Dynamic Rationality of Many-Valued Judgment Aggregation: Implications for Multi-Agent LLM Reasoning

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Abstract

Large Language Models and multi-agent systems inherently operate with continuous probabilities rather than binary truth values, effectively engaging in many-valued reasoning. Ensuring logical consistency in such systems, especially when they update their knowledge bases with new information, is a critical challenge. This paper addresses the problem of dynamic rationality—the requirement that collective judgments remain logically consistent regardless of whether aggregation occurs before or after information updates. Modeling the probabilistic outputs of AI agents as judgments in the interval [0,1], we investigate this property under Kleene-Zadeh logic, a formal system suitable for fuzzy reasoning. We characterize the revision process of many-valued judgments and prove that under specific conditions—order-compatible profiles and aggregation rules satisfying linear systematicity and unanimity—dynamic rationality is mathematically guaranteed. The results provide a theoretical foundation for designing robust, logically consistent multi-agent reasoning systems that can reliably integrate new information.

Keywords: Judgment Aggregation, Dynamic Rationality, Multi-Agent Systems, Fuzzy Reasoning, Many-Valued Logic

Introduction

The integration of Logic and Artificial Intelligence has become a pivotal area of research, particularly with the rise of Large Language Models (LLMs). While LLMs demonstrate remarkable capabilities, they face significant challenges in logical consistency and symbolic reasoning.(Friedman 2023) A particularly complex scenario arises in multi-agent LLM systems or ensemble approaches, where multiple models or a single model sampled multiple times generate probabilistic outputs (values in [0, 1]) regarding a set of logical propositions.

Consider a practical problem: a system of LLM agents aggregates their individual "judgments" (confidence scores) to form a collective conclusion. When an external tool provides new, verified information that necessitates a revision of these judgments, how should the system respond? Intuitively, there are two pathways as in Figure 1:

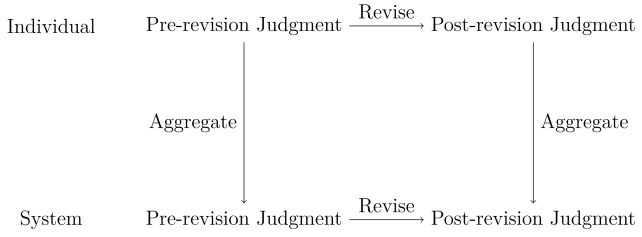

Figure 1: The order of Aggregation and Revision

1. Revise-then-Aggregate: Each agent updates its internal judgment state based on the new context, and the system aggregates these updated judgemnts.
2. Aggregate-then-Revise: The system first aggregates the initial judgments into a collective baseline, which is then revised globally based on the new information.

For a rational AI system, these two pathways must yield the same result. If they differ, the system exhibits dynamic inconsistency, leading to self-contradictory responses depending on when the information was processed. This property is known in social choice theory as "Dynamic Rationality." If the two ways yield the same aggregation result, the group does not need to choose between them. This implies that regardless of when the revision triggered by new information occurs, the group can aggregate individual judgments at any moment without affecting the final outcom. Dietrich and List pioneered the study of dynamic rationality in classical binary judgment aggregation, proving an impossibility theorem and characterizing pathways to resolve this impossibility (Dietrich and List 2024).

In this paper, we bridge the gap between social choice theory and AI logic by exploring dynamic rationality in many-valued judgment aggregation. We focus on Kleene-Zadeh (KZ) logic, which is widely used to model fuzzy reasoning and uncertainty in AI.

Judgment aggregation theory is an interdisciplinary field involving logic, economics, computer science, and mathematics, closely related to social choice and voting theory (Grossi and Pigozzi 2022)(List 2012). Clas-

sical judgment aggregation focuses on how to rationally aggregate individual binary judgments (acceptance/rejection) on propositions in an agenda (a set of logical propositions) into a group judgment. The primary factors evaluating rationality are whether judgment aggregation rules (e.g., majority rule, premise-based rule) can produce consistent results and simultaneously satisfy rational properties such as universality, unanimity, and non-dictatorship. Inspired by the doctrinal paradox (Kornhauser 1992) and the discursive dilemma (Pettit and Rabinowicz 2001), List et al. proposed formal models of judgment aggregation and proved several impossibility theorems, demonstrating that no judgment aggregation rule can simultaneously satisfy certain combinations of rational properties (List and Pettit 2002)(List and Pettit 2004). These results have been strengthened and extended, yielding further significant theorems (Dietrich and List 2008)(Elkind 2018).

