Formalization of QFT: Osterwalder–Schrader Axioms for the Free Field in Lean 4

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Abstract

A foundational result in constructive quantum field theory is the construction of the free bosonic field on four-dimensional Euclidean spacetime and the verification that it satisfies the Glimm-Jaffe axioms, a variant of the Osterwalder-Schrader axioms. We present a complete formalization of this result in the Lean 4 interactive theorem prover. The project serves as a proof of concept that extended arguments in mathematical physics can be translated into machine-checked proofs using currently available formalization tools and AI assistance.

We introduce the relevant background in interactive theorem proving and constructive quantum field theory, describe the structure of the formalization, and discuss the design choices and methods that made the project feasible. Because the work was carried out during a period of rapid change in AI coding capabilities, from July 2025 to March 2026, we also reflect on how these changes affected the formalization process. We conclude by discussing practical strategies for AI-assisted formalization and its possible role in the future of theoretical physics.

A more detailed version of this paper is available at arXiv:2603.15770, and the accompanying Lean 4 code is available at `https://github.com/mrdouglasny/OSforGFF`.

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1. Introduction

AI is beginning to have a major impact on science, in part through its use in interactive theorem proving (ITP), a technology for constructing and verifying mathematical proofs expressed in a formal language. Recent milestones demonstrate the practical maturity of machine-verified mathematics at scale, including Lean-verified solutions at IMO 2025 gold-medal level and end-to-end formal solution corpora for Putnam 2025 (Aristotle, 2025; har, 2025).

We believe ITP will prove equally valuable for theoretical physics. To demonstrate its current practicality, we have formalized the construction of the 4D Euclidean free massive boson theory in Lean, following Glimm and Jaffe (Glimm & Jaffe, 2012). [1]

1.1. What Is Formalization and Why?

ITP develops and checks mathematical proofs by expressing them in a formal language with precise logical semantics verifiable by a computer. The process of translating informal mathematics into such a language is called formalization (Harrison, 2014). Several proof assistants are well suited to this task, including Coq/Rocq (roc), Isabelle (Paulson, 1994), and Lean (Avigad, 2024), which has a large and growing library of standard results called Mathlib (DBL, 2019).

A technology that accelerates proof while guaranteeing complete rigor would be extremely valuable. Mathematics journals are notorious for lengthy review times; ITP could facilitate collaboration by removing the need to individually vet every contribution. Theoretical physicists should also take interest: formalization forces every mathematical object to be explicitly defined and every assumption stated, exposing hidden assumptions routinely taken for granted in the physics literature.

1.2. ITP in the Age of AI

As of early 2026, ITP has not yet had a major impact on mathematics research, partly because only a tiny fraction of mathematics has been formalized (Massot, 2024). ITP also

[1]Generalization

has a steep learning curve: historically it has taken roughly a week to formalize a single textbook page (Wiedijk, 2006). However, this situation is rapidly improving. Automated theorem proving, agentic interaction protocols (MCP) (lea, 2024; mcp, 2025), and AI for theorem proving (AITP) (ait, 2024) are all advancing quickly. AlphaProof combines reinforcement learning with formal verification (Hubert, 2025), and autoformalization leverages LLMs to translate informal mathematics into formal proofs (Lu, 2024). Our own productivity with mid-2025 tools was far greater than at the beginning of that year, and early 2026 tools are better still.

1.3. Status of Rigorous QFT

The mass gap problem asks whether quantum Yang–Mills theory can be defined rigorously in four dimensions with a positive spectral gap. The Clay Mathematics Institute formulates this as proving that, for any compact simple gauge group, a nontrivial quantum Yang–Mills theory exists on $\mathbb{R}^4$ with a mass gap $\Delta > 0$ (Clay Mathematics Institute). This gap is tied to the observed short-range, confining behavior of the strong force and the absence of free gluons, so a solution would place one of the Standard Model's most successful components on a fully rigorous mathematical foundation.

In the 1970s, mathematical physicists successfully constructed Euclidean scalar QFTs ($P(\Phi)_2$, Φ^4_3) satisfying the Osterwalder–Schrader axioms (Glimm & Jaffe, 2012; Osterwalder & Schrader, 1972). In four dimensions, heuristic calculations indicated that no nontrivial scalar field theories exist, and this was rigorously confirmed: Φ^4_d is trivial for $d \geq 5$ (Aizenman, 1981) and $d = 4$ (Aizenman & Duminil-Copin, 2021). Despite this, Yang–Mills theory is widely believed to be nontrivial with a mass gap.

Interest in constructive QFT has recently revived, partly due to regularity structures and paracontrolled calculus for singular stochastic PDEs (Hairer, 2014; Gubinelli et al., 2017). These breakthroughs have enabled the stochastic quantization program for Φ^4_3 and partial progress on pure Yang–Mills and Yang–Mills–Higgs in 2d and 3d (Chandra et al., 2022). On the lattice side, continuum-limit convergence for the Abelian theory in 4d was proven by Driver (Driver, 1987), and significant progress toward the non-Abelian theory was made by Balaban (Balaban, 1989; Bałaban, 1987). More recently, gauge–string duality and stochastic quantization have weakened the required coupling assumptions (Shen et al., 2023; Cao et al., 2025; Cao et al., 2025; Nissim, 2025), and the pure Yang–Mills $N = \infty$ limit has been shown to exist (Chatterjee, 2019; Borga et al., 2024). What remains missing is fully rigorous control of the renormalization and continuum-limit procedure for non-Abelian gauge theories in four dimensions.

2. Axiomatic QFT

Rigorously proving that a QFT exists requires first specifying the properties a mathematical object must satisfy, then constructing such an object. The main axiomatic frameworks are the Wightman axioms (Gårding & Wightman, 1964) (Minkowski spacetime, correlation functions), the Osterwalder–Schrader (OS) axioms (Osterwalder & Schrader, 1972) (Euclidean spacetime), and the Haag–Kastler axioms (Haag & Kastler, 1964) (algebraic QFT). Reconstruction theorems relate these formulations.

We formalized the Glimm–Jaffe (Glimm & Jaffe, 1987) variant of the OS axioms, which recasts the problem as constructing a functional measure (path integral). Since measure theory and probability are relatively well developed in Lean's Mathlib library, this was the most practical choice. By contrast, Mathlib currently lacks sufficient operator theory and complex analysis for the Wightman or 2d CFT approaches.

The axioms are stated in terms of the generating functional

$$S(f) := \int e^{i\phi(f)}\, d\mu[\phi], \tag{1}$$

where μ is a Borel probability measure on $\mathcal{S}'(\mathbb{R}^D)$ and $\phi(f)$ denotes the pairing of a distribution $\phi \in \mathcal{S}'(\mathbb{R}^D)$ with a test function $f \in \mathcal{S}(\mathbb{R}^D)$.

- **OS0 (Analyticity):** For any finite set of test functions $f_1, \ldots, f_n \in \mathcal{S}(\mathbb{R}^D)$, the map

$$(z_1, \ldots, z_n) \mapsto S\left(\sum_{j=1}^{n} z_j f_j\right) \tag{2}$$

  is entire on $\mathbb{C}^n$.

