ShapleyLaw: A Game-Theoretic Approach to Multilingual Scaling Laws

Anonymous ACL submission

Abstract

In multilingual pretraining, the test loss of a pretrained model is heavily influenced by the proportion of each language in the pretraining data, namely the *language mixture ratios*. Multilingual scaling laws can predict the test loss under different language mixture ratios and can therefore be used to estimate the optimal ratios. However, the current approaches to multilingual scaling laws do not measure the *cross-lingual transfer* effect, resulting in suboptimal mixture ratios. In this paper, we consider multilingual pretraining as a cooperative game in which each language acts as a player that jointly contributes to pretraining, gaining the resulting reduction in test loss as the payoff. Consequently, from the perspective of cooperative game theory, we quantify the cross-lingual transfer from each language by its contribution in the game, and propose a game-theoretic multilingual scaling law called *ShapleyLaw*. Our experiments show that ShapleyLaw outperforms baseline methods in model performance prediction and language mixture optimization. [1]

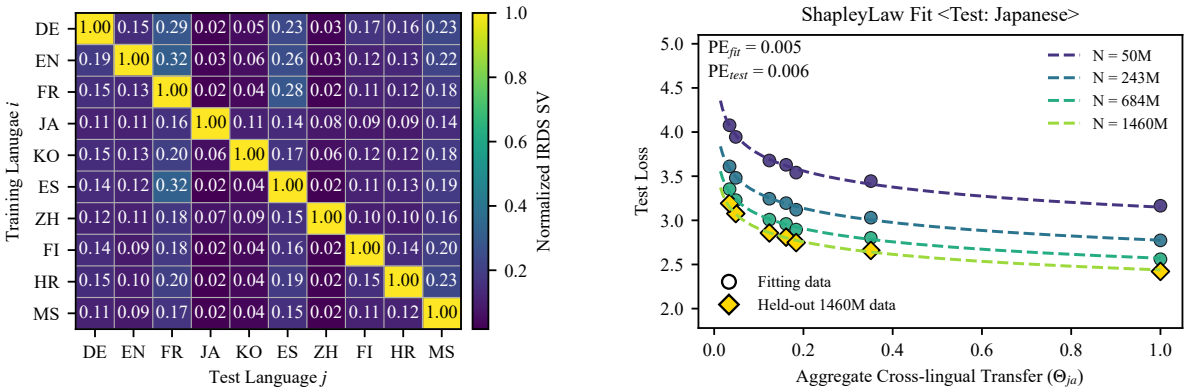
1 Introduction

As large language models (LLMs) serve increasingly diverse users, multilingual pretraining on mixed-language corpora has become standard practice for modern foundation models (Liu et al., 2020; Lin et al., 2020; Huang et al., 2023; Zhao et al., 2024; Xu et al., 2025; Wang et al., 2025a). The performance of a pretrained model on a given language j depends not only on the model size N and the pretraining data size D, but also critically on the proportion of each language i in the pretraining corpus, i.e., the mixture ratios $\mathbf{p}$. This is because knowledge learned from language i can transfer to language j, a phenomenon known as *cross-lingual transfer* (Weiss et al., 2016; Conneau and Lample, 2019; Schuster et al., 2019; Pfeiffer et al., 2020; Fernandes et al., 2023; Isonuma and Titov, 2024; He et al., 2025; Wang et al., 2025c). Therefore, identifying an optimal set of language mixture ratios that minimizes the test loss is a crucial problem for multilingual pretraining.

Recently, He et al. (2025) proposed using *multilingual scaling laws* to predict the test loss of pretrained models under different language mixture ratios, thereby optimizing language mixture. A central challenge in constructing such laws is how to measure cross-lingual transfer and relate mixture ratios to test loss. To simplify the complicated transfer effect among languages, the authors grouped languages by language family and assumed independence across families: *cross-lingual transfer occurs only among languages within the same family.* Then, they constructed multilingual scaling laws for each language family, referred to as *FamilyLaw*. However, FamilyLaw can merely be used to optimize mixture ratios across language families rather than across individual languages; the authors simply used the smoothed sampling strategy to allocate language mixture ratios within each family.

Moreover, the family-independence assumption underlying FamilyLaw does not hold. In our experiments, we find that cross-lingual transfer also exists between languages from different families, and that its strength varies substantially. For example, as Japanese and Chinese share substantial lexical overlap, although Chinese, Japanese, and Spanish belong to different language families, Japanese data improves performance on the Chinese task much more than Spanish data does (see Section 2.2). This result is consistent with the findings in Lin et al. (2019), which showed that the cross-lingual transfer effect from language i to language j largely depends on linguistic distances such as geographic distance and syntactic distance. By ignoring cross-family transfer, FamilyLaw may inform language mixtures that deviate substantially from the optimum.

[1] Our code and data are available at https://anonymous.4open.science/r/ShapleyLaw-DE3F/.



(a) Cross-lingual transfer measured by the SV.

(b) Test loss prediction by ShapleyLaw.

Figure 1: **(a)** Cross-lingual transfer effects quantified by normalized SVs estimated on a small-scale configuration ($N = 50$M, $D = 50$B), shown as an asymmetric matrix. In contrast, the FamilyLaw (He et al., 2025) baseline yields zero off-diagonal entries. **(b)** Across diverse pre-training mixtures and model scales $N \in \{50\text{M}, 243\text{M}, 684\text{M}, 1460\text{M}\}$ with total data size $D = 100$B, the proposed ShapleyLaw $\mathcal{L}_j(N, D, \Theta_{ja})$ accurately fits and predicts Japanese performance.

In this work, we propose quantifying cross-lingual transfer from the perspective of cooperative game theory (CGT). We formulate multilingual pretraining as a cooperative game, in which each language is treated as a player and the payoff is defined by the reduction in test loss on a target evaluation language. We then use the Shapley value (SV) to measure the contribution of each training language to the target language, and interpret this contribution as cross-lingual transfer, as illustrated in Figure 1a.

Based on this transfer measure, we propose *ShapleyLaw*, a game-theoretic multilingual scaling law that predicts test loss from model size, data size, and language mixture ratios. To capture the effect of multilingual mixture ratios on a target language, we introduce an *aggregate cross-lingual transfer* quantity that summarizes the transfer effect of a mixture through SV-based transfer strengths. In this way, ShapleyLaw extends standard scaling law formulations with a language-level characterization of cross-lingual transfer. As shown in Figure 1b, it accurately predicts test loss on various mixtures and model scales, enabling mixture optimization in individual languages.

We summarize our contributions as follows.

- **SV-based measure of cross-lingual transfer.** We formulate multilingual pretraining as a cooperative game, which allows us to apply the SV to quantify the cross-lingual transfer effect across languages.
- **ShapleyLaw for multilingual scaling laws.** ShapleyLaw is the first framework to characterize the functional relationship between language-level mixture ratios $\mathbf{p}$ and test loss, thereby informing optimal mixture ratios across languages.
- **Empirical validation.** We evaluate ShapleyLaw on multilingual pretraining tasks covering 10 languages. The results show that ShapleyLaw consistently outperforms baseline methods in test loss prediction, mixture ratios optimization, and downstream task performance prediction.

2 Preliminaries

2.1 Problem Statement

We consider pretraining a multilingual model $\theta_{\mathcal{K}}$ that supports a given set of languages $\mathcal{K} = \{1, \ldots, K\}$. For each language $i \in \mathcal{K}$, we have a corpus of size $\bar{D}_i$. To construct a pretraining dataset $\mathcal{D}_{\mathcal{K}}$ of size D for pretraining $\theta_{\mathcal{K}}$, for each language i, we sample a training subset $\mathcal{D}_i$ from the corpus according to a mixture ratio p_i, where the subset size is $D_i = p_i \cdot D$. The union of all the languages' training subsets forms the pretraining dataset, i.e., $\mathcal{D}_{\mathcal{K}} = \bigcup_{i=1}^{K} \mathcal{D}_i$.

Performance prediction. Because pretraining a large language model is computationally expensive, we aim to predict the test loss of the pretrained model $\theta_{\mathcal{K}}$ on a target evaluation task $\mathcal{V}_{\mathcal{K}}$ by constructing a multilingual scaling law. Specifically, the target task $\mathcal{V}_{\mathcal{K}} = \{\mathcal{V}_1, \ldots, \mathcal{V}_K\}$ consists of multiple language-specific tasks, each denoted by $\mathcal{V}_j$, where $\mathcal{V}_j$ evaluates the model's performance on language $j \in \mathcal{K}$. For each task $\mathcal{V}_j$, we estimate a language-specific scaling law $\mathcal{L}_j(N, D, \mathbf{p})$ to predict the loss of model $\theta_{\mathcal{K}}$ on that task, where N is

the model size, and $\mathbf{p} = [p_1, \ldots, p_K]$ is the vector of all the language mixture ratios. Then, we can predict the model's performance on the target task $\mathcal{V}_\mathcal{K}$ by aggregating all the language-specific scaling laws into a multilingual scaling law:

$$\mathcal{L}(N, D, \mathbf{p}) = \sum_{j=1}^{K} w_j \cdot \mathcal{L}_j(N, D, \mathbf{p})$$

where w_j denotes a weight representing the relative importance of task $\mathcal{V}_j$. For example, we can determine $w_1, \ldots, w_K$ by estimating the real-world distribution of language tokens across languages.

Mixture optimization. In addition to predicting test loss, which is the traditional function of scaling laws, multilingual scaling laws can further be used to inform optimal mixture ratios. Specifically, given a model size N, a total data size D, and importance weights $\mathbf{w} = [w_1, \ldots, w_K]$, we can estimate the optimal mixture ratios $\mathbf{p}^\star$ by solving the following optimization problem:

$$\mathbf{p}^\star = \arg\min_{\mathbf{p}} \sum_{j=1}^{K} w_j \cdot \mathcal{L}_j(N, D, \mathbf{p}). \quad (1)$$

In summary, a multilingual scaling law should (1) accurately predict test losses for different parameter configurations $(N, D, \mathbf{p})$, and (2) enable the model pretrained with the optimized mixture ratios $\mathbf{p}^\star$ to achieve a test loss as lower as possible.

2.2 Limitations of the FamilyLaw

He et al. (2025) proposed the current SOTA multilingual scaling law, which we refer to as FamilyLaw. To simplify the modeling of cross-lingual transfer, FamilyLaw groups the languages into M language families and predicts the test loss $\mathcal{L}_m^{\mathbf{fam}}$ for each family $m \in [M]$ as follows:

$$\mathcal{L}_m^{\mathbf{fam}}(N, D, \mathbf{p}) = (E_m + \frac{A_m}{N^{\alpha_m}} + \frac{B_m}{D^{\beta_m}})(\mathbf{p}_m)^{-\gamma_m}$$

where $\mathbf{p}_m$ is the sum of the mixture ratios for languages in family m, and $A_m, B_m, \alpha_m, \beta_m$ are fitting parameters.

However, we find that FamilyLaw has the following three limitations. **(1)**, for performance prediction, FamilyLaw does not construct language-specific scaling laws, making it difficult to predict model performance on single-language tasks. **(2)**, for mixture ratio optimization, FamilyLaw only optimizes mixture ratios across language families, but does not inform the optimal mixture ratios within each family (He et al. (2025) simply applied smoothed sampling to determine the intra-family ratios). **(3)**, FamilyLaw relies on a core assumption that cross-lingual transfer occurs only within the same language family. However, as shown in Figure 2, this assumption does not hold in practice: Although both Japanese and Spanish belong to language families different from Chinese, Japanese data exhibits substantially stronger cross-lingual transfer for the Chinese task because Japanese and Chinese share substantial lexical overlap.

