LLMs Faithfully and Iteratively Compute Answers During CoT: A Systematic Analysis With Multi-hop Arithmetics

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Abstract

This study investigates the internal information flow of large language models (LLMs) while performing chain-of-thought (CoT) style reasoning. Specifically, with a particular interest in the faithfulness of the CoT explanation to LLMs' final answer, we explore (i) when the LLMs' answer is (pre)determined, especially before the CoT begins or after, and (ii) how strongly the information from CoT specifically has a causal effect on the final answer. Our experiments with controlled arithmetic tasks reveal a systematic internal reasoning mechanism of LLMs. They have not derived an answer at the moment when input was fed into the model. Instead, they compute (sub-)answers while generating the reasoning chain on the fly. Therefore, the generated reasoning chains can be regarded as faithful reflections of the model's internal computation.

1 Introduction

Modern large language models (LLMs), given a query from users, typically produce intermediate reasoning steps as well as a final answer. Such reasoning steps to achieve the answer, i.e., explanation, are helpful to grasp the model's underlying reasoning process and the plausible justification for the produced answer. One critical concern is how *faithful* the explanation, typically with a chain-of-thought (CoT) style, is to their final answer, i.e., the causal relationship between the CoT process and the final answer. In particular, how models *internally* associate the CoT part and its following answer is an intriguing question in the interpretability field.

In this study, we analyze the faithfulness between the CoT explanation and the model's final answer through the lens of (mechanistic) interpretability. This question involves two key subquestions—when, in the first place, models come up with (sub-)answers during CoT-style reasoning, and how these are referred to internally. One possible strategy, for example, would be to reach the final answer even during reading problem statements before CoT generation (the first pass), and then the model produces a fluent explanation just to follow an expected format, and ultimately, the model simply refers to their predetermined answers. In this case, the final answer is no longer strictly faithful to the CoT explanation, and users should be aware of this limitation in interpreting the model's outputs.

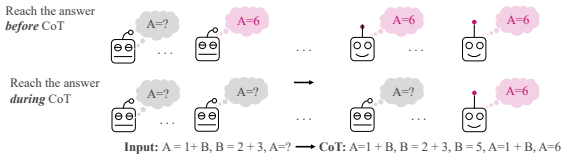

Figure 1: Using linear probes, we investigated at which time during the LLM's problem-solving process it is possible to determine the values of each variable, illustrating the model's problem-solving process. Our analysis indicates that LLMs come up with (sub-)answers during CoT. This conclusion is also consistent with the findings from the causal experiments in § 4.

In our first experiments, we exploratorily analyze *when* (at each layer at each timestep) models represent answers internally, using linear probes. To prevent situations where CoT is not required for deriving the answer, we set up controlled testbeds of symbolic arithmetic reasoning and observed when/where probes could accurately extract and control the answer (Figure 1). By comparing accuracies across each timestep, one can observe at which point models internally start being informative (the values of the variables are encoded in the hidden states as linearly separable representations) to the probes, illustrating the model's internal reasoning pattern. We found that models have not derived an answer at the moment they first read the problem; instead, they obtain (sub-)answers while generating the reasoning chain, suggesting the faithfulness of CoT to the final answer.

Based on the probing results, we further conducted causal intervention analyses to clarify the causal relationship between the model's internal representations and answers (§ 4). We found that, after CoT generation, the model's final answer can be changed by causal intervention to internal representations for the CoT explanation part, while not for the first pass (problem state reading) part. The discovered causal graph follows recency bias, i.e., each reasoning step in CoT causally depends on its previous step, and the final answer heavily relies on the latter part of CoT. Thus, we tentatively conclude that models derive an answer during CoT on the fly, and the generated reasoning chains can be regarded as faithful reflections of the model's final answer. [1]

2 General settings

2.1 Arithmetic problems

We prepared a synthetic dataset of multi-hop arithmetic problems similarly to Kudo et al. (2023) and Yu (2025). Each problem instance consists of strings of assignments (e.g., `A=1`) and operations (e.g., `B=1+3` or `B=1+A`) ending with a query for a variable's value (e.g., `B=?`). By using this synthetic dataset, we can conduct controlled experiments across different levels of problem complexity and confirm the generality of the obtained results. We specifically defined five complexity levels (Table 1), depending on (i) how many equations need to be resolved to reach the answer (#Step in Table 1), (ii) how many variables' values cannot be immediately resolved (and thus pended to a stack) in their first appearance when incrementally reading the problem from left to right (#Stack), (iii) and the number of unnecessary distractor equations (#Dist.). For example, in the level 5 example in Table 1, where #Step is three and #Stack is two, calculating `A` requires at least three steps of reasoning: `C(=1+2)=3`, `B(=2+3)=5`, and then `A(=1+5)=6`, and two variables need to be resolved before reaching `A`: `B` and `C`.

Notation. Formally, let v denote a variable name sampled from the 26 letters of the English alphabet $\Sigma = \{\mathrm{a}, \mathrm{b}, \mathrm{c}, \ldots, \mathrm{z}\}$, and d denote a number sampled from the set of decimal digits $D = \{0, 1, 2, \ldots, 9\}$. Each instance consists of multiple equations $[e_1, e_2, \cdots, e_n]$ followed by a final query q. Each equation follows the format $v = d$, $v = d \pm d$, or $v = d \pm v$. We denote i-th variable in an instance from the left as v_i; e.g., in the Level 5 example in Table 1, $v_1 =$ `A`, $v_2 =$ `B`, and $v_3 =$ `C`.[2] The value assigned to a variable v_i is denoted as $\$\{v_i\} \in D$.

Data constraints. We ensure that $\$\{v_i\}$ for any v_i is also a single-digit number, and $\$\{v_i\}$ is constant within the same instance (i.e., we exclude cases such as `A=1+2,A=B+2,B=6`). All samples of the same complexity level follow the exact same format except for the actual numbers, variable names, and operators. In other words, an answer $\$\{v_i\}$ in each level can be obtained with exactly the same procedure. For example, first calculate $\$\{v_3\}$, and then calculate $\$\{v_2\}$ with $\$\{v_3\}$, and then finally calculate $\$\{v_1\}$ with $\$\{v_2\}$. Non-duplicated instances are created for each level by varying the variable names, numbers, and operators appearing in the equations. We created 12,000 unique instances in total, of which 10,000 are used for training probing classifiers and 2,000 for testing their accuracy. To prevent the probe from simply memorizing (sub-)answers, the training and test sets were constructed with no overlap in arithmetic expressions.[3]

2.2 CoT-style inference

Given the input described in § 2.1, the model generates CoT z and a final answer y. Henceforth, we refer to the part before CoT as the INPUT x and the CoT-reasoning part (y, z) as the OUTPUT of an instance, as shown in the right part of Table 1. For example, for the Level 1 instance in Table 1 (topmost), $x =$ "`A=1+B, B=2,`" $z =$ "`A=1+B, B=2, A=1+2, A=`", and $y =$ "`3`." We demonstrate three examples in the same problem level for the model to inform the problem setting and expected CoT-style reasoning style (Table 1). That is, the task is, given a demonstration of three examples, and an INPUT x, to generate intermediate steps z and derive a final answer y. As a sanity check, we confirmed that the target models could follow the expected OUTPUT format and solve the tasks with nearly 100% accuracy in this setting (see § B.1 for

[1] Code and data are available at https://anonymous.4open.science/r/TTT-under-review/README.org

[2] For brevity, all examples in this paper use the uppercased variables `A`, `B`, `C`, and only the operator `+`, but the actual instances have a variety in the variable names and operator types.

[3] The overlap is computed at the equation-level. For example, if `1+2` appears in the training set, then `1+2` does not appear in any test instances.

Level	INPUT	OUTPUT	#Step	#Stack	#Dist.
1	$\underline{\mathrm{A}=1+\mathrm{B}}_{-3}, \underline{\mathrm{B}=2}_{-2}; \underline{\mathrm{A}=?}_{-1}$	$\underline{\mathrm{A}=1+\mathrm{B}}_{0}, \underline{\mathrm{B}=2}_{1}, \underline{\mathrm{A}=1+\mathrm{B}}_{2}, \underline{\mathrm{A}=1+2}_{3}, \underline{\mathrm{A}=3}_{4}$	1	1	0
2	$\underline{\mathrm{A}=2+3}_{-3}, \underline{\mathrm{B}=1+\mathrm{A}}_{-2}; \underline{\mathrm{B}=?}_{-1}$	$\underline{\mathrm{B}=1+\mathrm{A}}_{0}, \underline{\mathrm{A}=2+3}_{1}, \underline{\mathrm{A}=5}_{2}, \underline{\mathrm{B}=1+\mathrm{A}}_{3}, \underline{\mathrm{B}=1+5}_{4}, \underline{\mathrm{B}=6}_{5}$	2	0	0
3	$\underline{\mathrm{A}=1+\mathrm{B}}_{-3}, \underline{\mathrm{B}=2+3}_{-2}; \underline{\mathrm{A}=?}_{-1}$	$\underline{\mathrm{A}=1+\mathrm{B}}_{0}, \underline{\mathrm{B}=2+3}_{1}, \underline{\mathrm{B}=5}_{2}, \underline{\mathrm{A}=1+\mathrm{B}}_{3}, \underline{\mathrm{A}=1+5}_{4}, \underline{\mathrm{A}=6}_{5}$	2	1	0
4	$\underline{\mathrm{A}=1+\mathrm{B}}_{-4}, \underline{\mathrm{B}=2+3}_{-3}, \underline{\mathrm{C}=4+5}_{-2}; \underline{\mathrm{A}=?}_{-1}$	$\underline{\mathrm{A}=1+\mathrm{B}}_{0}, \underline{\mathrm{B}=2+3}_{1}, \underline{\mathrm{B}=5}_{2}, \underline{\mathrm{A}=1+\mathrm{B}}_{3}, \underline{\mathrm{A}=1+5}_{4}, \underline{\mathrm{A}=6}_{5}$	2	1	1
5	$\underline{\mathrm{A}=1+\mathrm{B}}_{-4}, \underline{\mathrm{B}=2+\mathrm{C}}_{-3}, \underline{\mathrm{C}=1+2}_{-2}; \underline{\mathrm{A}=?}_{-1}$	$\underline{\mathrm{A}=1+\mathrm{B}}_{0}, \underline{\mathrm{B}=2+\mathrm{C}}_{1}, \underline{\mathrm{C}=1+2}_{2}, \underline{\mathrm{C}=3}_{3}, \underline{\mathrm{B}=2+\mathrm{C}}_{4}, \underline{\mathrm{B}=2+3}_{5}, \underline{\mathrm{B}=5}_{6}, \underline{\mathrm{A}=1+\mathrm{B}}_{7}, \underline{\mathrm{A}=1+5}_{8}, \underline{\mathrm{A}=6}_{9}$	3	2	0

Table 1: Examples of arithmetic reasoning tasks used in our experiments at each complexity level. #Step indicates the number of required operations to reach the final answer. #Stack indicates how many variables' values are not immediately determined in their first appearing equation. #Dist. is the number of unnecessary distractor equations. The number (e.g., $-\mathbf{3}$) indicated in the lower right corner of each equation represents the equation's position. This position is used as a reference point for calculating t^*_{eq} in § 3.2.

more details).

2.3 Research scope

We are particularly concerned about the faithfulness between the CoT part z and its following final answer y, since these two parts are typically shown to users, and thus are of primary concern regarding the model's reliability from the user's perspective. In contrast, some recent studies (Afzal et al., 2025; Cox, 2025; Ye et al., 2025) have investigated whether the CoT z depends on the internal state of the model x before CoT begins (whether the CoT z is *post hoc* or not). This causal relationship $x \to z$ is out of scope in this study, while we are still interested in whether the model x has a predetermined answer y in order to distinguish the possibilities of $x \to y$ and $z \to y$, i.e., which part is causally relied on to the final answer y.

3 Linear probing

To begin with the analysis, it will be good to know where the final answer, or the necessary sub-answer for it, can possibly come from. We first analyze where final-/sub- answers can be extracted from the model's internal representations, using linear probes.

Position t. We denote a token position within the entire concatenated sequence $x \oplus z \oplus y$ with $t \in \mathbb{Z}$. The position t is relative to CoT; that is, t is zero at the moment CoT begins, negative within the INPUT, and positive within the OUTPUT. Similarly, we assign an equation position $t_{\mathrm{eq}} \in \mathbb{Z}$ to each equation in the INPUT and OUTPUT (subscripts on the underlines in Table 1).

3.1 Training linear probes

Linear probes (Alain and Bengio, 2017a) are trained to identify when the model derives the answer internally. We separately train a probe for each combination of token position $t \in \mathbb{Z}$, layer depth $l \in \mathbb{N}$, and the variable of interest $v_i \in \Sigma$ in each level of the problem. That is, each probe is solely responsible for the position (t, l, v_i) to identify the answer $\$\{\hat{v}_i\}$ represented. Formally, given a model's d-dimensional hidden state $\boldsymbol{h}_{t,l} \in \mathbb{R}^d$, the probing classifier $f_{t,l,v_i}(\cdot) : \mathbb{R}^d \to D$ is trained to predict the correct answer $\$\{v_i\}$. If a probe f_{t,l,v_i} achieves high accuracy, this suggests that the answer for a particular variable $\$\{v_i\}$ is already computed at the corresponding position (t, l). The probe f_{t,l,v_i} is a single linear transformation:

$$\begin{aligned} \$\{\hat{v}_i\}_{t,l} &= f_{t,l,v_i}(\boldsymbol{h}_{t,l}) \\ &= \arg\max_{D} \boldsymbol{W}_{t,l,v_i} \boldsymbol{h}_{t,l} + \boldsymbol{b}_{t,l,v_i} \quad , \end{aligned} \tag{1}$$

where, $\boldsymbol{W}_{t,l,v_i} \in \mathbb{R}^{|D| \times d}$ and $\boldsymbol{b}_{t,l,v_i} \in \mathbb{R}^{|D|}$ are the trainable weight and bias parameters of the probe, respectively. The symbol $\hat{\cdot}$ refers to the model's predicted answer. We train the probes using stochastic gradient descent (Robbins, 1951) to minimize the cross entropy loss. The hyperparameters are listed in Table 8 in the appendix.

Each probe classifier is trained with 10,000 of $\boldsymbol{h}_{t,l}$ from training instances and then computes the accuracy with 2,000 hidden states from test instances.

3.2 Evaluation metrics

The probing results from all the token positions t and layers l are aggregated as follows:

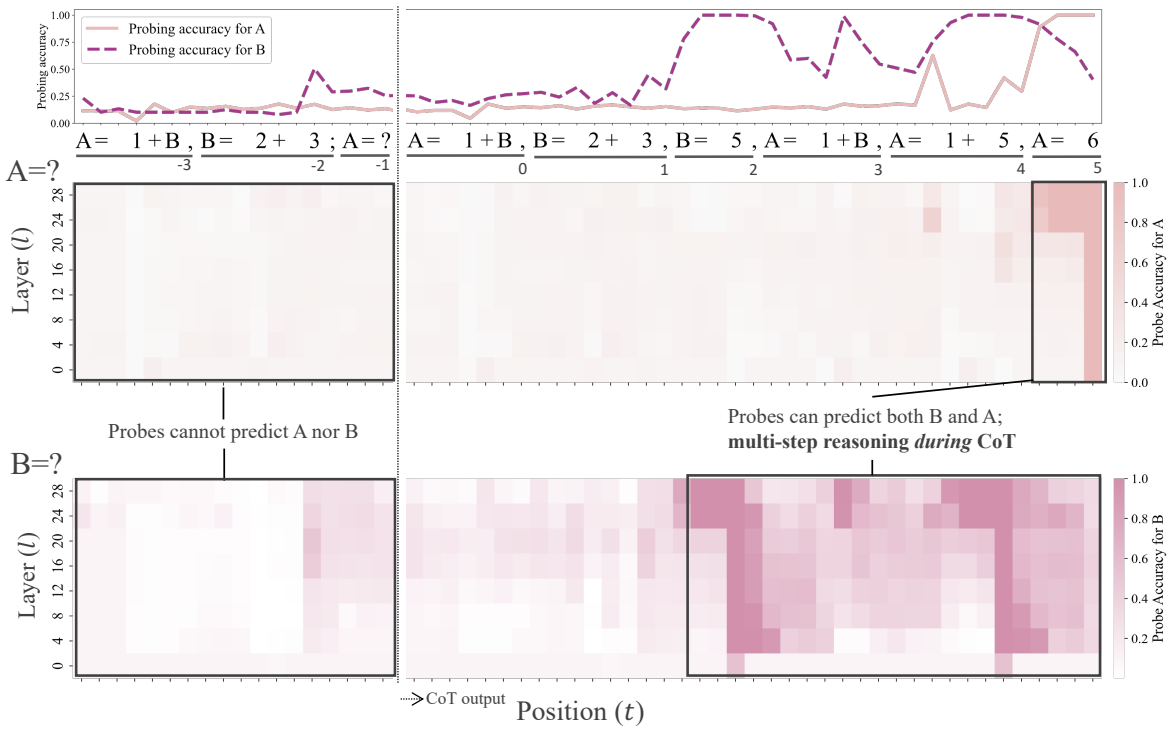

Figure 2: Probing results for Qwen2.5-7B at the Level 3 task. The heatmaps in the lower section represent the accuracy of probes computed on the evaluation set. Each cell shows the probing accuracies in each token t, layer l. The upper part indicates the maximum probing accuracy achieved at each token position t. The input sequence below the line graphs is just an example; in the actual evaluation set, each variable name, number, and operator are randomly sampled from $(D, \Sigma, \{+, -\})$.

$$t^*(v_i) = \min\{t \mid \max_l \mathrm{acc}(t, l, v_i) > \tau\}, \quad (2)$$

where $\mathrm{acc}(t, l, v_i) \in [0, 1]$ denotes the probing accuracy of f_{t,l,v_i}. The position $t^*(v_i) \in \mathbb{Z}$ indicates when the first time the probing classifier achieved a reasonable accuracy above τ. We set $\tau = 0.9$ in our main experiment, and we show additional results with different thresholds in Appendix B.3. As a more coarse but comprehensive value, we also report $t^*_{\mathrm{eq}}(v_i)$, indicating which equation t_{eq} in the input x the position $t^*(v_i)$ falls into. Given that the t (and t_{eq}) is relative to the CoT-beginning position, if $t^*_{(\mathrm{eq})}(v_i)$ is negative, the value $\$\{v_i\}$ is computed before CoT begins. In Figure 2, for example, both $t^*(\texttt{A})$ and $t^*(\texttt{B})$ are positive.

As complementary information of the position, we also report two types of accuracy:

$$\mathrm{Acc}_{\prec \mathrm{CoT}}(v_i) = \max_{t<0,l} \mathrm{acc}(t, l, v_i), \quad (3)$$

$$\mathrm{Acc}_{\succ \mathrm{CoT}}(v_i) = \max_{t\geq 0,l} \mathrm{acc}(t, l, v_i). \quad (4)$$

If $\mathrm{Acc}_{\prec \mathrm{CoT}}(v_i)$ is sufficiently high, $\$\{v_i\}$ is resolved internally before CoT begins. Conversely, if $\mathrm{Acc}_{\prec \mathrm{CoT}}(v_i)$ is low and $\mathrm{Acc}_{\succ \mathrm{CoT}}(v_i)$ is high, the answer is derived while performing CoT reasoning.

3.3 Results

Across task levels. We first analyze Qwen2.5-7B (Qwen Team, 2024) across the five task levels. Table 2 reports the position t^*_{eq} for each variable and its lower bound $t^{\mathrm{inf}}_{\mathrm{eq}}$ where the answer can be first identifiable. In most cases, (sub)answers are represented in the model' s hidden states corresponding to the OUTPUT part ($t^*_{\mathrm{eq}} > 0$) in a linearly separable form, even when deriving relatively simple sub-answers, e.g., $\$\{\texttt{B}\}$ in Level 2. The exceptions are: (i) v_2 in Level 1, and (ii) v_3 in Level 4. Here, v_2 in Level 1 requires no computation (#Steps= 0), and v_3 in Level 4 is a distractor that is not needed to derive the final answer. To sum up, all calculations (at least 1 step) required to obtain the final answer are resolved **during CoT**, suggesting that it's unlikely to refer to a predetermined answer as model's final answer.

Across models. We analyzed nine models (Table 3) on the Level 3 task. Similarly to Table 2, we observed $t^*_{\mathrm{eq}} > 0$. This pattern holds for

	Variable		When (↓)		Acc. (↑)	
Level	variable	#Step	t^*_{eq}	$t^{\text{inf}}_{\text{eq}}$	$\prec$ CoT	$\succ$ CoT
1	v_1 (A)	1	4	-2	35.8	100
	v_2 (B)	0	-2	-2	100	100
2	v_1 (A)	1	2	-3	49.2	100
	v_2 (B)	2	5	-2	21.6	94.7
3	v_1 (A)	2	5	-2	17.9	97.4
	v_2 (B)	1	2	-2	50.5	100
4	v_1 (A)	2	5	-3	17.2	100
	v_2 (B)	1	2	-3	47.7	100
	v_3 (C)	1	N/A	-2	43.7	23.7
5	v_1 (A)	3	9	-2	18.1	100
	v_2 (B)	2	6	-2	22.6	100
	v_3 (C)	1	3	-2	50.6	100

Table 2: The results of Qwen2.5-7B on the five levels. The t^*_{eq} is the time when the model comes up with the correct answer (see § 3.2). The $t^{\text{inf}}_{\text{eq}}$ column indicates the lower bound of t^*_{eq} score. The $\prec$ CoT and $\succ$ CoT scores correspond to the accuracies introduced in § 3.2. N/A indicates that the threshold τ was not exceeded at any position t.

		When (↓)		Acc (↑)	
	Var.	t^*_{eq}	t^*	$\prec$ CoT	$\succ$ CoT
Qwen2.5 (7B)	v_1 (A)	5	36	17.9	100
(Qwen Team, 2024)	v_2 (B)	2	16	50.5	100
Qwen2.5 (14B)	v_1 (A)	5	35	17.8	100
(Qwen Team, 2024)	v_2 (B)	2	16	50.5	100
Qwen2.5 (32B)	v_1 (A)	5	36	17.8	100
(Qwen Team, 2024)	v_2 (B)	2	15	67.4	100
Qwen2.5-Math (7B)	v_1 (A)	5	35	18.6	100
(Yang et al., 2024a)	v_2 (B)	2	15	56.1	100
Yi1.5 (9B)	v_1 (A)	5	41	17.8	100
(Young et al., 2024)	v_2 (B)	2	18	36.9	100
Yi1.5 (34B)	v_1 (A)	5	41	22.4	100
(Young et al., 2024)	v_2 (B)	2	18	37.4	100
Llama3.1 (8B)	v_1 (A)	5	35	26.0	100
(Dubey et al., 2024)	v_2 (B)	2	16	29.6	100
Llama3.2 (3B)	v_1 (A)	5	36	17.8	93.2
(Dubey et al., 2024)	v_2 (B)	2	17	33.2	95.4
Mistral-Nemo (12B)	v_1 (A)	5	36	17.8	100
(Mistral AI Team, 2024)	v_2 (B)	2	16	28.9	100

Table 3: Results for various models on the Level 3 task. The t^* column shows the token-wise time (described in § 3.2), and the other columns are the same as Table 2. The t^* and t^*_{eq} scores that are the same as their lower bounds are bolded.

other tasks and for different thresholds τ (Tables 9–23 in § B.3). On the other hand, in the INPUT part—particularly the latter half of equation -2 ($\underline{\text{B} = 2 + 3}, \mathbf{-2}$)—the pre-CoT accuracy $\text{Acc}_{\prec\text{CoT}}(v_i)$ increases modestly, peaking around 60%.

3.4 Analysis

In this section, we present the results of a prompt-sensitivity analysis and an error analysis. We also analyze the effects of factors such as equation order, distractors, and the reasoning-chain format (including implicit reasoning). For details, see § A.

