Power and Limitations of Aggregation in Compound AI Systems

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Abstract

A common use of compound AI systems is querying multiple homogeneous copies of a model under different prompts and aggregating their outputs. We ask when this unlocks outputs that no single query can elicit. Within a stylized principal-agent framework that captures both prompt-engineering and model-capability limitations, we identify three natural mechanisms by which aggregation expands the set of elicitable outputs—*feasibility expansion*, *support expansion*, and *binding-set contraction*. We prove that strengthened versions of these mechanisms exactly characterize when aggregation adds power. We complement the characterization with an empirical illustration on a toy reference-generation task with LLMs.

1. Introduction

Compound AI systems, which leverage multiple AI components rather than a single model in isolation, present a powerful paradigm for tackling complex tasks [4]. A common setup is to create many copies of the same model, give them different prompts or tools, and aggregate the outputs at test-time. Examples include multi-agent research [2], debate protocols [20, 34], and prompt ensembling [3]. This raises a basic conceptual question: when does aggregating across homogeneous copies of a model unlock outputs that a single query cannot? At first glance, aggregation may seem redundant when models are homogenous. One source of improvement is at the prompting level: aggregation lets the system designer replace complex prompt engineering with simple-but-diverse prompts, overcoming *prompt engineering* limitations [3]. Another is at the output level: aggregation can correct errors such as hallucinations, overcoming *model capability* limitations [20].

To study this question, we extend the principal-agent framework of Kleinberg and Raghavan [35] (Section 2) to capture both prompt-engineering and model-capability limitations explicitly: prompt-engineering limitations are captured by letting reward functions operate on a coarser N-dimensional feature space, and model-capability limitations by conic constraints on the agents' feasible sets. Within this framework, a system designer specifies a reward function and budget for each of K agents; each agent maximizes its reward over its feasible set of M-dimensional outputs; the system designer aggregates the resulting outputs into a synthesized output $\boldsymbol{x}^{(A)}$.

We formalize three natural mechanisms by which aggregation expands the set of elicitable outputs (Section 3, Section 3): *feasibility expansion* (the aggregate $\boldsymbol{x}^{(A)}$ lies outside any agent's feasible set), *support expansion* (the aggregate has support richer than any input's), and *binding-set contraction* (the aggregate lies in the interior of the feasibility region while the inputs are on its boundary).

Our main results characterize when aggregation expands elicitability. Implementing at least one of these mechanisms is *necessary* (Theorem 18): if none are, aggregation cannot expand elicitability

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for any feature map. The mechanisms are not sufficient on their own, but *strengthened* versions of them are jointly necessary and sufficient, giving a full characterization (Theorem 9).

We complement the theory with an empirical illustration on a toy reference-generation task using LLMs (GPT-4o-mini, GPT-5-mini, GPT-5.4): the aggregations we construct certifiably expand elicitability, even though LLMs violate the simplifying assumptions of our model (e.g., deterministic reward maximization). Related work is in Section B.

2. Model

We extend the principal-agent framework in [35] to capture a compound AI system with K agents (who represent LLMs) and a single principal (the system designer) who aggregates the outputs of the agents. We define the components of this model in the rest of this section. In Section C we instantiate these components in different ways for a reference-generation task [45, 52], which we use as a running example throughout this section. We defer a discussion of the limitations of our framework to Section A.

2.1. Output space

We embed outputs of agents into M-dimensional vectors with non-negative coordinates. We view each output dimension as capturing a different characteristic of the output. The vector representation $\boldsymbol{x}$ quantifies the degree to which the output captures each characteristic. We note that some dimensions may capture undesirable characteristics (e.g., hallucinations). The system designer seeks a specific output $\boldsymbol{x}^{(A)} \in \mathbb{R}^M_{\geq 0}$. Concrete instantiations of this output space for the reference-generation task are given in Section 5 and Section C.

Model capability limitations as conic constraints. Our framework can capture restrictions on the outputs agents produce, for example due to capability limitations. We study restrictions requiring output vectors to satisfy conic constraints. These conic constraints capture the types of outputs that the agent can produce: for example, some agents may not be able to avoid producing hallucinations without facing capability degradation along other characteristics.

We let L denote the number of conic constraints, and we let $\boldsymbol{C} \in \mathbb{R}^{L \times M}$ denote the conic constraints themselves. Let $\boldsymbol{C}_i \in \mathbb{R}^M$ denote the ith row of $\boldsymbol{C}$ for $i \in [L]$, and let $\boldsymbol{C}_V \in \mathbb{R}^{|V| \times M}$ denote the set of rows corresponding to indices $V \subseteq [L]$. We denote by $\boldsymbol{C}_\emptyset$ the zero-vector, to capture how $\{\boldsymbol{d} : C_\emptyset \boldsymbol{d} \leq 0, \boldsymbol{d} \geq 0\} = \mathbb{R}^M_{\geq 0}$. This restriction defines a feasible set $\mathcal{B}^{\text{feasible}}(\boldsymbol{C})$ of output vectors: $\mathcal{B}^{\text{feasible}}(\boldsymbol{C}) := \left\{\boldsymbol{x} \in \mathbb{R}^M_{\geq 0} \mid \boldsymbol{C}\boldsymbol{x} \leq \boldsymbol{0}\right\}$.

We assume that membership in $\mathcal{B}^{\text{feasible}}(\boldsymbol{C})$ does not implicitly require any output dimension to always be zero. This assumption on $\boldsymbol{C}$ is stated below.

Assumption 1 *We assume that given any output dimension $i \in [M]$, there exists an output vector $\boldsymbol{x} \in \mathbb{R}^M_{\geq 0}$ satisfying $x_i > 0$ and $\boldsymbol{C}\boldsymbol{x} \leq 0$.*

2.2. Reward and Budget Specification

The system designer designs a reward function specification $R^{(k)}$ and a budget level $E^{(k)}$ for each agent $k \in [K]$. The reward function specification represents the reward function implicit in the prompt given to the agent, and the budget level represents the level of test-time compute that the agent is allowed to use.

Prompt engineering limitations as coarser features. To capture prompt engineering limitations, we model the reward function specification as operating over a coarser N-dimensional feature space than the outputs. We adopt the same form of how output vectors map to features as in [35]. Here, the features $\boldsymbol{F}(\boldsymbol{x}) = [F_1(\boldsymbol{x}), \ldots, F_N(\boldsymbol{x})]$ take the form $F_j(\boldsymbol{x}) = f_j\left(\sum_{i=1}^{M} \alpha_{ji}\boldsymbol{x}_i\right)$. We call the α_{ji}'s *feature weights*. We will denote by $\boldsymbol{\alpha} \in \mathbb{R}_{\geq 0}^{N\times M}$ the matrix with entries α_{ji} and call this the *feature weights matrix*. We borrow the following assumptions on the feature mapping functions F_j, $j \in [N]$ from [35].

Assumption 2 (Assumptions on feature mapping) *We assume that each $f_j(\cdot)$ for $j \in [N]$ is strictly increasing, nonnegative, smooth, and weakly concave (i.e., diminishing returns from increasing quality on this dimension). We also assume each $\alpha_{ji} \geq 0$, for $i \in [M], j \in [N]$. Finally, we assume that $\boldsymbol{\alpha}$ has no zero rows and no zero columns, which rules out trivial features and output dimensions that affect no feature, respectively.*

Reward functions. We consider reward functions $R^{(1)}, \ldots, R^{(K)} : \mathbb{R}^N \to \mathbb{R}$ which operate on the features $(F_j)_{j=1}^N$. Following prior work [35], we restrict to *monotone* reward functions R satisfying the notion of monotonicity stated below.

Assumption 3 (Monotonicity of reward functions) *We assume reward functions R are monotone. That is, R does not decrease if all features are weakly increased i.e., if $F_j(\boldsymbol{x}') \geq F_j(\boldsymbol{x})$ for every $j \in [N]$, then $R(x') \geq R(x)$. Additionally, there exists a feature F_j such that increasing the value of F_j, keeping all other features fixed, strictly increases the value of R.*

Agent optimization program. Given a monotone reward function $R^{(k)}$ and a positive budget level $E^{(k)} > 0$, each agent k produces an output that maximizes its reward $R^{(k)}$, meets its budget constraint, and is within the feasible set $\mathcal{B}^{\text{feasible}}(\boldsymbol{C})$: that is,

$$\boldsymbol{x} \in \mathbf{X}^*(R^{(k)}, E^{(k)}; \boldsymbol{C}, \boldsymbol{F}) := \text{argmax}_{\boldsymbol{x}\in\mathcal{B}^{\text{feasible}}(\boldsymbol{C}), \|\boldsymbol{x}\|_1 \leq E^{(k)}} R^{(k)}(\boldsymbol{F}(\boldsymbol{x})).$$

This captures how even though agents are homogeneous and solve the same optimization program, they can be given different reward functions and thus produce different outputs. In Section 5, we empirically show our findings to translate to stochastic LLMs, despite the reward-maximization assumption being violated.

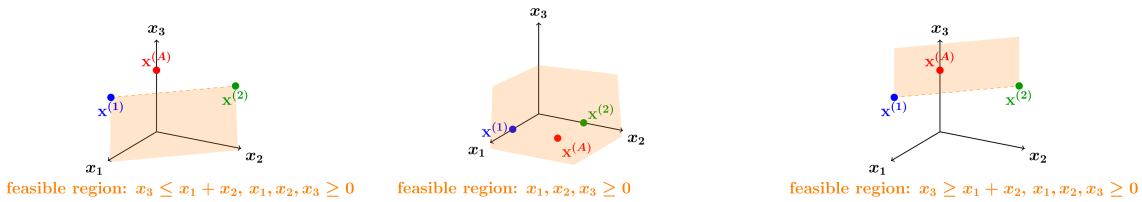
2.3. Elicitability

Given constraints $\boldsymbol{C}$ and features $\boldsymbol{F}$, we say that an output $\boldsymbol{x}$ is elicitable by a single agent if there exist a monotone reward function R and a positive budget level E such that $\boldsymbol{x} \in \mathbf{X}^*(R, E; \boldsymbol{C}, \boldsymbol{F})$.

Elicitability-expansion. When the system designer can aggregate the outputs of different agents, this may expand the set of elicitable outputs. The following definition captures when aggregation expands the set of elicitable outputs.

Definition 4 (Elicitability-expansion) *We call $\boldsymbol{x}^{(1)} \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ an **elicitability-expanding operation*** relative to constraints $\boldsymbol{C}$ and features $\boldsymbol{F}$ *if*

- *There exist monotone reward functions $R^{(1)}, \ldots, R^{(K)}$ and positive budget levels $E^{(1)}, \ldots, E^{(K)}$ such that $\boldsymbol{x}^{(k)} \in X^*(R^{(k)}, E^{(k)}; \boldsymbol{C}, \boldsymbol{F})$ for all $k \in [K]$.*



(a) Feasibility expansion (b) Support expansion (c) Binding set contraction

Figure 1: Three mechanisms by which the aggregation operation $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)} \to \boldsymbol{x}^{(A)}$ expands the set of outputs that the system designer can elicit. Feasibility expansion captures when two feasible vectors are aggregated into an infeasible vector (left; Theorem 5). Support expansion captures when two vectors are aggregated into a vector with richer support (middle; Theorem 6). Binding set contraction captures when two vectors on the boundary of the feasible set are aggregated into a vector in the interior (right; Theorem 7). Any aggregation operation must implement one of these mechanisms to offer power to the system designer (Theorem 18), and strengthened versions of these mechanisms characterize when aggregation adds power (Theorem 9, Theorem 9). See Figure 2 for an empirical illustration of these mechanisms for LLMs in a reference-generation task.

- *There does not exist a monotone reward function R and budget level $E > 0$ such that $x^{(A)} \in X^*(R, E; \boldsymbol{C}, \boldsymbol{F})$.*

We say $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ is elicitability-expanding relative to constraints $\boldsymbol{C}$ if there exist *features $\boldsymbol{F}$ such that $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ is elicitability-expanding relative to $\boldsymbol{C}$ and $\boldsymbol{F}$.*

We only consider aggregation operations $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ where each $\boldsymbol{x}^{(k)}$, for $k \in [K]$ is feasible. Other operations are clearly not useful to the system designer.

Intuitively, if an aggregation operation is elicitability-expanding, then there is a prompt engineering limitation under which this operation offers power. When an aggregation operation is not elicitability-expanding it is not useful for *any* form of prompt engineering limitation.

As we will show later in our analysis, the condition for whether $\boldsymbol{x}$ is elicitable only depends on $\boldsymbol{x}$ through the following sufficient statistic $(\mathcal{S}(\boldsymbol{x}), \mathcal{V}(\boldsymbol{x}))$, where $\mathcal{S}(\boldsymbol{x}) = \{i \in [M] : x_i > 0\}$ denotes the support of x and where $\mathcal{V}(\boldsymbol{x}) = \{l \in [L] : C_l \boldsymbol{x} = 0\}$ denotes the set of indices of conic constraints that are binding at $\boldsymbol{x}$.

Aggregation rules. An aggregation rule is a mapping from a list of output vectors $(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)})$ to an aggregated output vector $\boldsymbol{x}^{(A)}$. Two natural aggregation rules we will refer to are *intersection* aggregation $\mathcal{A}_{\text{intersect}}$ (the coordinate-wise minimum of the inputs) and *addition* aggregation $\mathcal{A}_{\text{add}}$ (a non-negative weighted sum of the inputs); formal definitions appear in Section D.1 where these rules are first used.

3. Mechanisms for Elicitability-Expansion

We formalize three natural mechanisms by which aggregation can expand the set of elicitable outputs. Section 3 illustrates each mechanism on a representative example; the full examples (with feature weights, constraints, and the elicitability argument) are deferred to Section D.1.

Mechanism 1: Feasibility Expansion. Aggregation can produce outputs outside any agent's feasible set $\mathcal{B}^{\text{feasible}}(\boldsymbol{C})$, overcoming the conic constraints faced by each agent.

Definition 5 (Feasibility Expansion) *Given a constraint matrix $\boldsymbol{C}$, an aggregation operation $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ implements **feasibility expansion** if $\boldsymbol{x}^{(A)}$ is infeasible i.e. $\boldsymbol{C}\boldsymbol{x}^{(A)} \not\leq 0$.*

Figure 1a depicts a representative example (Example 1 in Section D.1) where intersection aggregation produces a vector outside the feasible set, expanding elicitability.

Mechanism 2: Support Expansion. Aggregation can combine outputs with smaller supports to produce an output with a larger support, overcoming the (single-agent) impossibility of eliciting large-support outputs under coarse features.

Definition 6 (Support expansion) *An aggregation operation $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ implements **support expansion relative to** k if $\mathcal{S}(\boldsymbol{x}^{(A)}) \not\subseteq \mathcal{S}(\boldsymbol{x}^{(k)})$.*

Figure 1b depicts a representative example (Example 2 in Section D.1) where addition aggregation combines two single-coordinate vectors into a richer-support target that is otherwise inelicitable.

Mechanism 3: Binding Set Contraction. Aggregation can combine outputs with binding constraints into an output with fewer binding constraints—an interior point may be easier to elicit because binding constraints disable reward-increasing directions for the agent.

Definition 7 (Binding set contraction) *An aggregation operation $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ implements **binding set contraction relative to** k if $\mathcal{V}(\boldsymbol{x}^{(A)}) \not\supseteq \mathcal{V}(\boldsymbol{x}^{(k)})$.*

Figure 1c depicts a representative example (Example 3 in Section D.1) where intersecting two boundary-binding vectors yields an interior target that is otherwise inelicitable.

Connections to Elicitability-Expansion. We show a fundamental connection between the mechanisms and elicitability-expansion. Specifically, if an aggregation operation expands elicitability for some feature weights matrix, it must implement at least one of the three mechanisms (Theorem 18, deferred to Section D.5.1). However, these mechanisms do not provide sufficient conditions (Section D.3.2).

4. Characterizing Elicitability-Expansion

In this section, we will provide a characterizing condition that is both necessary and sufficient for an aggregation operation to have power i.e., expand elicitability under some feature map.

4.1. Characterizing condition

Feasible, budget-reducing directions. We first define a key object—the set of feasible, budget-reducing directions—and discuss how it relates to the elicitability of output vectors.

Definition 8 (Feasible, budget-reducing directions) *The set of feasible, budget-reducing directions for an output vector $\boldsymbol{x}$ depends on the sufficient statistics of the support and binding conic constraints indices $(\mathcal{S}(\boldsymbol{x}), \mathcal{V}(\boldsymbol{x}))$ of $\boldsymbol{x}$. It is defined as $\mathcal{B}_{\mathcal{S}(\boldsymbol{x}),\mathcal{V}(\boldsymbol{x})} =$*

$$\left\{\boldsymbol{d} \in \mathbb{R}^M \,\middle|\, \boldsymbol{C}_{\mathcal{V}(\boldsymbol{x})}\boldsymbol{d} \leq 0,\ d_j \geq 0\ \forall j \in \mathcal{S}(\boldsymbol{x})^c,\ \mathbb{1}^\top \boldsymbol{d} < 0\right\}.$$

$\mathcal{B}_{\mathcal{S}(\boldsymbol{x}),\mathcal{V}(\boldsymbol{x})}$ captures viable directions to move from $\boldsymbol{x}$ while maintaining feasibility, support, and budget constraints; $\boldsymbol{x}$ is elicitable iff no reward-improving direction lies in this set.

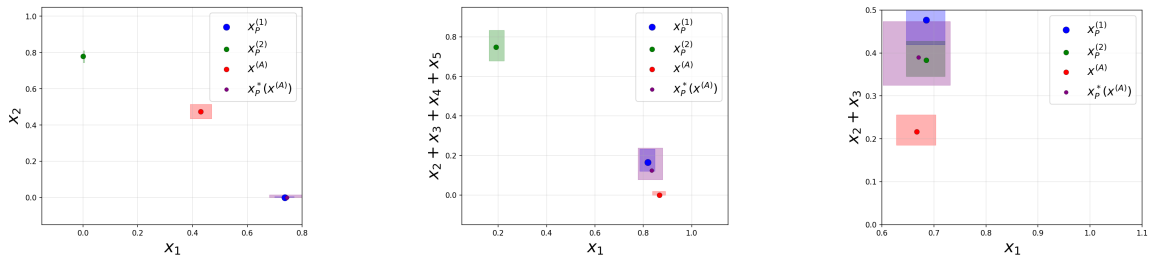


Figure 2: Empirical illustration of *support expansion* (left), *binding-set contraction* (middle), and *feasibility expansion* (right) for GPT-4o-mini (temp 0.7). Each plot shows individual outputs $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}$, the aggregate $\boldsymbol{x}^{(A)}$, and the closest single-prompt output $\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$ (shaded confidence sets).

Power-characterizing condition (informal). The **power-characterizing condition** for $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ holds if either (i) $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ is feasibility-expanding, or (ii) there is a single $\boldsymbol{d} \in \mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(A)}),\mathcal{V}(\boldsymbol{x}^{(A)})}$ that, for every $k \in [K]$, witnesses a strengthened support-expansion or binding-set-contraction relative to $\boldsymbol{x}^{(k)}$. The strengthening sharpens the mechanisms of Section 3 in two ways: a *single* $\boldsymbol{d}$ works jointly across all agents (rather than per-agent), and the violation must exceed a margin proportional to $|\mathbb{1}^\top \boldsymbol{d}|$ measured against a worst-case non-negative weighting of $\boldsymbol{x}^{(k)}$'s binding constraints. The formal statement is Theorem 19 (Section E.1); Section E.3.1 elaborates.

4.2. Characterization results

Theorem 9 (Characterization) *Fix constraints $\boldsymbol{C}$ and aggregation operation $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ where each $\boldsymbol{x}^{(k)}$ is feasible. Then $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ is elicitability-expanding for some feature map if and only if the power-characterizing condition (Theorem 19) is satisfied.*

Proof ideas. Our main technical tool (Theorem 22, proved in Section E.4.2) extends [35] to handle conic constraints and aggregated outputs by reducing elicitability under a fixed $\boldsymbol{\alpha}$ to whether $\mathcal{B}_{\mathcal{S}(\boldsymbol{x}),\mathcal{V}(\boldsymbol{x})}$ intersects the cone $\{\boldsymbol{d} : \boldsymbol{\alpha}\boldsymbol{d} \geq \boldsymbol{0}\}$, certified via duality. To extend to all $\boldsymbol{\alpha}$, the strengthened condition (joint margin) supplies the robustness needed against scaling and translation in the feature-improving cone. Full proofs are in Sections E.4.5 and E.4.6.

5. Empirical Illustration using LLMs

We illustrate the three mechanisms for LLMs (GPT-4o-mini, temp 0.7) on a toy reference-generation task: each output $\boldsymbol{x} \in \mathbb{R}^M_{\geq 0}$ is the topic histogram of an LLM-produced reference list (judged by an LLM into output topics $\{\mathcal{T}_i^O\}_{i\in[M]}$); prompts are inclusion/exclusion subsets of a coarser prompt-topic set $\{\mathcal{T}_j^P\}_{j\in[N]}$ joined by `and`/`or` connectors, aggregated by union ($\mathcal{A}^\cup$) or intersection ($\mathcal{A}^\cap$). For each mechanism, we construct $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)} \to \boldsymbol{x}^{(A)}$ mirroring the corresponding example in Section 3 (Examples 1 to 3), average over 30 trials, and certify elicitability-expansion via a confidence bound on $\|\boldsymbol{x}^{(A)} - \boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})\|_1$, where $\boldsymbol{x}_P^*$ is the closest single-prompt counterpart by brute-force search. Results are summarized in Figure 2 and Table 2; full details, per-mechanism setup, and model variations are in Sections F.1 to F.3.

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References

[1] Tal Alon, Magdalen Dobson, Ariel Procaccia, Inbal Talgam-Cohen, and Jamie Tucker-Foltz. Multiagent evaluation mechanisms. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 34, pages 1774–1781, 2020.

[2] Anthropic. How we built our multi-agent research system. https://www.anthropic.com/engineering/multi-agent-research-system, 2025. Blog post.

[3] Simran Arora, Avanika Narayan, Mayee F Chen, Laurel Orr, Neel Guha, Kush Bhatia, Ines Chami, Frederic Sala, and Christopher Ré. Ask me anything: A simple strategy for prompting language models. *arXiv preprint arXiv:2210.02441*, 2022.

[4] BAIR Research Blog. Compound ai systems. https://bair.berkeley.edu/blog/2024/02/18/compound-ai-systems/, 2024. Blog post.

[5] Gagan Bansal, Tongshuang Wu, Joyce Zhou, Raymond Fok, Besmira Nushi, Ece Kamar, Marco Tulio Ribeiro, and Daniel Weld. Does the whole exceed its parts? the effect of ai explanations on complementary team performance. In *Proceedings of the 2021 CHI conference on human factors in computing systems*, pages 1–16, 2021.

[6] Omer Ben-Porat and Moshe Tennenholtz. Regression equilibrium. In *Proceedings of the 2019 ACM Conference on Economics and Computation*, pages 173–191, 2019.

[7] Philip Bond and Armando Gomes. Multitask principal–agent problems: Optimal contracts, fragility, and effort misallocation. *Journal of Economic Theory*, 144(1):175–211, 2009.

[8] Arthur Campbell, Moshe Cohen, Florian Ederer, and Johannes Spinnewijn. *Solutions Manual to Accompany Contract Theory*, volume 1. The MIT Press, 2007.

[9] Lingjiao Chen, Matei Zaharia, and James Zou. Frugalgpt: How to use large language models while reducing cost and improving performance. *arXiv preprint arXiv:2305.05176*, 2023.

[10] Keertana Chidambaram, Karthik Vinary Seetharaman, and Vasilis Syrgkanis. Direct preference optimization with unobserved preference heterogeneity: The necessity of ternary preferences. *arXiv preprint arXiv:2510.15716*, 2025.

[11] Paul F Christiano, Jan Leike, Tom Brown, Miljan Martic, Shane Legg, and Dario Amodei. Deep reinforcement learning from human preferences. In *Advances in Neural Information Processing Systems 30 (NeurIPS)*, volume 30, 2017.

[12] Natalie Collina, Surbhi Goel, Aaron Roth, Emily Ryu, and Mirah Shi. Emergent alignment via competition. *arXiv preprint arXiv:2509.15090*, 2025.

