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Joint Consistency of a Multidimensional Nonparametric Continuous Response Model

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Abstract

Ramsay, 1991 aimed to estimate the ICC using kernel nonparametric regression, a functional data analysis technique. This approach allows greater flexibility in ICC shapes, requires fewer assumptions, and avoids estimation issues from correlated parameters. typically introduced in traditional parametric models. Douglas, 1997 proved joint consistency in the one-dimensional case. Luzardo et al., 2026 extended this result to the multidimensional setting, establishing conditions for joint consistency in the dichotomous case. Separately, Luzardo and Rodríguez, 2015 built on the multidimensional Ramsay model to develop an isotonic estimator of the ICC, ensuring monotonicity. Here, we present a multidimensional continuous-response nonparametric model. The ICC is defined as the conditional expectation of the item response given the multidimensional latent ability vector, assuming the response is an absolutely continuous random variable. We establish the conditions required (kernel functions, bandwidth selection, sample size, test length) to ensure joint consistency in trait and ICC estimation. To analyze joint consistency, we employ a triangular array of ICCs and ability estimates, considering the number of examinees as a function of the number of items; in the asymptotic setting, the number of examinees increases with the number of items. This framework provides a unified theoretical treatment for multidimensional continuous-response nonparametric IRT models and clarifies the asymptotic conditions for jointly consistent.

Keywords: continuous response models, nonparametric IRT, estimation of item characteristic curves

1. Introduction

Item Characteristic Curves (ICCs) play a central role in Item Response Theory (IRT), as they describe the relationship between a latent trait and the probability of a given item response. Ramsay, 1991 proposed estimating ICC as smooth functions of a unidimensional latent trait without imposing specific parametric forms, such as the logistic family. The proposed framework leverages the inherent rank-order information in test data, explicitly acknowledging the fundamental identifiability constraints of IRT, wherein only the ordinal ordering of examinees is recoverable and the metric of the latent ability scale remains arbitrary. Douglas, 1997 derives conditions to test the joint consistency of the estimators for both the latent trait and the ICCs when both, test length and the number of subjects increase simultaneously in the unidimensional case. The analysis is conducted under a set of general, nonparametric assumptions designed to be minimally restrictive while ensuring psychometric plausibility. Key conditions include the postulation of a unidimensional latent trait with local independence (A1), the adoption of a uniform (0,1) ability scale for theoretical convenience (A2), and the critical requirement that the test characteristic curve is strictly increasing and differentiable, guaranteeing

the test provides discrimination across the entire ability range (A3). Further regularity is imposed by assuming bounded score distributions (A4), a polynomial growth rate linking sample size to test length (A5), and smoothness conditions on the ICCs, specifically the existence and boundedness of their first derivatives on compact intervals within the ability continuum (A6, A7). Complementing these model assumptions are technical conditions on the kernel smoother, requiring a bounded, Lipschitz-continuous kernel with compact support (C1-C3) and a bandwidth sequence (h_n) that decays at a controlled rate, specified as $O(n^{-\alpha})$ for $0 < \alpha < 1/2$ (C4). This bandwidth condition is meticulously chosen to balance the error from smoothing with the error inherent in using estimated ability ranks.

To address limitations of parametric models, Luzardo and Rodríguez, 2015 propose a two-stage estimator. The first stage employs Ramsay, 1991 nonparametric kernel regression to obtain a preliminary ICC estimate. In the second stage, this estimate is used to compute the density function of the inverse ICC. Integrating this density yields an isotonic estimator of the inverse function, which is then symmetrized to produce the final monotone ICC estimator. The framework is developed for the unidimensional case and subsequently generalized to multidimensional latent traits, operating coordinate-wise. The method is highlighted for its flexibility, non-iterative nature, and ability to provide smooth estimates without restrictive parametric assumptions. In subsequent work, Luzardo et al., 2026 extends this approach to a fully multidimensional setting using local polynomial regression, encompassing both Nadaraya–Watson and local linear estimators, and establishes conditions for joint consistency of trait and ICC estimation. These conditions are formulated in terms of asymptotically Lipschitz distribution functions, and the analysis further provides exponential concentration bounds for the estimation error