Prompt differences. We investigated whether the format of few-shot prompting influences the model's internal reasoning patterns. We analyzed the differences between two scenarios: *1. Same level prompting* and *2. General prompting*. In the same level prompting scenario, we provided three equations that matched the level of the evaluation task, similar to the general setup (§ 2). Conversely, in the general prompting scenario, the model received randomly generated samples, each containing three equations. See § A.3 for more details.

Behavior in incorrect instances. Figure 3 shows the top-1 predictions of the probe for B in instances where Llama3.2-3B produced *incorrect* answers for level 3 (where B happened to be 4). When the model produced an incorrect answer, we often found that the correct answer had appeared at an earlier decoding step (see § B.2 for the probe's predictions on additional samples). The underlying reason for these mistakes remains unclear, but through our probing framework, one can track how such an incorrect answer is propagated to the subsequent generation, which might be practically helpful in debugging the model internals.

4 Causal interventions

From our probing experiments in § 3, we tentatively concluded that the final answer is determined after CoT-style generation begins. That is, CoT is faithful to the final answer. In this section, we corroborate this conclusion using causal intervention analysis and further clarify how input information flows internally while generating CoT reasoning and the final answer.

4.1 Settings

Activation patching. We employ activation patching (Vig et al., 2020; Meng et al., 2022; Zhang and Nanda, 2024), which is a widely adopted technique for causal intervention analysis. To inspect the causal relationship between specific hidden states $\boldsymbol{h}_{t,l}$ and a final answer y, we compare two

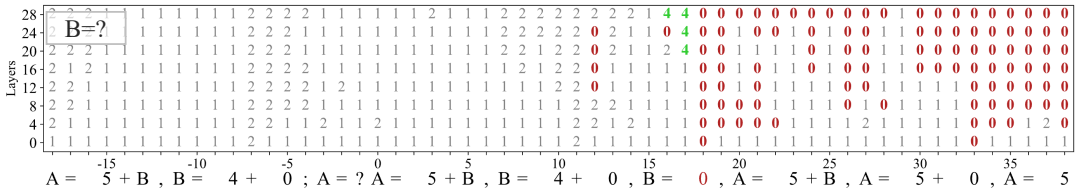

Figure 3: Error analysis of cases where Llama3.2-3B generated *incorrect* answers. The vertical axis represents the index of the transformer layer. The horizontal axis represents the tokens input to the model over time. The numbers highlighted in green represent the gold labels for the predictions, while those highlighted in red denote the values incorrectly generated by the model.

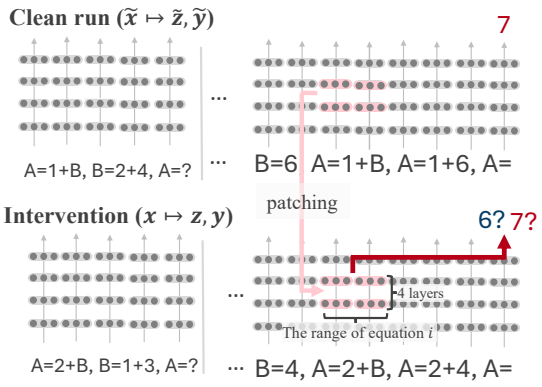

Figure 4: Overview of the causal intervention experiment. First, we perform normal inference (Clean run) and cache its hidden states. Subsequently, we evaluate whether the output changes by replacing some of the hidden states of a model solving a different problem with the cached hidden states.

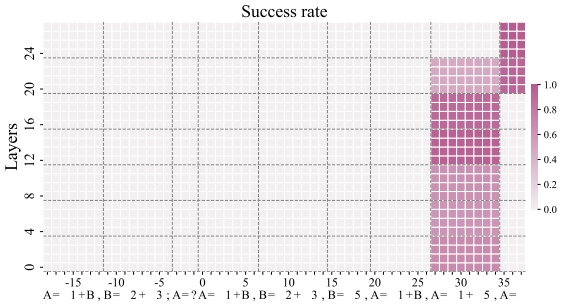

Figure 5: Results of the causal intervention on Qwen2.5-7B. Each grid cell shows the success rate when the final answer y (A =$\mathbf{6}_5$) is the target token.

generation scenarios: (i) the ordinary inference and (ii) the intervened inference. In the latter scenario, we replace the specific hidden states $\boldsymbol{h}_{t,l}$ with other variants $\tilde{\boldsymbol{h}}_{t,l}$ obtained from the same model but with a different input $\tilde{x}$ (Clean run in Figure 4).

The input x and $\tilde{x}$ have different correct answer y and $\tilde{y}$ as well as different chains z and $\tilde{z}$, respectively. For example, for the triple (x = "A=1+B,B=2+4;A=?," z = "A=1+B,B=2+4,B=6,A=1+B,A=1+6,A=7," y =7), one may use ($\tilde{x}$ = "A=2+B,B=1+3;A=?," $\tilde{z}$ = "A=2+B,B=1+3,B=4,A=2+B,A=2+4,A=6," $\tilde{y}$ =6). If the model's output turns from y into $\tilde{y}$ or z_t into $\tilde{z}_t$ due to the intervention to $\boldsymbol{h}_{t,l}$ with $\tilde{\boldsymbol{h}}_{t,l}$, we can confirm the causal relationship between $\boldsymbol{h}_{t,l}$ and the original answer y. Henceforth, to precisely explain the setting, we denote the model's final output without intervention as $\hat{\tilde{y}}$ and that with intervention as $\hat{y}$, respectively ($\tilde{y}$ and y denote respective gold answers). In the same way, we denote the generated reasoning chain without intervention as $\hat{\tilde{z}}_t$ and that with intervention as $\hat{z}_t$, respectively.

Evaluation metrics. We report *Success Rate* as a metric for this experiment. The Success rate indicates how frequently (%) the intervened output $\hat{\tilde{y}}$ aligns with the correct answer $\tilde{y}$. For reasoning chains, we report the Success Rate for $\hat{\tilde{z}}_t$ as well.

Patching targets. We specifically focus on Level 3 tasks and Qwen2.5-7B. The experimental results for the other models are reported in § B.4. Inspired by sliding window patching (Zhang and Nanda, 2024), we partition the hidden states into coarse grids, corresponding to each equation and every four layers, and perform activation patching on each grid separately (illustrated in Figure 6).[4] For every grid, we compute the *Success rate* by applying activation patching. We also examine multiple target tokens, specifically, at (i) the end of the equation 2 (z_{17} in B =$\mathbf{5}_2$ in Figure 2), (ii) the end of the equation 4 (z_{32} in A = 1+$\mathbf{5}_4$ in Figure 2), and (iii) the final answer (y). When we apply activation patching, we generate only the target token with greedy decoding while forced-decoding the context. Note that the above equations are examples. We create a test set of 2,000 instances for evaluation.

[4] All the hidden states in each grid are intervened at once.

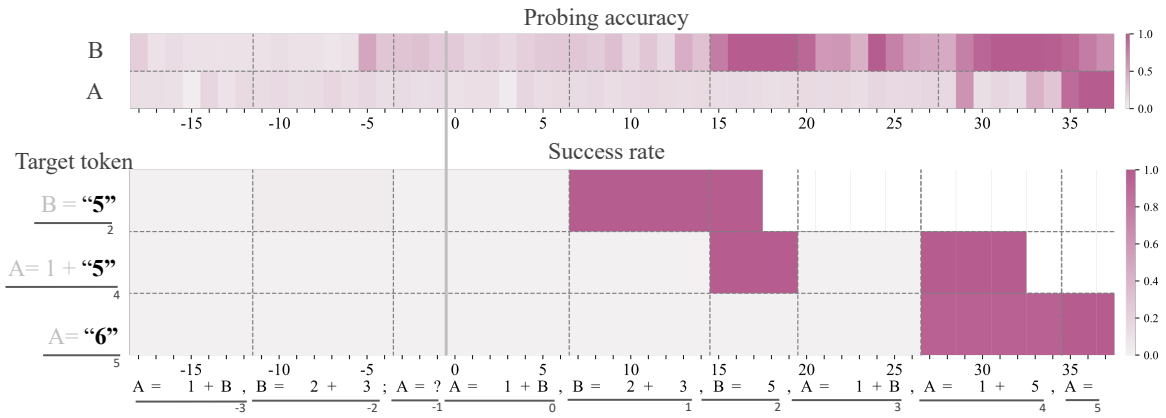

Figure 6: The upper part is the accuracy of probes, as shown in Figure 2. The lower part is the result of max pooling the Success rates from Figure 5 in the layer direction.

4.2 Results.

CoT is faithful to the final answer. Figure 5 presents the success rate at each grid position when the final answer y ($\underline{\text{A} = \mathbf{6}}_{\mathbf{2}}$) is the target token. The results show a strong causal dependence of the final answer on the OUTPUT part, whereas the INPUT part has only a limited effect. We observe the same pattern for other target tokens (e.g., z_{17} and z_{32}). This trend is broadly consistent across models (§ B.4). Furthermore, we focus on the generation of $\underline{\text{B} = \mathbf{5}}_{\mathbf{2}}$, which is a necessary sub-answer to reach the final answer (see Figure 6). Intervention to the immediately preceding equation $\underline{\text{B} = 2 + 3}_{\mathbf{1}}$ affects the generation of $\underline{\text{B} = \mathbf{5}}_{\mathbf{2}}$, but patching the equation $\underline{\text{B} = 2 + 3}_{\mathbf{-2}}$ in the problem statement x (the region where a slight improvement in probing accuracy before the CoT was observed; $t = -11$ to $t = -4$) had almost no effect on the output of $\underline{\text{B} = \mathbf{5}}_{\mathbf{2}}$. Similarly, when we focus on the case where the target token is $\underline{\text{A} = 1{+}\mathbf{5}}_{\mathbf{4}}$, patching the equation $\underline{\text{B} = 2 + 3}_{\mathbf{-2}}$ again resulted in almost no change in the output. Therefore, the influence of the information processed in the INPUT part on the sub-answers (B $=\mathbf{5}$) is limited. This tendency is especially strong in the Qwen family (except for Qwen2.5-Math-7B). By contrast, other model families show a weak causal relationship between $\underline{\text{B} = \mathbf{5}}_{\mathbf{2}}$ and the INPUT part (§ B.4). This aligns with the results observed in probing experiments § 3.3.

Recency bias. We further investigate which parts of CoT output causally depend on which input. The bottom part of Figure 6 suggests strong *recency bias* in the causal relationship between hidden states and output tokens for Qwen2.5-7B. That is, intervention succeeded only when the target hidden state is (i) in the same equation as the target token, (ii) in the last equation where necessary information is stated (e.g., `B=2+3`→`B=5`), or (iii) in the last equation where a value of a relevant variable is explicitly stated (e.g., `B=5`→`A=1+5`). This finding suggests strong recency bias in the internal process of LLMs' multi-hop reasoning. This aligns with our finding that models derive an answer during CoT on the fly, and the generated reasoning chains can be regarded as faithful reflections of the model's final answer.

5 Related Work

Post hoc nature of CoT. Several prior studies investigate similar questions (Afzal et al., 2025; Ye et al., 2025; Cox, 2025; Lewis-Lim et al., 2025). For instance, Cox (2025) examined whether CoT explanations are post-hoc or not, specifically, confirmed that, in a binary classification task, the correct label can be predicted from the hidden state at the end of the input (prompt). In their setting, the CoT z is (freely) produced from the intervened hidden state x, while we force-decode the CoT chain. In this sense, their scope is rather on the causality of $x \to z$, and thus, as stated in § 2.3, the scope is slightly different. They also showed that, in some tasks (e.g., sports understanding), the final answer can be predicted with the model's final state in the prompt, apparently contradicting our probing experiments, but their tasks would be answerable with factual or commonsense knowledge without CoT, and in their specific logical deduction task—where on-the-fly reasoning is required, similarly to our arithmetic tasks—the probe indeed

struggles to predict the final answer, similarly to our results. Afzal et al. (2025) also suggests the model reaches the answer internally before CoT begins, but their experimental design is somewhat different from ours. Specifically, they try to predict whether the model will succeed/fail the task before CoT begins as a binary problem, rather than tracing how models reach the correct answer during CoT or identifying where the concrete answer comes up with during CoT reasoning. Ye et al. (2025) also find that the model, in advance, anticipates which components (variables) will be required to reach the final answer by tracking whether a variable has been computed, and how models retain computed values during reasoning. Although the experiments are similar, they do not investigate causal relationships and therefore do not fully answer the question of whether the reasoning chain is faithful.

Interpreting multi-hop reasoning in language models. Interpreting internal mechanisms of LLMs has been actively investigated (Conneau et al., 2018; Tenney et al., 2019; Niven and Kao, 2019; nostalgebraist, 2020; Geva et al., 2023; Lieberum et al., 2024; Ghandeharioun et al., 2024; Ferrando and Voita, 2024). They revealed, for example, specialized attention heads responsible for specific operations (Cabannes et al., 2024) or decision-making, copying, and induction (Dutta et al., 2024). With a more concrete example, Yang et al. (2024c) showed that, even during the first pass of the problem statements such as *The mother of the singer of Thriller is* ___, language models first resolve a *bridge entity*, Stevie Wonder in this case, then identify the final answer. This study is more focused on the difference between the first pass of the problem statements (before CoT generation) and their second pass involving explicit problem solving (while CoT generation).

Arithmetic representations in LLMs. How models handle numerical information has also been closely studied. For instance, Heinzerling and Inui (2024) used partial least squares regression (Wold et al., 2001) to demonstrate that numeric attributes, such as birth years and population numbers, are encoded as monotonic representations in the activation space of LLMs and can be manipulated with interventions. In turn, Stolfo et al. (2023) showed that, in autoregressive LLMs, the operations and numerical information necessary for solving quantitative reasoning are processed in the lower layers of the model, and these results are used by the attention layers to predict the final calculation outcomes. Zhu et al. (2025) studied the representation of numbers in language models' hidden states during single-hop arithmetic tasks (e.g., `What is the sum of 12 and 34?`). Their analysis revealed that numerical information is encoded linearly within the hidden states and demonstrated that individual digits could be manipulated independently. In this study, we add to this literature by introducing incremental arithmetic problem solving, i.e., what numerical information is contained in the model's hidden states at each time step of multi-hop arithmetic reasoning.

Model interpretability methods. Linear probing (Alain and Bengio, 2017b) is one of the representative methods for analyzing the internal representations of neural models—a small model predicts a specific feature from them, thereby determining whether the input contains information about that feature. In this study, we use them to derive the models' intermediate answers. The causality with the model's output can be further verified by examining if a model's predictions change when the hidden states are intervened (Li et al., 2023; Wu et al., 2023). One representative intervention method is activation patching (Vig et al., 2020; Meng et al., 2022; Zhang and Nanda, 2024), where hidden states obtained from one model instance are transplanted onto another during inference to change its predictions. Such techniques can be applied as a way to control model behavior in practical scenarios such as mitigating inherent biases (Zhao et al., 2019; Ganguli et al., 2023; Yang et al., 2024b). Here, we employ activation patching to validate the plausibility of the probing results.

6 Conclusions

We conducted probing experiments to determine when (sub)answers are derived during the CoT-style reasoning, using synthetic arithmetic problems as a controlled testbed. Across multiple models and task difficulties, we found that models tend to resolve the necessary (sub)answer after the CoT begins, i.e., computing the answer on the fly during CoT generation. Moreover, causal experiments support that the intervention to the CoT part specifically and causally impacts the final answer; that is, we conclude that the CoT is faithful to the final answer at least in our controlled experimental settings.

Limitations

Variety of tasks We analyzed the internal reasoning patterns of language models using synthetic arithmetic reasoning tasks. The use of synthetic data allows for more detailed control compared to experiments on natural language tasks. However, vocabulary and expression diversity, for example, are limited compared to natural language tasks. Therefore, conducting similar analyses on more realistic reasoning tasks in natural language will verify whether the results of this study apply to other broader, realistic contexts as well. In addition, we focused on the tasks that presumably require step-by-step reasoning as the CoT process, but as suggested in existing studies (§ 5), the situation may be different if the task is, in the first place, so simple that CoT is no longer needed.

Probing methods Interpreting internal mechanisms of LLMs using probing has been actively conducted in our field (Conneau et al., 2018; Tenney et al., 2019; Campbell et al., 2023; Li et al., 2023); however, there are criticisms regarding the validity of some probing approaches (Liu et al., 2023; Burns et al., 2023). One way to overcome such concerns will be to analyze the generality of obtained results through more diverse methodologies (Gurnee et al., 2023; Bricken et al., 2023).

Causal intervention purity Hidden states carry mixed information (e.g., features of the input text itself). Thus, even though activation patching is standard in prior work, we cannot fully rule out noise introduced by this intervention.

Ethics statement

This paper will not raise particular ethical concerns, considering that (i) no human experiments were conducted, and (ii) our tasks do not involve ethically sensitive topics.

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A Additional analysis of the probing experiments

A.1 Discussion on task-level differences

Equation order. The difference between these two tasks lies in the order of the equations. Level 3 tasks involve forward references, whereas Level 2 tasks do not. An autoregressive language model can attend only to past inputs. Therefore, the model may be able to reach the final answer more quickly on Level 2 tasks than on Level 3 tasks. In practice, Table 2 indicates that the probe predicts the answer during CoT at both levels.Therefore, the impact of equation order (dependency structure) on the model' s internal mechanism is thought to be limited.

Distractors. In the Level 4 task (see Table 1), the variable v_3 (C) is a distractor, that is, $\${v_3}$ is not necessary to derive the final answer. The models can infer this fact from the in context examples. According to Table 2, the $\mathrm{Acc}(v_3)$ in Level 4 was at most 44%, a relatively low accuracy. From this result, we can see that, unlike the variables required to derive the final answer, v_3 is not encoded in a simple form that can be extracted by a linear transformation alone. This suggests the possibility that the model employs an efficient internal mechanism that does not derive variables unnecessary for obtaining the final answer. It is also consistent with the finding that the CoT is faithful to the final answer.

Effect of depth of reasoning. The difference between Level 3 and Level 5 is whether two or three computational steps are required to reach the final answer. Qualitatively, the probe' s prediction accuracy for each variable increases a few tokens before the answer appears in the output text, indicating alignment between the internal state and the output. This finding supports the conclusion that the generated explanations are faithful to the internal state and that this conclusion generalizes across levels.

A.2 Alternative CoT formats.

For comparison, we also ran probing experiments on Qwen2.5-7B under two reasoning chain formats. Specifically, we defined two formats.

The first is the Simple CoT setting, in which the reasoning chain outputs only the intermediate subresults. The second is the Implicit (reasoning) setting, which omits intermediate computation steps in the output text. Table 4 lists example equations.

The experimental results are shown in Table 5 and Figures 54 and 55. Under the Simple CoT setting, the model' s task accuracy was 99.5%, whereas under the Implicit reasoning setting it was 77.8%[5]. As in the default setting, intermediate results are linearly separable in the OUTPUT segment ($t^*_{\mathrm{eq}} > 0$).In the implicit reasoning case, the accuracy was not reached the threshold at all positions, but we observed a slight tendency for $\mathrm{Acc}_{\prec\mathrm{CoT}}(v_i)$ to increase.

A.3 Details on the prompt-format comparison.

As described in § A, we examined whether the format of few-shot prompts affects the model's internal reasoning patterns. We consider two strategies: *1. Same level prompting* and *2. General prompting*. In same level prompting, we provide three equations that match the level of the evaluation task, following the general setting (§ 2). In contrast, under general prompting the model receives 50 randomly generated demonstrations (three equations each).[6] Table 6 reports t^*_{eq} for both strategies. We observe no difference in t^*_{eq}, suggesting that the few-shot prompt format has minimal effect on the model's internal reasoning patterns.

B Supplemental results

B.1 Performance of language models in the arithmetic tasks

Table 7 shows the accuracy of language models on arithmetic reasoning tasks for each experimental setting. We computed the accuracy based on exact matches between the output, including the chain ($\hat{z} \oplus \hat{y}$), and the gold labels ($z \oplus y$). The accuracy for all models is nearly 100%, indicating that they are capable of solving the arithmetic reasoning tasks used in this experiment.

B.2 Additional probing predictions for correct/incorrect instances.

Figure 7 shows the top-1 predictions of the probe for B in instances where Llama3.2-3B produced *incorrect* answers for Task 3. In contrast, Figure 8 illustrates the probe predictions for instances where

[5]Please note that, unlike the other experimental configurations, under the implicit reasoning setting the model is unable to answer the problems with sufficiently high accuracy.

[6]We increased the number of demonstrations in the general prompts because task performance was otherwise insufficient.

Reasoning chain	INPUT	OUTPUT
Simple CoT	$\underline{\mathrm{A} = 1 + \mathrm{B}}_{-3}, \underline{\mathrm{B} = 2 + 3}_{-2}; \underline{\mathrm{A} = ?}_{-1}$	$\underline{\mathrm{B} = 5}_{0}, \underline{\mathrm{A} = 6}_{1}$
Implicit	$\underline{\mathrm{A} = 1 + \mathrm{B}}_{-3}, \underline{\mathrm{B} = 2 + 3}_{-2}; \underline{\mathrm{A} = ?}_{-1}$	$\underline{\mathrm{A} = 6}_{0}$

Table 4: Examples of arithmetic reasoning tasks used for reasoning chain format comparison. This position is used as a reference point for calculating t^*_{eq} in § 3.2.

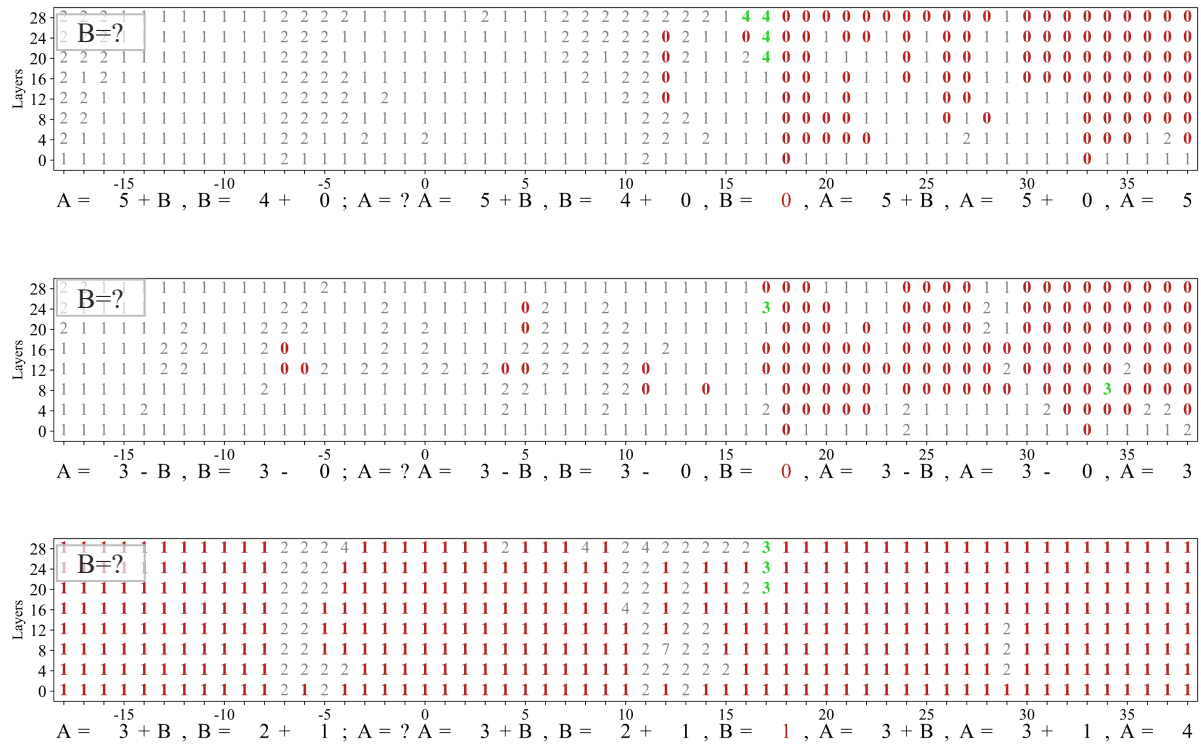

Figure 7: Error analysis of cases where Llama3.2-3B generated *incorrect* answers. The vertical axis represents the index of the transformer layer. The horizontal axis represents the tokens input to the model over time. The numbers in the figure indicate the labels predicted by each probe (top-1) at each layer and time step. The numbers highlighted in green represent the gold labels for the predictions, while those highlighted in red denote the values incorrectly generated by the model.

