta^2 P_\star^2\mathcal{H}_t(\mathcal{K}_t+\mathcal{K}_t^2) + C\eta\delta_q P_\star\left(P_\star + \mathcal{K}_t + \eta P_\star\mathcal{K}_t + \eta\mathfrak{e}_{q,r}\right)\mathcal{H}_t + C\eta^2\delta_q^2 P_\star^2\mathcal{H}_t(\mathcal{K}_t+\mathcal{K}_t^2), \\ \mathcal{E}_{t+1} =& (1-4\rho)\mathcal{E}_t + \Omega_t, \\ \mathcal{A}_{t+1} =& \mathcal{A}_t + \eta\mathfrak{e}_{q,r} + C\eta\delta_q\left[P_\star(\mathcal{E}_t + P_\star\mathcal{A}_t) + (P_\star\mathcal{B}_t + \mathcal{Z}_t)\mathcal{B}_t\right], \\ \mathcal{B}_{t+1} =& \mathcal{B}_t + \eta\mathfrak{e}_{q,r} + C\eta\delta_q P_\star(P_\star\mathcal{B}_t + \mathcal{Z}_t), \qquad \mathcal{Z}_{t+1} = \mathcal{A}_{t+1}\mathcal{B}_{t+1}. \end{aligned} \tag{38}$$

Suppose

$$\begin{aligned} &0 < \epsilon \le \frac{1}{2}, \qquad 0 \le \epsilon_{\max} \le 1, \\ &\eta \le \min\left\{\frac{c_0}{\kappa\lambda_1^*}, \frac{1}{4P_\star^2}\right\}, \qquad \mathcal{A}_0 \le c_0\frac{\lambda_{r^*}^*}{\sqrt{\lambda_1^*}}, \\ &C(1+\delta_q)P_\star\sum_{j=0}^{\tau-1}\eta(3\mathcal{E}_j + \mathcal{Z}_j) + C\tau\eta\mathfrak{e}_{q,r} \le c_0\min\left\{\sqrt{\lambda_1^*}, \frac{\lambda_{r^*}^*}{\sqrt{\lambda_1^*}}\right\}, \end{aligned} \tag{39}$$

Then with probability at least $1 - 2C'\exp(-\epsilon^2 N)$ *for a universal constant* $C' > 0$*, on the same high-probability event, for every integer* $0 \le t \le \tau$*,*

$$\begin{aligned} \mathcal{M}_t \le \mathcal{E}_t, &\qquad \|\mathbf{R}_t^{(3)}\|_{op} \le \mathcal{Z}_t, \\ \|\mathbf{A}_t^\perp\|_{op} \le \mathcal{A}_t, &\qquad \|\mathbf{B}_t^\perp\|_{op} \le \mathcal{B}_t, \\ \|\mathbf{A}_t\|_{op} \le P_\star, &\qquad \|\mathbf{B}_t\|_{op} \le P_\star, \\ \lambda_{r^*}(\mathbf{A}_t^{\mathbf{U}}) \ge \frac{\lambda_{r^*}^*}{4\sqrt{\lambda_1^*}}, &\qquad \lambda_{r^*}(\mathbf{B}_t^{\mathbf{V}}) \ge \frac{\lambda_{r^*}^*}{4\sqrt{\lambda_1^*}}. \end{aligned} \tag{40}$$

Proof By Lemma 8, with probability at least $1 - 2C'\exp(-\epsilon^2 N)$,

$$\delta = \left\|\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d\right\|_{op} \le \delta_q, \qquad \|\widetilde{\mathbf{X}}\|_{op} \le \sqrt{N(1+\epsilon)}, \qquad \|\widetilde{\mathbf{X}}\|_F \le \sqrt{Nd(1+\epsilon)}.$$

On this event, Assumption 2.2 gives

$$\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op} \le \sqrt{N(1+\epsilon)}(1+\epsilon_{\max}\sqrt{d}).$$

We prove (40) by induction.

