TRA: Better Length Generalisation with Threshold Relative Attention

Anonymous authors
Paper under double-blind review

Abstract

Transformers struggle with length generalisation, displaying poor performance even on basic tasks. We test whether these limitations can be explained through two key failures of the self-attention mechanism. The first is the inability to fully remove irrelevant information. The second is tied to position, even if the dot product between a key and query is highly negative (i.e. an irrelevant key) learned positional biases may unintentionally up-weight such information - dangerous when distances become out of distribution. Put together, these two failure cases lead to compounding generalisation difficulties. We test whether they can be mitigated through the combination of a) **selective sparsity** - completely removing irrelevant keys from the attention softmax and b) **contextualised relative distance** - distance is only considered as between the query and *the keys that matter*. We show how refactoring the attention mechanism with these two mitigations in place can substantially improve generalisation capabilities of decoder only transformers.

1 Introduction

Transformers (Vaswani et al., 2017) have sparked a revolution in artificial intelligence. When pre-trained at scale, these models display impressive broad-spectrum capabilities, particularly in the domains of natural language and code (Bubeck et al., 2023; Dubey et al., 2024). This includes the ability to perform complex abstract symbolic processing through in-context learning (ICL) (Brown et al., 2020; Sinha et al., 2024; Smolensky et al., 2024). On the other hand, these abilities are far from robust (Dziri et al., 2023; Mirzadeh et al., 2024) and display dramatic failure modes such as self-contradiction and hallucination (Liu et al., 2023a; Ji et al., 2023; Guerreiro et al., 2023). Similar failure modes are also observed in controlled synthetic settings where the correct solution *should be trivial*. Decoder-only transformers consistently display poor generalisation to unseen lengths and dependency distances (Zhou et al., 2023; Liu et al., 2023a).

Addressing these issues can provide crucial benefits regarding both long context utilisation and length generalisation more broadly. Firstly, current approaches generally involve training at a certain sequence length and then adapting the positional encoding to longer context. Most commonly this entails additional fine-tuning for best effect, thereby incurring a further compute cost penalty. Even then current SoTA foundation models still struggle with long context (Hsieh et al., 2024). Ideally, we would want a solution that simply works out the box. Put another way, if the transformer is in fact learning circuits (Merrill et al., 2022; Olsson et al., 2022; Liu et al., 2023b), then their correct execution should be general purpose and robust, rather than contingent on specific quirks of the training data.

To understand how to get there, we should first look at the core capability that the attention mechanism was supposed to enable in the first place, namely the ability to selectively retrieve and propagate information (Bahdanau et al., 2014; Luong et al., 2015; Vaswani et al., 2017). We can break this capability down into three broad categories. (1) Purely positional: attending to ones local neighbourhood (i.e. contextualisation within a sentence or phrase). (2) Purely based on content: attending to the most relevant item irrespective of position. (3) Content followed by position: of the relevant items apply a positional bias (e.g. select the most recent). (3) is a crucial capability, it enables tracking the state of an entity over time, and is a pre-requisite for robust reasoning. Our contention is that standard attention mechanisms can perform (1) and (2), but

struggle with (3). This due to the way that semantic content and positional information combine. Position is generally treated as independent of content. Which means that favouring one must come at the expense of the other. This leaves the two types of information in conflict, fundamentally limiting self-attention.

Our solution involves a two step process. We first mask out irrelevant keys based on threshold applied to the raw attention weights, and then compute the relative distance but only as between the query and the keys that remain. We call this mechanism **T**hreshold **R**elative **A**ttention (**TRA**). TRA is a simple, modular addition that allows self-attention to learn general solutions to all three behaviours outlined above. We find that TRA's introduction leads to considerably improved length generalisation on synthetic benchmarks. As well as robust, and even improved, language modeling perplexity for out-of-distribution sequence lengths, even up to an increase of 32x. Our findings contribute to the development of more robust attention mechanisms.

2 Model

2.1 Background: Relative Positional Encodings

Relative positional encodings (Dai et al., 2019; Raffel et al., 2019) decompose the attention operation into two steps. First the standard $\boldsymbol{QK}^T$ dot product is calculated, but this purely based on semantic relevance. Let's call that matrix $\boldsymbol{S}$ (semantic). This is then added with a learned positional bias matrix $\boldsymbol{D}$ (distance) to produce the final attention weight.

$$\boldsymbol{S} = \frac{\boldsymbol{QK}^T}{\sqrt{d_{\boldsymbol{k}}}} \tag{1}$$

$$\boldsymbol{A} = \mathrm{softmax}\left(\boldsymbol{S} + \boldsymbol{D}\right) \tag{2}$$

Our contention is that a central cause of length generalisation issues stems from $\boldsymbol{S}$ and $\boldsymbol{D}$ being treated as independent of each other. If $\boldsymbol{D}$ is fit to the training distribution, then distribution shifts will necessarily begin to introduce errors. $\boldsymbol{S}$ is more robust in this regard, because relevance isn't impacted by changing lengths. We want to make $\boldsymbol{D}$ more robust by making it contingent on $\boldsymbol{S}$.