- **OS1 (Regularity):** There exist $p \in [1, 2]$ and $c > 0$ such that

$$|S(f)| \leq \exp\Big[c\big(\|f\|_{L^1} + \|f\|^p_{L^p}\big)\Big]. \tag{3}$$

  For $p = 2$, the two-point Schwinger function is locally integrable.

- **OS2 (Euclidean Invariance):** $S(f) = S(f \circ E^{-1})$ for every Euclidean isometry E of $\mathbb{R}^D$.

Before stating OS3 and OS4, we introduce the algebra of exponential functionals on $\mathcal{S}'(\mathbb{R}^D)$:

$$\mathcal{A} = \Big\{A(\phi) = \sum_{j=1}^{N} c_j \exp\big(\phi(f_j)\big) : \; c_j \in \mathbb{C}, \; f_j \in \mathcal{S}(\mathbb{R}^D)\Big\}. \tag{4}$$

Let $\theta : (x_0, \vec{x}) \mapsto (-x_0, \vec{x})$ denote time reflection and $T_s : (x_0, \vec{x}) \mapsto (x_0+s, \vec{x})$ time translation by $s \in \mathbb{R}$. Define $\mathcal{A}^+ \subset \mathcal{A}$ to be the subalgebra in which every f_j is supported in the positive-time half-space $\mathbb{R}^D_+ := \{(x_0, \vec{x}) : x_0 > 0\}$.

- **OS3 (Reflection Positivity):** For all $A \in \mathcal{A}^+$,

$$\langle \theta A,\, A\rangle_{L^2(\mu)} \;=\; \int \overline{A(\theta\,\phi)}\; A(\phi)\; d\mu(\phi) \;\geq\; 0. \quad (5)$$

- **OS4 (Ergodicity):** The time translation subgroup acts ergodically on μ: for all $A \in \mathcal{A}$,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t A(T_s\,\phi)\, ds \;=\; \int A(\phi)\, d\mu(\phi), \quad (6)$$

  where the limit is taken in $L^2(\mu)$.

Remark 2.1. In our formalization we work with Schwartz functions $\mathcal{S}(\mathbb{R}^D)$ and tempered distributions $\mathcal{S}'(\mathbb{R}^D)$ rather than $\mathcal{D}(\mathbb{R}^D)$ and $\mathcal{D}'(\mathbb{R}^D)$. Since $\mathcal{D} \subset \mathcal{S}$ and $\mathcal{S}' \subset \mathcal{D}'$, our axioms imply those of Glimm–Jaffe. For OS4, we state and prove a stronger polynomial clustering condition, from which ergodicity follows (see Appendix B.5).

3. Mathematical Framework

We treat a free massive scalar field in $D = 4$ Euclidean dimensions, following Glimm and Jaffe (Glimm & Jaffe, 1987). The physical goal is to define a functional measure corresponding to the action $S[\phi] = \frac{1}{2}\int(\partial\phi)^2 + m^2\phi^2$ with $m^2 > 0$. The naive expression $d\mu[\phi] \sim \prod_x d\phi(x)\exp(-S[\phi])$ cannot be interpreted literally, since the infinite-dimensional Lebesgue measure does not exist.

Construction via Minlos' theorem. We construct the Gaussian free field (GFF) by applying Minlos' theorem to the characteristic functional

$$\chi(f) = \exp\!\left(-\tfrac{1}{2}C(f,f)\right), \quad (7)$$

where $C(f,f) = \int \frac{m}{4\pi^2|x-y|} K_1(m|x-y|) f(x) f(y)\, dx\, dy$ is the covariance form defined using the Bessel function K_1.

Theorem 3.1 (Minlos)**.** *Let E be a real nuclear locally convex topological vector space, and E' its continuous dual. If $\chi : E \to \mathbb{C}$ is continuous, positive definite, and $\chi(0) = 1$, then there exists a unique Borel probability measure μ on E' such that $\chi(f) = \int_{E'} e^{i\phi(f)}\, d\mu(\phi)$ for all $f \in E$.*

Schwartz space and distribution. Physicists usually build an infinite-dimensional field space through finite-dimensional approximations (e.g., lattice discretization with spacing $a = 2^{-N}$). While perfectly rigorous, a lattice approximation breaks Euclidean invariance (OS2) from the outset. Since there are no nontrivial finite-dimensional representations of the Euclidean group, preserving OS2 naturally leads to measures on infinite-dimensional function spaces.

A conceptually simple construction is to diagonalize the covariance and write $\phi(x) = \sum_{k\in\mathbb{N}} c_k \phi_k(x)$ with c_k independent Gaussians. However, in $D = 4$ the covariance is not trace class ($\int d^4p/(p^2+m^2) = \infty$), so $K(x,y)$ is singular on the diagonal and the field cannot be an ordinary function—only a random distribution.

The Schwartz space $\mathcal{S}(\mathbb{R}^D)$ satisfies the requirements: free choice on finite point sets, finite integrals against $K(x-y)$, and finite total integrals. The key structural fact is that $\mathcal{S}(\mathbb{R}^D)$ is a *nuclear* Fréchet space, meaning its topology can be described by nested Hilbert norms with trace-class inclusion maps. By the Bochner–Minlos theorem, every continuous positive-definite functional on a nuclear space is the characteristic functional of a unique probability measure on its dual. Nuclearity plays the role that tightness plays in finite dimensions: it prevents probability mass from escaping to infinity (Glimm & Jaffe, 2012).

In our initial formalization, we postulated Minlos' theorem and nuclearity of Schwartz space as axioms, carefully checked by hand. These have since been proven or circumvented, so the axioms are now established using only Lean and Mathlib.

Bessel function covariance. The standard physics definition of the covariance as a Fourier transform of $1/(k^2+m^2)$ is a conditionally convergent integral whose integrability is hard to prove and can even be interpreted as zero in certain contexts. We instead used the absolutely convergent Bessel function representation, which is more amenable to formal verification. This experience led us to introduce systematic checks for absolute convergence throughout the project.

4. Details of the Formalization

Full proofs are given in Appendix B. Here we highlight key choices and design decisions.

Euclidean group. We defined the Euclidean group action largely from scratch, specifying the group structure, its elements, and their actions on the measure and test functions. These definitions are specialized to 4D spacetime and scalar test functions; generalizing to arbitrary dimension and non-scalar fields is left to future work.

Fourier transform conventions. We used two slightly different Fourier transform definitions—one following the physics convention, the other following Mathlib—differing by a factor of 2π in the phase. Different conventions proved more convenient for different theorems. Unlike symbolic

mathematics, the prover catches any mismatch in conventions, so using both simultaneously does not introduce errors.

Positivity via Schur–Hadamard. We prove both the positivity required by Minlos and reflection positivity (OS3) via the Schur–Hadamard theorem, which reduces positivity of the generating functional to positivity of the free covariance. Positivity here is defined in the sense of probability theory (details in Appendix B).

OS3: Schwinger parametrization. The naive Fourier-space proof of reflection positivity requires exchanging integration orders for an integrand that is only conditionally convergent. We resolved this using the Schwinger parametrization (heat kernel expansion), which yields an absolutely convergent representation amenable to Fubini's theorem (see Appendix B).