For $t = 0$, Lemma 9 gives

$$\mathcal{M}_0 \le \mathcal{E}_0, \qquad \|\mathbf{A}_0^{\perp}\|_{op} \le \mathcal{A}_0, \qquad \mathbf{B}_0^{\perp} = \mathbf{0}, \qquad \mathbf{R}_0^{(3)} = \mathbf{0}.$$

Also,

$$\|\mathbf{A}_0\|_{op},\ \|\mathbf{B}_0\|_{op} \le \sqrt{(1+\epsilon)\lambda_1^*} \le P_\star.$$

Since $\epsilon \le 1/2$ and $\lambda_{r^*}^* \le \lambda_1^*$,

$$\sqrt{(1-\epsilon)\lambda_{r^*}^*} \ge \frac{1}{\sqrt{2}} \frac{\lambda_{r^*}^*}{\sqrt{\lambda_1^*}}.$$

Choosing $c_0 \le 1/\sqrt{2} - 1/2$ in (39), Lemma 9 and the condition on $\mathcal{A}_0$ imply the stronger base signal bound

$$\lambda_{r^*}(\mathbf{A}_0^{\mathbf{U}}),\ \lambda_{r^*}(\mathbf{B}_0^{\mathbf{V}}) \ge \frac{\lambda_{r^*}^*}{2\sqrt{\lambda_1^*}}.$$

This proves the base case, since the theorem only requires the weaker lower bound with constant $1/4$.

Envelope consequences at time t. Assume the claims hold for every time $0 \le j \le t < \tau$. Then, in the notation of Corollary 14,

$$P_t = \max\{1, \|\mathbf{A}_t\|_{op}, \|\mathbf{B}_t\|_{op}\} \le P_\star,\ e_t = \mathcal{M}_t \le \mathcal{E}_t,\ \|\mathbf{A}_t^{\perp}\|_{op} \le \mathcal{A}_t,\ \|\mathbf{B}_t^{\perp}\|_{op} \le \mathcal{B}_t,\ \|\mathbf{R}_t^{(3)}\|_{op} \le \mathcal{Z}_t,$$

and

$$s_t = \min\left\{\lambda_{r^*}^2(\mathbf{A}_t^{\mathbf{U}}), \lambda_{r^*}^2(\mathbf{B}_t^{\mathbf{V}})\right\} \ge \frac{(\lambda_{r^*}^*)^2}{16\lambda_1^*} = 4\gamma.$$

The rank of $\mathbf{A}_t$ and $\mathbf{B}_t$ is at most r, so the induction hypothesis also gives

$$\|\mathbf{A}_t\|_F,\ \|\mathbf{B}_t\|_F \le \sqrt{r} P_\star.$$

Consequently,

$$\|\mathbf{A}_t\|_{op} + \epsilon_A \|\mathbf{A}_t\|_F,\ \|\mathbf{B}_t\|_{op} + \epsilon_B \|\mathbf{B}_t\|_F \le P_\star(1 + \epsilon_{\max}\sqrt{r}),$$

and the same estimate with $\mathbf{A}_t$ or $\mathbf{B}_t$ replaced by its Frobenius norm costs at most another factor $\sqrt{r}$. Also,

$$\|\Delta\|_{op} = \lambda_1^*, \qquad \|\widetilde{\mathbf{Y}}\|_F = \|\widetilde{\mathbf{X}}\mathbf{W}^*\|_F \le \sqrt{N(1+\epsilon)}\|\mathbf{W}^*\|_F,$$

and

$$\frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N} \le \frac{\sqrt{1+\epsilon}(1+\epsilon_{\max}\sqrt{d})}{\sqrt{N}}, \qquad \frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N} \le (1+\epsilon)(1+\epsilon_{\max}\sqrt{d})^2.$$

Quantization-error bound. Since $\epsilon_i \le \epsilon_{\max}$ and $\epsilon_1, \epsilon_2, \epsilon_3 \le C\epsilon_{\max}(1+\epsilon_{\max})^3$, the first group in the bound for $\mathbf{E}_{t,\mathbf{A}}$ in Lemma 7 satisfies

$$\frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}}{N}(\|\mathbf{B}_t\|_{op}+\epsilon_B\|\mathbf{B}_t\|_F)\left(\epsilon_1\|\widetilde{\mathbf{Y}}\|_F+\epsilon_d\|\widetilde{\mathbf{X}}\|_F\|\mathbf{W}^*\|_{op}+\epsilon_2\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}\|\mathbf{W}^\natural\|_F\right)$$
$$\le C\epsilon_{\max}(1+\epsilon_{\max}\sqrt{r})^3(1+\epsilon)(1+\epsilon_{\max}\sqrt{d})^2P_\star\left(\|\mathbf{W}^*\|_F+\sqrt{d}\|\mathbf{W}^*\|_{op}+\|\mathbf{W}^\natural\|_F\right).$$

