Landmark-Based Node Representations for Shortest Path Distance Approximations in Random Graphs

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Abstract

Learning node representations is a fundamental problem in graph machine learning. While existing embedding methods effectively preserve local similarity measures, they often fail to capture global functions like graph distances. Inspired by Bourgain's seminal work on Hilbert space embeddings of metric spaces [1985], we study the performance of local distance-preserving node embeddings. Known as landmark-based algorithms, these embeddings approximate pairwise distances by computing shortest paths from a small subset of reference nodes called landmarks. Our main theoretical contribution shows that random graphs, such as Erdős–Rényi random graphs, require lower dimensions in landmark-based embeddings compared to worst-case graphs. Empirically, we demonstrate that the GNN-based approximations for the distances to landmarks generalize well to larger real-world networks, offering a scalable and transferable alternative for graph representation learning.

1 Introduction

Learning representations for network data has long been central to graph machine learning with a key objective to learn low-dimensional *node embeddings* that map structurally similar nodes to nearby points. These embeddings facilitate the application of machine learning methods to graph data, enabling a wide range of downstream tasks such as node classification, link prediction, and community detection [Hamilton et al., 2017b, Grover and Leskovec, 2016].

Traditional methods such as DeepWalk [Perozzi et al., 2014] and Node2Vec [Grover and Leskovec, 2016] use random walks to preserve local graph structures like node neighborhoods, while extensions such as GraRep [Cao et al., 2015] and PRONE [Zhang et al., 2021] capture higher-order relationships via k-hop transition factorizations. Spectral methods like Laplacian Eigenmaps [Belkin and Niyogi, 2003] preserve global geometry by embedding graphs into low-dimensional spaces that approximate their underlying manifold. Cauchy embeddings [Tang et al., 2019] further improve spectral methods by increasing their sensitivity to edge weight differences. While effective at capturing local graph structure, these methods often fail to preserve global topology and functionals such as shortest

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path distances, especially in large, complex graphs [Goyal and Ferrara, 2018, Tsitsulin et al., 2018, Brunner, 2021].

A rich body of theoretical works has focused on the minimum dimension k_ε required to embed worst-case graphs into $\mathbb{R}^k$ while preserving all pairwise distances up to a factor of $(1 \pm \varepsilon)$. In a seminal work, Bourgain [1985] showed $k_\varepsilon = \Omega((\log n)^2/(\log\log n)^2)$, providing a negative answer to Johnson and Lindenstrauss's Problem 3 [1984]. This was later strengthened to $k_\varepsilon = \Omega((\log n)^2)$ [Linial et al., 1995], and further to $k_\varepsilon = \Omega(n^{c/(1+\varepsilon)})$ for some universal $c > 0$ [Matoušek, 1996]. The latter was also proven recently by Naor [2016] and Naor [2021] using expanders, showing that low-distortion embeddings of graphs with strong expansion properties require polynomial dimensionality.

From an algorithmic perspective, finding embeddings with minimum distortion is NP-hard; see Sidiropoulos et al. [2019] for a survey of approximation algorithms and hardness results. Practical methods often rely on landmark-based algorithms [Goldberg and Harrelson, 2005, Sarma et al., 2010, Potamias et al., 2009, Tretyakov et al., 2011, Akiba et al., 2013, Rizi et al., 2018, Qi et al., 2020], which preselect a subset of landmark nodes and compute distances to them via local message passing (see Sommer [2014] for a review). The resulting landmark distances can be viewed as an embedding useful for approximating graph distances. Yet these methods inherit worst-case limitations of local message passing, often requiring prohibitively large dimensions for general graphs [Sarma et al., 2012, Loukas, 2020].

In this work, we focus on the problem of learning node embeddings that preserve both local similarities and global graph distances. Motivated by Bourgain's seminal results on metric embeddings [1985], we analyze a landmark-based algorithm that approximates graph distances via shortest paths from a small set of landmarks. Our study analyzes its performance on random graphs—particularly Erdős–Rényi (ER) graphs—compared to worst-case instances. Of particular interest is the dimension of the embedding space.

Theoretical Contributions. Our main theoretical contribution is a detailed analysis of the dimensionality requirements for landmark-based embeddings. We show that random graphs, with high probability (w.h.p.), require lower embedding dimension than worst-case graphs when using the lower bound $\underline{d}(u_1, u_2)$ and the upper bound $\bar{d}(u_1, u_2)$ to approximate the shortest path distance $d(u_1, u_2)$.

Methodological and Empirical Contributions. Building upon this theory, we propose a GNN-augmented variant that predicts landmark distances from graph structure, avoiding explicit shortest-path computations. This reduces computational cost while retaining embedding quality, as the GNN is trained to approximate landmark distances in a supervised manner.

2 The Shortest Path Problem and Landmark-Based Embeddings

Given a graph $G = (V, E)$ and nodes $u_1, u_2 \in V$, the shortest distance problem is to find the minimum number of edges connecting u_1 and u_2, i.e., the *shortest path distance* $d(u_1, u_2)$. The classical solution to this fundamental graph problem is Dijkstra's algorithm, with running time $O(n^2)$ for a single pair and $O(n^3)$ for all pairs using naive data structures, reducible to $O(m \log n)$ and $O(m + n \log n)$ with heaps and Fibonacci heaps [Schrijver, 2012]. More refined variants for single source include S-Dial ($O(m + nl_{\max})$, $l_{\max}$ is the maximum arc length), S-Heap ($O(m \log n)$ or $O(n \log n)$ in sparse graphs), and Fredman–Willard's implementation ($O(m + n \log n/ \log\log n)$) [Gallo and Pallottino, 1988, Fredman and Willard, 1990]. For all pairs, Floyd–Warshall and primal sequential algorithms run in $O(n^3)$ [Gallo and Pallottino, 1988], while hidden-path achieves $O(mn + n^2 \log n)$ [Karger et al., 1993]. Despite these advances, exact computation remains costly on large graphs.

While computing exact shortest path distances is expensive, we can afford to compute local paths. Sarma et al.'s offline sketch algorithm [2010] leverages this principle in its local step to construct landmark embeddings (see Algorithm 1). The resulting local embeddings are then saved in memory and later retrieved to quickly estimate $d(u_1, u_2)$ via the bounds $\underline{d}(u_1, u_2)$ and $\bar{d}(u_1, u_2)$, which is common practice to reduce computational and memory demands in shortest path distance approximation [Bourgain, 1985, Matoušek, 1996, Sarma et al., 2010, Gubichev et al., 2010, Sommer, 2014, Akiba et al., 2014, Meng et al., 2015, Jiang et al., 2021, Awasthi et al., 2022].

Algorithm 1: Landmark Algorithm Adapted From Sarma et al. [2010]

Input: Connected graph $G = (V, E)$ with $|V| = n$. Set cardinalities $|S_0| = 1$, $|S_1|$, $|S_2|$, ..., $|S_r|$ for some positive integer r.

Output: Shortest path lower bound $\underline{d}(u, v)$ and upper bound $\bar{d}(u, v)$ for all $u, v \in V$.

for $i = 0, 1, \ldots, r$ **do** /* LOCAL STEP */

  $S_i \leftarrow \{s_1, \ldots, s_{|S_i|} \sim \text{Uniform}(V)\}$

  for $u = 1, \ldots, n$ **do**

    $[\mathbf{x}_u]_i = \min_{s \in S_i} d(s, u)$

    $[\boldsymbol{\sigma}_u]_i = \text{argmin}_{s \in S_i} d(s, u)$

  end

end

for $u = 1, \ldots, n$ **do** /* GLOBAL STEP */

  for $v = 1, \ldots, n$ **do**

    $\underline{d}(u, v) = \|\mathbf{x}_u - \mathbf{x}_v\|_\infty$ /* Lower Bound */

    $\bar{d}(u, v) = \min\{[\mathbf{x}_u]_i + [\mathbf{x}_v]_j : (i, j) \text{ s.t. } [\boldsymbol{\sigma}_u \mathbf{1}^T - \mathbf{1}\boldsymbol{\sigma}_v^T]_{ij} = 0\}$ /* Upper Bound */

  end

end

3 Lower and Upper Bound Distortions on Sparse Erdős–Rényi

In this section, we show our main results on the performance of $\underline{d}(u_1, u_2)$ and $\bar{d}(u_1, u_2)$ outputted by Algorithm 1 as shortest path approximations in sparse ER graphs, where each edge appears independently with a fixed probability. We write $G \sim \text{ER}_n(\lambda/n)$ to denote this distribution over the space of all graphs on n nodes with probability λ/n, $\lambda \in [0, n]$. Based on a classical result in random graph theory [van der Hofstad, 2017, Theorems 4.4, 4.8 and Corollary 4.13], we consider $\lambda > 1$ since otherwise the giant component dies out in probability, making most pairs of nodes not connected.

Theorem 3.1 (***Lower Bound Distortion***). *Let $G \sim \text{ER}_n(\lambda/n)$ with $\lambda > 1$. Let u_1, u_2 be chosen independently and uniformly at random with replacement from G. Fix $\varepsilon \in (0, 1)$, an integer $M > 1$, $\theta \in (0, \varepsilon)$, and $r = \lfloor \frac{\theta}{\log M} \log n \rfloor$. With embedding dimension $D = \Omega\left(M n^{1 - \frac{\varepsilon}{2} - \min\{\frac{\varepsilon}{2}, \theta\} + \varsigma} \frac{\theta}{\log M} \log n\right)$ resulting from $R = \Omega\left(M n^{1 - \frac{\varepsilon}{2} - \min\{\frac{\varepsilon}{2}, \theta\} + \varsigma}\right)$ runs of the local step with set cardinalities $|S_0| = M^0, |S_1| = M^1, \ldots, |S_r| = M^r$ and any arbitrarily small $\varsigma > 0$, $\underline{d}(u_1, u_2)$ provides a $(1 - \varepsilon)$-approximation of $d(u_1, u_2)$ (i.e. $d(u_1, u_2) \geq \underline{d}(u_1, u_2) \geq (1 - \varepsilon) d(u_1, u_2)$) w.h.p.*