In recent years, results from classical judgment aggregation have been extended to the domain of fuzzy logic, a form of non-classical logic (Richard and Maurice 2011)(Uhl 2023). One core difference is that individual judgments on propositions are generalized from binary values $\{0, 1\}$ to arbitrary values in $[0, 1]$, representing the degree of support for a proposition, thus better capturing scenarios involving vague attitudes. Duddy et al. characterized the boundary between possibility and impossibility in many-valued judgment aggregation by defining t-norms and zero divisors (Duddy and Piggins 2013). Herzberg and Esteban abstracted the many-valued judgment aggregation process as an algebraic problem, deriving impossibility conclusions similar to those in the classical case (Herzberg 2013)(Esteban, Palmigiano, and Zhao 2015). These findings prompted scholars to explore the possibility of consistent many-valued judgment aggregation results. Fermüller first proved sufficient conditions for the average rule to generate consistent results in common many-valued logics such as Kleene-Zadeh (KZ) logic and Łukasiewicz (Ł) logic (Fermüller 2023). Later, working with Uhl, he extended this result to arbitrary aggregation rules satisfying certain properties and to Gödel (G) logic, providing a possibility theorem based on median aggregation rules for arbitrary many-valued logics (Uhl and Fermüller 2023). The work of Fermüller and Uhl has significantly inspired this paper.

The outcome of many-valued judgment aggregation is influenced by the choice of the many-valued logic system. Broadly speaking, KZ logic and Ł logic treat negation, disjunction, and conjunction as basic propositional connectives. KZ logic calculates the truth values of disjunction and conjunction using $max(x, y)$ and $min(x, y)$ respectively (Sakharov 2024)(Fermüller 2014), while Ł logic uses $min(1, x + y)$ and $max(0, x + y - 1)$ (Gerla 2012)(Jeřábek 2010). Both use 1-x for negation. G logic aligns with KZ logic for disjunction and conjunction but introduces implication as a connective, with semantics defined as:

$$v(\phi \to \psi) = \begin{cases} 1 & \text{if } v(\phi) < v(\psi) \\ v(\psi) & \text{otherwise} \end{cases}$$

and defines negation as $\neg\phi = \phi \to \bot$ (Preining 2010)(Van Atten and Kennedy 2009). Furthermore, many-valued logic in judgment aggregation applies to two scenarios: (1) individuals make binary judgments, and the group aggregates using many-valued logic; (2) individuals make many-valued judgments, and the group aggregates using many-valued logic. Clearly, (1) is a special case of (2). This paper primarily investigates dynamic rationality under KZ logic in scenario (2); dynamic rationality in other many-valued logics remains a topic for future discussion.

Previous judgment aggregation theory has only studied "static" aggregation processes, focusing on whether aggregation rules remain rational at a given point in time. However, dynamic rationality has been extensively studied in related fields. Belief revision arguably offers the most in-depth theoretical treatment of the dynamic processes of revision and contraction, characterizing how individuals update belief systems in the face of new information, starting from AGM theory (Goodman and Salow 2024)(Van Benthem 2007)(Gärdenfors 2003). Probability aggregation, using non-binary subjective probabilities to represent judgments, has shown several (im)possibility theorems, such as the finding that arithmetic mean aggregation cannot maintain rationality after revision, whereas geometric mean aggregation can (Genest 1984)(Dietrich 2010)(Russell, Hawthorne, and Buchak 2015). Abstract argumentation theory, based on argumentation frameworks constituted by abstract arguments and attack relations, discusses conditions for dynamic rationality in acyclic argumentation frameworks (Chen and Ju 2024)(Doutre and Mailly 2018)(Doutre, Herzig, and Perrussel 2014). Results from these fields offer valuable references for researching dynamic rationality in judgment aggregation, yet current research on this specific topic remains sparse.

The problem of dynamic rationality in judgment aggregation was systematically proposed by Dietrich and List[1]. They provided basic definitions for revision operators in classical binary judgment aggregation and proved that aggregation rules simultaneously satisfying certain rational properties cannot be dynamically rational for specific agendas (Dietrich and List 2024). The "dynamic" aspect in the judgment aggregation process manifests primarily in two ways: first, allowing groups or individuals to revise judgments when facing new information; and second, permitting this revision to occur at any moment, including before or after aggregation. "Rationality" focuses on whether consistent group

[1]Judgment revision has notably been studied in aggregation scenarios involving "peer disagreement," where revision arises from differing opinions among others rather than the acquisition of new information discussed here (Pettit 2006)(List 2011).

judgments can be obtained after aggregating individual judgments, and whether the timing of the revision affects the aggregation result. This paper attempts to extend the findings of dynamic rationality in binary judgment aggregation to the many-valued context. By defining many-valued judgment revision operators, we derive a relatively optimistic possibility theorem, demonstrating that under the premise of achieving consistent aggregation results, KZ logic many-valued judgment aggregation can be dynamically rational.