2.2 Contextual Relative Distance

We want to condition $\boldsymbol{D}$ on $\boldsymbol{S}$. In other words we only want to consider position over keys that matter. To achieve this we first introduce **selective sparsity**. We will allow certain keys to be fully removed from $\boldsymbol{S}$, but do not force it to be one hot. Instead we introduce a **threshold** and eliminate all keys which fall below it. Lets call that matrix $\boldsymbol{S'}$. The simplest way to get $\boldsymbol{S'}$ is by applying a ReLU.

$$\boldsymbol{S'} = \mathrm{ReLU}\left(\frac{\boldsymbol{QK}^T}{\sqrt{d_{\boldsymbol{k}}}}\right) \tag{3}$$

Next calculate a boolean mask $\boldsymbol{M}$ that is true only for relevant positions:

$$M_{i,j} = \begin{cases} 1 & \text{if } S'_{i,j} > 0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

We are now going to compute our contextualised distance matrix, but only consider the relative distance to keys which make it through the threshold, this give us the **contextualised relative distance**. To do this we first initialise distances $\hat{\boldsymbol{D}}$ as $\boldsymbol{M}$. Consider this example:

$$\hat{\boldsymbol{D}} = \begin{matrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{matrix} \tag{5}$$

We next perform a right-to-left cumulative sum within each row giving us our updated distances[1] $\bar{\boldsymbol{D}}$:

$$\bar{\boldsymbol{D}} = \begin{matrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 0 \end{matrix} \tag{6}$$

2.3 Parameterising $\boldsymbol{D}$

Now we have arrived at our contextualised relative distances, but we still need some way of turning them into an actual weight. In order to do so we utilise a parameterised forget gate. Lets denote the residual stream for position i as $\boldsymbol{x}_i$, with $\boldsymbol{W}_f \in \mathbb{R}^{E\times 1}$ as the forget projection, b as a scalar bias, and σ as the sigmoid non-linearity. Then the weight for each position is given as:

$$\delta_i = \sigma\left(W_f x_i + b\right) \tag{7}$$

$$D'_{ij} = \delta_i^{\bar{D}_{ij}} \tag{8}$$

As each δ_i is a sigmoided scalar value the more times it is multiplied with itself the smaller it gets. The closer δ_i is to 1 the longer the decay takes, the closer to zero the shorter. This provides the model a temporal memory. Enabling it to forget irrelevant past information if necessary, or to treat all timesteps uniformly if temporal ordering is not a concern. We opt for a recency bias for several reasons. Firstly, it has been generally shown to be useful when modeling natural language (Shen et al., 2018; Press et al., 2021; Zhu et al., 2021; Clark et al., 2025). Secondly, it is unclear whether a token occupying the ith position provides useful information beyond causal ordering. The first occurrence can be identified by proximity to the start of sequence token, and middle positions are better differentiated via their unique semantic context as described in Ebrahimi et al. (2024). Finally, decaying memory is the mechanism empowering the resurgence in state space models (Peng et al., 2023; Orvieto et al., 2023; De et al., 2024), recent successful recursive networks (Opper & Siddharth, 2025), and has shown promise with transformers (Csordás et al., 2021; Lin et al., 2025).

2.4 The Final Attention Weights

With τ as a head specific learnable temperature parameter, we compute the final attention weights $\boldsymbol{A'}$:

$$\hat{\boldsymbol{A}} = \boldsymbol{S'} + \boldsymbol{D'} \tag{9}$$

$$\bar{A}_{ij} = \begin{cases} -1e11 & \text{if } M_{ij} = 0 \\ \hat{A}_{ij} & \text{otherwise} \end{cases} \tag{10}$$

$$\boldsymbol{A'} = \text{softmax}\left(\bar{\boldsymbol{A}}/\tau\right) \tag{11}$$

This gives us the full TRA attention mechanism. Note that keys which do not meet the threshold are completely masked out from the final attention softmax, and distances are only considered between the query and the keys that matter. This means that TRA exhibits both selective sparsity and allows semantic content to determine and consequently synergise with the positional bias.

3 Experiments

3.1 Tasks

Our core hypothesis is that length generalisation failures occur in transformers because the standard attention mechanism puts semantic and positional information in conflict. TRA is designed to mitigate this. To test our hypothesis, we first turn to controlled synthetic tasks, followed by language modeling.

[1] In this example we zero all indices that don't meet the threshold for clarity, but note that this is not strictly necessary as they are masked in the operation from (10).