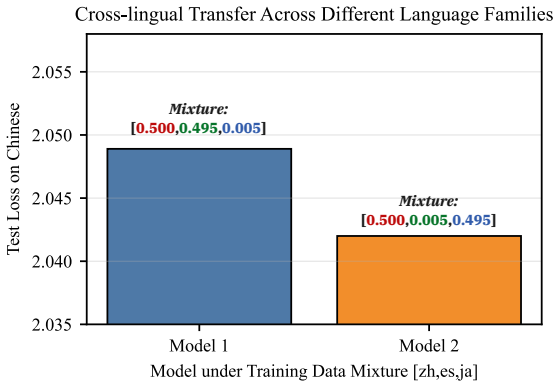


Figure 2: Consider pretraining two models for a Chinese task with training data mixtures in Chinese (zh), Japanese (es), and Spanish (es), which belong to different language families. We first pretrain Model 1 by sampling Chinese, Spanish, and Japanese data with ratios $[0.500, 0.495, 0.005]$, respectively. We then swap the mixture ratios of Japanese and Spanish to pretrain Model 2. Although Models 1 and 2 share the same model size N=684M and data size D=100B, they exhibit significantly different losses on the Chinese task.

Our objectives. Considering the above limitations, we aim to develop a multilingual scaling law that (1) predicts language-specific test loss, (2) informs optimal mixture ratios across individual languages rather than language families, and (3) models cross-lingual transfer across languages.

3 Shapley Value-Based Scaling Law

3.1 Multilingual Pretraining Game

The core challenge in developing multilingual scaling laws lies in quantifying cross-lingual transfers across languages. We address this challenge using CGT, which studies how to fairly assess the contribution of each player when multiple players jointly complete a task. In the context of multilingual pretraining, each language $i \in \mathcal{K}$ can likewise be viewed as a player. These players jointly pretrain

a model, which is used to complete an evaluation task $\mathcal{V}_j$ in a target language j. This formulation allows us to use contribution metrics from CGT to quantify the contribution of each language i to the target language j, and thereby assess cross-lingual transfer. We refer to this cooperative game as *multilingual pretraining game*, defined as follows.

Definition 3.1 (Multilingual Pretraining Game (MPG))**.** *An MPG $\mathcal{G} = (\mathcal{K}, N, D, \mathcal{V})$ is specified by a language set $\mathcal{K}$, a model size N, a data size D, and a target evaluation task $\mathcal{V}$. For any language subset $S \subseteq \mathcal{K}$, the payoff $v_j^{\mathcal{G}}(S)$ of S on any language-specific task $\mathcal{V}_j \in \mathcal{V}$ is the decreased test loss by pretraining $\theta_{\mathcal{K}}$ using a pretraining dataset $\mathcal{D}_S^{uni}$, where $\mathcal{D}_S^{uni}$ consists of training data from languages S with uniform mixture ratios. A contribution metric ϕ to the game $\mathcal{G}$ returns a $K \times K$ contribution matrix $\phi(\mathcal{G}) = \{\phi_{i,j(\mathcal{G})}\}_{i,j\in\mathcal{K}}$ that distributes the payoff $v_j^{\mathcal{G}}(\mathcal{K})$ among all the languages $\mathcal{K}$ for all task $\mathcal{V}_j \in \mathcal{V}$, i.e., $\forall j, \sum_{i=1}^{K} \phi_{i,j}(\mathcal{G}) = v_j^{\mathcal{G}}(\mathcal{K})$.*

Intuitively, in the MPG $\mathcal{G}$, all the languages $\mathcal{K}$ as players participate in pretraining with the same amount of data. For each task $\mathcal{V}_j$, they gain a payoff of $v_j^{\mathcal{G}}(\mathcal{K})$, which is the reduction in the test loss of the pretrained model. The contribution of language i to task $\mathcal{V}_j$, denoted as $\phi_{i,j}(\mathcal{G})$, measures the portion of this test loss reduction attributable to language i. Consequently, we can measure the cross-lingual transfer from language i to language j as the contribution $\phi_{i,j}$ in MPG.

3.2 Shapley Value for Measuring Cross-Lingual Transfer

The Shapley value (SV) is a widely-adopted contribution metric because it is the only metric that satisfies four important properties: *linearity*, *efficiency*, *symmetry*, and *null player* (see Appendix A.1 for details). In MPG, we can also employ the SV ϕ^{SV} to compute the contribution matrix as follows:

$$\phi_{i,j}^{SV}(\mathcal{G}) = \textstyle\sum_{S\subseteq\mathcal{K}\setminus\{i\}} W(S) \cdot \Delta_{i,j}^{\mathcal{G}}(S)$$

where $W(S) = \frac{|S|!(K-|S|-1)!}{K!}$, and $\Delta_{i,j}^{\mathcal{G}}(S) = v_j^{\mathcal{G}}(S \cup \{i\}) - v_j^{\mathcal{G}}(S)$.

Intuitively, the SV measures the marginal contribution of language i to the reduction in test loss on task $\mathcal{V}_j$, averaged over all possible subsets of languages. Specifically, the SV (1) enumerates all language subsets $S \subseteq \mathcal{K} \setminus \{i\}$, (2) calculates the marginal improvement $\Delta_{i,j}^{\mathcal{G}}(S)$ in the test loss by adding language i into S for pretraining, and (3) finally aggregates the improvements by language i across all subsets S as language i's contribution to task $\mathcal{V}_j$, where $W(S)$ is the weight for aggregation. To ensure comparability across different tasks, we further apply exponential normalization to compute the normalized SV as follows:

$$\phi_{i,j}^{NSV}(\mathcal{G}) = \exp\left(\phi_{i,j}^{SV}(\mathcal{G}) - \max_{i'} \phi_{i',j}^{SV}(\mathcal{G})\right)$$

which ensures that the language i with the largest contribution to task $\mathcal{V}_j$ has a normalized SV of 1, while the values for other languages lie in the range $(0, 1]$. Consequently, the normalized SV can serve as a measure of the cross-lingual transfer from any language i to any language j. For simplicity, we denote $\phi_{i,j}^{NSV}(\mathcal{G})$ as $\phi_{i,j}^{NSV}$ in the following, when no ambiguity arises.

3.3 Construction of ShapleyLaw

Finally, we construct a multilingual scaling law, named *ShapleyLaw*, based on the SV. Formally, given a set of parameter configurations $(N, D, \mathbf{p})$ and a set of normalized SVs $\{\phi_{i,j}^{NSV}\}$ that represent cross-lingual transfers among languages in $\mathcal{K}$, ShapleyLaw predicts test loss on each task $\mathcal{V}_j$ as:

$$\mathcal{L}_j(N, D, \mathbf{p} \mid \{\phi_{i,j}^{NSV}\}) = \mathcal{L}_j^{\mathbf{chin}}(N, D) \cdot (\Theta_j)^{-\gamma_j}$$

where $E_j, A_j, B_j, \alpha_j, \beta_j, \gamma_j$ are fitting parameters; $\mathcal{L}_j^{\mathbf{chin}}(N, D) = E_j + \frac{A_j}{N^{\alpha_j}} + \frac{B_j}{D^{\beta_j}}$ corresponds to the Chinchilla scaling law (Hoffmann et al., 2022) that does not consider multilingual mixture ratios; and $\Theta_j = \sum_{i\in\mathcal{K}} p_i \cdot \phi_{i,j}^{NSV}$ measures the *aggregate cross-lingual transfer* from languages $\mathcal{K}$ to language j weighted by their mixture ratios. Intuitively, ShapleyLaw rescales the Chinchilla scaling law using Θ_j, thereby incorporating the effect of cross-lingual transfer on pretraining.

Note that SV-based cross-lingual transfer exhibits transferability (see Appendix C.1). This implies that we can construct an MPG $\mathcal{G} = (\mathcal{K}, N', D', \mathcal{V})$ using a small model size N' and data size D' to compute the normalized SVs. These normalized SVs can then be treated as fixed parameters when constructing ShapleyLaw for larger model sizes $N \gg N'$ and data sizes $D \gg D'$. As illustrated in Figure 1a, we compute the normalized SVs under the setting $N' = 50$M and $D' = 50$B; ShapleyLaw accurately fits the test losses across in the settings with $N \in \{243\text{M}, 684\text{M}, 1460\text{M}\}$ and $D = 100$B.

Besides predicting language-specific test loss, ShapleyLaw can also be used to estimate *optimal mixture ratios* by solving the optimization problem

in Formula 1 (see Appendix A.2). Therefore, ShapleyLaw achieves our three objectives.

SV approximation. Since the SV $\phi_{i,j}^{SV}$ enumerates all non-empty language subsets $S \subseteq \mathcal{K}$ to evaluate their payoff $v_j^{\mathcal{G}}(S)$, we need to pretrain $2^K - 1$ models for exact SV computation, which is NP-hard. To accelerate SV computation, we employ the In-Run Data Shapley (IRDS) approximation technique (Wang et al., 2025b) (see Appendix A.3) in our experiments, which pretrains only one model to approximate the SVs. With IRDS, ShapleyLaw achieves $O(1)$ computational complexity and is therefore a tractable multilingual scaling law that can scale up to dozens or even hundreds of languages. Our experiments in Appendix C.4 show that IRDS accurately approximates the exact SV.

4 Experiments

Setup. We describe the basic experimental setup here. More details are provided in Appendix B. We study $K = 10$ languages from eight language families, as summarized in Table 5. We sample 0.1% of the corpora as a fixed test set for evaluating all pretrained models. We consider four model scales with non-embedding parameter counts $N \in \{50\text{M}, 243\text{M}, 684\text{M}, 1460\text{M}\}$, following the open-source llm-jp configurations. All models share the same architecture family and tokenizers. We consider the cross-entropy (CE) loss for evaluation. The SVs are computed in an MPG with model size $N = 50\text{M}$ and $D = 50\text{B}$ across all experiments.

Research questions. Our experiments address the following research questions. **Q1** (Performance Prediction): Can ShapleyLaw more accurately predict the model's test loss? **Q2** (Mixture Optimization): Can ShapleyLaw achieve more optimal language mixtures? **Q3** (Downstream Tasks): Can ShapleyLaw's ability to predict model performance and optimal mixtrue ratios generalize to downstream tasks? **Q4** (SV Approximation): How is the fitting performance of ShapleyLaw based on approximate SVs? (see Appendix C.4)

4.1 Experiments on Performance Prediction

Experiment design. We pretrain models with $N \in \{50\text{M}, 243\text{M}, 684\text{M}, 1460\text{M}\}$ and $D = 100\text{B}$. At each model scale, we consider both *controlled mixtures* and *empirical mixtures*. Controlled mixtures isolate the effect of language mixtures under fixed ratios, including monolingual training, uniform mixtures over selected languages (e.g., zh–ja–ko, fi-hr-ms and en–de–fr–es, abbreviated as *ZJK*, *FHM* and *EDFS*), and the uniform mixture over all languages (*Uni*). Empirical mixtures mirror multilingual data distributions adopted in prior LLM studies, including the language distributions of Common Crawl[2] and FineWeb-2 (Penedo et al., 2025), the English-centric multilingual model LLaMA-2 (Touvron et al., 2023), and the Japanese-focused pretraining corpus LLM-JP-V4 (LLM-jp et al., 2024) (abbreviated as *CC*, *FW2*, *L2*, and *JPv4*). To evaluate the fitting performance, we also use the normalized MAE to quantify the prediction error (PE).