	Variable		When (↓)		Acc (↑)	
Setting	Variable	#Step	t^*_{eq}	$t^\dagger_{\text{eq}}$	$\prec$ CoT	$\succ$ CoT
Simple CoT	v_1	2	1	5	34.8	99.7
	v_2	1	0	1	69.7	1.0
Implicit	v_1	2	N/A	N/A	40.8	80.7
	v_2	1	N/A	N/A	69.1	79.2

Table 5: Probing evaluation results with different reasoning chains. Each column is the same as Table 2.

	Variable		When (↓)		Acc (↑)	
Setting	Variable	#Step	t^*_{eq}	$t^\dagger_{\text{eq}}$	$\prec$ CoT	$\succ$ CoT
Same level	v_1	2	5	-3	17.2	100
	v_2	1	2	-3	47.7	100
	v_3	1	N/A	-2	43.7	23.7
General	v_1	2	5	-3	18.1	94.7
	v_2	1	2	-3	49.4	100
	v_3	1	N/A	-2	43.3	28.1

Table 6: Evaluation results of same level prompting and general prompting. Each column is the same as Table 2.

Llama3.2-3B generates correct responses. In Figure 3 the model should output (B=4) but instead produces 0. We observe the same pattern in other examples in Figure 7. One possible cause is that digits for computed results (e.g., 4) and digits from the input (e.g., 0) are represented in neighboring subspaces, which may cause confusion; we leave a detailed analysis for future work.

B.3 All probing results

Figures 9 through 53 present the probing results for all models and tasks discussed in this paper. For

Model	Level	Task
Qwen2.5 (7B)	1	100
	2	100
	3	100
	4	100
	5	100
Qwen2.5 (14B)	3	100
Qwen2.5 (32B)	3	100
Qwen2.5-Math (7B)	3	100
Yi1.5 (9B)	3	100
Yi1.5 (34B)	3	100
Llama3.1 (8B)	3	100
Llama3.2 (3B)	3	97.6
Mistral-Nemo (12B)	3	99.6

Table 7: The performance of language models on the arithmetic reasoning tasks. The Task column shows the accuracy for the evaluation set (exact match).

each figure, the input sequence below the graphs is one example (the results are averaged over the test set). The upper part indicates the maximum probing accuracy achieved at each token position t. The bottom part shows the probing accuracies in each token t, layer l, and variable v_i. Tables 9 through 23 summarize these results for thresholds (τ) ranging from 0.85 to 0.95.

From these results, we observe trends similar to those described in § 3.3 across many settings. However, for the smaller model Llama3.2 (3B), increasing the threshold τ often leads to cases where the accuracy does not reach the threshold (N/A). Nonetheless, a consistent pattern remains: $\mathrm{Acc}_{\prec \mathrm{CoT}}(v_i)$ is low whereas $\mathrm{Acc}_{\succ \mathrm{CoT}}(v_i)$ is high.

B.4 Additional causal intervention results

Figures 56- 108 show the causal intervention results for the target tokens z_{17} and z_{32}, respectively.

Here, in addition to the Success rate, we also present the *Unchanged rate* as a metric. The Unchanged rate indicates how frequently (%) the intervened output $\hat{\tilde{y}}$ remains the same as y. If this value is small, it indicates that the patched hidden states do not affect the output.

C Hyperparameters

Table 8 shows the hyperparameters used for training the probes.

D Computational resources

We used NVIDIA A100 GPUs (40GB and 80GB memory) and NVIDIA H100 GPUs to conduct this study.

Train instances	10,000
Optimizer	SGD (Robbins, 1951)
Learning rate	1.0×10^{-3} (constant)
Batch size	10,000
Epochs	10,000

Table 8: Hyperparameters for training the probe

E Usage of AI assistants

For writing this paper and the source code for the experiments, we use AI assistants (e.g., ChatGPT, GitHub Copilot). However, the use is limited to purposes such as code completion, translation, text editing, and table creation, and all content is solely based on the authors' ideas.

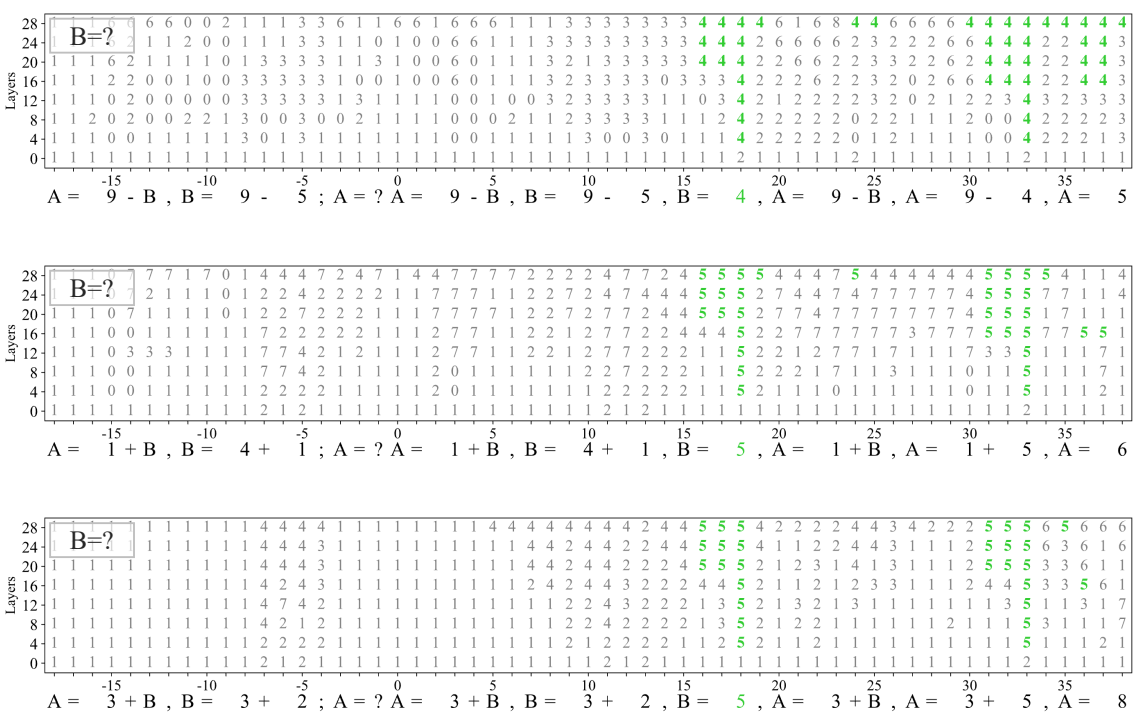

Figure 8: Top 1 predictions of probes where Llama3.2-3B generated *correct* answers. The vertical axis represents the index of the transformer layer. The horizontal axis represents the tokens input to the model over time. The numbers in the figure indicate the labels predicted by each probe (top-1) at each layer and time step. The numbers highlighted in green represent the gold labels for the predictions.

Model	Variable	When (↓) t^*_{eq}	When (↓) t^*	Acc (↑) $\prec$ CoT	Acc (↑) $\succ$ CoT
Qwen2.5 (7B)	v_1 (A)	4	27	35.8	100
(Qwen Team, 2024)	v_2 (B)	**−2**	**−5**	100	100
Qwen2.5 (14B)	v_1 (A)	4	27	36.9	100
(Qwen Team, 2024)	v_2 (B)	**−2**	**−5**	100	100
Qwen2.5 (32B)	v_1 (A)	4	28	30.5	100
(Qwen Team, 2024)	v_2 (B)	**−2**	**−5**	100	100
Qwen2.5-Math (7B)	v_1 (A)	4	27	41.8	100
(Yang et al., 2024a)	v_2 (B)	**−2**	**−5**	100	100
Yi1.5 (9B)	v_1 (A)	4	32	28.1	100
(Young et al., 2024)	v_2 (B)	**−2**	**−5**	100	100
Yi1.5 (34B)	v_1 (A)	4	31	22.9	100
(Young et al., 2024)	v_2 (B)	**−2**	**−5**	100	100
Llama3.1 (8B)	v_1 (A)	4	27	20.6	100
(Dubey et al., 2024)	v_2 (B)	**−2**	**−5**	100	100
Llama3.2 (3B)	v_1 (A)	4	28	21.8	100.0
(Dubey et al., 2024)	v_2 (B)	**−2**	**−5**	100	100
Mistral-Nemo (12B)	v_1 (A)	4	27	18.0	100
(Mistral AI Team, 2024)	v_2 (B)	**−2**	**−5**	100	100

Table 9: Results for various models on the Level 1 task ($\tau = 0.85$).

Model	Variable	When (↓) t^*_{eq}	When (↓) t^*	Acc (↑) $\prec$ CoT	Acc (↑) $\succ$ CoT
Qwen2.5 (7B)	v_1 (A)	2	16	49.2	100
(Qwen Team, 2024)	v_2 (B)	5	35	21.2	100
Qwen2.5 (14B)	v_1 (A)	2	15	48.8	100
(Qwen Team, 2024)	v_2 (B)	5	36	21.5	100
Qwen2.5 (32B)	v_1 (A)	2	15	66.4	100
(Qwen Team, 2024)	v_2 (B)	5	36	21.3	100
Qwen2.5-Math (7B)	v_1 (A)	2	15	53.7	100
(Yang et al., 2024a)	v_2 (B)	5	35	22.1	100
Yi1.5 (9B)	v_1 (A)	2	18	40.2	100
(Young et al., 2024)	v_2 (B)	5	41	17.8	100
Yi1.5 (34B)	v_1 (A)	2	18	35.6	100
(Young et al., 2024)	v_2 (B)	5	41	18.3	100
Llama3.1 (8B)	v_1 (A)	2	15	31.9	100
(Dubey et al., 2024)	v_2 (B)	5	35	17.8	100
Llama3.2 (3B)	v_1 (A)	2	16	36.2	99.9
(Dubey et al., 2024)	v_2 (B)	5	36	17.8	99.9
Mistral-Nemo (12B)	v_1 (A)	2	16	30.8	100
(Mistral AI Team, 2024)	v_2 (B)	5	36	17.8	100

Table 10: Results for various models on the Level 2 task ($\tau = 0.85$).

Model	Variable	When (↓) t^*_{eq}	When (↓) t^*	Acc (↑) $\prec$ CoT	Acc (↑) $\succ$ CoT
Qwen2.5 (7B)	v_1 (A)	5	35	17.9	100
(Qwen Team, 2024)	v_2 (B)	2	16	50.5	100
Qwen2.5 (14B)	v_1 (A)	5	35	17.8	100
(Qwen Team, 2024)	v_2 (B)	2	15	50.5	100
Qwen2.5 (32B)	v_1 (A)	5	36	17.8	100
(Qwen Team, 2024)	v_2 (B)	2	15	67.4	100
Qwen2.5-Math (7B)	v_1 (A)	5	35	18.6	100
(Yang et al., 2024a)	v_2 (B)	2	15	56.1	100
Yi1.5 (9B)	v_1 (A)	5	41	17.8	100
(Young et al., 2024)	v_2 (B)	2	18	36.9	100
Yi1.5 (34B)	v_1 (A)	5	41	22.4	100
(Young et al., 2024)	v_2 (B)	2	18	37.4	100
Llama3.1 (8B)	v_1 (A)	5	35	26.0	100
(Dubey et al., 2024)	v_2 (B)	2	16	29.6	100
Llama3.2 (3B)	v_1 (A)	5	36	17.8	93.2
(Dubey et al., 2024)	v_2 (B)	2	16	33.2	95.4
Mistral-Nemo (12B)	v_1 (A)	5	36	17.8	100
(Mistral AI Team, 2024)	v_2 (B)	2	16	28.9	100

Table 11: Results for various models on the Level 3 task ($\tau = 0.85$).

Model	Variable	When (↓) t^*_{eq}	When (↓) t^*	Acc (↑) $\prec$ CoT	Acc (↑) $\succ$ CoT
Qwen2.5 (7B)	v_1 (A)	4	29	30.4	100
(Qwen Team, 2024)	v_2 (B)	2	15	27.2	100
	v_3 (C)	N/A	N/A	18.5	17.6
Qwen2.5 (14B)	v_1 (A)	5	35	18.9	100
(Qwen Team, 2024)	v_2 (B)	2	15	44.3	100
	v_3 (C)	N/A	N/A	40.4	26.8
Qwen2.5 (32B)	v_1 (A)	5	36	17.4	100
(Qwen Team, 2024)	v_2 (B)	2	15	62.8	100
	v_3 (C)	N/A	N/A	64.4	32.6
Qwen2.5-Math (7B)	v_1 (A)	5	35	17.2	100
(Yang et al., 2024a)	v_2 (B)	2	15	55.6	100
	v_3 (C)	N/A	N/A	47.8	29.4
Yi1.5 (9B)	v_1 (A)	5	41	17.8	100
(Young et al., 2024)	v_2 (B)	2	18	43.5	100
	v_3 (C)	N/A	N/A	36.7	21.2
Yi1.5 (34B)	v_1 (A)	5	40	19.3	100
(Young et al., 2024)	v_2 (B)	2	18	40.8	100
	v_3 (C)	N/A	N/A	27.9	26.2
Llama3.1 (8B)	v_1 (A)	4	29	30.4	100
(Dubey et al., 2024)	v_2 (B)	2	15	27.2	100
	v_3 (C)	N/A	N/A	18.5	17.6
Llama3.2 (3B)	v_1 (A)	5	36	26.2	91.7
(Dubey et al., 2024)	v_2 (B)	2	16	29.1	98.7
	v_3 (C)	N/A	N/A	18.3	17.3
Mistral-Nemo (12B)	v_1 (A)	5	36	17.2	100
(Mistral AI Team, 2024)	v_2 (B)	2	16	29.9	100
	v_3 (C)	N/A	N/A	22.0	19.8

Table 12: Results for various models on the Level 4 task ($\tau = 0.85$).

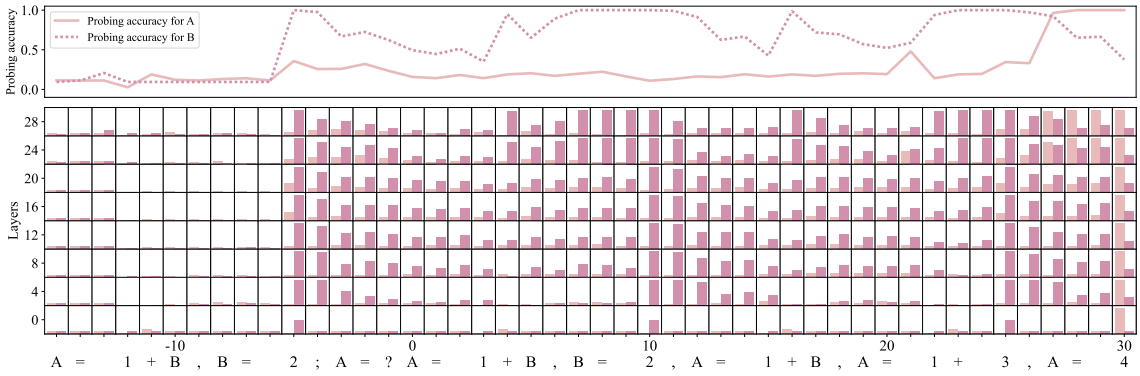

Figure 9: Probing results when Qwen2.5-7B solves Level 1.

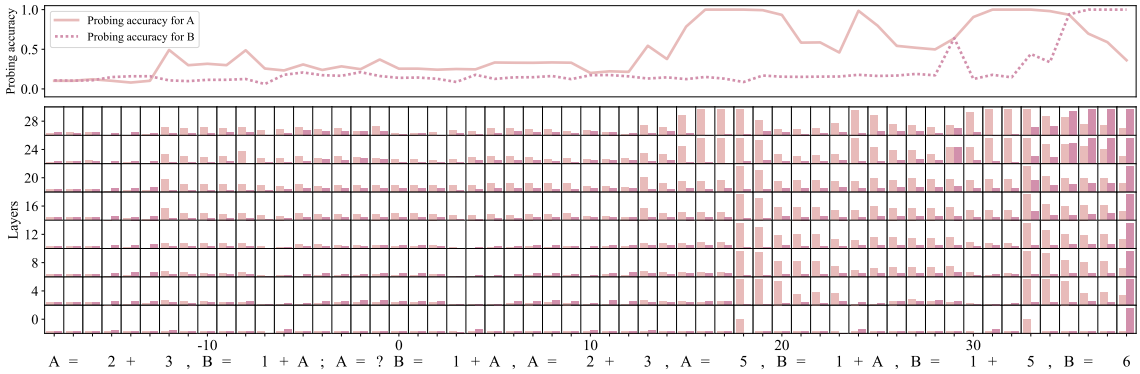

Figure 10: Probing results when Qwen2.5-7B solves Level 2.

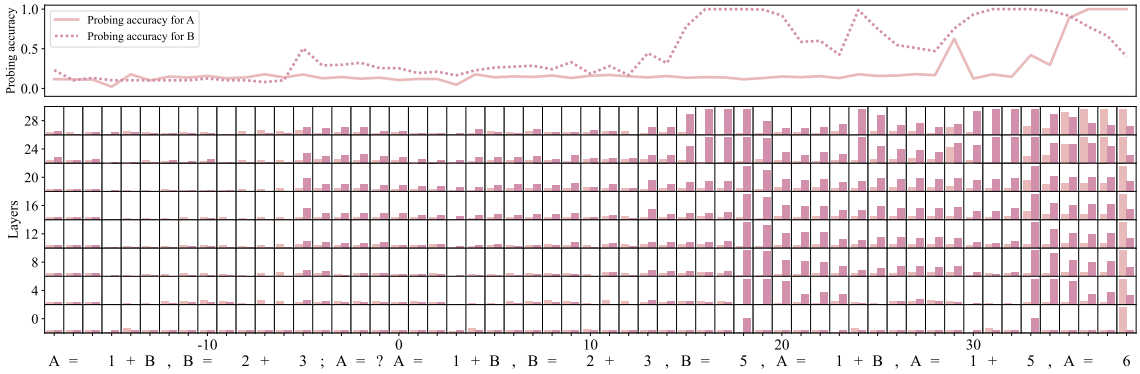

Figure 11: Probing results when Qwen2.5-7B solves Level 3.

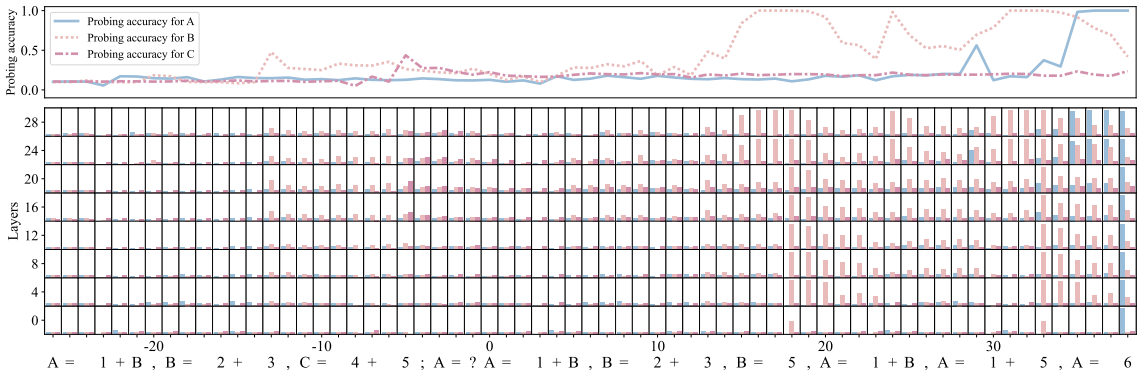

Figure 12: Probing results when Qwen2.5-7B solves Level 4.

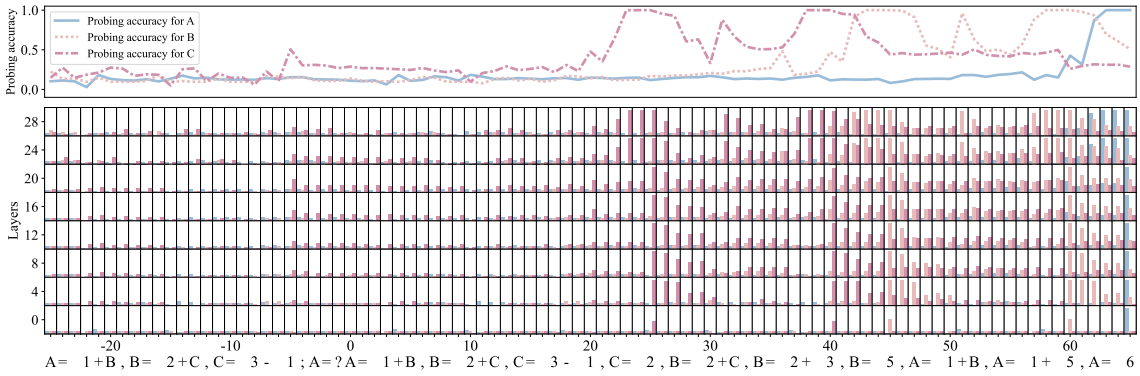

Figure 13: Probing results when Qwen2.5-7B solves Level 5.

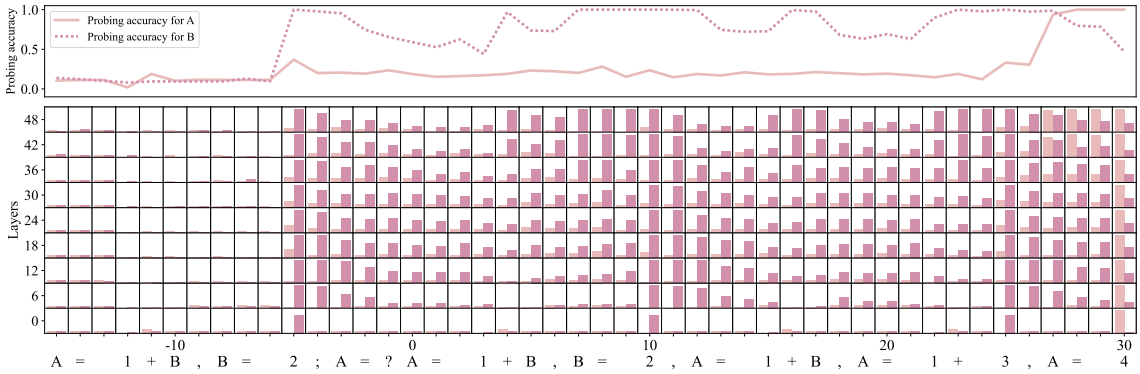

Figure 14: Probing results when Qwen2.5-14B solves Level 1.

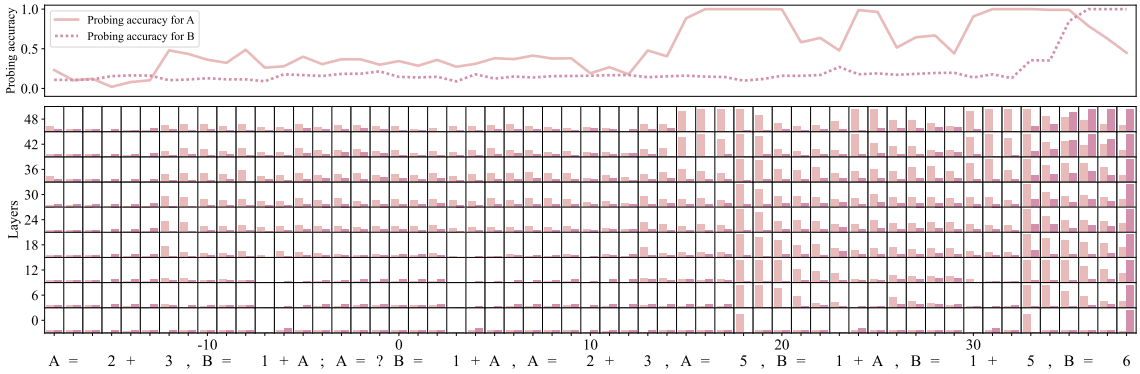

Figure 15: Probing results when Qwen2.5-14B solves Level 2.