Flip-Flop Language Modelling: Introduced by Liu et al. (2023a) Flip-Flop language modelling is a algorithmic reasoning benchmark designed to test transformers ability to glitchlessly handle sequential dependencies. Sequences consist of symbol alphabet of three instructions write (w), ignore (i) and read (r) each of which is followed by a bit. To solve the task the model has to recall the bit that follows the nearest write instruction while ignoring all irrelevant information (e.g. in: w1i0i1i1i0i1i0i0i1i0r out: 1). Memorising sequences will lead to failure. The model has to learn a program to succeed. The ability to model a flip-flop is the foundation for all syntactic parsing and algorithmic reasoning capabilities (it necessitates the propagation of information according to abstract rules), and is a pre-requisite to being able to maintain a state hierarchically (Merrill et al., 2022; Liu et al., 2023b). The task consists of three test sets: IID, and two out of distribution sets that vary the number of intervening ignore instructions. OOD sparse increases the number of ignore instructions and requires the ability to handle increased dependency distance. OOD dense decreases the number of ignore instructions and consequently probes whether the model retain focus in the presence of an increased number of attractors (i.e. write instructions). Input examples consist of strings of length 512 with the probability of an ignore instruction being generated set to 0.98 in the sparse set and 0.1 in the dense set. The criterion for success is perfection (i.e. 100% accuracy across all test sets). The authors find that transformers exhibit reasoning errors across a wide range of architectural variants and that issue is *scale-invariant*, persisting even up to GPT-4.

Induction Heads: This is a test of associative memory. Given a string of unique symbols the model must learn the following reasoning pattern: a $\rightarrow$ b ... a $\rightarrow$? out: b, otherwise known as an induction head. This ability has been shown to emerge for specific heads and for specific input-output pairs after exposure to roughly two billion tokens of pre-training data (Olsson et al., 2022), and is hypothesised to be a crucial mechanism for in-context learning. However, when tested in controlled synthetic settings, transformers' ability to learn the generalised pattern has been limited. Zhou et al. (2023) show that models are only able to weakly generalise (from length $\approx$40 to 50) despite the fact that they *should* in theory easily be able to learn the circuit. Beyond basic positional reasoning the induction heads task is also a test of the attention mechanisms capacity for specificity. As length increases the number of irrelevant features grows and the model must be able to sharply discriminate so as to select the correct successor.

Copy: A more difficult version of the induction heads task is copying non-unique strings. Here we evaluate using a vocabulary of 10 symbols. The model must learn to absorb the history for each token in order to predict the correct successor. In out of distribution length settings difficulty is increased because the level of history required to distinguish between potential successors increases. It is also a task where our current models of transformer programs, which treat symbol manipulation as fully discrete (i.e. one-hot selection), cannot find a solution (Weiss et al., 2021; Zhou et al., 2023), and therefore predict generalisation failure. Copying is a test of the models capacity to maintain a memory. A given tokens capacity to memorise its past determines successful disambiguation of its successor.

For all tasks we use full sequence accuracy as the criterion (exact match). For the copy and induct tasks we train on sequences with input length 50 and evaluate on buckets of increased OOD lengths up to 300.

3.2 Baselines and Experimental Setup

For both TRA and the baselines we use the same core transformer backbone based on the Llama 3 variant of the architecture (Dubey et al., 2024) consisting on SwiGLU layers Shazeer (2020) coupled with RMSNorm (Zhang & Sennrich, 2019). We compare TRA with the following baselines:

No positional encoding (NoPE): Kazemnejad et al. (2023) claim that decoder only transformers are implicitly able to represent positional information and that therefore explicitly encoding it is unnecessary. Our first baseline is therefore simply standard causal attention with no positional embedding.

Absolute positional encoding (APE): Positional information for each index is represented by a learned embedding which is added to the residual stream at layer zero. This was the standard in the original variant (Vaswani et al., 2017) as well as GPT-2 (Radford et al., 2019). Lengths not encountered during training will not have the corresponding embeddings trained.

Relative Positional Encodings (REL): Introduced by Transformer-XL Dai et al. (2019) and also adopted by T5 (Raffel et al., 2019). Relative Positional Encodings model position via a learnable additive bias term. Position is considered as to the relative distance of key to the query. OOD distances are all assigned the same learned value for maximum length.

Rotational Positional Encoding (RoPE): Introduced by Su et al. (2021). This approach models positional information by applying a rotation to the keys proportional to the distance from the query. Recent work by Barbero et al. (2025) demonstrates that rather than encoding a gradual distance based decay RoPE actually learns either fully positional attention (attend to predecessor or attend to first token) or full semantic attention. One coming at the expense of the other.

Label Encoding (Label): Introduced by Li & McClelland (2023) these seek to mitigate the OOD issues faced by APE by randomly selecting and then sorting indices from a range greater than sequence length. The authors demonstrate that this aides length generalisation under a limited OOD setting ($L25 \rightarrow 50$), with the contention that this is due to the encodings learning to respect relative ordering rather than absolute value. The same mechanism is also proposed in concurrent work by Ruoss et al. (2023).

Contextual Positional Encoding (CoPE): Perhaps conceptually the most related to TRA, CoPE (Golovneva et al., 2024) utilises spikes in the $\boldsymbol{S}$ matrix in order to calculate positions. Tokens are assigned fractional positions by interpolating betweeen a fixed set of absolute embeddings. Unlike TRA it does not completely remove irrelevant items from the softmax and uses an absolute rather than a relative scheme. This incurs further computational cost because *each query in each head* needs to be matrix multiplied with a layer specific absolute positional embedding matrix. The authors attempt to mitigate this issue by limiting the number of absolute positions to a small number. TRA allows for similar behaviour but in a more computationally efficient manner, and allows removal of noise through its thresholding mechanism.