A multidimensional nonparametric continuous-response approach extends classical nonparametric IRT (Mokken, 1971; Sijtsma and Molenaar, 2002) and kernel smoothing methods for item response estimation (Ramsay, 1991) to the setting of continuous responses and multidimensional latent traits (Samejima, 1973; Samejima, 1974, Reckase, 2009). Here, the item response function is defined as the conditional expectation of the continuous response given the latent ability vector, yielding a flexible model under weaker assumptions than traditional parametric formulations.

In this work we extend previous results to a multidimensional nonparametric continuous-response IRT framework. We employ nonparametric regression with multidimensional covariates, relying on kernel-based local polynomial estimators for their analytic tractability, and estimate latent traits via empirical-ordering procedures based on, for example, total score; a step that introduces specific technical challenges in the multidimensional case. We present the model, specify estimators for traits and ICCs, and derive three classes of asymptotic results establishing joint consistency and exponential concentration bounds.

1.1 Presentation of the Multidimensional Model

As shown in Luzardo et al., 2026, the multidimensional model first estimates the latent traits, and based on this, the ICCs are estimated, relying on multivariate regression using kernels. In this regard, we consider the trait $\Theta = (\Theta_1, ..., \Theta_d)$ as a random vector in $\mathbb{R}^d$

To estimate each component of the vector representing a specific trait, we will use statistics to rank the subjects and thus take the corresponding empirical quantile as the estimate of that trait component for each subject. In practice, these statistics may be thought of as dimension-specific scores: for example, the sum, average, or weighted average of the responses to items primarily related to a given latent trait. Ranking subjects according to these scores provides an empirical ordering for each dimension, from which the corresponding empirical quantiles are used as trait estimates.

Let $(\Omega, \mathcal{A}, P)$ be a probability space and consider, without loss of generality, that the latent trait Θ has uniform marginal distributions $U[0, 1]$. It is clear that anything we deduce under this assumption will work for any type of marginal distribution. It is advantageous to remain independent of the

possible distribution of the trait, as this is typically unknown.

In the unidimensional case, let us denote the trait as τ and suppose it has distribution F. Since $F(\tau)$ is uniformly distributed on $[0, 1]$, we can estimate, instead of τ, a monotonic transformation of it, $\theta = F(\tau)$. Thus, we can assume that the trait has a uniform distribution and then estimate it via the empirical proportion. If we know the distribution F of the trait, we obtain the original scale through $\tau = F^{-1}(\theta)$, which has distribution F. In the multidimensional case, we resort to copulas.

Let $(\Theta_1, \ldots, \Theta_d)$, suppose that their marginals are known and continuous. The vector $(U_1, U_2, \ldots, U_d) = (F_1(\Theta_1), \ldots, F_d(\Theta_d))$ has uniform marginals. So, the copula of $(\Theta_1, \ldots, \Theta_d)$ is defined as

$$C(u_1, u_2, \ldots, u_d) = \mathbb{P}[U_1 \leq u_1, U_2 \leq u_2, \ldots, U_d \leq u_d] = P(\Theta_1 \leq F_1^{-1}(u_1), \ldots, \Theta_d \leq F_d^{-1}(u_d))$$

This function contains all the dependency information of the trait vector. Applying Sklar's theorem, we can reconstruct the joint distribution function of the traits from the marginals. If we denote by $H(\theta_1, \ldots, \theta_d) = P[\Theta_1 \leq \theta_1, \ldots, \Theta_d \leq \theta_d]$ the joint distribution of the trait, then $H(\theta_1, \ldots, \theta_d) = C\,(F_1(\theta_1), \ldots, F_d(\theta_d))$. Henceforth, we will work with uniform marginals.