The structure of the subsequent sections is as follows: Section 2 reviews the basic definitions of many-valued judgment aggregation and possibility theorems for KZ logic. Section 3 defines many-valued judgment revision operators and dynamic rationality. Section 4 investigates the conditions for the existence of dynamic rationality in KZ logic, proving that for specific agendas and order-compatible judgment profiles, aggregation rules possessing linear systematicity and unanimity are dynamically rational. Section 5 summarizes the main results of the paper.

Many-Valued Judgment Aggregation

The truth values in classical judgment aggregation are binary, and existing results are helpful for understanding many-valued judgment aggregation (Grossi and Pigozzi 2022). This chapter discusses how judgment aggregation should be conducted if truth values are extended to the $[0,1]$ interval.

Definition 1 • Propositional Language $\mathcal{L}$ is a set of formulas generated from atomic propositions (propositional variables) in At and propositional connectives in $Op_{\mathcal{L}}$ according to the following rules: (1) $At \subseteq \mathcal{L}$, (2) If $\phi_1, ..., \phi_n \in \mathcal{L}$, then for any n-ary connective $\circ \in Op_{\mathcal{L}}$, $\circ(\phi_1, ..., \phi_n) \in \mathcal{L}$.

- A Many-Valued Logic Λ on the truth value interval $[0,1]$ is determined by the propositional language $\mathcal{L}$ and a truth function $\tilde{\circ} : [0,1]^n \to [0,1]$ corresponding to each $\circ \in Op_{\mathcal{L}}$.
- A Λ-valuation of a set of formulas $X \subseteq \mathcal{L}$ is a truth value assignment to all formulas in X based on the truth functions in Λ. An assignment $f : X \to [0,1]$ is called Λ-consistent if f is a Λ-valuation; otherwise, f is called Λ-inconsistent.

Some many-valued logic systems use $\vee$ (disjunction), $\wedge$ (conjunction), and $\neg$ (negation) as propositional connectives, such as KZ logic and Ł logic, while others introduce $\to$ (implication) or other symbols, such as G logic. The main theorems of this paper relate to KZ logic, defined as follows:

Definition 2 The set of propositional connectives for KZ Logic is $Op_{\mathcal{L}} = \{\vee, \wedge, \neg\}$. The valuation function $v : At \to [0,1]$ is extended to arbitrary KZ formulas using the following rules: $v(\phi \vee \psi) = max(v(\phi), v(\psi))$, $v(\phi \wedge \psi) = min(v(\phi), v(\psi))$, $v(\neg\phi) = 1 - v(\phi)$.

In many-valued logic, the analysis of judgment aggregation problems is similar to the classical case. However, since truth values extend from {0,1} to the $[0,1]$ interval, discussions regarding agenda characteristics, definitions of judgments, and rational properties differ significantly.

Definition 3 • A Judgment Aggregation Problem on a propositional language $\mathcal{L}$ is a tuple $(N, \mathcal{A})$, where $N = \{1, ..., n\}$ is a finite non-empty set of individuals, and $\mathcal{A}$ is a finite subset of $\mathcal{L}$, called the Agenda.

- An agenda $\mathcal{A}$ is called closed if for any formula $\phi \in \mathcal{A}$, all sub-formulas of ϕ are also elements of $\mathcal{A}$. Let $At(\mathcal{A})$ denote the set of all propositional variables in $\mathcal{A}$, and $sub(\phi)$ denote the set of all sub-formulas of ϕ.
- An agenda $\mathcal{A}$ is called internally positive if for any formula $\phi \in \mathcal{A}$, if ϕ contains a negation symbol, the negation symbol appears only at the outermost layer of ϕ.

Unless otherwise specified, we will consider a given many-valued logic Λ, a closed agenda $\mathcal{A}$, and a judgment aggregation problem $(N, \mathcal{A})$.[2] The following are core definitions for many-valued judgment aggregation.

Definition 4 • A Λ-judgment $J : \mathcal{A} \to [0,1]$ is a valuation function on $\mathcal{A}$. Let $\mathbf{J}_\Lambda$ denote the set of all judgments on Λ.

- A Judgment Profile on Λ, $P = (J_1, ..., J_n) \in \mathbf{J}_\Lambda^n$, is an n-tuple of Λ-judgments, where each J_i represents the judgment of individual $i \in N$. The profile P is called Λ-consistent if $\forall i \in N$, J_i is Λ-consistent. Let $\mathbf{P}_\Lambda$ denote the set of all judgment profiles on Λ.
- A many-valued logic Judgment Aggregation Rule is a function $F_\Lambda : \mathbf{P}_\Lambda \to [0,1]^{\mathcal{A}}$. F_Λ maps each judgment profile $P \in \mathbf{J}_\Lambda^n$ to a group judgment $F(P)$. If $F(P)$ is a Λ-valuation, then the group judgment is called Λ-consistent (or rational).