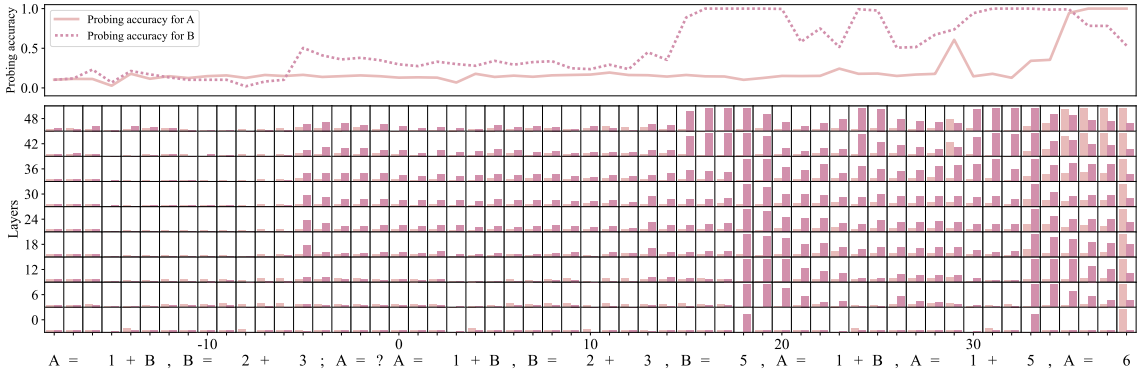

Figure 16: Probing results when Qwen2.5-14B solves Level 3.

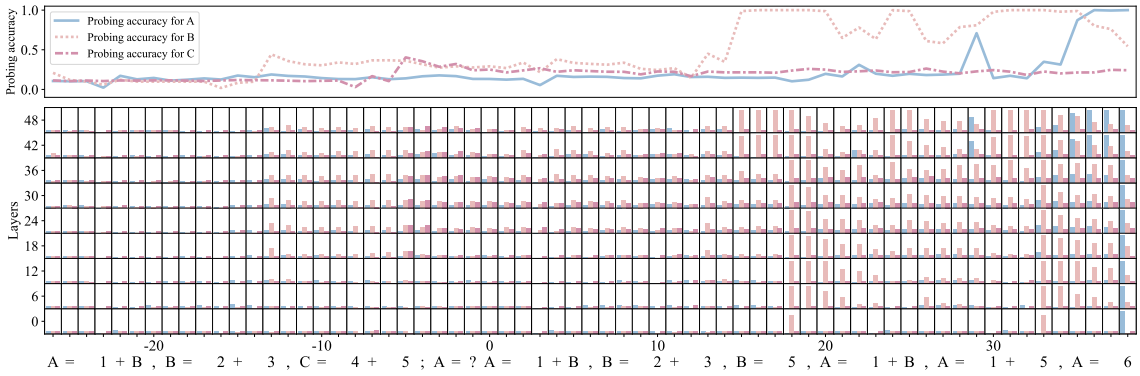

Figure 17: Probing results when Qwen2.5-14B solves Level 4.

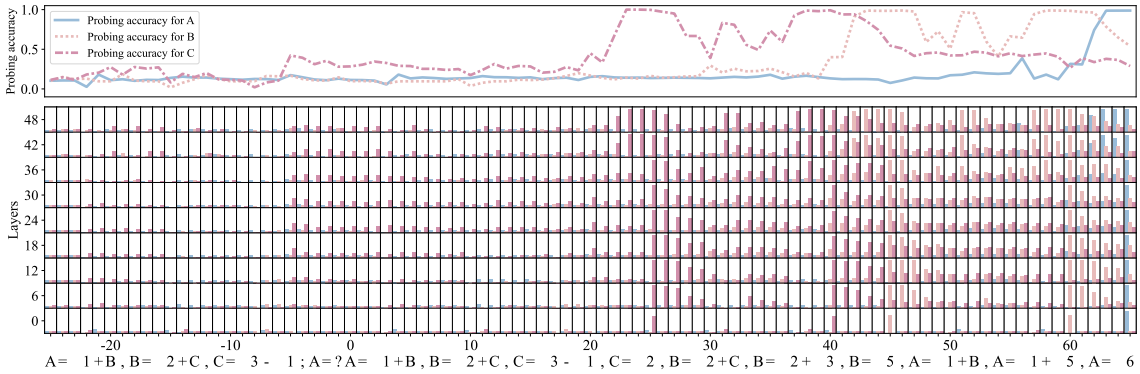

Figure 18: Probing results when Qwen2.5-14B solves Level 5.

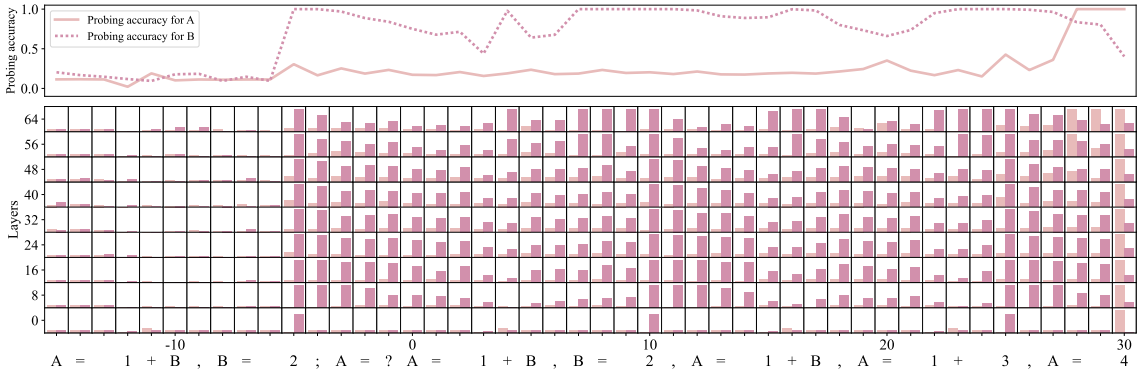

Figure 19: Probing results when Qwen2.5-32B solves Level 1.

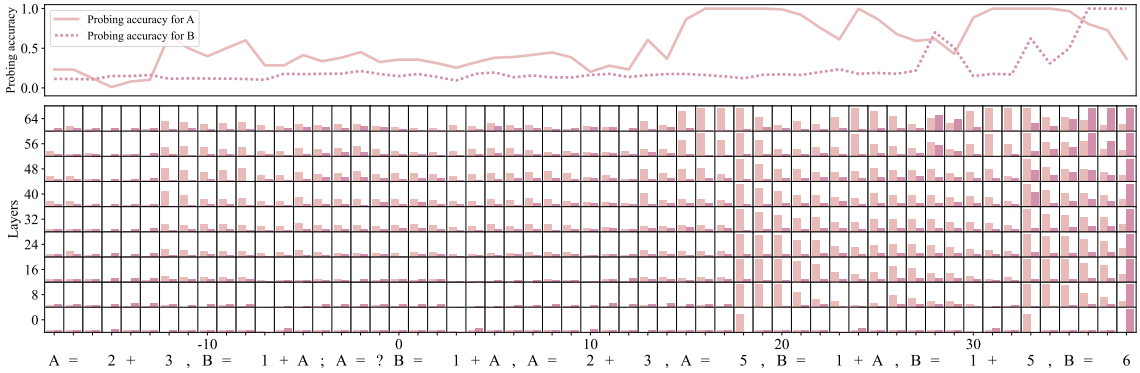

Figure 20: Probing results when Qwen2.5-32B solves Level 2.

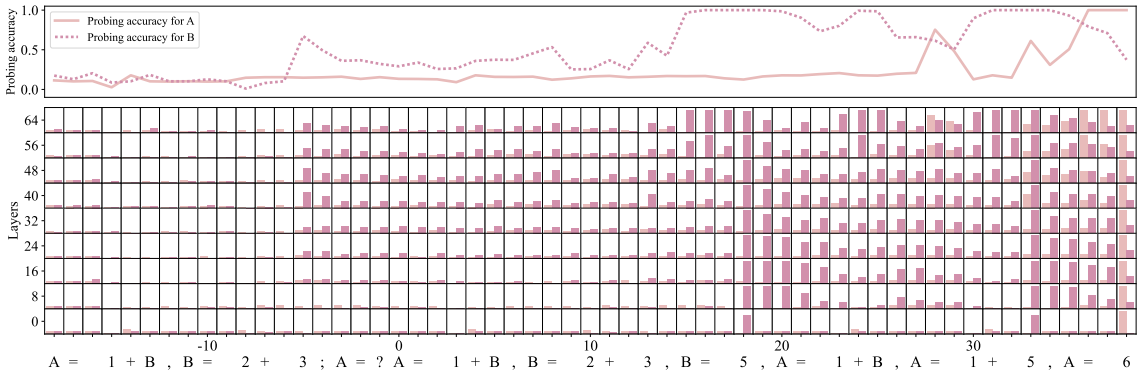

Figure 21: Probing results when Qwen2.5-32B solves Level 3.

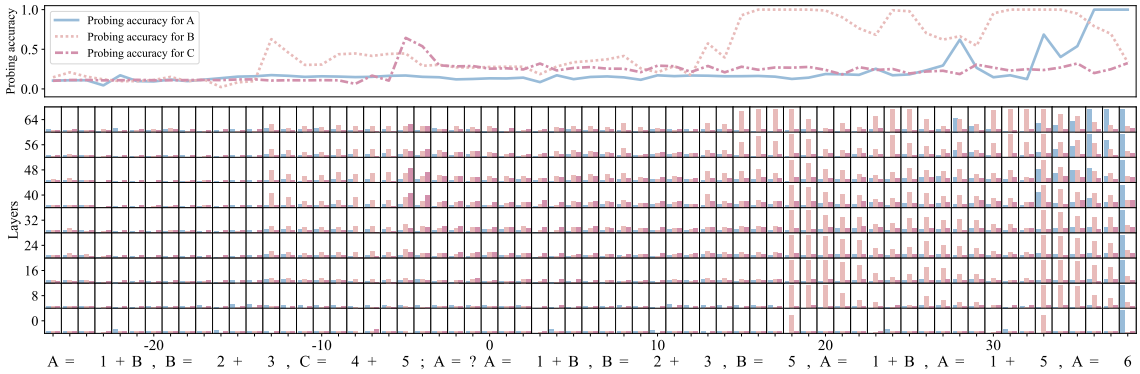

Figure 22: Probing results when Qwen2.5-32B solves Level 4.

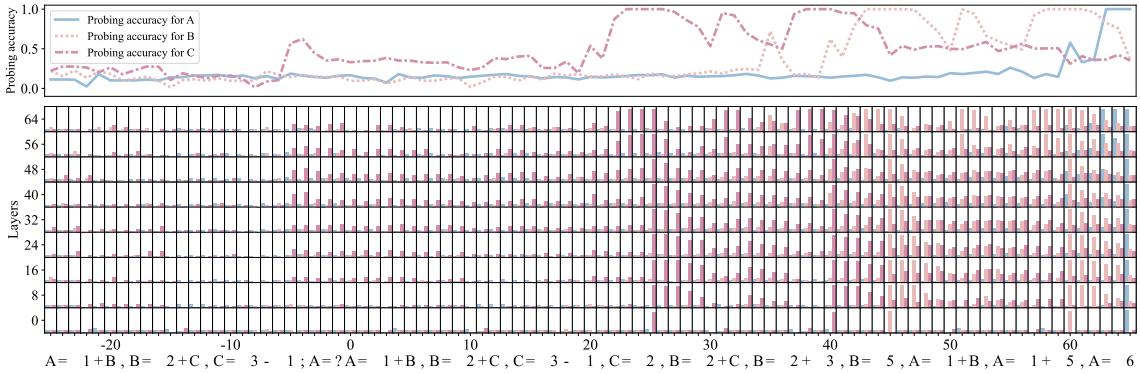

Figure 23: Probing results when Qwen2.5-32B solves Level 5.

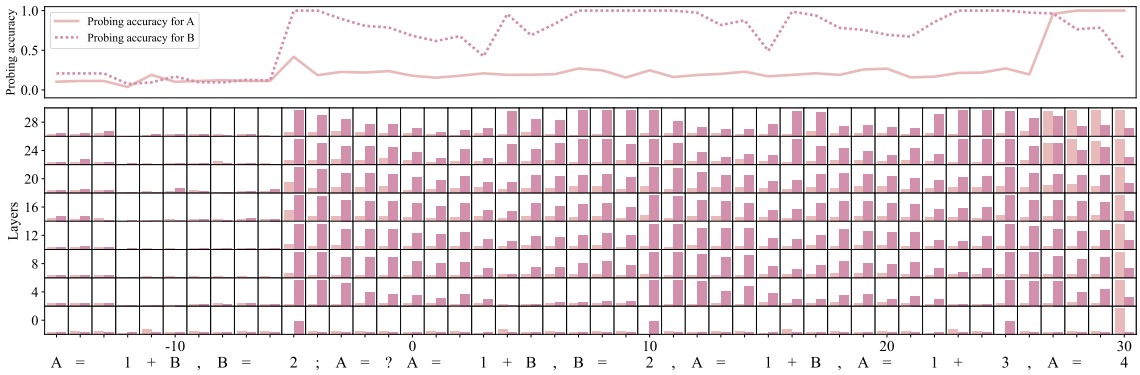

Figure 24: Probing results when Qwen2.5-Math-7B solves Level 1.

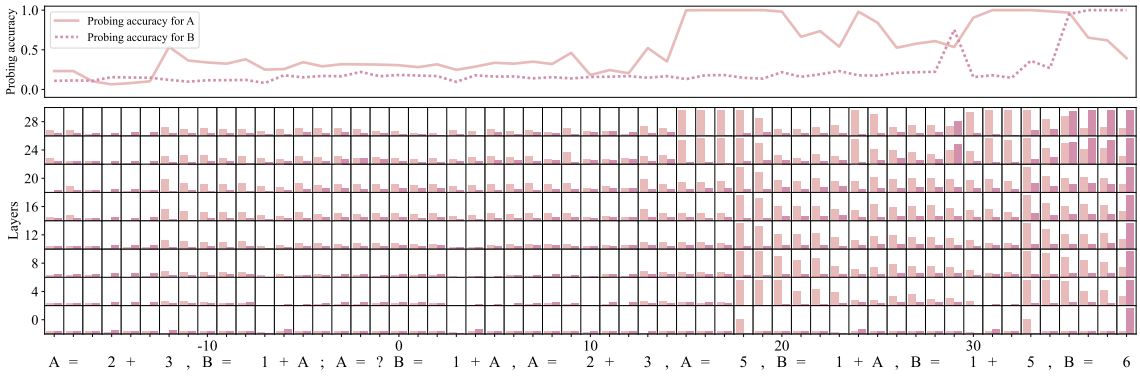

Figure 25: Probing results when Qwen2.5-Math-7B solves Level 2.

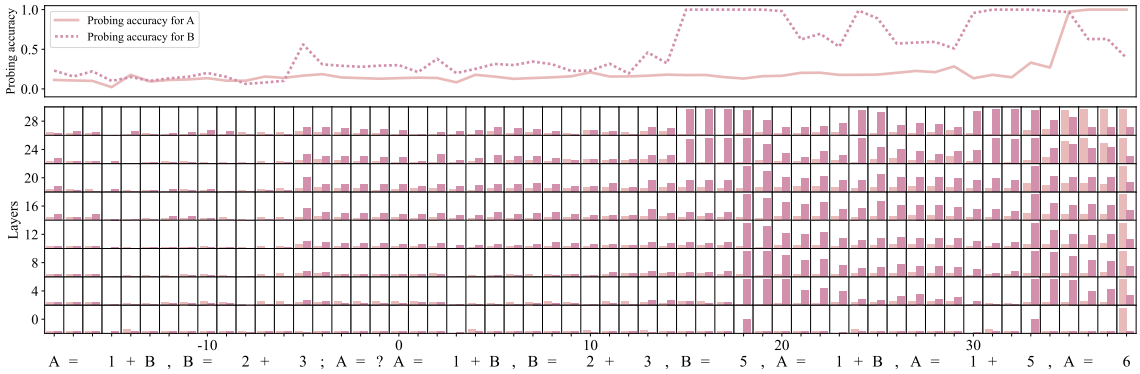

Figure 26: Probing results when Qwen2.5-Math-7B solves Level 3.

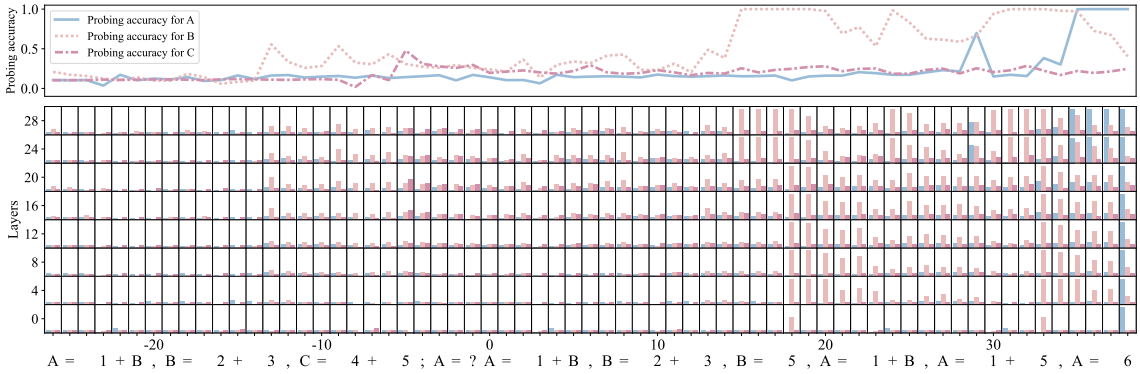

Figure 27: Probing results when Qwen2.5-Math-7B solves Level 4.

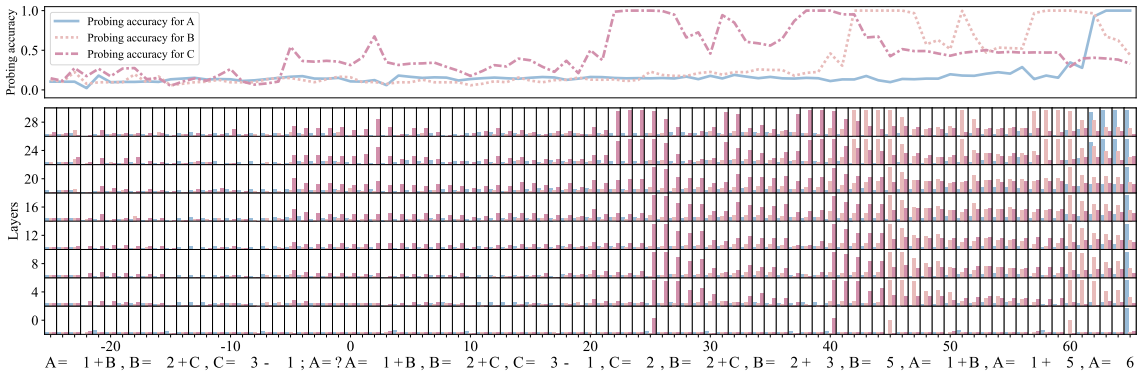

Figure 28: Probing results when Qwen2.5-Math-7B solves Level 5.

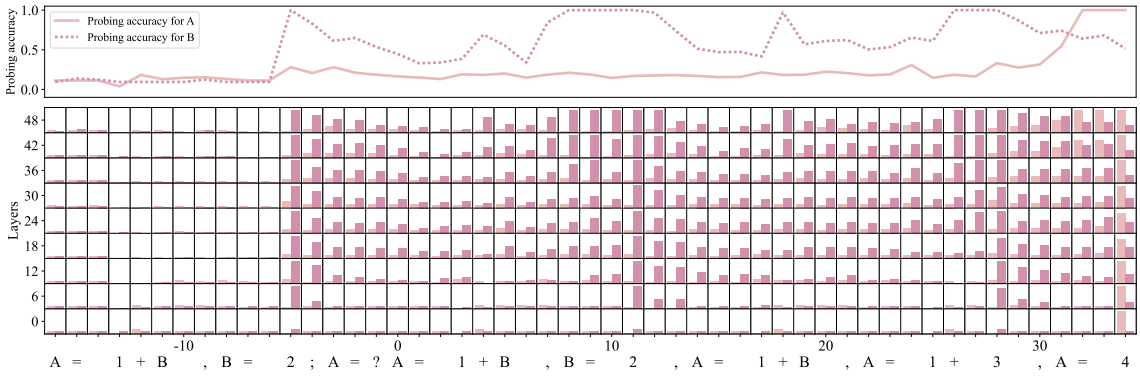

Figure 29: Probing results when Yi-1.5-9B solves Level 1.

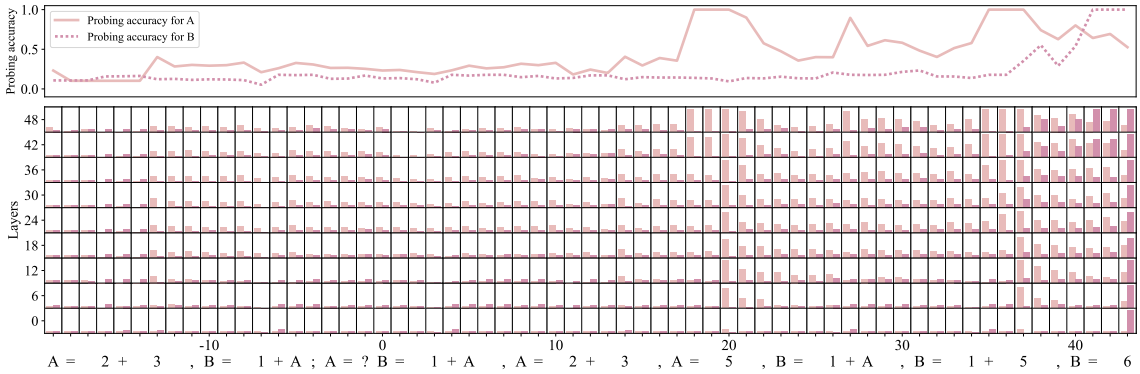

Figure 30: Probing results when Yi-1.5-9B solves Level 2.

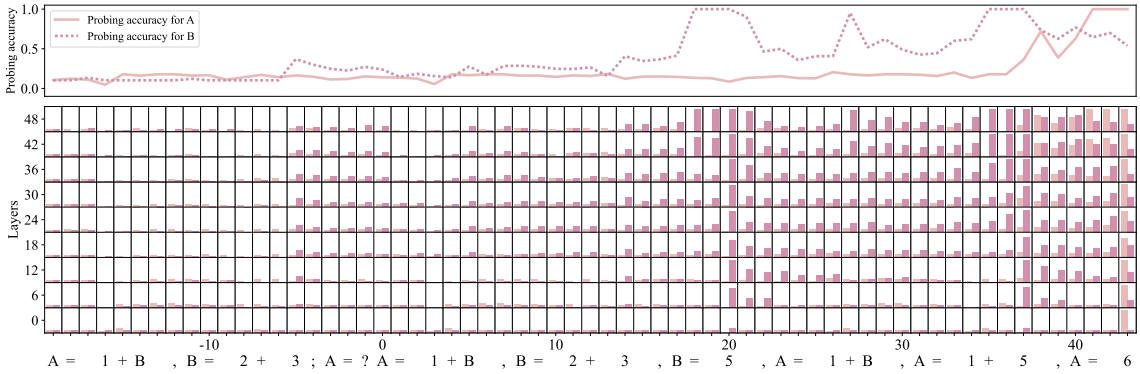

Figure 31: Probing results when Yi-1.5-9B solves Level 3.