Forgetting Transformer (FoT): Utilised by the recent Forgetting Transformer (FoT) (Lin et al., 2025) this mechanism shares the clostest similarity to TRA. The crucial difference is that the forgetting transformer uses standard relative distance compared with TRA contextualised alternative. It therefore serves as the truest test as to whether the issues we hypothesise regarding position are valid.

Differential Transformer (Diff): Introduced by Ye et al. (2024) the differential transformer utilises a twin heads approach where an auxiliary attention heads looks to perform noise reduction by down-weighting attention to irrelevant keys. It tests to what extent simply the ability to remove noise from the attention dot product is beneficial - without any further considerations regarding position.

Hyper-parameters: For all models we use the mini configuration from Turc et al. (2019): four heads, four layers and a 256 dimensional embedding size. The MLP is set 2x embedding size for both the linear and gate hidden units. Our focus is on small models following prior work (Zhou et al., 2023). Furthermore, attention glitches have been shown to persist at scale (Liu et al., 2023a), and the true solution for these tasks should not require additional layers. We use a linear warm-up for the first 5% of steps coupled with cosine decay. Other task and model specific hyper-parameters can be found in Appendix A.

3.3 Results

Results are in Table 1. We draw the following conclusions:

Most PE methods struggle OOD: Absolute and relative positional embeddings consistently struggle in out of distribution settings. Positional information is attached to specific indices. Best case they learn a bias in a particular direction (recency or the opposite), but even then this bias is input solely index dependent and only applicable for IID indices. Consequently, they struggle with OOD settings. RoPE also appears to learn a strategy which is fit to the training distribution and therefore struggles with out of distribution settings. This issue extends even to the differential transformer as it is based on RoPE. Consequently, despite having a more flexible selection mechanism it is still limited by its underlying positional encoding.

NoPE Generalises Weakly: The NoPE hypothesis (Kazemnejad et al., 2023) states that decoder only transformers do not need positional embeddings because they are able to approximate a counter. The theory is that they do so by selecting a token which serves as an anchor and sends some signal via the value message.

Table 1: Results represent the average across four random initialisations. Metric is full sequence accuracy (exact match). The numbers next to Copy and Induct represent input length ranges.

Model	NoPE	APE	REL	RoPE	Label	FoT	Diff	CoPE	TRA
Flip Flop IID	$\mathbf{100 \pm 0.0}$	$\mathbf{100 \pm 0.0}$	$\mathbf{100 \pm 0.0}$	$\mathbf{100 \pm 0.0}$	$\mathbf{100 \pm 0.0}$	$\mathbf{100 \pm 0.0}$	$\mathbf{100 \pm 0.0}$	$\mathbf{100 \pm 0.0}$	$\mathbf{100 \pm 0.0}$
Flip Flop OOD (Dense)	0.15 ± 0.0	27.38 ± 13.1	41.2 ± 47.29	$\mathbf{100 \pm 0.0}$	12.58 ± 15.87	$\mathbf{100 \pm 0.0}$	$\mathbf{100 \pm 0.0}$	$\mathbf{100 \pm 0.0}$	$\mathbf{100 \pm 0.0}$
Flip Flop OOD (Sparse)	99.97 ± 0.1	90.61 ± 5.64	70.48 ± 10.92	72.82 ± 1.27	86.93 ± 9.93	93.4 ± 4.21	77.8 ± 6.73	95.1 ± 4.4	$\mathbf{100 \pm 0.0}$
Induct IID (0-50)	$\mathbf{100 \pm 0.0}$	$\mathbf{100 \pm 0.0}$	$\mathbf{100 \pm 0.0}$	$\mathbf{100 \pm 0.0}$	$\mathbf{99.96 \pm 0.04}$	$\mathbf{100 \pm 0.0}$	$\mathbf{100 \pm 0.0}$	$\mathbf{100 \pm 0.0}$	$\mathbf{100 \pm 0.0}$
Induct OOD (50-100)	13.42 ± 1.96	0.0 ± 0.0	0.0 ± 0.0	7.85 ± 11.87	45.36 ± 43.99	15.00 ± 2.83	4.43 ± 0.0	20.42 ± 2.74	$\mathbf{100 \pm 0.0}$
Induct OOD (100-200)	0.0 ± 0.0	0.0 ± 0.0	0.0 ± 0.0	0.0 ± 0.0	0.0 ± 0.0	0.0 ± 0.0	0.0 ± 0.0	0.0 ± 0.0	$\mathbf{99.90 \pm 0.0}$
Induct OOD (200-300)	0.0 ± 0.0	0.0 ± 0.0	0.0 ± 0.0	0.0 ± 0.0	0.0 ± 0.0	0.0 ± 0.0	0.0 ± 0.0	0.0 ± 0.0	$\mathbf{99.33 \pm 0.0}$
Copy IID (0-50)	$\mathbf{100 \pm 0.0}$	$\mathbf{100 \pm 0.0}$	$\mathbf{100 \pm 0.0}$	$\mathbf{100 \pm 0.0}$	$\mathbf{100 \pm 0.0}$	$\mathbf{100 \pm 0.0}$	$\mathbf{100 \pm 0.0}$	$\mathbf{100 \pm 0.0}$	$\mathbf{100 \pm 0.0}$
Copy OOD (50-100)	12.43 ± 1.66	0.0 ± 0.0	3.1 ± 2.5	4.44 ± 4.79	26.97 ± 6.11	$\mathbf{97.65 \pm 4.7}$	1.61 ± 0.29	86.47 ± 21.17	$\mathbf{100 \pm 0.0}$
Copy OOD (100-200)	0.0 ± 0.0	0.0 ± 0.0	0.0 ± 0.0	0.0 ± 0.0	0.0 ± 0.0	66.04 ± 40.49	0.0 ± 0.0	40.89 ± 31.27	$\mathbf{99.87 \pm 0.14}$
Copy OOD (200-300)	0.0 ± 0.0	0.0 ± 0.0	0.0 ± 0.0	0.0 ± 0.0	0.0 ± 0.0	3.54 ± 4.35	0.0 ± 0.0	2.08 ± 2.14	$\mathbf{98.16 \pm 1.82}$