Common aggregation rules in classical judgment aggregation (Lang and Slavkovik 2013)(List 2012) can be viewed as special cases of many-valued aggregation. The Majority Rule and Average Rule in many-valued aggregation can be defined as:

Majority Rule (Maj)

$$F_{maj}(\phi) = \begin{cases} 1 & \text{if } \sum_{i \in N} J_i(\phi) > |N|/2 \\ 0 & \text{otherwise} \end{cases}$$

Average Rule (Avg) $\quad F_{avg}(\phi) = \sum_{i \in N} J_i(\phi)/|N|$

The "discursive dilemma" also exists in many-valued judgment aggregation and is heavily influenced by different aggregation rules and logic systems. The following example illustrates this phenomenon.

Example 1 Consider a judgment aggregation problem $(N, \mathcal{A})$ where $N = \{1, 2, 3\}$ and $\mathcal{A} = \{p, q, p \wedge q, p \vee q\}$. If individuals maintain their judgments on atomic propositions p and q, the group will obtain different consistent judgment profiles under KZ logic and Ł logic. As

[2] Considering a closed agenda does not result in a loss of generality, as we can simply ignore valuations of irrelevant propositions if necessary.

previously mentioned, KZ logic calculates disjunction and conjunction values using $max(x, y)$ and $min(x, y)$, while Ł logic uses $min(1, x + y)$ and $max(0, x + y-1)$. The resulting truth table is shown below.

			KZ Logic		Ł Logic	
	p	q	$p \wedge q$	$p \vee q$	$p \wedge q$	$p \vee q$
1	0.1	0.7	0.1	0.7	0	0.8
2	0.2	0.1	0.1	0.2	0	0.3
3	0.3	0.4	0.3	0.4	0	0.7
Maj	0	0	0	0	0	1
Avg	0.2	0.4	1/6	13/30	0	0.6

Table 1: Discursive Dilemma in Many-Valued Judgment Aggregation

In this example, we observe that even if all individuals provide consistent judgments under different logics, aggregation produces different results. The majority aggregation result is consistent under KZ logic but inconsistent under Ł logic because $F_{maj}(p \vee q) \neq 0$. Conversely, the average aggregation result is inconsistent under KZ logic because $F_{avg}(p \wedge q) \neq 0.2$ and $F_{avg}(p \vee q) \neq 0.4$, but it is consistent under Ł logic.

To better discuss how many-valued aggregation rules affect aggregation results, we need to consider several rational properties.

Definition 5 Given a judgment aggregation problem $(N, \mathcal{A})$, a set of judgment profiles $\mathbf{P}$, and an aggregation rule $F : \mathbf{P} \to [0, 1]^{\mathcal{A}}$, some rational properties of F are defined as follows:

Unanimity $\forall \phi \in \mathcal{A}\ \forall P \in \mathbf{P}$, if $\forall\ i \in N$, $J_i(\phi) = m$, then $F(P)(\phi) = m$, where $m \in [0, 1]$.

Systematicity $\exists f : [0, 1]^n \to [0, 1]$ such that $\forall P = (J_1, ..., J_n) \in \mathbf{P}\ \forall \phi \in \mathcal{A}$, $F(P)(\phi) = f(J_1(\phi), ..., J_n(\phi))$.

Linear Systematicity[3] $\exists f : [0, 1]^n \to [0, 1]$ satisfying $\forall (x_1, ..., x_n)$, $f(1 - x_1, ..., 1 - x_n) = 1 - f(x_1, ..., x_n)$, such that $\forall P = (J_1, ..., J_n) \in \mathbf{P}\ \forall \phi \in \mathcal{A}$, $F(P)(\phi) = f(J_1(\phi), ..., J_n(\phi))$.

Collective Rationality $\forall P \in \mathbf{P}$, $F(P)$ is a Λ-valuation.

Non-dictatorship There exists no $i \in N$ such that $\forall P \in \mathbf{P}$, $F(P) = J_i$.

Herzberg demonstrated that the inconsistency in Example 1 is pervasive, proving that for non-trivial agendas, any many-valued aggregation rule in KZ logic and Ł logic that simultaneously satisfies collective rationality, unanimity, and systematicity is dictatorial (Herzberg 2013). Esteban extended this impossibility theorem to arbitrary many-valued logics using abstract algebraic methods (Esteban, Palmigiano, and Zhao 2015). However, Fermüller identified specific agendas and judgment profiles that allow the average aggregation rule to produce KZ-consistent or Ł-consistent aggregation results (Fermüller 2023). Later, with Uhl, he extended this possibility theorem to arbitrary aggregation rules satisfying certain properties and proposed median aggregation rules capable of producing consistent aggregation results in arbitrary many-valued logics (Uhl and Fermüller 2023). Uhl's possibility theorem regarding KZ logic involves the following type of judgment profile, which forms the basis of our study on KZ logic dynamic rationality.