The second group in the same bound satisfies

$$\frac{\|\mathcal{Q}_d(\widetilde{\mathbf{X}})\|_{op}^2}{N}\left[\epsilon_3(\|\mathbf{B}_t\|_{op}+\epsilon_B\|\mathbf{B}_t\|_F)^2\|\mathbf{A}_t\|_F+\epsilon_B\|\mathbf{B}_t\|_F\left(\|\Delta\|_{op}+\|\mathbf{A}_t\|_{op}(2\|\mathbf{B}_t\|_{op}+\epsilon_B\|\mathbf{B}_t\|_F)\right)\right]$$
$$\le C\epsilon_{\max}(1+\epsilon_{\max}\sqrt{r})^3(1+\epsilon)(1+\epsilon_{\max}\sqrt{d})^2P_\star\left(\sqrt{r}\,\lambda_1^*+rP_\star^2\right).$$

Hence

$$\|\mathbf{E}_{t,\mathbf{A}}\|_{op} \le C\epsilon_{\max}(1+\epsilon_{\max}\sqrt{r})^3(1+\epsilon)(1+\epsilon_{\max}\sqrt{d})^2P_\star$$
$$\cdot\Big[(1+\sqrt{r})\|\mathbf{W}^*\|_F+\sqrt{d}\|\mathbf{W}^*\|_{op}+(1+\sqrt{r})\|\mathbf{W}^\natural\|_F+\sqrt{r}\lambda_1^*+rP_\star^2\Big].$$

For $\mathbf{E}_{t,\mathbf{B}}$, the terms containing $\|\Delta\|_{op}$ are bounded by the $\sqrt{r}\lambda_1^*$ term, the two response terms by $(1+\sqrt{r})\|\mathbf{W}^*\|_F$, the data-error term by $\sqrt{d}\|\mathbf{W}^*\|_{op}$, the two $\mathbf{W}^\natural$ terms by $(1+\sqrt{r})\|\mathbf{W}^\natural\|_F$, and the remaining parameter-only terms by $rP_\star^2$. Thus Lemma 7 gives

$$\|\mathbf{E}_{t,\mathbf{B}}\|_{op} \le C\epsilon_{\max}(1+\epsilon_{\max}\sqrt{r})^3(1+\epsilon)(1+\epsilon_{\max}\sqrt{d})^2P_\star$$
$$\cdot\Big[(1+\sqrt{r})\|\mathbf{W}^*\|_F+\sqrt{d}\|\mathbf{W}^*\|_{op}+(1+\sqrt{r})\|\mathbf{W}^\natural\|_F+\sqrt{r}\lambda_1^*+rP_\star^2\Big].$$

Increasing the universal constant C in (37) if necessary, this yields

$$q_t = \max\{\|\mathbf{E}_{t,\mathbf{A}}\|_{op}, \|\mathbf{E}_{t,\mathbf{B}}\|_{op}\} \le \mathfrak{e}_{q,r}. \tag{41}$$

The same argument applies at every earlier time $0 \le j \le t$ under the induction hypotheses, so $q_j \le \mathfrak{e}_{q,r}$ for all such j.

Residual bound for $\mathcal{M}_{t+1}$. The step-size hypothesis in Corollary 14 holds because

$$\eta\left(\|\mathbf{B}_t^{\mathbf{V}}\|_{op}^2+\|\mathbf{B}_t^{\perp}\|_{op}^2\right)+\eta\left(\|\mathbf{A}_t^{\mathbf{U}}\|_{op}^2+\|\mathbf{A}_t^{\perp}\|_{op}^2\right) \le 4\eta P_\star^2 \le 1.$$

Apply Corollary 14 with

$$\alpha_t = \mathcal{A}_t, \qquad \beta_t = \mathcal{B}_t, \qquad \zeta_t = \mathcal{Z}_t.$$

Since every term in $\widetilde{\Phi}_t$ has nonnegative coefficients and is monotone in $P_t, q_t, e_t, \delta, \alpha_t, \beta_t, \zeta_t$, the induction bounds, $\delta \le \delta_q$, and (41) imply

$$\widetilde{\Phi}_t \le \Omega_t.$$

Moreover $1-4\rho \ge 0$ after taking c_0 small enough, because $\eta \le c_0/(\kappa\lambda_1^*)$ and $\gamma = \lambda_{r^*}^*/(64\kappa) \le \lambda_1^*$. Therefore Corollary 14 gives

$$\mathcal{M}_{t+1} \le (1-\eta s_t)e_t + \widetilde{\Phi}_t \le (1-4\rho)\mathcal{E}_t + \Omega_t = \mathcal{E}_{t+1}.$$