Intervening tokens send a null signal. The position of a given token is then determined by the extent to which the anchor token's signal is diminished by the softmax. From Table 1 we can see that this strategy generalises better than absolute, relative and rotary PE, consistent with the author's findings. However, it still does very poorly overall. NoPE fails at copy and induct indicating that when the anchor signal becomes too diluted due to increased length it fails to convey position accurately. The total failure of NoPE on the flip flop dense OOD set indicates issues when there are a number of possible anchor tokens in close proximity to one another. Too many anchor signals lead to errors in the count. Results show that NoPE is insufficient for true generalisation.

Flexible PE Yields Improvements but is Insufficient: The strongest contenders of the baselines utilise more flexible positonal encodings. Label, which randomly selects from a larger set of positions, is the weakest among them, but performs better than simply using absolute positions. However, alone the approach is insufficient for strong generalisation. FoT performs well on the copy and flip flop tasks, but is not able to fully generalise to either. Similar performance patterns are observed with CoPE. However, both methods struggle to generalise to the induct task which requires as its first step sharp attention to the predecessor token. This operation, also known as token shift (Peng et al., 2023), is purely positional and requires the a token to completely remove attention from itself and then apply a sharp recency bias to capture the predecessor. TRA can use its threshold mechanism to perform this operation while CoPE and FoT lack this capability and consequently suffer.

TRA succeeds: it is able to robustly generalise to all tasks. Its performance on copy and induct demonstrate strong capability for memory and associative recall. Its ability to solve flip-flops shows that it can combine positional and semantic information in a complementary fashion. To the best of our knowledge it is the first transformer network to fully solve the flip-flop benchmark of Liu et al. (2023a). It achieves 100% accuracy across seeds, which till this point has been only been achievable by LSTM networks. However, both FoT and CoPE come very close to full generalisation. To further tease apart performance differences on this core capability we generate an additional synthetic test in the same spirit as the original benchmark.

3.4 Flip-Flops++

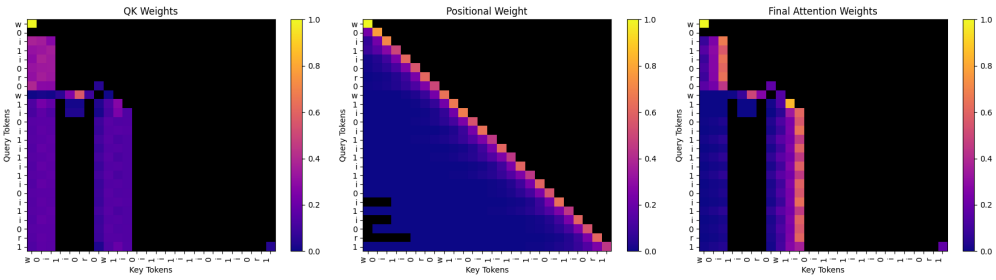
Task: To further test the differences between contextualised positions and the baseline approaches we create a further synthetic test which we refer to as Flip-Flops++. The task is a follows: the input consists of instruction followed by a sequence of random symbols combined with a trigger symbol. The set of instructions are as follows: return the symbol that occurs immediately after the *first occurrence* of the trigger, immediately after it, and immediately before or after the last occurence. For example, if we use a as the trigger symbol an example could consist of: (instruction: before first, sequence: bcxaklcaztyab, out: x). The training set contains examples from each of the instructions sequence lengths 0-50. The test set consists of of sequences of lengths 51-500. This is a more challenging scenario than the original benchmark as dependency differences can be starker than in the original sparse set, and the model is required to behave differently depending on the instruction. Moreover, because we include having to retrieve from the first occurrence among the instructions it lets us investigate whether techniques which employ a recency bias are limited to only being able to attend to the most recent relevant item. For training we employ the same hyperparameters as before.