Theorem 3.2 (***Upper Bound Distortion***). *Let G, λ, u_1, u_2 be as in Theorem 3.1. Fix $\varepsilon \in (0, 1)$, an integer $M > 1$, $\theta \in \left(0, \frac{1-\varepsilon}{2}\right)$, and $r = \lfloor \frac{\theta}{\log M} \log n \rfloor$. With embedding dimension $D = \Omega\left(n^{1 - \varepsilon + \varsigma}\right)$ resulting from $R = \Omega\left(\frac{\log M}{\theta \log n} n^{1 - \varepsilon + \varsigma}\right)$ runs of the local step with set cardinalities $|S_0| = M^0, |S_1| = M^1, \ldots, |S_r| = M^r$ and any arbitrarily small $\varsigma > 0$, $\bar{d}(u_1, u_2)$ provides a $(1 + \varepsilon)$-approximation of $d(u_1, u_2)$ (i.e. $d(u_1, u_2) \leq \bar{d}(u_1, u_2) \leq (1 + \varepsilon) d(u_1, u_2)$) w.h.p.*

While Bourgain [1985], Matoušek [1996], and Sarma et al. [2010] showed that, in the worst case, Algorithm 1 with $M = 2$ requires embedding dimension $\Omega(n^{1/c} \log n)$ for a $\frac{1}{2c-1}$-factor lower bound and a $(2c - 1)$-factor upper bound, our results offer a more efficient alternative for ER graphs by loosening the dimensionality restrictions, specifically $\Omega\left(n^{\frac{1}{2c-1} + \varsigma} \theta \log n\right)$ with $\theta \in \left[\frac{c-1}{2c-1}, \frac{2(c-1)}{2c-1}\right)$ for a $\frac{1}{2c-1}$-factor lower bound and $\Omega(n^{3 - 2c + \varsigma})$ for a $(2c - 1)$-factor upper bound for any $\varsigma > 0$ and $c > 1$. The proof of Theorems 3.1 and 3.2 along with the intermediate results that leverage Poisson branching process to approximate local neighborhood expansions are provided in Appendix A.

4 GNN-Based Landmark Embeddings and Experimental Results

Although Algorithm 1 outperforms traditional methods, its landmark distance calculations rely on one run of Breadth-First Search (BFS) for each landmark set, which is costly for large graphs ($O(n + m)$

per pair, $O(n(n+m))$ for all pairs [Cormen et al., 2009]). We propose replacing BFS with a GNN to approximate shortest-path distances between nodes and landmarks, which comes with three advantages: (i) embeddings are computed automatically once the GNN is trained, (ii) inference is cheaper than exact distance calculations, and (iii) the GNN's transferability [Ruiz et al., 2020, 2023] enables generalization to larger graphs from the same graphon model.

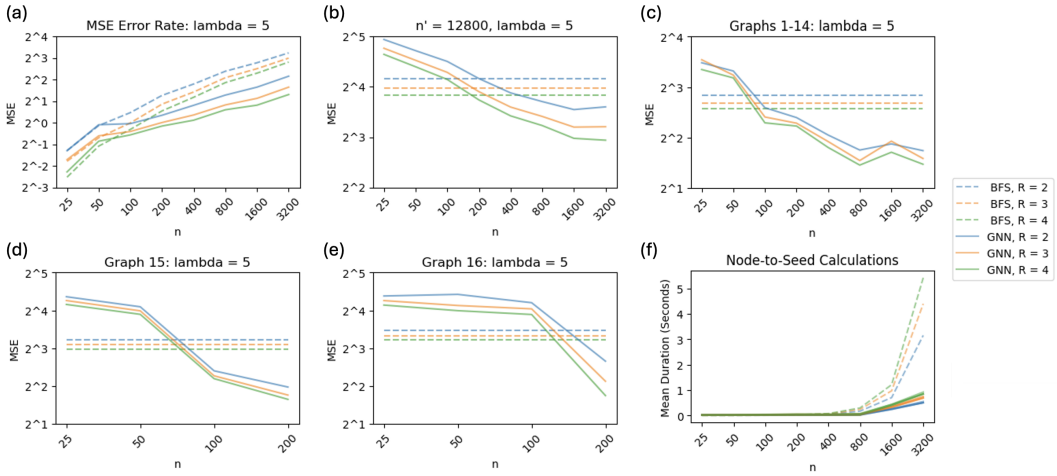

Figure 1: Error rates of BFS-based and GNN-based lower bounds on (a) test ER graphs generated from the same $\mathrm{ER}_n(\lambda/n)$ as the training graphs, (b) test ER graphs generated by $\mathrm{ER}_{n'}(\lambda/n')$ with larger graph size n', (c) real-world networks with 3,892 to 28,281 nodes, (d) Brightkite social network with 56,739 nodes, and (e) ER-AVGDEG10-100K-L2 labeled network with 99,997 nodes. (f) Duration of generating all landmark distances by NetworkX's highly optimized BFS compared with our widest and deepest GNNs—GCN, GraphSage, GAT, and GIN models were examined and are represented by solid lines of the same color for the same number of local step R. See Appendices B and C for further details and discussions on the experiments and benchmark networks.

GNNs are well-suited for this task as they align with dynamic programming strategies that are used in shortest-path algorithms [Xu et al., 2019b, Dudzik and Veličković, 2022]. As shown in Figure 1(a), the GNN-based approach outperforms exact landmark embeddings in approximating shortest-path distances with global lower bounds. Figures 1(b-e) further demonstrate the transferability of GNNs to larger ER graphs and real-world benchmark datasets, even when trained on smaller synthesized ER graphs. The GNN-based embeddings not only provide better distance approximations but also scale more efficiently in time and space, as illustrated in Figure 1(f), making them a promising tool for large-scale graph representation learning in practical applications.

5 Conclusion

Our analysis, focused on average-case random graphs, provides a simplified framework for developing theoretical tools and insights into landmark-based embedding algorithms. Particularly, Algorithm 1 achieves $(1 \pm \varepsilon)$-factor approximations of shortest-path distances on random graphs w.h.p. even with reduced embedding dimensionality, complementing Bourgain's worst-case results [1985]. By modifying the algorithm to integrate GNNs into the embedding construction, we further improve its generalizability and transferability while still reducing time and space complexity, as demonstrated by empirical results on ER graphs and benchmark datasets.

Limitations and Future Work. While our results improve upon existing landmark-based algorithms, several limitations remain. Our theoretical analysis focuses on ER graphs, a simplified model; extending it to more complex graphs (e.g., inhomogeneous random graphs, configuration models, planar graphs) is a key direction for future work. The approach also relies on GNNs generalizing from smaller to larger graphs; further studies are needed to assess robustness across diverse graph properties. Finally, additional improvements in memory and inference efficiency may be possible with advanced GNN architectures or alternative models.

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A Supporting Results and Proofs

The proofs of Theorems 3.1 and 3.2 rely on local neighborhood expansions in ER graphs $G \sim \mathrm{ER}_n(\lambda/n)$, which can be accessed via the Poisson branching process with mean offspring λ. Let $N_k(u)$ denote the set of nodes with graph distance at most k from u and $\partial N_k(u)$ denote the set of nodes at distance exactly k from u. By Lemmas A.1 and A.2, $\partial N_k(u)$ grows as λ^k for any fixed u. Furthermore, by Proposition A.3, $|\partial N_k(u_1) \cap \partial N_k(u_2)|$ grows as $\frac{\lambda^{2k}}{n}$ for any fixed u_1, u_2. These growth rates imply that, w.h.p., the local step selects a landmark set that intersects $N_{k_1}(u_1)$ but not the disjoint $N_{k_2}(u_2)$, with $k_2 - k_1 \geq (1-\varepsilon)d(u_1, u_2)$, yielding

$$\underline{d}(u_1, u_2) \;\geq\; (1-\varepsilon)\, d(u_1, u_2) \text{ w.h.p.}$$

For the upper bound, we show that w.h.p. there is a landmark set intersecting $N_k(u_1) \cap N_k(u_2)$ but not $\big(N_k(u_1) \cup N_k(u_2)\big) \setminus \big(N_k(u_1) \cap N_k(u_2)\big)$, where $k \leq \frac{1+\varepsilon}{2} d(u_1, u_2)$. This ensures

$$\overline{d}(u_1, u_2) \;\leq\; (1+\varepsilon)\, d(u_1, u_2) \text{ w.h.p.}$$

Finally, $\underline{d}(u_1, u_2) \leq d(u_1, u_2) \leq \bar{d}(u_1, u_2)$ follows directly from the formulation of $\underline{d}(u_1, u_2)$ and $\bar{d}(u_1, u_2)$ in Algorithm 1. The upper bound $\bar{d}(u_1, u_2)$ is trivial by triangle inequality: $\bar{d}(u_1, u_2) = d(u_1, v) + d(u_2, v) \geq d(u_1, u_2)$ for some landmark node v. To prove the lower bound, without loss of generality, assume $\underline{d}(u_1, u_2) = d(u_1, S) - d(u_2, S)$ for some landmark set S with $d(u_2, S) = d(u_2, v_1)$ and $d(u_1, S) = d(u_1, v_2) \leq d(u_1, v_1)$. Then triangle inequality implies $\underline{d}(u_1, u_2) \leq d(u_1, v_1) - d(u_2, v_1) \leq d(u_1, u_2)$.

Notation. We let $G = (V, E)$ denote an undirected, unweighted graph, where V is the set of nodes and E is the set of edges, with $|V| = n$ and $|E| = m$, where $|X|$ denotes the cardinality of any discrete set X. We consider only one graph at a time and use $\mathcal{C}_{(i)}$ to denote the i-th largest connected component of the graph. We write $u_1 \leftrightarrow u_2$ to mean that there exists a path between u_1 and u_2 (i.e., u_1 and u_2 are in the same connected component). We often use the Bachmann–Landau asymptotic notation $o(1), O(1), \omega(1), \Omega(1), \Theta(1)$, etc. to describe the asymptotic behavior of functions. Given a sequence of probability measures $(\mathbb{P}_n)_{n\geq 1}$, a sequence of events $(\mathcal{E}_n)_{n\geq 1}$ is said to hold *with high probability (w.h.p.)* if $\lim_{n\to\infty} \mathbb{P}_n(\mathcal{E}_n) = 1$. For a sequence of random variables $(X_n)_{n\geq 1}$, $X_n \xrightarrow{\mathbb{P}} c$ means that X_n converges to c in probability. We write statements such as $X_n = f(n)^{o(1)}$ w.h.p. to abbreviate that $\log X_n / \log f(n) \xrightarrow{\mathbb{P}} 0$. Also, we write $X_n = O(1)$ w.h.p. to mean that $\mathbb{P}(X_n \geq K) \to 0$ for a sufficiently large K.