OS4: Polynomial clustering. Rather than proving ergodicity directly, we established a stronger polynomial clustering bound: for any $f, g \in \mathcal{S}(\mathbb{R}^D)$ and $\alpha > 0$, $|S[f + T(s)g] - S[f]S[g]| \leq C(1 + |s|)^{-\alpha}$. OS4 then follows by an L^2 argument (see Appendix B.5). Even the seemingly trivial reduction to single exponentials via Minkowski's inequality required nontrivial work in Lean, particularly for verifying measurability of basic operations.

5. Methods

We started in July 2025 with an informal outline, which was given to GPT-5 and Gemini 2.5 to translate into Lean. The results were low quality but useful for suggesting definitions. We then hand-coded the basic definitions and OS axiom statements.

From that point we primarily used coding assistants, initially GPT and Gemini in GitHub Copilot with Lean-LSP MCP. The introduction of Claude Code represented the single largest jump in capability; by the project's end, Claude Code Opus 4.6 was our primary agent, substantially stronger than the tools we started with.

All observations below reflect the state of AI tooling as of early 2026. Given the pace of improvement, some of these limitations may be short-lived. Indeed, perhaps the most striking aspect of this project was how noticeably the models improved over its course. At the outset, it was a struggle to accomplish anything nontrivial. By around December 2025, generating proofs was no longer the main difficulty; the real obstacles were false analogies, context limits, and stale documentation. By the time of writing, we are obtaining fairly nontrivial proofs with minimal guidance.

Backward chaining. We developed a backward-chaining strategy: when agents could not prove a lemma, rather than leaving `sorry`s, we had them propose simpler "helper lemmas" to assume as axioms. These were validated through cross-model verification (a conjecture proposed by one model would be reviewed by the others). A particularly effective variation was asking agents to propose helper lemmas formulated as generally as possible, without project-specific definitions. This reduced false conjectures and produced lemmas that were easier to prove independently.

Correctness of definitions. LLMs have a reasonable intuition for mathematical "correctness": when two models are independently asked whether a conjectured axiom is true, at least one will usually flag an outright falsehood—which is why our cross-model validation protocol is valuable. The more insidious failure mode is subtler: models can be led astray by analogies that are nearly but not quite right. For instance, a model may confidently apply results about complex matrices to matrices over a Grassmann algebra, or treat a conditionally convergent integral as absolutely convergent, because these are the "standard" settings most common in its training data. Even with explicit instructions about proof structure, a model's prior intuitions can override guidance. During the Lean code-writing phase, the most important human role was ensuring that key definitions were correct and that the mathematical context was not being silently shifted. Subtle slips are hard to detect by inspection, especially when types align and everything compiles. A reliable countermeasure was to build small unit tests continually: take a lemma just proved, or a definition just stated, and use it to prove an additional statement not dictated by the original scaffold. In practice, one such auxiliary check was usually enough to surface hidden mistakes.

Decomposition of long proofs. Context limits prevent models from simultaneously "seeing" all definitions, prior lemmas, and local proof state, and nor can they reliably produce a long proof script without losing earlier constraints. As the project progressed, refactoring became more manageable—for example, the proof of Bochner's theorem was completed by Claude Code in February 2026 over two days, nearly autonomously. Still, when models are pushed to plan and execute long-horizon proofs, the resulting Lean code tends to be "merely true" rather than legible and production-ready. Introducing new definitions and using them across long proofs remains a challenge; human input is valuable for formulating definitions, planning decomposition, and providing proof outlines. With current tooling, formalization tends to follow known human proofs, and it remains difficult to use these systems to produce genuinely new arguments.

Stale docstrings and unreliable summaries. LLMs navigate the codebase primarily through docstrings and comments rather than by genuinely parsing Lean code. These annotations may reference deleted definitions or outdated strategies. When we asked agents to summarize files, the resulting documents were often polluted with stale information. Worse, models showed little ability to distinguish essential proof components from routine bookkeeping. Summaries would dwell on type coercions and import boilerplate while glossing over key mathematical ideas. Human-written or human-edited summaries remain necessary for reliable high-level overviews.

Unproductive search. For certain theorems, models worked for hours, repeatedly adding auxiliary lemmas without progress. They would propose alternatives that failed to resolve the core obstruction, unable to identify the central argument. When a human intervened with a clearer proof route, the model could usually implement the new direction quickly.

Workflow and complexity management. As the project grew, several late-stage difficulties emerged: not getting lost in helper lemmas, tracking which axioms contribute to the main proof line, and managing two or three simultaneously developing proof strategies. This was aided by dependency extraction tools (Cihal, 2023), but the main organizational technique was frequently asking agents to write and update auxiliary documents describing active proof chains. For example, we maintained files listing all `sorry`s and assumed axioms. Planning broadly was essential and deserved even more investment than we gave it.

6. Going Forward

We outline several directions for near-term development.

Existing results in mathematical physics. A natural near-term goal is the OS reconstruction theorem, which connects the Euclidean approach we followed to the original Wightman definition. Ideally one would use OS axioms that are both provable and state necessary and sufficient conditions for reconstruction, but whether the current axioms suffice is not entirely clear (see (Kravchuk et al., 2021) for a recent discussion). Formalizing the reconstruction theorem could help settle these questions.

Among the simplest interacting QFTs is $P(\phi)_2$, the 2d boson with polynomial potential, constructed in (Glimm & Jaffe, 2012; Simon, 1974). Progress on formalizing this is underway. Rigorous construction of fermionic field theories is generally easier, as they do not suffer from the large-field problem; combined with the abelian Higgs model, one could aspire to a direct QFT argument for mirror symmetry.

Quantum information theory is also being formalized in Lean (Meiburg et al., 2025). Recent work formally verified the Generalized Quantum Stein's Lemma, clarifying implicit assumptions and contributing to a growing QuantumInfo library (Meiburg & collaborators, 2025).

From the perspective of formal theorists, it would also be appealing to apply these ideas to quantum gravity. The gravitational path integral $Z = \int \mathcal{D}g\, e^{iS[g]}$ involves integration over an infinite-dimensional space of metrics modulo diffeomorphisms, and the functional measure is not rigorously defined. Clarifying the appropriate configuration space would constitute a significant conceptual advance.

Non-rigorous theoretical physics. The low-hanging fruit is improving verification of calculations. Many algebraic calculations in physics are essentially rigorous, they only need to be placed in the correct mathematical context. Formalizing that context, even without proving it from foundations, would catch errors due to incompatible conventions, as exemplified by the long-standing muon $g-2$ sign-convention disagreement (Knecht et al., 2002).

There is little to gain by formalizing symbolic algebra in Lean, since a good computer algebra system should be fully reliable. Instead, one could extend formalization's scope by implementing communication between ITP verifiers and symbolic algebra systems (Rowney et al., 2026). For theories far beyond present-day mathematical physics, such as 4d Yang–Mills and its supersymmetric extensions, ITP would make it far easier to postulate axioms capturing current understanding, test them, and explore their consequences.