[13] Vincent Conitzer, Rachel Freedman, Jobst Heitzig, Wesley H Holliday, Bob M Jacobs, Nathan Lambert, Milan Mossé, Eric Pacuit, Stuart Russell, Hailey Schoelkopf, et al. Social choice should guide ai alignment in dealing with diverse human feedback. *arXiv preprint arXiv:2404.10271*, 2024.

[14] Justin Cui, Wei-Lin Chiang, Ion Stoica, and Cho-Jui Hsieh. Or-bench: An over-refusal benchmark for large language models. *arXiv preprint arXiv:2405.20947*, 2024.

[15] Jessica Dai and Eve Fleisig. Mapping social choice theory to rlhf. *arXiv preprint arXiv:2404.13038*, 2024.

[16] Krishna Dasaratha, Benjamin Golub, and Anant Shah. Incentive design with spillovers. *arXiv preprint arXiv:2411.08026*, 2024.

[17] Thomas G Dietterich. Ensemble methods in machine learning. In *International workshop on multiple classifier systems*, pages 1–15. Springer, 2000.

[18] Kate Donahue and Manish Raghavan. Impacts of aggregation on model diversity and consumer utility. *CoRR*, abs/2602.23293, 2026. doi: 10.48550/ARXIV.2602.23293. URL https://doi.org/10.48550/arXiv.2602.23293.

[19] Kate Donahue, Alexandra Chouldechova, and Krishnaram Kenthapadi. Human-algorithm collaboration: Achieving complementarity and avoiding unfairness. In *Proceedings of the 2022 ACM Conference on Fairness, Accountability, and Transparency*, pages 1639–1656, 2022.

[20] Yilun Du, Shuang Li, Antonio Torralba, Joshua B Tenenbaum, and Igor Mordatch. Improving factuality and reasoning in language models through multiagent debate. In *International Conference on Machine Learning*, pages 11733–11763. PMLR, 2024.

[21] Ohad Einav and Nir Rosenfeld. A market for accuracy: Classification under competition. In *Forty-second International Conference on Machine Learning, ICML 2025, Vancouver, BC, Canada, July 13-19, 2025*, volume 267 of *Proceedings of Machine Learning Research*. PMLR / OpenReview.net, 2025.

[22] Matthew Gentzkow and Emir Kamenica. Bayesian persuasion with multiple senders and rich signal spaces. *Games and Economic Behavior*, 104:411–429, 2017.

[23] Paul Gölz, Nika Haghtalab, and Kunhe Yang. Distortion of ai alignment: Does preference optimization optimize for preferences? *arXiv preprint arXiv:2505.23749*, 2025.

[24] Sanford J Grossman and Oliver D Hart. An analysis of the principal-agent problem. *Econometrica*, 51(1):7–46, 1983.

[25] Bengt Holmström. Moral hazard and observability. *The Bell journal of economics*, pages 74–91, 1979.

[26] Bengt Holmstrom. Moral hazard in teams. *The Bell journal of economics*, pages 324–340, 1982.

[27] Bengt Holmstrom and Paul Milgrom. Multitask principal–agent analyses: Incentive contracts, asset ownership, and job design. *The Journal of Law, Economics, and Organization*, 7 (special_issue):24–52, 1991.

[28] Safwan Hossain, Evi Micha, Yiling Chen, and Ariel Procaccia. Strategic classification with externalities. *arXiv preprint arXiv:2410.08032*, 2024.

[29] Audrey Huang, Adam Block, Qinghua Liu, Nan Jiang, Akshay Krishnamurthy, and Dylan J Foster. Is best-of-n the best of them? coverage, scaling, and optimality in inference-time alignment. In *Forty-second International Conference on Machine Learning*, 2025.

[30] Athul Paul Jacob, Yikang Shen, Gabriele Farina, and Jacob Andreas. The consensus game: Language model generation via equilibrium search. *arXiv preprint arXiv:2310.09139*, 2023.

[31] Meena Jagadeesan, Michael Jordan, Jacob Steinhardt, and Nika Haghtalab. Improved bayes risk can yield reduced social welfare under competition. *Advances in Neural Information Processing Systems*, 36:66940–66952, 2023.

[32] Erik Jones, Anca Dragan, and Jacob Steinhardt. Adversaries can misuse combinations of safe models. In *Forty-second International Conference on Machine Learning*, 2025.

[33] Erik Jones, Arjun Patrawala, and Jacob Steinhardt. Uncovering gaps in how humans and llms interpret subjective language. In *The Thirteenth International Conference on Learning Representations*, 2025.

[34] Akbir Khan, John Hughes, Dan Valentine, Laura Ruis, Kshitij Sachan, Ansh Radhakrishnan, Edward Grefenstette, Samuel R Bowman, Tim Rocktäschel, and Ethan Perez. Debating with more persuasive llms leads to more truthful answers. In *International Conference on Machine Learning*, pages 23662–23733. PMLR, 2024.

[35] Jon Kleinberg and Manish Raghavan. How do classifiers induce agents to invest effort strategically? *ACM Transactions on Economics and Computation (TEAC)*, 8(4):1–23, 2020.

[36] Krishna K Ladha. The condorcet jury theorem, free speech, and correlated votes. *American Journal of Political Science*, pages 617–634, 1992.

[37] Jean-Jacques Laffont and David Martimort. *The Theory of Incentives: The Principal-Agent Model*. Princeton University Press, Princeton, NJ, 2002.

[38] Edward P Lazear and Sherwin Rosen. Rank-order tournaments as optimum labor contracts. *Journal of political Economy*, 89(5):841–864, 1981.

[39] Xuechen Li, Tianyi Zhang, Yann Dubois, Rohan Taori, Ishaan Gulrajani, Carlos Guestrin, Percy Liang, and Tatsunori B Hashimoto. Alpacaeval: An automatic evaluator of instruction-following models, 2023.

[40] Lydia T Liu, Nikhil Garg, and Christian Borgs. Strategic ranking. In *International Conference on Artificial Intelligence and Statistics*, pages 2489–2518. PMLR, 2022.

[41] Wenhao Liu, Zhengkang Guo, Mingchen Xie, Jingwen Xu, Zisu Huang, Muzhao Tian, Jianhan Xu, Muling Wu, Xiaohua Wang, Changze Lv, et al. Recast: Strengthening llms' complex instruction following with constraint-verifiable data. *arXiv preprint arXiv:2505.19030*, 2025.

[42] Nancy A Lynch. *Distributed algorithms*. Elsevier, 1996.

[43] Abhilash Mishra. Ai alignment and social choice: Fundamental limitations and policy implications. *arXiv preprint arXiv:2310.16048*, 2023.

[44] Long Ouyang, Jeffrey Wu, Xu Jiang, Diogo Almeida, Carroll Wainwright, Pamela Mishkin, Chong Zhang, Sandhini Agarwal, Katarina Slama, Alex Ray, et al. Training language models to follow instructions with human feedback. *Advances in neural information processing systems*, 35:27730–27744, 2022.

[45] Ori Press, Andreas Hochlehnert, Ameya Prabhu, Vishaal Udandarao, Ofir Press, and Matthias Bethge. Citeme: Can language models accurately cite scientific claims? *Advances in Neural Information Processing Systems*, 37:7847–7877, 2024.

[46] Manish Raghavan. Competition and diversity in generative ai. *arXiv preprint arXiv:2412.08610*, 2024.

[47] Nils Reimers and Iryna Gurevych. Sentence-bert: Sentence embeddings using siamese bert-networks. *arXiv preprint arXiv:1908.10084*, 2019.

[48] Ali Shirali, Arash Nasr-Esfahany, Abdullah Alomar, Parsa Mirtaheri, Rediet Abebe, and Ariel Procaccia. Direct alignment with heterogeneous preferences. *arXiv preprint arXiv:2502.16320*, 2025.

[49] Ritwik Sinha, Zhao Song, and Tianyi Zhou. A mathematical abstraction for balancing the trade-off between creativity and reality in large language models. *arXiv preprint arXiv:2306.02295*, 2023.

[50] Margaret E Slade. Multitask agency and contract choice: An empirical exploration. *International Economic Review*, pages 465–486, 1996.

[51] Ming Tan. Multi-agent reinforcement learning: Independent vs. cooperative agents. In *Proceedings of the tenth international conference on machine learning*, pages 330–337, 1993.

[52] Xiaoxuan Wang, Ziniu Hu, Pan Lu, Yanqiao Zhu, Jieyu Zhang, Satyen Subramaniam, Arjun R Loomba, Shichang Zhang, Yizhou Sun, and Wei Wang. Scibench: Evaluating college-level scientific problem-solving abilities of large language models. *arXiv preprint arXiv:2307.10635*, 2023.

[53] Xuezhi Wang, Jason Wei, Dale Schuurmans, Quoc V Le, Ed H Chi, Sharan Narang, Aakanksha Chowdhery, and Denny Zhou. Self-consistency improves chain of thought reasoning in language models. In *The Eleventh International Conference on Learning Representations*, 2023.

[54] Bosi Wen, Pei Ke, Xiaotao Gu, Lindong Wu, Hao Huang, Jinfeng Zhou, Wenchuang Li, Binxin Hu, Wendy Gao, Jiaxing Xu, et al. Benchmarking complex instruction-following with multiple constraints composition. *Advances in Neural Information Processing Systems*, 37: 137610–137645, 2024.

[55] Renzhe Xu, Kang Wang, and Bo Li. Heterogeneous data game: Characterizing the model competition across multiple data sources. *arXiv preprint arXiv:2505.07688*, 2025.

[56] Chenyang Yang, Yike Shi, Qianou Ma, Michael Xieyang Liu, Christian Kästner, and Tongshuang Wu. What prompts don't say: Understanding and managing underspecification in llm prompts. *arXiv preprint arXiv:2505.13360*, 2025.

[57] Bohan Zhang, Xiaokang Zhang, Jing Zhang, Jifan Yu, Sijia Luo, and Jie Tang. Cot-based synthesizer: Enhancing llm performance through answer synthesis, 2025.

[58] Simon Zhuang and Dylan Hadfield-Menell. Consequences of misaligned ai. *Advances in Neural Information Processing Systems*, 33:15763–15773, 2020.

Appendix A. Discussion

In this work, we theoretically study how aggregating multiple copies of the same model gives access to a greater set of outputs than using only a single model. Building on a principal-agent framework, our results show how aggregation must implement one of three mechanisms—feasibility-expansion, support expansion, and binding set contraction—in order to expand the set of elicitable outputs. We also show a more precise condition formed from strengthening the mechanisms is sufficient. Finally, we empirically illustrate these mechanisms by deploying LLMs in a toy reference-generation task, demonstrating the robustness of our findings beyond the assumptions of our stylized model. We discuss how each mechanism connects to more broadly observed empirical phenomena (e.g., safety vs. overrefusal trade-offs, prompt underspecification, multi-requirement elicitation) in Section D.2.

Model limitations and extensions. Our stylized model, which builds on a classical principal-agent framework [35], makes simplifying assumptions for tractability. First, we assume for simplicity that the agent chooses an output deterministically. Nonetheless, our empirical findings robustly generalize to LLMs with nonzero temperature which exhibit stochasticity (Section 5). It would be interesting to extend our theoretical results to capture this randomness and to allow the system designer to choose a temperature for each agent. Moreover, while our analysis allows for nonlinear rewards R, we restrict the output constraints (i.e., model limitations) and the feature map (i.e., prompt engineering limitations) to linear functional forms. Extending our model to allow for nonlinear limitations, which would complicate the structure of the agent's optimization program, is an interesting direction for future work. Furthermore, we also assume each agent's reward depends only on its own outputs, though richer interdependencies may arise in repeated, multi-turn interactions [20]. Finally, while our analysis focuses on steering agents through reward design, it would be interesting to incorporate other choices, such as tool use and fine-tuning, that enable specialization in compound AI systems [4].

Appendix B. Related Work

Aggregation across multiple models. Aggregating multiple LLM outputs is a common strategy for complex tasks. Approaches based on resampling a single model use reward models [11], self-consistency [53], or synthesis [57], with coverage as a key property [29]. Closest to our setting are systems running multiple model copies under different reward specifications: LLM debate [20], generator–discriminator consensus games [30], prompt ensembling [3], and multi-agent research [2]. Other lines combine heterogeneous models via routing [9] or adversarial composition [32]. Motivated by these systems, we study the fundamental conceptual question of when aggregating multiple models elicits strictly more outputs than querying a single model.

Beyond LLMs, aggregation has been studied in ensembling [17], voting [36], distributed algorithms [42], and multi-agent reinforcement learning [51], among other domains. Some works on aggregating heterogeneous agents (e.g., [5, 12, 19, 22]) demonstrate alignment benefits, but focus on a prespecified set of distinct agents, whereas we focus on eliciting different behaviors from the same agent through different rewards. Turning to market-level aggregation arising from users choosing between models, a line of work studies when model providers are incentivized to train heterogeneous models [6, 18, 21, 31, 46, 55]. A growing line of work on alignment under plurality uses voting and social choice to aggregate diverse human preferences and train models to optimize the aggregated objective (e.g., [10, 13, 15, 23, 43, 48]).

Principal-agent models and reward design. Our model extends the principal-agent framework of Kleinberg and Raghavan [35], originally developed for a strategic classification setting, by introducing multiple agents and conic output constraints. Our goal also differs: we introduce and characterize a novel notion of elicitability-expansion to capture the power afforded by aggregation for some level of prompting limitation. Alon et al. [1] also generalize Kleinberg and Raghavan [35] to multiple agents, but their setup—heterogeneous feature maps with a shared reward and a single target structure—differs from ours. These works fall under the broader principal-agent framework [8, 24, 25, 37], which captures the challenge of designing rewards based on imperfect proxies; Zhuang and Hadfield-Menell [58] use this framework to study misalignment between the AI agent's reward and the human's reward under reward underspecification. Multi-task principal-agent settings [7, 27, 50] study cost dependencies between tasks—substitutability (effort on one task increases marginal cost of effort on another) and complementarity (it decreases it)—which are similar to the dependencies among output dimensions captured by our conic constraints. Multi-agent principal-agent theory [16, 26, 38] has focused mainly on joint reward design across agents; our work differs in considering aggregation to synthesize new outputs. In strategic classification, a few papers [28, 40] study strategic interactions between multiple agents.

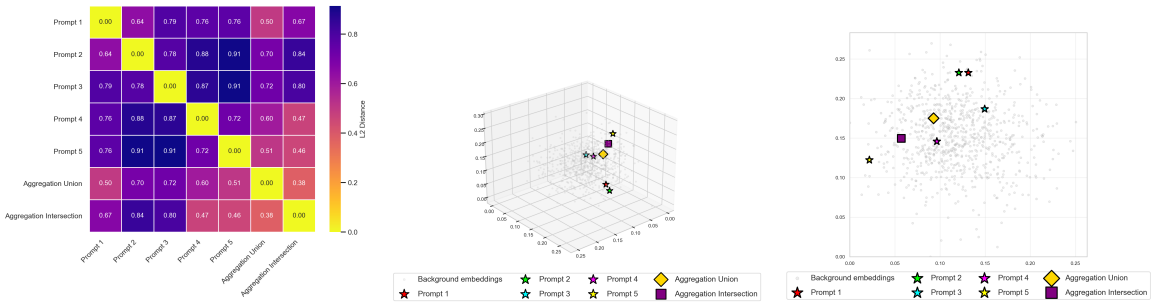
Appendix C. Additional details for Section 2

This appendix elaborates on how the abstract components of the model in Section 2—the M-dimensional output space, the coarser feature space inducing prompt-engineering limitations, and the aggregation rules—can be instantiated for a concrete reference-generation task with LLMs. Throughout, we use the running example of a system designer who queries multiple copies of an LLM with different prompts and aggregates the resulting reference lists into a single synthesized list. The empirical setup we describe below is purely illustrative: we visualize how different prompts elicit semantically different outputs (when each output is encoded as a high-dimensional sentence embedding) and how intersection-style and addition-style aggregation rules combine them into outputs that resemble none of the originals. This complements the controlled topic-histogram experiments in Section 5 (which target the framework's mechanisms directly) by showing that the framework's output-space and aggregation primitives map naturally onto realistic LLM outputs.

Output vector space. We specify two different instantiations of the output vector space $\mathbb{R}^M_{\geq 0}$ in our framework, depending on the level of specificity of the output which the system designer aims to elicit.

1. Suppose that the system designer aims to elicit a list of papers that overall reflects a specific balance of different topics (e.g., machine learning, economics, etc.). To capture this, let each dimension $i \in [M]$ of the output capture a different topic, so the value x_i captures the fraction of papers on the list that reflect the corresponding topic. We build on this instantiation in our empirical analysis in Section 5.

2. Suppose that the system designer aims to elicit a specific list of references. To capture this, we take the output vector to be a high-dimensional embedding coming from a text embedding model. We design a simple experiment using LLMs to illustrate this: we prompt GPT-4o-mini in different ways, perform aggregation also using GPT-4o-mini, and use a sentence-transformers model to produce 768-dimensional output vectors (see the empirical details below for the setup).



(a) ℓ_2 distances (b) Projection onto top 3 dims (c) Projection onto top 2 dims

Figure 3: Visualization of output vectors for a reference-generation task (Section 2). Output vectors are computed using the $M = 768$-dimensional embeddings from all-mpnet-base-v2, shifted to be in the nonnegative orthant. Embeddings are shown for GPT-4o-mini outputs from five different prompts, as well as two different aggregated outputs based on additional-style and intersection-style aggregation rules. The ℓ_2-distances (left) and projections onto the top 3 highest-variance (middle) and top 2 highest-variance dimensions (right), are shown. The plots show that the five prompts produce semantically different outputs, and each aggregation operation results in a combination of the five outputs that does not resemble any output in isolation.

Figure 3 shows the embeddings for outputs to the five prompts, as well as the outputs produced by the intersection-style and addition-style aggregation rules. The five prompt outputs vary substantially, and the aggregated outputs differ markedly from both the originals and from each other, demonstrating how different prompts and aggregation rules can shape the embedding-space representation.

Reward function specification limitations. In our framework, recall that the reward function operates over coarsenings of the output dimensions (as captured by the feature weights matrix $\boldsymbol{\alpha}$) rather than directly on the output dimensions. This captures two types of prompt engineering limitations that we describe below.

1. First, the system designer may struggle to precisely express what they truly want in the prompt, leading to underspecified prompts omitting some of the system designer's requirements [56]. For example, the system designer may prompt the model to focus on combinations of high-level topics, rather than fully specifying the fine-grained topic that they wish to see. We build on this instantiation in our empirical analysis in Section 5.

2. Second, the model may not correctly interpret the system designer's prompt. For example, the model might map two dissimilar words in the prompt to the same word [33]. In the citation task, this could surface as the model interpreting "papers with high attribute 'X'" similarly for many different attributes "X".

C.1. Empirical details

Task and Setup. We study a citation task where the system designer seeks a list of 10 influential LLM papers spanning five perspectives: (1) ML theory, (2) NLP/CL, (3) cognitive science, (4)

AI alignment and human–AI interaction, and (5) multi-agent systems. The system designer issues five prompts, each targeting one perspective, to gpt-4o-mini-2024-07-18 and then aggregates the resulting lists. We prompt another LLM (also gpt-4o-mini-2024-07-18) to aggregate these five lists, instantiating aggregation rules that are inspired by intersection aggregation $\mathcal{A}_{\text{intersect}}$ and union aggregation $\mathcal{A}_{\text{add}}$. Specifically, the model is prompted with *aggregation instructions* along with the five different lists of 10 references, and produces an aggregated list of 10 references. The *intersection-style aggregation instructions* ask for references which are central and broadly relevant across all five perspectives, thus approximating intersection even when the literal overlap of references is empty. The *addition-style* aggregation instructions ask for references that jointly cover and reflect the combined topical space of all five perspectives.

Model output generation. The outputs are generated using gpt-4o-mini-2024-07-18 with the temperature set to 1.0. These are the five prompts that are used to produce model outputs:

1. "From a machine learning theory perspective, list 10 influential papers that have shaped our current understanding of large language models."
2. "From the perspective of natural language processing and computational linguistics, list 10 key research papers that have been most influential in the development of modern large language models."
3. "From a cognitive science and psycholinguistics standpoint, list 10 important papers that inform our understanding of how large language models represent, process, or acquire linguistic and conceptual structure."
4. "From the standpoint of AI alignment and human–AI interaction, list 10 important papers that have shaped how large language models are aligned, instructed, or trained with feedback."
5. "From a multi-agent and game-theoretic perspective, list 10 influential papers that contribute to the development or understanding of large language models"

These prompts produce five outputs $X_1, \ldots, X_5$, each a list of 10 papers tailored to its respective perspective. Next, we pass the concatenated outputs $(X_1, \ldots, X_5)$ to gpt-4o-mini-2024-07-18 by prompting the model with *aggregation instructions* followed by the concatenation of the 5 lists of papers, where each list is preceded by "List of papers: [insert output number]". The intersection-style and addition-style aggregation operations are performed using the following *aggregation instructions*.

- *Addition-style aggregation:* "Each of the following lists contains influential papers on large language models in specializing in different areas: machine learning theory, natural language processing, computational linguistics, AI alignment, human–AI interaction, and multi-agent systems. Based on these lists, generate a new list of 10 papers that reflects the union of their themes and coverage. Your list should be freshly generated (not a literal set union), but it should include papers that plausibly come from any of the provided lists, covering as much of the combined topical space as possible."
- *Intersection-style aggregation:* "Each of the following lists contains influential papers on large language models in specializing in different areas: machine learning theory, natural language processing, computational linguistics, AI alignment, human–AI interaction, and multi-agent

systems. Based on these lists, generate a new list of 10 papers that reflects their intersection. That is, papers belonging to many of these areas of specialization. Your list should be freshly generated (not a literal intersection), selecting papers that could plausibly appear in all of the lists. If the literal intersection is empty, still generate the best possible list of papers that are central, broadly relevant, and thematically compatible with all lists."

These aggregation prompts produce outputs $X_{\text{addition}}, X_{\text{intersection}}$.

Output vector computation. We now describe in more detail how we compute the embeddings shown in Figure 3. We embed and visualize the set $\{X_1, \ldots, X_5, X_{\text{addition}}, X_{\text{intersection}}\}$. We calculate the 768-dimensional embeddings using all-mpnet-base-v2 [47], which is built into the sentence-transformers package in pytorch. To make these embeddings fit into our framework, we translate them to the nonnegative orthant by applying an additive shift $\mathbf{s} \in \mathbb{R}^{768}_{\geq 0}$. To do this, we compute the embeddings of the 805 gpt-4o-mini-2024-07-18 outputs from the helpful-base dataset in AlpacaEval [39]. The additive shift $\mathbf{s}$ is taken to be negative of the minimum coordinate along each dimension in this set of 805 embeddings. We translate all 5 outputs and the aggregated outputs by adding $\mathbf{s}$. We compute the variance across the 5 translated output vectors along each of the 768 dimensions, and select the top 2 and top 3 dimensions according to variance. We also compute the ℓ_2-distance between outputs, which is invariant to the additive shift.

Appendix D. Additional details for Section 3 (Mechanisms)

D.1. Examples of mechanisms

We give examples that illustrate each of the three mechanisms in Section 3 (feasibility expansion, support expansion, and binding-set contraction). Our examples focus on a 3-dimensional output ($M = 3$) with 2-dimensional features ($N = 2$). We focus on feature weights matrices $\boldsymbol{\alpha}(q) := \begin{bmatrix} 1 & 0 & q \\ 0 & 1 & q \end{bmatrix}$.

The examples use two aggregation rules. *Intersection* aggregation is the coordinate-wise minimum: $\mathcal{A}_{\text{intersect}}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)}) = \boldsymbol{x}^{(1)} \wedge \ldots \wedge \boldsymbol{x}^{(K)}$. *Addition* aggregation takes a non-negative weighted sum: for a weight vector $\boldsymbol{w} \in \mathbb{R}^K_{\geq 0}$, $\mathcal{A}_{\text{add}}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)}; \boldsymbol{w}) = \sum_{k=1}^K \boldsymbol{w}_k \boldsymbol{x}^{(k)}$.

The following example, depicted in Figure 1a, illustrates how aggregation operations which implement feasibility expansion (Theorem 5) can in turn expand elicitability.