Lemma A.1. *Let G, λ, u_1, u_2 be as in Theorems 3.1 and 3.2. Let $\kappa_0 \in (0, \frac{1}{2})$, $L = \kappa_0 \log_\lambda n$, and $\varepsilon \in (0, \kappa_0)$. Let A_n denote the event that $n^{-\varepsilon}\lambda^L \leq |\partial N_L(u_1)|, |\partial N_L(u_2)| \leq n^{\varepsilon}\lambda^L$ and B_n denote the event that u_1 and u_2 are in the same connected component. Then $\mathbb{P}(A_n \setminus B_n) \to 0$ and $\mathbb{P}(B_n \setminus A_n) \to 0$ as $n \to \infty$.*

Proof. If A_n occurs but B_n does not, then $|\mathcal{C}_{(2)}| \geq n^{\kappa_0 - \varepsilon}$, which occurs with probability tending to zero since $|\mathcal{C}_{(2)}| = O(\log n)$ w.h.p. On the other hand, if B_n occurs and A_n does not, then $|\partial N_L(u_1)| \notin [n^{\kappa_0-\varepsilon}, n^{\kappa_0+\varepsilon}]$ or $|\partial N_L(u_2)| \notin [n^{\kappa_0-\varepsilon}, n^{\kappa_0+\varepsilon}]$.

To bound the probabilities of these events, consider a branching process with progeny distribution being Poisson(λ)n and let $\mathcal{X}_l$ be the number of children at generation l. We first claim that, for any fixed node u in G,

$$\lim_{n\to\infty} \mathbb{P}(|\partial N_L(u)| = \mathcal{X}_L) = 1 \tag{1}$$

for any $\kappa_0 \in (0, \frac{1}{2})$ and $L = \kappa_0 \log_\lambda n$. Indeed, this is a consequence of Lemma 3.13 from Bordenave [2016]. Next, classical theory of branching processes shows that, on the event of survival, the growth rate of a branching process is exponential. More precisely, Theorem 5.5 (iii) from Tanny [1977] together with Theorem 2 from Athreya and Ney [1972] yields that

$$\lim_{L\to\infty} \mathbb{P}\big(L(1-\varepsilon) \leq \log_\lambda \mathcal{X}_L \leq L(1+\varepsilon), \mathcal{X}_L > 0\big) = 1.$$

Since $\kappa_0 - \varepsilon < \kappa_0(1-\varepsilon)$ and $\kappa_0 + \varepsilon > \kappa_0(1+\varepsilon)$,

$$\lim_{L\to\infty} \mathbb{P}\big(n^{\kappa_0-\varepsilon} \leq \mathcal{X}_L \leq n^{\kappa_0+\varepsilon}, \mathcal{X}_L > 0\big) = 1. \tag{2}$$

Combining (1) and (2), it follows that

$$\mathbb{P}(B_n \setminus A_n) \leq \mathbb{P}(|\partial N_L(u_1)| \notin [n^{\kappa_0-\varepsilon}, n^{\kappa_0+\varepsilon}]) + \mathbb{P}(|\partial N_L(u_2)| \notin [n^{\kappa_0-\varepsilon}, n^{\kappa_0+\varepsilon}]) \to 0.$$

□

Lemma A.2. *Let $G, \lambda, u_1, u_2, k_0, L$ be as in Lemma A.1. Let $\varepsilon \in (0, \kappa_0)$ and $\kappa \in (0, 1-\kappa_0)$. Let A_{b_m,b_M} be the event that $|\partial N_L(u_i)| \in [b_m, b_M]$ for $i \in \{1,2\}$ and $\mathcal{E}_{n,k}$ be the good event that $|\partial N_{L+k}(u_i)| \in [b_m\lambda^k, b_M\lambda^k]$ for $i \in \{1,2\}$, where $b_m = n^{-\varepsilon}\lambda^L$ and $b_M = n^{\varepsilon}\lambda^L$. Then, there exists $\delta > 0$ such that $\mathbb{P}(\cap_{l=0}^{k}\mathcal{E}_{n,l} \mid A_{b_m,b_M}) \geq 1 - 3kn^{-\delta}$ for any $k \leq \kappa \log_\lambda n$ for all sufficiently large n.*

Proof. The proof is adapted from Section 2.6.4 in van der Hofstad [2024]. Since we need an exponential bound on the probability and L grows with n, the proof does not follow from van der Hofstad [2024].

Note that for any fixed node u and any $k \geq 1$, $|\partial N_k(u)| \leq \sum_{x \notin N_{k-1}(u)} \sum_{y \in \partial N_{k-1}(u)} I_{xy}$, where I_{xy} is the indicator random variable for the edge $\{x, y\}$ being present. Therefore, $\mathbb{E}(|\partial N_k(u)|) \leq \lambda E(|\partial N_{k-1}(u)|)$. Proceeding inductively, we have $\mathbb{E}(|\partial N_k(u)|) \leq \lambda^k$ and consequently,

$$\mathbb{E}(|N_k(u)|) \leq \frac{\lambda^{k+1}-1}{\lambda-1} = O(\lambda^k). \tag{3}$$

Then with Markov's inequality, there exists $\delta > 0$ for any $\gamma \in (\kappa_0 + \kappa, 1)$ such that

$$\mathbb{P}(|N_k(u_i)| \geq n^\gamma) \leq \frac{O(\lambda^k)}{n^\gamma} \leq \frac{O\left(n^{\kappa_0+\kappa}\right)}{n^\gamma} \leq n^{-\delta}$$

for $i = 1, 2$ and $k \leq (\kappa_0+\kappa)\log_\lambda n$ with sufficiently large n. Then for each fixed $k \leq (\kappa_0+\kappa)\log_\lambda n$,

$$\mathbb{P}(|N_k(u_i)| \leq n^\gamma : i = 1,2) \geq 1 - \sum_{i=1,2} \mathbb{P}(|N_k(u_i)| \geq n^\gamma) \geq 1 - 2n^{-\delta}. \tag{4}$$

Let $\delta_n = n^{-\beta}$ with $0 < \beta < \frac{\kappa_0-\varepsilon'}{2}$. Also define

$$\bar{\mathcal{E}}_{n,k} = \{|\partial N_{L+k}(u_i)| \in [b'_m(1-\delta_n)^k(1-n^{\gamma-1})^k\lambda^k, b'_M(1+\delta_n)^k\lambda^k] : i = 1,2\}$$

with $b'_m = n^{-\varepsilon'}\lambda^L$ and $b'_M = n^{\varepsilon'}\lambda^L$ for some $0 < \varepsilon' < \min\{\varepsilon, 1-\kappa_0-\kappa\}$. Conditionally on A_{b_m,b_M},

$$\begin{aligned}
\mathbb{E}(|\partial N_{L+k}(u_i)| \mid N_{L+k-1}(u_i)) &= \mathbb{E}\left(\sum_{x \notin N_{L+k-1}(u_i)} \mathbb{1}_{\{\exists y \in \partial N_{L+k-1}(u_i): I_{xy}=1\}} \mid N_{L+k-1}(u_i)\right) \\
&= (n - |N_{L+k-1}(u_i)|)\mathbb{P}(\exists y \in \partial N_{L+k-1}(u_i) : I_{xy} = 1 \mid N_{L+k-1}(u_i);\ x \notin N_{L+k-1}(u_i)) \\
&= (n - |N_{L+k-1}(u_i)|)\left(1 - \left(1 - \frac{\lambda}{n}\right)^{|\partial N_{L+k-1}(u_i)|}\right).
\end{aligned}$$

Since $\frac{\lambda}{n} \in [0, 1]$,

$$1 - |\partial N_{L+k-1}(u_i)|\frac{\lambda}{n} \leq \left(1 - \frac{\lambda}{n}\right)^{|\partial N_{L+k-1}(u_i)|} \leq 1 - |\partial N_{L+k-1}(u_i)|\frac{\lambda}{n} + \frac{|\partial N_{L+k-1}(u_i)|^2}{2}\left(\frac{\lambda}{n}\right)^2.$$

Conditionally on $\bar{\mathcal{E}}_{n,k-1}$,

$$|\partial N_{L+k-1}(u_i)|\frac{\lambda}{n} \leq n^{\varepsilon'}(1+\delta_n)^{k-1}\frac{\lambda^{L+k}}{n} \leq n^{\varepsilon'-1}(1+\delta_n)^{k-1}n^{\kappa_0+\kappa} \to 0 \text{ as } n \to \infty.$$

Since $\left(|\partial N_{L+k-1}(u_i)|\frac{\lambda}{n}\right)^2$ vanishes faster than $|\partial N_{L+k-1}(u_i)|\frac{\lambda}{n}$, we have w.h.p. that

$$\left(1 - \frac{\lambda}{n}\right)^{|\partial N_{L+k-1}(u_i)|} = 1 - |\partial N_{L+k-1}(u_i)|\frac{\lambda}{n}$$

and so

$$\mathbb{E}(|\partial N_{L+k}(u_i)| \mid N_{L+k-1}(u_i)) = (n - |N_{L+k-1}(u_i)|)|\partial N_{L+k-1}(u_i)|\frac{\lambda}{n}.$$

Conditionally on $\cap_{l=0}^{k-1}\bar{\mathcal{E}}_{n,l}$ and A_{b_m,b_M}, from (4) we have with probability at least $1-2n^{-\delta}$ that

$$\mathbb{E}(|\partial N_{L+k}(u_i)| \mid N_{L+k-1}(u_i)) \in \left[b'_m(1-\delta_n)^{k-1}(1-n^{\gamma-1})^k\lambda^k, b'_M(1+\delta_n)^{k-1}\lambda^k\right]$$

since $1-n^{\gamma-1} \le 1-\frac{|N_{L+k-1}(u_i)|}{n} \le 1$ for $i=1,2$ with sufficiently large n. Denote this event R_k. The fact that $\mathbb{P}(A) \le \mathbb{P}(A \mid B) + \mathbb{P}(B^c)$ implies

$$\mathbb{P}(\bar{\mathcal{E}}^c_{n,k} \mid \cap_{l=0}^{k-1}\bar{\mathcal{E}}_{n,l}, A_{b_m,b_M}) \le \mathbb{P}(\bar{\mathcal{E}}^c_{n,k} \mid R_k, \cap_{l=0}^{k-1}\bar{\mathcal{E}}_{n,l}, A_{b_m,b_M}) + 2n^{-\delta}.$$