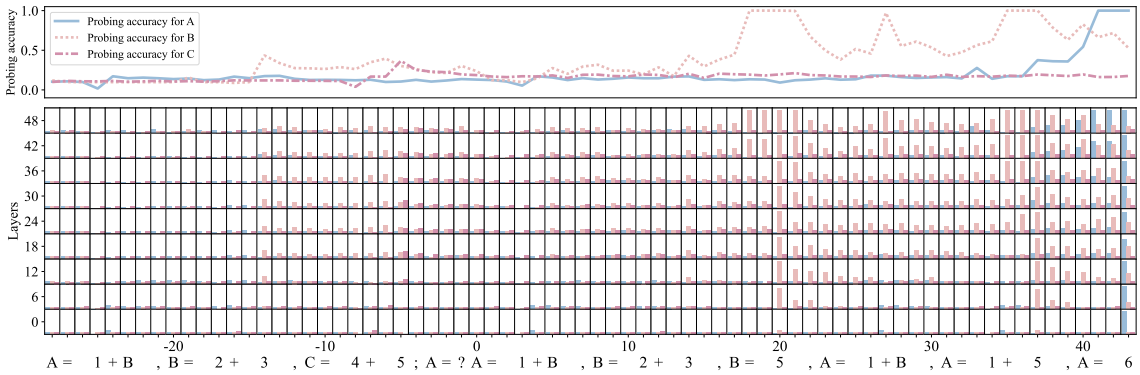

Figure 32: Probing results when Yi-1.5-9B solves Level 4.

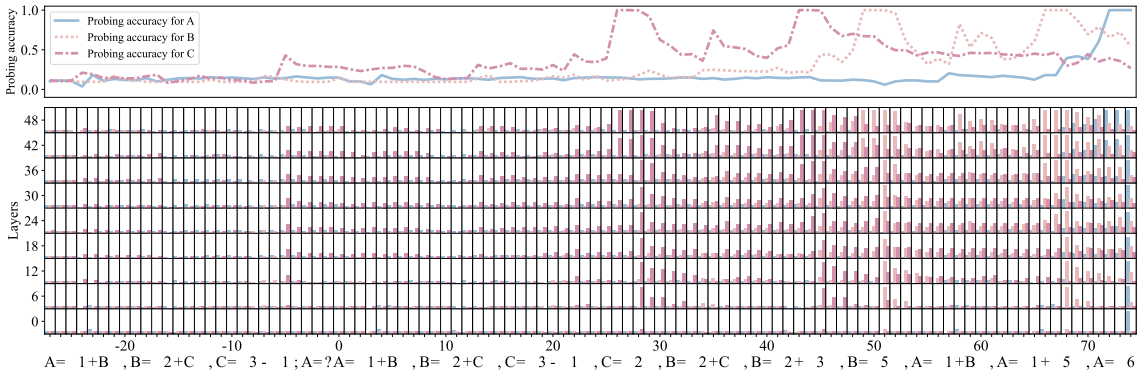

Figure 33: Probing results when Yi-1.5-9B solves Level 5.

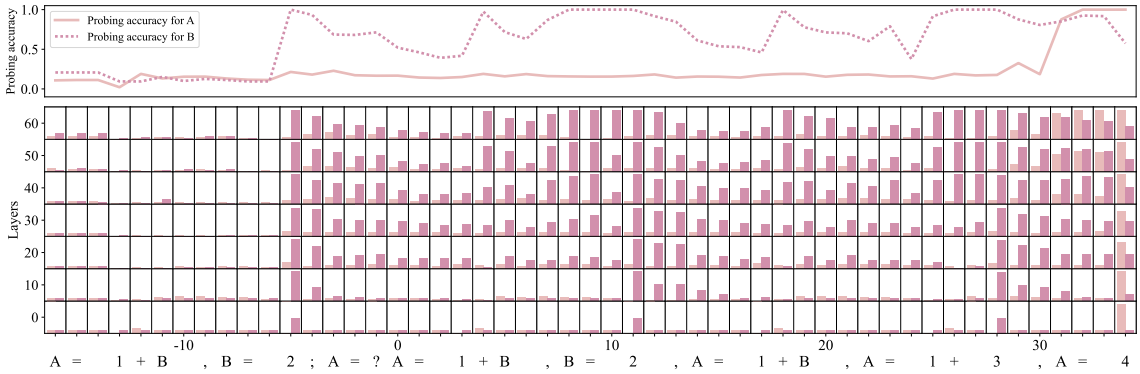

Figure 34: Probing results when Yi-1.5-34B solves Level 1.

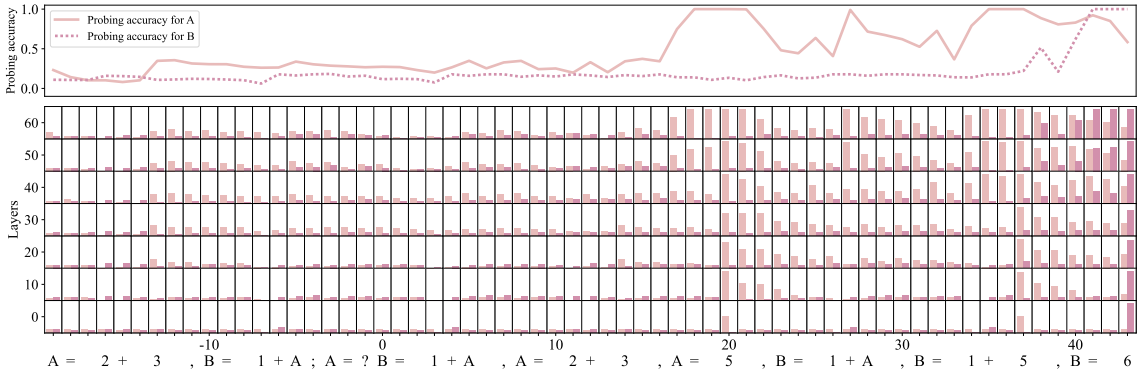

Figure 35: Probing results when Yi-1.5-34B solves Level 2.

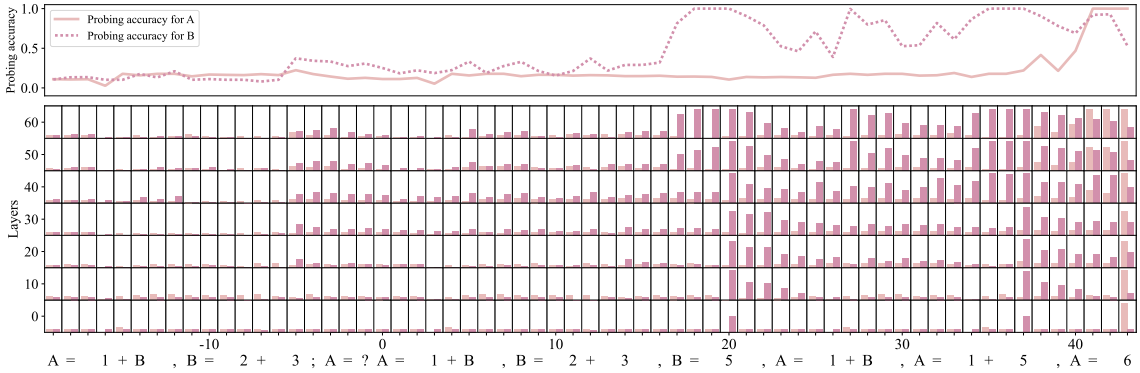

Figure 36: Probing results when Yi-1.5-34B solves Level 3.

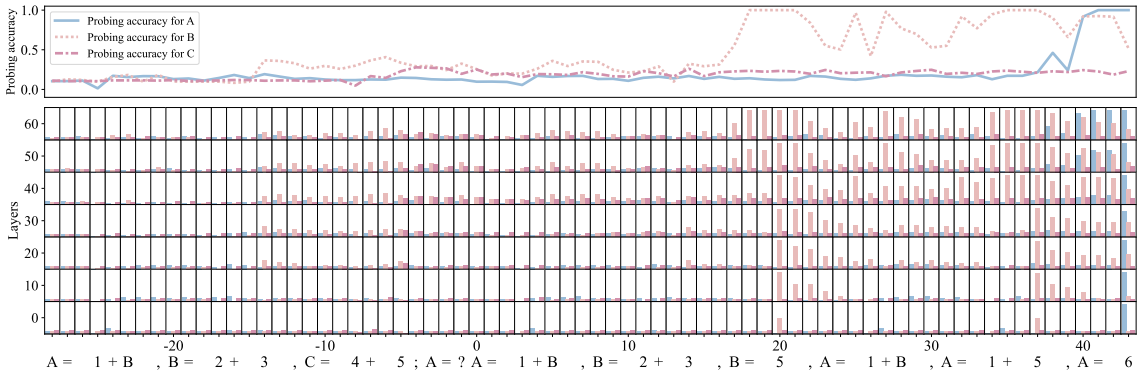

Figure 37: Probing results when Yi-1.5-34B solves Level 4.

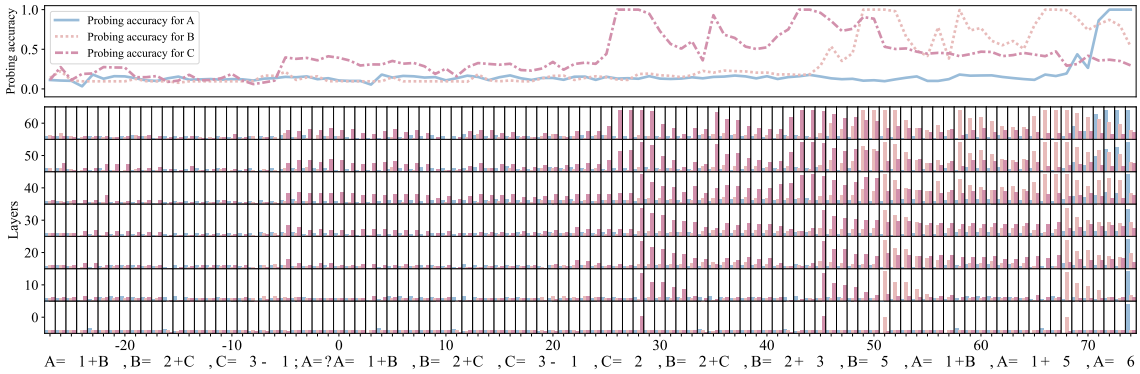

Figure 38: Probing results when Yi-1.5-34B solves Level 5.

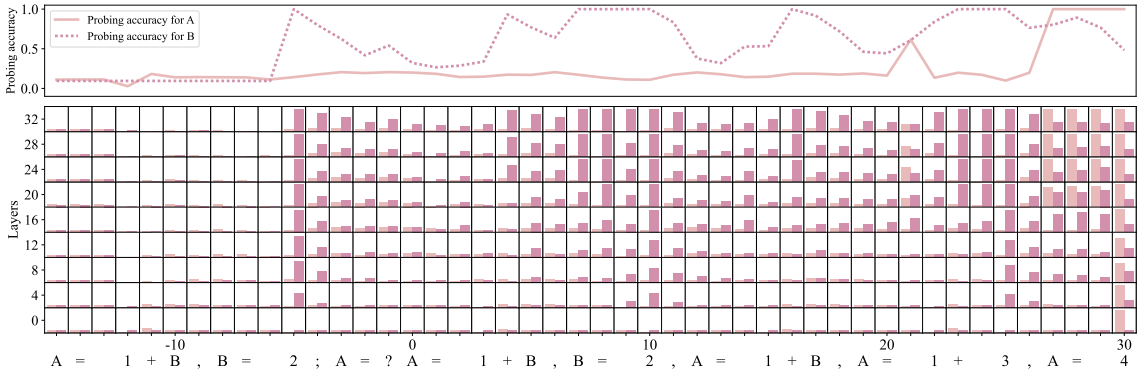

Figure 39: Probing results when Llama-3.1-8B solves Level 1.

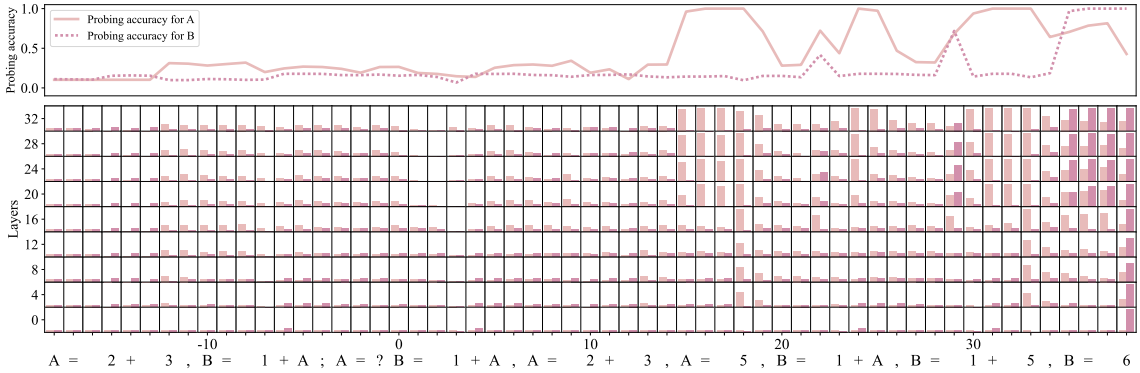

Figure 40: Probing results when Llama-3.1-8B solves Level 2.

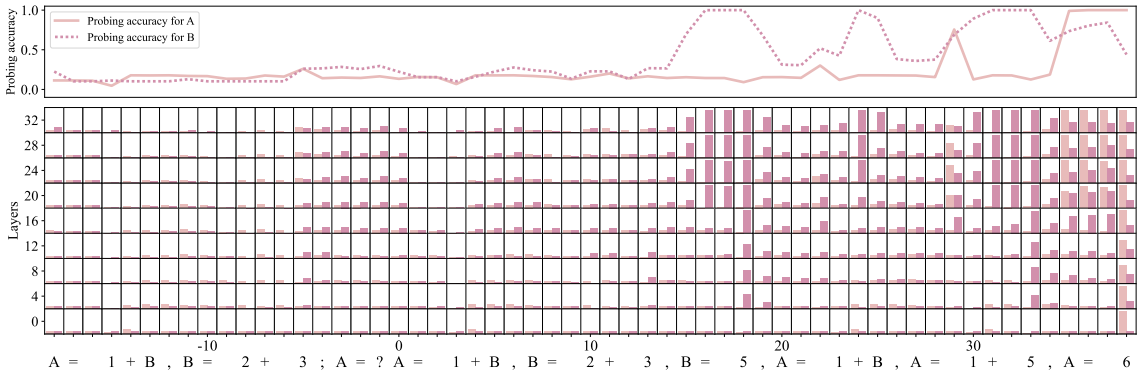

Figure 41: Probing results when Llama-3.1-8B solves Level 3.

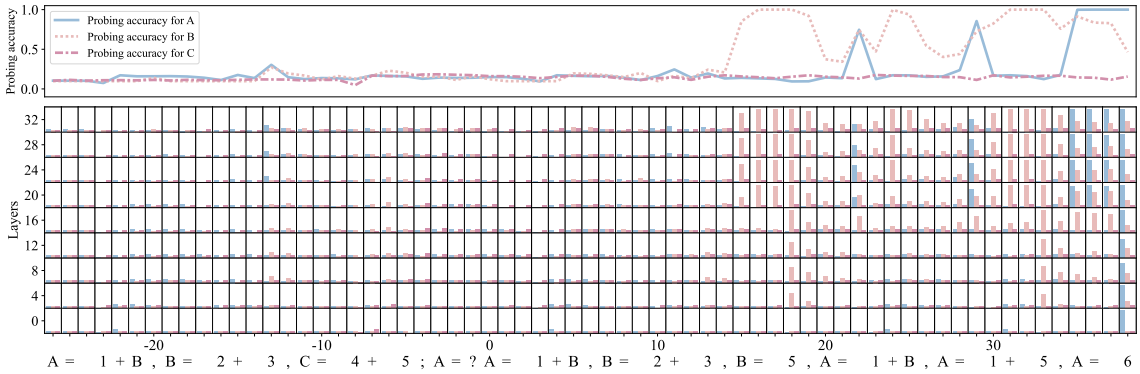

Figure 42: Probing results when Llama-3.1-8B solves Level 4.

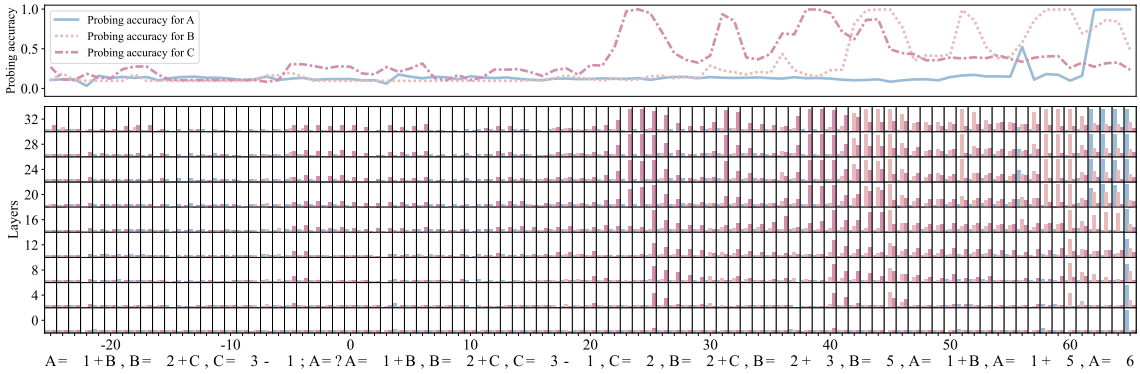

Figure 43: Probing results when Llama-3.1-8B solves Level 5.

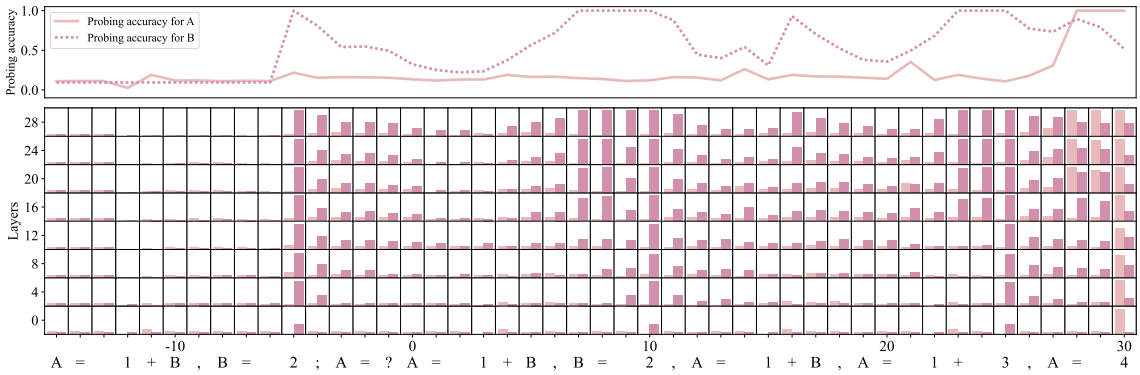

Figure 44: Probing results when Llama-3.2-3B solves Level 1.

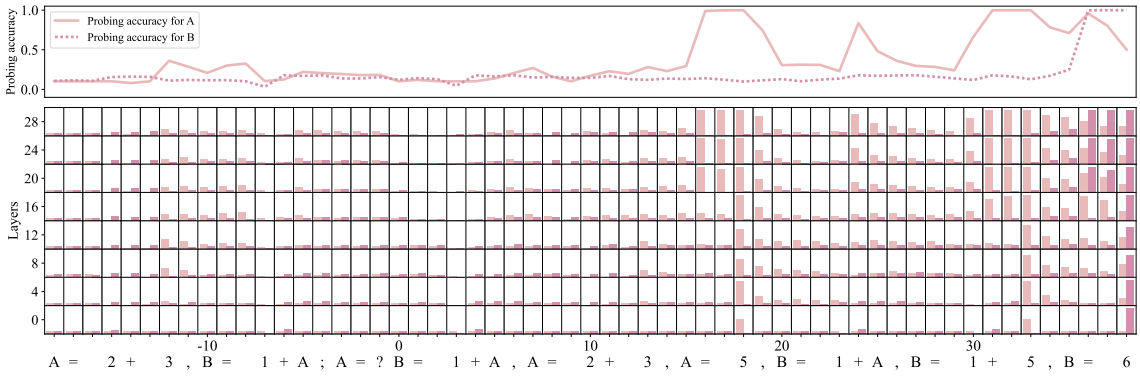

Figure 45: Probing results when Llama-3.2-3B solves Level 2.

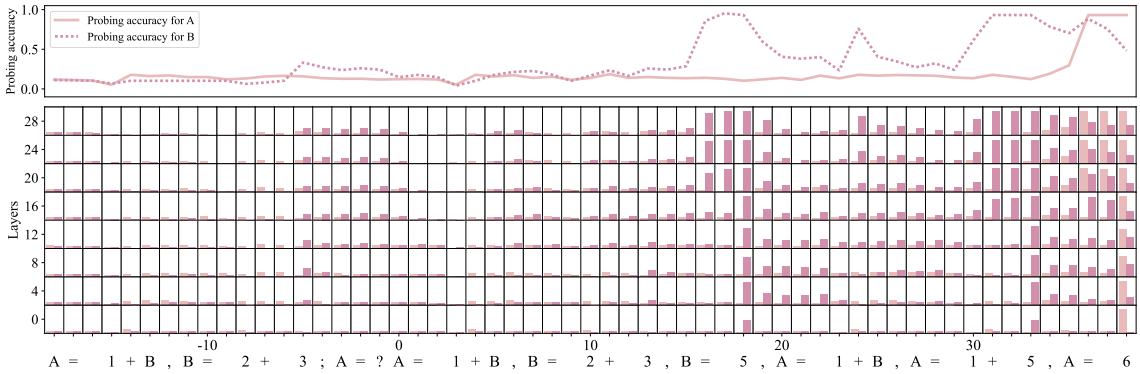

Figure 46: Probing results when Llama-3.2-3B solves Level 3.

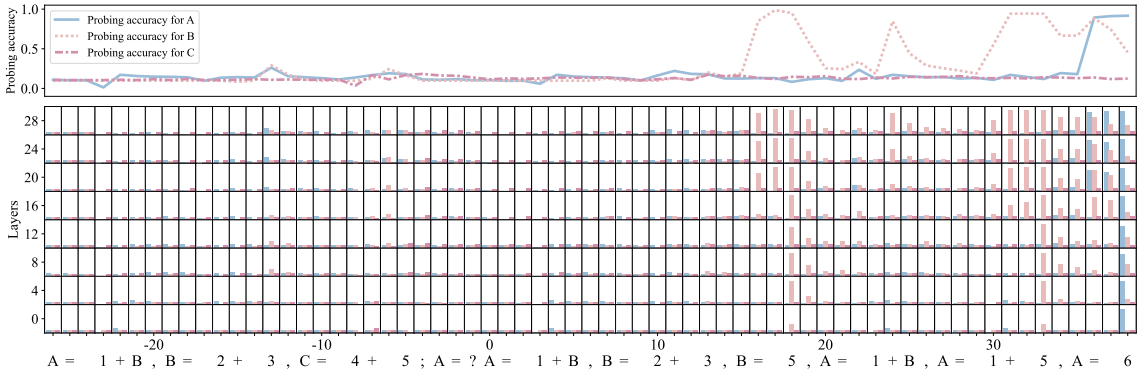

Figure 47: Probing results when Llama-3.2-3B solves Level 4.