In addition, we examine 50 randomly sampled language mixtures under $N = 243\text{M}$ to further evaluate generalization to unseen mixture ratios. To ensure reliable fitting, we exclude those random mixtures where a training language i has subset size $D_i \leq 1.5\text{B}$. This regime corresponds to extremely low data availability for language i, causing optimization to become unstable even for $N = 243\text{M}$ models. This threshold remains practically relevant: in FineWeb-scale corpora, a language budget of roughly 1.5B tokens already extends to many low-resource languages considered in current multilingual LLM studies, including Basque and Armenian. We then split the remaining runs into fitting and held-out evaluation subsets.

Results. Figure 3 shows the fitted results for Chinese, Spanish, and Japanese. On curated multilingual mixtures across four model scales, ShapleyLaw achieves strong held-out performance, with $\text{PE}_{test} = 0.021$ for Chinese, 0.010 for Spanish, and 0.006 for Japanese ($R^2 = 0.990, 0.997, 0.996$). On randomly sampled mixtures at $N = 243\text{M}$, the fitted curves remain smooth with respect to Θ, with training-set fits of $\text{PE}_{test} = 0.008, 0.005$, and 0.002 for Chinese, Spanish, and Japanese ($R^2 = 0.975, 0.942, 0.987$), respectively. These results indicate that the proposed method makes stable predictions across sampling regimes and typologically diverse languages. Additional results for French, German and Korean are provided in Appendix C.2.

We next compare ShapleyLaw with the family-level baseline FamilyLaw in a controlled setting that isolates cross-family transfer. As shown in Figure 4, we run experiments on the *ZJS* mixture. We fix the Chinese mixture ratio at $p_{\text{zh}} = 0.05$ and vary only the Japanese–Spanish proportions, keeping the total token budget fixed. Under FamilyLaw, both

[2]https://commoncrawl.org

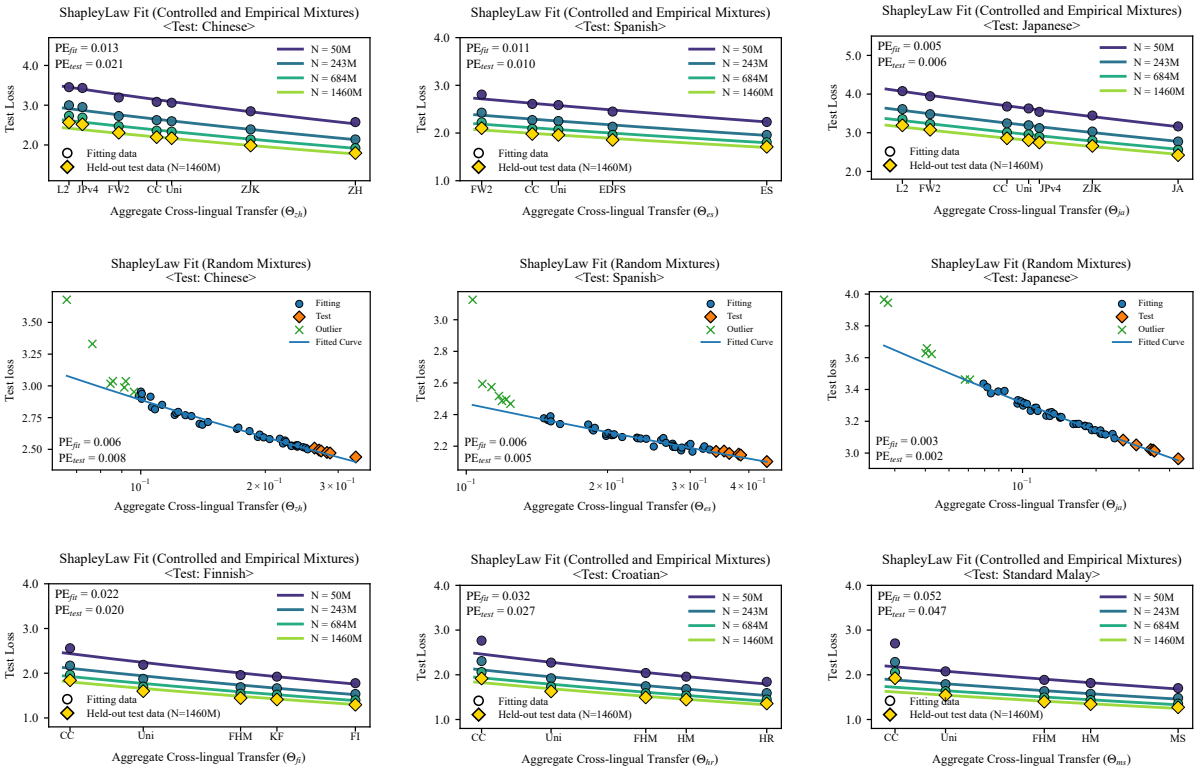


Figure 3: **Fitting ShapleyLaw across multilingual mixtures and target languages. Top row**: fits with respect to aggregate cross-lingual transfer Θ on curated multilingual mixtures for Chinese (ZH), Spanish (ES), and Japanese (JA) across four model scales $\{50\text{M}, 243\text{M}, 684\text{M}, 1460\text{M}\}$; circles denote fitting points and yellow diamonds denote held-out evaluation points. **Middle row**: fits under randomly sampled mixtures at model scale $N = 243\text{M}$; circles indicate fitting points, yellow diamonds indicate held-out points, and orange crosses denote removed outliers. **Bottom row**: fits with respect to aggregate cross-lingual transfer Θ on curated multilingual mixtures for low-resource languages Finnish (FI), Croatian (HR), and Standard Malay (MS).

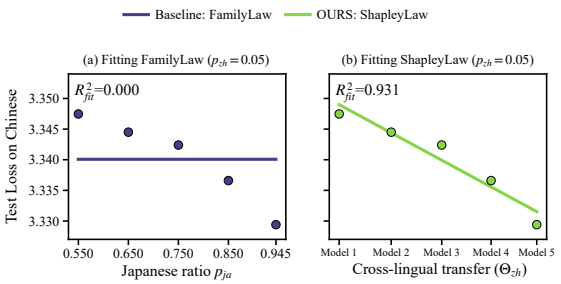


Figure 4: **Controlled comparison between FamilyLaw and ShapleyLaw on a Chinese evaluation task.** We pretrain multilingual models with $N = 50\text{M}$ and $D = 100\text{B}$ using Chinese, Japanese, and Spanish corpora. Starting from the mixture $[0.05, 0.55, 0.40]$, we vary the Spanish–Japanese proportions up to $[0.05, 0.945, 0.05]$ while keeping the Chinese ratio fixed. Performance is evaluated on the Chinese test set. **Left:** FamilyLaw, where x-axis stands for language ratio, yields a poor fit. **Right:** ShapleyLaw, where x-axis stands for different Chinese aggregate transfer Θ, fits accurately.

Japanese and Spanish lie outside the Chinese language family, so changing their ratios should not affect Chinese performance; accordingly, the fitted relation is nearly constant and fails to explain the observed variation ($R^2 \leq 0$). In contrast, ShapleyLaw captures the cross-language transfer and fits the loss on Chinese much more accurately ($R^2 = 0.931$). This result shows that family-level aggregation is too coarse to model important cross-lingual interactions in multilingual pretraining, such as shared script and lexical overlap.

4.2 Experiments on Mixture Optimization

Experiment design. We compare the ShapleyLaw-based mixture optimization against four representative baselines chosen to reflect the main strategies used in multilingual data allocation, including *Uniform Sampling*, *Smoothed Sampling*, *FamilyLaw Optimization*, and *Empirical Mixtures* (see Appendix B for baseline introduction and Appendix A.4 for baseline selection clarification). We pretrain a model ($N = 243\text{M}$ and $D = 100\text{B}$) using the mixture ratios obtained by each method.

We consider two settings of w_j for weighting the importance of each language-specific task $\mathcal{V}_j$. **Unweighted Sum:** Each task $\mathcal{V}_j$ receives equal weight,

Strategy	Language Mixture Ratio (%)										CE Loss ↓
	en	de	fr	es	zh	ja	ko	fi	hr	ms	$\mathcal{L}$
Uniform	10.0	10.0	10.0	10.0	10.0	10.0	10.0	10.0	10.0	10.0	24.440
$\alpha = 0.5$	13.2	14.5	12.6	13.6	19.1	11.4	4.9	4.7	3.7	2.4	24.683
Common Crawl	61.4	8.3	6.4	6.4	7.2	8.2	1.2	0.5	0.3	0.1	25.491
FineWeb-2	80.6	3.3	2.3	2.7	6.7	3.1	0.5	0.4	0.3	0.1	27.133
FamilyLaw	14.1	14.1	1.8	1.8	12.1	11.6	10.2	12.6	11.0	10.7	24.749
ShapleyLaw	26.0	13.5	1.4	1.5	20.2	13.4	10.4	5.3	6.9	1.4	**24.108**

Strategy	Language Mixture Ratio (%)										CE Loss ↓
	en	de	fr	es	zh	ja	ko	fi	hr	ms	$\mathcal{L}$
Uniform	10.0	10.0	10.0	10.0	10.0	10.0	10.0	10.0	10.0	10.0	11.985
$\alpha = 0.5$	13.2	14.5	12.6	13.6	19.1	11.4	4.9	4.7	3.7	2.4	12.180
Common Crawl	61.4	8.3	6.4	6.4	7.2	8.2	1.2	0.5	0.3	0.1	12.715
FineWeb-2	80.6	3.3	2.3	2.7	6.7	3.1	0.5	0.4	0.3	0.1	13.613
FamilyLaw	9.9	9.9	1.5	1.5	11.6	8.6	9.3	17.4	14.9	15.4	12.153
ShapleyLaw	18.1	15.4	1.5	1.5	19.7	9.3	9.6	10.3	12.0	2.6	**11.813**

Table 1: Comparison of mixture optimization strategies and their CE loss under the Unweighted Sum (**Left**) and Normalized Sum (**Right**) settings.

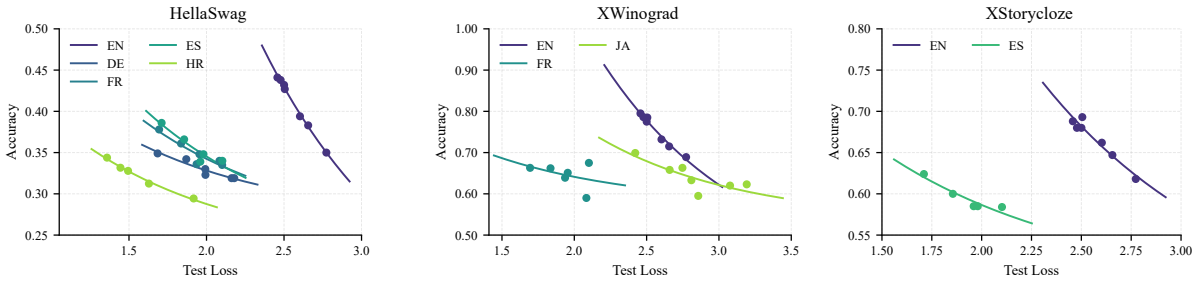


Figure 5: **Fitting Relationship between Downstream Performance and Test CE Loss.** Experiments are conducted on six downstream tasks for English, German, French, Spanish, Croatian, and Japanese. Across all tasks, we observe strong *negative correlations* ($|r| > 0.90$, with an average $R^2 = 0.933$), indicating that lower test loss consistently corresponds to higher downstream performance.

so the objective is to minimize the average multilingual loss. **Normalized Sum:** Each task $\mathcal{V}_j$ is weighted by the inverse of its monolingual loss. This ensures that intrinsically harder languages do not dominate the aggregate objective in Formula 1 and provides a more balanced evaluation of multilingual trade-offs across languages.