Perpendicular and $\mathbf{R}_{t+1}^{(3)}$ bounds. The perpendicular bounds in the same corollary give

$$\|\mathbf{A}_{t+1}^{\perp}\|_{op} \le \mathcal{A}_t + \eta\mathfrak{e}_{q,r} + C\eta\delta_q\left[P_\star(\mathcal{E}_t + P_\star\mathcal{A}_t) + (P_\star\mathcal{B}_t + \mathcal{Z}_t)\mathcal{B}_t\right] = \mathcal{A}_{t+1},$$

and

$$\|\mathbf{B}_{t+1}^{\perp}\|_{op} \le \mathcal{B}_t + \eta\mathfrak{e}_{q,r} + C\eta\delta_q P_\star(P_\star\mathcal{B}_t + \mathcal{Z}_t) = \mathcal{B}_{t+1}.$$

Therefore

$$\|\mathbf{R}_{t+1}^{(3)}\|_{op} = \|\mathbf{A}_{t+1}^{\perp}\mathbf{B}_{t+1}^{\perp}\|_{op} \le \mathcal{A}_{t+1}\mathcal{B}_{t+1} = \mathcal{Z}_{t+1}.$$

Bounded and nonsingular tube at $t+1$. It remains to keep the iterates inside the bounded, nonsingular tube. For every $0 \le j \le t$, the already proved block bounds give

$$\|\mathbf{A}_j\mathbf{B}_j - \Delta\|_{op} \le \|\mathbf{R}_j^*\|_{op} + \|\mathbf{R}_j^{(1)}\|_{op} + \|\mathbf{R}_j^{(2)}\|_{op} + \|\mathbf{R}_j^{(3)}\|_{op} \le 3\mathcal{E}_j + \mathcal{Z}_j.$$

Projecting the component updates in Lemma 5 back to the ambient coordinates gives

$$\begin{aligned}
\mathbf{A}_{j+1} - \mathbf{A}_j &= -\eta(\mathbf{A}_j\mathbf{B}_j - \Delta)\mathbf{B}_j^\top - \eta(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d)(\mathbf{A}_j\mathbf{B}_j - \Delta)\mathbf{B}_j^\top + \eta\mathbf{E}_{j,\mathbf{A}},\\
\mathbf{B}_{j+1} - \mathbf{B}_j &= -\eta\mathbf{A}_j^\top(\mathbf{A}_j\mathbf{B}_j - \Delta) - \eta\mathbf{A}_j^\top(\widehat{\mathbf{\Sigma}}_{\mathcal{Q}} - \mathbf{I}_d)(\mathbf{A}_j\mathbf{B}_j - \Delta) + \eta\mathbf{E}_{j,\mathbf{B}}.
\end{aligned}$$

Hence, using $P_j \le P_\star$ and $q_j \le \mathfrak{e}_{q,r}$,

$$\|\mathbf{A}_{j+1} - \mathbf{A}_j\|_{op} + \|\mathbf{B}_{j+1} - \mathbf{B}_j\|_{op} \le C\eta(1+\delta_q)P_\star(3\mathcal{E}_j + \mathcal{Z}_j) + C\eta\mathfrak{e}_{q,r}.$$

Summing from $j = 0$ to t and using (39) gives

$$\|\mathbf{A}_{t+1} - \mathbf{A}_0\|_{op} + \|\mathbf{B}_{t+1} - \mathbf{B}_0\|_{op} \le c_0 \min\left\{\sqrt{\lambda_1^*}, \frac{\lambda_{r^*}^*}{\sqrt{\lambda_1^*}}\right\}.$$

Therefore, by Lemma 9 and Weyl's singular-value inequality,

$$\|\mathbf{A}_{t+1}\|_{op},\ \|\mathbf{B}_{t+1}\|_{op} \le \sqrt{(1+\epsilon)\lambda_1^*} + c_0\sqrt{\lambda_1^*} \le P_\star,$$

and, after taking c_0 small enough,

$$\lambda_{r^*}(\mathbf{A}_{t+1}^{\mathbf{U}}),\ \lambda_{r^*}(\mathbf{B}_{t+1}^{\mathbf{V}}) \ge \frac{\lambda_{r^*}^*}{2\sqrt{\lambda_1^*}} - c_0\frac{\lambda_{r^*}^*}{\sqrt{\lambda_1^*}} \ge \frac{\lambda_{r^*}^*}{4\sqrt{\lambda_1^*}}.$$

This closes the induction. ■

Corollary 16 (Whole Frobenius error from the general-rank induction) *Under the conditions of Theorem 15, on the same high-probability event, for every integer $0 \le t \le \tau$,*

$$\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F \le \left(3r^*\mathcal{E}_t^2 + r\mathcal{Z}_t^2\right)^{1/2} \le \sqrt{3r^*}\,\mathcal{E}_t + \sqrt{r}\,\mathcal{Z}_t.$$