Library development. The most immediate enabling factor is the formalization of more existing mathematics. Mathlib, while impressive, is still small relative to the known mathematical corpus. As of this writing, roughly 2,300 pull requests await review, each requiring substantial human effort. We hope AI tools will help in two ways: assisting the review process itself (checking conventions, verifying integration, flagging issues) and helping authors bring contributions into compliance with library standards before submission.

A closely related issue is the fragmentation of independent formalization projects. Each group may define its own versions of basic objects—Euclidean transformations, QFT axioms, function spaces—leading to mutually incompatible formalizations. It is unclear that a single monolithic library is the best or even a scalable solution. An alternative would be to design lightweight but rigorous standards: shared definitions, interface conventions, and compatibility requirements, together with search tools to discover and adopt prior work. We view this as a very important infras-

tructure problem.

Benchmarking formalization progress. Existing benchmarks such as miniF2F and ProofNet evaluate AI proving capability, but they measure the AI rather than the state of formalized knowledge. What is missing is a way to track how much mathematics has actually been formalized and where gaps remain. In mathematics, formalization proceeds bottom-up from definitions to theorems. In physics, the natural direction is often reversed: one starts from a conjecture and works backward to identify the precise conditions needed. A meaningful benchmark for physical formalization would need to capture not just how many theorems have been proved, but how far the top-down analysis has reached: how many assumptions have been made explicit, validated, or eliminated.

Further AI development. With rapid improvement, it is becoming possible to formalize entire textbook chapters at a pace that would have been impractical a year ago. If coding agents continue to improve at the current rate, autoformalization may well become practical soon.

One might suppose that if AI can easily translate between informal and formal reasoning, it will also achieve fully reliable informal reasoning, obviating the need for formalization. Recent progress can be read both ways: at the 2025 IMO, both formal and informal AI reasoning led to strong performance.

We find it useful to draw an analogy from computer science: the distinction between interpreted and compiled languages. A compiler translates code to machine code in advance, demanding more rigid source but yielding faster and more secure execution. In practice, interpreted languages serve smaller, short-lived projects, while compiled languages dominate larger, long-lived software. Similarly, formalization does more verification work upfront, avoiding the need to repeat it later. Its advantages will therefore be most significant for large theories under sustained development. Even if reliable informal AI reasoning becomes possible, formalization will still bring substantial benefits.

Advice for now (early 2026). This reflects our impressions as of March 2026 and may go out of date quickly.

- The most important resource for coding agents is context, and reading many large files rapidly exhausts it. Keep the project well organized with supplementary documentation files (summaries, plans, lists of sorrys). Markdown format works better than LaTeX or PDF for these.
- Frontier models' abilities differ and their ranking changes frequently; use several and follow their performance. A stable pattern: coding agents with harness (Claude Code, Codex) are better at coding but will try hard to prove false statements; chat/query agents (especially Gemini) are better at evaluating definitions for correctness.
- The most difficult phase is the beginning: developing project-specific definitions and a proof sketch. Formulating precise final theorems from the start, with minimal use of untested definitions, is very advantageous.
- Give models the actual math/physics papers (LaTeX is better but PDF works) and ask them to study and summarize.
- Backward chaining works well: first drafts can use `sorry`s; ask agents to fill them; if too hard, ask for a "textbook axiom" that would suffice, then carefully vet with other models and human review.
- Keeping supplementary docs and Lean docstrings up to date pays off.

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A. Notations

Here we provide the notations we used in this paper.

1. Function Spaces
   (a) For $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $C^k(\mathbb{R}^D; \mathbb{F})$ will denote the space of $\mathbb{F}$-valued functions on $\mathbb{R}^D$ for which all derivatives of order $\leq k$ exist and are continuous, and we define the set of smooth functions as $C^\infty(\mathbb{R}^D; \mathbb{F}) = \bigcup_{k \geq 0} C^k(\mathbb{R}^n; \mathbb{F})$. For $\mathbb{F} = \mathbb{C}$ we will simply denote the spaces $C^k(\mathbb{R}^D; \mathbb{F})$ ($C^\infty(\mathbb{R}^D; \mathbb{F})$) by $C^\infty(\mathbb{R}^D)$ ($C^k(\mathbb{R}^D)$).
   (b) $C_c^\infty(\mathbb{R}^D)$ will denote the set of smooth functions with compact support. We frequently use the notation $\mathcal{D}(\mathbb{R}^D) = C_c^\infty(\mathbb{R}^D)$ for the set of test functions. The topology on $\mathcal{D}(\mathbb{R}^D)$ is specified by saying a sequence of functions $f_n \to f$ in $\mathcal{D}(\mathbb{R}^n)$ if $\partial^\alpha f_n \to \partial f$ uniformly on K, for any multi-index α and compact set $K \subset \mathbb{R}^D$.
   (c) The space of Schwartz functions $\mathcal{S}(\mathbb{R}^D; \mathbb{F})$ denotes the set of $f \in C^\infty(\mathbb{R}^D; \mathbb{F})$ for which all derivatives $\partial^\alpha f$ decay faster than any polynomial, i.e. $\sup_{x \in \mathbb{R}^n} |(1 + |x|)^{-k} \partial^\alpha f(x)| < \infty$ for all $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$. The seminorms $\|f\|_{k,\alpha} := \sup_{x \in \mathbb{R}^n} |(1 + |x|)^{-k} \partial^\alpha f(x)|$ turn $\mathcal{S}(\mathbb{R}^D; \mathbb{F})$ into a Fréchet space.
   (d) For $p \in [1, \infty)$, and a measure space $(X, \mathcal{F}, \mu)$ and $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$,
   $L^p(X; \mathbb{F}) = L^p(\mu; \mathbb{F}) = \{\text{measurable } f \text{ on } \mathbb{R}^n \text{ taking values in } \mathbb{F} : \int_X |f|^p dx < \infty\}$, and
   $L^\infty(X; \mathbb{F}) = L^\infty(\mu; \mathbb{F}) = \{\text{measurable } f \text{ on } \mathbb{R}^n \text{ taking values in } \mathbb{F} : \operatorname{ess\,sup}|f| < \infty\}$ Once again when $\mathbb{F} = \mathbb{C}$ we omit $\mathbb{F}$ from the notation. $L^p(X; \mathbb{F})$ is a Banach space with norm $\|f\|_{L^p} := (\int_X |f|^p dx)^{1/p}$ for $p < \infty$ and $\|f\|_{L^\infty} := \operatorname{ess\,sup}|f|$.
2. Distributions
   (a) The space of distributions $\mathcal{D}'(\mathbb{R}^D)$ is the dual space of $\mathcal{D}(\mathbb{R}^D)$.
   (b) The space of tempered distributions $\mathcal{S}'(\mathbb{R}^D)$ is the dual space of $\mathcal{S}(\mathbb{R}^D)$.
3. Complex Analysis
   (a) A function $f : \mathbb{C}^n \to \mathbb{C}$ is *holomorphic* at a point $(z_1, \ldots, z_n) \equiv \vec{z}$ if all of its first order complex derivatives exist as complex linear maps at $\vec{z}$. Complex linearity requires $uD_{u\vec{z}} f = D_{\vec{z}} f$ for complex u, which implies $\bar{\partial}_{\vec{z}} f = 0$.
   (b) A function $f : \mathbb{C}^n \to \mathbb{C}$ is *analytic* at a point $(z_1, \ldots, z_n)$ if it can be represented by a convergent power series in a neighborhood of the point. One can prove that holomorphic functions are analytic (Goursat's theorem).
   (c) A function $f : \mathbb{C}^n \to \mathbb{C}$ is *entire* if it is analytic at every point in $\mathbb{C}^n$.