Results. Table 1 shows the optimized mixtures and their resulting CE loss. Under both preference settings, the mixtures obtained from ShapleyLaw outperform all baselines in the final weighted loss. The gains are especially meaningful because they are obtained under actual pretraining rather than only from the fitted objective, indicating that the learned law transfers effectively from prediction to optimization. By contrast, the mixture obtained from FamilyLaw is consistently less effective in the per-language setting, suggesting that family-level aggregation is too coarse to capture the trade-offs needed for fine-grained multilingual allocation.

4.3 Experiments on Downstream Tasks

Experiment design. We focus on three multilingual benchmarks that cover complementary NLU abilities: **HellaSwag** (Zellers et al., 2019; Lai et al., 2023) for commonsense reasoning through contextually appropriate ending prediction, **XStoryCloze** (Lin et al., 2022; Mostafazadeh et al., 2017) for cross-lingual story ending prediction, and **XWinograd** for cross-lingual coreference resolution (Tikhonov and Ryabinin, 2021).

We evaluate pretrained models under the same setting as Section 4.1, using $N = 1.46$B and $D = 100$B, and the same multilingual mixture families (controlled and empirical mixtures). We report 5-shot accuracy on the standard benchmark splits. Specifically, we evaluate HellaSwag in {en, fr, es, de, hr}, XWinograd in {en, fr, ja}, and XStoryCloze in {en, es}. For each task-language pair, we compare downstream accuracy against the corresponding pretraining test cross-entropy on the same language, measured on the held-out test corpus. This yields one calibration line per task-language pair.

Results. We fit a simple calibration model $s = f(\ell) = a \cdot g(\ell) + b$ where ℓ is the test loss, s is downstream accuracy, and a, b are fitting parameters (Chen et al., 2024). We choose $g(\ell) = \exp(-\ell)$ as the transformation function $g(\cdot)$ (see more details in Appendix C.3). Figure 5 shows the resulting fits for the three multilingual benchmarks, with one line per task-language pair and points colored by language. Across all evaluated languages, lower pretraining loss consistently corresponds to higher downstream accuracy, and this relationship is generally well captured by a transformed linear fit. The fitted results show that this trend holds not only within a single benchmark, but across multilingual commonsense reasoning, story completion, and coreference resolution tasks. Table 2 reports the corresponding R^2 values for each task-language pair.

Task	de	en	fr	es	ja	hr
HellaSwag	0.918	0.996	0.864	0.897	–	0.992
XWinograd	–	0.980	0.975	–	0.675	–
XStorycloze	–	0.947	–	0.905	–	–

Table 2: Corresponding R^2 values for the fitted relationship between downstream performance and test CE loss for each task–language pair. English achieves the highest R^2, while Japanese is the lowest. The results consistently show a negative correlation between downstream performance and test CE loss.

Strategy	Language Mixture Ratio (%)					Overall Acc. ↑
	en	de	fr	es	hr	HellaSwag
Uniform	20.0	20.0	20.0	20.0	20.0	0.343
$\alpha = 0.5$	25.2	22.9	21.9	23.6	6.4	0.344
Common Crawl	73.9	10.0	7.7	7.8	0.6	0.354
FineWeb-2	90.3	3.7	2.5	3.0	0.5	0.355
FamilyLaw	31.7	31.7	4.1	4.1	28.4	0.346
ShapleyLaw	95.7	0.1	0.1	0.1	4.0	**0.356**

Table 3: Comparison of mixture optimization strategies and average HellaSwag downstream-task accuracies across five languages under the unweighted-sum setting. ShapleyLaw achieves the best performance, with an average accuracy of 0.356.

These results show that for the multilingual NLU benchmarks considered here, language-specific pre-training loss remains strongly informative for downstream accuracy across diverse mixtures. As ShapleyLaw predicts this loss accurately across language mixtures, it provides a useful intermediate signal for downstream-oriented mixture design.

Optimal Ratio Downstream Experiment for HellaSwag We further evaluated downstream mixture optimization for HellaSwag in {en, fr, es, de, hr}. To assess the effectiveness of ShapleyLaw on downstream tasks, we pretrained six models using different sampling strategies and evaluated them on HellaSwag. The overall performance was measured by the average accuracy across the five languages.

Results. As shown in Table 3, ShapleyLaw achieves the highest overall accuracy on multilingual HellaSwag task, reaching 0.356 under the unweighted-sum setting. This result outperforms all baseline strategies. Notably, ShapleyLaw assigns most of the mixture to English while retaining a small proportion of Croatian, suggesting that the optimal downstream mixture ratio is highly English-dependent. These results demonstrate that ShapleyLaw can identify more effective language mixture ratio for downstream performance than including controlled and empirical mixture strategies.

5 Related Work

Neural scaling laws describe predictable relationships between model performance and scale variables such as model parameters, training tokens, and compute (Kaplan et al., 2020; Hoffmann et al., 2022), but standard formulations do not account for *data composition*. Recent work extends scaling-law analysis to data mixtures in domain-specific pretraining (Que et al., 2024; Shukor et al., 2025). These works highlight the importance of data composition, but they do not provide a multilingual scaling law that directly optimizes *language-level* mixture ratios. In multilingual settings, prior work either models mixture ratios at the family level (He et al., 2025) or studies bilingual transfer in solely multilingual machine translation (Fernandes et al., 2023; Isik et al., 2025). To the best of our knowledge, ShapleyLaw is the first multilingual scaling law that enables optimization of language-level mixture ratios through explicit modeling of cross-lingual transfer between individual languages.

6 Conclusion

In this paper, we proposed *ShapleyLaw*, a game-theoretic multilingual scaling law that explicitly incorporates aggregate cross-lingual transfer into multilingual pre-training. Across experiments on ten languages from eight language families, we show that the proposed law accurately predicts per-language pre-training loss across diverse model sizes, data budgets, and language mixtures, while capturing transfer effects that are missed by prior family-level approaches. We further demonstrate that ShapleyLaw supports practical optimization of multilingual sampling ratios, producing better mixtures than heuristic and prior-law baseline under real pre-training. In addition, the predicted language-specific loss was shown to remain informative for downstream multilingual NLU performance, highlighting the practical utility of the framework beyond pre-training loss alone. We also examine the efficiency of approximate Shapley value estimation and show that it preserves strong empirical performance, highlighting the potential scalability of our method to larger multilingual settings. Taken together, these findings suggest that multilingual scaling behavior is better understood through fine-grained, transfer-aware language interactions, and that our method provide a useful foundation for efficient multilingual mixture design.

Limitations

Due to constraints on computational resources, the largest model scale validated in the main text of this paper is $N = 1460$M parameters. However, we emphasize that this scale already exceeds the largest model size ($N = 1.2$B) validated in FamilyLaw (He et al., 2025), the SOTA multilingual scaling law. Therefore, we believe that our empirical validation covers a sufficiently broad range of model scales to demonstrate the effectiveness of our method.

To further address potential concerns regarding the extrapolation of ShapleyLaw to larger models, we additionally trained four models at $N = 3.2$B parameters, with results reported in Appendix C.5. ShapleyLaw continues to predict the test loss of these larger models with high accuracy, providing strong evidence that our proposed scaling law generalizes well beyond the scales used for fitting.

Ethical Statements

We do not anticipate any ethical concerns arising from this work.

Use of Generative AI

AI assistants, such as ChatGPT, were used in this work for grammar checking and language editing.

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A More Technical Details

A.1 Theoretical Properties of the SV

The SV uniquely satisfies the following properties.

- *Efficiency:* The payoff (i.e., loss reduction) $v_j^{\mathcal{G}}(\mathcal{K})$ for each task $\mathcal{V}_j$ is fully distributed among all the languages $\mathcal{K}$, i.e., $\forall j, \sum_{i=1}^{K} \phi_{i,j}^{SV}(\mathcal{G}) = v_j^{\mathcal{G}}(\mathcal{K})$.
- *Linearity:* The SVs of each language to different evaluation tasks are linearly additive. Let $\phi_{i,j+j'}^{SV}$ denotes the Shapley value of language i for task $\mathcal{V}_{j+j'} = \mathcal{V}_j \cup \mathcal{V}_{j'}$. Then, for any language i, and for any two tasks $\mathcal{V}_j$ and $\mathcal{V}_{j'}$, we have $\phi_{i,j+j'}^{SV}(\mathcal{G}) = \phi_{i,j}^{SV}(\mathcal{G}) + \phi_{i,j'}^{SV}(\mathcal{G})$. That means that the contribution of language i to a combined task $\mathcal{V}_{j+j'}$ should be the sum of its contributions to the individual tasks $\mathcal{V}_j$ and $\mathcal{V}_{j'}$.
- *Symmetry:* If two languages i, i' are equivalent for improving model performance, then their SVs are identical. Formally, if $v_j^{\mathcal{G}}(S \cup \{i\}) = v_j^{\mathcal{G}}(S \cup \{i'\})$ for any language subset $S \subseteq \mathcal{K}$, then $\phi_{i,j}^{SV} = \phi_{i',j}^{SV}$.
- *Null player:* If a language i never improves the model performance when its data is included for pretraining, its contribution should be zero. That is, if $v_j^{\mathcal{G}}(S \cup \{i\}) = v_j^{\mathcal{G}}(S)$ for any language subset $S \subseteq \mathcal{K}$, then $\phi_{i,j}^{SV} = 0$.

A.2 ShapleyLaw Mixture Optimization

Optimal mixture ratios for language preferences. In practice, multilingual pre-training is often guided by an application-specific evaluation suite spanning multiple languages, where different languages may carry different importance. To model this, suppose we have nonnegative preference weights $\{w_j\}_{j=1}^{K}$ (e.g., proportional to user demand or benchmark priority) and consider the weighted objective

$$\mathcal{J}(\mathbf{p}) \;=\; \sum_{j=1}^{K} w_j \, \mathcal{L}_j(N, D, \mathbf{p})$$

where $\mathcal{L}_j$ is the test loss for target language j. Introducing ShapleyLaw into Eq. A.2 yields

$$\mathcal{J}(\mathbf{p}) \;=\; \sum_{j=1}^{K} w_j \cdot C_j \cdot \big(\Theta_j\big)^{-\gamma_j} \qquad (2)$$

where $C_j \triangleq E_j + \frac{A_j}{N^{\alpha_j}} + \frac{B_j}{D^{\beta_j}}$, which makes explicit how mixture influences the weighted loss through Θ_j.