Proof The orthogonal decompositions induced by $\mathbf{U}, \mathbf{U}_\perp, \mathbf{V}, \mathbf{V}_\perp$ give

$$\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F^2 = \|\mathbf{R}_t^*\|_F^2 + \|\mathbf{R}_t^{(1)}\|_F^2 + \|\mathbf{R}_t^{(2)}\|_F^2 + \|\mathbf{R}_t^{(3)}\|_F^2.$$

The first three blocks have rank at most r^*, while $\mathbf{R}_t^{(3)} = \mathbf{A}_t^\perp \mathbf{B}_t^\perp$ has rank at most r. Hence $\|\mathbf{X}\|_F \le \sqrt{\mathrm{rank}(\mathbf{X})}\|\mathbf{X}\|_{op}$ gives

$$\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F^2 \le r^* \left(\|\mathbf{R}_t^*\|_{op}^2 + \|\mathbf{R}_t^{(1)}\|_{op}^2 + \|\mathbf{R}_t^{(2)}\|_{op}^2\right) + r\|\mathbf{R}_t^{(3)}\|_{op}^2.$$

Theorem 15 gives

$$\|\mathbf{R}_t^*\|_{op},\ \|\mathbf{R}_t^{(1)}\|_{op},\ \|\mathbf{R}_t^{(2)}\|_{op} \le \mathcal{E}_t, \qquad \|\mathbf{R}_t^{(3)}\|_{op} \le \mathcal{Z}_t.$$

Substituting these estimates proves the first inequality, and the second follows from $\sqrt{x+y} \le \sqrt{x} + \sqrt{y}$. ■

Corollary 17 (Simplified whole Frobenius error, general rank) *Under Assumptions 2.1 and 2.2, consider the spectral initialization* (spectral initialization) *and the quantized GD iteration* (quantized GD) *with $\eta_1 = \eta_2 = \eta > 0$. Assume $r \ge r^*$, write $\lambda_i^* = \lambda_i(\Delta)$, $\kappa = \lambda_1^*/\lambda_{r^*}^*$, and*

$$\gamma = \frac{\lambda_{r^*}^*}{64\kappa}.$$

Suppose the signal quantities are at constant level: λ_1^*, $1/\lambda_{r^*}^*$, κ, r, and $\|\mathbf{W}^\natural\|_{op}$ are bounded by absolute constants. Define

$$P_\star = 1 + 2\sqrt{\lambda_1^*}.$$

There exist constants $c \in (0,1)$ and $C > 1$, depending only on the preceding constant-level bounds and the universal constants in Theorem 15, such that the following holds. For an integer $t \ge 0$, define

$$z = \epsilon_{\max}\sqrt{d}, \qquad T_t = t\eta, \qquad D_t = z + \epsilon T_t z + (T_t z)^2.$$

Choose ϵ, η satisfying

$$0 < \epsilon \le c \min\left\{1,\ \frac{1}{\kappa},\ \frac{\lambda_{r^*}^*}{\kappa(\lambda_1^* + \kappa^2)},\ \frac{\gamma}{P_\star^2},\ \frac{(\lambda_{r^*}^*)^{3/2}}{(\lambda_1^*)^{3/2}}\right\},$$

and

$$0 < \eta \le c \min\left\{1,\ \frac{1}{\kappa\lambda_1^*},\ \frac{1}{\lambda_1^*},\ \frac{1}{P_\star^2}\right\}.$$

If the low-precision errors and horizon obey

$$z \le c \min\left\{1,\ \frac{1}{\kappa},\ \frac{\lambda_{r^*}^*}{\kappa(\lambda_1^* + \kappa^2)},\ \frac{\gamma}{P_\star^2}\right\},$$

$$T_t(\epsilon + z) \le c, \qquad \epsilon + T_t z + D_t \le c\frac{\gamma}{P_\star},$$

and

$$\epsilon + T_t z + T_t D_t \le \frac{c}{P_\star} \min\left\{ \sqrt{\lambda_1^*}, \frac{\lambda_{r^*}^*}{\sqrt{\lambda_1^*}} \right\},$$

then with probability at least $1 - C\exp(-c\epsilon^2 N)$*, on the same high-probability event,*

$$\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F \le C\sqrt{r}\left[(1-\eta\gamma)^t + D_t\right].$$

In particular, if additionally

$$D_t \le c(1-\eta\gamma)^t,$$

then the low-precision iterates match the full-precision linear rate at time t*:*

$$\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F \le C\sqrt{r}(1-\eta\gamma)^t.$$

Proof Let c_0 denote the small universal constant in Theorem 15. Throughout the proof, constants denoted by C depend only on the constant-level bounds in the statement and on the universal constants in Theorem 15. We apply Theorem 15 with horizon $\tau = t$.