B. Proofs of Glimm-Jaffe Axioms

We mostly followed the proof structure laid out in Glimm and Jaffe with slight variations.

B.1. Analyticity (OS0)

The strict definition of analyticity of the GFF generating functional requires that

$$S\left[\sum_i z_i f_i\right] = \int \exp\left(i \sum_i z_i \, \phi(f_i)\right) d\mu(\phi) \tag{8}$$

admits a converging power series expansion with respect to all z_i for each fixed ω. In practice, it would be hard to construct an expansion of $S\left[\sum_i z_i \, f_i\right]$ and prove its convergence; instead, we used Hartog's theorem to simplify the proof.

Hartog's theorem states that in finite dimension, holomorphy of a complex function ($\mathbb{C}$-differentiable) implies its analyticity. Therefore OS0 simplifies to proving

$$\frac{\partial}{\partial z_j} \int \exp\left(i \sum_i z_i \, \phi(f_i)\right) d\mu(\phi) = \int i \, \phi(f_j) \cdot \exp\left(i \sum_i z_i \, \phi(f_i)\right) d\mu(\phi) \tag{9}$$

is well-defined. This requires that
1. the integrand $\exp\left(i \sum_i z_i \, \phi(f_i)\right)$ is measurable, integrable (for $S\left[\sum_i z_i \, f_i\right]$ to be well-defined), and analytic (derivative

exists);
2. the z derivative is bounded by an integrable function $B(\phi)$, so that the derivative and integral can be exchanged;
3. the z derivative is measurable, and hence integrable when combined with (2).

The hardest part in the proof was to establish a bound on the derivative. As we can see on the RHS of 9, its norm is bounded by

$$|i\,\phi(f_j)\cdot\exp\left(i\sum_i z_i\,\phi(f_i)\right)| \le \sum_j |\,\phi(f_j)|\exp\left(-\,[\phi(f_i)]_{\text{im}}\right). \tag{10}$$

To prove integrability of the bound, we use Fernique's Theorem, which states that for any f, there exists $\alpha > 0$ such that

$$\int \exp\left(\alpha\,[\phi(f)]^2\right) d\mu(\phi) < \infty, \tag{11}$$

and hence we can show $|\,\phi(f_j)|, \exp\left(-[\phi(f)]_{\text{im}}\right) \in L^2$. Lastly, we can apply Hölder's inequality, which says that product of L^2 functions is in L^1, to show that the RHS of 9 is integrable. In the actual proof, the exact bound is slightly different because we require the bound to be uniform in a neighborhood of z_0 (when taking the ∂z_j derivatives), so we need to expand the bound function around z_0; still, the general logic follows the same way.

B.2. Regularity (OS1)

For GFF, we will prove **??** with $p = 2$ and $c = \frac{1}{2m^2}$. The local integrability follows from the close-form expression of two-point functions:

$$K(x,y) = S_2(x,y) = \frac{m}{4\pi^2|x-y|}K_1(m|x-y|). \tag{12}$$

Let $y = 0$ and take $x \to 0$, we have

$$\frac{m}{4\pi^2|x|}K_1(m|x|) \sim \frac{1}{4\pi^2|x|^2} \tag{13}$$

which is locally integrable in 4-dimension.

First, by Minlos construction of the GFF, we have

$$S(f) = \langle f, Kf\rangle \equiv \int d^4x\, d^4y\; f^*(x)\,K(x,y)\,f(y). \tag{14}$$

To obtain an exponential bound, first notice that

$$|S(f)| = \left|\exp\left(-\frac{1}{2}\langle f, Kf\rangle_{\mathbb{C}}\right)\right| = \exp\left(-\frac{1}{2}\text{Re}\langle f, Kf\rangle_{\mathbb{C}}\right), \tag{15}$$

then decompose f into its real and imaginary components

$$\langle f, Kf\rangle_{\mathbb{C}} = \langle f_{\text{re}}, Kf_{\text{re}}\rangle + i\langle f_{\text{re}}, Kf_{\text{im}}\rangle + i\langle f_{\text{im}}, Kf_{\text{re}}\rangle - \langle f_{\text{im}}, Kf_{\text{im}}\rangle. \tag{16}$$

Taking the real part of both sides we have

$$\text{Re}\langle f, Kf\rangle_{\mathbb{C}} = \langle f_{\text{re}}, Kf_{\text{re}}\rangle - \langle f_{\text{im}}, Kf_{\text{im}}\rangle \tag{17}$$

Since K is positive semi-definite, $\langle f_{\text{re}}, Kf_{\text{re}}\rangle \ge 0$, so

$$-\text{Re}\langle f, Kf\rangle_{\mathbb{C}} \le \langle f_{\text{im}}, Kf_{\text{im}}\rangle. \tag{18}$$

To bound $\langle f, Kf\rangle$, we can expand it in momentum space

$$\langle f, Kf\rangle = \int \frac{|\hat{f}(k)|^2}{|k|^2 + m^2}\,dk \tag{19}$$

where $\hat{f}(k)$ is the fourier transform of f. Notice that the denominator is bounded by

$$\frac{1}{|k|^2 + m^2} \leq \frac{1}{m^2} \tag{20}$$

and by Plancherel's theorem ($\int |f(x)|^2 dx = \int |\hat{f}(k)|^2 dk$), we get the bound

$$\langle f, Kf \rangle \leq \frac{1}{m^2} \|f\|_{L^2}^2, \tag{21}$$

and therefore

$$\langle f_{\text{im}}, Kf_{\text{im}} \rangle \leq \frac{1}{m^2} \|f_{\text{im}}\|_{L^2}^2 \leq \frac{1}{m^2} \|f\|_{L^2}^2 \tag{22}$$

and

$$|S(f)| = \exp\left(-\frac{1}{2}\text{Re}\langle f, Kf \rangle\right) \leq \exp\left(\frac{1}{2}\langle f_{\text{im}}, Kf_{\text{im}} \rangle\right) \leq \exp\left(\frac{1}{2m^2}\|f\|_{L^2}^2\right) \leq \exp\left(\frac{1}{2m^2}(\|f\|_{L^1} + \|f\|_{L^2}^2)\right). \tag{23}$$

B.3. Euclidean Invariance (OS2)

We would like to show that under any Euclidean transformation E (translation and rotation), $S(f)$ remains unchanged. First we perform a change of variable by $x \to E^{-1}x$ and $y \to E^{-1}y$:

$$S(Ef) = \int d^4x\, d^4y\; f^*(Ex)K(x,y)f(Ey) = \int d^4(E^{-1}x)\, d^4(E^{-1}y)\; f^*(x)K(E^{-1}x, E^{-1}y)f(y). \tag{24}$$