A statistic (denoted as $g_{n,l}$) will be defined for each trait. In each component, the scores obtained from the statistic $g_{n,l}$ will be ranked and then the empirical quantile will be taken to obtain the estimated trait of the subject. Consider a sequence of vector-valued, Borel measurable functions, $\mathbf{g_n}$, from $\mathbb{R}^n$ to $[0, 1]^d$. Let $g_{n,l}$ be the l-th component of $\mathbf{g_n}$, so for $x \in \mathbb{R}^n$, $\mathbf{g_n}(x) = (g_{n,1}(x), ..., g_{n,d}(x))$.

Consider a test of n items answered by N subjects and the random variables $X_{i,k}$, $i = 1, ..., n$, $k = 1, ...N$, indicating the response of the k-th subject to the i-th item. These variables depend on d latent traits (Θ_l) and let $\mathbf{X} = (X_1, .., X_n)$ be the random vector indicating the responses to the n items. Specifically, $\mathbf{X_k}$ will denote the response vector of subject k.

Assume that the functions $g_{n,l}(X)$ are independent of $\Theta_1, ..., \Theta_{l-1}, \Theta_{l+1}, .., \Theta_n$ for all $1 \leq l \leq n$. Define the sequence of functions $G_{g_n} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ such that for each $x = (x_1, ..., x_d)$ we have:

$$G_{g_n}(x) = (P(g_{n,1}(X) \leq x_1), ..., P(g_{n,d}(X) \leq x_d)) \tag{1}$$

Let the empirical distributions be: $\widehat{F}_{N,l}(x) = \frac{\#\{g_{n,l}(X) \leq x\}}{N} = \frac{\sum_{i=1}^{N} \chi_{\{g_{n,l}(X_i) \leq x\}}}{N}$ for $l = 1, .., d$ and define the function:

$$\widehat{G}_N(x_1, ..., x_d) = (\widehat{F}_N, 1(x_1), ..., \widehat{F}_{N,d}(x_d)) \tag{2}$$

Based on the above, to estimate the l-th component of the trait, the function $g_{n,l}$ is used to rank the subjects. Then we take the empirical distribution function, that is,

$$\widehat{\Theta}_n = \widehat{G}_N(g_n(X)) \tag{3}$$

Note that this estimation method does not depend on the type of response obtained, which can be dichotomous, polytomous, or continuous. If we know the marginals, we can obtain estimates on the original scale by applying the inverse of each of these distributions to the trait estimated via the uniform distribution.

2. ICC estimation

Since $P(\theta_i) = E(X|\Theta = \theta_i)$, we can consider using conditional expectation estimators to estimate the ICCs. For example:

$$E(\widehat{X \mid \Theta = \theta}) = \frac{\sum_{i=1}^{N} X_i K_H(\theta_i - \theta)}{\sum_{i=1}^{N} K_H(\theta_i - \theta)} \tag{4}$$

where K is a d-dimensional kernel and H is a d-dimensional symmetric, non-singular matrix. Note that the numerator is the convolution of K with the data and the denominator is the convolution of K with unity.

This estimator is a locally weighted average, with weights given by $w_i(x) = \frac{K_H(\theta_i - \theta)}{\sum_{i=1}^{N} K_H(\theta_i - \theta)}$. That is, it computes a weighted average of the responses of those whose θ_i lies in a ball or cube around θ. It is clear that in the general case, the values of $\theta_1 ... \theta_N$ are unknown. However, if we have a sequence of estimators $\widehat{\theta}_{n,i}$ that converges to θ_i for all $i = 1, .., N$ in probability, almost surely, or completely, it is natural to think of replacing θ_i with $\widehat{\theta}_i$. Luzardo et al., 2026 prove that this can be done with the trait estimators defined in the previous section. The matrix H contains the bandwidth parameters and thus governs the rate at which the weights decay to zero. If we wish to use a single bandwidth, we can take $H = hI_d$, where I is the $d \times d$ identity matrix. Conversely, if we want a different bandwidth for each dimension, we can take $H = diag(h_1, .., h_d)$.