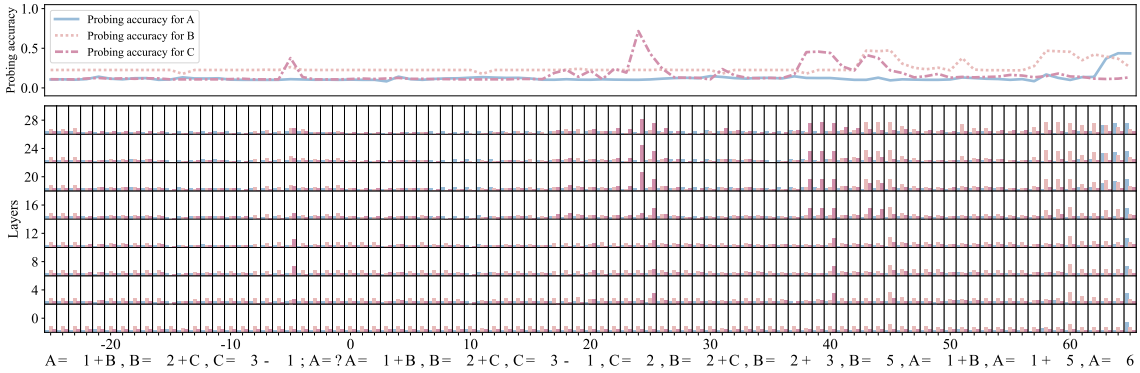

Figure 48: Probing results when Llama-3.2-3B solves Level 5.

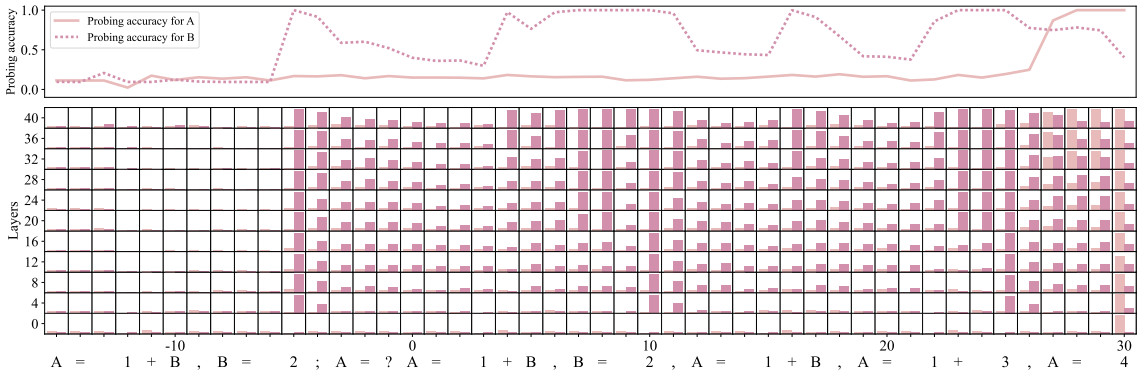

Figure 49: Probing results when Mistral-Nemo-Base-2407 solves Level 1.

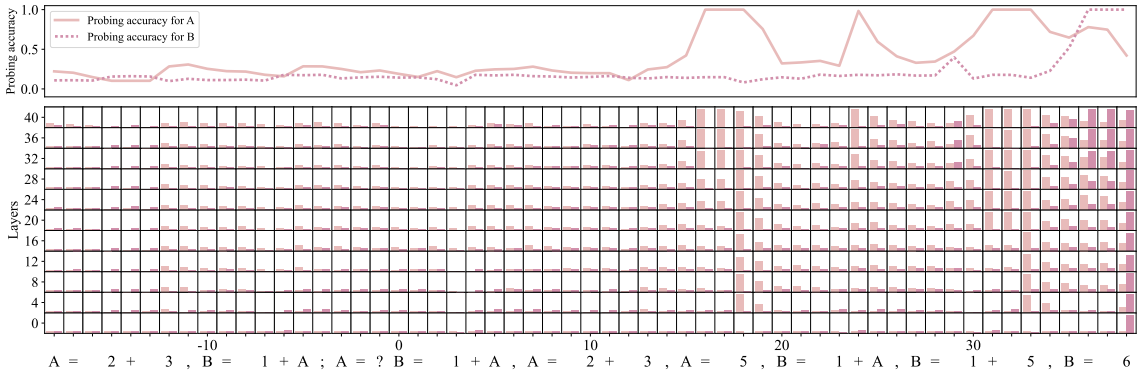

Figure 50: Probing results when Mistral-Nemo-Base-2407 solves Level 2.

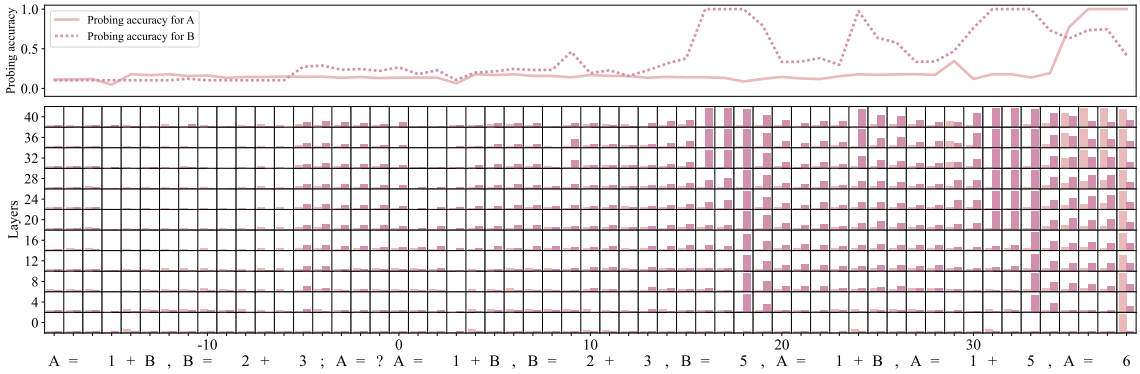

Figure 51: Probing results when Mistral-Nemo-Base-2407 solves Level 3.

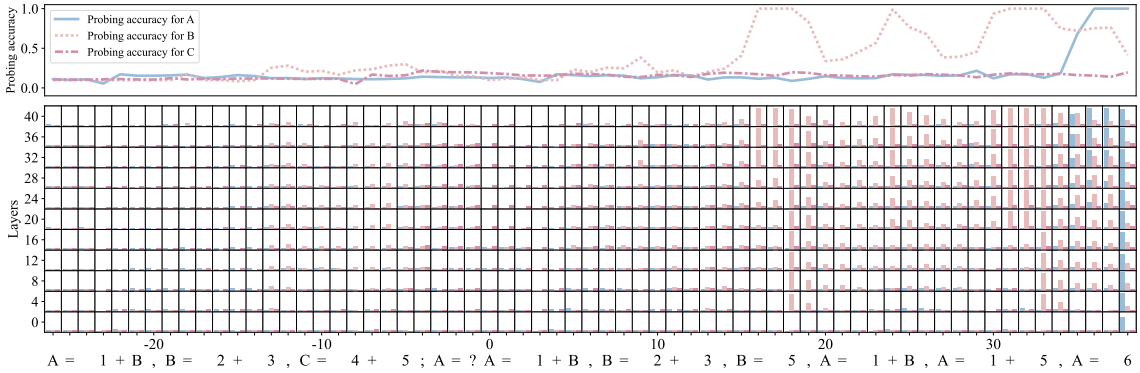

Figure 52: Probing results when Mistral-Nemo-Base-2407 solves Level 4.

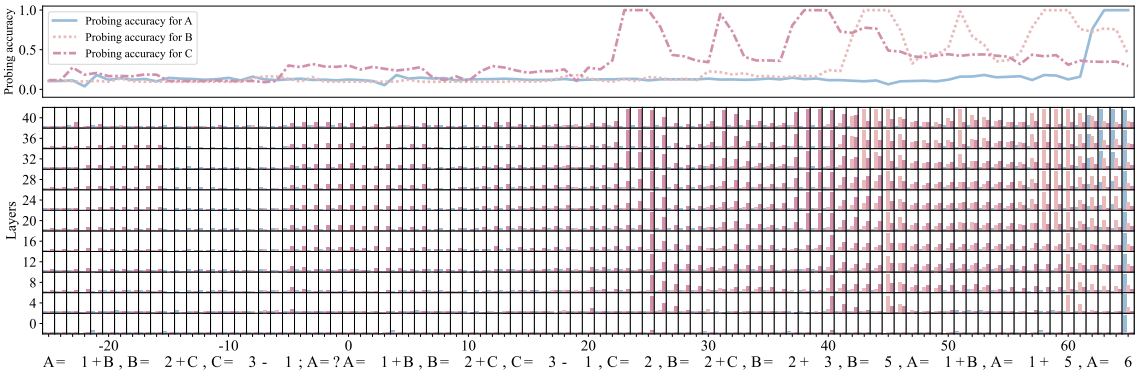

Figure 53: Probing results when Mistral-Nemo-Base-2407 solves Level 5.

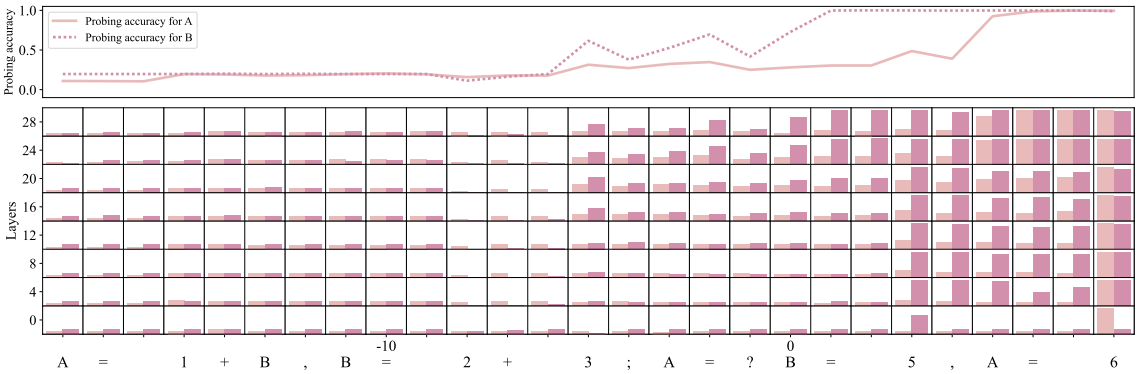

Figure 54: Probing results when Qwen2.5-7B solves Level 2 with **Simple CoT**.

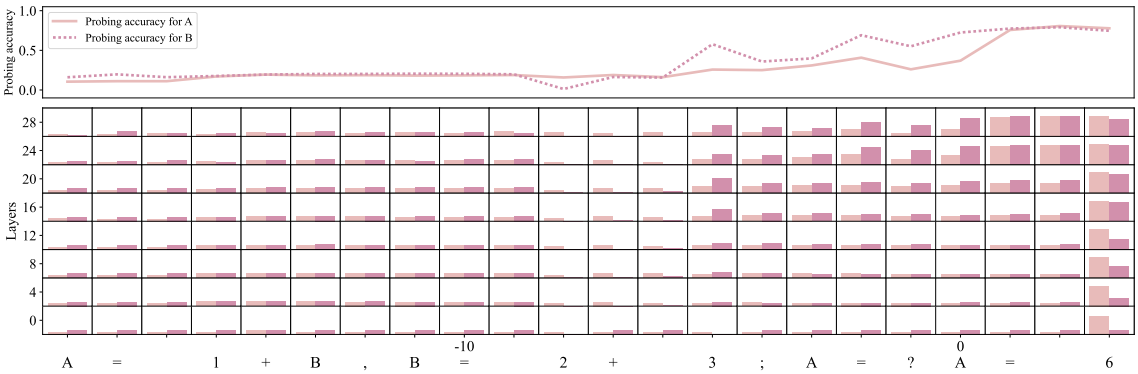

Figure 55: Probing results when Qwen2.5-7B solves Level 2 with **Implicit reasoning**.

Model	Variable	When (↓) t^*_{eq}	When (↓) t^*	Acc (↑) $\prec$ CoT	Acc (↑) $\succ$ CoT
Qwen2.5 (7B) (Qwen Team, 2024)	v_1 (A)	9	62	18.1	100
	v_2 (B)	6	42	22.6	100
	v_3 (C)	3	23	50.6	100
Qwen2.5 (14B) (Qwen Team, 2024)	v_1 (A)	9	63	18.1	98.8
	v_2 (B)	6	42	18.7	98.9
	v_3 (C)	3	23	42.2	100
Qwen2.5 (32B) (Qwen Team, 2024)	v_1 (A)	9	63	18.7	100
	v_2 (B)	6	43	22.6	100
	v_3 (C)	3	23	62.4	100
Qwen2.5-Math (7B) (Yang et al., 2024a)	v_1 (A)	9	62	18.1	100
	v_2 (B)	6	42	22.6	100
	v_3 (C)	3	22	54.5	100
Yi1.5 (34B) (Young et al., 2024)	v_1 (A)	9	71	18.1	100
	v_2 (B)	6	49	22.6	100
	v_3 (C)	3	26	41.2	100
Llama3.1 (8B) (Dubey et al., 2024)	v_1 (A)	9	62	16.0	99.5
	v_2 (B)	6	43	20.0	99.5
	v_3 (C)	3	23	30.6	99.8
Llama3.2 (3B) (Dubey et al., 2024)	v_1 (A)	N/A	N/A	14.2	43.7
	v_2 (B)	N/A	N/A	26.3	47.4
	v_3 (C)	N/A	N/A	37.7	71.7
Mistral-Nemo (12B) (Mistral AI Team, 2024)	v_1 (A)	9	63	18.1	99.9
	v_2 (B)	6	43	16.3	99.9
	v_3 (C)	3	23	32.0	99.9

Table 13: Results for various models on the Level 5 task ($\tau = 0.85$).

Model	Variable	When (↓) t^*_{eq}	When (↓) t^*	Acc (↑) $\prec$ CoT	Acc (↑) $\succ$ CoT
Qwen2.5 (7B) (Qwen Team, 2024)	v_1 (A)	4	27	35.8	100
	v_2 (B)	**−2**	**−5**	100	100
Qwen2.5 (14B) (Qwen Team, 2024)	v_1 (A)	4	27	36.9	100
	v_2 (B)	**−2**	**−5**	100	100
Qwen2.5 (32B) (Qwen Team, 2024)	v_1 (A)	4	28	30.5	100
	v_2 (B)	**−2**	**−5**	100	100
Qwen2.5-Math (7B) (Yang et al., 2024a)	v_1 (A)	4	27	41.8	100
	v_2 (B)	**−2**	**−5**	100	100
Yi1.5 (9B) (Young et al., 2024)	v_1 (A)	4	32	28.1	100
	v_2 (B)	**−2**	**−5**	100	100
Yi1.5 (34B) (Young et al., 2024)	v_1 (A)	4	32	22.9	100
	v_2 (B)	**−2**	**−5**	100	100
Llama3.1 (8B) (Dubey et al., 2024)	v_1 (A)	4	27	20.6	100
	v_2 (B)	**−2**	**−5**	100	100
Llama3.2 (3B) (Dubey et al., 2024)	v_1 (A)	4	28	21.8	100.0
	v_2 (B)	**−2**	**−5**	100	100
Mistral-Nemo (12B) (Mistral AI Team, 2024)	v_1 (A)	4	28	18.0	100
	v_2 (B)	**−2**	**−5**	100	100

Table 14: Results for various models on the Level 1 task ($\tau = 0.90$).

Model	Variable	When (↓) t^*_{eq}	When (↓) t^*	Acc (↑) $\prec$ CoT	Acc (↑) $\succ$ CoT
Qwen2.5 (7B) (Qwen Team, 2024)	v_1 (A)	2	16	49.2	100
	v_2 (B)	5	35	21.2	100
Qwen2.5 (14B) (Qwen Team, 2024)	v_1 (A)	2	16	48.8	100
	v_2 (B)	5	36	21.5	100
Qwen2.5 (32B) (Qwen Team, 2024)	v_1 (A)	2	16	66.4	100
	v_2 (B)	5	36	21.3	100
Qwen2.5-Math (7B) (Yang et al., 2024a)	v_1 (A)	2	15	53.7	100
	v_2 (B)	5	35	22.1	100
Yi1.5 (9B) (Young et al., 2024)	v_1 (A)	2	18	40.2	100
	v_2 (B)	5	41	17.8	100
Yi1.5 (34B) (Young et al., 2024)	v_1 (A)	2	18	35.6	100
	v_2 (B)	5	41	18.3	100
Llama3.1 (8B) (Dubey et al., 2024)	v_1 (A)	2	15	31.9	100
	v_2 (B)	5	35	17.8	100
Llama3.2 (3B) (Dubey et al., 2024)	v_1 (A)	2	16	36.2	99.9
	v_2 (B)	5	36	17.8	99.9
Mistral-Nemo (12B) (Mistral AI Team, 2024)	v_1 (A)	2	16	30.8	100
	v_2 (B)	5	36	17.8	100

Table 15: Results for various models on the Level 2 task ($\tau = 0.90$).

	Variable	When (↓) t^*_{eq}	When (↓) t^*	Acc (↑) $\prec$ CoT	Acc (↑) $\succ$ CoT
Qwen2.5 (7B) (Qwen Team, 2024)	v_1 (A)	5	36	17.9	100
	v_2 (B)	2	16	50.5	100
Qwen2.5 (14B) (Qwen Team, 2024)	v_1 (A)	5	35	17.8	100
	v_2 (B)	2	16	50.5	100
Qwen2.5 (32B) (Qwen Team, 2024)	v_1 (A)	5	36	17.8	100
	v_2 (B)	2	15	67.4	100
Qwen2.5-Math (7B) (Yang et al., 2024a)	v_1 (A)	5	35	18.6	100
	v_2 (B)	2	15	56.1	100
Yi1.5 (9B) (Young et al., 2024)	v_1 (A)	5	41	17.8	100
	v_2 (B)	2	18	36.9	100
Yi1.5 (34B) (Young et al., 2024)	v_1 (A)	5	41	22.4	100
	v_2 (B)	2	18	37.4	100
Llama3.1 (8B) (Dubey et al., 2024)	v_1 (A)	5	35	26.0	100
	v_2 (B)	2	16	29.6	100
Llama3.2 (3B) (Dubey et al., 2024)	v_1 (A)	5	36	17.8	93.2
	v_2 (B)	2	17	33.2	95.4
Mistral-Nemo (12B) (Mistral AI Team, 2024)	v_1 (A)	5	36	17.8	100
	v_2 (B)	2	16	28.9	100

Table 16: Results for various models on the Level 3 task ($\tau = 0.90$).

Model	Variable	When (↓) t^*_{eq}	When (↓) t^*	Acc (↑) $\prec$ CoT	Acc (↑) $\succ$ CoT
Qwen2.5 (7B) (Qwen Team, 2024)	v_1 (A)	5	35	17.2	100
	v_2 (B)	2	16	47.7	100
	v_3 (C)	N/A	N/A	43.7	23.7
Qwen2.5 (14B) (Qwen Team, 2024)	v_1 (A)	5	36	18.9	100
	v_2 (B)	2	15	44.3	100
	v_3 (C)	N/A	N/A	40.4	26.8
Qwen2.5 (32B) (Qwen Team, 2024)	v_1 (A)	5	36	17.4	100
	v_2 (B)	2	15	62.8	100
	v_3 (C)	N/A	N/A	64.4	32.6
Qwen2.5-Math (7B) (Yang et al., 2024a)	v_1 (A)	5	35	17.2	100
	v_2 (B)	2	15	55.6	100
	v_3 (C)	N/A	N/A	47.8	29.4
Yi1.5 (9B) (Young et al., 2024)	v_1 (A)	5	41	17.8	100
	v_2 (B)	2	18	43.5	100
	v_3 (C)	N/A	N/A	36.7	21.2
Yi1.5 (34B) (Young et al., 2024)	v_1 (A)	5	40	19.3	100
	v_2 (B)	2	18	40.8	100
	v_3 (C)	N/A	N/A	27.9	26.2
Llama3.1 (8B) (Dubey et al., 2024)	v_1 (A)	5	35	30.4	100
	v_2 (B)	2	16	27.2	100
	v_3 (C)	N/A	N/A	18.5	17.6
Llama3.2 (3B) (Dubey et al., 2024)	v_1 (A)	5	37	26.2	91.7
	v_2 (B)	2	17	29.1	98.7
	v_3 (C)	N/A	N/A	18.3	17.3
Mistral-Nemo (12B) (Mistral AI Team, 2024)	v_1 (A)	5	36	17.2	100
	v_2 (B)	2	16	29.9	100
	v_3 (C)	N/A	N/A	22.0	19.8

Table 17: Results for various models on the Level 4 task ($\tau = 0.90$).

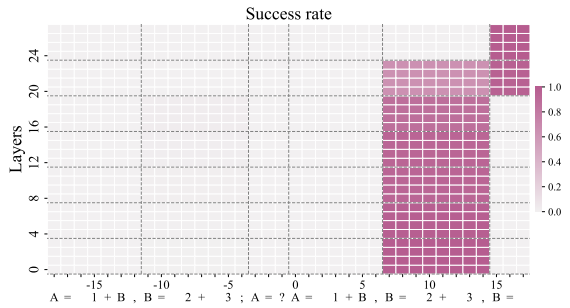

Figure 56: Results of the causal intervention on Qwen2.5-7B. Each grid cell shows the success rate when the intermediate token z_{17} (B =5 $_{\mathbf{2}}$) is the target token.

Model	Variable	When (↓) t^*_{eq}	When (↓) t^*	Acc (↑) $\prec$ CoT	Acc (↑) $\succ$ CoT
Qwen2.5 (7B) (Qwen Team, 2024)	v_1 (A)	9	63	18.1	100
	v_2 (B)	6	42	22.6	100
	v_3 (C)	3	23	50.6	100
Qwen2.5 (14B) (Qwen Team, 2024)	v_1 (A)	9	63	18.1	98.8
	v_2 (B)	6	42	18.7	98.9
	v_3 (C)	3	23	42.2	100
Qwen2.5 (32B) (Qwen Team, 2024)	v_1 (A)	9	63	18.7	100
	v_2 (B)	6	43	22.6	100
	v_3 (C)	3	23	62.4	100
Qwen2.5-Math (7B) (Yang et al., 2024a)	v_1 (A)	9	62	18.1	100
	v_2 (B)	6	42	22.6	100
	v_3 (C)	3	22	54.5	100
Yi1.5 (34B) (Young et al., 2024)	v_1 (A)	9	72	18.1	100
	v_2 (B)	6	49	22.6	100
	v_3 (C)	3	26	41.2	100
Llama3.1 (8B) (Dubey et al., 2024)	v_1 (A)	9	62	16.0	99.5
	v_2 (B)	6	43	20.0	99.5
	v_3 (C)	3	23	30.6	99.8
Llama3.2 (3B) (Dubey et al., 2024)	v_1 (A)	N/A	N/A	14.2	43.7
	v_2 (B)	N/A	N/A	26.3	47.4
	v_3 (C)	N/A	N/A	37.7	71.7
Mistral-Nemo (12B) (Mistral AI Team, 2024)	v_1 (A)	9	63	18.1	99.9
	v_2 (B)	6	43	16.3	99.9
	v_3 (C)	3	23	32.0	99.9

Table 18: Results for various models on the Level 5 task ($\tau = 0.90$).