From 12, we can see that $K(x,y)$ only depends $|x-y|$, and therefore $K(E^{-1}x, E^{-1}y) = K(x,y)$ is invariant under Euclidean group action. Furthermore, the Lebesgue measure is also invariant under translation and rotation ($|\det R| = 1$), so $d^4(E^{-1}x) = d^4x$. Combining these facts, we can conclude that

$$S(Ef) = \int d^4(x)\, d^4(y)\; f^*(x)K(x,y)f(y) = S(f). \tag{25}$$

B.4. Reflection Positivity (OS3)

The statement of OS3 as in 5 refers to the reflection positivity of the generating functional. First, we would want to use the Schur product theorem to reduce this statement to the reflection positivity of the free covariance. An equivalent statement with 5 is that for every finite sequence of functions f_j, $S\{f_i - \theta f_j\}$ is positive semi-definite. We want to show that

$$\langle \theta f, Kf \rangle \geq 0 \Rightarrow S\{f_i - \theta f_j\} = \exp\left(-\frac{1}{2}\langle f_i - \Theta f_j, K(f_i - \Theta f_j)\rangle\right) \succeq 0 \tag{26}$$

for any f defined on positive time. Rewriting the exponent on the RHS gives

$$\langle f_i - \Theta f_j, K(f_i - \Theta f_j)\rangle = \langle f_i, Kf_i \rangle + \langle f_j, Kf_j \rangle - 2\langle \Theta f_i, Kf_j \rangle \tag{27}$$

where we used the time-reflection symmetry of the inner product. Then we can expand the exponential

$$\exp\left(-\frac{1}{2}\langle f_i - \Theta f_j, K(f_i - \Theta f_j)\rangle\right) = \exp\left(-\frac{1}{2}\langle f_i, Kf_i \rangle\right) \times \exp\left(-\frac{1}{2}\langle f_j, Kf_j \rangle\right) \times \exp\left(R_{ij}\right) \tag{28}$$

where $R_{ij} \equiv \langle \theta f_i, Kf_j \rangle$. For any coefficient c, the quadratic form is

$$\sum_{i,j} c_i c_j R_{ij} = \left\langle \Theta\left(\sum_i c_i f_i\right), K\left(\sum_j c_j f_j\right)\right\rangle \geq 0 \tag{29}$$

where the last inequality is given by free covariance reflection positivity, treating $\sum c_i f_i$ as one function. Since R_{ij} is positive semi-definite, by Schur product theorem (product of PSD is also PSD), we conclude that $\exp(R_{ij})$ is PSD. Finally, looking at the quadratic form of 28, we have

$$\sum_{i,j} c_i c_j S(f_i - \theta f_j) = \sum_{i,j} \left[c_i \exp\left(-\frac{1}{2}\langle f_i, K f_i\rangle\right)\right]\left[c_j \exp\left(-\frac{1}{2}\langle f_j, K f_j\rangle\right)\right] \exp(R_{ij}) \geq 0 \tag{30}$$

which is exactly the condition for OS3.

Now the proof os OS3 simplifies to proving the reflection positivity of the free covariance $\langle \theta f, K f\rangle$. Expanding the inner product gives

$$\begin{aligned}\langle \theta f, K f\rangle &\equiv \int d^4x\, d^4y\, f^*(-x_0, \vec{x}) f(y_0, \vec{y}) K(x, y) \\ &= \int d^4x\, d^4y\, f^*(x_0, \vec{x}) f(y_0, \vec{y}) \int d^3k\, dk_0\, \frac{1}{k^2 + m^2} e^{-ik_0(x_0+y_0) + i\vec{k}\cdot\vec{x}},\end{aligned} \tag{31}$$

where θ is the time reflection operator and $K(x, y)$ is the free covariance, or the Green's function. Naively, we want to first perform the k_0 integral to get

$$\int k_0\, \frac{1}{k^2 + m^2} e^{-ik_0(x_0+y_0)} = \frac{\pi}{\omega_k} e^{-\omega_k |x_0+y_0|}, \tag{32}$$

where $\omega_k = \sqrt{\vec{k}^2 + m^2}$. Then we commute the $\vec{k}$ integral outside the x and y integrals, and perform the $\vec{x}$ and $\vec{y}$ integrals to get

$$\begin{aligned}&\int d^4x\, d^4y \int d^3k\, f^*(x_0, \vec{x}) f(y_0, \vec{y}) \frac{\pi}{\omega_k} e^{-\omega_k|x_0+y_0|} e^{i\vec{k}\cdot\vec{x}} \\ &= \int d^3k \int d^4x\, d^4y\, f^*(x_0, \vec{x}) f(y_0, \vec{y}) \frac{\pi}{\omega_k} e^{-\omega_p|x_0+y_0|} e^{i\vec{k}\cdot\vec{x}} \\ &= \int d^3k \int dx_0\, dy_0\, \frac{\pi}{\omega_p} \tilde{f}^*(x_0, \vec{k}) \tilde{f}(y_0, \vec{k}) e^{-\omega_k|x_0+y_0|} \\ &= \int d^3k\, \frac{\pi}{\omega_k} \left| \int dt\, f(t, \vec{k}) e^{-\omega_p t}\right|^2 \geq 0\end{aligned} \tag{33}$$

given that $\omega_k \geq 0$ and f is supported in positive time only. However, this did not work so simply in lean, as we require the integrand to be absolutely convergent to use Fubini theorem for exchanging order of integration. The integrand in the first line of 33 is NOT absolutely integrable - the integrand is finite for $x \neq y$, but it relies on the oscillating property. If we take the absolute value, the factor $1/\sqrt{k^2 + m^2}$ is divergent under the 3D k-integral. To resolve this, we need to make use of the Schwinger parametrization:

$$\begin{aligned}\langle \theta f, K f\rangle &\equiv \int d^4x\, d^4y\, f^*(x) f(y) K(\theta x, y) \\ &= \int_x \int_y f^*(x)\, f(y) \left[\int_0^\infty e^{-sm^2}\, H(s, |\Theta x - y|)\, ds\right] \\ &= \int_0^\infty e^{-sm^2} \left[\int_x \int_y f^*(x)\, f(y)\, H(s, |\Theta x - y|)\right] ds\end{aligned} \tag{34}$$

where

$$H(s, r) \equiv \frac{1}{(4\pi s)^2} \exp\left(-\frac{r^2}{4s}\right) \tag{35}$$

is the heat kernel, and bounded by s^{-2}. The integrand is absolute integrable in this case: the Schwartz functions $f^*(x)$ and $f(y)$ are bounded and integrable in x and y by definition; $H(s, r)$ is bounded, in addition to the exponential decay e^{-sm^2} on s.[2] Therefore, we are allowed to perform the change of integration order in 34.

[2]The exact proof can be found in OS3_foundations, theorem named Schwinger_bilinear_integrable.