By taking the statement constant c small enough, the displayed assumptions imply $\epsilon \le 1/2$, $\eta \le 1$, $0 \le \epsilon_{\max} \le z \le 1$, and the covariance perturbation satisfies

$$\delta_q = \epsilon + (1+\epsilon)(2z + z^2) \le \epsilon + 5z.$$

The constant-level assumptions imply

$$\|\Delta\|_F \le \sqrt{r}\,\lambda_1^* = \mathcal{O}(1), \qquad \|\mathbf{W}^*\|_{op} \le \|\mathbf{W}^\natural\|_{op} + \lambda_1^* = \mathcal{O}(1),$$

and, because $\mathbf{W}^\natural \in \mathbb{R}^{d\times k}$,

$$\|\mathbf{W}^\natural\|_F \le \sqrt{d}\,\|\mathbf{W}^\natural\|_{op} = \mathcal{O}(\sqrt{d}), \qquad \|\mathbf{W}^*\|_F \le \|\mathbf{W}^\natural\|_F + \|\Delta\|_F = \mathcal{O}(\sqrt{d}).$$

Substituting these bounds and $z \le c$ into (37) gives

$$\mathfrak{e}_{q,r} \le Cz.$$

We first record explicit bounds for the recursive envelopes in Theorem 15. For $0 \le s \le t$, set $T_s = s\eta$ and

$$D_s = z + \epsilon T_s z + (T_s z)^2.$$

We claim that, after decreasing c if necessary,

$$\mathcal{E}_s \le C\left[(1-\eta\gamma)^s\epsilon + D_s\right], \quad \mathcal{A}_s \le C(\epsilon + T_s z), \quad \mathcal{B}_s \le CT_s z, \quad \mathcal{Z}_s \le C\left[\epsilon T_s z + (T_s z)^2\right]. \tag{42}$$

The bounds hold at $s = 0$ by the initialization in Theorem 15. Suppose they hold up to time $s < t$. The assumptions imply

$$\mathcal{K}_s \le C\left[(1-\eta\gamma)^s\epsilon + \epsilon + T_t z + D_t\right], \qquad \mathcal{A}_s + \mathcal{B}_s \le C(\epsilon + T_t z).$$

Hence the displayed smallness assumptions, together with $\delta_q \le C(\epsilon + z)$ and $\mathfrak{e}_{q,r} \le Cz$, give

$$P_\star(\mathcal{A}_s + \mathcal{B}_s) \le c_0\gamma, \qquad \eta P_\star^2(\mathcal{K}_s + \mathcal{K}_s^2) \le c_0\gamma,$$

and

$$\delta_q P_\star(P_\star + \mathcal{K}_s + \eta P_\star \mathcal{K}_s + \eta \mathfrak{e}_{q,r}) + \eta \delta_q^2 P_\star^2(\mathcal{K}_s + \mathcal{K}_s^2) \le c_0\gamma,$$

after decreasing c by a universal factor. Here the first estimate uses $\epsilon + T_t z + D_t \le c\gamma/P_\star$, the second also uses $\eta \le c/P_\star^2$, and the covariance estimate also uses $\epsilon + z \le c\gamma/P_\star^2$. Substituting (42) into the recursions (38), using $\mathfrak{e}_{q,r} \le Cz$ and $\delta_q \le C(\epsilon + z)$, and applying the discrete Gronwall inequality with $T_t(\epsilon + z) \le c$, gives

$$\begin{aligned}
\mathcal{A}_{s+1} &\le \mathcal{A}_s + C\eta z + C\eta(\epsilon + z)(\mathcal{E}_s + \mathcal{A}_s + \mathcal{B}_s^2 + \mathcal{Z}_s\mathcal{B}_s) \le C(\epsilon + T_{s+1}z),\\
\mathcal{B}_{s+1} &\le \mathcal{B}_s + C\eta z + C\eta(\epsilon + z)(\mathcal{B}_s + \mathcal{Z}_s) \le CT_{s+1}z.
\end{aligned}$$

Therefore

$$\mathcal{Z}_{s+1} = \mathcal{A}_{s+1}\mathcal{B}_{s+1} \le C\left[\epsilon T_{s+1}z + (T_{s+1}z)^2\right].$$

For the signal-block envelope, the same substitutions yield

$$\Omega_s \le C\eta z + c_0\eta\gamma\left(\mathcal{E}_s + \mathcal{Z}_s\right).$$

Taking c_0 small enough so that the coefficient of $\mathcal{E}_s$ is absorbed into the contraction, and using the discrete convolution bound $\sum_{\ell=0}^{s} \eta(1-\eta\gamma)^\ell \le C/\gamma = C$, while bounding the convolved $\mathcal{Z}_s$ term by its monotone envelope, gives

$$\mathcal{E}_{s+1} \le C\left[(1-\eta\gamma)^{s+1}\epsilon + z + \epsilon T_{s+1}z + (T_{s+1}z)^2\right].$$

This proves (42).