Example 1 *Let the feature map be $\boldsymbol{\alpha} = \boldsymbol{\alpha}(2)$, so that increasing the third dimension contributes significantly to both features. Let $\boldsymbol{C}$ be a single constraint $x_3 \leq x_1 + x_2$. The output $[0, 0, 1]$ is infeasible. The system designer can produce this output through intersection aggregation $\boldsymbol{x}^{(1)} = [1, 0, 1], \boldsymbol{x}^{(2)} = [0, 1, 1] \to \mathcal{A}_{\textit{intersect}}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}) = [0, 0, 1]$ (Proposition 15 in Appendix D.4.1).*

Aggregation operations which implement support expansion (Theorem 6) can in turn expand elicitability. We illustrate this in the following example, which is depicted in Figure 1b.

Example 2 *Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}(0.6)$ and $\boldsymbol{C} = \emptyset$. The target $[1/2, 1/2, 0]$, supported on both dimensions 1 and 2, is not elicitable: a reward emphasizing F_j alone prefers x_j to x_3 (weight 1 vs. 0.6), while a reward valuing both features prefers x_3 (contributing 0.6 to each) over splitting between x_1 and x_2. Yet addition aggregation produces the target from two elicitable vectors: $\boldsymbol{x}^{(1)} = [1, 0, 0]$, $\boldsymbol{x}^{(2)} = [0, 1, 0]$, with $\mathcal{A}_{\textit{add}}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}; [1/2, 1/2]) = [1/2, 1/2, 0]$ (Theorem 16 in Section D.4.2).*

Aggregation operations which implement binding set contraction (Theorem 7) can expand elicitability. We illustrate this in the following example, which is depicted in Figure 1c.

Example 3 *Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}(0.2)$ and $\mathbf{C}$ be a single constraint of the form $x_1 + x_2 \leq x_3$. The value of $q = 0.2$ is small leading to any level of x_3 being inelicitable without the constraint (Proposition 17 in Appendix D.4.3). The target $[0, 0, 1]$ is still inelicitable because it is in the interior of the feasibility region. But it can be formed by intersecting the elicitable vectors $\boldsymbol{x}^{(1)} = [1, 0, 1]$ and $\boldsymbol{x}^{(2)} = [0, 1, 1]$.*

D.2. Connecting our mechanisms to empirical phenomena

To complement our empirical support for the three mechanisms in the toy reference-generation task, we briefly discuss how the mechanisms connect with more broadly observed empirical phenomena. Since aggregation is only powerful when individual models are limited on their own, we begin by outlining the single-model limitations underlying each mechanism and the empirical phenomena supporting them.

- The power of feasibility expansion traces back to limitations in the types of outputs that individual models can generate: specifically, when models can't exhibit certain (desirable) dimensions without exhibiting other (undesirable) dimensions as a side effect. This side effect has been empirically observed for safety versus overrefusal, where models which refuse a larger fraction of toxic outputs tend to refuse a larger fraction of safe outputs as a side effect [14]. Similar side effects have been observed for alignment and hedging [44], and theoretically studied for creativity and factuality [49].
- The power of support expansion traces back to challenges with eliciting outputs that perform along multiple dimensions at once in single-agent settings. This limitation has been empirically observed in cases where each dimension corresponds to a distinct user requirement. For example, prompts are often underspecified, since users may not include all of the requirements that they care about in the prompt [56]. Moreover, even when users specify all their requirements, LLMs struggle to satisfy many requirements simultaneously [41, 54].

We leave pinpointing empirical phenomena which support binding set contraction—whose emergence depends on the interaction between prompt-engineering and model limitations—to future work. More broadly, since our results identify when aggregation enables these mechanisms, an important direction is to connect them to practice by testing whether real aggregation methods (e.g., debate [20], prompt ensembling [3]) exhibit them. The single-model limitations discussed above suggest promising empirical settings where aggregation should add power.

D.3. Additional details for Section 3

D.3.1. How addition and intersection aggregation rules can or cannot implement the mechanisms

In Section 3, we showed examples of intersection and addition aggregation rules expanding elicitability by implementing each of the feasibility-expansion, support expansion or binding set contraction mechanisms. In this part, we will discuss the mechanisms that each aggregation rule can or cannot implement. These results are summarized in Table 1.

Intersection aggregation does not implement support expansion for any problem instance, as the following result formalizes.

	Feasibility Expansion	**Support Expansion**	**Binding Set Contraction**
Intersection aggregation	✓ (Example 1)	× (Theorem 10)	✓ (Example 3)
Addition aggregation	× (Theorem 11)	✓ (Example 2)	✓ (Example 4)

Table 1: Implementability of mechanisms in Section 3 for the intersection aggregation rule and addition aggregation rule. The symbol ✓ denotes that there exists a problem instance where the aggregation rule implements that mechanism. The symbol × denotes that the aggregation rule does not implement the mechanism for any problem instance.

Proposition 10 (Intersection does not expand support) *Consider any aggregation operation of the form* $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)} = \mathcal{A}_{intersect}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)})$*. For any* $k \in [K]$*, this aggregation operation does not implement support expansion relative to* k*.*

Proof Proposition 10 follows from the fact that the support of $\mathcal{A}_{\text{intersect}}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)})$ is always a subset of the support of each $\boldsymbol{x}^{(k)}$ for $k \in [K]$. ■

Intersection aggregation can implement feasibility-expansion as shown in Example 1 and binding set-contraction as shown in Example 3. In fact, these examples go one step further and demonstrate that elicitability expansion is achievable via these mechanisms.

Addition aggregation does not implement feasibility expansion for any problem instance, as the following result formalizes.

Proposition 11 (Addition cannot expand feasibility) *Consider constraints* $\boldsymbol{C}$*. Any aggregation operation of the form* $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)} = \mathcal{A}_{add}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)}; \boldsymbol{w})$ *does not implement feasibility expansion relative to* $\boldsymbol{C}$*.*

Proof Proposition 11 directly follows from the fact that the constraint set $\boldsymbol{C}$ is conic. ■

On the other hand, addition aggregation operations can implement the other two mechanisms. Example 2 already constructed a problem instance where addition aggregation implements support expansion. The next example constructs a problem instance where addition aggregation can implement binding set contraction (Theorem 7) and achieve elicitability-expansion for some feature mapping.

Example 4 (Addition can result in binding set contraction) *Consider the constraint matrix*

$$C = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -\frac{1}{4} & -1 \end{pmatrix},$$

and consider vectors $\boldsymbol{x}^{(1)} = (1, 1, 2)$ *and* $\boldsymbol{x}^{(2)} = (2, 4, 1)$*. Note that they are both feasible and* $\boldsymbol{x}^{(1)}$ *is binding in the first constraint and* $\boldsymbol{x}^{(2)}$ *in the second. Their sum is* $\mathcal{A}_{add}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)}; \boldsymbol{w})(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}; [1, 1]) \to \boldsymbol{x}^{(A)} = (3, 5, 3)$*, which is also feasible but does not have any binding constraints.*

D.3.2. INSUFFICIENCY OF NATURAL MECHANISMS IMPLEMENTATION FOR ELICITABILITY-EXPANSION

While implementing one of feasibility-expansion, support expansion, or binding set contraction is necessary for an aggregation operation to be elicitability-expanding as shown by Theorem 18, in this section, we will show that it is not sufficient.

Support expansion on its own or binding set contraction on its own is not sufficient for power. The following proposition states the insufficiency of support expansion relative to every input vector of the aggregation operation for elicitability-expansion.

Proposition 12 *Fix $\boldsymbol{C} = \emptyset$. There exists an aggregation operation $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \boldsymbol{x}^{(3)} \to \boldsymbol{x}^{(A)}$ such that $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \boldsymbol{x}^{(3)} \to \boldsymbol{x}^{(A)}$ implements support expansion relative to i for $i \in \{1, 2\}$. However, $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \boldsymbol{x}^{(3)} \to \boldsymbol{x}^{(A)}$ is not elicitability-expanding.*

Proof This follows mainly from Theorem 21. This Corollary shows that an additional condition beyond support expansion is required for an operation $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ to be elicitability-expanding when $\mathcal{V}(\boldsymbol{x}^{(1)}) = \ldots = \mathcal{V}(\boldsymbol{x}^{(K)}) = \mathcal{V}(\boldsymbol{x}^{(A)})$. This is the condition that if $\boldsymbol{x}^{(A)}$ has full support, then there is a $j \in [M]$ and $k \in [K]$ such that $[M] \setminus \{j\} \not\subseteq \mathcal{S}(\boldsymbol{x}^{(k)})$.

We will now construct a problem instance with three output dimensions and an aggregation operation that is support expanding but fails the additional condition. Additionally this problem instance has no conic constraints making the binding constraint set the empty set for all vectors. Consider the aggregation operation $\boldsymbol{x}^{(1)} = [0, 1, 1], \boldsymbol{x}^{(2)} = [1, 0, 1], \boldsymbol{x}^{(3)} = [1, 1, 0] \to \boldsymbol{x}^{(A)} = [1, 1, 1]$. This aggregation operation fails the necessary condition for elicitability-expansion stated in Theorem 21. ■

Similarly, binding set contraction relative to every input vector of the aggregation operation also does not guarantee that aggregation has power. The next proposition formalizes this.

Proposition 13 *There exists an aggregation operation $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)} \to \boldsymbol{x}^{(A)}$ and a set of conic constraints $\boldsymbol{C}$ such that $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ implements binding set contraction relative to i for $i \in \{1, 2\}$. However, $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)} \to \boldsymbol{x}^{(A)}$ is not elicitability-expanding.*

Proof Consider a problem with two output dimensions having the following two constraints: 1) $c_1 : x_1 - x_2 \leq 0$, 2) $c_2 : -2x_1 + x_2 \leq 0$. Consider an aggregation operation $\boldsymbol{x}^{(1)} = [1, 1], \boldsymbol{x}^{(2)} = [1, 2] \to \boldsymbol{x}^{(A)} = [5, 7]$, where the binding constraints sets are $\mathcal{V}_{\boldsymbol{x}^{(i)}} = \{c_i\}$ for $i \in \{1, 2\}$ and $\mathcal{V}_{\boldsymbol{x}^{(A)}} = \emptyset$ and the supports are the entire set of output dimensions. Hence the feasibility-improving and budget-reducing directions are $\mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(A)}),\mathcal{V}(\boldsymbol{x}^{(A)})} = \{\boldsymbol{d} : \mathbb{1}^\top \boldsymbol{d} < 0\}, \mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(1)}),\mathcal{V}(\boldsymbol{x}^{(1)})} = \{\boldsymbol{d} : \boldsymbol{d}_1 - \boldsymbol{d}_2 \leq 0, \mathbb{1}^\top \boldsymbol{d} < 0\}, \mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(2)}),\mathcal{V}(\boldsymbol{x}^{(2)})} = \{\boldsymbol{d} : -2\boldsymbol{d}_1 + \boldsymbol{d}_2 \leq 0, \mathbb{1}^\top \boldsymbol{d} < 0\}$.

First note that this operation is not feasibility-expanding since $\boldsymbol{x}^{(A)}$ satisfies the conic constraints. We will show that there is no $\boldsymbol{d} \in \mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(A)}),\mathcal{V}(\boldsymbol{x}^{(A)})}$ such that $\boldsymbol{d} \not\leq 0$ and $\boldsymbol{d}$ is in neither $\mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(1)}),\mathcal{V}(\boldsymbol{x}^{(1)})}$ nor in $\mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(2)}),\mathcal{V}(\boldsymbol{x}^{(2)})}$. This along with the lack of feasibility-expansion implies violation of the alternate power-characterizing condition in Theorem 23 which implies that the aggregation operation cannot be elicitability-expanding by Theorem 9 and Theorem 29.

Any $\boldsymbol{d} \in \mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(A)}),\mathcal{V}(\boldsymbol{x}^{(A)})}$ such that $\boldsymbol{d} \notin \mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(1)}),\mathcal{V}(\boldsymbol{x}^{(1)})} \cup \mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(2)}),\mathcal{V}(\boldsymbol{x}^{(2)})}$ satisfies $\boldsymbol{d}_1 - \boldsymbol{d}_2 > 0$ and $-2\boldsymbol{d}_1 + \boldsymbol{d}_2 > 0$. These inequalities satisfy $\boldsymbol{d}_2 < \boldsymbol{d}_1 < 0$. Hence such a $\boldsymbol{d}$ cannot satisfy $\boldsymbol{d} \not\leq 0$. ■

On the other hand, feasibility expansion on its own guarantees power under some feature mapping as stated by the following proposition.

Proposition 14 *Fix conic constraints $\boldsymbol{C}$. If an aggregation operation $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ implements feasibility-expansion, then it is elicitability-expanding.*

Proof From the definition of the power-characterizing condition (Theorem 19), this condition is satisfied if the aggregation operation implements feasibility-expansion. Theorem 9 showing the sufficiency of the power-characterizing condition for elicitability-expansion implies that implementing feasibility-expansion is sufficient for elicitability-expansion. ■

D.4. Proofs for Section 3

Recall that the examples in this section use the feature weights matrix $\boldsymbol{\alpha}(q) := \begin{bmatrix} 1 & 0 & q \\ 0 & 1 & q \end{bmatrix}$.

D.4.1. ANALYSIS OF EXAMPLE 1

Proposition 15 *For the feature weights matrix $\boldsymbol{\alpha}_2$ and constraint matrix with the row $x_3 \leq x_1 + x_2$ in Example 1, the aggregation operation $\boldsymbol{x}^{(1)} = [1, 0, 1], \boldsymbol{x}^{(2)} = [0, 1, 1] \to \boldsymbol{x}^{(A)} = [0, 0, 1]$ is elicitability-expanding.*

Proof From the construction, it is easy to see that $x^{(1)}$ can be elicited with a linear reward function $[1, 0, 0]$ equal to the F_1 and budget level $E = 1$ and $x^{(2)}$ can be elicited with a linear reward function $[0, 1, 0]$ equal to the F_1 and budget level $E = 2$.

Let us use our characterization Theorem 22 to formally demonstrate elicitability-expansion.

The support sets of $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}$ are $\mathcal{S}(\boldsymbol{x}^{(1)}) = \{1, 3\}$ and $\mathcal{S}(\boldsymbol{x}^{(2)}) = \{2, 3\}$ respectively. The single conic constraint is binding for both $x^{(1)}, x^{(2)}$. The set $\mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(1)}),\mathcal{V}(x^{(1)})}$ is $\{\boldsymbol{d} : \boldsymbol{d}_3 \leq \boldsymbol{d}_1 + \boldsymbol{d}_2, \boldsymbol{d}_2 \geq 0, \boldsymbol{d}_1 + \boldsymbol{d}_2 + \boldsymbol{d}_3 < 0\}$. For any $\boldsymbol{d}$ in this set, $\boldsymbol{d}_3 < 0$ and $\boldsymbol{d}_1 + \boldsymbol{d}_2 < -\boldsymbol{d}_3$. The set $\mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(2)}),\mathcal{V}(x^{(2)})}$ is $\{\boldsymbol{d} : \boldsymbol{d}_3 \leq \boldsymbol{d}_1 + \boldsymbol{d}_2, \boldsymbol{d}_1 \geq 0, \boldsymbol{d}_1 + \boldsymbol{d}_2 + \boldsymbol{d}_3 < 0\}$. For any $\boldsymbol{d}$ in this set, $\boldsymbol{d}_3 < 0$ and $\boldsymbol{d}_1 + \boldsymbol{d}_2 < -\boldsymbol{d}_3$.

Now consider the set of feature-improving direction $\{\boldsymbol{d} : \boldsymbol{d}_1 + 2\boldsymbol{d}_3 \geq 0, \boldsymbol{d}_2 + 2\boldsymbol{d}_3 \geq 0\}$. For any $\boldsymbol{d}$ in this set, $\boldsymbol{d}_1 + \boldsymbol{d}_2 \geq -4d_3$.

All three conditions $\boldsymbol{d}_3 < 0$, $\boldsymbol{d}_1 + \boldsymbol{d}_2 < -\boldsymbol{d}_3$, and $\boldsymbol{d}_1 + \boldsymbol{d}_2 \geq -4d_3$ cannot be satisfied since for $\boldsymbol{d}_3 < 0$, $-\boldsymbol{d}_3 < -4\boldsymbol{d}_3$. Hence there is no intersection between feasibility improving directions and features improving directions and $x^{(1)}$ is elicitable. Similarly, $x^{(2)}$ is also elicitable.

$\boldsymbol{x}^{(A)}$ is not feasible and hence not elicitable. This shows that $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)} \to \boldsymbol{x}^{(A)}$ is elicitability-expanding by implementing feasibility-expansion. ■

D.4.2. ANALYSIS OF EXAMPLE 2

Proposition 16 *For the feature weights matrix $\boldsymbol{\alpha}(0.6)$ and null constraint matrix in Example 2, the aggregation operation $\boldsymbol{x}^{(1)} = [1, 0, 0], \boldsymbol{x}^{(2)} = [0, 1, 0] \to \boldsymbol{x}^{(A)} = [1/2, 1/2, 0]$ is elicitability-expanding.*

Proof The set of directions $\mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(1)}),\mathcal{V}(\boldsymbol{x}^{(1)})} = \{\boldsymbol{d} : \boldsymbol{d}_2 \geq 0, \boldsymbol{d}_3 \geq 0, \boldsymbol{d}_1 + \boldsymbol{d}_2 + \boldsymbol{d}_3 < 0\}$. And the set of feature-improving directions is $\mathcal{A} = \{\boldsymbol{d} : \boldsymbol{d}_1 + 0.6\boldsymbol{d}_3 \geq 0, \boldsymbol{d}_2 + 0.6\boldsymbol{d}_3 \geq 0\}$.

$\boldsymbol{d} \in \mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(1)}),\mathcal{V}(\boldsymbol{x}^{(1)})}$ means that $\boldsymbol{d}_1 < -(\boldsymbol{d}_2 + \boldsymbol{d}_3) < -\boldsymbol{d}_3$ and $\boldsymbol{d}_3 \geq 0$. $\boldsymbol{d} \in \mathcal{A}_1$ means that $\boldsymbol{d}_1 \geq -0.6\boldsymbol{d}_3$. These three conditions cannot be simultaneously showing that $\boldsymbol{x}^{(1)}$ is elicitable due to empty intersection of $\mathcal{A}$ and $\mathcal{B}_1$. Symmetrically, we can also show that $\boldsymbol{x}^{(2)}$ is also elicitable.

Now let us argue that $\boldsymbol{x}^{(A)} = [1/2, 1/2, 0]$ is not elicitable. $\mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(A)}),\mathcal{V}(\boldsymbol{x}^{(A)})} = \{\boldsymbol{d} : \boldsymbol{d}_3 \geq 0, \boldsymbol{d}_1 + \boldsymbol{d}_2 + \boldsymbol{d}_3 < 0\}$. Consider $\boldsymbol{d} = [-0.6, -0.6, 1]$. $\boldsymbol{d} \in \mathcal{A} \cap \mathcal{B}_A$. This shows that $\boldsymbol{x}^{(A)}$ is not elicitable.

■

D.4.3. ANALYSIS OF EXAMPLE 3

Proposition 17 *For the feature weights matrix $\boldsymbol{\alpha}(0.2)$ and conic constraint matrix with one constraint $x_1 + x_2 \leq x_3$ from Example 3, the aggregation operation $\boldsymbol{x}^{(1)} = [1, 0, 1], \boldsymbol{x}^{(2)} = [0, 1, 1] \to \boldsymbol{x}^{(A)} = [0, 0, 1]$ is elicitability-expanding.*

Proof [Proof of Theorem 17] The feature-improving directions are the set $\mathcal{A} = \{\boldsymbol{d} : \boldsymbol{d}_1 + 0.2\boldsymbol{d}_3 \geq 0, \boldsymbol{d}_2 + 0.2\boldsymbol{d}_3 \geq 0\}$.

The constraint is binding at both $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$. The feasibility improving directions are $\mathcal{B}_{(\mathcal{S}(\boldsymbol{x}^{(1)}),\mathcal{V}(\boldsymbol{x}^{(1)})} = \{\boldsymbol{d} : \boldsymbol{d}_1 + \boldsymbol{d}_2 \leq \boldsymbol{d}_3, \boldsymbol{d}_2 \geq 0, \boldsymbol{d}_1 + \boldsymbol{d}_2 + \boldsymbol{d}_3 < 0\}$.

If $\boldsymbol{d} \in \mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(1)}),\mathcal{V}(\boldsymbol{x}^{(1)})} \cap \mathcal{A}$, then $\boldsymbol{d}_1 + \boldsymbol{d}_2 + \boldsymbol{d}_3 < 0$, $\boldsymbol{d}_1 + \boldsymbol{d}_2 + \boldsymbol{d}_3 \leq 2\boldsymbol{d}_3$, $\boldsymbol{d} \in \mathcal{A}$, $\boldsymbol{d}_1 \geq -0.2\boldsymbol{d}_3$, and $\boldsymbol{d}_2 \geq -0.2\boldsymbol{d}_3$. This implies that $\boldsymbol{d}_3 < 0$. If all the conditions are satisfied simultaneously, then $\boldsymbol{d}_1 > 0$ and $\boldsymbol{d}_2 > 0$. This contradicts $\boldsymbol{d}_1 + \boldsymbol{d}_2 \leq \boldsymbol{d}_3 < 0$.

The conic constraint is not binding at $\boldsymbol{x}^{(A)}$. Now consider the feasibility improving directions of $\boldsymbol{d}^{(A)}$: $\mathcal{B}_{(\mathcal{S}(\boldsymbol{x}^{(A)}),\mathcal{V}(\boldsymbol{x}^{(A)})} = \{\boldsymbol{d} : \boldsymbol{d}_1 \geq 0, \boldsymbol{d}_2 \geq 0, \boldsymbol{d}_1 + \boldsymbol{d}_2 + \boldsymbol{d}_3 < 0\}$. The vector $\boldsymbol{d} = (0.2, 0.2, -1) \in \mathcal{A} \cap \mathcal{B}_A$ demonstrating that $\boldsymbol{x}^{(A)}$ is not elicitable. ■

D.5. Proofs in Section 3

D.5.1. PROOF OF THEOREM 18

Theorem 18 *Fix conic constraints $\boldsymbol{C}$, and any aggregation operation $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ where each $\boldsymbol{x}^{(k)}$ is feasible i.e., $\boldsymbol{C}\boldsymbol{x}^{(k)} \leq 0$, for every $k \in [K]$. If $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ is elicitability-expanding, then at least one of the following conditions holds:*

- *$\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ is feasibility-expanding relative to $\boldsymbol{C}$ (Theorem 5).*
- *For each $k \in [K]$, $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ is either support-expanding relative to k (Theorem 6) or binding set-contracting relative to k (Theorem 7).*

Proof [Proof of Theorem 18] We show this as a corollary of Theorem 9. We will prove this by showing that when both of the conditions in Theorem 18 are violated, the power-characterizing condition (Theorem 19) is violated and hence $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ cannot be elicitability-expanding. The violation of the conditions in Theorem 18 correspond to lack of implementation of any of the natural mechanisms.

One way of satisfying the power-characterizing condition is through implementing feasibility-expansion. Violating the conditions in Theorem 18 means feasibility-expansion is not implemented. We will show that the other way of satisfying the power-characterizing condition also does not hold.