We then use the fourier representation of the heat kernel

$$H(s, |z|) = (2\pi)^{-4} \int_k \exp(-ik \cdot z) \exp(-s|k|^2) \tag{36}$$

to rewrite the full integral as

$$\int_{s\geq 0} \int_x \int_y \int_{\vec{k}} \int_{k_0} e^{-sm^2} f^*(x)\, f(y)\, \exp(-ik \cdot (\theta x - y)) \exp(-s|k|^2). \tag{37}$$

The k_0 integral yields

$$\int e^{ik_0(x_0+y_0)} e^{-sk_0^2}\, dk_0 = \sqrt{\frac{\pi}{s}} e^{-(x_0+y_0)^2/4s}. \tag{38}$$

In order to apply Fubini theorem, we will show that the absolute value of the integrand

$$\left| \sqrt{\frac{\pi}{s}} e^{-(x_0+y_0)^2/4s} e^{-s(m^2+\vec{k}^2)} f^*(x)\, f(y)\, e^{i\vec{k}\cdot\vec{x}} \right| \tag{39}$$

is bounded by an integrable function. When taking the norm, $\exp(i\vec{k} \cdot \vec{x})$ has norm 1 and drops out. We can bound the Schwartz functions by

$$|f(x)||f(y)| \leq C \cdot \frac{x_0 y_0}{(1+|\vec{x}|^2)^N (1+|\vec{y}|^2)^N} \tag{40}$$

for some constant C and any $N \in \mathbb{Z}^+$, given that $x_0, y_0 > 0$ as f is only defined on positive time. Then we can use the Gaussian moment formula on the x_0 and y_0 integrals, and use $N = 4$ for the spatial integrals to get

$$\begin{aligned} &\int_x \int_y \left| \sqrt{\frac{\pi}{s}} e^{-(x_0+y_0)^2/4s} e^{-s(m^2+\vec{k}^2)} f^*(x)\, f(y)\, e^{i\vec{k}\cdot\vec{x}} \right| \\ &\leq C \int_x \int_y \sqrt{\frac{\pi}{s}} \frac{x_0 y_0}{(1+|\vec{x}|^2)^4 (1+|\vec{y}|^2)^4} e^{-(x_0+y_0)^2/4s} e^{-s(m^2+\vec{k}^2)} \\ &= \frac{4C'}{3} \sqrt{\frac{\pi}{s}} e^{-s(m^2+\vec{k}^2)} \sqrt{\pi} s^2 \end{aligned} \tag{41}$$

where C' is C multiplied by the finite result from the spatial integrals. This shows that 39 is integrable under the (x, y) double integration. Then we can prove that this is still integrable under the $\vec{k}$ ans s integrals:

$$\int_{s\geq 0} \int_{\vec{k}} e^{-s(m^2+\vec{k}^2)} s^{3/2} = \int_{s\geq 0} e^{-sm^2} s^{3/2} (\pi/s)^{3/2} = \frac{\pi^{3/2}}{m^2} \tag{42}$$

is finite.

After rearranging the order of integration in 37, we have

$$\begin{aligned} &\int_{\vec{k}} \int_x \int_y \int_{s\geq 0} \sqrt{\frac{\pi}{s}} e^{-(x_0+y_0)^2/4s} e^{-s(m^2+\vec{k}^2)} f^*(x)\, f(y)\, e^{i\vec{k}\cdot\vec{x}} \\ &= \int_{\vec{k}} \int_x \int_y \frac{\pi}{\omega} e^{-\omega|x_0+y_0|} f^*(x)\, f(y)\, e^{i\vec{k}\cdot\vec{x}} \end{aligned} \tag{43}$$

where we used $\int_0^\infty s^{-1/2} \exp(-a/s - bs)\, ds = \sqrt{\pi/b} \exp(-2\sqrt{ab})$. This is essentially the exchange of integration order we wanted in 33, and the positivity simply follows as proposed.

B.5. Ergodicity (OS4)

We start by first assuming the polynomial clustering property that for all $f, g \in \mathcal{S}(\mathbb{R}^4)$ there exists a constant $C_{f,g,d,n}$

$$|S[f + T(s)g] - S[f]S[g]| \leq C_{f,g,d,n} (1 + |s|)^{-n}. \tag{44}$$

Instead of proving ergodicity for functions of the form $A(\phi) = \sum_j c_j e^{(\phi, f_j)}$, we can reduce the claim to working with single exponential functions $e^{(\phi,f)}$ by Minkowski's (L^2 triangle) inequality and linearity of the integral.

$$\left\| \frac{1}{t} \int_0^t A(T_s\phi) ds - \mathbb{E}_\mu[A(\phi)] \right\|_{L^2(\mu)} \leq \sum_j |c_j| \left\| \frac{1}{t} \int_0^t e^{(T_s\phi, f_j)} ds - \mathbb{E}_\mu[e^{(\phi, f_j)}] \right\|_{L^2(\mu)}. \tag{45}$$

Next using Fubini, the invariance $C(T_s g, T_s g) = C(g, g)$, and the assumed polynomial clustering property (44),

$$\begin{aligned}
\left\| \mathbb{E}_\mu[e^{(\phi,f)}] - \frac{1}{t} \int_0^t e^{(T_s\phi, f)} ds \right\|_{L^2}^2 &= \left\| \frac{1}{t} \int_0^t (\mathbb{E}_\mu[e^{(\phi,f)}] - e^{(T_s\phi,f)}) ds \right\|_{L^2}^2 \\
&= \frac{1}{t^2} \int_0^t \int_0^t \mathbb{E}_\mu[(\mathbb{E}_\mu[e^{(\phi,f)}] - e^{(T_s\phi,f)})(\overline{\mathbb{E}_\mu[e^{(\phi,f)}] - e^{(T_{s'}\phi,f)}})] ds ds' \\
&\leq \frac{1}{t^2} \int_0^t \int_0^t |\mathbb{E}_\mu[e^{(\phi,f)}]\mathbb{E}_\mu[e^{(\phi,\overline{f})}] - \mathbb{E}_\mu[e^{(T_s\phi,f)+(T_{s'}\phi,\overline{f})}]| ds ds' \\
&\leq \frac{1}{t^2} \int_0^t \int_0^t c(1 + |s - s'|)^{-2} ds ds' \\
&\leq c \sup_{s' \in [0,t]} \frac{1}{t} \int_0^t (1 + |s - s'|)^{-2} ds \\
&\leq c \frac{1}{t} \int_{-\infty}^{\infty} (1 + |s|)^{-2} ds \to 0
\end{aligned} \tag{46}$$

as $t \to \infty$. The justification for Fubini follows from the Fernique bound.

The remaining task for OS4 is now to prove the polynomial clustering version (44). This property follows from the following two key results

1. (Pointwise decay of correlation functions) The covariance function $K(x, y)$ satisfies a bound of the form,

$$|K(x, y)| \leq c|x - y|^{-2} e^{-m|x-y|} \tag{47}$$

   for some constant $c > 0$.