Convexity and optimality conditions. For $\gamma_j > 0$, the function $x \mapsto x^{-\gamma_j}$ is convex over $x > 0$; since Θ_j is affine in $\mathbf{p}$, $\mathcal{J}(\mathbf{p})$ is a convex objective over the simplex whenever $\Theta_j > 0$ for all j. Therefore, the mixture selection problem

$$\min_{\mathbf{p} \in \Delta^{K-1}} \; \mathcal{J}(\mathbf{p})$$

admits a global optimum and can be solved efficiently. Using a Lagrangian with simplex constraints and Karush-Kuhn-Tucker (KKT) conditions, the gradient of Eq. 2 is

$$\frac{\partial \mathcal{J}}{\partial p_i} \;=\; -\sum_{j=1}^{K} w_j C_j \gamma_j \, \phi_{i,j}^{NSV} \, \big(\Theta_j\big)^{-(\gamma_j+1)}.$$

At an interior optimum (all $p_i > 0$), these gradients are equal up to a shared Lagrange multiplier, implying an intuitive *marginal utility equalization* principle: optimal mixtures allocate probability mass so that the expected reduction in weighted loss per unit increase of p_i is balanced across languages.

Closed-form solution in a diagonal (self-dominant) approximation. A useful and interpretable special case arises when each target language is primarily influenced by its own data, i.e., $\phi_{i,j}^{NSV}$ is approximately diagonal so that $\Theta_j \approx p_j \, r_j$ with $r_j \triangleq \phi_{j,j}^{NSV}$. Then Eq. 2 becomes separable:

$$\mathcal{J}(\mathbf{p}) \approx \sum_{j=1}^{K} w_j C_j (p_j r_j)^{-\gamma_j}.$$

The KKT conditions yield a closed-form optimal ratio:

$$p_i^{\star} \;=\; \frac{\Big(w_i C_i \gamma_i \, r_i^{-\gamma_i}\Big)^{\frac{1}{\gamma_i+1}}}{\sum_{m=1}^{K} \Big(w_m C_m \gamma_m \, r_m^{-\gamma_m}\Big)^{\frac{1}{\gamma_m+1}}}. \qquad (3)$$

First-order correction under cross-lingual transfer. For realistic multilingual settings, cross-lingual transfer introduces non-negligible off-diagonal terms in ϕ^{NSV}. Let the Shapley matrix be decomposed as

$$\phi^{NSV} = I + E$$

where E captures cross-lingual interactions with relatively small off-diagonal entries. Then

$$\Theta = \phi^{NSV}\mathbf{p} = \mathbf{p} + E\mathbf{p}.$$

Expanding $\Theta_i^{-(\gamma_i+1)}$ to first order around the diagonal solution $p^{(0)}$ from Eq. 3 yields

$$p_i^\star \approx p_i^{(0)} \left(1 - \frac{1}{\gamma_i + 1} \sum_{j, j \neq i} \phi_{i,j}^{NSV} \right) \quad (4)$$

followed by renormalization to satisfy the simplex constraint. The correction term depends on the column sum $\sum_{j \neq i} \phi_{i,j}^{NSV}$, which measures how strongly language i contributes to other languages through cross-lingual transfer. Languages with larger column sums effectively provide training signal to many other languages, and therefore require less direct sampling mass in the optimal mixture. Conversely, languages with weaker transfer connections receive higher allocation to maintain balanced marginal utility across tasks.

In practice, the available corpus size for each language may be limited. Let $\bar{D}_i$ denote the maximum usable token budget for language i, and let the total training budget be D. Then the mixture ratio must satisfy the additional box constraint

$$p_i \leq \bar{p}_i \quad \text{where} \quad \bar{p}_i = \frac{\bar{D}_i}{D}. \quad (5)$$

The mixture optimization problem thus becomes

$$\min_{\mathbf{p} \in \Delta^{K-1}} \mathcal{J}(\mathbf{p}) \quad \text{s.t.} \quad 0 \leq p_i \leq \bar{p}_i.$$

Because $\mathcal{J}(\mathbf{p})$ is convex, the constrained problem remains convex and can be characterized via KKT conditions with complementary slackness. However, a simple and effective approximation can be derived by projecting the unconstrained optimum onto the feasible region.

Clipped closed-form approximation. Let $p_i^{(0)}$ denote the unconstrained solution from Eq. 3 (or its cross-lingual correction in Eq. 4). Define the *active set* of data-limited languages

$$\mathcal{A} = \{ i \mid p_i^{(0)} > \bar{p}_i \}.$$

For languages in $\mathcal{A}$, the optimal solution saturates the constraint:

$$p_i^\star = \bar{p}_i, \qquad i \in \mathcal{A}.$$

The remaining probability mass is

$$M = 1 - \sum_{i \in \mathcal{A}} \bar{p}_i$$

which is redistributed among the unconstrained languages $\mathcal{F} = \{1, \ldots, K\} \setminus \mathcal{A}$ proportionally to their original optimal ratios:

$$p_i^\star \approx M \cdot \frac{p_i^{(0)}}{\sum_{m \in \mathcal{F}} p_m^{(0)}}, \qquad i \in \mathcal{F}. \quad (6)$$

If redistribution causes additional violations ($p_i^\star > \bar{p}_i$ for some $i \in \mathcal{F}$), the clipping-and-redistribution step can be repeated. Since each iteration adds at least one index to $\mathcal{A}$, the procedure converges in at most K steps.

This approximation corresponds to projecting the unconstrained optimum onto the truncated simplex defined by Eq. 5. Intuitively, languages with insufficient data are trained to exhaustion, while the remaining training budget is allocated among data-rich languages according to their relative marginal utility. Notably, the structure of Eq. 6 preserves the optimal trade-off encoded by Shapley-aware scaling laws while ensuring practical feasibility under corpus-size limitations.

A.3 In-Run Data Shapley for SV Approximation

Why In-Run Data Shapley approximation is needed. For a multilingual pretraining game (MPG), the Shapley value (SV) of language i on task j is defined as

$$\phi_{i,j}^{SV}(\mathcal{G}) = \sum_{S \subseteq \mathcal{K} \setminus \{i\}} W(S) \cdot \Delta_{i,j}^{\mathcal{G}}(S)$$

where $W(S) = \frac{|S|!(K-|S|-1)!}{K!}$, and $\Delta_{i,j}^{\mathcal{G}}(S) = v_j^{\mathcal{G}}(S \cup \{i\}) - v_j^{\mathcal{G}}(S)$. Exact computation requires evaluating the payoff of all language coalitions, which entails pre-training $2^K - 1$ models for K languages.

To avoid repeated pre-training, we replace coalition-level retraining with an In-Run Data Shapley (IRDS) approximation. Instead of asking how the final payoff changes when language i is added to an independently trained coalition, IRDS measures how language i contributes to the validation loss reduction along a specific multilingual pre-training trajectory. Thus, the resulting value directly attributes the cross-lingual transfer observed in the actual training run used for SV estimation.

Local payoff in IRDS. Consider a single multilingual pre-training run with checkpoints $\{w_0, w_1, \ldots, w_T\}$. At training step t, let B_t be the sampled mini-batch and let $B_{t,i} \subseteq B_t$ denote the

subset of examples whose language is i. For a language subset $S \subseteq \mathcal{K}$, we define a counterfactual one-step update using only examples from languages in S:

$$\widetilde{w}_{t+1}(S) = w_t - \eta_t \sum_{r \in S} \sum_{z \in B_{t,r}} \nabla \ell(w_t, z),$$

where η_t is the learning rate and $\ell(w, z)$ is the training loss.

For a target validation language j, let $\mathcal{V}_j$ be its validation set and let

$$\ell_j(w) = \frac{1}{|\mathcal{V}_j|} \sum_{z^{val} \in \mathcal{V}_j} \ell(w, z^{val})$$

denote the validation loss on task j. We define the local payoff of coalition S at step t as the one-step validation loss reduction:

$$u_j^{(t)}(S) = \ell_j(w_t) - \ell_j(\widetilde{w}_{t+1}(S)). \tag{7}$$

This local payoff measures how much the current update, if performed using only languages in S, improves performance on target language j.

IRDS Shapley Value. The IRDS SV of language i on task j is obtained by computing the SV of i for each local payoff $u_j^{(t)}(\cdot)$ and accumulating it across training:

$$\hat{\phi}_{i,j}^{IRDS} = \sum_{t=0}^{T-1} \phi_i\left(u_j^{(t)}\right). \tag{8}$$

By the linearity of SV, this is equivalent to computing the SV of the cumulative payoff $\sum_{t=0}^{T-1} u_j^{(t)}(\cdot)$. Therefore, $\hat{\phi}_{i,j}^{IRDS}$ measures the cumulative contribution of language i to the loss reduction on target language j during the same pre-training run.

First-order approximation. Directly computing $\phi_i(u_j^{(t)})$ still requires enumerating language subsets within each step. We therefore approximate the local payoff in Eq. 7 by a first-order Taylor expansion. Define the validation gradient for target language j as

$$g_{t,j}^{val} = \nabla \ell_j(w_t),$$

and define the aggregated training gradient of language i at step t as

$$g_{t,i} = \sum_{z \in B_{t,i}} \nabla \ell(w_t, z).$$

Algorithm 1 In-Run Data Shapley Approximation

Require: Language set $\mathcal{K}$, uniform multilingual pre-training data, validation sets $\{\mathcal{V}_j\}_{j=1}^{K}$, learning rates $\{\eta_t\}_{t=0}^{T-1}$
Ensure: Approximate IRDS Shapley values $\{\hat{\phi}_{i,j}^{IRDS}\}_{i,j \in \mathcal{K}}$

```
for i = 1 to K do
    for j = 1 to K do
        φ̂^IRDS_{i,j} ← 0
    end for
end for
for t = 0 to T − 1 do
    Sample multilingual mini-batch B_t
    for i = 1 to K do
        B_{t,i} ← {z ∈ B_t : lang(z) = i}
        g_{t,i} ← Σ_{z∈B_{t,i}} ∇ℓ(w_t, z)
    end for
    for j = 1 to K do
        g^val_{t,j} ← ∇ℓ_j(w_t)
        for i = 1 to K do
            Δ^(t)_{i,j} ← η_t ⟨g^val_{t,j}, g_{t,i}⟩        ▷ First-order IRDS
            φ̂^IRDS_{i,j} ← φ̂^IRDS_{i,j} + Δ^(t)_{i,j}
        end for
    end for
    Update w_{t+1} using the full multilingual batch B_t
end for
return {φ̂^IRDS_{i,j}}_{i,j∈K}
```

Then,

$$u_j^{(t)}(S) \approx \eta_t \sum_{r \in S} \left\langle g_{t,j}^{val}, g_{t,r} \right\rangle.$$

Since the approximated utility is additive over languages, its Shapley value has a closed form:

$$\phi_i\left(u_j^{(t)}\right) \approx \eta_t \left\langle g_{t,j}^{val}, g_{t,i} \right\rangle.$$

Thus, the first-order IRDS Shapley estimator is

$$\hat{\phi}_{i,j}^{IRDS\text{-}1} = \sum_{t=0}^{T-1} \eta_t \left\langle g_{t,j}^{val}, g_{t,i} \right\rangle. \tag{9}$$

Intuitively, language i receives a positive contribution at step t when its training gradient is aligned with the validation gradient of target language j, because this update decreases the validation loss on task j. Conversely, a negative value indicates that the update from language i increases the validation loss on task j.