When the second condition of Theorem 18 theorem is violated, there exists $k \in [K]$ with respect to which $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ is neither support-expanding nor binding-set contracting. That is, there is a k such that $\mathcal{V}(\boldsymbol{x}^{(k)}) \subseteq \mathcal{V}(\boldsymbol{x}^{(A)})$ and $\mathcal{S}(\boldsymbol{x}^{(k)}) \supseteq \mathcal{S}(\boldsymbol{x}^{(A)})$. As a result, the rows of $\boldsymbol{C}_{\mathcal{V}(\boldsymbol{x}^{(k)})}$ are a subset of the rows of $\boldsymbol{C}_{\mathcal{V}(\boldsymbol{x}^{(A)})}$, and the rows of $I_{\mathcal{S}(\boldsymbol{x}^{(k)})^c}$ are a subset of the rows of $I_{\mathcal{S}(\boldsymbol{x}^{(A)})^c}$. So every $\boldsymbol{d}$ in the feasibility-improving and budget-reducing directions set of $\boldsymbol{x}^{(A)}$ (i.e., $\boldsymbol{d} \in \{\boldsymbol{C}_{\mathcal{V}(\boldsymbol{x}^{(A)})}\boldsymbol{d} \leq 0,\ \boldsymbol{d}_{\mathcal{S}(\boldsymbol{x}^{(A)})^c} \geq 0,\ \mathbb{1}^\top \boldsymbol{d} = -1\}$) satisfies $\boldsymbol{C}_{\mathcal{V}(\boldsymbol{x}^{(k)})}\boldsymbol{d} \leq 0$ and $\boldsymbol{d}_{\mathcal{S}(\boldsymbol{x}^{(k)})^c} \geq 0$. Hence for any $\boldsymbol{\gamma}^{(k)} \in \mathbb{R}_{\geq 0}^{|\mathcal{V}(\boldsymbol{x}^{(k)})|}$ and $\boldsymbol{\lambda}^{(k)} \in \mathbb{R}_{\geq 0}^{|\mathcal{S}(\boldsymbol{x}^{(k)})^c|}$, writing $\boldsymbol{w}^{(k)} := \left(\boldsymbol{\gamma}^{(k)}\right)^\top \boldsymbol{C}_{\mathcal{V}(\boldsymbol{x}^{(k)})} - \left(\boldsymbol{\lambda}^{(k)}\right)^\top I_{\mathcal{S}(\boldsymbol{x}^{(k)})^c}$, we have $\left(\boldsymbol{\gamma}^{(k)}\right)^\top \boldsymbol{C}_{\mathcal{V}(\boldsymbol{x}^{(k)})}\boldsymbol{d} \leq 0$ (a non-negative combination of non-positive scalars) and $\left(\boldsymbol{\lambda}^{(k)}\right)^\top I_{\mathcal{S}(\boldsymbol{x}^{(k)})^c}\boldsymbol{d} \geq 0$ (a non-negative combination of non-negative scalars), so $\boldsymbol{w}^{(k)}\boldsymbol{d} \leq 0$. Combined with $|\mathbb{1}^\top \boldsymbol{d}| \cdot \min_{j \in [M]} \min(\boldsymbol{w}_j^{(k)}, 0) \leq 0$, this gives

$$\boldsymbol{w}^{(k)}\boldsymbol{d} \ + \ |\mathbb{1}^\top \boldsymbol{d}| \cdot \min_{j \in [M]} \min(\boldsymbol{w}_j^{(k)}, 0) \ \leq \ 0$$

for every $\boldsymbol{\gamma}^{(k)}, \boldsymbol{\lambda}^{(k)} \geq 0$. That is, the second case of the power-characterizing condition is violated. ■

Appendix E. Additional details for Section 4 (Characterization)

E.1. Formal power-characterizing condition

Below is the formal version of the power-characterizing condition described informally in Section 4.

Definition 19 *[Power-characterizing condition] Fix conic constraints $\boldsymbol{C}$. We say that the* ***power-characterizing condition*** *is satisfied for $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ if and only if one of the following two conditions hold:*

1. Feasibility expansion: $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ *implements feasibility-expansion for $\boldsymbol{C}$ or*
2. Strengthened support expansion or strengthened binding set contraction: *There exists $\boldsymbol{d} \in \mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(A)}),\mathcal{V}(\boldsymbol{x}^{(A)})}$ with $\boldsymbol{d} \not\leq 0$ such that for every $k \in [K]$, there exist $\boldsymbol{\gamma}^{(k)} \in \mathbb{R}_{\geq 0}^{|\mathcal{V}(\boldsymbol{x}^{(k)})|}$ and $\boldsymbol{\lambda}^{(k)} \in \mathbb{R}_{\geq 0}^{|\mathcal{S}(\boldsymbol{x}^{(k)})^c|}$ such that the following inequality holds: $\boldsymbol{w}^{(k)}\boldsymbol{d} + \left|\mathbb{1}^\top \boldsymbol{d}\right| \cdot \min_{j \in [M]} \min\left(\boldsymbol{w}_j^{(k)}, 0\right) > 0$, where $\boldsymbol{w}^{(k)} := \left(\boldsymbol{\gamma}^{(k)}\right)^\top C_{\mathcal{V}(\boldsymbol{x}^{(k)})} - \left(\boldsymbol{\lambda}^{(k)}\right)^\top I_{\mathcal{S}(\boldsymbol{x}^{(k)})^c}$.*

In the second case, the term $(\boldsymbol{\gamma}^{(k)})^\top C_{\mathcal{V}(\boldsymbol{x}^{(k)})}$ corresponds to strengthened binding-set contraction, and the term $(\boldsymbol{\lambda}^{(k)})^\top I_{\mathcal{S}(\boldsymbol{x}^{(k)})^c}$ corresponds to strengthened support expansion. As discussed informally in Section 4 and elaborated in Section E.3.1, this condition is a strict strengthening of the mechanisms.

E.2. Conceptual insights

Our characterization results offer conceptual insights into when system designers benefit from specific aggregation operations in compound AI systems. As an illustrative example, consider intersection aggregation $\mathcal{A}_{\text{intersect}}$ and addition aggregation $\mathcal{A}_{\text{add}}$.

Intersection aggregation: Intersection aggregation combines outputs based on commonality among different output vectors, which is conceptually similar to debate protocols [20] that aim

to create agreement or inference scaling methods that aim to filter out incorrect information [57]. Intersection aggregation can implement feasibility expansion and binding-set contraction, but not support expansion (Table 1 in Section D.3.1). However, feasibility expansion and binding-set contraction fundamentally rely on model capability limitations. If models are very capable (i.e., if they face no conic constraints), Theorem 9 demonstrates that intersection aggregation never adds power, regardless of whether the system designer employs sophisticated or unsophisticated prompt engineering practices.

Addition aggregation: Addition aggregation interpolates among different output directions, which conceptually captures how system designers synthesize multiple outputs to delegate specialized subtasks to each agent and synthesize the outputs of these subtasks [2, 4]. Addition aggregation can implement support expansion (Table 1 in Section D.3.1) and can actually implement the strengthened form of support expansion as well (Example 2). Theorem 9 shows that addition aggregation adds power even when models face no capability limitations (i.e, no conic constraints), at least for some level of prompt engineering limitations.

E.3. Additional Details for Section 4

E.3.1. CONNECTION OF THE POWER-CHARACTERIZING CONDITION TO MECHANISMS

The power-characterizing condition requires one of two cases to hold. The first is implementation of feasibility expansion. We can interpret the second case as unifying a strengthening of support expansion and a strengthening of binding set contraction into a single inequality. To demonstrate the connection between the power-characterizing condition and the mechanisms, we will first show how the power-characterizing condition implies implementation of one of the mechanisms. We will then show how the power-characterizing condition strengthens the mechanisms.

Power-characterization condition implies the mechanisms. The following result shows that the power-characterizing condition implies that at least one of feasibility-expansion, support expansion, or binding set contraction is implemented. This result immediately implies Theorem 18 (i.e., that implementing at least one of these mechanisms is necessary for elicitability-expansion).

Proposition 20 *Fix conic constraints $\boldsymbol{C}$, and any aggregation operation $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ where each $\boldsymbol{x}^{(k)}$ is feasible i.e., $\boldsymbol{C}\boldsymbol{x}^{(k)} \leq 0$, for every $k \in [K]$. If $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ satisfies the power-characterizing condition (Theorem 19), then one of the following conditions holds:*

- *$\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ is feasibility-expanding relative to $\boldsymbol{C}$ (Theorem 5).*
- *For each $k \in [K]$, $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ is either support-expanding relative to k (Theorem 6) or binding set-contracting relative to k (Theorem 7).*

Proof The first case for the power-characterizing condition to hold is feasibility expansion. Suppose feasibility expansion does not hold, i.e., all vectors in the aggregation are feasible. Then the second case of Theorem 19 holds: there exists $\boldsymbol{d} \in \mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(A)}), \mathcal{V}(\boldsymbol{x}^{(A)})}$ with $\boldsymbol{d} \not\leq 0$ such that for every $k \in [K]$, there exist $\boldsymbol{\gamma}^{(k)} \in \mathbb{R}_{\geq 0}^{|\mathcal{V}(\boldsymbol{x}^{(k)})|}$ and $\boldsymbol{\lambda}^{(k)} \in \mathbb{R}_{\geq 0}^{|\mathcal{S}(\boldsymbol{x}^{(k)})^c|}$ with

$$\boldsymbol{w}^{(k)}\boldsymbol{d} \;+\; |\mathbb{1}^\top \boldsymbol{d}| \cdot \min_{j \in [M]} \min\big(\boldsymbol{w}_j^{(k)},\, 0\big) \;>\; 0,$$

where $\boldsymbol{w}^{(k)} := \left(\boldsymbol{\gamma}^{(k)}\right)^\top \boldsymbol{C}_{\mathcal{V}(\boldsymbol{x}^{(k)})} - \left(\boldsymbol{\lambda}^{(k)}\right)^\top I_{\mathcal{S}(\boldsymbol{x}^{(k)})^c}$. Since $\min_{j\in[M]} \min(\boldsymbol{w}_j^{(k)}, 0) \leq 0$ and $|\mathbb{1}^\top \boldsymbol{d}| \geq 0$, the second term on the left-hand side is non-positive, so the inequality implies $\boldsymbol{w}^{(k)}\boldsymbol{d} > 0$, i.e.,

$$\left(\boldsymbol{\gamma}^{(k)}\right)^\top \boldsymbol{C}_{\mathcal{V}(\boldsymbol{x}^{(k)})}\boldsymbol{d} \;>\; \left(\boldsymbol{\lambda}^{(k)}\right)^\top I_{\mathcal{S}(\boldsymbol{x}^{(k)})^c}\boldsymbol{d}.$$

We will show that for each $k \in [K]$, this strict inequality implies either binding set contraction relative to k or support expansion relative to k.

Case 1: $\left(\boldsymbol{\gamma}^{(k)}\right)^\top \boldsymbol{C}_{\mathcal{V}(\boldsymbol{x}^{(k)})}\boldsymbol{d} > 0$ *implies binding set contraction relative to* k. Since $\boldsymbol{\gamma}^{(k)} \geq 0$ and $\left(\boldsymbol{\gamma}^{(k)}\right)^\top \boldsymbol{C}_{\mathcal{V}(\boldsymbol{x}^{(k)})}\boldsymbol{d} = \sum_{\ell\in\mathcal{V}(\boldsymbol{x}^{(k)})} \boldsymbol{\gamma}_\ell^{(k)} \boldsymbol{C}_\ell \boldsymbol{d} > 0$, there exists $\ell \in \mathcal{V}(\boldsymbol{x}^{(k)})$ with $\boldsymbol{C}_\ell \boldsymbol{d} > 0$. However, $\boldsymbol{d} \in \mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(A)}),\mathcal{V}(\boldsymbol{x}^{(A)})}$ requires $\boldsymbol{C}_\ell \boldsymbol{d} \leq 0$ for every $\ell \in \mathcal{V}(\boldsymbol{x}^{(A)})$. So $\ell \in \mathcal{V}(\boldsymbol{x}^{(k)}) \setminus \mathcal{V}(\boldsymbol{x}^{(A)})$, which is evidence that $\mathcal{V}(\boldsymbol{x}^{(A)}) \not\supseteq \mathcal{V}(\boldsymbol{x}^{(k)})$, i.e., binding set contraction relative to k.

Case 2: $\left(\boldsymbol{\gamma}^{(k)}\right)^\top \boldsymbol{C}_{\mathcal{V}(\boldsymbol{x}^{(k)})}\boldsymbol{d} \leq 0$ *implies support expansion relative to* k. In this case, the strict inequality forces $\left(\boldsymbol{\lambda}^{(k)}\right)^\top I_{\mathcal{S}(\boldsymbol{x}^{(k)})^c}\boldsymbol{d} < 0$, i.e., $\sum_{j\in\mathcal{S}(\boldsymbol{x}^{(k)})^c} \boldsymbol{\lambda}_j^{(k)} \boldsymbol{d}_j < 0$. So there exists $j \in \mathcal{S}(\boldsymbol{x}^{(k)})^c$ with $\boldsymbol{d}_j < 0$. Since $\boldsymbol{d} \in \mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(A)}),\mathcal{V}(\boldsymbol{x}^{(A)})}$ requires $\boldsymbol{d}_j \geq 0$ for every $j \in \mathcal{S}(\boldsymbol{x}^{(A)})^c$, we have $j \notin \mathcal{S}(\boldsymbol{x}^{(A)})^c$, i.e., $j \in \mathcal{S}(\boldsymbol{x}^{(A)}) \setminus \mathcal{S}(\boldsymbol{x}^{(k)})$. This is evidence that $\mathcal{S}(\boldsymbol{x}^{(A)}) \not\subseteq \mathcal{S}(\boldsymbol{x}^{(k)})$, i.e., support expansion relative to k. ■

How the power-characterizing condition strengthens the mechanisms. While the power-characterizing condition implies the implementation of one of the mechanisms, it is a strictly stronger condition. This is evidenced by the fact that the power-characterizing condition is necessary and sufficient for elicitability-expansion while implementing one of the mechanisms is not sufficient for elicitability-expansion as shown by propositions in Section D.3.2.

The power-characterizing condition, specifically the second case of Theorem 19, strengthens the mechanisms in two ways.

- First, we require a mechanism to jointly be implemented across all of the agents, rather than implemented for each agent individually. That is, we place a *joint* requirement across all $k \in [K]$, requiring that the same $\boldsymbol{d} \in \mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(A)}),\mathcal{V}(\boldsymbol{x}^{(A)})}$ witnesses the violation of binding constraints for every $k \in [K]$.

- Second, we strengthen the mechanism for each agent individually, requiring there to be a sufficient gap between the original outputs $\boldsymbol{x}^{(k)}$ and the aggregated output $\boldsymbol{x}^{(A)}$. Specifically, implementing support expansion or binding set contraction only requires the existence of $\boldsymbol{d} \in \mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(A)}),\mathcal{V}(\boldsymbol{x}^{(A)})}$ that violates some binding non-negativity or conic constraint of each $\boldsymbol{x}^{(k)}$. This can allow $\boldsymbol{x}^{(A)}$ to be very close any of the $\boldsymbol{x}^{(k)}$ vectors. In contrast, the power-characterizing condition requires violation of these binding constraints by a minimum margin. For example, strengthened support expansion asks for $\boldsymbol{d}$ to violate a binding non-negativity constraint by a minimum margin of $|\mathbb{1}^\top \boldsymbol{d}|$, requiring $\boldsymbol{x}^{(A)}$ to be sufficiently distinct from each $\boldsymbol{x}^{(k)}$.

The power-characterizing condition is in general a strictly stronger condition than implementing one of the mechanisms, as reflected by counterexamples in Section D.3.2. However, in some special cases, the power-characterizing condition corresponds exactly to implementing one of the mechanisms, instead of implementing a strengthening. One special case is when no vector in

the aggregation operation has any binding conic constraints. This holds when there are no conic constraints. In this special case, even the regular, non-strengthened form of binding set contraction cannot kick in. We can show that in this special case, the power-characterizing condition reduces to either feasibility-expansion or the usual, non-strengthened support expansion as long as a particular edge case does not occur (the proof is deferred to Section E.4.3).

Corollary 21 *Fix conic constraints $\boldsymbol{C}$, and any $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$. Suppose that $\mathcal{V}(x^{(A)}) = \mathcal{V}(x^{(1)}) = \ldots = \mathcal{V}(x^{(K)}) = \emptyset$. Suppose also that either $\boldsymbol{x}^{(A)}$ is not full support (i.e., $\mathcal{S}(\boldsymbol{x}^{(A)}) \neq [M]$) or there exists $j \in [M]$ such that no $\boldsymbol{x}^{(k)}$ has support $[M] \setminus \{j\}$. Then $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ is elicitability-expanding relative to $\boldsymbol{C}$ if and only if at least one of the following two conditions holds:*

- *$\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ is feasibility-expanding.*
- *$\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ is support-expanding relative to every $k \in [K]$.*

Note that it appears harder to show an analogous result for binding-set contraction. This is due to the joint geometry of the constraints that appears in the power-characterizing condition.

E.3.2. MAIN LEMMA: ELICITABILITY-EXPANSION FOR A GIVEN FEATURE MAP

In this section, we will start building tools to prove our main characterization result. The main technical lemma we use for our characterization of elicitability-expansion is the characterization of elicitability-expansion under a given feature map $\boldsymbol{\alpha}$. This lemma extends the characterization of previous work [35] for elicitability of an output vector under a feature map α in a setting without conic constraints on the agent's output vector. We extend this characterization to allow for output limitations (i.e., nontrivial constraints $\boldsymbol{C}$) and aggregation of multiple outputs.

This characterization depends on the set $\mathcal{B}_{\mathcal{S}(\boldsymbol{x}),\mathcal{V}(\boldsymbol{x})}$ of feasible, budget-reducing directions (Theorem 8) of the vectors $\boldsymbol{x}$ in the aggregation operation. The condition checks for intersection between these sets and the set of feature-improving directions $\left\{\boldsymbol{d} \in \mathbb{R}^M \mid \boldsymbol{\alpha}\boldsymbol{d} \geq \boldsymbol{0}\right\}$ consisting of directions that weakly increase *all* feature values.

Lemma 22 *Fix conic constraints $\boldsymbol{C}$, feature weights matrix $\boldsymbol{\alpha}$, and aggregation operation $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ where each $\boldsymbol{x}^{(k)}$ is feasible i.e., $\boldsymbol{C}\boldsymbol{x}^{(k)} \leq 0$, for every $k \in [K]$. The aggregation operation $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ is elicitability-expanding if and only if both of the following conditions hold:*

- *For every $k \in [K]$, $\mathcal{B}_{\mathcal{S}(x^{(k)}),\mathcal{V}(\boldsymbol{x}^{(k)})} \cap \left\{\boldsymbol{d} \in \mathbb{R}^M \mid \boldsymbol{\alpha}\boldsymbol{d} \geq \boldsymbol{0}\right\} = \emptyset$ and $\boldsymbol{C}\boldsymbol{x}^{(k)} \leq 0$.*
- *$\mathcal{B}_{\mathcal{S}(x^{(A)}),\mathcal{V}(x^{(A)})} \cap \left\{\boldsymbol{d} \in \mathbb{R}^M \mid \boldsymbol{\alpha}\boldsymbol{d} \geq \boldsymbol{0}\right\} \neq \emptyset$ or $\boldsymbol{C}\boldsymbol{x}^{(A)} \not\leq 0$.*

Lemma 22 characterizes the power of aggregation for a given feature weights matrix $\boldsymbol{\alpha}$. As a result, the characterizing condition depends on both the reward function specification limitation (which reflects prompt engineering limitations) via $\boldsymbol{\alpha}$ and the output limitation (which reflect model capability limitations) via the conic constraints $\mathbf{C}$. This highlights the role that both forms of limitations play in determining the power of aggregation in specific contexts.

Proof ideas. The full proof of Theorem 22 appears in Section E.4.2. The key idea is that elicitability of an output vector $\boldsymbol{x}$ is characterized by whether the set of feasible perturbation directions, $\mathcal{B}_{\mathcal{S}(\boldsymbol{x}),\mathcal{V}(\boldsymbol{x})}$, intersects the set $\{\boldsymbol{d} \in \mathbb{R}^M : \boldsymbol{\alpha d} \geq \boldsymbol{0}\}$. Lemmas 26 and 27 establish the necessity and sufficiency of this condition. This characterization yields conditions under which each individual output $\boldsymbol{x}^{(k)}$ for $k \in [K]$ is elicitable, while the aggregate output $\boldsymbol{x}^{(A)}$ is not, which is precisely the notion of elicitability expansion.

To prove the lemma, one direction is straightforward: if the intersection is nonempty, moving in any direction $\boldsymbol{d}$ in the intersection maintains feasibility and strictly increases every monotone reward over the features, resulting in a feasible vector strictly preferred over $\boldsymbol{x}$, providing a certificate that $\boldsymbol{x}$ cannot be elicitable.

The other direction of the characterization is more involved and shows that whenever the intersection is empty, there is a reward function that elicits $\boldsymbol{x}$. We can write the empty intersection as infeasibility of a system of linear inequalities. We certify emptiness via duality, and the dual certificate directly yields a reward function that elicits the desired input vectors. Specifically, using Motzkin's transposition, we can equivalently write this as a certificate in terms of dual variables. The dual variables from this certificate along with a linear reward function constructed based on these variables demonstrate optimality of $\boldsymbol{x}$ for the reward-maximization program by satisfying the KKT conditions of this program.

E.3.3. PROOF IDEAS FOR MAIN THEOREMS

In this section, we will provide proof sketches for our main theorems: Theorem 9 and Theorem 9 that provide the characterization of elicitability-expansion. We will build on the technical lemma (Theorem 22), but move beyond specific feature weights matrices $\boldsymbol{\alpha}$ to reason about elicitability-expansion under *some* feature weights matrix $\boldsymbol{\alpha}$.

Based on Theorem 22, one might expect that elicitability-expansion occurs exactly when $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ either implements feasibility-expansion or if there exists a direction $\boldsymbol{d} \in \mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(A)}),\mathcal{V}(\boldsymbol{x}^{(A)})}$ that does not exist in $\mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(k)}),\mathcal{V}(\boldsymbol{x}^{(k)})}$ for any $k \in [K]$. We might hope that the existence of such a $\boldsymbol{d}$ allows us to construct feature weights matrix $\boldsymbol{\alpha}$ where $\boldsymbol{d}$ is the only feature-improving direction, which would make $\boldsymbol{x}^{(A)}$ not elicitable while each $\boldsymbol{x}^{(k)}$ is elicitable.

However, it is not possible to construct $\boldsymbol{\alpha}$ with $\boldsymbol{d}$ being the only feature-improving direction. If $\boldsymbol{d}$ is a feature-improving direction (i.e., $\boldsymbol{\alpha d} \geq 0$), then every direction $\boldsymbol{u} + \lambda \boldsymbol{d}$ for $\boldsymbol{u}, \lambda > 0$ is also a feature-improving direction. Due to this property, a stronger empty intersection condition than each $\mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(k)}),\mathcal{V}(\boldsymbol{x}^{(k)})}$ and $\mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(A)}),\mathcal{V}(\boldsymbol{x}^{(A)})}$ having empty intersection turns out to be necessary for elicitability expansion. This stronger condition is stated below and we will show that this is actually equivalent to the power-characterizing condition.

Definition 23 *Fix constraints $\boldsymbol{C}$ and aggregation operation $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$. We say that the* ***alternate power-characterizing condition*** *is satisfied for $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ if (1) $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ implements feasibility-expansion, or (2) there exists $\boldsymbol{d}^{(A)} \in \mathcal{B}_{\mathcal{S}(x^{(A)}),\mathcal{V}(x^{(A)})}$ with $\boldsymbol{d}^{(A)} \not\leq 0$ such that:*

$$\left\{\boldsymbol{u} + \lambda \boldsymbol{d}^{(A)} \mid \boldsymbol{u} \in \mathbb{R}^M_{\geq 0}, \lambda \geq 0\right\} \bigcap \left(\bigcup_{k \in [K]} \mathcal{B}_{\mathcal{S}(x^{(k)}),\mathcal{V}(x^{(k)})} \right) = \varnothing.$$

Equivalence of both forms of power-characterizing condition. The alternate power-characterizing condition turns out to be exactly equivalent to the power-characterizing condition in Theorem 19, as stated in Theorem 29.

We provide a brief proof sketch of Theorem 29 here. The full proof is in Section E.4.4. Note that Theorem 19 is stated in terms of a $\boldsymbol{d} \in \mathcal{B}_{\mathcal{S}\boldsymbol{x}^{(A)},\mathcal{V}\boldsymbol{x}^{(A)}}$ violating constraints by a margin, whereas Theorem 23 is in terms of all non-negative scaled translations $\boldsymbol{u} + \lambda\boldsymbol{d}$ violating constraints. Let us now describe how violation by a margin relates to violation after non-negative translation. For simplicity, let us assume there is a single constraint. That is $\boldsymbol{C}$ has a single row.