We now verify the conditions of Theorem 15. We have already observed that $0 < \epsilon \le 1/2$ and $0 \le \epsilon_{\max} \le 1$. The displayed step-size condition gives

$$\eta \le \min\left\{\frac{c_0}{\kappa\lambda_1^*}, \frac{1}{4P_\star^2}\right\}$$

after decreasing c if necessary. The initialization condition follows from

$$\mathcal{A}_0 = \frac{\epsilon\lambda_1^*}{\sqrt{(1-\epsilon)\lambda_{r^*}^*}} \le c_0\frac{\lambda_{r^*}^*}{\sqrt{\lambda_1^*}},$$

using the explicit upper bound on ϵ. Finally, by (42),

$$\sum_{s=0}^{t-1} \eta(3\mathcal{E}_s + \mathcal{Z}_s) + T_t\mathfrak{e}_{q,r} \le C\left[\epsilon + T_t z + T_t D_t\right].$$

Here we used $\sum_{s=0}^{t-1} \eta(1-\eta\gamma)^s \le C/\gamma = C$. Since $\delta_q \le C(\epsilon + z) \le 1$ after decreasing c, the displayed tube smallness assumption, with c decreased by another universal factor if necessary, gives the finite-horizon tube condition in (39).

Thus Corollary 16 applies. Combining it with (42) at time t and using $r^* \le r$ and $\mathcal{Z}_t \le CD_t$ gives

$$\|\mathbf{A}_t\mathbf{B}_t - \Delta\|_F \le C\sqrt{r}\left[(1-\eta\gamma)^t\epsilon + D_t\right].$$

Since $\epsilon \le 1$, the first displayed claim follows. If $D_t \le c(1-\eta\gamma)^t$, then the low-precision terms are absorbed into the geometric term, proving the final statement. ■

Appendix E. Technical Lemmas

Lemma 18 (Operator norm contraction) *For any compatible matrices* $\mathbf{X}, \mathbf{P}, \mathbf{Q}$ *and scalars* $\eta_1, \eta_2 \geq 0$ *such that* $\eta_1 \|\mathbf{P}\|_{op}^2 + \eta_2 \|\mathbf{Q}\|_{op}^2 \leq 1$*, we have*

$$\left\| \mathbf{X} - \eta_1 \mathbf{X}\mathbf{P}^\top \mathbf{P} - \eta_2 \mathbf{Q}\mathbf{Q}^\top \mathbf{X} \right\|_{op} \leq \left(1 - \eta_1 \lambda_{\min}(\mathbf{P}^\top \mathbf{P}) - \eta_2 \lambda_{\min}(\mathbf{Q}\mathbf{Q}^\top)\right) \|\mathbf{X}\|_{op}.$$

Proof Let $\mathcal{L}(\mathbf{X}) = \mathbf{X} - \eta_1 \mathbf{X}\mathbf{P}^\top \mathbf{P} - \eta_2 \mathbf{Q}\mathbf{Q}^\top \mathbf{X}$ denote the linear mapping. Since $\mathbf{P}^\top \mathbf{P}$ and $\mathbf{Q}\mathbf{Q}^\top$ are real symmetric and positive semi-definite (PSD) matrices, they admit orthogonal diagonalizations: $\mathbf{P}^\top \mathbf{P} = \mathbf{V}_P \mathbf{\Lambda}_P \mathbf{V}_P^\top$ and $\mathbf{Q}\mathbf{Q}^\top = \mathbf{V}_Q \mathbf{\Lambda}_Q \mathbf{V}_Q^\top$, where $\mathbf{V}_P, \mathbf{V}_Q$ are orthogonal matrices and $\mathbf{\Lambda}_P, \mathbf{\Lambda}_Q$ are diagonal matrices containing the non-negative eigenvalues.

Applying this decomposition to the linear mapping and multiplying by $\mathbf{V}_Q^\top$ on the left and $\mathbf{V}_P$ on the right, we obtain

$$\mathbf{V}_Q^\top \mathcal{L}(\mathbf{X}) \mathbf{V}_P = \widetilde{\mathbf{X}} - \eta_1 \widetilde{\mathbf{X}} \mathbf{\Lambda}_P - \eta_2 \mathbf{\Lambda}_Q \widetilde{\mathbf{X}},$$

where we define $\widetilde{\mathbf{X}} = \mathbf{V}_Q^\top \mathbf{X} \mathbf{V}_P$. Since orthogonal transformations preserve the operator norm, we have $\|\widetilde{\mathbf{X}}\|_{op} = \|\mathbf{X}\|_{op}$ and $\|\mathbf{V}_Q^\top \mathcal{L}(\mathbf{X}) \mathbf{V}_P\|_{op} = \|\mathcal{L}(\mathbf{X})\|_{op}$.