Using union bound and Chernoff-Hoeffding bound [Dubhashi and Panconesi, 2009, Theorem 1.1],

$$\begin{aligned}
&\mathbb{P}(\bar{\mathcal{E}}^c_{n,k} \mid R_k, \cap_{l=0}^{k-1}\bar{\mathcal{E}}_{n,l}, A_{b_m,b_M}) \\
&\quad\le \sum_{i=1,2} \mathbb{P}(||\partial N_{L+k}(u_i)| - \mathbb{E}(|\partial N_{L+k}(u_i)|)| \ge \delta_n \mathbb{E}(|\partial N_{L+k}(u_i)|) \mid R_k, \cap_{l=0}^{k-1}\bar{\mathcal{E}}_{n,l}, A_{b_m,b_M}) \\
&\quad\le \sum_{i=1,2} 2\exp\left(-\frac{\delta_n^2}{3}n^{-\varepsilon'}(1-\delta_n)^{k-1}(1-n^{\gamma-1})^k\lambda^{L+k}\right) \\
&\quad\le 4\exp\left(-\frac{n^{-2\beta}}{3}n^{-\varepsilon'}(1-\delta_n)^{k-1}(1-n^{\gamma-1})^k n^{\kappa_0}\right).
\end{aligned}$$

Since $(1-\delta_n)^{k-1}, (1-n^{\gamma-1})^k \to 1$ as $n \to \infty$ and $2\beta < \kappa_0 - \varepsilon'$, $4\exp\left(-\frac{n^{-2\beta}}{3}n^{-\varepsilon'}(1-\delta_n)^{k-1}(1-n^{\gamma-1})^k n^{\kappa_0}\right)$ vanishes faster than $2n^{-\delta}$. Then with sufficiently large n, $\mathbb{P}(\bar{\mathcal{E}}_{n,k} \mid \cap_{l=0}^{k-1}\bar{\mathcal{E}}_{n,l}, A_{b_m,b_M}) \ge 1-3n^{-\delta}$. Proceed inductively,

$$\begin{aligned}
&\mathbb{P}(\cap_{l=0}^{k}\bar{\mathcal{E}}_{n,l} \mid A_{b_m,b_M}) = \mathbb{P}(\bar{\mathcal{E}}_{n,k} \mid \cap_{l=0}^{k-1}\bar{\mathcal{E}}_{n,l}, A_{b_m,b_M}) \dots \mathbb{P}(\bar{\mathcal{E}}_{n,1} \mid \bar{\mathcal{E}}_{n,0}, A_{b_m,b_M})\mathbb{P}(\bar{\mathcal{E}}_{n,0} \mid A_{b_m,b_M}) \\
&\quad\ge (1-3n^{-\delta})\dots(1-3n^{-\delta})\mathbb{P}(\bar{\mathcal{E}}_{n,0} \mid A_{b_m,b_M}) = (1-3n^{-\delta})^k\mathbb{P}(\bar{\mathcal{E}}_{n,0} \mid A_{b_m,b_M}).
\end{aligned}$$

Since $b'_m(1-\delta_n)^k(1-n^{\gamma-1})^k \ge b_m$ and $b'_M(1+\delta_n)^k \le b_M$, $\bar{\mathcal{E}}_{n,0} \subseteq A_{b_m,b_M}$ and $\bar{\mathcal{E}}_{n,k} \subseteq \mathcal{E}_{n,k}$ for all $k \ge 0$. Hence, $\mathbb{P}(\bar{\mathcal{E}}_{n,0} \mid A_{b_m,b_M}) = 1$ and so

$$\mathbb{P}(\cap_{l=0}^{k}\mathcal{E}_{n,l} \mid A_{b_m,b_M}) \ge \mathbb{P}(\cap_{l=0}^{k}\bar{\mathcal{E}}_{n,l} \mid A_{b_m,b_M}) \ge (1-3n^{-\delta})^k \ge 1-3kn^{-\delta}.$$

□

Proposition A.3. *Let $G, \lambda, u_1, u_2, \kappa_0, \kappa, L$ be as in Lemma A.2 and $\varepsilon > 0$. Conditionally on u_1, u_2 being in the same connected component, $|\partial N_{k_1}(u_1) \cap \partial N_{k_2}(u_2)| \in \left[\frac{n^{-\varepsilon}\lambda^{k_1+k_2}}{2n}, \frac{n^{\varepsilon}\lambda^{k_1+k_2}}{n}\right]$ w.h.p. for any k_1, k_2 such that $L < k_1, k_2 \le (\kappa_0+\kappa)\log_\lambda n$ and $k_1+k_2 > (1+\zeta)\log_\lambda n$ for any small $\zeta > 0$.*

Proof. Recall all the notation from Lemma A.2 and its proof. Then for any k_1, k_2 such that $L < k_1, k_2 \le k = (\kappa_0+\kappa)\log_\lambda n$,

$$\begin{aligned}
&\mathbb{E}(|\partial N_{k_1}(u_1) \cap \partial N_{k_2}(u_2)| \mid N_{k_1}(u_1), N_{k_2-1}(u_2)) \\
&\quad= \mathbb{E}\left(\sum_{x \in \partial N_{k_1}(u_1) \setminus N_{k_2-1}(u_2)} \mathbb{1}_{\{\exists y \in \partial N_{k_2-1}(u_2): I_{xy}=1\}} \mid N_{k_1}(u_1), N_{k_2-1}(u_2)\right) \\
&\quad= \left(|\partial N_{k_1}(u_1)| - \sum_{j \le k_2-1} |\partial N_{k_1}(u_1) \cap \partial N_j(u_2)|\right)\left(1-\left(1-\frac{\lambda}{n}\right)^{|\partial N_{k_2-1}(u_2)|}\right).
\end{aligned}$$

Since $\frac{\lambda}{n} \in [0,1]$,

$$1-|\partial N_{k_2-1}(u_2)|\frac{\lambda}{n} \le \left(1-\frac{\lambda}{n}\right)^{|\partial N_{k_2-1}(u_2)|} \le 1-|\partial N_{k_2-1}(u_2)|\frac{\lambda}{n} + \frac{|\partial N_{k_2-1}(u_2)|^2}{2}\left(\frac{\lambda}{n}\right)^2.$$

Conditionally on u_1, u_2 being in the same connected component, Lemmas A.1 and A.2 imply that with probability at least $1 - 3n^{-\delta}(\lfloor k \rfloor - \lfloor L \rfloor) \geq 1 - 3n^{-\delta}(\kappa \log_\lambda n + 1)$ for some $\delta > 0$, $\cap_{l=0}^{k-L} \mathcal{E}_{n,l}$ occurs. Then with $\varepsilon' \in (0, \min\{\varepsilon, 1 - \kappa_0 - \kappa\})$ (ε' to be chosen later),

$$|\partial N_{k_2-1}(u_2)| \frac{\lambda}{n} \leq n^{\frac{\varepsilon'}{2}} \frac{\lambda^{k_2}}{n} \leq n^{\frac{\varepsilon'}{2}-1} n^{\kappa_0+\kappa} \to 0 \text{ as } n \to \infty.$$

Since $|\partial N_{k_2-1}(u_2)| \frac{\lambda}{n}$ vanishes and $\left(|\partial N_{k_2-1}(u_2)| \frac{\lambda}{n}\right)^2$ vanishes faster, we have w.h.p. that $\left(1 - \frac{\lambda}{n}\right)^{|\partial N_{k_2-1}(u_2)|} = 1 - |\partial N_{k_2-1}(u_2)| \frac{\lambda}{n}$, and so

$$\begin{aligned} &\mathbb{E}\big(|\partial N_{k_1}(u_1) \cap \partial N_{k_2}(u_2)| \,\big|\, N_{k_1}(u_1), N_{k_2-1}(u_2)\big) \\ &= \left(|\partial N_{k_1}(u_1)| - \sum_{j \leq k_2-1} |\partial N_{k_1}(u_1) \cap \partial N_j(u_2)| \right) |\partial N_{k_2-1}(u_2)| \frac{\lambda}{n}. \end{aligned} \tag{5}$$

Conditionally on $\cap_{l=0}^{k-L} \mathcal{E}_{n,l}$,

$$\begin{aligned} \mathbb{E}\big(|\partial N_{k_1}(u_1) \cap \partial N_{k_2}(u_2)| \,\big|\, N_{k_1}(u_1), N_{k_2-1}(u_2)\big) &\leq n^{\frac{\varepsilon'}{2}} \lambda^{k_1} n^{\frac{\varepsilon'}{2}} \frac{\lambda^{k_2}}{n} \\ &\leq n^{\varepsilon'} \lambda^{k_1} \frac{n^{\kappa_0+\kappa}}{n} \leq \lambda^{k_1} \frac{n^{-\gamma}}{7(\lfloor k \rfloor - \lfloor L \rfloor)} \end{aligned} \tag{6}$$

for all $L < k_2 \leq k$ with $0 < \gamma < \min\{\kappa_0, 1 - \kappa_0 - \kappa\}$ and sufficiently large n. Here we choose ε' small enough so that $\frac{\varepsilon'}{2} < \gamma$, $\frac{k_1}{\log_\lambda n} - \frac{\varepsilon'}{2} > \kappa_0$, and $k_1 + k_2 > (1 + \varepsilon') \log_\lambda n$.

Let A be the event that there exists $L < j \leq k_2$ such that $|\partial N_{k_1}(u_1) \cap \partial N_j(u_2)| \geq \lambda^{k_1} \frac{n^{-\gamma}}{\lfloor k \rfloor - \lfloor L \rfloor}$. Let B be the event that $\mathbb{E}(|\partial N_{k_1}(u_1) \cap \partial N_j(u_2)| \mid N_{k_1}(u_1), N_{k_2-1}(u_2)) \leq \lambda^{k_1} \frac{n^{-\gamma}}{7(\lfloor k \rfloor - \lfloor L \rfloor)}$ for all $L < j \leq k_2$. The fact that $\mathbb{P}(A) \leq \mathbb{P}(A \mid B) + \mathbb{P}(B^c)$ implies

$$\mathbb{P}(A \mid N_{k_1}(u_1), N_{k_2-1}(u_2)) \leq \mathbb{P}(A \mid B, N_{k_1}(u_1), N_{k_2-1}(u_2)) + 3n^{-\delta}(\kappa \log_\lambda n + 1).$$

By Theorem 2.8 and Corollary 2.4 from Janson et al. [2000] with union bound,

$$\begin{aligned} \mathbb{P}(A \mid B, N_{k_1}(u_1), N_{k_2-1}(u_2)) &\leq (\lfloor k \rfloor - \lfloor L \rfloor) \exp\left(-\lambda^{k_1} \frac{n^{-\gamma}}{\lfloor k \rfloor - \lfloor L \rfloor} \right) \\ &< (\kappa \log_\lambda n + 1) \exp\left(-\frac{n^{\kappa_0 - \gamma}}{\lfloor k \rfloor - \lfloor L \rfloor} \right) = (\kappa \log_\lambda n + 1) \exp\left(-n^{\gamma'} \right) \end{aligned}$$

for some $\gamma' = \kappa_0 - \gamma > 0$. It follows that

$$\begin{aligned} \mathbb{P}\left(A^c \mid N_{k_1}(u_1), N_{k_2-1}(u_2) \right) &\geq 1 - (\kappa \log_\lambda n + 1) \exp\left(-n^{\gamma'} \right) - 3n^{-\delta}(\kappa \log_\lambda n + 1) \\ &\geq 1 - 4n^{-\delta}(\kappa \log_\lambda n + 1) \end{aligned} \tag{7}$$

for sufficiently large n since $(\kappa \log_\lambda n + 1) \exp\left(-n^{\gamma'} \right)$ vanishes faster than $3n^{-\delta}(\kappa \log_\lambda n + 1)$.