To prevent against any budget-reducing $\boldsymbol{u} + \lambda\boldsymbol{d}$ satisfying the constraint, we need to prevent against the optimal choice of $\boldsymbol{u}$ picked for constraint satisfaction. This is the $\boldsymbol{u}$ that minimizes $\boldsymbol{C}\boldsymbol{u}$ subject to the budget-decreasing constraint, which is that $\|\boldsymbol{u}\|_1 \leq \lambda|\mathbb{1}^\top\boldsymbol{d}|$. The minimizing $\boldsymbol{u}$ is the zero vector if $\boldsymbol{C}$ has all positive coordinates. Otherwise, it places all of its weight on the most negative coordinate of $\boldsymbol{C}$. Ensuring that the minimum value of $\boldsymbol{C}\boldsymbol{u}$, which is $\lambda|\mathbb{1}^\top\boldsymbol{d}| \min_{j\in[M]} \min(0, \boldsymbol{C}_j)$, is lower than $\lambda\boldsymbol{C}\boldsymbol{d}$ is exactly the violation by a margin condition from the power-characterizing condition (with $\boldsymbol{w}^{(k)} = \boldsymbol{C}$).

The same reasoning applies to the binding non-negativity constraints $\boldsymbol{d}_j \geq 0$ for $j \in \mathcal{S}(\boldsymbol{x}^{(k)})^c$: the dual variable $\boldsymbol{\lambda}^{(k)}$ plays the analogous role of weighting these constraints, with $-I_{\mathcal{S}(\boldsymbol{x}^{(k)})^c}$ replacing $\boldsymbol{C}$. The unified row vector $\boldsymbol{w}^{(k)} = \left(\boldsymbol{\gamma}^{(k)}\right)^\top \boldsymbol{C}_{\mathcal{V}(\boldsymbol{x}^{(k)})} - \left(\boldsymbol{\lambda}^{(k)}\right)^\top I_{\mathcal{S}(\boldsymbol{x}^{(k)})^c}$ in Theorem 19 jointly certifies violation against both conic and non-negativity constraints defining $\mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(k)}),\mathcal{V}(\boldsymbol{x}^{(k)})}$.

Extending to multiple constraints (of either type) requires ensuring that no single translation $\boldsymbol{u}$ can satisfy all of them simultaneously. Rather than maximizing satisfaction of a single constraint, we must now consider the max-min problem: choosing $\boldsymbol{u}$ to maximize the minimum level of satisfaction across all constraints. By the minimax theorem, this max-min value equals the min-max value obtained by first choosing a weighting over constraints (i.e., choosing $\boldsymbol{\gamma}^{(k)}$ and $\boldsymbol{\lambda}^{(k)}$) and then choosing $\boldsymbol{u}$ to maximize satisfaction of the resulting weighted constraint. This duality reduces the multi-constraint case to the single-constraint analysis above, applied to the worst-case weighted combination $\boldsymbol{w}^{(k)}$.

Proof sketch of Theorem 9. The full proof is provided in Section E.4.5. It uses the ideas from the proof of Theorem 18 which shows the need for a feasible, budget-reducing direction of $\boldsymbol{x}^{(A)}$ that is not a feasible, budget-reducing direction for any other $\boldsymbol{x}^{(k)}$ for $k \in [K]$. Only then can there be a feature-improving direction that is viable for $\boldsymbol{x}^{(A)}$ but not any of the $\boldsymbol{x}^{(k)}$'s to result in elicitability of each $\boldsymbol{x}^{(k)}$ but not of $\boldsymbol{x}^{(A)}$. However, since all non-negative scaling and translations of feature-improving directions continue to be feature-improving, we further require positive scaling and translation of some direction $\boldsymbol{d}^{(A)} \in \mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(A)}),\mathcal{V}(\boldsymbol{x}^{(A)})}$ to not be present in any $\mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(k)}),\mathcal{V}(\boldsymbol{x}^{(k)})}$, $k \in [K]$. This is precisely the condition of the alternate power-characterizing condition.

Proof sketch of Theorem 9. The full proof is provided in Section E.4.6. We have shown why a stronger condition is still necessary for elicitability-expansion. Now let us show that this stronger condition is also sufficient. We show this by explicitly constructing a feature map $\boldsymbol{\alpha}$ for which the set $\{\boldsymbol{u} + \lambda\boldsymbol{d}^{(A)} \mid \boldsymbol{u} \in \mathbb{R}^M_{\geq 0}, \lambda \geq 0\}$ contains all feature-improving directions i.e., the set $\{\boldsymbol{d} : \boldsymbol{\alpha}\boldsymbol{d} \geq 0\}$. In the construction, feature improving directions can have any value on the positive coordinates of $\boldsymbol{d}^{(A)}$. But their negative coordinates cannot be too large compared to the positive coordinates.

E.4. Proofs for Section 4

E.4.1. CHARACTERIZATION OF SINGLE OUTPUT ELICITATION

A key technical tool for our characterization of elicitability through aggregation (provided by Theorem 22) is the characterization of direct elicitability of a vector $\boldsymbol{x}$ with a single agent, given a feature weights matrix $\boldsymbol{\alpha}$. This characterization is in terms of the intersection of feasible perturbation directions $\mathcal{B}_{\mathcal{S}(\boldsymbol{x}),\mathcal{V}(\boldsymbol{x})} = \{\boldsymbol{d} : C_{\mathcal{V}(\boldsymbol{x})}\boldsymbol{d} \leq 0\} \cap \{\boldsymbol{d} : \boldsymbol{d}_{\mathcal{S}(\boldsymbol{x})^c} \geq 0\} \cap \{\mathbb{1}^t\boldsymbol{d} < 0\}$ and feature-improving directions $d \in \mathbb{R}^M$: $\{\boldsymbol{d} : \boldsymbol{\alpha}\boldsymbol{d} \geq 0\}$. It is stated in the following proposition and it generalizes the characterization results in Kleinberg and Raghavan [35] to allow for conic constraints $\boldsymbol{C}$.

Proposition 24 (Single output elicitation characterization) *Given a feature weights matrix $\boldsymbol{\alpha}$ and conic constraints $\boldsymbol{C}$, an output vector $\boldsymbol{x} \gneq 0$ is elicitable if and only if it is feasible i.e., $\boldsymbol{C}\boldsymbol{x} \leq 0$ is and $\mathcal{B}_{\mathcal{S}(\boldsymbol{x}),\mathcal{V}(\boldsymbol{x})} \cap \{\boldsymbol{d} : \boldsymbol{\alpha}\boldsymbol{d} \geq 0\}$ is empty.*

We will prove this proposition by proving the necessary and sufficient directions through Theorem 26 and Theorem 27 respectively.

To prove these lemmas we will make use of the monotonicity property we assume our reward functions satisfy. Recall that we assume that reward functions do not decrease if all features are weakly increased, and strictly increase if some feature is increased.

A consequence of this monotonicity of rewards is that the budget necessary to elicit a vector $\boldsymbol{x}$ is its ℓ_1 norm $\|\boldsymbol{x}\|_1$. This is shown in the following lemma.

Lemma 25 *If a vector $\boldsymbol{x}$ is elicitable with budget E, then $\|\boldsymbol{x}\|_1 = E$.*

Proof If $\boldsymbol{x}$ is elicitable, then it must be feasible. So $\|\boldsymbol{x}\|_1 \leq E$ due to the budget constraint and $\boldsymbol{C}\boldsymbol{x} \leq 0$. For any monotone reward function R, there is a feature F_j such that increasing the value of feature F_j strictly increases the reward R. Since we assume that $\boldsymbol{\alpha}$ has no zero rows (Theorem 2), there is a coordinate i in the jth row of $\boldsymbol{\alpha}$ that is non-zero. Additionally, by Theorem 1, there is a vector $\boldsymbol{y} \geq 0$ with $\boldsymbol{y}_i > 0$ that satisfies $\boldsymbol{C}\boldsymbol{y} \leq 0$. Consider a scaled version $\boldsymbol{y}'$ of $\boldsymbol{y}$ with ℓ_1 norm less than $E - \|\boldsymbol{x}\|_1$. That is $\boldsymbol{y}' = (E - \|\boldsymbol{x}\|_1)\boldsymbol{y}/\|\boldsymbol{y}\|_1$. The vector $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{y}'$ has a strictly higher value of feature F_j and no lower values on other features. That is, $\boldsymbol{\alpha}\boldsymbol{x}' \geq \boldsymbol{\alpha}\boldsymbol{x}$ and $(\boldsymbol{\alpha}\boldsymbol{x}')_j > (\boldsymbol{\alpha}\boldsymbol{x})_j$. As a result, the reward function has a higher value on $\boldsymbol{x}'$ than on $\boldsymbol{x}$. $\boldsymbol{x}'$ satisfies the budget constraint since $\|\boldsymbol{x}\|_1 \leq \|\boldsymbol{x}\|_1 + \|\boldsymbol{y}'\|_1 \leq E$. It also satisfies conic constraints $\boldsymbol{C}$ since both $\boldsymbol{x}$ and $\boldsymbol{y}'$ satisfy $\boldsymbol{C}$. This shows that for any reward function, it is possible to construct another feasible vector with a higher reward if $\boldsymbol{x}$ has ℓ_1 norm less than the effort level E. Therefore, $\boldsymbol{x}$ is not elicitable if $\|\boldsymbol{x}\|_1 < E$. ■

Lemma 26 (Single output elicitation necessary) *Given a feature weights matrix $\boldsymbol{\alpha}$ and conic constraints $\boldsymbol{C}$, an output vector $\boldsymbol{x} \gneq 0$ is elicitable only if $\boldsymbol{x}$ is feasible and $\mathcal{B}_{\mathcal{S}(\boldsymbol{x}),\mathcal{V}(\boldsymbol{x})} \cap \{\boldsymbol{d} : \boldsymbol{\alpha}\boldsymbol{d} \geq 0\}$ is empty.*

Proof [Proof of Theorem 26] We will show that if $\mathcal{B}_{\mathcal{S}(\boldsymbol{x}),\mathcal{V}(\boldsymbol{x})} \cap \{\boldsymbol{\alpha}\boldsymbol{d} \geq 0\}$ is non-empty then $\boldsymbol{x}$ is not elicitable. If the intersection is non-empty, consider any $\boldsymbol{d} \in \mathcal{B}_{\mathcal{S}(\boldsymbol{x}),\mathcal{V}(\boldsymbol{x})} \cap \{\boldsymbol{\alpha}\boldsymbol{d} \geq 0\}$, we will use this $\boldsymbol{d}$ to construct a feasible output vector $\boldsymbol{y}$ with $\|\boldsymbol{y}\|_1 \leq \|\boldsymbol{x}\|_1$ and having strictly higher reward than $\boldsymbol{x}$ for every monotone reward function of the features. This vector $\boldsymbol{y}$ we construct is $\boldsymbol{y} = \|\boldsymbol{x}\|_1(\boldsymbol{x} + \lambda\boldsymbol{d})/\|\boldsymbol{x} + \lambda\boldsymbol{d}\|_1$ for an appropriate choice of λ that we will describe shortly.

First consider the vector $\boldsymbol{y}' = \boldsymbol{x} + \lambda \boldsymbol{d}$. Note that $\boldsymbol{y}'$ is feasible on all conic and non-negativity constraints that are binding at $\boldsymbol{x}$ due to $\boldsymbol{d}$'s membership in $\{\boldsymbol{d} : C_{\mathcal{V}(\boldsymbol{x})}\boldsymbol{d} \leq 0\} \cap \{\boldsymbol{d} : \boldsymbol{d}_{\mathcal{S}(\boldsymbol{x})^c} \geq 0\}$. We can choose λ to be small enough so that $\boldsymbol{y}'$ continues to meet all non-binding constraints. That is choose $\lambda < \min_{j \in \mathcal{V}(\boldsymbol{x})^c, C_j \boldsymbol{d} > 0} -\boldsymbol{C}_j \boldsymbol{x} / \boldsymbol{C}_j \boldsymbol{d}$ and $\min_{i \in \mathcal{S}(\boldsymbol{x}), \boldsymbol{d}_i < 0} -\boldsymbol{x}_i / \boldsymbol{d}_i$. This establishes that we have a positive choice of λ making $\boldsymbol{y}'$ satisfy the nonnegativity and conic constraints. Additionally, we have that $\mathbb{1}^t \boldsymbol{y}' = \|\boldsymbol{y}'\|_1 = \|\boldsymbol{x}\|_1 + \lambda \mathbb{1}^t d < \|\boldsymbol{x}\|_1$. That is, $\boldsymbol{y}'$ satisfies any budget constraint that is satisfied by $\boldsymbol{x}$, and satisfies with a strictly larger margin. Hence $\boldsymbol{y}'$ is feasible relative to the conic constraints and budget constraints from any level of specified budget.

We also have that $\boldsymbol{\alpha}^t \boldsymbol{y}' = \boldsymbol{\alpha}^t (\boldsymbol{x} + \lambda \boldsymbol{d}) \geq \boldsymbol{\alpha}^t \boldsymbol{x}$ since $\boldsymbol{\alpha}^t \boldsymbol{d} \geq 0$. Hence $\boldsymbol{y}'$ satisfies feasibility constraints and has at least as high values on all features. By Theorem 25, y' is not elicitable with budget $\|\boldsymbol{x}\|_1$ since $\|\boldsymbol{y}'\|_1 < \|\boldsymbol{x}\|_1$. This means that for any monotone reward function R, there is a feasible vector $\boldsymbol{x}'$ with $\|\boldsymbol{x}'\| \leq \|\boldsymbol{x}\|_1$ having $R(\boldsymbol{x}') > R(\boldsymbol{y}')$. Since $\boldsymbol{\alpha}\boldsymbol{y}' \geq \boldsymbol{\alpha}\boldsymbol{x}$, $R(\boldsymbol{y}') \geq R(\boldsymbol{x})$. In particular, $\boldsymbol{x}'$ has a strictly higher reward than $\boldsymbol{x}$. This means that $\boldsymbol{x}$ cannot be elicited with effort level $\|\boldsymbol{x}\|_1$ which is the only possible level at which $\boldsymbol{x}$ can be elicited by Theorem 25. So, $\boldsymbol{x}$ cannot be elicited if $\mathcal{B}_{\mathcal{S}(\boldsymbol{x}), \mathcal{V}(\boldsymbol{x})} \cap \{\boldsymbol{\alpha}\boldsymbol{d} \geq 0\} \neq \emptyset$. ■

Lemma 27 (Single output elicitation sufficient) *Given feature weights matrix $\boldsymbol{\alpha}$ and conic constraints, $\boldsymbol{C}$, an output vector $\boldsymbol{x} \geq 0$ is elicitable if $\boldsymbol{x}$ satisfies the conic constraints i.e., $\boldsymbol{C}\boldsymbol{x} \leq 0$ and $\mathcal{B}_{\mathcal{S}(\boldsymbol{x}), \mathcal{V}(\boldsymbol{x})} \cap \{\boldsymbol{d} : \boldsymbol{\alpha}\boldsymbol{d} \geq 0\}$ is empty.*

Proof Let us denote $S := \mathcal{S}(\boldsymbol{x})$ and $V := \mathcal{V}(\boldsymbol{x})$. And let $D_{\boldsymbol{\alpha}}$ be the set of feature-improving directions $\{\boldsymbol{\alpha}\boldsymbol{d} \geq 0\}$. Under the condition that $\boldsymbol{x}$ is feasible and $\mathcal{B}_{S,V} \cap D_{\boldsymbol{\alpha}}$ is empty, we will show that $\boldsymbol{x}$ is elicitable by explicitly constructing a reward function that elicits $\boldsymbol{x}$ with budget $\|\boldsymbol{x}\|_1$.

Dual certificate of empty intersection. The condition of empty intersection of the sets $\mathcal{B}_{S,V}$ and $D_{\boldsymbol{\alpha}}$ can equivalently be written as the infeasibility of the system :

$$C_V d \leq 0, \qquad I_{S^c} d \geq 0, \qquad \alpha^\top d \geq 0, \qquad \mathbf{1}^\top d < 0, \tag{1}$$

where $I_{S^c} \in \mathbb{R}^{|S^c| \times M}$ is the subset of rows of the $M \times M$ identity matrix indexed by the set S^c.

By Motzkin's transposition theorem, infeasibility of (1) implies the existence of dual variables

$$\gamma \in \mathbb{R}^{|V|}_{\geq 0}, \quad \lambda \in \mathbb{R}^{|S^c|}_{\geq 0}, \quad \nu \in \mathbb{R}^N_{\geq 0}, \quad \tau > 0$$

such that

$$C_V^\top \gamma \; - \; I_{S^c}^\top \lambda \; + \; \tau \mathbf{1} \; - \; \boldsymbol{\alpha}\, \nu \; = \; 0 \tag{2}$$

Reward function construction. We will now define a reward function that is linear in the features to elicit a vector $\boldsymbol{x}$. We will later show how this reward function along with budget $\|\boldsymbol{x}\|_1$ elicits the vector $\boldsymbol{x}$ when the empty intersection condition is satisfied. The reward function is

$$R(\boldsymbol{u}) \; = \; \sum_{i=1}^{N} \beta_i f_i((\boldsymbol{\alpha}^\top \boldsymbol{u})_i) \qquad \text{with} \qquad \beta_i \; := \; \frac{\nu_i}{f_i'\big((\boldsymbol{\alpha}^\top \boldsymbol{x})_i\big)},$$

which is well-defined since each f_i is strictly increasing, hence $f_i'((\boldsymbol{\alpha}^\top \boldsymbol{x})_i) > 0$. Because each f_i is concave and increasing, R is concave. Its gradient at x is

$$\nabla R(\boldsymbol{x}) \; = \; \sum_{i=1}^{N} \beta_i f_i'\big((\boldsymbol{\alpha}^\top \boldsymbol{x})_i\big)\, \alpha_{\cdot,i} \; = \; \alpha\, \nu,$$

where $\alpha_{\cdot,i}$ is the i-th column of α.

Elicitability. From Theorem 25 we know that a vector $\boldsymbol{x}$ can only be elicited with a budget of $E = \|\boldsymbol{x}\|_1$. So let us consider the constrained reward maximization program with this budget E and a reward function R. We will show that when $\mathcal{B}_{S,V}$ and $D_{\boldsymbol{\alpha}}$ have empty intersection, the dual certificate of this empty intersection that we computed before along with $\boldsymbol{x}$ satisfies the KKT conditions of the reward-maximizing optimization program.

$$\max_{\boldsymbol{u}\in\mathbb{R}^M} \quad R(\boldsymbol{f}(\boldsymbol{\alpha}^\top \boldsymbol{u})) \quad \text{s.t.} \quad \boldsymbol{C}\boldsymbol{u} \le 0, \;\; \boldsymbol{u} \ge 0, \;\; \boldsymbol{1}^\top \boldsymbol{u} \le E.$$

This is a concave program, and its Lagrangian is

$$\mathcal{L}(\boldsymbol{u}, \lambda_0, \boldsymbol{\mu}, \tilde{\boldsymbol{\gamma}}) \;=\; R(\boldsymbol{f}(\boldsymbol{\alpha}^\top \boldsymbol{u})) \;+\; \lambda_0\,(E - \boldsymbol{1}^\top \boldsymbol{u}) \;+\; \boldsymbol{\mu}^\top \boldsymbol{u} \;-\; \tilde{\boldsymbol{\gamma}}^\top (\boldsymbol{C}\boldsymbol{u}),$$

with dual variables $\lambda_0 \ge 0$, $\boldsymbol{\mu} \ge 0$, $\tilde{\boldsymbol{\gamma}} \ge 0$. Evaluate the KKT conditions at $\boldsymbol{u} = \boldsymbol{x}$ with the choice of dual variables from the dual certificate of the empty intersection condition. That is, the dual variables are

$$\lambda_0 := \tau, \qquad \mu_S := 0, \;\; \mu_{S^c} := \lambda, \qquad \tilde{\gamma}_V := \gamma, \;\; \tilde{\gamma}_{V^c} := 0.$$

Primal feasibility holds due to feasibility of $\boldsymbol{x}$ by definition of S, V. Complementary slackness holds since $x_j = 0$ for $j \in S^c$ and $(Cx)_\ell = 0$ for $\ell \in V$, while $\mu_S = 0$. For stationarity,

$$\nabla R(x) \;-\; \lambda_0 \boldsymbol{1} \;+\; \mu \;-\; C^\top \tilde{\gamma} \;=\; \alpha\nu \;-\; \tau \boldsymbol{1} \;+\; I_{S^c}^\top \lambda \;-\; C_V^\top \gamma \;=\; 0$$

by (2).

Since R is concave and the constraints are linear, the KKT conditions are sufficient to certify optimality; hence x maximizes R over the feasible region and is therefore elicitable. ■

Proof [Proof of Theorem 24] This follows from Theorem 26, Theorem 27 showing necessity and sufficiency of the condition of non-emptiness of $\mathcal{B}_{\mathcal{S}(\boldsymbol{x}),\mathcal{V}(\boldsymbol{x})} \cap \{\boldsymbol{\alpha}\boldsymbol{d} \ge 0\}$ for elicitability of $\boldsymbol{x}$. ■

An immediate consequence of the condition characterizing elicitability of a single output vector from Theorem 24 is that the role of the budget is only in scaling the ℓ_1 norm of elicitable output vectors.

Corollary 28 *A vector $\boldsymbol{x}$ is elicitable with some budget E if and only $\boldsymbol{x}/\|\boldsymbol{x}\|_1$ is elicitable with budget 1 for the same reward function.*

Proof The condition characterizing the elicitability of $\boldsymbol{x}$ given a feature weights matrix $\boldsymbol{\alpha}$ is whether or not the sets $\mathcal{B}_{\mathcal{S}(\boldsymbol{x}),\mathcal{V}(\boldsymbol{x})}$ and $\{\boldsymbol{d} : \boldsymbol{\alpha}\boldsymbol{d} \ge 0\}$ intersect (Theorem 22). Note that the sets $\mathcal{B}_{\mathcal{S}(\boldsymbol{x}),\mathcal{V}(\boldsymbol{x})}$ and $\mathcal{B}_{\mathcal{S}(\boldsymbol{x}/\|\boldsymbol{x}\|_1),\mathcal{V}(\boldsymbol{x}/\|\boldsymbol{x}\|_1)}$ are equal because the supports and binding sets of $\boldsymbol{x}$ and $\boldsymbol{x}/\|\boldsymbol{x}\|_1$ are the same. As a result, $\boldsymbol{x}$ is elicitable if and only if $\boldsymbol{x}/\|\boldsymbol{x}\|_1$ is elicitable. From Theorem 25, $\boldsymbol{x}$, if elicitable, is elicitable with budget $\|\boldsymbol{x}\|_1$ and similarly, $\boldsymbol{x}/\|\boldsymbol{x}\|_1$, if elicitable, is elicitable with budget 1. ■

E.4.2. PROOF OF THEOREM 22

Proof [Proof of Theorem 22] This follows directly from the characterization of elicitability of a single output vector $\boldsymbol{x}$ for a given feature weights matrix $\boldsymbol{\alpha}$ provided by Theorem 24. The condition characterizing whether an aggregation operation $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ is elicitability-expanding given a feature weights matrix $\boldsymbol{\alpha}$ is the condition that each $\boldsymbol{x}^{(k)}$ is elicitable under $\boldsymbol{\alpha}$ and $\boldsymbol{x}^{(A)}$ is not elicitable. Each of these conditions are provided by Theorem 24. The condition for $\boldsymbol{x}^{(k)}$ to be elicitable under $\boldsymbol{x}^{(k)}$ to be elicitable is that $\mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(k)}),\mathcal{V}(\boldsymbol{x}^{(k)})}$ and $\{\boldsymbol{d} : \boldsymbol{\alpha}\boldsymbol{d} \geq 0\}$ have empty intersection and the condition for $\boldsymbol{x}^{(A)}$ to not be elicitable is that $\mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(A)}),\mathcal{V}(\boldsymbol{x}^{(A)})}$ and $\{\boldsymbol{d} : \boldsymbol{\alpha}\boldsymbol{d} \geq 0\}$ have non-empty intersection. ■

E.4.3. PROOF OF THEOREM 21

Proof [Proof of Theorem 21]

The assumptions are (i) no conic constraint is binding for these vectors, and (ii) $\boldsymbol{x}^{(A)}$ is not full support (i.e., $\mathcal{S}(\boldsymbol{x}^{(A)}) \neq [M]$) or there exists $j \in [M]$ such that no $\boldsymbol{x}^{(k)}$ has support $[M] \setminus \{j\}$.