Table 2: Length generalisation results by instruction on Flip-Flop++, taken over four random initialisations. Metric is full sequence accuracy (exact match).

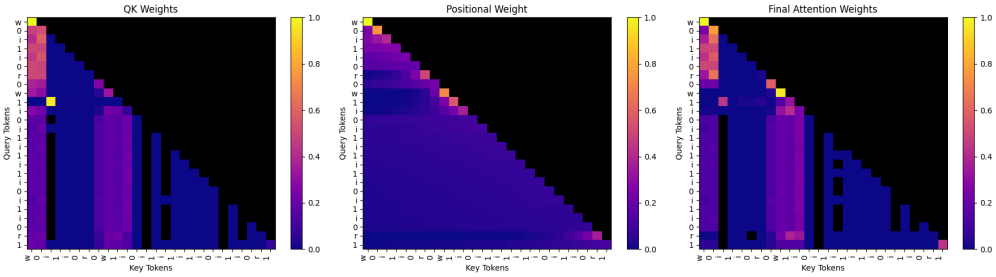
Model	NoPE	APE	REL	RoPE	Label	FoT	Diff	CoPE	TRA
After First Match	64.53 ± 19.36	70.78 ± 18.37	95.78 ± 5.1	99.27 ± 0.96	96.80 ± 3.91	51.74 ± 18.08	92.01 ± 2.07	**100 ± 0.0**	95.64 ± 4.87
After Last Match	20 ± 4.0	11.77 ± 1.84	25.16 ± 6.09	38.06 ± 7.09	37.58 ± 10.98	65.0 ± 22.86	27.74 ± 1.82	89.52 ± 10.21	**99.84 ± 0.28**
Before First Match	58.23 ± 18.15	71.17 ± 15.67	86.32 ± 21.67	98.68 ± 0.87	87.94 ± 12.47	57.8 ± 15.07	93.53 ± 2.39	**100 ± 0.0**	98.97 ± 1.78
Before Last Match	13.45 ± 0.69	10.12 ± 2.24	27.06 ± 9.02	33.86 ± 15.61	34.34 ± 12.97	76.42 ± 13.72	23.25 ± 4.28	91.3 ± 6.32	**100 ± 0.0**

Results are shown in Table 2. As before most PE methods struggle to generalise, though RoPE performs quite strongly with first match, echoing the findings of Barbero et al. (2025). However, in this more challenging setting, only the contextualised schemes, CoPE and TRA, are able to robustly generalise. Though TRA performs better at last match while CoPE holds a slight edge at first match, due to their use of different positional schemes. For natural language it is likely that the former is preferable as it better reflects causal structure (i.e. it allows understanding of an entities *current* state). Notably FoT, which in the original Flip-Flop benchmark of Liu et al. (2023a) appeared to be a strong contender, suffers substantial performance degradation in this more challenging setting. This is because, as we hypothesised in the introduction, it is unable to synergise positional and semantic information and consequently fails to generalise to capability (3) from earlier. This shortcoming is clearly visible by contrasting the attention weights of FoT and TRA.

3.5 Why does contextualised relative distance matter?



(a) TRA final layer attention in FFLM task. Black means null attention weight.



(b) FoT final layer attention in FFLM task. Black means null attention weight.

Figure 1: Contrasting Attention Heatmaps between TRA and FoT. TRA synergises semantic and positional information while FoT must trade one for the other.

In order to analyse the impact of contextualised relative distance we train two layer one head TRA and FoT models on the flip-flop language modeling task and compare the attention heatmaps for the final layer in Figure 1. TRA fully generalises in this limited setting (the only model to be able to do so out of all the baselines) while FoT fails. Figure 1a shows how TRA is able to combine both semantic content with relative positions. It first filters out irrelevant keys fully and is then able to apply a strong positional bias to the remainder that truly matter, allowing it to cleanly select the most recent write instruction. Its recency bias (1a centre) is applied in a shifted manner only to what is relevant, which allows it express a strong positional preference. FoT on the other hand (1b), has no such mechanism and as a result must diminish

its recency bias to be able to focus on semantic content (contrast the strength of 1b centre with 1a centre). This fact, coupled with its inability to completely remove irrelevant tokens, means that it is unable to handle long distance dependencies and consequently cannot learn semantic and positional preference such that they complement each other. We use FoT here as a clean illustrative example, but note that the same failure mode *must* be true of all schemes that do not employ contextualised distance.

3.6 Language Modeling:

Table 3: Test Perplexity on WikiText-103 (↓). Mean and standard deviation taken over four random initalisations. Models are trained for 100k steps with a context window of 128 tokens.