3. Joint Consistency of the Multivariate Model

As described in the introduction, Douglas, 1997 finds the conditions for joint consistency for dichotomous items in the unidimensional case, assuming certain hypotheses about the items, the ICCs, the sample sizes N_n, the kernels, and the bandwidths h_n. Joint consistency requires that the sample size and the test length grow together. Luzardo et al., 2026 present the conditions for the joint consistency in the multivariate case. The hipotesis are:

1) Local Independence.

2) The trait Θ has uniform marginal distributions $U[0, 1]$.

3) For the functions $g_{n,l}$ there exist constants $c_{n,l,1}, ..., c_{n,l,n}$ such that:

$$sup_{x_1,..,x_n,x'_1,..x'_n} \sum_{i=1}^{n} \mid g_{n,l}(x_1, ..x_i, ..x_n) - g_{n,l}(x_1, ..x'_i, ..x_n) \mid^2 \leq \sum_{i=1}^{n} c^2_{n,l,i} \tag{5}$$

for $1 \leq l \leq d$.

4) The functions $F_{n,l}$ are asymptotically Lipschitz, (L_n, γ_n) for all l such that $1 \leq l \leq n$

A sequence of functions $f_n : V \longrightarrow R^d$ with $V \subset R^k$ bounded, is asymptotically Lipschitz if there exist sequences L_n and $\gamma_n > 0$ such that for some constants $\delta > 0$ and $\lambda > 0$ with $\delta < L_n < \lambda$ for all n and $\gamma_n \longrightarrow 0$ the following inequality holds: $\| f_n(x) - f_n(y) \| \leq L_n \| x - y \| + \gamma_n \forall x, y \in V$.

This notion is introduced to approximate Lipschitz continuity for sequences of distribution functions arising from discrete variables or empirical distributions. Although such functions are inherently stepwise and therefore not Lipschitz, they may satisfy the above inequality, which is sufficient for our purposes. It follows from the definition that, if the sequence of functions converges, continuity of the norm implies that the limiting function is Lipschitz continuous.

5) $\frac{\partial E(g_{n,l}(X)|\Theta_l)}{\partial \Theta_l} > M_l > 0$.

6) For $0 < \alpha < 1/2$, $\frac{\gamma_n}{\max_l(\sum_{i=1}^{n} c^2_{n,l,i})^{\alpha}} \to 0$

7) $H \max_l(\sum_{i=1}^{n} c^2_{n,l,i})^{-\frac{d+2}{2}} < N < Mn^r$ with $r > d + 1$

8) $\lim \log n \max_l(\sum_{i=1}^{n} c^2_{n,l,i})^{1-2\alpha} = 0$

9) The kernel K is symmetric, bounded, and with compact support.

10) The ICCs have continuous partial derivatives in $(0, 1)^d$ and moreover on each compact $[a, b]^d \subset (0, 1)^d$ and every pair (n,i) they are uniformly bounded.

11) For $0 < \alpha < 1/2$, $\max_l(\sum_{i=1}^{n} c^2_{n,l,i})^{\alpha} = O(h_n)$

12) The kernel K is Lipschitz with constant J.

Under the conditions 1) to 12) the joint consistency of the traits and the ICCs is obtained, proving

$$max\{max_{k=1,\dots,N} \parallel \widehat{\Theta}_{n,k} - \Theta_{n,k} \parallel, sup_{\theta\in(a,b)^d} max_{1\leq i\leq n} \mid \widehat{P}_{n,i}(\theta) - P_{n,i}(\theta) \mid\} \to 0$$

if $n \to +\infty$ with probability 1

That is, both the maximum parameter estimation error and the uniform approximation error of the ICCs converge almost surely to zero as n grows infinitely

4. Conditions for the joint Consistency in the continuous case

We present below the conditions on functions $g_{n,l}$ to obtain the joint consistency when $X_{i,j}$ are continuous variables. For the case of continuous variables, it is necessary to ensure that the conditions required for assumptions 3) and 4) are satisfied. Regarding assumption 3), it is generally sufficient, for smooth functions, that the variables X_i be bounded. Particular attention must be paid to the verification of assumption 4).