The first term measures the direct alignment between language i and the target language j. The paper also provides a second-order approximation that captures interactions between language i and the full multilingual batch under the local curvature of the validation loss. In practice, we use the first-order estimator by default because it is simpler, more stable, and incurs minimal additional cost. The second-order estimator can be used when within-batch language interactions are important.

Normalization. After obtaining the IRDS Shapley matrix $\{\hat{\phi}_{i,j}^{IRDS}\}_{i,j\in\mathcal{K}}$, we apply the same normalization as in the main text:

$$\hat{\phi}_{i,j}^{NSV} = \exp\left(\hat{\phi}_{i,j}^{IRDS} - \max_{i'} \hat{\phi}_{i',j}^{IRDS}\right).$$

The normalized values are then used as cross-lingual transfer strengths in ShapleyLaw:

$$\Theta_j = \sum_{i\in\mathcal{K}} p_i \cdot \hat{\phi}_{i,j}^{NSV}.$$

Computational complexity. Unlike exact SV computation, IRDS requires only one multilingual pre-training run for SV estimation. During this run, the method accumulates language-level gradient contributions at each update step. The first-order estimator requires gradient dot-products between the validation gradient of each target language and the aggregated training gradient of each source language. These dot-products can be computed efficiently during backpropagation and do not require pre-training additional coalition models. Therefore, the number of pretrained models is reduced from $O(2^K)$ for exact SV to $O(1)$ pre-training runs for IRDS.

A.4 Clarification on Baseline Selection

The goal of this paper is to develop a multilingual scaling law for performance prediction and language mixture optimization. Accordingly, appropriate baselines are methods that can optimize language mixtures for large-scale training configurations *from observations made under small-scale training configurations*. Uniform Sampling, Smoothed Sampling, FamilyLaw Optimization, and Empirical Mixtures all satisfy this criterion, and we therefore adopt them as our mixture optimization baselines.

In contrast, some recent data mixture optimization methods such as DoReMi and DML (Xie et al., 2023; Ye et al., 2025) address a different problem: optimizing data mixtures under a fixed data budget. As a result, they cannot predict optimal mixtures for unseen data scales. Since the core contribution of ShapleyLaw is mixture optimization across both data and model scales, a direct comparison with these methods would be unfair and would not evaluate the capability that ShapleyLaw is designed to provide. We therefore do not include them as baselines.

B Experimental Setup Details

Corpora. We use the English $sample$-$350BT$ subset [3] from *FineWeb* (Penedo et al., 2024), and the German, French, Spanish, Chinese, Japanese, Korean, Finnish, Croatian, and Standard Malay subsets from *FineWeb-2 v2.0.0* (Penedo et al., 2025). The total tokens are summarized in following Table 5.

Training Configuration. We pre-train multilingual LLMs with Megatron-LM (Shoeybi et al., 2019) on 8-node 8×H200 clusters. We adopt the open-source model architectures and training hyperparameters provided by the *llm-jp* project[4].

Table 4 summarizes the configurations of three transformer language models with increasing non-embedding parameter scales (50M, 243M, 684M, and 1.46B). As the model size grows, the number of layers and hidden dimensions increase, while the number of attention heads and context length remain constant. Larger models allocate substantially more parameters to both embedding and non-embedding components, with non-embedding parameters growing more rapidly, indicating that most additional capacity comes from deeper and wider transformer blocks rather than the embedding layer.

We adopt a decoder-only Transformer with RMSNorm, RoPE, and SwiGLU. Tokenization uses the LLaMA-2–style `llm-jp-tokenizer-100k.ver3.0b1`[5] tokenizer with a 100k multilingual vocabulary. Models are trained with learning rate 3×10^{-4}, sequence length 4,096, and global batch size 512 using Adam (β_1=0.9, β_2=0.95), cosine decay with warmup, and standard stabilization techniques (weight decay, gradient clipping, Z-loss). Training efficiency is improved through bf16 precision, FlashAttention, fused kernels, and Megatron-LM parallelism, and enabling stable large-scale multilingual pre-training.

Mixture Optimization Baselines. We consider the following baselines for mixture optimization.

- *Uniform Sampling:* All languages are assigned equal proportions. This balanced heuristic serves as the simplest non-parametric baseline.
- *Smoothed Sampling (Conneau and Lample, 2019):* This strategy downweights dominant languages

[3] A subset randomly sampled from the whole dataset of around 350B gpt2 tokens (388GB).

[4] https://huggingface.co/llm-jp

[5] https://github.com/llm-jp/llm-jp-tokenizer

Params	Layers	Hidden size	Heads	Context length	Embedding parameters	Non-embedding parameters
150M (50M)	12	512	8	4,096	101,874,688	50,344,448
440M (243M)	16	1,024	8	4,096	203,749,376	243,303,424
980M (684M)	20	1,536	8	4,096	305,624,064	684,258,816
1.8B (1460M)	24	2,048	16	4,096	407,498,752	1,459,718,144
3.7B (3171M)	28	3,072	24	4,096	611,248,128	3,171,068,928

Table 4: Model configurations and parameter breakdown.

Family	Languages	Tokens (B)
Germanic	English, German	373 & 450
Romance	French, Spanish	340 & 397
Sino-Tibetan	Chinese	788
Japonic	Japanese	281
Korean	Korean	52
Uralic	Finnish	48
Slavic	Croatian	29
Austronesian	Standard Malay	12

Table 5: **Corpora used in our experiments.** The study covers ten languages from eight language families, the available token counts is listed for each language.

while still reflecting corpus size through a smoothing parameter α, i.e., $p_i = \frac{(\bar{D}_i/D)^\alpha}{\sum_{i'=1}^{K}(\bar{D}_{i'}/D)^\alpha}$. We adopt the common setting $\alpha = 0.5$ for all the related experiments.

- *FamilyLaw Optimization (He et al., 2025):* This strategy first optimizes family-level mixture ratios using FamilyLaw and then allocates language-level mixture ratios using the Smoothed Sampling (see Appendix F in (He et al., 2025) for details).

- *Empirical Mixtures:* Mixture ratios are drawn from existing open-source corpora, such as *Common Crawl* and *FineWeb*. We include this setting to evaluate whether the proposed method improves over corpus-formed sampling strategies.

Downstream Tasks. We evaluate all seven pretraining languages on publicly available benchmarks covering linguistic competence, natural language understanding, and commonsense reasoning. The tasks are: *HellaSwag* (commonsense completion) (Zellers et al., 2019; Lai et al., 2023), *XStoryCloze* (cross-lingual story ending prediction) (Lin et al., 2022; Mostafazadeh et al., 2017), and *XWinograd* (multilingual coreference resolution) (Tikhonov and Ryabinin, 2021). Following standard multilingual evaluation practice, we use a *five-shot* setting and report accuracy.

C Additional Experimental Results

C.1 Experiments on Transferability of Shapley Values.

Experiment design. A key assumption of ShapleyLaw is that the SVs used to quantify cross-lingual transfer are transferable across model sizes. Under this assumption, the SVs estimated from a relatively small model can be reused as fixed transfer coefficients when fitting scaling laws for larger models. To empirically verify this property, we conduct a controlled comparison across two model sizes, $N = 50\text{M}$ and $N = 243\text{M}$, while keeping the pretraining data size fixed at $D = 50\text{B}$. We consider six language sets, $\mathcal{K} = \{\text{de}, \text{en}, \text{fr}, \text{ja}, \text{ko}, \text{zh}\}$. For each model size, we construct a multilingual pretraining game over the same language set and estimate the normalized SV matrix $\phi^{\text{NSV}} \in \mathbb{R}^{|\mathcal{K}|\times|\mathcal{K}|}$, where each entry $\phi^{\text{NSV}}_{i,j}$ measures the transfer contribution from training language i to evaluation language j. We then compare the two normalized SV matrices obtained at different model sizes.

We use cosine similarity and Pearson correlation coefficient as the main comparison metrics. Specifically, we vectorize the two SV matrices in the same language order and compute their similarity over all language pairs. Since the diagonal entries correspond to self-transfer and are normalized to one, we also report the similarity after removing diagonal entries, which provides a stricter evaluation of cross-lingual transfer patterns.

Results. Figure 6 shows the normalized SV matrices estimated from the 50M and 243M models. The two heatmaps display highly similar structures. In both cases, self-transfer has the largest contribution, while the off-diagonal entries reveal stable asymmetric cross-lingual transfer patterns. For example, English provides relatively strong transfer to several European languages, and the transfer patterns among Japanese, Korean, and Chinese remain qualitatively consistent across the two model sizes.

Quantitatively, when all entries are included, the

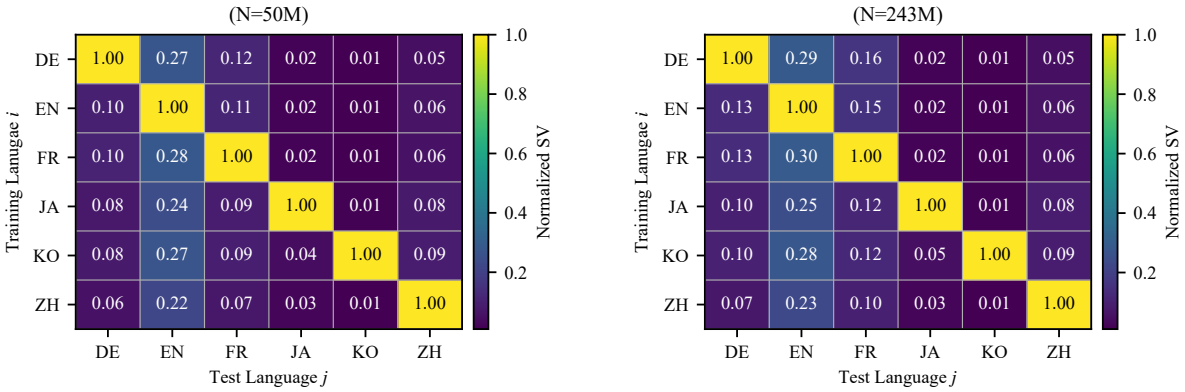

Figure 6: **Comparison of Shapley values across model sizes. Left**: normalized Shapley values estimated with $N = 50$M and $D = 50$B. **Right**: normalized Shapley values estimated with $N = 243$M and $D = 50$B. The two heatmaps exhibit highly consistent cross-lingual transfer patterns across model sizes.

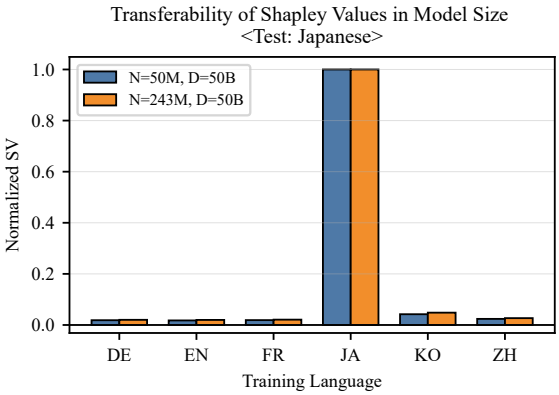

Figure 7: **Comparison of Shapley values across model sizes <Japanese>.** We compare the Shapley values for the Japanese evaluation language across two model sizes. The values obtained from the $N = 50$M and $N = 243$M models are highly consistent, with a cosine similarity of 0.999 and a Pearson correlation coefficient of 0.999.

cosine similarity between the two vectorized SV matrices is 0.999, and the Pearson correlation coefficient is 0.999. After excluding the diagonal entries, the cosine similarity remains high at 0.995, and the Pearson correlation coefficient is 0.989. These results indicate that the cross-lingual transfer structure captured by SVs is highly stable when scaling the model from 50M to 243M parameters.