By Theorem 9 and Theorem 9, the aggregation operation is elicitability-expanding if and only if the power-characterizing condition is satisfied. Because of assumption (i), $\mathcal{V}(\boldsymbol{x}^{(k)}) = \emptyset$ for every $k \in [K]$, so $\boldsymbol{\gamma}^{(k)} \in \mathbb{R}_{\geq 0}^{|\mathcal{V}(\boldsymbol{x}^{(k)})|}$ is trivially zero in the second case of Theorem 19. The condition therefore reduces to either feasibility expansion or a strengthened form of support expansion (involving only $\boldsymbol{\lambda}^{(k)}$). It suffices to show that under assumptions (i) and (ii), strengthened support expansion is equivalent to support expansion.

We know that (ii) coupled with support expansion implies that either

$$\mathcal{S}(\boldsymbol{x}^{(A)}) \not\subseteq \mathcal{S}(\boldsymbol{x}^{(k)}) \text{ for all } k \in [K] \quad \text{and} \quad \mathcal{S}(\boldsymbol{x}^{(A)}) \neq [M],$$

or

$$\mathcal{S}(\boldsymbol{x}^{(A)}) = [M] \text{ and } \mathcal{S}(\boldsymbol{x}^{(k)}) \neq [M] \forall k \in [K] \text{ and there exists } j \in [M] \text{ s.t. } \mathcal{S}(\boldsymbol{x}^{(k)}) \neq [M] \setminus \{j\} \forall k \in [K].$$

A useful equivalent form of (ii) coupled with support expansion. We first show that (ii) coupled with support expansion is equivalent to the existence of indices $j(k) \in \mathcal{S}(\boldsymbol{x}^{(A)}) \setminus \mathcal{S}(\boldsymbol{x}^{(k)})$ for each $k \in [K]$ such that the set

$$J := \bigcup_{k \in [K]} \{j(k)\}$$

is a *strict* subset of $[M]$. Indeed, for any selection of $j(k)$ we always have $J \subseteq \mathcal{S}(\boldsymbol{x}^{(A)})$. If $\mathcal{S}(\boldsymbol{x}^{(A)}) \neq [M]$, then necessarily $J \subsetneq [M]$. Otherwise, $\mathcal{S}(\boldsymbol{x}^{(A)}) = [M]$, and the existence of some j with $[M] \setminus \{j\} \not\subseteq \mathcal{S}(\boldsymbol{x}^{(k)})$ for all k is equivalent to being able to choose all $j(k)$ from $[M] \setminus \{j\}$, which forces $J \subseteq [M] \setminus \{j\} \subsetneq [M]$.

(ii) coupled with support expansion $\Rightarrow$ Strengthened support expansion. Assume the above condition holds: there exists $J \subsetneq [M]$ and indices $j(k) \in \mathcal{S}(\boldsymbol{x}^{(A)}) \setminus \mathcal{S}(\boldsymbol{x}^{(k)})$ with $J = \bigcup_{k \in [K]} \{j(k)\}$. Fix any $c > 0$ and define $\boldsymbol{d} \in \mathbb{R}^M$ coordinate-wise by

$$\boldsymbol{d}_j := \begin{cases} -1, & j \in J, \\ c, & j \notin J, \end{cases}$$

where c is a constant chosen such that

$$\frac{|J|-1}{M-|J|} < c < \frac{|J|}{M-|J|}.$$

Since $J \subseteq \mathcal{S}(\boldsymbol{x}^{(A)})$, setting $\boldsymbol{d}_j < 0$ on J does not violate feasibility for coordinates outside the support, and we have $\boldsymbol{d}_{\mathcal{S}(\boldsymbol{x}^{(A)})^c} \geq 0$. Moreover,

$$\mathbb{1}^\top \boldsymbol{d} = -|J| + (M-|J|)c < 0.$$

Thus $\boldsymbol{d} \in \mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(A)}),\emptyset}$. Finally, for each $k \in [K]$ we have $\boldsymbol{d}_{j(k)} = -1$ and therefore

$$-|\mathbb{1}^\top \boldsymbol{d}| = -|J| + (M-|J|)c > -1 = \boldsymbol{d}_{j(k)},$$

so strengthened support expansion holds.

Strengthened support expansion ⇒ support expansion. This follows by the same argument as Proposition 20.

■

E.4.4. PROOF OF EQUIVALENCE BETWEEN THE VERSIONS OF THE POWER-CHARACTERIZING CONDITION DEFINED IN THEOREM 19 AND THEOREM 23

To prove Theorem 9 and Theorem 9, we will use an alternate but equivalent way of expressing the power-characterizing condition defined in Theorem 19. This equivalent condition is defined in Theorem 23. The equivalence between both conditions is stated in the following proposition.

Proposition 29 *The conditions defined in Theorem 19 and Theorem 23 are equivalent.*

Proof

For ease of notation, let $V_k := \mathcal{V}(\boldsymbol{x}^{(k)})$, $S_k := \mathcal{S}(\boldsymbol{x}^{(k)})$ for $k \in [K]$, and let $V_A := \mathcal{V}(\boldsymbol{x}^{(A)})$, $S_A := \mathcal{S}(\boldsymbol{x}^{(A)})$. Both definitions agree in the feasibility-expansion case, so it suffices to compare their second cases. We will show the following per-k equivalence: for any fixed $\boldsymbol{d} \in \mathcal{B}_{S_A,V_A}$ with $\boldsymbol{d} \not\leq 0$ and any $k \in [K]$,

(1) $\left\{\boldsymbol{u} + \lambda \boldsymbol{d} : \boldsymbol{u} \in \mathbb{R}^M_{\geq 0},\ \lambda \geq 0\right\} \cap \mathcal{B}_{S_k,V_k} = \varnothing,$ if and only if

(2) there exist $\boldsymbol{\gamma}^{(k)} \in \mathbb{R}^{|V_k|}_{\geq 0}$ and $\boldsymbol{\lambda}^{(k)} \in \mathbb{R}^{|S_k^c|}_{\geq 0}$ such that, writing $\boldsymbol{w}^{(k)} := \left(\boldsymbol{\gamma}^{(k)}\right)^\top \boldsymbol{C}_{V_k} - \left(\boldsymbol{\lambda}^{(k)}\right)^\top I_{S_k^c}$,

$$\boldsymbol{w}^{(k)}\boldsymbol{d} \ + \ \left|\mathbb{1}^\top \boldsymbol{d}\right| \cdot \min_{j\in[M]} \min\left(\boldsymbol{w}_j^{(k)},\, 0\right) \ > \ 0.$$

The per-k equivalence implies equivalence of the two definitions: the second case of Theorem 23 requires (1) to hold for every $k \in [K]$ for a single $\boldsymbol{d}$, and the second case of Theorem 19 requires (2) to hold for every $k \in [K]$ for a single $\boldsymbol{d}$.

Reformulating empty intersection as feasibility of an LP. The intersection in (1) is non-empty if and only if there exist $\boldsymbol{u} \in \mathbb{R}^M_{\geq 0}$ and $\lambda \geq 0$ such that

$$\boldsymbol{C}_{V_k}(\boldsymbol{u} + \lambda \boldsymbol{d}) \leq \boldsymbol{0}, \qquad -I_{S_k^c}(\boldsymbol{u} + \lambda \boldsymbol{d}) \leq \boldsymbol{0}, \qquad \mathbb{1}^\top(\boldsymbol{u} + \lambda \boldsymbol{d}) < 0.$$

Both (1) and the inequality in (2) are positively homogeneous in $\boldsymbol{d}$: scaling $\boldsymbol{d} \mapsto t\boldsymbol{d}$ for $t > 0$ rescales λ in (1) and rescales both terms equally in (2). We may therefore normalize $\boldsymbol{d}$ so that $\mathbb{1}^\top \boldsymbol{d} = -1$; under this normalization, $|\mathbb{1}^\top \boldsymbol{d}| = 1$. Note that λ must be strictly positive: $\boldsymbol{u} \geq \boldsymbol{0}$ implies $\mathbb{1}^\top \boldsymbol{u} \geq 0$, while $\mathbb{1}^\top(\boldsymbol{u} + \lambda \boldsymbol{d}) < 0$ together with $\mathbb{1}^\top \boldsymbol{d} = -1$ forces $\lambda > 0$. Setting $\boldsymbol{v} := \boldsymbol{u}/\lambda \in \mathbb{R}^M_{\geq 0}$ and dividing the inequalities by λ, the non-empty intersection is equivalent to the existence of $\boldsymbol{v} \in \mathbb{R}^M_{\geq 0}$ with $\mathbb{1}^\top \boldsymbol{v} < 1$ such that

$$\boldsymbol{C}_{V_k}(\boldsymbol{d} + \boldsymbol{v}) \leq \boldsymbol{0} \qquad \text{and} \qquad -I_{S_k^c}(\boldsymbol{d} + \boldsymbol{v}) \leq \boldsymbol{0}.$$

Dualizing the inequality system. A finite system of linear inequalities $A\boldsymbol{y} \leq \boldsymbol{0}$ holds if and only if every non-negative weighted combination of its rows holds, i.e., $\boldsymbol{\gamma}^\top A\boldsymbol{y} \leq 0$ for every $\boldsymbol{\gamma} \geq \boldsymbol{0}$. Applying this to the system above, the non-empty intersection is equivalent to the existence of $\boldsymbol{v} \in \mathbb{R}^M_{\geq 0}$ with $\mathbb{1}^\top \boldsymbol{v} \leq 1$ such that

$$\big(\boldsymbol{\gamma}^{(k)\top} \boldsymbol{C}_{V_k} - \boldsymbol{\lambda}^{(k)\top} I_{S_k^c}\big)(\boldsymbol{d} + \boldsymbol{v}) \;\leq\; 0 \qquad \forall \boldsymbol{\gamma}^{(k)} \in \mathbb{R}^{|V_k|}_{\geq 0},\; \boldsymbol{\lambda}^{(k)} \in \mathbb{R}^{|S_k^c|}_{\geq 0}.$$

We have replaced the strict inequality $\mathbb{1}^\top \boldsymbol{v} < 1$ with the closed inequality ≤ 1; this is without loss of generality because the value of the objective is continuous in $\boldsymbol{v}$ on a closed feasible set. We may also restrict $\boldsymbol{\gamma}^{(k)}, \boldsymbol{\lambda}^{(k)}$ to bounded norm $\|\boldsymbol{\gamma}^{(k)}\|_1 \leq 1$, $\|\boldsymbol{\lambda}^{(k)}\|_1 \leq 1$ without loss of generality, since the objective is positively homogeneous in $(\boldsymbol{\gamma}^{(k)}, \boldsymbol{\lambda}^{(k)})$. So the non-empty intersection is equivalent to

$$\inf_{\substack{\boldsymbol{v} \in \mathbb{R}^M_{\geq 0} \\ \mathbb{1}^\top \boldsymbol{v} \leq 1}} \; \sup_{\substack{\boldsymbol{\gamma}^{(k)} \in \mathbb{R}^{|V_k|}_{\geq 0},\, \|\boldsymbol{\gamma}^{(k)}\|_1 \leq 1 \\ \boldsymbol{\lambda}^{(k)} \in \mathbb{R}^{|S_k^c|}_{\geq 0},\, \|\boldsymbol{\lambda}^{(k)}\|_1 \leq 1}} \boldsymbol{w}^{(k)}(\boldsymbol{d} + \boldsymbol{v}) \;\leq\; 0,$$

where $\boldsymbol{w}^{(k)} := \boldsymbol{\gamma}^{(k)\top} \boldsymbol{C}_{V_k} - \boldsymbol{\lambda}^{(k)\top} I_{S_k^c}$.

Applying the minimax theorem. The objective $\boldsymbol{w}^{(k)}(\boldsymbol{d} + \boldsymbol{v})$ is bilinear in $(\boldsymbol{\gamma}^{(k)}, \boldsymbol{\lambda}^{(k)})$ and $\boldsymbol{v}$, and both feasible sets are convex and compact. Sion's minimax theorem applies, so we may swap the order of $\inf$ and $\sup$:

$$\sup_{\substack{\boldsymbol{\gamma}^{(k)} \in \mathbb{R}^{|V_k|}_{\geq 0},\, \|\boldsymbol{\gamma}^{(k)}\|_1 \leq 1 \\ \boldsymbol{\lambda}^{(k)} \in \mathbb{R}^{|S_k^c|}_{\geq 0},\, \|\boldsymbol{\lambda}^{(k)}\|_1 \leq 1}} \; \inf_{\substack{\boldsymbol{v} \in \mathbb{R}^M_{\geq 0} \\ \mathbb{1}^\top \boldsymbol{v} \leq 1}} \boldsymbol{w}^{(k)}(\boldsymbol{d} + \boldsymbol{v}) \;\leq\; 0.$$

Closed form of the inner infimum. For fixed $\boldsymbol{\gamma}^{(k)}, \boldsymbol{\lambda}^{(k)}$ (and hence fixed $\boldsymbol{w}^{(k)}$), we compute $\inf_{\boldsymbol{v} \in \mathbb{R}^M_{\geq 0},\, \mathbb{1}^\top \boldsymbol{v} \leq 1} \boldsymbol{w}^{(k)} \boldsymbol{v}$. If $\boldsymbol{w}^{(k)}$ has any strictly negative coordinate, the infimum is $\min_j \boldsymbol{w}^{(k)}_j$, achieved by placing all of $\boldsymbol{v}$'s mass on the most negative coordinate of $\boldsymbol{w}^{(k)}$. If $\boldsymbol{w}^{(k)} \geq \boldsymbol{0}$, the infimum is 0, achieved at $\boldsymbol{v} = \boldsymbol{0}$. In both cases, $\inf_{\boldsymbol{v}} \boldsymbol{w}^{(k)} \boldsymbol{v} = \min_{j \in [M]} \min(\boldsymbol{w}^{(k)}_j, 0)$, so

$$\inf_{\substack{\boldsymbol{v} \in \mathbb{R}^M_{\geq 0} \\ \mathbb{1}^\top \boldsymbol{v} \leq 1}} \boldsymbol{w}^{(k)}(\boldsymbol{d} + \boldsymbol{v}) \;=\; \boldsymbol{w}^{(k)} \boldsymbol{d} + \min_{j \in [M]} \min(\boldsymbol{w}^{(k)}_j, 0).$$

Identifying the certificate. Combining, the non-empty intersection is equivalent to

$$\sup_{\substack{\boldsymbol{\gamma}^{(k)}\in\mathbb{R}_{\geq 0}^{|V_k|},\ \|\boldsymbol{\gamma}^{(k)}\|_1\leq 1\\ \boldsymbol{\lambda}^{(k)}\in\mathbb{R}_{\geq 0}^{|S_k^c|},\ \|\boldsymbol{\lambda}^{(k)}\|_1\leq 1}} \Big(\boldsymbol{w}^{(k)}\boldsymbol{d} + \min_{j\in[M]} \min(\boldsymbol{w}_j^{(k)}, 0)\Big) \ \leq\ 0,$$

i.e., $\boldsymbol{w}^{(k)}\boldsymbol{d} + \min_{j\in[M]} \min(\boldsymbol{w}_j^{(k)}, 0) \ \leq\ 0$ for every bounded-norm $\boldsymbol{\gamma}^{(k)}, \boldsymbol{\lambda}^{(k)} \ \geq\ \mathbf{0}$. Negating and using positive homogeneity in $(\boldsymbol{\gamma}^{(k)}, \boldsymbol{\lambda}^{(k)})$ to drop the bounded-norm restriction, the empty intersection (1) (under the normalization $\mathbb{1}^\top \boldsymbol{d} = -1$) is equivalent to: there exist $\boldsymbol{\gamma}^{(k)} \in \mathbb{R}_{\geq 0}^{|V_k|}$ and $\boldsymbol{\lambda}^{(k)} \in \mathbb{R}_{\geq 0}^{|S_k^c|}$ such that $\boldsymbol{w}^{(k)}\boldsymbol{d} + \min_{j\in[M]} \min(\boldsymbol{w}_j^{(k)}, 0) \ >\ 0$. Undoing the normalization (using positive homogeneity in $\boldsymbol{d}$ to restore the factor $|\mathbb{1}^\top \boldsymbol{d}|$), (1) is equivalent to: there exist $\boldsymbol{\gamma}^{(k)} \in \mathbb{R}_{\geq 0}^{|V_k|}$ and $\boldsymbol{\lambda}^{(k)} \in \mathbb{R}_{\geq 0}^{|S_k^c|}$ such that

$$\boldsymbol{w}^{(k)}\boldsymbol{d} \ +\ \big|\mathbb{1}^\top \boldsymbol{d}\big| \cdot \min_{j\in[M]} \min\big(\boldsymbol{w}_j^{(k)},\ 0\big) \ >\ 0,$$

which is exactly (2). ■

E.4.5. PROOF OF THEOREM 9

Using the equivalence of the power-characterizing condition in Theorem 19 and the alternative power-characterizing condition in Theorem 23 that we established in Theorem 29, we will show the necessity of the power-characterizing condition for elicitability-expansion by showing the necessity of the alternate condition. In the proof of Theorem 9, we will use the single-agent characterization of elicitability (Theorem 26) rather than the multi-agent characterization of elicitability-expansion (Theorem 22).

Proof [Proof of Theorem 9] We will prove the contrapositive. That is, if $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ does not satisfy the alternative power-characterizing condition (Theorem 23), then $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ is not elicitability-expanding. Violation of the power-characterizing condition means that $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ is not feasibility-expanding. That is, $\boldsymbol{x}^{(A)}$ and each $\boldsymbol{x}^{(k)}$ for $k \in [K]$ is feasible.

We will show that $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ is not elicitability-expanding when violating the power-characterizing condition by showing that for any feature-weights matrix $\boldsymbol{\alpha}$ that makes $\boldsymbol{x}^{(A)}$ not elicitable, there exists some $k \in [K]$ such that $\boldsymbol{x}^{(k)}$ is also not elicitable under $\boldsymbol{\alpha}$.

For any feature weights matrix $\boldsymbol{\alpha}$ that makes $\boldsymbol{x}^{(A)}$ inelicitable, by Lemma 27, there is a $\boldsymbol{d}^{(A)} \in \mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(A)}),\mathcal{V}(\boldsymbol{x}^{(A)})}$ such that $\boldsymbol{\alpha}\boldsymbol{d}^{(A)} \geq 0$. We claim that $\boldsymbol{d}^{(A)} \not\leq 0$. Suppose for contradiction that $\boldsymbol{d}^{(A)} \leq 0$. Since $\mathbb{1}^\top \boldsymbol{d}^{(A)} < 0$, $\boldsymbol{d}^{(A)}$ has some strictly negative coordinate i; combined with $\boldsymbol{d}_j^{(A)} \geq 0$ for $j \in \mathcal{S}(\boldsymbol{x}^{(A)})^c$ (from $\boldsymbol{d}^{(A)} \in \mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(A)}),\mathcal{V}(\boldsymbol{x}^{(A)})}$), this coordinate satisfies $i \in \mathcal{S}(\boldsymbol{x}^{(A)})$. Since $\boldsymbol{\alpha} \geq 0$ and $\boldsymbol{d}^{(A)} \leq 0$, every entry of $\boldsymbol{\alpha}\boldsymbol{d}^{(A)}$ is non-positive; for $\boldsymbol{\alpha}\boldsymbol{d}^{(A)} \geq 0$ to hold we would need $\alpha_{ji}\boldsymbol{d}_i^{(A)} = 0$ for every $j \in [N]$, forcing column i of $\boldsymbol{\alpha}$ to be zero, which contradicts that $\boldsymbol{\alpha}$ has no zero columns (Theorem 2). Hence $\boldsymbol{d}^{(A)} \not\leq 0$.

Since the alternate power-characterizing condition is violated (Proposition 29), there exists $\boldsymbol{x}^{(k)}$ with $\mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(k)}),\mathcal{V}(\boldsymbol{x}^{(k)})}$ having non-empty intersection with $\{\boldsymbol{u} + \lambda\boldsymbol{d}^{(A)}\}$. It suffices to show that $\boldsymbol{x}^{(k)}$

is not elicitable under feature mapping $\boldsymbol{\alpha}$ that makes $\boldsymbol{x}^{(A)}$ inelicitable. To see this, let $\boldsymbol{d}^{(k)}$ denote an element of the intersection $\mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(k)}),\mathcal{V}(\boldsymbol{x}^{(k)})} \cap \{\boldsymbol{u} + \lambda \boldsymbol{d}^{(A)}\}$. We can then write $\boldsymbol{d}^{(k)} = \boldsymbol{u} + \lambda \boldsymbol{d}^{(A)}$. Note that $\boldsymbol{\alpha d}_i = \boldsymbol{\alpha u} + \lambda \boldsymbol{\alpha d}^{(A)}$. We know that $\boldsymbol{\alpha u} \geq 0$ since $\boldsymbol{u} \geq 0$ and $\boldsymbol{\alpha}$ has non-negative entries. Additionally, $\boldsymbol{\alpha d}^{(A)} \geq 0$. Hence $\boldsymbol{\alpha d}^{(k)} \geq 0$ for $\boldsymbol{d}^{(k)} \in \mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(k)}),\mathcal{V}(\boldsymbol{x}^{(k)})}$. By Lemma 26, this means that $\boldsymbol{x}^{(k)}$ is not elicitable.

■

E.4.6. PROOF OF THEOREM 9

Proof [Proof of Theorem 9] Suppose the power-characterizing condition is satisfied. By Proposition 29, this means that the alternate power-characterizing condition is also satisfied. Then we know that we are in one of two cases.

Case 1: $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)} \to \boldsymbol{x}^{(A)}$ implements feasibility expansion. Consider a feature mapping with a single feature and all dimensions contribute equal weights of one to this feature. All output vectors with the same ℓ_1 norm result in the same reward for all reward functions, and thus all feasible outcomes are elicitable. That is any output vector is elicitable if and only if it is feasible. A feasible output is elicitable with budget equal to its ℓ_1 norm. Under this construction, feasibility-expansion implies elicitability-expansion.

Case 2: there exists $\boldsymbol{d}^{(A)} \in \mathcal{B}_{S(\boldsymbol{x}^{(A)},\mathcal{V}(\boldsymbol{x}^{(A)})}$ with $\boldsymbol{d}^{(A)} \not\leq 0$ such that for all $\boldsymbol{u} \geq 0, \lambda \geq 0$, $\boldsymbol{u} + \lambda \boldsymbol{d}^{(A)} \notin \mathcal{B}_{\mathcal{S}(\boldsymbol{x}^{(k)}),\mathcal{V}(\boldsymbol{x}^{(k)})}$ for $k \in [K]$. We will construct a feature mapping $\boldsymbol{\alpha}$ based on $\boldsymbol{d}^{(A)}$ such that the set of directions weakly increasing feature values i.e., the set $D_{\boldsymbol{\alpha}} = \{\boldsymbol{d} : \boldsymbol{\alpha d} \geq 0\}$ is a subset of $\{\boldsymbol{u} + \lambda \boldsymbol{d}^{(A)} : \boldsymbol{u} \geq 0, \lambda \geq 0\}$. By Theorem 22, this implies that $\boldsymbol{x}^{(A)}$ is not elicitable but for all other outputs $\boldsymbol{x}^{(k)}$, $D_{\boldsymbol{\alpha}} \cap \mathcal{B}_{S(\boldsymbol{x}^{(k)},\mathcal{V}(\boldsymbol{x}^{(k)})}$ is empty and hence each $\boldsymbol{x}^{(k)}$ is elicitable under $\boldsymbol{\alpha}$.

To complete this argument, we will explicitly construct such an $\boldsymbol{\alpha}$ based on $\boldsymbol{d}^{(A)}$. Let $P_0 = \{i \in [m] : \boldsymbol{d}_i^{(A)} > 0\}$ denote the positive coordinates of $\boldsymbol{d}^{(A)}$ and let $N_0 = \{i \in [m] : \boldsymbol{d}_i^{(A)} \leq 0\}$ denote the negative or zero coordinates. Note that P_0 is non-empty since $\boldsymbol{d}^{(A)} \not\leq 0$. We construct two sets of features:

- For every $p \in P_0$, there is a corresponding feature F_p whose row in $\boldsymbol{\alpha}$ is the vector e_p which is the vector with 1 at coordinate p and zero everywhere else. That is, dimension p has weight 1 on feature F_p and all other dimensions have zero weight.
- The next set of features are defined for every pair $p \in P_0$, $q \in N_0$. This feature $F_{p,q}$ has a corresponding row in $\boldsymbol{\alpha}$ that is the vector $\boldsymbol{d}_p^{(A)} e_q - \boldsymbol{d}_q^{(A)} e_p$. That is, the only dimensions with possible non-zero weights to $F_{p,q}$ are dimensions p, q. The weight from dimension p is $|\boldsymbol{d}_q^{(A)}|$ and the weight from dimension q is $|\boldsymbol{d}_p^{(A)}|$.