We will show the conditions required for the mean and the weighted mean to meet hypothesis 4).

Theorem 1 *Suppose that $X_1, \dots, X_n$ are continuous random variables. Let $S_n = \sum_{i=1}^n X_i$ and $\overline{X} = S_n/n$. Let F_n denote the distribution function of $\overline{X}$.*

Assume that there exists a constant $K > 0$ such that, for all n, the random variable S_n admits a density f_{S_n} and the following uniform bound holds:

$$\sup_{t\in\mathbb{R}} f_{S_n}(t) \;\leq\; \frac{K}{n}.$$

Then the sequence $\{F_n\}$ is Lipschitz (and therefore asymptotically Lipschitz), with constant $L_n = K$ and $\gamma_n = 0$.

Proof.

Indeed, since $\overline{X} = S_n/n$ and S_n has density f_{S_n}, the random variable $\overline{X}$ admits the density

$$f_{\overline{X}}(x) = n f_{S_n}(nx),$$

and therefore

$$\sup_{x\in\mathbb{R}} f_{\overline{X}}(x) = \sup_{x\in\mathbb{R}} n f_{S_n}(nx) \leq n \cdot \frac{K}{n} = K.$$

Thus, for any $x \leq x'$,

$$F_n(x') - F_n(x) = \int_x^{x'} f_{\overline{X}}(u)\, du \leq \int_x^{x'} K\, du = K(x' - x),$$

that is,

$$|F_n(x') - F_n(x)| \leq K|x' - x|.$$

□

Theorem 2 (Asymptotically Lipschitz via conditional control) *Let Θ be a random variable, $S_n(\theta) = \sum_{i=1}^n X_i$, and let $\overline{X}_n = S_n/n$.*

Let $F_n(\cdot \mid \theta)$ denote the conditional distribution function of $\overline{X}_n$ given $\Theta = \theta$, and let F_n denote the unconditional distribution function.

Suppose that, for all n and all θ:

1. *$S_n \mid \Theta = \theta$ admits a density $f_{S_n|\Theta}$.*

2. *There exist a sequence $B_n > 0$, $L > 0$ and $B_n \leq L$ such that*

$$\sup_{\theta} \sup_{t \in \mathbb{R}} f_{S_n|\theta}(t) \leq \frac{B_n}{n},$$

Then $\{F_n\}$ is Lipschitz (on $\mathbb{R}$) that is, for all $x \leq x'$,

$$0 \leq F_n(x') - F_n(x) \leq B_n(x' - x).$$

therefore is asymptotically Lipschitz

Proof. *Since $\overline{X}_n = S_n/n$, by a change of variables, for each fixed θ, the conditional density of $\overline{X}_n$ is given by*

$$f_{\overline{X}_n|\theta}(x) = n f_{S_n|\theta}(nx).$$

Then,

$$\sup_x f_{\overline{X}_n|\theta}(x) \leq n \sup_t f_{S_n|\theta}(t) \leq n \cdot \frac{B_n}{n} = B_n.$$

Therefore, for $x \leq x'$,

$$F_n(x' \mid \theta) - F_n(x \mid \theta) = \int_x^{x'} f_{\overline{X}_n|\theta}(u)\, du \leq \int_x^{x'} B_n\, du = B_n(x' - x).$$

Finally, using the law of total probability,

$$F_n(x') - F_n(x) = \int \left(F_n(x' \mid \theta) - F_n(x \mid \theta)\right) dP_\Theta(\theta) \leq \int B_n(x' - x)\, dP_\Theta(\theta) = B_n(x' - x) \leq L(x' - x),$$

which proves the result: □

Since we are dealing with a multidimensional case, it is appropriate to consider weighted averages as functions for ordering latent traits in a given dimension.