Definition 6 A judgment profile $P = (J_1, ..., J_n)$ is called order-compatible if there exists a sequence $\langle p_1, ..., p_m \rangle$ such that $\forall i \in N : J_i(p_1) \leq ... \leq J_i(p_m)$, where $p_1, ..., p_m$ are all atomic propositions appearing in the agenda $\mathcal{A}$.

Theorem 1 (Possibility Theorem for KZ Logic) Let $(N, \mathcal{A})$ be a judgment aggregation problem, where $\mathcal{A}$ is closed and internally positive. Then, every linear systematic aggregation function F yields a KZ-consistent judgment $F(P)$ for every order-compatible KZ-judgment profile $P \in \mathbf{P}$.(Uhl and Fermüller 2023)

Judgment Revision and Dynamic Rationality

The discussion above regarding the consistency of many-valued judgment aggregation essentially investigates how to achieve static rationality at the group level—that is, consistency of the aggregation result at a specific point in time. This paper focuses on dynamic rationality. The primary meaning of "dynamic" is that when an individual or group receives new information in the form of a proposition, they should incorporate this learned information into their judgments. We use judgment revision operators to describe this process.

Given a many-valued logic Λ and a many-valued judgment aggregation problem $(N, \mathcal{A})$.

Definition 7 A Judgment Revision Operator is a function $|: \mathbf{J}_\Lambda \times At(\mathcal{A}) \to [0, 1]^{\mathcal{A}}$. It maps each ordered pair consisting of a judgment J and an atomic proposition p from the agenda $\mathcal{A}$ to a new valuation over all propositions in the agenda, denoted as $J \mid p$.

Definition 8 Given a judgment revision operator $|$, let $(J, p) \in \mathbf{J}_\Lambda \times At(\mathcal{A})$.

- $|$ is successful if $J \mid p\ (p) = 1$.
- $|$ is conservative if $J(p) = 1$ implies $J \mid p = J$.
- $|$ is independent of irrelevant formulas if $\forall \phi \in \mathcal{A}$, $p \notin sub(\phi)$ implies $J(\phi) = J \mid p\ (\phi)$.
- $|$ is regular if it is successful, conservative, and independent of irrelevant formulas.
- $|$ is rationality-preserving if J being Λ-consistent implies $J \mid p$ is Λ-consistent.

In binary judgment aggregation, Dietrich defined the revision operator as $2^{\mathcal{A}} \times \mathcal{A} \to 2^{\mathcal{A}}$, allowing revision of arbitrary judgment sets by arbitrary formulas (Dietrich and List 2024). However, the situation in many-valued aggregation is significantly different. We desire that judgment revision based on new information only considers judgments that are themselves consistent. While this might impose high demands on group judgment in classical binary aggregation, Theorem 1 makes this achievable in many-valued aggregation. Furthermore,

we restrict revision to atomic propositions rather than arbitrary formulas because truth values in many-valued aggregation are much more complex. If we wish for a consistent judgment to remain consistent after revision, we must limit the learnable new information to atomic forms; otherwise, immense uncertainty arises, as illustrated below:

Example 2 Suppose a KZ-consistent judgment J is:

$$J(p) = 0.4, J(q) = 0.7, J(p \vee q) = 0.7, J(p \wedge q) = 0.4$$

If the new information learned is $\phi = p \vee q$, then based on a regular judgment revision operator $|$, we have $J \mid \phi\ (p \vee q) = 1$. Now, to obtain a KZ-consistent judgment after revision, we must have $J \mid \phi\ (p) = 1$ or $J \mid \phi\ (q) = 1$. However, we lack sufficient reason to make a determinate choice between these two scenarios, and either choice would directly affect the truth value of $J \mid \phi\ (p \wedge q)$. But if the learned new information is an atomic proposition p or q, we can obtain a unique consistent judgment after revision.

It would be irresponsible to ignore this uncertainty solely to generalize the form of learned new information, as it would lose practical applicability. Therefore, we restrict the new information used for revision to the form of atomic propositions.

Properties that revision behavior should satisfy have been deeply studied in distance-based belief revision theory. It requires that when a belief is revised based on new information, the revised belief should remain as similar as possible to the original belief, provided the new information is incorporated and inconsistency is avoided (Katsuno and Mendelzon 1991)(Lehmann, Magidor, and Schlechta 2001). The definition of regularity in Definition 8 serves this purpose. In judgment revision: success implies the revised judgment fully supports the learned information p (accepting the learned information); conservativeness implies that if the original judgment already fully supports the information, the revision produces no change (no new information, no change); independence of irrelevant formulas implies that revision does not affect judgments on propositions unrelated to the new information. These three properties form our basic requirements for judgment revision operators. The rationality-preserving property is also an intuitive expectation for revision behavior: if a consistent judgment becomes inconsistent after revision, the revision lacks practical value. How to achieve the rationality-preserving property is a key factor distinguishing different judgment revision operators.