Now let us show that the set $D_{\boldsymbol{\alpha}} = \{\boldsymbol{d} : \boldsymbol{\alpha d} \geq 0\}$ is a subset of $B_A = \{\boldsymbol{u} + \lambda \boldsymbol{d}^{(A)}\}$. Take any $\boldsymbol{d} \in D_{\boldsymbol{\alpha}}$.

For every $p \in P_0$, since $\boldsymbol{d}$ weakly improves value of F_p, it holds that $\boldsymbol{d}_p \geq 0$. By ensuring that $\lambda \leq \boldsymbol{d}_p / \boldsymbol{d}_p^{(A)}$ for all $p \in P_0$, we can ensure that $\boldsymbol{d}_p - \lambda \boldsymbol{d}_p^{(A)} \geq 0$.

For every $p \in P_0, q \in N_0$, since $\boldsymbol{d}$ weakly improves value of $F_{p,q}$, it holds that $-\boldsymbol{d}_p \boldsymbol{d}_q^{(A)} + \boldsymbol{d}_q \boldsymbol{d}_p^{(A)} \geq 0$. In other words, $\boldsymbol{d}_q \geq \boldsymbol{d}_p \boldsymbol{d}_q^{(A)} / \boldsymbol{d}_p^{(A)}$. By choosing λ less than $\boldsymbol{d}_q / \boldsymbol{d}_q^{(A)}$ for every $q \in N_0$, we get $\boldsymbol{d}_q - \lambda \boldsymbol{d}_q^{(A)} \geq 0$ for every $q \in N_0$.

■

Appendix F. Additional details for Section 5 (Empirical)

F.1. Empirical instantiations of each mechanism

This appendix gives the per-mechanism details for the empirical illustration in Section 5. Each instantiation is the empirical analogue of the corresponding example in Section 3 (Example 2, Example 3, Example 1).

Support expansion. We have $M = 3$ output topics ($\mathcal{T}_1^O$: complexity theory, $\mathcal{T}_2^O$: macroeconomics, $\mathcal{T}_3^O$: mechanism design), $N = 2$ prompt topics ($\mathcal{T}_1^P$: CS theory, $\mathcal{T}_2^P$: economics), and aggregation $\mathcal{A}^\cup$. The prompts for $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ each focus on one prompt topic. The aggregate $\boldsymbol{x}^{(A)} = [0.43, 0.47, 0.04]$ has substantial weight on both $\mathcal{T}_1^O$ and $\mathcal{T}_2^O$, while $\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)}) = [0.74, 0.00, 0.07]$ misses $\mathcal{T}_2^O$ (ℓ_1 confidence bound $[0.63, 1.02]$). No single prompt over $\mathcal{T}^P$ activates both $\mathcal{T}_1^O$ and $\mathcal{T}_2^O$ simultaneously. Prompting directly on output topics ($G^{\text{inc}} = \{\mathcal{T}_1^O, \mathcal{T}_2^O\}$ activates both $\mathcal{T}_1^O$ and $\mathcal{T}_2^O$ producing $[0.52, 0.51, 0.00]$. So needing aggregation to expand support is not due to feasibility expansion.

Binding-set contraction. We have $M = 5$ output topics ($\mathcal{T}_1^O$: deep learning + $\mathcal{T}_{2:5}^O$: subareas plus a non-overlapping background topic), $N = 2$ prompt topics ($\mathcal{T}_1^P$:NLP, $\mathcal{T}_2^P$:CV), and aggregation $\mathcal{A}^\cap$. Each prompt for $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ focuses on one prompt topic and excludes the other. The aggregate $\boldsymbol{x}^{(A)} = [0.87, 0, 0, 0, 0]$ concentrates on the umbrella deep-learning topic, excluding the subareas, while $\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)}) = [0.83, 0.11, 0, 0.01, 0.01]$ includes the subareas as well (ℓ_1 bound $[0.07, 0.35]$). Both $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ are binding under the semantic constraint that $\mathcal{T}_1^O$ subsumes the deep-learning subareas $\mathcal{T}_{2:4}^O$, while $\boldsymbol{x}^{(A)}$ lies in its interior. Prompting directly on output topics ($G^{\text{inc}} = \{\mathcal{T}_1^O\}$, $G^{\text{exc}} = \mathcal{T}_{2:5}^O$) produces $[0.96, 0, 0.01, 0.01, 0]$, filtering our subareas. So feasibility expansion is not driving the need for aggregation to filter subareas. The smaller support of $\boldsymbol{x}^{(A)}$ also rules out support expansion.

Feasibility expansion. We have the same set of output topics and prompt topics of size $M = N = 3$. The output topics are $\mathcal{T}_1^O$: blockchain, $\mathcal{T}_2^O$:cryptography, $\mathcal{T}_3^O$:distributed systems and $\mathcal{T}^P = \mathcal{T}^O$. We use aggregation $\mathcal{A}^\cap$. Each prompt asks for blockchain or one of the other two topics while excluding the third. The aggregate $\boldsymbol{x}^{(A)} = [0.67, 0, 0.22]$ has lower distributed-systems content than $\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)}) = [0.67, 0, 0.39]$ (ℓ_1 bound $[0.07, 0.39]$, with 0 outside the confidence set). This is consistent with the natural constraint that blockchain papers are typically connected to cryptography or distributed systems.

F.2. Other empirical details for Section 5

F.2.1. PER-MECHANISM DETAILS

We provide the full topic descriptions, prompt specifications, and numerical results for each of the three mechanisms shown in Figure 2 of the main body.

	x_1	x_2	x_3
$\boldsymbol{x}^{(1)}$	$[0.70, 0.77]$	$[0.00, 0.00]$	$[0.04, 0.08]$
$\boldsymbol{x}^{(2)}$	$[0.00, 0.00]$	$[0.74, 0.81]$	$[0.03, 0.06]$
$\boldsymbol{x}^{(A)}$	$[0.39, 0.47]$	$[0.43, 0.51]$	$[0.03, 0.06]$
$\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$	$[0.68, 0.80]$	$[0.00, 0.01]$	$[0.05, 0.12]$

(a) Support Expansion

	x_1	x_2	x_3
$\boldsymbol{x}^{(1)}$	$[0.65, 0.72]$	$[0.36, 0.44]$	$[0.06, 0.10]$
$\boldsymbol{x}^{(2)}$	$[0.65, 0.72]$	$[0.00, 0.00]$	$[0.35, 0.42]$
$\boldsymbol{x}^{(A)}$	$[0.63, 0.70]$	$[0.00, 0.00]$	$[0.19, 0.25]$
$\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$	$[0.60, 0.73]$	$[0.00, 0.01]$	$[0.33, 0.46]$

(b) Feasibility Expansion

	x_1	x_2	x_3	x_4	x_5
$\boldsymbol{x}^{(1)}$	$[0.79, 0.85]$	$[0.09, 0.14]$	$[0.00, 0.02]$	$[0.00, 0.02]$	$[0.03, 0.06]$
$\boldsymbol{x}^{(2)}$	$[0.16, 0.22]$	$[0.00, 0.01]$	$[0.61, 0.69]$	$[0.04, 0.08]$	$[0.02, 0.05]$
$\boldsymbol{x}^{(A)}$	$[0.84, 0.89]$	$[0.00, 0.00]$	$[0.00, 0.00]$	$[0.00, 0.00]$	$[0.00, 0.00]$
$\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$	$[0.78, 0.88]$	$[0.07, 0.16]$	$[0.00, 0.01]$	$[0.00, 0.03]$	$[0.00, 0.04]$

(c) Binding-set contraction

Experiment	$\lVert \boldsymbol{x}_P^*(\boldsymbol{x}^{(A)}) - \boldsymbol{x}^{(A)} \rVert_1$
Support expansion	[0.63, 1.02]
Feasibility expansion	[0.07, 0.39]
Binding-set contraction	[0.07, 0.35]

(d) ℓ_1 distance between $\boldsymbol{x}^{(A)}$ and $\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$

Table 2: Empirical illustration of the three mechanisms in a toy reference-generation task with GPT-4o-mini at temperature 0.7 (Section 5). We show the exact values from Figure 2 for support expansion, feasibility expansion, and binding-set contraction. We also show the ℓ_1 distance between $\boldsymbol{x}^{(A)}$ and $\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$. All of our quantities are reported as 95% confidence intervals.

Support Expansion. We construct a setup that resembles Example 2 shown in Figure 1b. Let there be $M = 3$ output dimensions and $N = 2$ prompt features. The output dimension topics are:

$\mathcal{T}_1^O$: complexity theory (computational complexity, P vs NP, complexity classes, hardness results, reductions, circuit complexity, space/time complexity bounds)

$\mathcal{T}_2^O$: macroeconomics (GDP, inflation, monetary policy, fiscal policy, business cycles, economic growth, unemployment, central banking, aggregate demand/supply)

$\mathcal{T}_3^O$: mechanism design (auction design, incentive compatibility, social choice, matching markets, market design, algorithmic game theory, incentive mechanisms)

The prompt topics are $\mathcal{T}_1^P$: computer science theory and $\mathcal{T}_2^P$: economics. We take the aggregation rule to be $\mathcal{A}^\cup$. We let $\boldsymbol{x}^{(1)}$ capture the output from prompt specification $(G_1^{\text{inc}} = \{\mathcal{T}_1^P\}, G_1^{\text{exc}} = \emptyset)$, and we let $\boldsymbol{x}^{(2)}$ capture the output from prompt specification $(G_2^{\text{inc}} = \{\mathcal{T}_2^P\}, G_2^{\text{exc}} = \emptyset)$.

The results are shown in the left panel of Figure 2 and Table 2a. We find on average that $\boldsymbol{x}^{(A)} = [0.43, 0.47, 0.04]$, that $\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)}) = [0.74, 0.00, 0.07]$. We obtain a confidence bound of $[0.63, 1.02]$ for ℓ_1 distance between $\boldsymbol{x}^{(A)}$ and $\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$. The vector $\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$ is obtained via brute-force search over prompt specifications.

These results suggest that $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)} \to \boldsymbol{x}^{(A)}$ implements support expansion and is elicitability-expanding. Specifically, $\boldsymbol{x}^{(1)}$ has most of its nonzero weight on topic $\mathcal{T}_1^O$, $\boldsymbol{x}^{(2)}$ has most of its nonzero weight on topic $\mathcal{T}_2^O$, and $\boldsymbol{x}^{(A)}$ reflects both topics $\mathcal{T}_1^O$ and $\mathcal{T}_2^O$. $\boldsymbol{x}^{(A)}$ is not elicitable by a single model, since $\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$ misses out on $\mathcal{T}_2^O$. This illustrates that no prompt over the prompt topics $\mathcal{T}^P$ is able to simultaneously activate topics $\mathcal{T}_1^O$ and $\mathcal{T}_2^O$, but aggregation is able to overcome this limitation by separately considering $\mathcal{T}_1^P$ and $\mathcal{T}_2^P$. On the other hand, if we prompt using output dimension topics $\mathcal{T}^O$ as opposed to the prompt topics $\mathcal{T}^P$, it is possible to simultaneously activate

both topics: we find on average that the output vector equals $[0.52, 0.51, 0.00]$ if we use the prompt specification $(G^{inc} = \{\mathcal{T}_1^O, \mathcal{T}_2^O\}, \text{op}^{\text{inc}} = \text{or}, G^{exc} = \emptyset)$.

Binding set contraction. We construct an example that resembles Example 3 shown in Figure 1c, but with a greater number of output dimensions. Let there be $M = 5$ output dimension topics and $N = 2$ prompt dimension topics. The output dimension topics are:

$\mathcal{T}_1^O$: deep learning (transformers, attention mechanisms, deep neural networks, modern architectures like BERT, GPT, ViT)

$\mathcal{T}_2^O$: non-transformer NLP methods (RNN, LSTM, GRU, word2vec, seq2seq without attention, traditional NLP)

$\mathcal{T}_3^O$: non-transformer CV methods (CNN, ConvNets, ResNet, VGG, pooling, convolutional architectures)

$\mathcal{T}_4^O$: multimodal methods bridging multiple modalities (combining text and images, vision-language models, image captioning, VQA)

$\mathcal{T}_5^O$: statistical machine learning (non-neural methods like SVM, random forests, logistic regression, Bayesian methods, traditional ML))

The prompt dimension topics are $\mathcal{T}_1^P$: natural language processing (NLP) and $\mathcal{T}_2^P$: computer vision (CV). We take the aggregation rule to be $\mathcal{A}^{\cap}$. We let $\boldsymbol{x}^{(1)}$ be the output from prompt specification $(G_1^{inc} = \{\mathcal{T}_1^P\}, G_1^{exc} = \{\mathcal{T}_2^P\})$, and we let $\boldsymbol{x}^{(2)}$ be the output from prompt specification $(G_2^{inc} = \{\mathcal{T}_2^P\}, G_2^{exc} = \{\mathcal{T}_1^P\})$.

The results are shown in the middle panel of Figure 2 and Table 2c. We find on average that $\boldsymbol{x}^{(A)} = [0.87, 0.00, 0.00, 0.00, 0.00]$, that $\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)}) = [0.83, 0.11, 0.00, 0.01, 0.01]$. We get a confidence bound of $[0.07, 0.35]$ for ℓ_1 distance between $\boldsymbol{x}^{(A)}$ and $\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$. The vector $\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$ is obtained via brute-force search over prompt specifications.

The results suggest that $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)} \to \boldsymbol{x}^{(A)}$ implements binding-set contraction and it is elicitability-expanding. Specifically, since $\mathcal{T}_1^O$: deep learning is an umbrella topic covering many papers in topics $\mathcal{T}_2^O, \mathcal{T}_3^O, \mathcal{T}_4^O$, we expect that there is a semantic constraint capturing how lists of papers which include topics $\mathcal{T}_2^O, \mathcal{T}_3^O$ or $\mathcal{T}_4^O$ also tend to include topic $\mathcal{T}_1^O$. Both $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ appear to be binding relative to this constraint, while $\boldsymbol{x}^{(A)}$ is in the interior. $\boldsymbol{x}^{(A)}$ is not elicitable by a single model: while $\boldsymbol{x}^{(A)}$ predominantly reflects topic $\mathcal{T}_1^O$, the output vector $\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$ also reflects some topics outside of $\mathcal{T}_1^O$. On the other hand, if we prompt using output dimension topics $\mathcal{T}^O$ rather than the prompt topics $\mathcal{T}^P$, it is possible to further reduce the fraction of papers reflecting topics outside of $\mathcal{T}_1^O$: we find on average that the output vector equals $[0.96, 0.00, 0.01, 0.01, 0.00]$ if we use the prompt specification $(G^{inc} = \{\mathcal{T}_1^O\}, G^{exc} = \{\mathcal{T}_2^O, \mathcal{T}_3^O, \mathcal{T}_4^O, \mathcal{T}_5^O\}, \text{op}^{\text{exc}} = \text{or})$.

Feasibility Expansion. We construct a setup that resembles Example 1 shown in Figure 1a, but without the reward specification limitations. Let there be $M = 3$ output dimensions, corresponding to topics:

$\mathcal{T}_1^O$: blockchain(consensus protocols, smart contracts, decentralized ledgers, proof-of-work, proof-of-stake)

$\mathcal{T}_2^O$: cryptography (encryption, zero-knowledge proofs, hash functions, digital signatures, key exchange)

$\mathcal{T}_3^O$: distributed systems (consensus algorithms, fault tolerance, replication, distributed databases, CAP theorem)

We take $\mathcal{T}^P = \mathcal{T}^O$, which means that $M = N = 3$ and also that feasibility and elicitability are equivalent. We take the aggregation rule to be $\mathcal{A}^{\cap}$. We let $\boldsymbol{x}^{(1)}$ capture the output from prompt specification $(G_1^{\text{inc}} = \{\mathcal{T}_1^O, \mathcal{T}_2^O\}, \text{op}^{\text{inc}} = \text{or}, G_1^{\text{exc}} = \{\mathcal{T}_3^O\})$, and we let $\boldsymbol{x}^{(2)}$ capture the output from prompt specification $(G_2^{\text{inc}} = \{\mathcal{T}_1^O, \mathcal{T}_3^O\}, \text{op}^{\text{inc}} = \text{or}, G_2^{\text{exc}} = \{\mathcal{T}_2^O\})$.

The results are shown in the right panel of Figure 2 and Table 2b. We find on average that $\boldsymbol{x}^{(A)} = [0.67, 0.00, 0.22]$, that $\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)}) = [0.67, 0.00, 0.39]$. We get a confidence bound of $[0.07, 0.39]$ for the ℓ_1 distance between $\boldsymbol{x}^{(A)}$ and $\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$, which notably does not contain 0, illustrating a clear gap. The vector $\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$ is obtained via brute-force search over prompt specifications.

The results suggest that $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)} \to \boldsymbol{x}^{(A)}$ implements feasibility expansion and is elicitability-expanding. Intuitively, we expect there to be the following constraint faced by agents (whether humans or LLMs) in this task: many blockchain papers ($\mathcal{T}_1^O$) are related to cryptography ($\mathcal{T}_2^O$) or to distributed systems ($\mathcal{T}_3^O$). Both $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ reflect this constraint, but $\boldsymbol{x}^{(A)}$ circumvents this constraint through aggregation. $\boldsymbol{x}^{(A)}$ is not elicitable by a single model: $\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$ has substantially more weight on $\mathcal{T}_3^O$ (distributed systems) than $\boldsymbol{x}^{(A)}$ does, while matching on $\mathcal{T}_1^O$. The aggregation operation produces a vector whose distributed-systems content is markedly lower than what any single prompt over $\mathcal{T}^O$ can achieve.

F.2.2. SETUP DETAILS

We run different exoeriment setups using different models (GPT-4o-mini, GPT-5-mini, and GPT-5.4) and different temperatures for generating the output vectors.

Prompting. A prompt is defined by an inclusion set $G^{inc} \subseteq G$, exclusion set $G^{exc} \subseteq G$ that is disjoint from the inclusion set G^{inc}, and operators $\text{op}^{\text{inc}}, \text{op}^{\text{exc}} \in \{\text{and}, \text{or}\}$. If the inclusion list has more than one element, we choose an inclusion list operator op^{inc} that is one of $\{\text{and}, \text{or}\}$ to combine the topics in the list for inclusion. Similarly, if the exclusion list has more than one element, we choose an exclusion list operator op^{exc} that is AND or OR. The prompt based on $G^{inc} = \{g_1^{inc}, \ldots, g_a^{inc}\}, G_{exc} = \{g_1^{exc}, \ldots, g_b^{exc}\}$, $\text{op}^{\text{inc}}, \text{op}^{\text{exc}}$ is "List up to 20 papers on g_1^{inc} op^{inc} $\ldots$ op^{inc} g_a^{inc}, but EXCLUDE any papers about g_1^{exc} op^{exc} $\ldots$ op^{exc} g_b^{exc}. For example, the prompt with $G^{inc} = \{\text{Topic 1}, \text{Topic 2}\}$, $G^{exc} = \{\text{Topic 3}, \text{Topic 4}\}$, $\text{op}^{\text{inc}} : \text{AND}$, $\text{op}^{\text{exc}} : \text{OR}$ is *List up to 20 papers on Topic 1 AND Topic 2, but EXCLUDE any papers about Topic 3 or Topic 4.*

LLM-as-judge setup. We use an LLM-as-judge to measure whether a list L exhibits a given topic T by prompting it. We use GPT-4o-mini at temperature 0 for the judge.

```
For each of the following research papers, determine whether it belongs
to each topic.
Treat each paper independently - do not let other papers in the list
influence your classification.

Papers:
{paper_list}

Topics:
{topic_list}

For each paper, independently answer yes/no for each topic.
```

```
Output format:
1. {topic_keys_str.replace(', ', ': yes/no, ')}: yes/no
2. {topic_keys_str.replace(', ', ': yes/no, ')}: yes/no
...

SUMMARY (count of "yes" for each topic):
{chr(10).join(f'{key}: [count]' for key in topics.keys())}
```

Aggregation. To perform aggregation, we first reformat the titles produced by an LLM's response by making everything lower case, removing punctuations and removing any extra whitespaces. We further remove duplicates by considering two titles equivalent if one is a substring of another. The aggregation rule $\mathcal{A}^{\cap}$ intersects the lists and $\mathcal{A}^{\cup}$ takes the union of the lists. Note that in contrast with Appendix C, we consider this form of literal set intersection and union as it allows us to consider a non-LLM based aggregation that does not inherit the behavior of the LLMs.

Feasibility and elicitability. To analyze feasibility and elicitability, we brute force over all possible prompt specifications. This brute force search enables us to identify the closest output vector $\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$ to the aggregated output vector.

Confidence intervals. For each output vector, we average over 30 trials and report a coordinate-wise 95% confidence interval. We use a Wilson confidence interval. To measure a confidence interval for the ℓ_1 distance between two output vectors, we use the confidence intervals for the two vectors themselves, and then we compute minimum possible and maximum possible ℓ_1 distance for those confidence sets. We use these confidence sets to compute $\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$. Specifically, for each candidate output vector x, we compute the *lower* bound in the confidence set for the ℓ_1 distance between $\boldsymbol{x}$ and $\boldsymbol{x}^{(A)}$, and then we choose the $\boldsymbol{x}$ that minimizes this lower bound.

F.3. Additional experiments

Building on the setup in Section 5, we conduct the following additional experiments where we vary our model configuration in three ways.

1. Experimental setup **E1** considers different settings for the temperature (0.3, 0.5, 0.7), and we report our findings in Tables 3-5.

2. Experimental setup **E2** considers different models (GPT-5.4 at temperatures 0 and 0.7, and GPT-5-mini at default temperature), and we report our findings in Tables 6-8.

3. Experimental setup **E3** considers the aggregation of heterogeneous models (GPT-4o-mini and GPT-5-mini; GPT-4o-mini and GPT-5.4; GPT-5-mini and GPT-5.4), and we report our findings in Tables 9-11.

We summarize our key findings.

- We find that support expansion readily generalizes. In fact, these mechanisms hold for the same instances (i.e., task, aggregation operation, and prompt/output specification) as in our original experiments in all of the cases.

- We find that binding-set contraction generalizes in nearly all cases. The exception is one case (Table 8; **E2** for GPT-5-mini), where intersection aggregation produces an empty list ($\boldsymbol{x}^{(A)} = \mathbf{0}$), even after switching to a slightly modified prompt specification intended to encourage a non-empty intersection. As with the feasibility expansion failures below, this failure mode is driven by intersection of the produced lists being empty. In all other cases, binding-set contraction holds for the same task, aggregation operation, output specification, and prompt specifications as in our original experiments.
- We find that feasibility expansion also generalizes to **E1** and **E2**. Again, these mechanisms hold for the same tasks and instances as our original experiments. However, feasibility expansion is no longer exhibited by the instance that we constructed for **E3** in two out of the three cases. The failure mode is that intersection aggregation produces an empty list likely due to model heterogeneity. We defer constructing instances that exhibit feasibility expansion for other models to future work.

Altogether, these experiments provide further support for these mechanisms, and additionally demonstrate the robustness of our experiments across almost all the generalized settings.