Theorem 3 (Weighted average) *Let Θ be a random variable and, conditional on $\Theta = \theta$, let $X_1, \ldots, X_n$ be* independent *random variables (not necessarily identically distributed) with finite means. For weights $\omega_{1,n}, \ldots, \omega_{n,n}$, define the weighted average*

$$\overline{X}_n = \sum_{i=1}^n \omega_{i,n} X_i, \qquad a_n = \sum_{i=1}^n \omega_{i,n}^2.$$

Assume that, for all n and all θ, the random variable $\overline{X}_n \mid (\Theta = \theta)$ admits a density $f_{n,\theta}$ and there exist a constant $C < \infty$ such that

$$\sup_{\theta \in [0,1]^d} \|f_{n,\theta}\|_\infty \leq \frac{C}{\sqrt{a_n}} \qquad \text{for all } n. \tag{6}$$

If, in addition, there exists $c > 0$ such that

$$\inf_{n \geq 1} a_n \geq c,$$

then the sequence of unconditional distribution functions

$$F_n(x) = P(\overline{X}_n \leq x)$$

is asymptotically Lipschitz (in fact, uniformly Lipschitz) with constants

$$L_n \equiv \frac{C}{\sqrt{c}}, \qquad \gamma_n = 0,$$

that is, for all $x \leq x'$,

$$0 \leq F_n(x') - F_n(x) \leq \frac{C}{\sqrt{c}}\,(x' - x) \qquad \textit{for all } n.$$

Proof. Let $x \leq x'$. By the law of total probability,

$$F_n(x') - F_n(x) = \int \Big(P(\overline{X}_n \leq x' \mid \Theta = \theta) - P(\overline{X}_n \leq x \mid \Theta = \theta) \Big)\, dP_\Theta(\theta).$$

For each θ, since $\overline{X}_n \mid (\Theta = \theta)$ admits a density $f_{n,\theta}$,

$$P(\overline{X}_n \leq x' \mid \Theta = \theta) - P(\overline{X}_n \leq x \mid \Theta = \theta) = \int_x^{x'} f_{n,\theta}(t)\, dt \leq (x' - x)\, \|f_{n,\theta}\|_\infty.$$

Then, using (6) and $\inf_n a_n \geq c$,

$$F_n(x') - F_n(x) \leq (x' - x) \int \|f_{n,\theta}\|_\infty \, dP_\Theta(\theta) \leq (x' - x) \sup_\theta \|f_{n,\theta}\|_\infty \leq (x' - x)\frac{C}{\sqrt{a_n}} \leq (x' - x)\frac{C}{\sqrt{c}}.$$

This holds for all n, and therefore $\{F_n\}$ is uniformly Lipschitz; in particular, it is asymptotically Lipschitz with $L_n = C/\sqrt{c}$ y $\gamma_n = 0$. □

In the previous theorem if $Y_n \mid \theta$ admits a density $f_n(\cdot \mid \theta)$ and that there exists a constant $K > 0$ such that, for all n,

$$\sup_\theta \sup_{y \in \mathbb{R}} f_n(y \mid \theta) \;\leq\; K.$$

Then, for all $y \leq y'$ and all θ,

$$0 \leq F_n(y' \mid \theta) - F_n(y \mid \theta) = \int_y^{y'} f_n(u \mid \theta)\, du \leq (y' - y) \sup_u f_n(u \mid \theta) \leq K\,(y' - y).$$

Therefore, the sequence $\{F_n(\cdot \mid \theta)\}$ is Lipschitz (and in particular asymptotically Lipschitz) with $L_n = K$ and $\gamma_n = 0$.

Theorem 4 (Continuous version for weighted averages via standardization) *Let Θ be a random variable and, conditional on $\Theta = \theta$, supose that $X_{1,n}, \ldots, X_{n,n}$ are independent (not necessarily identically distributed) and absolutely continuous.*

Consider the weighted sum

$$Y_n = \sum_{i=1}^n \omega_{i,n} X_{i,n}, \qquad a_n = \sum_{i=1}^n \omega_{i,n}^2.$$

Let $F_n(\cdot \mid \theta)$ denote the distribution function of Y_n given $\Theta = \theta$ and define

$$\mu_n(\theta) = E(Y_n \mid \Theta = \theta), \qquad \sigma_n^2(\theta) = Var(Y_n \mid \Theta = \theta), \qquad \widetilde{Y}_n(\theta) = \frac{Y_n - \mu_n(\theta)}{\sigma_n(\theta)}.$$