The mapping can now be expressed as a Hadamard product (element-wise multiplication):

$$\mathbf{V}_Q^\top \mathcal{L}(\mathbf{X}) \mathbf{V}_P = \mathbf{C} \circ \widetilde{\mathbf{X}},$$

where the coefficient matrix $\mathbf{C}$ has entries $C_{i,j} = 1 - \eta_2 (\mathbf{\Lambda}_Q)_{ii} - \eta_1 (\mathbf{\Lambda}_P)_{jj}$.

Let $a_i = \eta_2 (\mathbf{\Lambda}_Q)_{ii}$ and $b_j = \eta_1 (\mathbf{\Lambda}_P)_{jj}$. By the assumption $\eta_1 \|\mathbf{P}\|_{op}^2 + \eta_2 \|\mathbf{Q}\|_{op}^2 \leq 1$, we know $a_i \geq 0, b_j \geq 0$ and $\max_i(a_i) + \max_j(b_j) \leq 1$. We define the maximum contraction factor as $\mu_{\max} = 1 - \min_i(a_i) - \min_j(b_j) = 1 - \eta_2 \lambda_{\min}(\mathbf{Q}\mathbf{Q}^\top) - \eta_1 \lambda_{\min}(\mathbf{P}^\top \mathbf{P})$. If $\mu_{\max} = 0$, then $\max_i(a_i) + \max_j(b_j) = \min_i(a_i) + \min_j(b_j) = 1$, so $C_{i,j} = 0$ for every i, j and the contraction map is zero; the claim is immediate.

To bound the operator norm of the Hadamard product, we normalize the coefficients. Let $s_i = \frac{a_i - \min(a)}{\mu_{\max}}$ and $t_j = \frac{b_j - \min(b)}{\mu_{\max}}$. It is straightforward to verify that $s_i \geq 0$, $t_j \geq 0$, and $s_i + t_j \leq 1$. We can then rewrite $C_{i,j}$ as:

$$C_{i,j} = \mu_{\max} \cdot (1 - s_i - t_j).$$

Crucially, the term $(1 - s_i - t_j)$ can be represented as an integral of the product of two indicator functions over $x \in [0, 1]$:

$$1 - s_i - t_j = \int_0^1 \mathbb{1}_{\{x \geq s_i\}} \cdot \mathbb{1}_{\{x \leq 1 - t_j\}} dx.$$

This integral representation reveals that $\mathbf{C}$ is a Schur multiplier. We can substitute this back into the Hadamard product:

$$\mathbf{C} \circ \widetilde{\mathbf{X}} = \mu_{\max} \int_0^1 \mathbf{D}_1(x) \widetilde{\mathbf{X}} \mathbf{D}_2(x) dx,$$

where $\mathbf{D}_1(x) = \text{diag}(\mathbb{1}_{\{x \geq s_i\}})$ and $\mathbf{D}_2(x) = \text{diag}(\mathbb{1}_{\{x \leq 1 - t_j\}})$ are diagonal matrices with entries being either 0 or 1. Consequently, their operator norms are strictly bounded by 1, i.e., $\|\mathbf{D}_1(x)\|_{op} \leq 1$ and $\|\mathbf{D}_2(x)\|_{op} \leq 1$ for any $x \in [0, 1]$.

Finally, taking the operator norm on both sides and applying the triangle inequality for integrals (Minkowski's inequality) alongside the sub-multiplicativity of the operator norm, we obtain:

$$\begin{aligned}
\|\mathcal{L}(\mathbf{X})\|_{op} &= \|\mathbf{C} \circ \widetilde{\mathbf{X}}\|_{op} \\
&\leq \mu_{\max} \int_0^1 \|\mathbf{D}_1(x)\|_{op} \|\widetilde{\mathbf{X}}\|_{op} \|\mathbf{D}_2(x)\|_{op} dx \\
&\leq \mu_{\max} \int_0^1 1 \cdot \|\mathbf{X}\|_{op} \cdot 1 dx \\
&= \mu_{\max} \|\mathbf{X}\|_{op}.
\end{aligned}$$

Substituting the definition of $\mu_{\max}$ completes the proof. ■