EXPERIMENT 1 (E1): CHANGING TEMPERATURE

	x_1	x_2	x_3
$\boldsymbol{x}^{(1)}$	$[0.63, 0.70]$	$[0.00, 0.00]$	$[0.04, 0.08]$
$\boldsymbol{x}^{(2)}$	$[0.00, 0.00]$	$[0.75, 0.82]$	$[0.03, 0.06]$
$\boldsymbol{x}^{(A)}$	$[0.34, 0.41]$	$[0.43, 0.51]$	$[0.03, 0.06]$
$\boldsymbol{x}^*_P(\boldsymbol{x}^{(A)})$	$[0.00, 0.01]$	$[0.78, 0.88]$	$[0.01, 0.05]$

(a) Support Expansion

	x_1	x_2	x_3
$\boldsymbol{x}^{(1)}$	$[0.66, 0.73]$	$[0.35, 0.42]$	$[0.01, 0.02]$
$\boldsymbol{x}^{(2)}$	$[0.65, 0.72]$	$[0.00, 0.00]$	$[0.32, 0.40]$
$\boldsymbol{x}^{(A)}$	$[0.70, 0.77]$	$[0.00, 0.00]$	$[0.14, 0.20]$
$\boldsymbol{x}^*_P(\boldsymbol{x}^{(A)})$	$[0.63, 0.75]$	$[0.00, 0.01]$	$[0.29, 0.42]$

(b) Feasibility Expansion

	x_1	x_2	x_3	x_4	x_5
$\boldsymbol{x}^{(1)}$	$[0.77, 0.84]$	$[0.09, 0.15]$	$[0.00, 0.00]$	$[0.00, 0.02]$	$[0.02, 0.05]$
$\boldsymbol{x}^{(2)}$	$[0.17, 0.23]$	$[0.00, 0.00]$	$[0.64, 0.72]$	$[0.04, 0.08]$	$[0.02, 0.05]$
$\boldsymbol{x}^{(A)}$	$[0.93, 0.96]$	$[0.00, 0.00]$	$[0.00, 0.00]$	$[0.01, 0.03]$	$[0.00, 0.00]$
$\boldsymbol{x}^*_P(\boldsymbol{x}^{(A)})$	$[0.78, 0.88]$	$[0.07, 0.15]$	$[0.00, 0.01]$	$[0.00, 0.01]$	$[0.02, 0.07]$

(c) Binding-set contraction

Experiment	$\lVert \boldsymbol{x}^*_P(\boldsymbol{x}^{(A)}) - \boldsymbol{x}^{(A)} \rVert_1$
Support expansion	[0.59, 0.91]
Feasibility expansion	[0.09, 0.43]
Binding-set contraction	[0.12, 0.45]

(d) ℓ_1 distance from $\boldsymbol{x}^{(A)}$

Table 3: Results for **gpt-4o-mini** and **temperature 0.3**. The empirical setup and construction of the instance (i.e., task, aggregation operation, and prompt/output specification) is the same as in Section 5. These results demonstrate support expansion, binding-set contraction, and feasibility expansion.

	x_1	x_2	x_3
$\boldsymbol{x}^{(1)}$	$[0.65, 0.72]$	$[0.00, 0.00]$	$[0.03, 0.07]$
$\boldsymbol{x}^{(2)}$	$[0.00, 0.00]$	$[0.73, 0.80]$	$[0.02, 0.05]$
$\boldsymbol{x}^{(A)}$	$[0.34, 0.42]$	$[0.44, 0.52]$	$[0.03, 0.06]$
$\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$	$[0.00, 0.01]$	$[0.76, 0.86]$	$[0.03, 0.09]$

(a) Support Expansion

	x_1	x_2	x_3
$\boldsymbol{x}^{(1)}$	$[0.67, 0.74]$	$[0.36, 0.44]$	$[0.04, 0.07]$
$\boldsymbol{x}^{(2)}$	$[0.65, 0.72]$	$[0.00, 0.00]$	$[0.33, 0.41]$
$\boldsymbol{x}^{(A)}$	$[0.56, 0.64]$	$[0.00, 0.00]$	$[0.14, 0.20]$
$\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$	$[0.61, 0.74]$	$[0.00, 0.01]$	$[0.30, 0.43]$

(b) Feasibility Expansion

	x_1	x_2	x_3	x_4	x_5
$\boldsymbol{x}^{(1)}$	$[0.80, 0.86]$	$[0.09, 0.14]$	$[0.00, 0.00]$	$[0.00, 0.02]$	$[0.01, 0.04]$
$\boldsymbol{x}^{(2)}$	$[0.18, 0.24]$	$[0.00, 0.01]$	$[0.62, 0.69]$	$[0.07, 0.11]$	$[0.02, 0.05]$
$\boldsymbol{x}^{(A)}$	$[0.91, 0.95]$	$[0.00, 0.00]$	$[0.00, 0.00]$	$[0.00, 0.00]$	$[0.00, 0.00]$
$\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$	$[0.78, 0.88]$	$[0.05, 0.13]$	$[0.00, 0.01]$	$[0.00, 0.01]$	$[0.00, 0.04]$

(c) Binding-set contraction

Experiment	$\|\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)}) - \boldsymbol{x}^{(A)}\|_1$
Support expansion	[0.56, 0.91]
Feasibility expansion	[0.10, 0.49]
Binding-set contraction	[0.08, 0.37]

(d) ℓ_1 distance from $\boldsymbol{x}^{(A)}$

Table 4: Results for **gpt-4o-mini** and **temperature 0.5**. The empirical setup and construction of the instance (i.e., task, aggregation operation, and prompt/output specification) is the same as in Section 5. These results demonstrate support expansion, binding-set contraction, and feasibility expansion.

	x_1	x_2	x_3
$\boldsymbol{x}^{(1)}$	$[0.70, 0.77]$	$[0.00, 0.00]$	$[0.04, 0.08]$
$\boldsymbol{x}^{(2)}$	$[0.00, 0.00]$	$[0.74, 0.81]$	$[0.03, 0.06]$
$\boldsymbol{x}^{(A)}$	$[0.39, 0.47]$	$[0.43, 0.51]$	$[0.03, 0.06]$
$\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$	$[0.68, 0.80]$	$[0.00, 0.01]$	$[0.05, 0.12]$

(a) Support Expansion

	x_1	x_2	x_3
$\boldsymbol{x}^{(1)}$	$[0.65, 0.72]$	$[0.36, 0.44]$	$[0.06, 0.10]$
$\boldsymbol{x}^{(2)}$	$[0.65, 0.72]$	$[0.00, 0.00]$	$[0.35, 0.42]$
$\boldsymbol{x}^{(A)}$	$[0.63, 0.70]$	$[0.00, 0.00]$	$[0.19, 0.25]$
$\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$	$[0.60, 0.73]$	$[0.00, 0.01]$	$[0.33, 0.46]$

(b) Feasibility Expansion

	x_1	x_2	x_3	x_4	x_5
$\boldsymbol{x}^{(1)}$	$[0.79, 0.85]$	$[0.09, 0.14]$	$[0.00, 0.02]$	$[0.00, 0.02]$	$[0.03, 0.06]$
$\boldsymbol{x}^{(2)}$	$[0.16, 0.22]$	$[0.00, 0.01]$	$[0.61, 0.69]$	$[0.04, 0.08]$	$[0.02, 0.05]$
$\boldsymbol{x}^{(A)}$	$[0.84, 0.89]$	$[0.00, 0.00]$	$[0.00, 0.00]$	$[0.00, 0.00]$	$[0.00, 0.00]$
$\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$	$[0.78, 0.88]$	$[0.07, 0.16]$	$[0.00, 0.01]$	$[0.00, 0.03]$	$[0.00, 0.04]$

(c) Binding-set contraction

Experiment	$\|\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)}) - \boldsymbol{x}^{(A)}\|_1$
Support expansion	[0.63, 1.02]
Feasibility expansion	[0.07, 0.39]
Binding-set contraction	[0.07, 0.35]

(d) ℓ_1 distance from $\boldsymbol{x}^{(A)}$

Table 5: Results for **gpt-4o-mini** and **temperature 0.7** (the default model and temperature used in the main text). This is the same data as in Table 2.

EXPERIMENT 2 (E2): CHANGING MODEL

	x_1	x_2	x_3
$\boldsymbol{x}^{(1)}$	$[0.65, 0.78]$	$[0.00, 0.01]$	$[0.00, 0.03]$
$\boldsymbol{x}^{(2)}$	$[0.00, 0.03]$	$[0.40, 0.53]$	$[0.16, 0.28]$
$\boldsymbol{x}^{(A)}$	$[0.34, 0.48]$	$[0.42, 0.56]$	$[0.02, 0.08]$
$\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$	$[0.24, 0.31]$	$[0.29, 0.36]$	$[0.21, 0.27]$

(a) Support Expansion

	x_1	x_2	x_3
$\boldsymbol{x}^{(1)}$	$[0.47, 0.61]$	$[0.45, 0.59]$	$[0.20, 0.32]$
$\boldsymbol{x}^{(2)}$	$[0.36, 0.50]$	$[0.00, 0.01]$	$[0.70, 0.81]$
$\boldsymbol{x}^{(A)}$	$[0.99, 1.00]$	$[0.00, 0.01]$	$[0.74, 0.85]$
$\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$	$[0.96, 0.99]$	$[0.05, 0.12]$	$[0.70, 0.82]$

(b) Feasibility Expansion

	x_1	x_2	x_3	x_4	x_5
$\boldsymbol{x}^{(1)}$	$[0.73, 0.85]$	$[0.20, 0.32]$	$[0.00, 0.01]$	$[0.00, 0.01]$	$[0.00, 0.01]$
$\boldsymbol{x}^{(2)}$	$[0.31, 0.45]$	$[0.00, 0.01]$	$[0.51, 0.65]$	$[0.09, 0.18]$	$[0.00, 0.01]$
$\boldsymbol{x}^{(A)}$	$[0.15, 0.26]$	$[0.00, 0.01]$	$[0.00, 0.01]$	$[0.00, 0.01]$	$[0.00, 0.01]$
$\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$	$[0.82, 0.91]$	$[0.00, 0.04]$	$[0.00, 0.04]$	$[0.00, 0.01]$	$[0.07, 0.15]$

(c) Binding-set contraction

Experiment	$\|\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)}) - \boldsymbol{x}^{(A)}\|_1$
Support expansion	[0.22, 0.76]
Feasibility expansion	[0.03, 0.31]
Binding-set contraction	[0.61, 0.99]

(d) ℓ_1 distance from $\boldsymbol{x}^{(A)}$

Table 6: Results for **gpt-5.4** and a temperature of 0. The empirical setup and construction of the instance (i.e., task, aggregation operation, and prompt/output specification) is the same as in Section 5. These results demonstrate support expansion, binding-set contraction, and feasibility expansion.

	x_1	x_2	x_3
$\boldsymbol{x}^{(1)}$	$[0.72, 0.79]$	$[0.00, 0.00]$	$[0.01, 0.04]$
$\boldsymbol{x}^{(2)}$	$[0.00, 0.00]$	$[0.46, 0.54]$	$[0.17, 0.23]$
$\boldsymbol{x}^{(A)}$	$[0.40, 0.48]$	$[0.44, 0.52]$	$[0.03, 0.07]$
$\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$	$[0.29, 0.37]$	$[0.37, 0.45]$	$[0.15, 0.21]$

(a) Support Expansion

	x_1	x_2	x_3
$\boldsymbol{x}^{(1)}$	$[0.49, 0.57]$	$[0.48, 0.56]$	$[0.17, 0.24]$
$\boldsymbol{x}^{(2)}$	$[0.37, 0.45]$	$[0.00, 0.00]$	$[0.77, 0.83]$
$\boldsymbol{x}^{(A)}$	$[0.98, 1.00]$	$[0.11, 0.16]$	$[0.88, 0.93]$
$\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$	$[0.93, 0.96]$	$[0.02, 0.04]$	$[0.68, 0.75]$

(b) Feasibility Expansion

	x_1	x_2	x_3	x_4	x_5
$\boldsymbol{x}^{(1)}$	$[0.83, 0.89]$	$[0.12, 0.17]$	$[0.00, 0.00]$	$[0.00, 0.00]$	$[0.00, 0.02]$
$\boldsymbol{x}^{(2)}$	$[0.33, 0.41]$	$[0.00, 0.00]$	$[0.55, 0.63]$	$[0.11, 0.16]$	$[0.01, 0.02]$
$\boldsymbol{x}^{(A)}$	$[0.30, 0.37]$	$[0.00, 0.00]$	$[0.00, 0.00]$	$[0.00, 0.00]$	$[0.00, 0.00]$
$\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$	$[0.71, 0.78]$	$[0.02, 0.04]$	$[0.01, 0.03]$	$[0.00, 0.01]$	$[0.14, 0.20]$

(c) Binding-set contraction

Experiment	$\|\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)}) - \boldsymbol{x}^{(A)}\|_1$
Support expansion	[0.12, 0.52]
Feasibility expansion	[0.22, 0.47]
Binding-set contraction	[0.49, 0.77]

(d) ℓ_1 distance from $\boldsymbol{x}^{(A)}$

Table 7: Results for **gpt-5.4** and **temperature 0.7**. The empirical setup and construction of the instance (i.e., task, aggregation operation, and prompt/output specification) is the same as in Section 5. These results demonstrate support expansion, binding-set contraction, and feasibility expansion.

	x_1	x_2	x_3
$\boldsymbol{x}^{(1)}$	$[0.83, 0.89]$	$[0.00, 0.00]$	$[0.05, 0.09]$
$\boldsymbol{x}^{(2)}$	$[0.00, 0.00]$	$[0.40, 0.48]$	$[0.26, 0.33]$
$\boldsymbol{x}^{(A)}$	$[0.41, 0.49]$	$[0.40, 0.48]$	$[0.12, 0.18]$
$\boldsymbol{x}^*_P(\boldsymbol{x}^{(A)})$	$[0.36, 0.49]$	$[0.13, 0.23]$	$[0.34, 0.47]$

(a) Support Expansion

	x_1	x_2	x_3
$\boldsymbol{x}^{(1)}$	$[0.33, 0.40]$	$[0.71, 0.78]$	$[0.01, 0.03]$
$\boldsymbol{x}^{(2)}$	$[0.32, 0.40]$	$[0.00, 0.02]$	$[0.79, 0.85]$
$\boldsymbol{x}^{(A)}$	$[0.02, 0.05]$	$[0.02, 0.05]$	$[0.02, 0.05]$
$\boldsymbol{x}^*_P(\boldsymbol{x}^{(A)})$	$[0.00, 0.01]$	$[0.04, 0.10]$	$[0.01, 0.04]$

(b) Feasibility Expansion

	x_1	x_2	x_3	x_4	x_5
$\boldsymbol{x}^{(1)}$	$[0.60, 0.68]$	$[0.28, 0.35]$	$[0.03, 0.06]$	$[0.00, 0.00]$	$[0.08, 0.13]$
$\boldsymbol{x}^{(2)}$	$[0.25, 0.32]$	$[0.00, 0.01]$	$[0.66, 0.74]$	$[0.02, 0.05]$	$[0.01, 0.03]$
$\boldsymbol{x}^{(A)}$	$[0.00, 0.00]$	$[0.00, 0.00]$	$[0.00, 0.00]$	$[0.00, 0.00]$	$[0.00, 0.00]$
$\boldsymbol{x}^*_P(\boldsymbol{x}^{(A)})$	$[0.00, 0.00]$	$[0.00, 0.00]$	$[0.00, 0.00]$	$[0.00, 0.00]$	$[0.00, 0.00]$

(c) Binding-set contraction

Experiment	$\|\boldsymbol{x}^*_P(\boldsymbol{x}^{(A)}) - \boldsymbol{x}^{(A)}\|_1$
Support expansion	[0.32, 0.83]
Feasibility expansion	[0.01, 0.18]
Binding-set contraction	[0.00, 0.00]

(d) ℓ_1 distance from $\boldsymbol{x}^{(A)}$

Table 8: Results for **gpt-5-mini** with the default temperature. The empirical setup and construction of the instance (i.e., task, aggregation operation, and output specification) is the same as in Section 5. For support expansion and feasibility expansion, we use the same prompt specification as in Section 5. For binding-set contraction, the prompts used to generate $x^{(1)}$ and $x^{(2)}$ slightly differ: we set $G_1^{\text{inc}} = \left\{\mathcal{T}_1^P\right\}, G_1^{\text{exc}} = \left\{\mathcal{T}_2^P\right\}$, and we set $G_2^{\text{inc}} = \left\{\mathcal{T}_1^P, \mathcal{T}_2^P\right\}, \text{op}_2^{\text{inc}} = \{\text{or}\}$. These results demonstrate support expansion and feasibility expansion (modestly), but not binding-set contraction: intersection aggregation produces an empty list ($\boldsymbol{x}^{(A)} = \mathbf{0}$), even with the modified prompt specifications described above.

EXPERIMENT 3 (E3): COMBINING HETEROGENEOUS MODELS

	x_1	x_2	x_3
$\boldsymbol{x}^{(1)}$	$[0.62, 0.71]$	$[0.00, 0.01]$	$[0.01, 0.04]$
$\boldsymbol{x}^{(2)}$	$[0.00, 0.01]$	$[0.80, 0.87]$	$[0.01, 0.03]$
$\boldsymbol{x}^{(A)}$	$[0.29, 0.38]$	$[0.45, 0.54]$	$[0.01, 0.03]$
$\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$	$[0.24, 0.31]$	$[0.29, 0.36]$	$[0.21, 0.27]$

(a) Support Expansion

	x_1	x_2	x_3
$\boldsymbol{x}^{(1)}$	$[0.66, 0.75]$	$[0.32, 0.42]$	$[0.01, 0.04]$
$\boldsymbol{x}^{(2)}$	$[0.37, 0.47]$	$[0.00, 0.01]$	$[0.72, 0.80]$
$\boldsymbol{x}^{(A)}$	$[0.00, 0.01]$	$[0.00, 0.01]$	$[0.00, 0.01]$
$\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$	$[0.01, 0.04]$	$[0.00, 0.01]$	$[0.00, 0.01]$

(b) Feasibility Expansion

	x_1	x_2	x_3	x_4	x_5
$\boldsymbol{x}^{(1)}$	$[0.80, 0.87]$	$[0.08, 0.14]$	$[0.00, 0.01]$	$[0.00, 0.01]$	$[0.00, 0.02]$
$\boldsymbol{x}^{(2)}$	$[0.30, 0.39]$	$[0.00, 0.01]$	$[0.56, 0.66]$	$[0.09, 0.16]$	$[0.00, 0.02]$
$\boldsymbol{x}^{(A)}$	$[0.21, 0.29]$	$[0.00, 0.01]$	$[0.00, 0.01]$	$[0.00, 0.01]$	$[0.00, 0.01]$
$\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$	$[0.02, 0.08]$	$[0.00, 0.01]$	$[0.01, 0.04]$	$[0.00, 0.03]$	$[0.09, 0.18]$

(c) Binding-set contraction

Experiment	$\lVert \boldsymbol{x}_P^*(\boldsymbol{x}^{(A)}) - \boldsymbol{x}^{(A)} \rVert_1$
Support expansion	[0.26, 0.66]
Feasibility expansion	[0.00, 0.07]
Binding-set contraction	[0.21, 0.54]

(d) ℓ_1 distance from $\boldsymbol{x}^{(A)}$

Table 9: Results for aggregating two different models: **gpt-4o-mini** (temp 0) and **gpt-5.4** (temp 0). Support expansion and binding set contraction seems to work. The empirical setup and construction of the instance (i.e., task, aggregation operation, and prompt/output specification) is the same as in Section 5. $\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$ is chosen via brute force search over prompt topics and over the two models. These results demonstrate support expansion, binding-set contraction. Feasibility expansion does not appear to occur for this instance due to intersection resulting in empty lists.

	x_1	x_2	x_3
$\boldsymbol{x}^{(1)}$	$[0.70, 0.78]$	$[0.00, 0.01]$	$[0.04, 0.09]$
$\boldsymbol{x}^{(2)}$	$[0.00, 0.01]$	$[0.76, 0.83]$	$[0.01, 0.04]$
$\boldsymbol{x}^{(A)}$	$[0.32, 0.41]$	$[0.45, 0.55]$	$[0.03, 0.08]$
$\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$	$[0.00, 0.01]$	$[0.42, 0.55]$	$[0.21, 0.33]$

(a) Support Expansion

	x_1	x_2	x_3
$\boldsymbol{x}^{(1)}$	$[0.66, 0.75]$	$[0.31, 0.41]$	$[0.01, 0.05]$
$\boldsymbol{x}^{(2)}$	$[0.30, 0.40]$	$[0.00, 0.02]$	$[0.74, 0.82]$
$\boldsymbol{x}^{(A)}$	$[0.00, 0.01]$	$[0.00, 0.01]$	$[0.00, 0.01]$
$\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$	$[0.01, 0.04]$	$[0.00, 0.01]$	$[0.00, 0.01]$

(b) Feasibility Expansion

	x_1	x_2	x_3	x_4	x_5
$\boldsymbol{x}^{(1)}$	$[0.59, 0.69]$	$[0.28, 0.37]$	$[0.01, 0.04]$	$[0.00, 0.01]$	$[0.07, 0.12]$
$\boldsymbol{x}^{(2)}$	$[0.15, 0.22]$	$[0.00, 0.01]$	$[0.68, 0.76]$	$[0.05, 0.10]$	$[0.00, 0.02]$
$\boldsymbol{x}^{(A)}$	$[0.99, 1.00]$	$[0.00, 0.01]$	$[0.00, 0.01]$	$[0.00, 0.01]$	$[0.00, 0.01]$
$\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$	$[0.78, 0.88]$	$[0.07, 0.16]$	$[0.00, 0.01]$	$[0.00, 0.03]$	$[0.00, 0.04]$

(c) Binding-set contraction

Experiment	$\|\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)}) - \boldsymbol{x}^{(A)}\|_1$
Support expansion	[0.43, 0.84]
Feasibility expansion	[0.00, 0.07]
Binding-set contraction	[0.18, 0.46]

(d) ℓ_1 distance from $\boldsymbol{x}^{(A)}$

Table 10: Results for aggregating two different models: **gpt-4o-mini** (temp 0) and **gpt-5-mini** (temp default). The empirical setup and construction of the instance (i.e., task, aggregation operation, and prompt/output specification) is the same as in Section 5. $\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$ is chosen via brute force search over prompt topics and over the two models. These results demonstrate support expansion, binding-set contraction. Feasibility expansion does not appear to occur for this instance due to intersection resulting in empty lists.

	x_1	x_2	x_3
$\boldsymbol{x}^{(1)}$	$[0.70, 0.78]$	$[0.00, 0.01]$	$[0.04, 0.09]$
$\boldsymbol{x}^{(2)}$	$[0.00, 0.01]$	$[0.36, 0.45]$	$[0.15, 0.23]$
$\boldsymbol{x}^{(A)}$	$[0.32, 0.41]$	$[0.45, 0.54]$	$[0.05, 0.10]$
$\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$	$[0.24, 0.31]$	$[0.29, 0.36]$	$[0.21, 0.27]$

(a) Support Expansion

	x_1	x_2	x_3
$\boldsymbol{x}^{(1)}$	$[0.34, 0.43]$	$[0.72, 0.80]$	$[0.01, 0.03]$
$\boldsymbol{x}^{(2)}$	$[0.39, 0.48]$	$[0.00, 0.01]$	$[0.69, 0.77]$
$\boldsymbol{x}^{(A)}$	$[0.30, 0.40]$	$[0.21, 0.29]$	$[0.21, 0.29]$
$\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$	$[0.41, 0.55]$	$[0.25, 0.37]$	$[0.36, 0.50]$

(b) Feasibility Expansion

	x_1	x_2	x_3	x_4	x_5
$\boldsymbol{x}^{(1)}$	$[0.62, 0.71]$	$[0.25, 0.34]$	$[0.02, 0.05]$	$[0.00, 0.01]$	$[0.05, 0.10]$
$\boldsymbol{x}^{(2)}$	$[0.32, 0.41]$	$[0.00, 0.01]$	$[0.55, 0.64]$	$[0.09, 0.16]$	$[0.00, 0.01]$
$\boldsymbol{x}^{(A)}$	$[0.16, 0.24]$	$[0.00, 0.01]$	$[0.00, 0.01]$	$[0.00, 0.01]$	$[0.00, 0.01]$
$\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$	$[0.15, 0.26]$	$[0.00, 0.01]$	$[0.00, 0.01]$	$[0.00, 0.01]$	$[0.41, 0.55]$

(c) Binding-set contraction

Experiment	$\|\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)}) - \boldsymbol{x}^{(A)}\|_1$
Support expansion	[0.20, 0.66]
Feasibility expansion	[0.08, 0.70]
Binding-set contraction	[0.40, 0.69]

(d) ℓ_1 distance from $\boldsymbol{x}^{(A)}$

Table 11: Results for aggregating two different models: **gpt-5.4** (temp 0) and **gpt-5-mini** (temp default). The empirical setup and construction of the instance (i.e., task, aggregation operation, and prompt/output specification) is the same as in Section 5. $\boldsymbol{x}_P^*(\boldsymbol{x}^{(A)})$ is chosen via brute force search over prompt topics and over the two models. These results demonstrate support expansion, binding-set contraction, and feasibility expansion.