Model	NoPE	APE	Rel	RoPE	Label	FoT	Diff	CoPE	TRA
Perplexity	31.74 ± 0.11	31.61 ± 0.16	31.27 ± 0.11	30.11 ± 0.04	32.09 ± 0.07	30.30 ± 0.04	29.46 ± 0.12	30.20 ± 0.09	30.04 ± 0.32

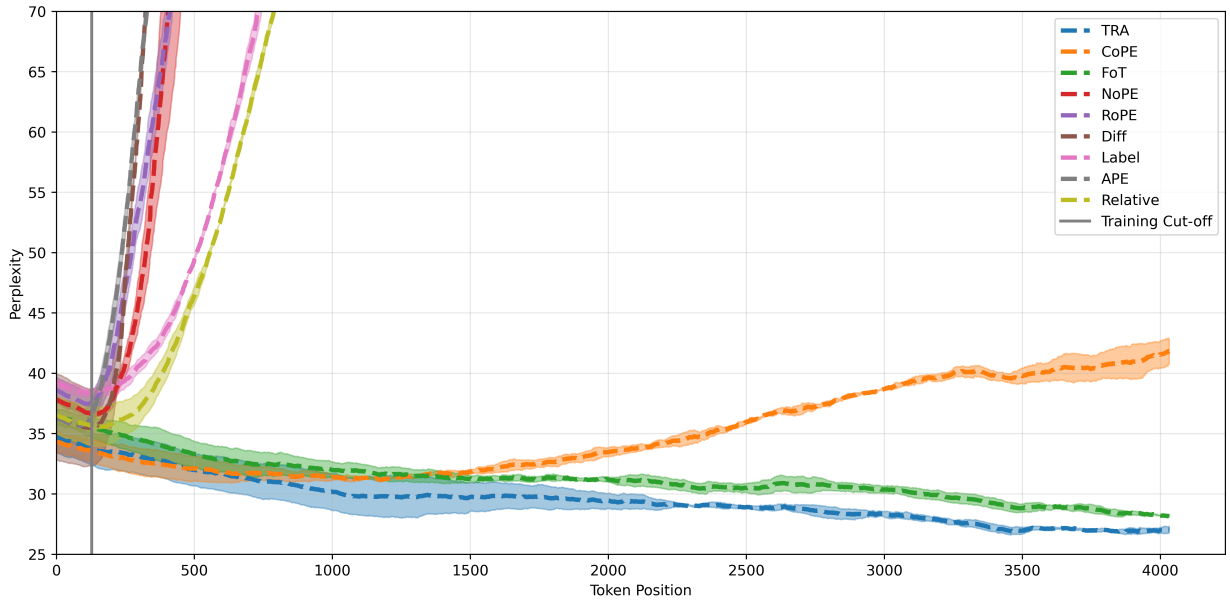


Figure 2: WikiText-103 Test Perplexity (↓) on OOD sequences lengths. Models trained for 100k steps with a window size of 128, and evaluated with window size 4098. Results taken over four random seeds.

TRA enables improved generalisation in controlled synthetic settings, which we attribute to its capacity for contextualised relative distance. What happens when we expose it to more complex data like natural language? Investigating this question has two core motivations. Firstly, contextualised relative distance requires selective sparsity to operate. This operation cuts the connection between columns and we must make sure that this does not harm learning in more complex domains by potentially limiting exploration. Secondly, it allows us to investigate whether the generalisation patterns observed in synthesis also occur in natural language. For our experiments we turn to the WikiText-103 benchmark (Merity et al., 2016), which consists of full Wikipedia articles comprising circa 100 million tokens, probing both scale and ability to handle long distance dependencies.

Hyper-parameters: For language modelling we increase model size to the medium configuration of Turc et al. (2019). This is eight layers, eight heads and a 512 dimensional embedding size. The MLP is set 2x embedding size for both the linear and gate hidden units as before. We use the GPT-2 tokenizer (Radford et al., 2019) which leads to a vocabulary size of roughly 52k. This leads to the total number of parameters for both TRA and the baselines to be circa 100 million. We train using a window size of 128 and a batch size of 64 for 100k steps. Scheduling regime remains the same as before. Our evaluation consist of two parts. In distribution: we measure perplexity on the test set using the training window of size of 128. Out of

distribution: we increase the window size to 4096 and observe the extent to which perplexity changes with increased length.

Results: In-distribution results are shown in table 3. TRA remains competitive with the strongest baselines. This means that we can introduce selective sparsity without hurting learning. Notably the strongest performing model *in-distribution* is the differential transformer (Ye et al., 2024), which contains a more sophisticated mechanism for relevancy determination. Replacing the simple ReLU we use here with such a more sophisticated approach could prove a fruitful avenue for future work. The out of distribution case is shown in figure 2. Most PE methods very rapidly begin to decline in quality, with similar degradation patterns emerging compared to the synthetic settings. The three chief exceptions to this rule are TRA, CoPE and FoT. CoPE is able to maintain/slightly improve perplexity for up to roughly a 10x increase in context length, before performance begins to deteriorate. We believe this is because its use of fractional positions only temporarily alleviates generalisation difficulties, but ultimately leaves the core issue unaddressed. Furthermore, increasing the number of positions in CoPE may delay degradation further, but any increase comes with a heavy computational burden and consequently makes CoPE difficult to scale as the sole mechanism for representing position. On the other hand, both FoT and TRA are capable of strong length generalisation (up to 32x greater than training length). Moreover, they display a promising trend of improved perplexity with increased length which indicates that they are able to *make use of* increased context rather than simply being robust to it. Between the two, TRA demonstrates a consistent perplexity edge compared with FoT. We believe this is due to its improved capacity for handling out of distribution dependency lengths as demonstrated by the flip-flop language modeling results. In sum, we show that TRA can introduce selective sparsity to the attention weights without compromising performance, and that its improved generalisation capabilities in synthetic settings also appear to be mirrored in the more complex setting of natural language.