The judgment revision operator characterizes how new information influences judgments. Based on this, we ideally hope that regardless of whether the revision occurs before or after aggregation, the same result is obtained. We define this dynamic rationality requirement as follows.

Definition 9 Given a judgment aggregation rule F and a judgment revision operator $|$, F is called dynamically rational if for any judgment profile $(J_1, ..., J_n) \in \mathbf{P}_\Lambda$ and learned new information $p \in At(\mathcal{A})$, it holds that $F(J_1 \mid p, ..., J_n \mid p) = F(J_1, ..., J_n) \mid p$.

However, without any restrictions, dynamic rationality is difficult to achieve.

Example 3 Consider the judgment profile under KZ logic and the average aggregation rule from Example 1. Assume the new information learned is p. The left side of the table below presents the situation without revision. Let the judgment revision operator be regular and rationality-preserving. If aggregation is performed before revision, the result is $(1, 0.4, 1/6)$[4], whereas revision before aggregation yields the situation on the right side of the table, resulting in $(1, 0.4, 0.4)$. Clearly, $F(J_1 \mid p, ..., J_n \mid p) \neq F(J_1, ..., J_n) \mid p$.[5]

	Pre-revision			Post-revision		
	p	q	$p \wedge q$	p	q	$p \wedge q$
1	0.1	0.7	0.1	1	0.7	0.7
2	0.2	0.1	0.1	1	0.1	0.1
3	0.3	0.4	0.3	1	0.4	0.4
Avg	0.2	0.4	1/6	1	0.4	0.4

Table 2: Example of Revision and Aggregation in KZ Logic

Dietrich proved an impossibility theorem in binary judgment aggregation, showing that for non-simple agendas, no judgment aggregation rule satisfying certain rational properties can be dynamically rational based on a regular and rationality-preserving revision operator (Dietrich and List 2024). What about the case in many-valued judgment aggregation? Observing the "aggregate then revise" case in Example 3, the inconsistent result produced by prior aggregation prevents the rationality-preserving property of the revision operator from functioning. This is a significant cause of non-dynamic rationality. However, Theorem 1 provides sufficient conditions for generating KZ-consistent many-valued aggregation results, making the realization of dynamic rationality possible.

Dynamic Rationality of KZ Logic Judgment Aggregation

The possibility theorem for KZ logic relies on order-compatible judgment profiles, meaning all individuals rank their support for atomic propositions in the agenda identically. This property is not destroyed by regular revision operators.

Lemma 1 Given a judgment profile $P = (J_1, ..., J_n)$ and a regular judgment revision operator $|$. For any atomic

[4] Although this aggregation result is KZ-inconsistent, the original aggregation result was also KZ-inconsistent, so this does not conflict with the rationality-preserving property.

[5] This mirrors the phenomenon where LLMs produce inconsistent answers when the context is presented in a different order.

proposition $p \in At(\mathcal{A})$, if P is order-compatible, then $(J_1 \mid p, ..., J_n \mid p)$ is also order-compatible.

Proof 1 Since P is order-compatible, there exists a sequence $\langle p_1, ..., p_m \rangle$ such that for any $i \in N$, $J_i(p_1) \leq ... \leq J_i(p_m)$, where $p_1, ..., p_m$ are all atomic propositions appearing in $\mathcal{A}$.

Let $j \in \{1, ..., m\}$ be arbitrary. We have $J_i(p_1) \leq ... \leq J_i(p_j) \leq ... \leq J_i(p_m)$.

Since the judgment revision operator $\mid$ is successful, $J_i \mid p_j\ (p_j) = 1$.

- Case 1: If $j = m$, then the sequence $\langle p_1, ..., p_m \rangle$ ensures that $(J_1 \mid p_j, ..., J_n \mid p_j)$ is order-compatible.
- Case 2: If $j \neq m$, since $\mid$ is independent of irrelevant formulas and $\forall k \neq j : p_j \notin sub(p_k)$, we have $\forall k \neq j : J_i(p_k) = J_i \mid p_j\ (p_k)$, where $k \in \{1, ..., m\}$.
  Therefore, $J_i \mid p_j\ (p_1) \leq ... \leq J_i \mid p_j\ (p_{j-1}) \leq J_i \mid p_j\ (p_{j+1}) \leq ... \leq J_i \mid p_j\ (p_m) \leq J_i \mid p_j\ (p_j)$.
  Thus, the sequence $\langle p_1, ..., p_{j-1}, p_{j+1}, ..., p_m, p_j \rangle$ ensures that $(J_1 \mid p_j, ..., J_n \mid p_j)$ is order-compatible.

In summary, by the arbitrariness of i and j, it is proven that $(J_1 \mid p_j, ..., J_n \mid p_j)$ is order-compatible.