Model	Variable	When (↓) t^*_{eq}	When (↓) t^*	Acc (↑) $\prec$ CoT	Acc (↑) $\succ$ CoT
Qwen2.5 (7B) (Qwen Team, 2024)	v_1 (A)	4	27	35.8	100
	v_2 (B)	$-\mathbf{2}$	$-\mathbf{5}$	100	100
Qwen2.5 (14B) (Qwen Team, 2024)	v_1 (A)	4	28	36.9	100
	v_2 (B)	$-\mathbf{2}$	$-\mathbf{5}$	100	100
Qwen2.5 (32B) (Qwen Team, 2024)	v_1 (A)	4	28	30.5	100
	v_2 (B)	$-\mathbf{2}$	$-\mathbf{5}$	100	100
Qwen2.5-Math (7B) (Yang et al., 2024a)	v_1 (A)	4	27	41.8	100
	v_2 (B)	$-\mathbf{2}$	$-\mathbf{5}$	100	100
Yi1.5 (9B) (Young et al., 2024)	v_1 (A)	4	32	28.1	100
	v_2 (B)	$-\mathbf{2}$	$-\mathbf{5}$	100	100
Yi1.5 (34B) (Young et al., 2024)	v_1 (A)	4	32	22.9	100
	v_2 (B)	$-\mathbf{2}$	$-\mathbf{5}$	100	100
Llama3.1 (8B) (Dubey et al., 2024)	v_1 (A)	4	27	20.6	100
	v_2 (B)	$-\mathbf{2}$	$-\mathbf{5}$	100	100
Llama3.2 (3B) (Dubey et al., 2024)	v_1 (A)	4	28	21.8	100.0
	v_2 (B)	$-\mathbf{2}$	$-\mathbf{5}$	100	100
Mistral-Nemo (12B) (Mistral AI Team, 2024)	v_1 (A)	4	28	17.9	100
	v_2 (B)	$-\mathbf{2}$	$-\mathbf{5}$	100	100

Table 19: Results for various models on the Level 1 task ($\tau = 0.95$).

Model	Variable	When (↓) t^*_{eq}	When (↓) t^*	Acc (↑) $\prec$ CoT	Acc (↑) $\succ$ CoT
Qwen2.5 (7B)	v_1 (A)	2	16	49.2	100
(Qwen Team, 2024)	v_2 (B)	5	36	21.2	100
Qwen2.5 (14B)	v_1 (A)	2	16	48.8	100
(Qwen Team, 2024)	v_2 (B)	5	36	21.5	100
Qwen2.5 (32B)	v_1 (A)	2	16	66.4	100
(Qwen Team, 2024)	v_2 (B)	5	36	21.3	100
Qwen2.5-Math (7B)	v_1 (A)	2	15	53.7	100
(Yang et al., 2024a)	v_2 (B)	5	36	22.1	100
Yi1.5 (9B)	v_1 (A)	2	18	40.2	100
(Young et al., 2024)	v_2 (B)	5	41	17.8	100
Yi1.5 (34B)	v_1 (A)	2	18	35.6	100
(Young et al., 2024)	v_2 (B)	5	41	18.3	100
Llama3.1 (8B)	v_1 (A)	2	15	31.9	100
(Dubey et al., 2024)	v_2 (B)	5	35	17.8	100
Llama3.2 (3B)	v_1 (A)	2	16	36.2	99.9
(Dubey et al., 2024)	v_2 (B)	5	36	17.8	99.9
Mistral-Nemo (12B)	v_1 (A)	2	16	30.8	100
(Mistral AI Team, 2024)	v_2 (B)	5	36	17.8	100

Table 20: Results for various models on the Level 2 task ($\tau = 0.95$).

Model	Variable	When (↓) t^*_{eq}	When (↓) t^*	Acc (↑) $\prec$ CoT	Acc (↑) $\succ$ CoT
Qwen2.5 (7B)	v_1 (A)	5	36	17.9	100
(Qwen Team, 2024)	v_2 (B)	2	16	50.5	100
Qwen2.5 (14B)	v_1 (A)	5	36	17.8	100
(Qwen Team, 2024)	v_2 (B)	2	16	50.5	100
Qwen2.5 (32B)	v_1 (A)	5	36	17.8	100
(Qwen Team, 2024)	v_2 (B)	2	15	67.4	100
Qwen2.5-Math (7B)	v_1 (A)	5	35	18.6	100
(Yang et al., 2024a)	v_2 (B)	2	15	56.1	100
Yi1.5 (9B)	v_1 (A)	5	41	17.8	100
(Young et al., 2024)	v_2 (B)	2	18	36.9	100
Yi1.5 (34B)	v_1 (A)	5	41	22.4	100
(Young et al., 2024)	v_2 (B)	2	18	37.4	100
Llama3.1 (8B)	v_1 (A)	5	35	26.0	100
(Dubey et al., 2024)	v_2 (B)	2	16	29.6	100
Llama3.2 (3B)	v_1 (A)	N/A	N/A	17.8	93.2
(Dubey et al., 2024)	v_2 (B)	2	17	33.2	95.4
Mistral-Nemo (12B)	v_1 (A)	5	36	17.8	100
(Mistral AI Team, 2024)	v_2 (B)	2	16	28.9	100

Table 21: Results for various models on the Level 3 task ($\tau = 0.95$).

Model	Variable	When (↓) t^*_{eq}	When (↓) t^*	Acc (↑) $\prec$ CoT	Acc (↑) $\succ$ CoT
Qwen2.5 (7B)	v_1 (A)	5	36	17.2	100
(Qwen Team, 2024)	v_2 (B)	2	16	47.7	100
	v_3 (C)	N/A	N/A	43.7	23.7
Qwen2.5 (14B)	v_1 (A)	5	36	18.9	100
(Qwen Team, 2024)	v_2 (B)	2	15	44.3	100
	v_3 (C)	N/A	N/A	40.4	26.8
Qwen2.5 (32B)	v_1 (A)	5	36	17.4	100
(Qwen Team, 2024)	v_2 (B)	2	16	62.8	100
	v_3 (C)	N/A	N/A	64.4	32.6
Qwen2.5-Math (7B)	v_1 (A)	5	35	17.2	100
(Yang et al., 2024a)	v_2 (B)	2	15	55.6	100
	v_3 (C)	N/A	N/A	47.8	29.4
Yi1.5 (9B)	v_1 (A)	5	41	17.8	100
(Young et al., 2024)	v_2 (B)	2	18	43.5	100
	v_3 (C)	N/A	N/A	36.7	21.2
Yi1.5 (34B)	v_1 (A)	5	41	19.3	100
(Young et al., 2024)	v_2 (B)	2	18	40.8	100
	v_3 (C)	N/A	N/A	27.9	26.2
Llama3.1 (8B)	v_1 (A)	5	35	30.4	100
(Dubey et al., 2024)	v_2 (B)	2	16	27.2	100
	v_3 (C)	N/A	N/A	18.5	17.6
Llama3.2 (3B)	v_1 (A)	N/A	N/A	26.2	91.7
(Dubey et al., 2024)	v_2 (B)	2	17	29.1	98.7
	v_3 (C)	N/A	N/A	18.3	17.3
Mistral-Nemo (12B)	v_1 (A)	5	36	17.2	100
(Mistral AI Team, 2024)	v_2 (B)	2	16	29.9	100
	v_3 (C)	N/A	N/A	22.0	19.8

Table 22: Results for various models on the Level 4 task ($\tau = 0.95$).

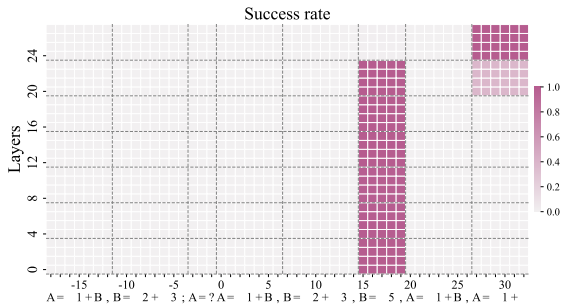

Figure 57: Results of the causal intervention on Qwen2.5-7B. Each grid cell shows the success rate when z_{32} ($\underline{\text{A} = 1{+}5}_{\,4}$) is the target token.

Model	Variable	When (↓) t^*_{eq}	When (↓) t^*	Acc (↑) $\prec$ CoT	Acc (↑) $\succ$ CoT
Qwen2.5 (7B)	v_1 (A)	9	63	18.1	100
(Qwen Team, 2024)	v_2 (B)	6	43	22.6	100
	v_3 (C)	3	23	50.6	100
Qwen2.5 (14B)	v_1 (A)	9	63	18.1	98.8
(Qwen Team, 2024)	v_2 (B)	6	43	18.7	98.9
	v_3 (C)	3	23	42.2	100
Qwen2.5 (32B)	v_1 (A)	9	63	18.7	100
(Qwen Team, 2024)	v_2 (B)	6	43	22.6	100
	v_3 (C)	3	23	62.4	100
Qwen2.5-Math (7B)	v_1 (A)	9	63	18.1	100
(Yang et al., 2024a)	v_2 (B)	6	42	22.6	100
	v_3 (C)	3	22	54.5	100
Yi1.5 (34B)	v_1 (A)	9	72	18.1	100
(Young et al., 2024)	v_2 (B)	6	49	22.6	100
	v_3 (C)	3	26	41.2	100
Llama3.1 (8B)	v_1 (A)	9	62	16.0	99.5
(Dubey et al., 2024)	v_2 (B)	6	43	20.0	99.5
	v_3 (C)	3	23	30.6	99.8
Llama3.2 (3B)	v_1 (A)	N/A	N/A	14.1	43.7
(Dubey et al., 2024)	v_2 (B)	N/A	N/A	26.3	47.4
	v_3 (C)	N/A	N/A	37.7	71.7
Mistral-Nemo (12B)	v_1 (A)	9	63	18.1	99.9
(Mistral AI Team, 2024)	v_2 (B)	6	43	16.3	99.9
	v_3 (C)	3	23	32.0	99.9

Table 23: Results for various models on the Level 5 task ($\tau = 0.95$).

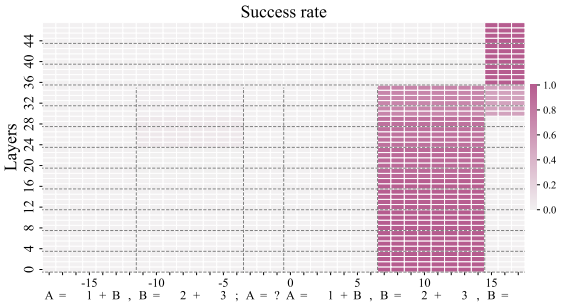

Figure 58: Results of the causal intervention on Qwen2.5-14B. Each grid cell shows the success rate when the intermediate token z_{17} (B =5 $_2$) is the target token.

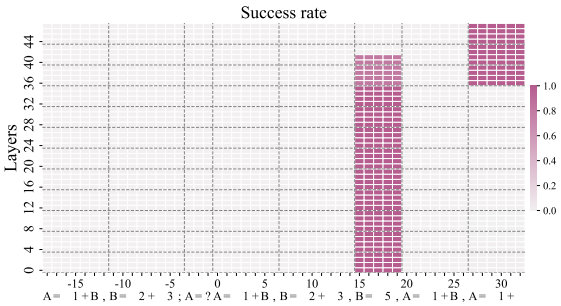

Figure 59: Results of the causal intervention on Qwen2.5-14B. Each grid cell shows the success rate when z_{32} (A = 1+5 $_4$) is the target token.

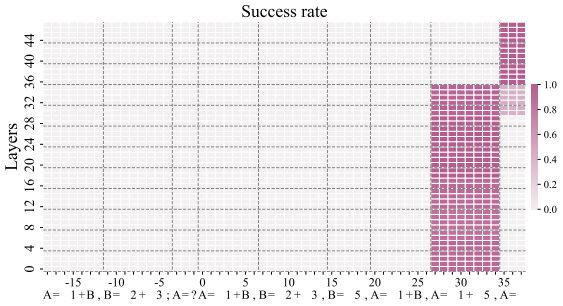

Figure 60: Results of the causal intervention on Qwen2.5-14B. Each grid cell shows the success rate when the final answer y (A =**6** $_5$) is the target token.

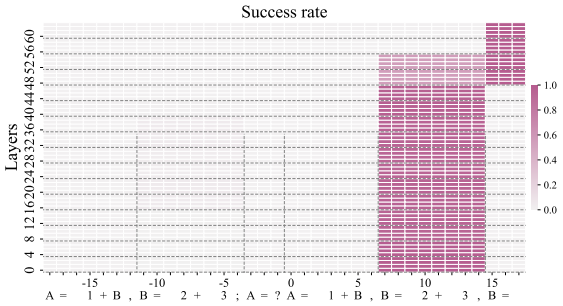

Figure 61: Results of the causal intervention on Qwen2.5-32B. Each grid cell shows the success rate when the intermediate token z_{17} (B =5 $_2$) is the target token.

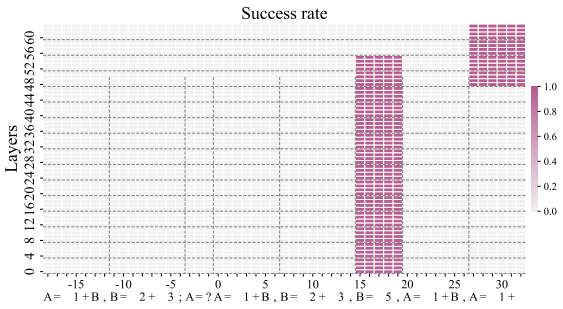

Figure 62: Results of the causal intervention on Qwen2.5-32B. Each grid cell shows the success rate when z_{32} (A = 1+5 $_4$) is the target token.

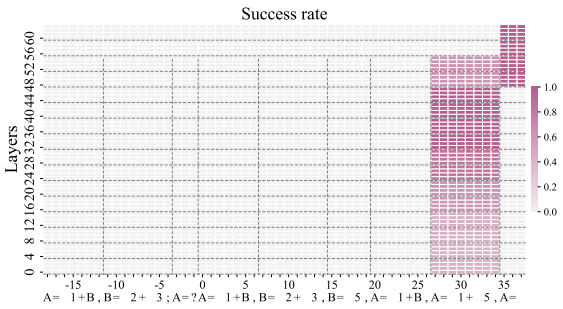

Figure 63: Results of the causal intervention on Qwen2.5-32B. Each grid cell shows the success rate when the final answer y (A =**6** $_5$) is the target token.

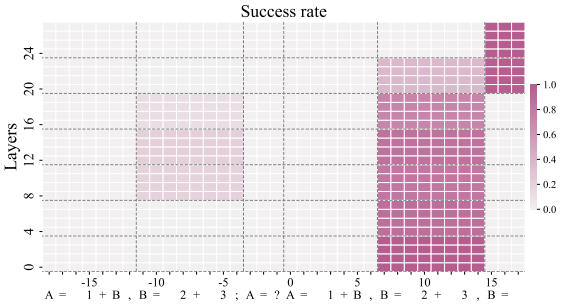

Figure 64: Results of the causal intervention on Qwen2.5-Math-7B. Each grid cell shows the success rate when the intermediate token z_{17} (B =5$_{\mathbf{2}}$) is the target token.

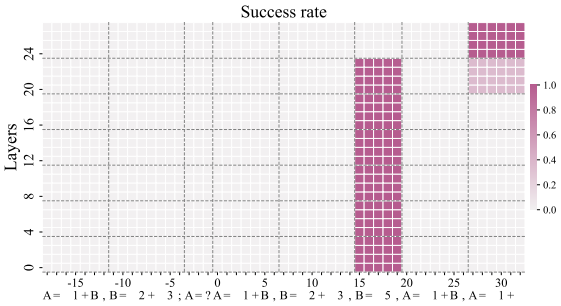

Figure 65: Results of the causal intervention on Qwen2.5-Math-7B. Each grid cell shows the success rate when z_{32} (A = 1+5$_{\mathbf{4}}$) is the target token.

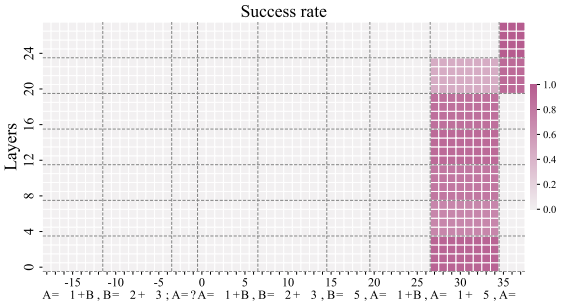

Figure 66: Results of the causal intervention on Qwen2.5-Math-7B. Each grid cell shows the success rate when the final answer y (A =$\mathbf{6}_{\mathbf{5}}$) is the target token.

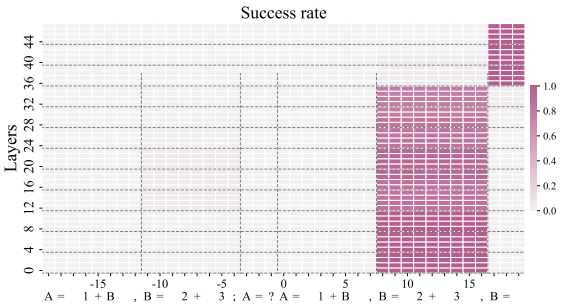

Figure 67: Results of the causal intervention on Yi-1.5-9B. Each grid cell shows the success rate when the intermediate token z_{17} (B =5$_{\mathbf{2}}$) is the target token.

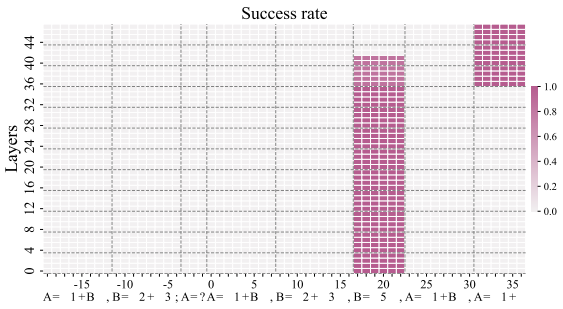

Figure 68: Results of the causal intervention on Yi-1.5-9B. Each grid cell shows the success rate when z_{32} (A = 1+5$_{\mathbf{4}}$) is the target token.

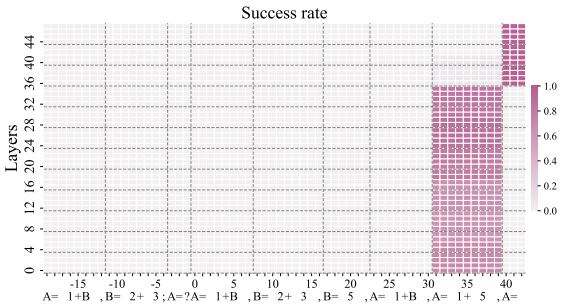

Figure 69: Results of the causal intervention on Yi-1.5-9B. Each grid cell shows the success rate when the final answer y (A =$\mathbf{6}_{\mathbf{5}}$) is the target token.

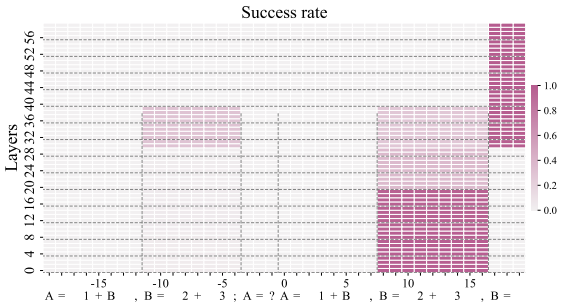

Figure 70: Results of the causal intervention on Yi-1.5-34B. Each grid cell shows the success rate when the intermediate token z_{17} (B =5$_{\mathbf{2}}$) is the target token.

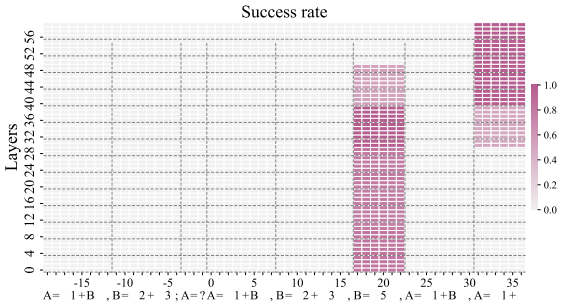

Figure 71: Results of the causal intervention on Yi-1.5-34B. Each grid cell shows the success rate when z_{32} (A = 1+5$_{\mathbf{4}}$) is the target token.

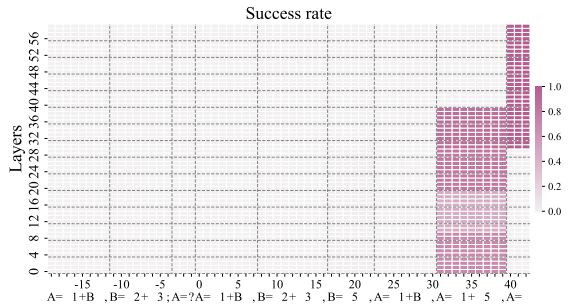

Figure 72: Results of the causal intervention on Yi-1.5-34B. Each grid cell shows the success rate when the final answer y (A =**6**$_5$) is the target token.

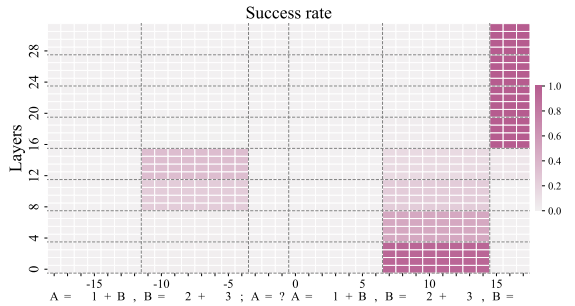

Figure 73: Results of the causal intervention on Llama-3.1-8B. Each grid cell shows the success rate when the intermediate token z_{17} (B =5$_2$) is the target token.

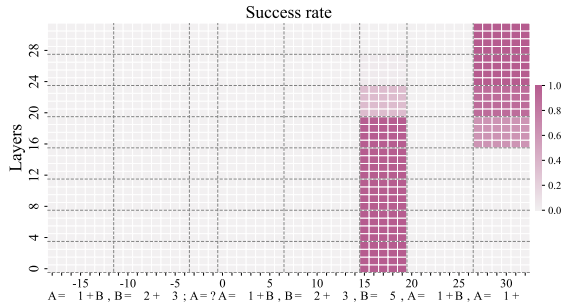

Figure 74: Results of the causal intervention on Llama-3.1-8B. Each grid cell shows the success rate when z_{32} (A = 1+5$_4$) is the target token.

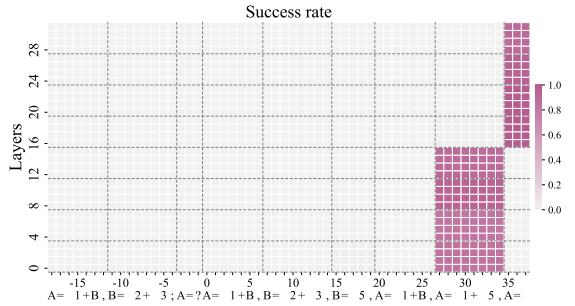

Figure 75: Results of the causal intervention on Llama-3.1-8B. Each grid cell shows the success rate when the final answer y (A =**6**$_5$) is the target token.

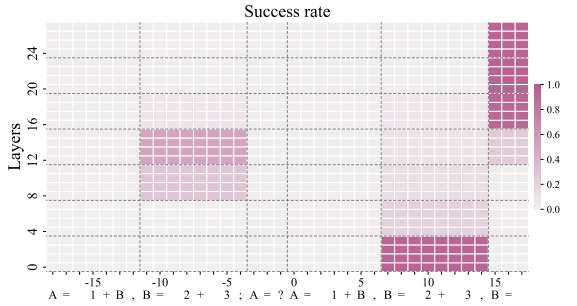

Figure 76: Results of the causal intervention on Llama-3.2-3B. Each grid cell shows the success rate when the intermediate token z_{17} (B =5$_2$) is the target token.

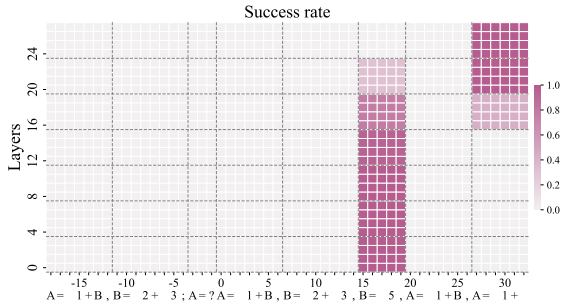

Figure 77: Results of the causal intervention on Llama-3.2-3B. Each grid cell shows the success rate when z_{32} (A = 1+5$_4$) is the target token.