2. (Decay of averaged correlation functions) Let E be a finite-dimensional real normed vector space with $d = \dim E$. Let $f, g \in \mathcal{S}(E; \mathbb{C})$ be Schwartz functions. Let $K : E \to \mathbb{R}$ be measurable and locally integrable. Assume there exist $m > 0$, $C_K > 0$, $R_0 > 0$ such that for all $z \in E$ with $\|z\| \geq R_0$,

$$|K(z)| \leq C_K e^{-m\|z\|}.$$

   Then for every $\alpha > 0$ there exists $c \geq 0$ such that for all $a \in E$,

$$\left\| \int_{x \in E} \int_{y \in E} f(x)\, K(x - y)\, g(y - a)\, dy\, dx \right\| \leq c\,(1 + \|a\|)^{-\alpha}. \tag{48}$$

For the bound on the covariance (47), recall the covariance $K(x, y) := \frac{m}{4\pi^2|x-y|} K_1(m|x - y|)$ where $K_1(z) := \int_0^\infty e^{-z\cosh t} \cosh t dt$ is the modified Bessel function of the first kind, and so it suffices to prove that $|K_1(z)| \leq Cz^{-1}e^{-z}$ for $z \in (0, \infty)$. The proof of this bound on the Bessel function proceeds by splitting the integral into the intervals $(0, 1)$ and $(1, \infty)$. For the $(0, 1)$ integral since $\cosh(t)$ is increasing,

$$\int_0^1 e^{-z\cosh t} \cosh t dt \leq e^{-z} \cosh(1). \tag{49}$$

And for the $(1, \infty)$ integral we use the trivial bound $\frac{1}{2}e^t \leq \cosh t \leq e^t$ and the change of variables $s = e^t$ to bound,

$$\begin{aligned}\int_1^\infty e^{-z\cosh t}\cosh t dt &\leq \int_1^\infty e^{-z\frac{1}{2}e^t} e^t dt \\ &= \int_e^\infty e^{-\frac{1}{2}zs} ds \\ &= \frac{2}{z}e^{-\frac{e}{2}z} \leq \frac{2}{z}e^{-z}.\end{aligned} \tag{50}$$

Instead of providing the full proof of (48), we state and prove a lemma appearing in the proof which highlights the key ideas.

Lemma B.1 (Convolution polynomial decay)**.** *Assume $N > d$. Let $F, G : E \to \mathbb{C}$ be integrable and satisfy*

$$|F(y)| \leq \frac{C_F}{(1 + \|y\|)^N}, \qquad |G(y)| \leq \frac{C_G}{(1 + \|y\|)^N}.$$

Define $h(x) := \int_E F(y)\, G(x - y)\, dy$. Then for all $x \in E$,

$$|h(x)| \leq \frac{C}{(1 + \|x\|)^N}$$

for $C := 2^N \big(C_F \|G\|_{L^1} + C_G \|F\|_{L^1}\big)$.

Proof (domain split at $\|y\| = \|x\|/2$). Fix $x \in E$ and split $E = A \cup B$ with

$$A := \{y : \|y\| \geq \|x\|/2\}, \qquad B := \{y : \|y\| < \|x\|/2\}.$$

Then

$$|h(x)| \leq \int_A |F(y)|\, |G(x - y)|\, dy + \int_B |F(y)|\, |G(x - y)|\, dy.$$

On A we have $1 + \|y\| \geq (1 + \|x\|)/2$, hence $(1 + \|y\|)^{-N} \leq 2^N (1 + \|x\|)^{-N}$ and

$$|F(y)| \leq \frac{C_F}{(1 + \|y\|)^N} \leq \frac{2^N C_F}{(1 + \|x\|)^N}.$$

Therefore

$$\int_A |F(y)|\, |G(x - y)|\, dy \leq \frac{2^N C_F}{(1 + \|x\|)^N} \int_E |G(x - y)|\, dy = \frac{2^N C_F}{(1 + \|x\|)^N} \|G\|_{L^1},$$

using translation invariance in the last equality.

On B, the triangle inequality gives $\|x - y\| \geq \|x\| - \|y\| > \|x\|/2$, hence similarly

$$|G(x - y)| \leq \frac{2^N C_G}{(1 + \|x\|)^N}.$$

Thus

$$\int_B |F(y)|\, |G(x - y)|\, dy \leq \frac{2^N C_G}{(1 + \|x\|)^N} \int_E |F(y)|\, dy = \frac{2^N C_G}{(1 + \|x\|)^N} \|F\|_{L^1}.$$

Adding the two bounds gives the claimed estimate. □

The rest of the proof of (48) involves similar arguments decomposing the domain of integration into 'singular' and 'tail' regions so we assume it for the rest of the argument.

Proof sketch of OS4 (polynomial clustering) for the massive GFF. Fix $\alpha > 0$ and test functions $f, g \in \mathcal{S}(\mathbb{R}^4; \mathbb{C})$. By definition of $T_s\omega$ as the dual action on distributions, and bilinearity of the inner product,

$$\mathbb{E}_\mu\big[e^{\langle \omega, f\rangle + \langle T_s\omega, g\rangle}\big] = \mathbb{E}_\mu\big[e^{\langle \omega, f\rangle + \langle \omega, T_{-s}g\rangle}\big] = \mathbb{E}_\mu\big[e^{\langle \omega, f + T_{-s}g\rangle}\big].$$

Next, for a centered Gaussian field with covariance bilinear form $S_2(\cdot,\cdot)$ (the Schwinger 2-point function), one has the standard identity

$$\mathbb{E}_\mu\big[e^{\langle\omega, f+T_{-s}g\rangle}\big] = \mathbb{E}_\mu\big[e^{\langle\omega,f\rangle}\big]\,\mathbb{E}_\mu\big[e^{\langle\omega,T_{-s}g\rangle}\big]\,\exp\big(S_2(f,T_{-s}g)\big).$$

Moreover, by time-translation invariance of the GFF, $\mathbb{E}_\mu\big[e^{\langle\omega,T_{-s}g\rangle}\big] = \mathbb{E}_\mu\big[e^{\langle\omega,g\rangle}\big]$. So defining the constants, $E_f := \mathbb{E}_\mu\big[e^{\langle\omega,f\rangle}\big], \qquad E_g := \mathbb{E}_\mu\big[e^{\langle\omega,g\rangle}\big]$, we have,

$$\mathbb{E}_\mu\big[e^{\langle\omega,f\rangle+\langle T_s\omega,g\rangle}\big] - E_f E_g = E_f E_g\big(\exp(S_2(f,T_{-s}g)) - 1\big), \tag{51}$$

and using the elementary inequality (valid for all $z \in \mathbb{C}$), $|\exp(z) - 1| \le |z|\, e^{|z|}$,

$$\Big|\mathbb{E}_\mu\big[e^{\langle\omega,f\rangle+\langle T_s\omega,g\rangle}\big] - E_f E_g\Big| \le |E_f|\,|E_g|\,|S_2(f,T_{-s}g)|\,\exp(|S_2(f,T_{-s}g)|). \tag{52}$$

So it suffices to prove a polynomial bound

$$|S_2(f,T_{-s}g)| \le c_{\text{decay}}\,(1+s)^{-\alpha} \qquad (s \ge 0), \tag{53}$$

with some constant $c_{\text{decay}} \ge 0$ depending on (f, g, α), but this is precisely the bound one obtains by plugging in (47) into (48). □

Finally, for completeness the file `OS/NonTrivial` verifies that the correlation functions are nontrivial, as taking $\phi(x) = 0$ would satisfy the axioms.