Let $\gamma'' \in \left(\kappa_0, \frac{k_1}{\log_\lambda n} - \frac{\varepsilon'}{2} \right)$. Markov's inequality and Lemma 3 imply that there exists $\delta' > 0$ such that

$$\mathbb{P}(|N_L(u_2)| \geq n^{\gamma''}) \leq \frac{O(\lambda^L)}{n^{\gamma''}} \leq \frac{O\left(n^{\kappa_0}\right)}{n^{\gamma''}} \leq n^{-\delta'} \tag{8}$$

for sufficiently large n.

Combining (5), (6), (7), (8) with Lemmas A.1, A.2, we have w.h.p. that

$$\begin{aligned} &\mathbb{E}\big(|\partial N_{k_1}(u_1) \cap \partial N_{k_2}(u_2)| \,\big|\, N_{k_1}(u_1), N_{k_2-1}(u_2)\big) \\ &\geq \left(|\partial N_{k_1}(u_1)| - \sum_{L < j \leq k_2-1} |\partial N_{k_1}(u_1) \cap \partial N_j(u_2)| - |N_L(u_2)| \right) |\partial N_{k_2-1}(u_2)| \frac{\lambda}{n} \\ &\geq \left(n^{-\frac{\varepsilon'}{2}} \lambda^{k_1} - (\lfloor k_2 \rfloor - 1 - \lfloor L \rfloor) \lambda^{k_1} \frac{n^{-\gamma}}{\lfloor k \rfloor - \lfloor L \rfloor} - n^{\gamma''} \right) \frac{n^{-\frac{\varepsilon'}{2}} \lambda^{k_2}}{n} > \frac{n^{-\varepsilon'} \lambda^{k_1+k_2}}{2n} \end{aligned}$$

and

$$\mathbb{E}\big(|\partial N_{k_1}(u_1)\cap\partial N_{k_2}(u_2)| \;\big|\; N_{k_1}(u_1), N_{k_2-1}(u_2)\big) \le n^{\varepsilon'}\frac{\lambda^{k_1+k_2}}{n},$$

where the last "$\geq$" holds since $\lambda^{k_1}n^{-\gamma}$ and $n^{\gamma''}$ grow strictly slower than $n^{-\frac{\varepsilon'}{2}}\lambda^{k_1}$ as $\frac{\varepsilon'}{2}<\gamma$ and $\gamma''<\frac{k_1}{\log_\lambda n}-\frac{\varepsilon'}{2}$. Therefore, $\mathbb{E}(|\partial N_{k_1}(u_1)\cap\partial N_{k_2}(u_2)|)\in\left[\frac{n^{-\varepsilon'}\lambda^{k_1+k_2}}{2n},\frac{n^{\varepsilon'}\lambda^{k_1+k_2}}{n}\right]$ and we denote this event S.

Let R denote the event that $|\partial N_{k_1}(u_1)\cap\partial N_{k_2}(u_2)|\notin\left[\frac{(1-\varepsilon)n^{-\varepsilon'}\lambda^{k_1+k_2}}{2n},\frac{(1+\varepsilon)n^{\varepsilon'}\lambda^{k_1+k_2}}{n}\right]$. Using Chernoff-Hoeffding bound [Dubhashi and Panconesi, 2009, Theorem 1.1],

$$\mathbb{P}(R)\le\mathbb{P}(R\mid S)+\mathbb{P}(S^c)\le 2\exp\left(-\frac{\varepsilon^2}{3}\frac{n^{-\varepsilon'}\lambda^{k_1+k_2}}{2n}\right)+\mathbb{P}(S^c).$$

Since $k_1+k_2>(1+\varepsilon')\log_\lambda n$ and S occurs w.h.p., $\mathbb{P}\left(R^c\right)$ converges to 1. Then w.h.p.,

$$|\partial N_{k_1}(u_1)\cap\partial N_{k_2}(u_2)|\in\left[\frac{(1-\varepsilon)n^{-\varepsilon'}\lambda^{k_1+k_2}}{2n},\frac{(1+\varepsilon)n^{\varepsilon'}\lambda^{k_1+k_2}}{n}\right]\subseteq\left[\frac{n^{-\varepsilon}\lambda^{k_1+k_2}}{2n},\frac{n^{\varepsilon}\lambda^{k_1+k_2}}{n}\right].$$

□

Proof of Theorem 3.1. Let $k_1=\varepsilon' d(u_1,u_2)$ and $k_2=(1-\varepsilon+\varepsilon')d(u_1,u_2)$, where $\varepsilon'=\min\left\{\frac{\varepsilon}{2},\varepsilon-\theta\right\}-\varepsilon''\in\left(0,\min\left\{\frac{\varepsilon}{2},\varepsilon-\theta\right\}\right)$ ($\varepsilon''\in\left(0,\min\left\{\frac{\varepsilon}{2},\varepsilon-\theta\right\}\right)$ to be chosen later). Since $\varepsilon'<\frac{\varepsilon}{2}$, $k_1+k_2<d(u_1,u_2)$, and so $N_{k_1}(u_1)\cap N_{k_2}(u_2)=\varnothing$. Conditionally on u_1,u_2 being in the same connected component, Theorem 2.36 from van der Hofstad [2024] implies that $d(u_1,u_2)/\log_\lambda n\xrightarrow{\mathbb{P}}1$. In other words, $(1-\epsilon)\log_\lambda n\le d(u_1,u_2)\le(1+\epsilon)\log_\lambda n$ w.h.p. for any fixed $\epsilon>0$. With ϵ small enough so that $\varepsilon'(1+\epsilon)<1$, $k_1\le\varepsilon'(1+\epsilon)\log_\lambda n<\log_\lambda n$ w.h.p., allowing us to apply Lemmas A.1 and A.2 on $|\partial N_{k_1}(u_1)|$.

Let S_{ij} be the landmark set of size M^i sampled in the j-th round and Z_{ij} denote the event that $S_{ij}\cap N_{k_1}(u_1)\neq\varnothing$ but $S_{ij}\cap N_{k_2}(u_2)=\varnothing$. If Z_{ij} happens for some $i\le r$ and $j\le R$, then $d(u_1,S_{ij})\le k_1$ and $d(u_2,S_{ij})\ge k_2$, and consequently, $\underline{d}(u_1,u_2)\ge k_2-k_1=(1-\varepsilon)d(u_1,u_2)$. Thus, denoting $Z=\cup_{i\le r,j\le R}Z_{ij}$, it suffices to prove that $\mathbb{P}(Z\mid u_1\leftrightarrow u_2)\xrightarrow{\mathbb{P}}1$. Since $\mathbb{P}(u_1\leftrightarrow u_2$ but $u_1,u_2\notin\mathcal{C}_{(1)})=\frac{1}{n^2}\sum_{i\ge2}|\mathcal{C}_{(i)}|^2\le\frac{|\mathcal{C}_{(2)}|}{n}\xrightarrow{\mathbb{P}}0$, it suffices to show that $\mathbb{P}(Z\mid u_1,u_2\in\mathcal{C}_{(1)})\xrightarrow{\mathbb{P}}1$ (or equivalently $\mathbb{P}(Z^c\mid u_1,u_2\in\mathcal{C}_{(1)})\xrightarrow{\mathbb{P}}0$). The fact that $\mathbb{P}(A^c\cap B)=\mathbb{P}(B)-\mathbb{P}(A\cap B)$ implies, for each (i,j), that

$$\begin{aligned}\mathbb{P}(Z_{ij}\mid u_1,u_2\in\mathcal{C}_{(1)})&=\mathbb{P}(S_{ij}\cap N_{k_1}(u_1)\neq\varnothing,S_{ij}\cap N_{k_2}(u_2)=\varnothing\mid u_1,u_2\in\mathcal{C}_{(1)})\\&=\left(1-\frac{|N_{k_2}(u_2)|}{n}\right)^{M^i}-\left(1-\frac{|N_{k_1}(u_1)|+|N_{k_2}(u_2)|}{n}\right)^{M^i}.\end{aligned}$$

By independence of Z_{ij}'s,

$$\begin{aligned}&\mathbb{P}(Z^c\mid u_1,u_2\in\mathcal{C}_{(1)})\\&=\left(\prod_{i=0}^{r}\left(1-\left(1-\frac{|N_{k_2}(u_2)|}{n}\right)^{M^i}+\left(1-\frac{|N_{k_1}(u_1)|+|N_{k_2}(u_2)|}{n}\right)^{M^i}\right)\right)^R\\&\le\exp\left(-R\sum_{i=0}^{r}\left(\left(1-\frac{|N_{k_2}(u_2)|}{n}\right)^{M^i}-\left(1-\frac{|N_{k_1}(u_1)|+|N_{k_2}(u_2)|}{n}\right)^{M^i}\right)\right)\\&=\exp\left(-R\sum_{i=0}^{r}\frac{|N_{k_1}(u_1)|}{n}\sum_{j=0}^{M^i-1}\left(1-\frac{|N_{k_2}(u_2)|}{n}\right)^{M^i-1-j}\left(1-\frac{|N_{k_1}(u_1)|+|N_{k_2}(u_2)|}{n}\right)^{j}\right)\\&\le\exp\left(-R\frac{|N_{k_1}(u_1)|}{n}\sum_{i=0}^{r}M^i\left(1-\frac{|N_{k_1}(u_1)|+|N_{k_2}(u_2)|}{n}\right)^{M^i-1}\right)\\&<\exp\left(-R\frac{|\partial N_{k_1}(u_1)|}{n}M^r\left(1-\frac{|N_{k_1}(u_1)|+|N_{k_2}(u_2)|}{n}\right)^{M^r}\right)\end{aligned}$$

where the first "$\leq$" uses $1-x \leq \exp(-x)$ and "$<$" uses $\sum_{i=0}^{r} M^i = \frac{M^{r+1}-1}{M-1} > \frac{M^{r+1}-M^r}{M-1} = M^r$.