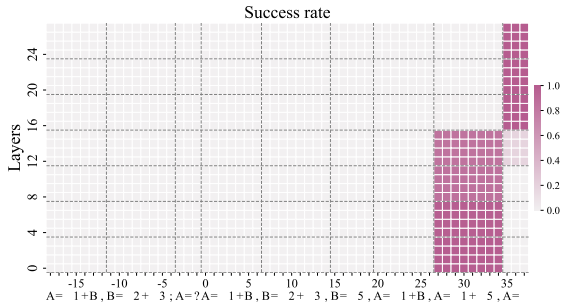

Figure 78: Results of the causal intervention on Llama-3.2-3B. Each grid cell shows the success rate when the final answer y (A =**6**$_5$) is the target token.

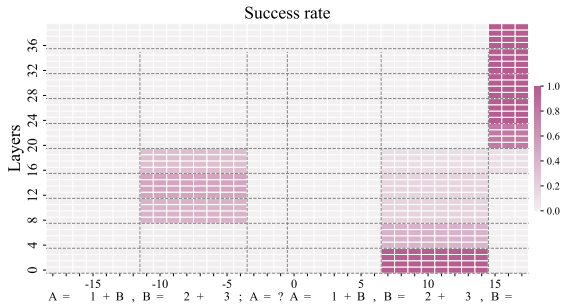

Figure 79: Results of the causal intervention on Mistral-Nemo-Base-2407. Each grid cell shows the success rate when the intermediate token z_{17} (B =5$_2$) is the target token.

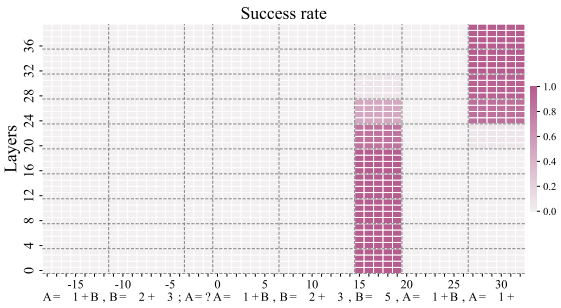

Figure 80: Results of the causal intervention on Mistral-Nemo-Base-2407. Each grid cell shows the success rate when z_{32} (A = 1+5 $_{\mathbf{4}}$) is the target token.

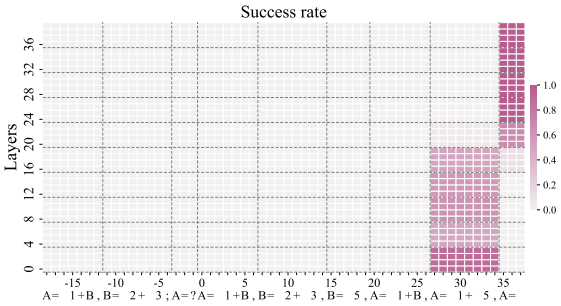

Figure 81: Results of the causal intervention on Mistral-Nemo-Base-2407. Each grid cell shows the success rate when the final answer y (A = $\mathbf{6}_{\mathbf{5}}$) is the target token.

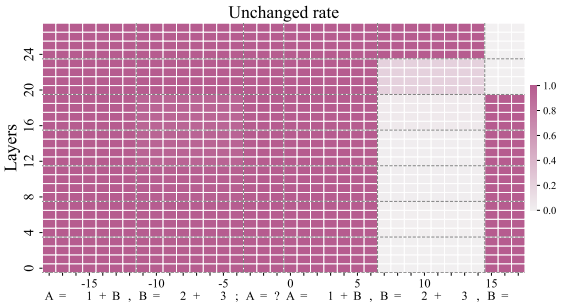

Figure 82: Results of the causal intervention on Qwen2.5-7B. Each grid cell shows the unchanged rate when the intermediate token z_{17} (B = 5 $_{\mathbf{2}}$) is the target token.

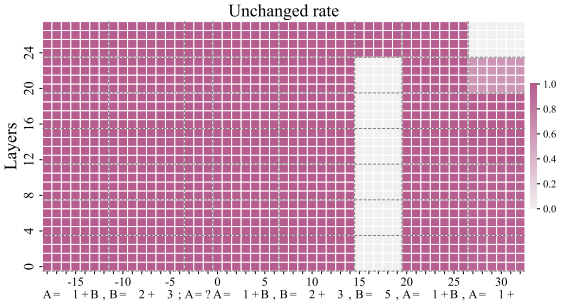

Figure 83: Results of the causal intervention on Qwen2.5-7B. Each grid cell shows the unchanged rate when z_{32} (A = 1+5 $_{\mathbf{4}}$) is the target token.

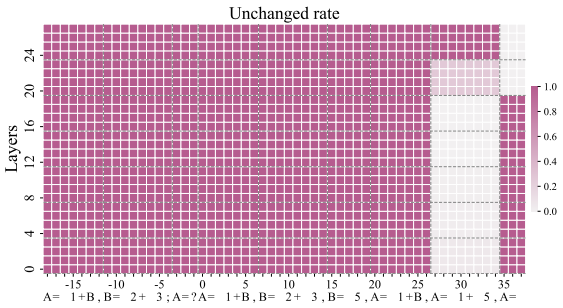

Figure 84: Results of the causal intervention on Qwen2.5-7B. Each grid cell shows the unchanged rate when the final answer y (A = $\mathbf{6}_{\mathbf{5}}$) is the target token.

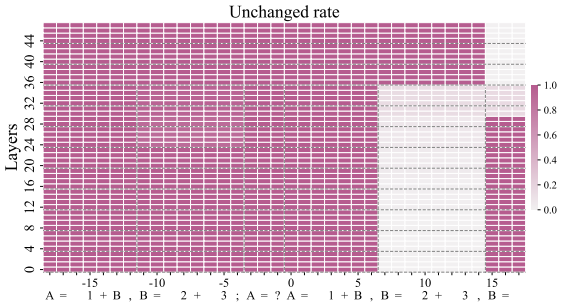

Figure 85: Results of the causal intervention on Qwen2.5-14B. Each grid cell shows the unchanged rate when the intermediate token z_{17} (B = 5 $_{\mathbf{2}}$) is the target token.

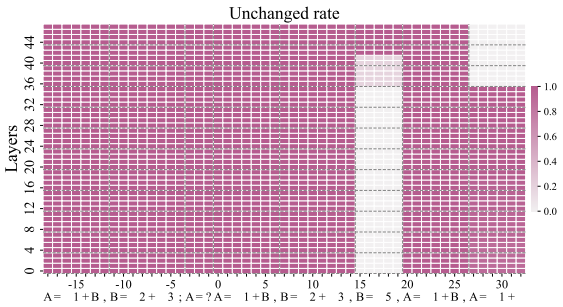

Figure 86: Results of the causal intervention on Qwen2.5-14B. Each grid cell shows the unchanged rate when z_{32} (A = 1+5 $_{\mathbf{4}}$) is the target token.

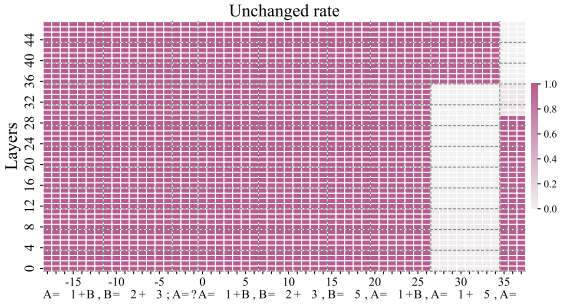

Figure 87: Results of the causal intervention on Qwen2.5-14B. Each grid cell shows the unchanged rate when the final answer y (A = $\mathbf{6}_{\mathbf{5}}$) is the target token.

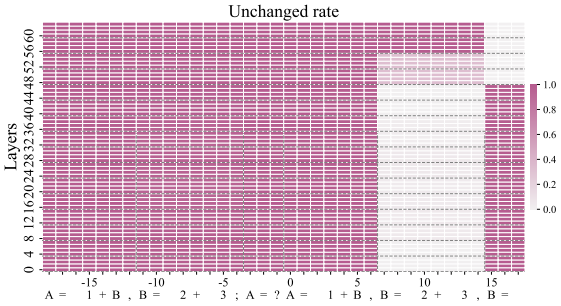

Figure 88: Results of the causal intervention on Qwen2.5-32B. Each grid cell shows the unchanged rate when the intermediate token z_{17} (B =5$_{\mathbf{2}}$) is the target token.

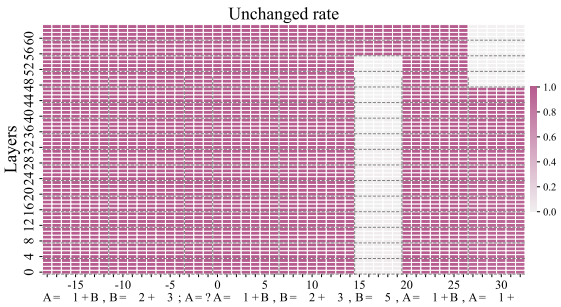

Figure 89: Results of the causal intervention on Qwen2.5-32B. Each grid cell shows the unchanged rate when z_{32} (A = 1+5$_{\mathbf{4}}$) is the target token.

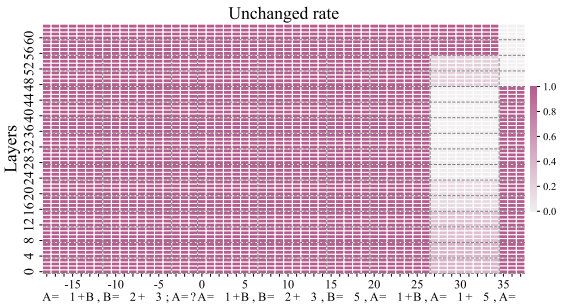

Figure 90: Results of the causal intervention on Qwen2.5-32B. Each grid cell shows the unchanged rate when the final answer y (A =$\mathbf{6}_{\mathbf{5}}$) is the target token.

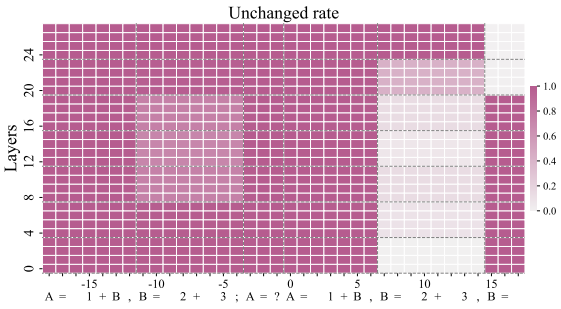

Figure 91: Results of the causal intervention on Qwen2.5-Math-7B. Each grid cell shows the unchanged rate when the intermediate token z_{17} (B =5$_{\mathbf{2}}$) is the target token.

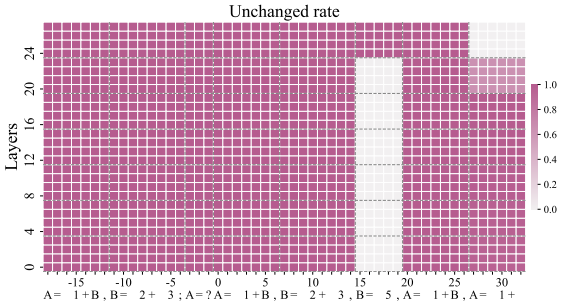

Figure 92: Results of the causal intervention on Qwen2.5-Math-7B. Each grid cell shows the unchanged rate when z_{32} (A = 1+5$_{\mathbf{4}}$) is the target token.

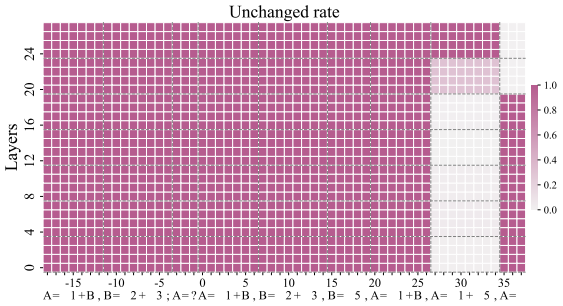

Figure 93: Results of the causal intervention on Qwen2.5-Math-7B. Each grid cell shows the unchanged rate when the final answer y (A =$\mathbf{6}_{\mathbf{5}}$) is the target token.

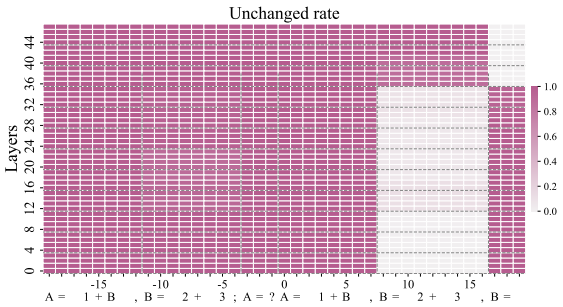

Figure 94: Results of the causal intervention on Yi-1.5-9B. Each grid cell shows the unchanged rate when the intermediate token z_{17} (B =5$_{\mathbf{2}}$) is the target token.

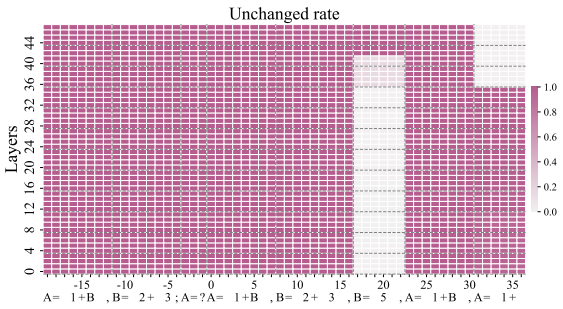

Figure 95: Results of the causal intervention on Yi-1.5-9B. Each grid cell shows the unchanged rate when z_{32} (A = 1+5$_{\mathbf{4}}$) is the target token.

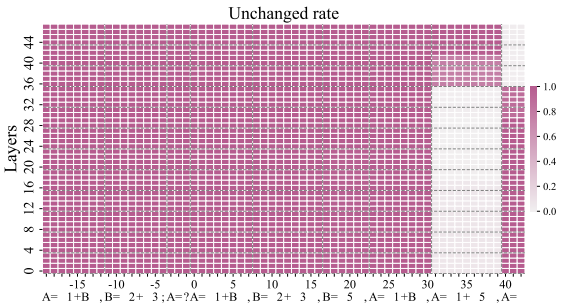

Figure 96: Results of the causal intervention on Yi-1.5-9B. Each grid cell shows the unchanged rate when the final answer y (A =$\mathbf{6}_{\mathbf{5}}$) is the target token.

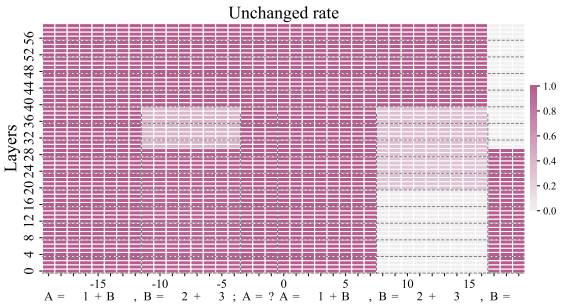

Figure 97: Results of the causal intervention on Yi-1.5-34B. Each grid cell shows the unchanged rate when the intermediate token z_{17} (B =$5_{\mathbf{2}}$) is the target token.

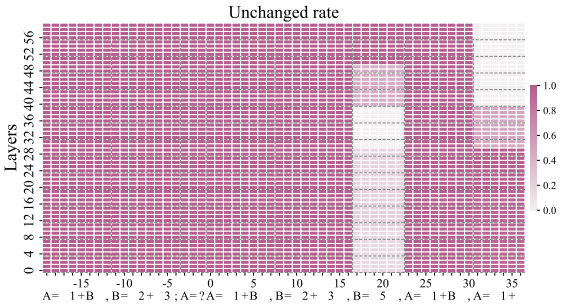

Figure 98: Results of the causal intervention on Yi-1.5-34B. Each grid cell shows the unchanged rate when z_{32} (A = 1+$5_{\mathbf{4}}$) is the target token.

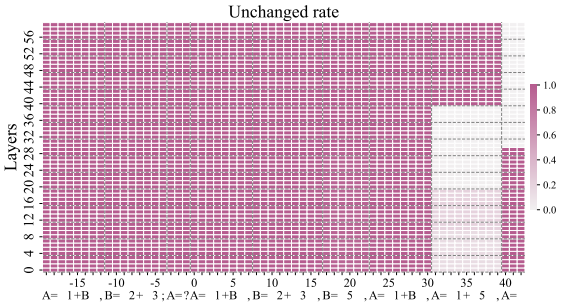

Figure 99: Results of the causal intervention on Yi-1.5-34B. Each grid cell shows the unchanged rate when the final answer y (A =$\mathbf{6}_{\mathbf{5}}$) is the target token.

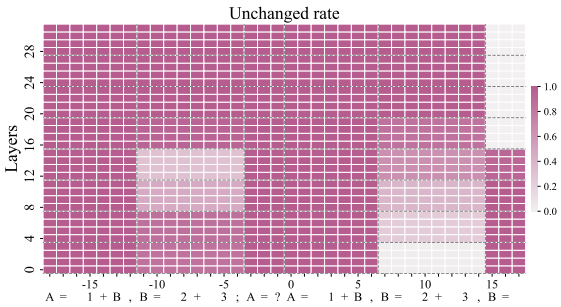

Figure 100: Results of the causal intervention on Llama-3.1-8B. Each grid cell shows the unchanged rate when the intermediate token z_{17} (B =$5_{\mathbf{2}}$) is the target token.

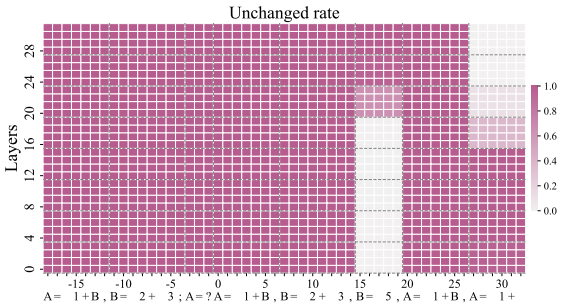

Figure 101: Results of the causal intervention on Llama-3.1-8B. Each grid cell shows the unchanged rate when z_{32} (A = 1+$5_{\mathbf{4}}$) is the target token.

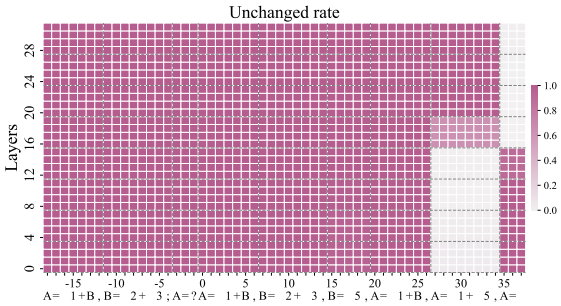

Figure 102: Results of the causal intervention on Llama-3.1-8B. Each grid cell shows the unchanged rate when the final answer y (A =$\mathbf{6}_{\mathbf{5}}$) is the target token.

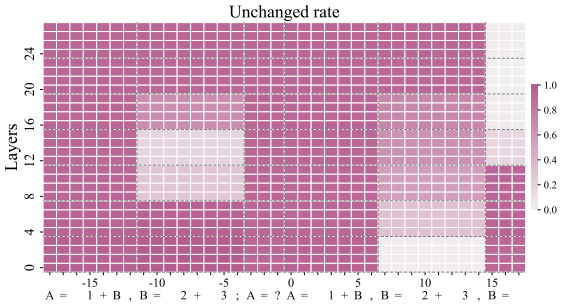

Figure 103: Results of the causal intervention on Llama-3.2-3B. Each grid cell shows the unchanged rate when the intermediate token z_{17} (B =$5_{\mathbf{2}}$) is the target token.

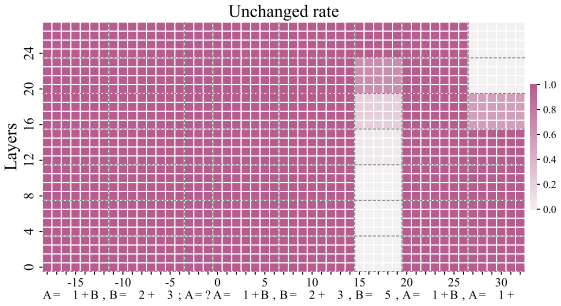

Figure 104: Results of the causal intervention on Llama-3.2-3B. Each grid cell shows the unchanged rate when z_{32} (A = 1+5 $_{\mathbf{4}}$) is the target token.

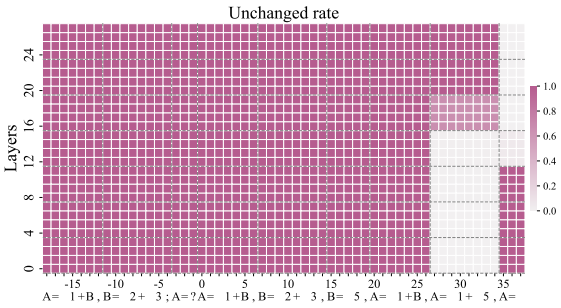

Figure 105: Results of the causal intervention on Llama-3.2-3B. Each grid cell shows the unchanged rate when the final answer y (A = $\mathbf{6}_{\mathbf{5}}$) is the target token.

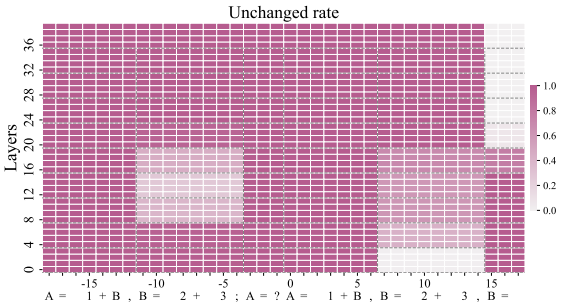

Figure 106: Results of the causal intervention on Mistral-Nemo-Base-2407. Each grid cell shows the unchanged rate when the intermediate token z_{17} (B = 5 $_{\mathbf{2}}$) is the target token.

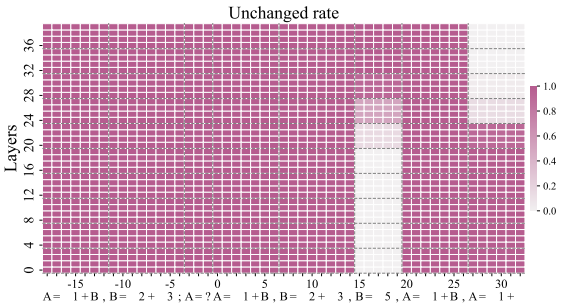

Figure 107: Results of the causal intervention on Mistral-Nemo-Base-2407. Each grid cell shows the unchanged rate when z_{32} (A = 1+5 $_{\mathbf{4}}$) is the target token.

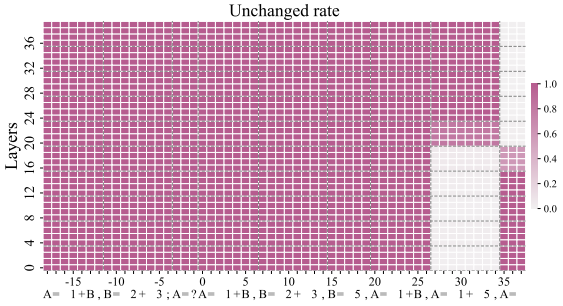

Figure 108: Results of the causal intervention on Mistral-Nemo-Base-2407. Each grid cell shows the unchanged rate when the final answer y (A = $\mathbf{6}_{\mathbf{5}}$) is the target token.