We further examine the Japanese evaluation language as a representative case. As shown in Figure 7, the SV profiles for Japanese estimated from the two model sizes are almost identical. The cosine similarity reaches 0.999, and the Pearson correlation coefficient is also 0.999. This confirms that not only the global transfer matrix but also individual target-language transfer profiles remain stable across model sizes.

Overall, these results support the transferability of SVs with respect to model size. Therefore, in the main experiments, we estimate SVs using a small-scale multilingual pretraining game and reuse them as fixed cross-lingual transfer coefficients when fitting ShapleyLaw for larger model sizes. This design substantially reduces the computational cost of estimating transfer effects while preserving the transfer structure needed for accurate multilingual scaling-law fitting.

C.2 Additional Experiments on Performance Prediction.

To further validate the robustness of the proposed Shapley-based scaling law, we report additional fitting results on four target languages: EN, FR, DE, and KO. For each target language, we fit the proposed relation using both curated multilingual mixtures and randomly sampled mixtures, and evaluate whether the aggregate cross-lingual transfer Θ consistently explains the variation in held-out test CE loss. Representative examples for EN, and FR are shown in Figure 8: the top row presents fits across predefined mixture settings and model scales, while the bottom row shows results under random mixture sampling. In both cases, empirical losses align closely with the predicted curves, and held-out points remain near the fitted trend, indicating that the scaling law generalizes beyond the mixtures used for fitting.

The same pattern is observed for DE, and KO (see Figure 9), where losses across different model sizes and sampling strategies can also be well described by the ShapleyLaw. Across these additional languages, the fitted curves remain smooth and monotonic with respect to Θ, and the resulting high R^2 values ranging from 0.985 to 0.998. These results provide further evidence that the proposed param-

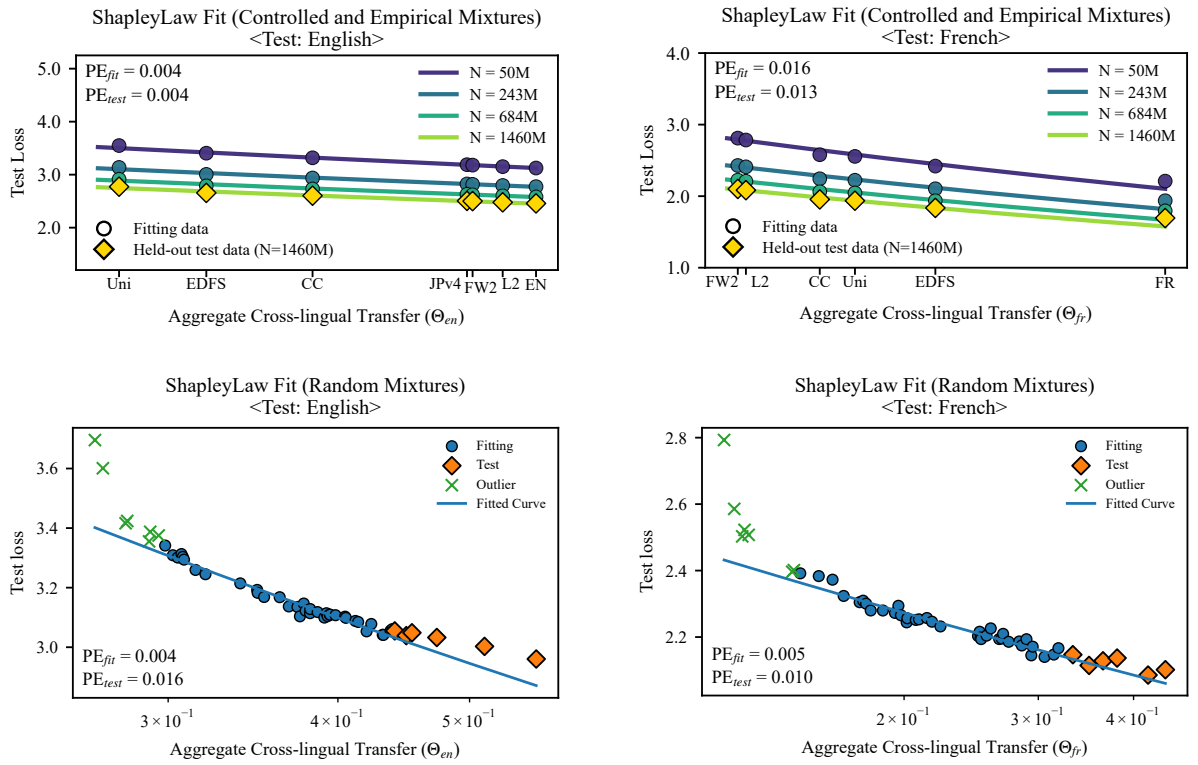

Figure 8: **Additional fitting results for Shapley-based scaling laws on <EN,FR>. Top row**: Fits on curated multilingual mixtures for EN, and FR across four model scales $\{50\text{M}, 243\text{M}, 684\text{M}, 1460\text{M}\}$; gray circles denote training points and yellow diamonds denote held-out points. **Bottom row**: Fits under randomly sampled mixtures under model scale N=243M. For both languages, the fitted curves remain smooth and monotonic with respect to Θ, and held-out points closely follow predicted trends, demonstrating strong generalization to unseen mixtures.

Task	ℓ	$-\ell$	$\log \ell$	$\exp(-\ell)$	$1/\ell$
HellaSwag	0.994	0.994	0.995	**0.996**	**0.996**
XWinograd	0.971	0.971	0.975	**0.980**	0.978
XStorycloze	**0.947**	**0.947**	0.943	0.936	0.939

Table 6: **Model Selection.** Model comparison for $s = a \cdot g(\ell) + b$ using different transformations of loss. Numbers denote R^2. Across tasks except XStorycloze, $g(\ell) = \exp(-\ell)$ consistently achieves the highest explanatory power, suggesting that accuracy scales approximately linearly with token-level probability$p = \exp(-\ell)$.

eterization is not specific to a single language or mixture family, but instead captures a common regularity in multilingual pre-training. Overall, the additional results strengthen the conclusion that aggregate cross-lingual transfer serves as an effective mixture variable for modeling per-language scaling behavior across diverse target languages.

C.3 Additional Experiments on Downstream Tasks

Results of Model Selection. To determine an appropriate form for $g(\ell)$, we evaluate five candidate transformations: $\{\ell, -\ell, \log \ell, \exp(-\ell), 1/\ell\}$. For each candidate, we fit a linear model of the form $s = a \cdot g(\ell) + b$ that maps loss values to predicted scores. Table 6 reports the resulting model comparisons. Among these transformations, $\exp(-\ell)$ always exhibits the strongest explanatory power, achieving the highest R^2 and consistently providing the best fit to the observed relationship between loss and score. These results indicate that an exponential transformation of loss most effectively captures the empirical mapping between the two quantities, and we therefore adopt $g(\ell) = \exp(-\ell)$ in the subsequent experiments.

Fitting Results on Downstream Tasks. We next model the relationship between pre-training loss ℓ and downstream performance using the functional form $s = a \cdot \exp(-\ell) + b$. Table 7 reports the fitting results across six English downstream tasks, comparing pre-training loss with few-shot (5-shot) accuracy. Across all tasks, we observe consistently strong *negative correlations*, with Pearson coefficients ranging from -0.909 to -0.997 (p-value < 0.01 in all cases), indicating that lower pre-training loss is strongly associated with higher downstream performance. The fitted models also achieve high explanatory power, with R^2 values be-

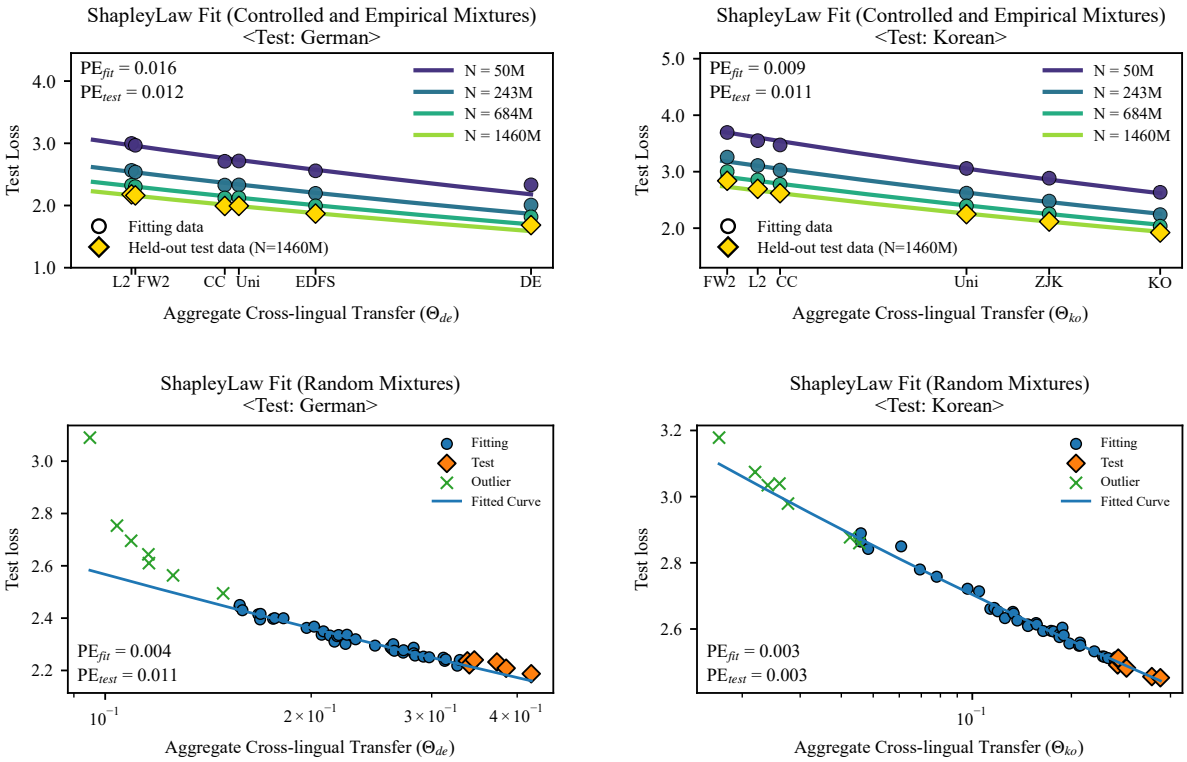

Figure 9: **Additional fitting results for Shapley-based scaling laws on <DE,KO>. Top row**: Fits on curated multilingual mixtures for DE and KO across the same four model scales. **Bottom row**: Fits under randomly sampled mixtures with model scale N=243M. For both languages, the fitted curves remain smooth and monotonic with respect to aggregate cross-lingual transfer Θ, and held-out points closely follow predicted trends, demonstrating strong generalization to unseen mixtures.