Assume that there exists a constant $C < \infty$ such that, for all n,

$$\sup_{\theta} \sup_{t \in \mathbb{R}} f_{\widetilde{Y}_n|\Theta}(t) \leq C, \tag{7}$$

where $f_{\widetilde{Y}_n|\Theta}$ denoted the density of $\widetilde{Y}_n(\theta)$ given $\Theta = \theta$. Assume moreover that there exists a constant $\underline{\nu} > 0$ such that for all n,

$$\sigma_n^2(\theta) = Var(Y_n \mid \Theta = \theta) \geq \underline{\nu}\, a_n \qquad \text{uniformly in } \theta. \tag{8}$$

Then, for all $\gamma \leq \gamma'$ and all θ,

$$0 \leq F_n(\gamma' \mid \theta) - F_n(\gamma \mid \theta) \leq \frac{C}{\sigma_n(\theta)}(\gamma' - \gamma) \leq \frac{C}{\sqrt{\underline{\nu}\, a_n}}(\gamma' - \gamma),$$

and therefore $\{F_n(\cdot \mid \theta)\}$ is Lipschitz (and hence asymptotically Lipschitz) in γ, with

$$L_n(\theta) = \frac{C}{\sigma_n(\theta)} \leq \frac{C}{\sqrt{\underline{\nu}\, a_n}}, \qquad \gamma_n = 0.$$

Moreover, the same bound holds for the unconditional distribution function $F_n(\gamma) = \int F_n(\gamma \mid \theta)\, dP_\Theta(\theta)$:

$$0 \leq F_n(\gamma') - F_n(\gamma) \leq \frac{C}{\sqrt{\underline{\nu}\, a_n}}(\gamma' - \gamma).$$

In particular, if $\inf_{n \geq 1} a_n > 0$, then the sequence $\{F_n(\cdot \mid \theta)\}$ (and $\{F_n\}$) is asymptotically Lipschitz with uniformly bounded constants L_n and $\gamma_n = 0$.

Proof. By change of variables,

$$f_{Y_n|\Theta}(\gamma) = \frac{1}{\sigma_n(\theta)} f_{\widetilde{Y}_n|\Theta}\left(\frac{\gamma - \mu_n(\theta)}{\sigma_n(\theta)}\right),$$

and hence, using (7),

$$\sup_{\gamma} f_{Y_n|\Theta}(\gamma) \leq \frac{1}{\sigma_n(\theta)} \sup_{t} f_{\widetilde{Y}_n|\Theta}(t) \leq \frac{C}{\sigma_n(\theta)}.$$

Therefore, for $\gamma \leq \gamma'$,

$$F_n(\gamma' \mid \theta) - F_n(\gamma \mid \theta) = \int_{\gamma}^{\gamma'} f_{Y_n|\Theta}(u)\, du \leq (\gamma' - \gamma) \sup_{u} f_{Y_n|\Theta}(u) \leq \frac{C}{\sigma_n(\theta)}(\gamma' - \gamma).$$

The final bound follows from (8). Integrating with respect to θ yields the analogous inequality for F_n. □

5. Conclusion

This study establishes a theoretical framework for the joint estimation of multidimensional latent traits and item characteristic curves within a fully nonparametric, continuous-response model. By extending the kernel smoothing approach described by Ramsay, 1991 to a multidimensional context with continuous responses, we have derived the conditions, encompassing the choice of ranking statistics, kernel functions, bandwidth sequences, and the growth relationship between sample size and test length, under which the trait and ICC estimators converge strongly and uniformly. This work not only generalizes the nonparametric paradigm to a broader class of psychometric data but also provides a theoretical justification for its use in complex settings where parametric assumptions are untenable. Future research should focus on implementing the estimator in practice, developing efficient algorithms for high-dimensional applications, and empirically validating its performance against parametric alternatives in real-world testing and measurement scenarios.

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