Simply put, if a sequence of atomic propositions makes a judgment profile order-compatible, then based on a regular judgment revision operator, after learning new information p, one simply needs to move p to the rightmost end of the sequence. The resulting new sequence will render the revised judgment profile order-compatible as well. Combining this with Theorem 1, we obtain the following corollary.

Corollary 1 Let $(N, \mathcal{A})$ be a judgment aggregation problem where $\mathcal{A}$ is closed and internally positive. Given a regular and rationality-preserving judgment revision operator $\mid$, for every order-compatible KZ-consistent judgment profile $P = (J_1, ..., J_n) \in \mathbf{P}$ and any information $p \in At(\mathcal{A})$, any linearly systematic judgment aggregation rule F will produce a KZ-consistent judgment aggregation result $F((J_1 \mid p, ..., J_n \mid p))$ after revision.

This corollary provides crucial support for our investigation into the dynamic rationality of KZ logic many-valued judgment aggregation. It implies that based on a regular and rationality-preserving revision operator $\mid$, for the same judgment profile, the result obtained by "revise then aggregate" $F((J_1 \mid p, ..., J_n \mid p))$ and the result obtained by "aggregate then revise" $F(J_1, ..., J_n) \mid p$ are both KZ-consistent, avoiding the inconsistency observed in Example 3. Based on this, we prove the sufficient conditions for achieving dynamic rationality in KZ logic many-valued judgment aggregation.

Theorem 2 Let $(N, \mathcal{A})$ be a judgment aggregation problem where $\mathcal{A}$ is closed and internally positive. Given a regular and rationality-preserving judgment revision operator $\mid$, for every order-compatible KZ-consistent judgment profile $P = (J_1, ..., J_n) \in \mathbf{P}$, any judgment aggregation rule F satisfying linear systematicity and unanimity is dynamically rational.

Proof 2 Let $(N, \mathcal{A})$ and $\mid$ satisfy the properties described above. Let the judgment aggregation rule F satisfy linear systematicity and unanimity. Take any order-compatible KZ-consistent judgment profile $P = (J_1, ..., J_n) \in \mathbf{P}$ and any information $p \in At(\mathcal{A})$. From Theorem 1 and Corollary 1, we know that $F(J_1, ..., J_n)$ and $F(J_1 \mid p, ..., J_n \mid p)$ are both KZ-consistent. Since $\mid$ is rationality-preserving, $F(J_1, ..., J_n) \mid p$ is also KZ-consistent. We need to prove $F(J_1 \mid p, ..., J_n \mid p) = F(J_1, ..., J_n) \mid p$.

1. If $\forall i \in N : J_i(p) = 1$, then by conservativeness, $J_i \mid p = J_i$, so $F(J_1 \mid p, ..., J_n \mid p) = F(J_1, ..., J_n)$. By unanimity, $F(J_1, ..., J_n)(p) = 1$. Thus, by conservativeness again, $F(J_1, ..., J_n) \mid p = F(J_1, ..., J_n)$. Therefore, $F(J_1 \mid p, ..., J_n \mid p) = F(J_1, ..., J_n) \mid p$.
2. If $\exists j \in N : J_j(p) \neq 1$, since $\mathcal{A}$ is closed, we proceed by induction on the size of $\mathcal{A}$.
   (a) First, we prove that F is dynamically rational on $At(\mathcal{A})$.
   - For $p \in At(\mathcal{A})$, by success, $\forall i \in N$, $J_i \mid p\ (p) = 1$, and $F(J_1, ..., J_n) \mid p\ (p) = 1$. By unanimity, $F(J_1 \mid p, ..., J_n \mid p)(p) = 1$. Thus, $F(J_1, ..., J_n) \mid p\ (p) = F(J_1 \mid p, ..., J_n \mid p)(p)$.
   - For any $q \in At(\mathcal{A})$ such that $q \neq p$, by systematicity, $\exists f : [0,1]^n \to [0,1]$ such that
     $$F(J_i \mid p, ..., J_i \mid p)(q) = f(J_1 \mid p\ (q), ..., J_n \mid p\ (q)) \quad (1)$$
     Since $q \notin sub(p)$, by independence of irrelevant formulas, $\forall i \in N, J_i \mid p\ (q) = J_i(q)$, and $F(J_1, ..., J_n) \mid p\ (q) = F(J_1, ..., J_n)(q)$.
     Therefore, equation (1) becomes $= f(J_1(q), ..., J_n(q)) = F(J_1, ..., J_n)(q)$.

   Consequently, $\forall q \in At(\mathcal{A}) : F(J_1 \mid p, ..., J_n \mid p)(q) = F(J_1, ..., J_n) \mid p\ (q)$, so F is dynamically rational on $At(\mathcal{A})$.

   (b) Take any closed subset X of $\mathcal{A}$ such that $At(\mathcal{A}) \subset X \subseteq \mathcal{A}$. Assume F is dynamically rational on $Y = X - \{\phi\}$, where $\phi \in \mathcal{A}$. We need to prove that F is also dynamically rational on X.