Recall that $(1-\epsilon)\log_\lambda n \leq d(u_1,u_2) \leq (1+\epsilon)\log_\lambda n$ w.h.p. for any fixed $\epsilon > 0$. Choosing ϵ small enough so that $(1-\varepsilon+\varepsilon')(1+\epsilon) < 1-\theta$, we have that $k_2 \leq (1-\varepsilon+\varepsilon')(1+\epsilon)\log_\lambda n < (1-\theta)\log_\lambda n$ w.h.p., and so there exists $\gamma \in (0, 1-\theta)$ such that $k_1 < k_2 < \gamma \log_\lambda n$. By Markov's inequality and (3), there exists $\delta > 0$ such that $\mathbb{P}(|N_{k_i}(u_i)| \geq n^\gamma) \leq \frac{O(\lambda^{k_i})}{n^\gamma} \leq n^{-\delta}$ for $i = 1, 2$ with sufficiently large n. Therefore,

$$\mathbb{P}(|N_{k_i}(u_i)| \leq n^\gamma : i = 1,2) \geq 1 - \sum_{i=1,2} \mathbb{P}(|N_{k_i}(u_i)| \geq n^\gamma) \geq 1 - 2n^{-\delta},$$

and so $|N_{k_i}(u_i)| \leq n^\gamma$ for $i = 1, 2$ w.h.p.

By Lemmas A.1 and A.2, $|\partial N_{k_1}(u_1)| \geq n^{-\varepsilon'''}\lambda^{k_1} \geq n^{-\varepsilon'''} n^{\varepsilon'(1-\epsilon)}$ w.h.p. for any $\varepsilon''' > 0$, and so

$$\begin{aligned}
&\mathbb{P}(Z^c \mid u_1, u_2 \in \mathcal{C}_{(1)}) \\
&< \exp\left(- R \frac{n^{-\varepsilon'''} n^{\left(\min\left\{\frac{\varepsilon}{2}, \varepsilon-\theta\right\} - \varepsilon''\right)(1-\epsilon)}}{n} M^{\frac{\theta}{\log M}\log n - 1} \left(1 - \frac{2n^\gamma}{n}\right)^{M^{\frac{\theta}{\log M}\log n}} \right) \\
&= \exp\left(- R \frac{n^{-\varepsilon'''} n^{\min\left\{\frac{\varepsilon}{2}, \varepsilon-\theta\right\} - \epsilon\min\left\{\frac{\varepsilon}{2}, \varepsilon-\theta\right\} - \varepsilon'' + \varepsilon''\epsilon}}{nM} n^\theta \left(1 - \frac{2n^\gamma}{n}\right)^{n^\theta} \right).
\end{aligned}$$

Since $\gamma < 1-\theta$, $\left(1 - \frac{2n^\gamma}{n}\right)^{n^\theta} \geq 1 - \frac{2n^{\gamma+\theta}}{n} \to 1$ as $n \to \infty$. Since $\varepsilon'', \varepsilon''', \epsilon$ can be chosen small enough so that $-\varepsilon''' - \epsilon\min\left\{\frac{\varepsilon}{2}, \varepsilon-\theta\right\} - \varepsilon'' + \varepsilon''\epsilon < \varsigma$ for any $\varsigma > 0$, $R = \Omega\left(Mn^{1-\theta-\min\left\{\frac{\varepsilon}{2}, \varepsilon-\theta\right\}+\varsigma}\right)$ is sufficient for the final bound to tend to 0. Since $\theta \in (0, \varepsilon)$, R can be further simplified to $\Omega\left(Mn^{1-\frac{\varepsilon}{2}-\min\left\{\frac{\varepsilon}{2}, \theta\right\}+\varsigma}\right)$.

Proof of Theorem 3.2. Let $k = \varepsilon' d(u_1,u_2)$ with $\varepsilon' = \frac{1+\varepsilon}{2} - \varepsilon'' \in \left(0, \frac{1+\varepsilon}{2}\right)$ ($\varepsilon'' \in \left(0, \frac{1+\varepsilon}{2}\right)$ to be chosen later). Conditionally on u_1, u_2 being in the same connected component, Theorem 2.36 from van der Hofstad [2024] implies that $d(u_1,u_2)/\log_\lambda n \xrightarrow{\mathbb{P}} 1$. In other words, $(1-\epsilon)\log_\lambda n \leq d(u_1,u_2) \leq (1+\epsilon)\log_\lambda n$ w.h.p. for any fixed $\epsilon > 0$. With ε'', ϵ small enough so that $\varepsilon'(1+\epsilon) < 1$ and $2\left(\frac{1+\varepsilon}{2} - \varepsilon''\right)(1-\epsilon) > 1$, $k \leq \varepsilon'(1+\epsilon)\log_\lambda n < \log_\lambda n$ and $k + k \geq 2\varepsilon'(1-\epsilon)\log_\lambda n > \log_\lambda n$ w.h.p. This allows us to apply Lemmas A.1 and A.2 and Proposition A.3.

Let S_{ij} be the landmark set of size M^i sampled in the j-th round and Z_{ij} be the event that S_{ij} contains at least one landmark node in $N_k(u_1) \cap N_k(u_2)$ and none in $(N_k(u_1) \cup N_k(u_2)) \setminus (N_k(u_1) \cap N_k(u_2))$. If Z_{ij} happens for some $i \leq r$ and $j \leq R$, the landmarks in the intersection will be the common landmarks for calculating $\bar{d}(u_1,u_2)$, and so $\bar{d}(u_1,u_2) \leq 2k \leq (1+\varepsilon)d(u_1,u_2)$. Thus, denoting $Z = \cup_{i\leq r, j\leq R} Z_{ij}$, it suffices to prove that $\mathbb{P}(Z \mid u_1 \leftrightarrow u_2) \xrightarrow{\mathbb{P}} 1$. Since $\mathbb{P}(u_1 \leftrightarrow u_2 \text{ but } u_1, u_2 \notin \mathcal{C}_{(1)}) = \frac{1}{n^2}\sum_{i\geq 2}|\mathcal{C}_{(i)}|^2 \leq \frac{|\mathcal{C}_{(2)}|}{n} \xrightarrow{\mathbb{P}} 0$, it suffices to show that $\mathbb{P}(Z \mid u_1, u_2 \in \mathcal{C}_{(1)}) \xrightarrow{\mathbb{P}} 1$ (or equivalently $\mathbb{P}(Z^c \mid u_1, u_2 \in \mathcal{C}_{(1)}) \xrightarrow{\mathbb{P}} 0$). Note that for each (i,j),

$$\begin{aligned}
&\mathbb{P}(Z_{ij} \mid u_1, u_2 \in \mathcal{C}_{(1)}) \\
&= \frac{|N_k(u_1) \cap N_k(u_2)|}{n} \left(\frac{|N_k(u_1) \cap N_k(u_2)|}{n} + 1 - \frac{|N_k(u_1) \cup N_k(u_2)|}{n} \right)^{M^i - 1}.
\end{aligned}$$

By independence of Z_{ij}'s,

$$\begin{aligned}
&\mathbb{P}(Z^c \mid u_1, u_2 \in \mathcal{C}_{(1)}) \\
&= \left(\prod_{i=0}^{r} \left(1 - \frac{|N_k(u_1) \cap N_k(u_2)|}{n} \left(\frac{|N_k(u_1) \cap N_k(u_2)|}{n} + 1 - \frac{|N_k(u_1) \cup N_k(u_2)|}{n} \right)^{M^i - 1} \right) \right)^R \\
&\leq \exp\left(-R \sum_{i=0}^{r} \frac{|\partial N_k(u_1) \cap \partial N_k(u_2)|}{n} \left(\frac{|\partial N_k(u_1) \cap \partial N_k(u_2)|}{n} + 1 - \frac{|N_k(u_1)| + |N_k(u_2)|}{n} \right)^{M^i - 1} \right).
\end{aligned}$$

Choosing $L \in (0, \min\{k, \gamma \log_\lambda n\})$ for some $\gamma \in (0, 1-\theta)$, we obtain from Markov's inequality and Lemma 3 that $\mathbb{P}(|N_L(u_i)| \geq n^\gamma) \leq \frac{O(\lambda^L)}{n^\gamma} \leq n^{-\delta}$ for $i = 1, 2$ with some $\delta > 0$ and sufficiently large n. Therefore,

$$\mathbb{P}(|N_L(u_i)| \leq n^\gamma : i = 1, 2) \geq 1 - \sum_{i=1,2} \mathbb{P}(|N_L(u_i)| \geq n^\gamma) \geq 1 - 2n^{-\delta},$$

and so $|N_L(u_i)| \leq n^\gamma$ for $i = 1, 2$ w.h.p. Then by Lemmas A.1 and A.2,

$$|N_k(u_i)| = |N_L(u_i)| + \sum_{l=L+1}^{k} |\partial N_l(u_i)| \leq n^\gamma + \sum_{l=L+1}^{k} n^{\varepsilon'''} \lambda^l < n^\gamma + (\log_\lambda n) n^{\varepsilon'''} \lambda^k$$

for $i = 1, 2$ w.h.p. with $0 < \varepsilon''' < 1 - \kappa_0 - \kappa$. Combining these with Proposition A.3, we have w.h.p. that

$$\begin{aligned}
&\mathbb{P}(Z^c \mid u_1, u_2 \in \mathcal{C}_{(1)}) \\
&\quad \leq \exp\left(-R \frac{n^{-\varepsilon'''} \lambda^{2k}}{2n^2} \sum_{i=0}^{r} \left(\frac{n^{-\varepsilon'''} \lambda^{2k}}{2n^2} + 1 - \frac{2n^\gamma}{n} - \frac{2(\log_\lambda n) n^{\varepsilon'''} \lambda^k}{n} \right)^{M^i - 1} \right).
\end{aligned}$$