Task	Pearson r	p-value	Best $g(\ell)$	R^2	a	b
HellaSwag	-0.997	8.15×10^{-7}	$\exp(-\ell)$	0.996	4.030	0.098
XWinograd	-0.985	4.92×10^{-5}	$\exp(-\ell)$	0.980	4.824	0.382
XStorycloze	-0.973	2.25×10^{-4}	$\exp(-\ell)$	0.980	3.034	0.433

Table 7: **Fitted Results (English Downstream Tasks).** Pearson correlation between pretraining loss ℓ and few-shot accuracy. All tasks exhibit strong negative correlation ($|r| > 0.90$), indicating that lower loss consistently corresponds to higher downstream performance. Correlation between pretraining loss ℓ and few-shot accuracy, and best-fit results for $\hat{s} = ag(\ell) + b$. Across all tasks, $\exp(-\ell)$ consistently achieves the highest R^2.

tween 0.843 and 0.996, suggesting that the exponential transformation $\exp(-\ell)$ captures the empirical relationship between loss and accuracy well. These results support the use of $s = a \cdot \exp(-\ell) + b$ as a simple yet effective mapping for predicting downstream task performance from pre-training loss.

Additional Fitting Results on Other Language Tasks. We conduct additional experiments in downstream tasks for Chinese and Japanese to explore feasibility of ShapleyLaw. In addition, we fit results of downstream performance vs test loss of 50 models ($N = 243$M, $D = 100$B) with randomly sampled language mixtures. As shown in Figure. 10, we report fitting results for Chinese, and Japanese.

Comparison	Cosine Similarity	Pearson
Exact SV vs. IRDS SV	0.9584	0.9390

Table 8: Similarity comparison between exact SV and IRDS SV.

Target	de	en	fr	es	ja	ko	zh
Exact	0.015	0.004	0.015	0.014	0.006	0.009	0.015
IRDS	0.015	0.003	0.036	0.017	0.008	0.023	0.013

Table 9: Comparison of corresponding PE_{fit} values for the fitted results with exact and IRDS SVs.

C.4 Additional Experiments on SV Approximation

Experiment Design. Exact SV computation requires evaluating all non-empty language coalitions, resulting in $2^K - 1$ pretrained models for K lan-

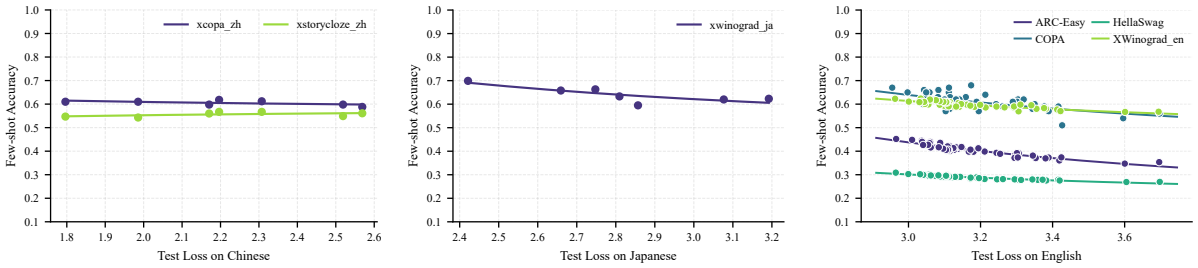

Figure 10: **Left.** Downstream performance versus test cross-entropy (CE) loss on Chinese tasks (*XCOPA_zh* and *XStoryCloze_zh*). **Middle.** Downstream performance versus test CE loss on the Japanese task (*XWinograd*). **Right.** Downstream performance for English versus test CE loss across 50 models ($N = 243$M, $D = 100$B) trained with randomly sampled data mixtures.

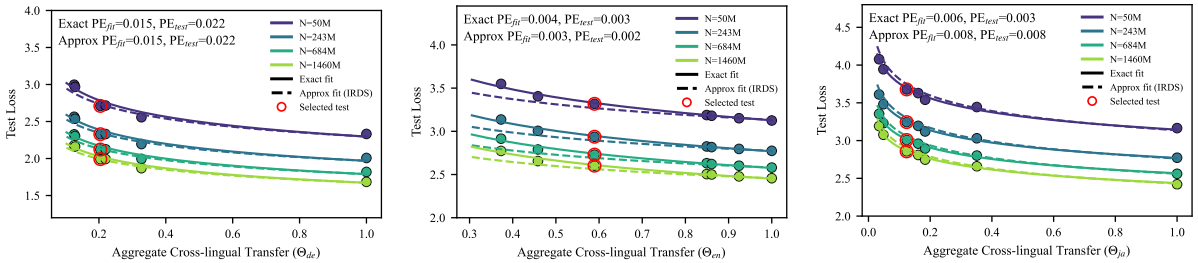

Figure 11: **Fitting results with exact and IRDS-approximated SVs.** To reduce the workload of SV computation, we use the IRDS approximation by pretraining a small-scale model ($N = 50$M, $D = 50$B) with a uniform mixture ratio over seven languages. The resulting IRDS SVs produce fitting curves that closely match those obtained with exact SVs across target languages.

guages. This cost grows exponentially with K and becomes prohibitive in large multilingual settings. To assess whether ShapleyLaw remains effective without exact SVs, we evaluate IRDS as a scalable approximation that estimates language contributions from a single model pretrained with a uniform multilingual mixture. We use the same MPG as in the main experiments. Specifically, we pretrain a small-scale model with $N = 50$M and $D = 50$B using a uniform mixture over seven languages, estimate the IRDS matrix, and construct the aggregate transfer variable Θ_j. We then fit ShapleyLaw under the same settings as the exact-SV variant, isolating the effect of SV approximation on both transfer estimation and scaling-law fitting quality.

Similarity between Exact and IRDS SVs. Table 8 reports the similarity between exact SVs and IRDS-estimated SVs. The two SV matrices are highly aligned, achieving a cosine similarity of 0.9584 and a Pearson correlation of 0.9390. This indicates that IRDS preserves the global geometry of the cross-lingual transfer matrix: languages that receive large contributions under exact SV estimation also tend to receive large contributions under IRDS. Therefore, although IRDS does not explicitly enumerate all language coalitions, it provides a close approximation to the relative transfer strengths needed by ShapleyLaw.

Fitting Results. Figure 11 shows that the IRDS-based fitting curves closely match those obtained with exact SVs across target languages, indicating that ShapleyLaw is robust to approximation noise in the SV matrix. The close alignment of the curves suggests that IRDS preserves this transfer structure sufficiently for loss prediction. Table 9 further confirms this trend. IRDS achieves fitting errors comparable to exact SVs for most languages, matching or slightly improving the exact-SV results for German, Chinese, and English. The degradation is modest for Japanese and Spanish but more pronounced for French and Korean. Nevertheless, the PEs remain low, and the fitted trends are broadly consistent with the exact-SV results.

Discussion. These results show that exact SV computation is not essential for ShapleyLaw's practical effectiveness. In the seven-language setting, exact SV estimation requires $2^7 - 1 = 127$ pretrained models, whereas IRDS requires only one. Despite this large cost reduction, IRDS achieves SV estimates with high similarity to exact SVs and yields comparable fitting quality for most languages. The remaining gaps mainly reflect approximation

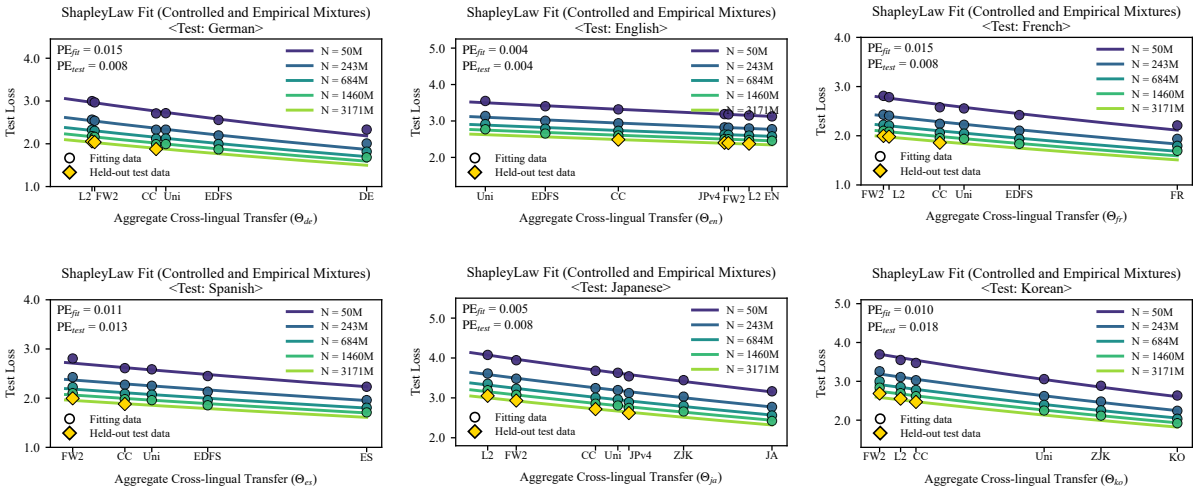

Figure 12: **Additional fitting results for ShapleyLaw at a larger model scale.** We extend the evaluation to $N = 3171$M and $D = 100$B. Fits are shown for DE, EN, FR, ES, JA, and KO across model scales $\{50\text{M}, 243\text{M}, 684\text{M}, 1460\text{M}, 3171\text{M}\}$. Gray circles denote fitting points and yellow diamonds denote held-out points.

noise and do not materially affect the fitted scaling behavior, supporting ShapleyLaw's scalability to larger multilingual settings.

C.5 Additional Experiments on Larger Model Scale

Experiment Design. To further evaluate the extrapolation ability of ShapleyLaw beyond the model scales used in the main experiments, we conduct additional experiments at a larger model size. Specifically, we pretrain models with $N = 3171$M and $D = 100$B under four empirical multilingual mixtures: *CC*, *FW2*, *L2*, and *JPv4*. These mixtures correspond to commonly used multilingual data distributions and provide a realistic evaluation setting for large-scale multilingual pretraining.

We treat all $N = 3171$M results as held-out points and fit ShapleyLaw using the smaller-scale observations. This setting directly tests whether the learned scaling relationship can generalize to a larger model scale that is not used for fitting. We evaluate six target languages, including DE, EN, FR, ES, JA, and KO. For each target language, the input variable remains the SV-based aggregate cross-lingual transfer Θ_j, while the scaling law jointly captures the effects of model size, data size, and language mixture.

Results. Figure 12 shows that ShapleyLaw accurately predicts the held-out $N = 3171$M test losses across all six languages. The held-out points generally lie close to the fitted curves, indicating that the learned relationship between aggregate cross-lingual transfer and test loss remains stable when extrapolated to a larger model scale.

ShapleyLaw achieves low held-out prediction errors across languages: $\text{PE}_{test} = 0.008$ for German, 0.004 for English, 0.008 for French, 0.013 for Spanish, 0.008 for Japanese, and 0.018 for Korean. These values are comparable to the fitting errors obtained on smaller-scale models, suggesting that the proposed law does not merely interpolate among observed configurations but can also provide reliable predictions for larger-scale models.

The results are particularly encouraging because the held-out points combine two sources of distribution shift: they use a larger model size and empirical multilingual mixtures. Despite this harder setting, ShapleyLaw preserves accurate predictions across typologically diverse target languages. This provides additional evidence that SV-based aggregate transfer is a stable predictor of multilingual test loss and that ShapleyLaw can support practical mixture evaluation at larger model scales without requiring exhaustive large-scale pretraining for every candidate mixture.