   Since $\mathcal{A}$ is internally positive, we proceed by induction on the structure of ϕ.
   - If $\phi = \psi_1 \vee \psi_2$: Since X is closed, $\psi_1, \psi_2 \in Y$. By the inductive hypothesis that F is dynamically rational on Y, we have $F(J_1 \mid p, ..., J_n \mid p)(\psi_1) = F(J_1, ...J_n) \mid p\ (\psi_1)$ and $F(J_1 \mid p, ..., J_n \mid p)(\psi_2) = F(J_1, ...J_n) \mid p\ (\psi_2)$.
     In KZ logic, $F(J_1 \mid p, ..., J_n \mid p)(\phi) = max\{F(J_1 \mid p, ..., J_n \mid p)(\psi_1), F(J_1 \mid p, ..., J_n \mid p)(\psi_2)\} = max\{F(J_1, ..., J_n) \mid p(\psi_1), ..., F(J_1, ..., J_n) \mid p(\psi_2)\} = F(J_1, ..., J_n) \mid p(\phi)$.
   - If $\phi = \psi_1 \wedge \psi_2$: Similarly, in KZ logic, $F(J_1 \mid p, ..., J_n \mid p)(\phi) = min\{F(J_1 \mid p, ..., J_n \mid p)(\psi_1), F(J_1 \mid p, ..., J_n \mid p)(\psi_2)\} = min\{F(J_1, ..., J_n) \mid p(\psi_1), ..., F(J_1, ..., J_n) \mid p(\psi_2)\} = F(J_1, ..., J_n) \mid p(\phi)$.

- If $\phi = \neg\psi$: Similarly, in KZ logic, $F(J_1 \mid p, ..., J_n \mid p)(\phi) = 1 - F(J_1 \mid p, ..., J_n \mid p)(\psi) = 1 - F(J_1, ..., J_n) \mid p(\psi) = F(J_1, ..., J_n) \mid (\phi)$.

  Through these three steps, we prove that F is dynamically rational on X.

In conclusion, by induction, F is dynamically rational on the agenda $\mathcal{A}$.

Conclusion

As LLMs increasingly interact with external tools and dynamic knowledge bases, guaranteeing the logical consistency of their reasoning processes becomes paramount. This paper extends the theoretical framework of dynamic rationality from classical binary logic to the many-valued domain, offering a rigorous mathematical foundation for consistent probabilistic reasoning in multi-agent systems.

We defined judgment revision operators and dynamic rationality in the context of many-valued judgment aggregation. Based on the possibility theorem for KZ logic, we established sufficient conditions for achieving dynamic rationality in KZ-judgment aggregation. We proved that for closed and internally positive agendas, if the judgment revision operator is regular and rationality-preserving, then for every order-compatible KZ-consistent judgment profile, any judgment aggregation rule satisfying linear systematicity and unanimity is dynamically rational. Although this conclusion involves several preconditions, each has a reasonable basis, making this work a valuable reference for researching dynamic rationality in many-valued judgment aggregation.

However, several limitations remain, and many related issues warrant deeper discussion. First, this paper only discussed dynamic rationality under KZ logic judgment aggregation. Uhl also provided a possibility theorem for G logic and, by defining median aggregation rules, offered a possibility theorem for arbitrary many-valued logics (Uhl and Fermüller 2023). Extending the work of this paper to other or even arbitrary many-valued logic systems using these results is an important topic. Second, despite the relatively optimistic possibility conclusion, we did not provide a strong response to the failure of dynamic rationality observed in Example 3. Dietrich provided a general impossibility theorem in the study of binary judgment aggregation dynamic rationality (Dietrich and List 2024). Can a similar conclusion be drawn in many-valued judgment aggregation? A breakthrough on this question would significantly broaden the scope of research. Third, the understanding of "dynamic" in this paper is limited to accepting new information and does not involve rejecting it. Chen Weiwei et al. discussed "contraction operators" in abstract argumentation theory, allowing individuals or groups to exclude unacceptable propositions (Chen and Ju 2024). This offers a new perspective for dynamic rationality in judgment aggregation.

For implications for AI and Logic, our theorem suggests that "order-compatibility" (agents ranking atomic propositions similarly) is a key feature for consistent ensembles. This could inform how we select or fine-tune agents for collaborative reasoning tasks. And the failure cases of dynamic rationality (as seen in Example 3) can serve as benchmarks (or "adversarial examples") to test whether current LLM aggregation methods are logically robust.

Future work should extend these results to other non-classical logics pertinent to AI and explore "contraction" operators—modeling how LLMs should discard hallucinatory information. By grounding AI aggregation mechanisms in formal logic, we move closer to systems that are not only statistically powerful but also logically sound.

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