Since $\gamma < 1$ and $0 < \lambda^k \leq n^{\kappa_0 + \kappa} < n^{1-\varepsilon'''} < n$, $\frac{2n^\gamma}{n} + \frac{2(\log_\lambda n) n^{\varepsilon'''} \lambda^k}{n} - \frac{n^{-\varepsilon'''} \lambda^{2k}}{2n^2} \in (0, 1)$ when n is large, and so

$$\begin{aligned}
&\mathbb{P}(Z^c \mid u_1, u_2 \in \mathcal{C}_{(1)}) \\
&\quad \leq \exp\left(-R \frac{n^{-\varepsilon'''} \lambda^{2k}}{2n^2} \sum_{i=0}^{r} \left(1 - \left(\frac{2n^\gamma}{n} + \frac{2(\log_\lambda n) n^{\varepsilon'''} \lambda^k}{n} - \frac{n^{-\varepsilon'''} \lambda^{2k}}{2n^2} \right) (M^i - 1) \right) \right) \\
&\quad < \exp\left(-R \frac{n^{-\varepsilon'''} \lambda^{2k}}{2n^2} \left(r - \left(\frac{2n^\gamma}{n} + \frac{2(\log_\lambda n) n^{\varepsilon'''} \lambda^k}{n} - \frac{n^{-\varepsilon'''} \lambda^{2k}}{2n^2} \right) \frac{n^\theta M}{M-1} \right) \right)
\end{aligned}$$

since $\sum_{i=0}^{r} (M^i - 1) < \sum_{i=0}^{r} M^i = \frac{M^{r+1} - 1}{M-1} \leq \frac{n^\theta M}{M-1}$. Since $\gamma < 1 - \theta$, $0 < \frac{2n^{\gamma+\theta}}{n} \to 0$ as $n \to \infty$. Since we can choose $\varepsilon'', \varepsilon''', \epsilon$ small enough so that $\varepsilon''' + \left(\frac{1+\varepsilon}{2} - \varepsilon''\right)(1+\epsilon) < 1 - \theta$, we then obtain $0 < n^{-\varepsilon'''} \frac{\lambda^{2k}}{n^2} n^\theta < n^{\varepsilon'''} \frac{\lambda^k}{n} n^\theta \leq \frac{n^{\varepsilon''' + \left(\frac{1+\varepsilon}{2} - \varepsilon''\right)(1+\epsilon) + \theta}}{n} < 1$ for sufficiently large n. Then w.h.p.,

$$\begin{aligned}
\mathbb{P}(Z^c \mid u_1, u_2 \in \mathcal{C}_{(1)}) &< \exp\left(-R \frac{n^{-\varepsilon'''} n^{2\left(\frac{1+\varepsilon}{2} - \varepsilon''\right)(1-\epsilon)}}{2n^2} \left(\frac{\theta}{\log M} \log n - 1 - (1 + 1 - 0) \right) \right) \\
&< \exp\left(-R \frac{n^{-\varepsilon'''} n^{1+\varepsilon+2\left(\varepsilon''\epsilon - \varepsilon'' - \epsilon\frac{1+\varepsilon}{2}\right)}}{2n^2} \frac{\theta}{2\log M} \log n \right).
\end{aligned}$$

Since $\varepsilon'', \varepsilon''', \epsilon$ can be chosen small enough so that $-\varepsilon''' + 2\left(\varepsilon''\epsilon - \varepsilon'' - \epsilon\frac{1+\varepsilon}{2}\right) < \varsigma$ for any $\varsigma > 0$, $R = \Omega\left(\frac{\log M}{\theta \log n} n^{1-\varepsilon+\varsigma}\right)$ is sufficient for the final bound to tend to 0.

B Experimental Setup

In our experiments, we train GNNs to approximate the landmark distances in sparse, undirected, unweighted random graphs. We consider four standard GNN architectures (GCN [Kipf and Welling, 2017], GraphSAGE [Hamilton et al., 2017a], GAT [Veličković et al., 2018], and GIN [Xu et al., 2019a]) with sum aggregation, dropout, and ReLU activations. For each architecture, we test nine models with $\lfloor \sqrt{n} \rfloor$ nodes in the first and last layers and hidden layers varying in depth and width:

- Depth-6: 128-64-32-16, 64-32-16-8, 32-16-8-4

- Depth-5: 128-64-32, 64-32-16, 32-16-8
- Depth-4: 128-64, 64-32, 32-16

The training data for the GNNs are graphs generated by $\mathrm{ER}_n(\lambda/n)$ with $1 < \lambda \ll n$, which ensures sparsity and the existence of a giant component w.h.p. In particular, we consider $\lambda \in \{3, 4, 5, 6\}$ and $n \in \{25, 50, 100, 200, 400, 800, 1600, 3200\}$. Each graph is treated as a batch of nodes with a 200-50-50 train-validation-test split to generate random input signals $\mathbf{X} \in \mathbb{R}^{n \times r}$, where each column one-hot encodes a landmark node. The outputs $\mathbf{Y} \in \mathbb{R}^{n \times r}$ have the same dimensions as the inputs and represent shortest path distances between nodes u and landmarks s, i.e., $[\mathbf{Y}]_{us} = d(u, s)$. Training runs for 1000 epochs with early stopping (100 epochs), MSE loss, Adam optimizer (lr=0.01, weight decay=0.0001), and a cyclic-cosine learning rate schedule (0.001–0.1 for 10 cycles, with default cosine annealing for up to 20 iterations).

All experiments use PyTorch Geometric [Fey and Lenssen, 2019] on a Lambda Vector 1 machine (AMD Ryzen Threadripper PRO 5955WX CPU, 16 cores, 128 GB RAM, 2× NVIDIA RTX 4090 GPUs, no parallel training). Code is available at `https://github.com/ruiz-lab/shortest-path`.

C More Experimental Results

We provide additional results with more detailed explanations offering deeper insights into GNNs and their use for generating landmark embeddings in shortest path approximations. The experiments are divided into three categories: learning the predictive power of GNNs, comparing the performance of the GNN-augmented approach with the vanilla landmark-based algorithm, and evaluating the transferability of both methods to larger random graphs and real-world benchmarks.

C.1 Experiment 1: Learning the GNNs

In the first experiment, we evaluate the ability of trained GNNs to compute end-to-end shortest paths. We consider $n = 50$ and set the GNN depth to be larger than $\lceil \log_\lambda n \rceil$. Figure 2 plots the actual shortest path distances versus those predicted by our selected GNN architectures. Predictions for distances beyond the GNN depth saturate, indicating that GNNs cannot capture longer distances even with depth exceeding the expected path length. As expected, GNNs are not suitable for computing end-to-end shortest path distances, especially on sparser graphs with $\lambda \in \{3, 4\}$, which tend to exhibit longer paths.

C.2 Experiment 2: Comparing BFS-Based and GNN-Based Landmark Embeddings

In this experiment, we compare the lower bounds (LBs) resulting from BFS-based and GNN-based landmark embeddings against the actual shortest path distances. *Only LBs are compared to ensure a fair evaluation, as computing the upper bounds (UBs) requires storing additional information—namely, the indices of the closest landmarks from the landmark sets to each node. Moreover, unlike in LB computations, the saturation effect inherent in GNNs cannot be mitigated in UB computations, making the UB an unreliable metric for shortest path approximation when calculated upon GNN-based landmark distances.*

To construct the landmark embeddings, we sample $r+1$ landmark sets $S_0, S_1, \ldots, S_r$ of cardinalities $2^0, 2^1, \ldots, 2^r$ with $r = \lfloor \log n \rfloor$ for R repetitions. In Figure 3(a-d), GNN-based lower bounds underperform the vanilla lower bounds for smaller $\lambda \in \{3, 4\}$, but yield substantial improvements for larger $\lambda \in \{5, 6\}$ across all three tested values of R. Although both λ values are in the supercritical regime ($\lambda > 1$), several factors explain this difference. As shown in Figure 2, the GNN learns poorer landmark embeddings for $\lambda \in \{3, 4\}$, even on small 50-node graphs. Additionally, for large n, graphs are almost surely connected when $\lambda \in \{5, 6\}$ but not when $\lambda \in \{3, 4\}$. Finally, Figure 3(e) illustrates that GNN-based embeddings can be generated faster than BFS-based embeddings, particularly on large graphs as exact local embedding computations via BFS scale poorly with graph size.

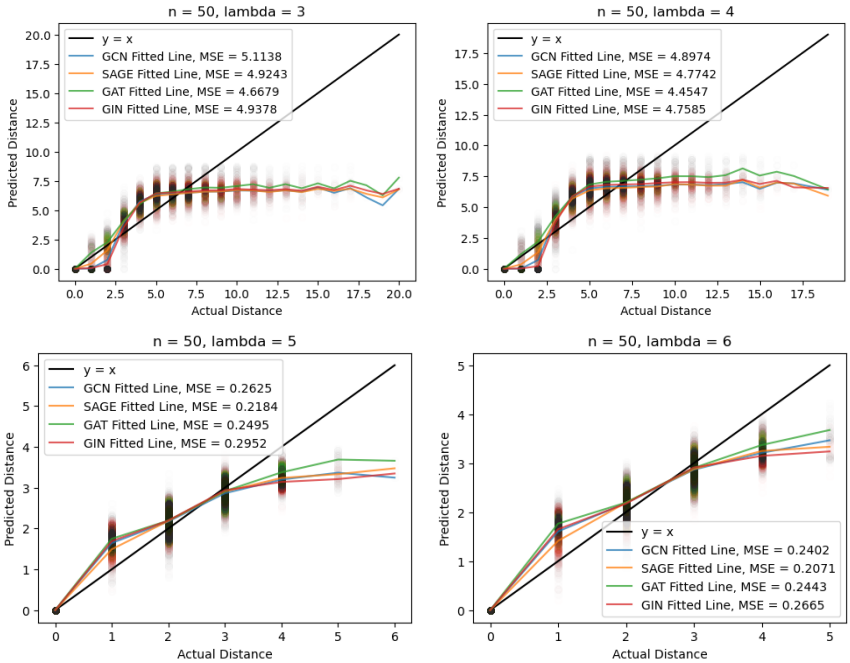

Figure 2: End-to-end shortest path distance predictions from $\lfloor\sqrt{n}\rfloor$-64-32-16-$\lfloor\sqrt{n}\rfloor$ GNNs trained on graphs generated by $\mathrm{ER}_n(\lambda/n)$. The evaluation data consists of graphs from the same model.

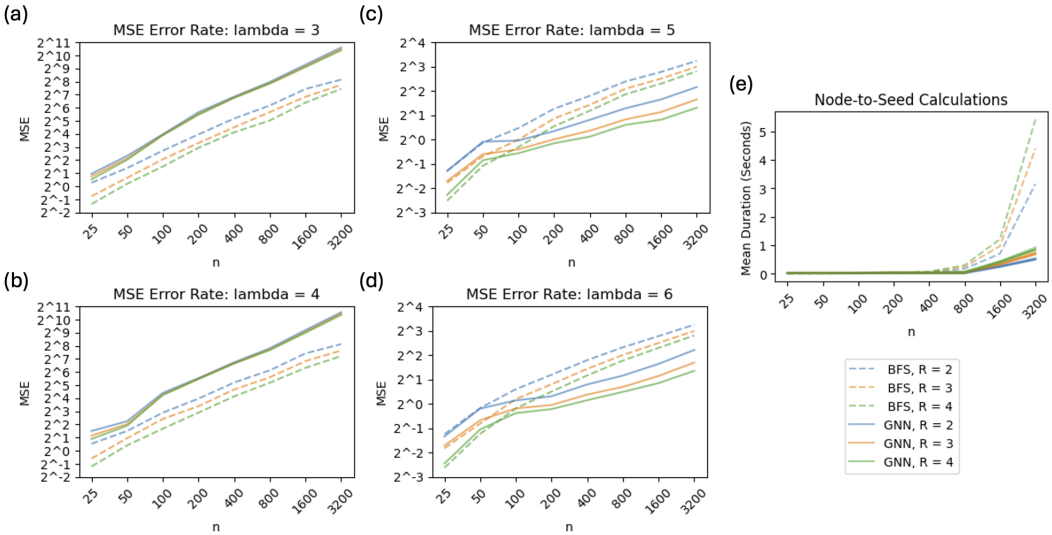

Figure 3: Error rates of BFS-based and GNN-based lower bounds on graphs generated by $\mathrm{ER}_n(\lambda/n)$, with the GNNs trained on graphs from the same model.

C.3 Experiment 3: Transferability

In our last experiment, we investigate whether GNNs trained on small graphs can be transferred to compute landmark embeddings on larger networks for downstream shortest path approximation via LBs. This is motivated by Ruiz et al. [2020] and Ruiz et al. [2023], which show that GNNs are transferable as their outputs converge on convergent graph sequences. This, in turn, allows models trained on smaller graphs to generalize to similar larger graphs.

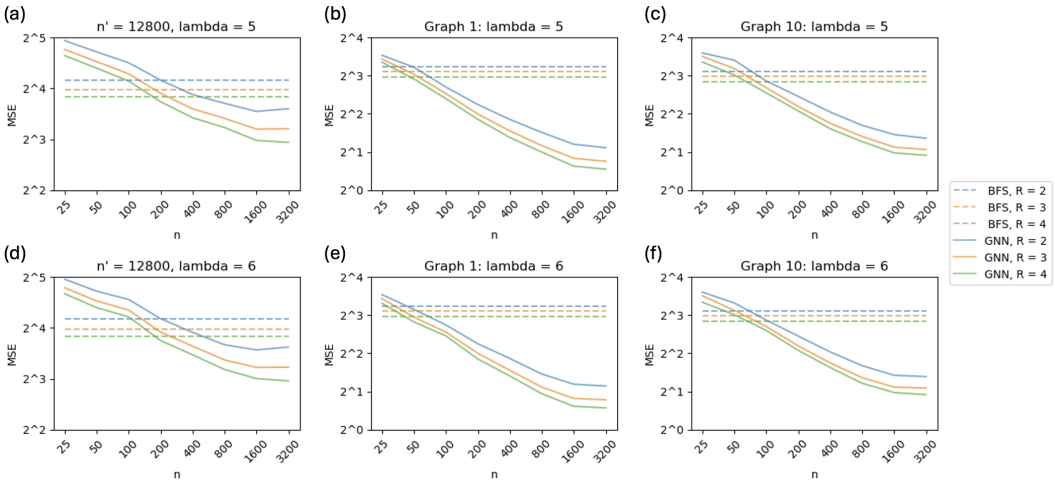

Figure 4: Error rates of BFS-based and GNN-based lower bounds on (a,d) test ER graphs generated by $\mathrm{ER}_{n'}(\lambda/n')$, (b,e) Arxiv COND-MAT collaboration network with 21,364 nodes, and (c,f) GEMSEC company network with 14,113 nodes, with the GNNs trained on graphs from $\mathrm{ER}_n(\lambda/n)$.

Here, we focus on $\lambda \in \{5, 6\}$ and train a sequence of eight GNNs on ER graphs ranging from $n = 25$ to $n = 3200$ nodes. These GNNs are then used to generate local node embeddings on graphs from the ER model with the same λ and $n' = 12800$ nodes. Figure 4(a,d) shows the MSE for each instance as the training graph size increases, with flat dashed lines indicating the MSE of BFS-based LBs on the n'-node graph. We observe a steady decrease in MSE as n grows, with GNN-based embeddings matching BFS-based performance when trained on graphs of $n = 100$, which is 128 times smaller than the target graph.

When examining the transferability of the same set of GNNs on sixteen real-world networks listed in Table 1, we again observe that MSE improves with training graph size and that GNN-based lower bounds outperform BFS-based lower bounds, even though the landmark embeddings are learned on much smaller graphs (see Figures 4 and 5). This can be explained as random graphs can model real-world networks in certain scenarios, and networks with similar sparsity likely exhibit similar local structures which local message-passing in GNNs can learn with sufficient training.

Table 1: Details on the largest connected component of selected benchmark networks.

#	Name	Category	# of Nodes	# of Edges
1	Arxiv COND-MAT [Leskovec et al., 2007]	Collaboration Network	21,364	91,315
2	Arxiv GR-QC [Leskovec et al., 2007]	Collaboration Network	4,158	13,425
3	Arxiv HEP-PH [Leskovec et al., 2007]	Collaboration Network	11,204	117,634
4	Arxiv HEP-TH [Leskovec et al., 2007]	Collaboration Network	8,638	24,817
5	Oregon Autonomous System 1 [Leskovec et al., 2005]	Autonomous System	11,174	23,409
6	Oregon Autonomous System 2 [Leskovec et al., 2005]	Autonomous System	11,461	32,730
7	GEMSEC Athletes [Rozemberczki et al., 2019b]	Social Network	13,866	86,858
8	GEMSEC Public Figures [Rozemberczki et al., 2019b]	Social Network	11,565	67,114
9	GEMSEC Politicians [Rozemberczki et al., 2019b]	Social Network	5,908	41,729
10	GEMSEC Companies [Rozemberczki et al., 2019b]	Social Network	14,113	52,310
11	GEMSEC TV Shows [Rozemberczki et al., 2019b]	Social Network	3,892	17,262
12	Twitch-EN [Rozemberczki et al., 2019a]	Social Network	7,126	35,324
13	Deezer Europe [Rozemberczki and Sarkar, 2020]	Social Network	28,281	92,752
14	LastFM Asia [Rozemberczki and Sarkar, 2020]	Social Network	7,624	27,806
15	Brightkite [Rossi and Ahmed, 2015]	Social Network	56,739	212,945
16	ER-AVGDEG10-100K-L2 [Rossi and Ahmed, 2015]	Labeled Network	99,997	499,359

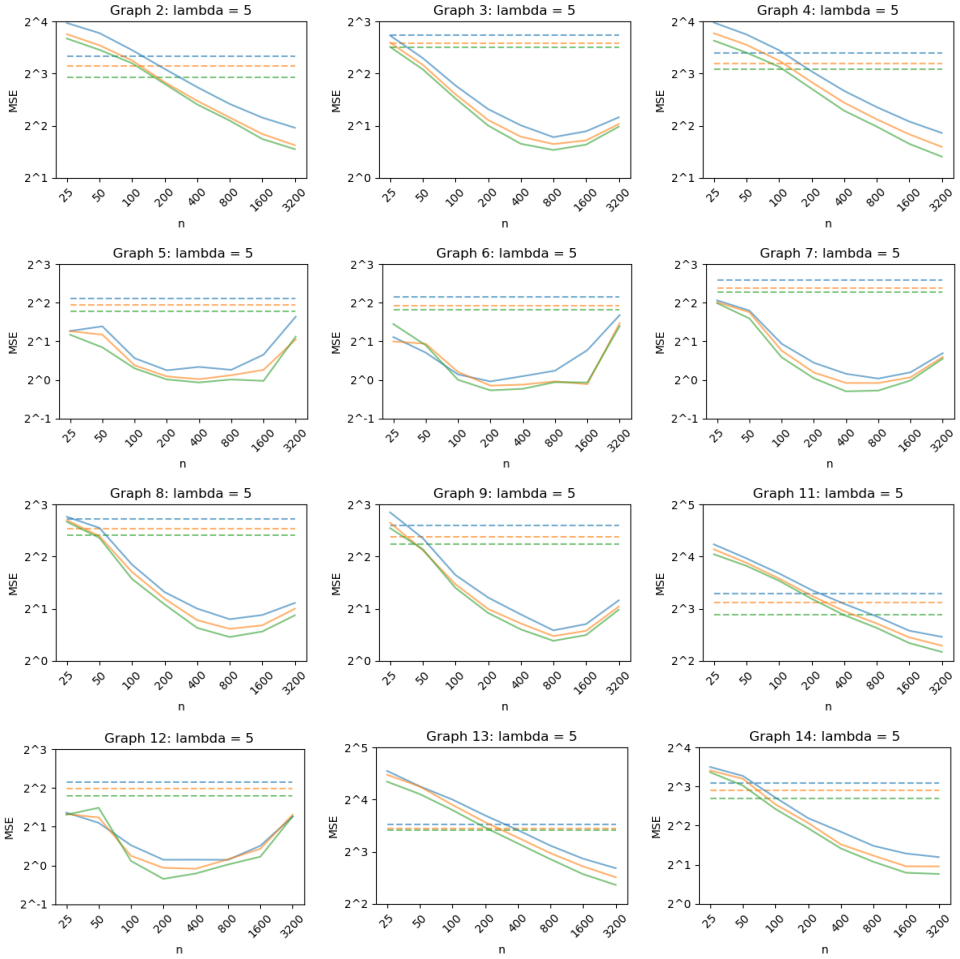

Figure 5: Additional transferability results on real networks, with the GNNs trained on graphs from $\mathrm{ER}_n(\lambda/n)$. Legend is the same as in Figure 4.