4 Related Work:

Generalisation in Neural Networks: Neural networks, and particularly transformers, struggle with out of distribution generalisation (Hupkes et al., 2019; 2023). Partially this is architecture dependent. Significant efforts have been made towards creating architectures that enable a more flexible information flow and consequently have access to greater space of possible solutions (Fan et al., 2020; Chowdhury & Caragea, 2021; Opper et al., 2023; Soulos et al., 2024; DuSell & Chiang, 2024). However, these approaches tend to come at the cost of either flexibility or parallelisability - the key properties which made transformers, and particularly their decoder only variants, take over the world.

Transformer Programs: A parallel effort has been made to understand what kind of finite depth solutions transformers *should* be able to represent (Weiss et al., 2021; Friedman et al., 2023; Smolensky et al., 2024). One motivation being the conjecture that transformers will learn to generalise iff they are able to represent a program which does so (Zhou et al., 2023). However, empirical validation for this conjecture has thus far remained limited. Even when we expect the models to generalise they often only do so weakly, and rigorously bridging the gap between theory and practice remains a long standing challenge. A key motivation for TRA came from the theoretical model presented by Smolensky et al. (2024) which draws analogy between the transformer and classical production system architecture from cognitive science. Their theoretical model has the attention mechanism first exactly match on specified conditions before applying a hard positional tie-breaker to perform one-hot selection. TRA arose out of an ongoing effort to create a learnable equivalent to their theoretical model.

Length Generalisation in Transformers: Several mechanisms have been posisted to improve transformer length generalisation. We partition this work into two categories. The first deals with trying to learn positional weights that are flexible w.r.t. the length distribution of the training data (Dubois et al., 2019; Ruoss et al., 2023; Li & McClelland, 2023; Ray Chowdhury & Caragea, 2023; Golovneva et al., 2024). Orthogonally, recent work has demonstrated that when extending to longer sequences the softmax tends towards uniformity and therefore introduces errors the attention signal (Ye et al., 2024; Veličković et al., 2024). TRA is intended to demonstrate that these two streams of research should in fact combine. It

both attempts to mitigate noise in the attention through selectively sparsity and also learn general purpose positional information through its forget gate.

5 Conclusion

In this work we investigate whether fundamental issues with length generalisation occur due to a conflict between positional and semantic information. As a probe for this hypothesis we present TRA, a simple modification to the attention mechanism designed to allow both types of information to operate in tandem. We find that the introduction of such a change significantly improves out of distribution generalisation. Moreover, the core innovation of TRA comes from marrying selectively sparsity with contextualised relative distance. We see substantial opportunity for development of future architectures along this axis. Selective sparsity allows columns to focus on what's important, while the forget gate enables a memory over past time steps. While this latter mechanism does not enable full state tracking behaviour, temporal memory mechanisms have shown enhanced ability to approximate NC^1 solutions compared with standard transformers (Merrill et al., 2024). This means that combining a more sophisticated memory with more flexible sparsity selection should yield models which are able to approximate sophisticated algorithmic reasoning more cleanly and with fewer layers. Pursuing research in this direction has the promise of bridging the gap between our theoretical notions of what transformers should be able to do and observed empirical realities, and in so doing yield models which are capable of learning more robust, generalising attentive circuits.

References

Dzmitry Bahdanau, Kyunghyun Cho, and Yoshua Bengio. Neural machine translation by jointly learning to align and translate. *CoRR*, abs/1409.0473, 2014. URL https://api.semanticscholar.org/CorpusID:11212020.

Federico Barbero, Alex Vitvitskyi, Christos Perivolaropoulos, Razvan Pascanu, and Petar Veličković. Round and round we go! what makes rotary positional encodings useful? In *The Thirteenth International Conference on Learning Representations*, 2025. URL https://openreview.net/forum?id=GtvuNrk58a.

Tom B. Brown, Benjamin Mann, Nick Ryder, Melanie Subbiah, Jared Kaplan, Prafulla Dhariwal, Arvind Neelakantan, Pranav Shyam, Girish Sastry, Amanda Askell, Sandhini Agarwal, Ariel Herbert-Voss, Gretchen Krueger, Tom Henighan, Rewon Child, Aditya Ramesh, Daniel M. Ziegler, Jeffrey Wu, Clemens Winter, Christopher Hesse, Mark Chen, Eric Sigler, Mateusz Litwin, Scott Gray, Benjamin Chess, Jack Clark, Christopher Berner, Sam McCandlish, Alec Radford, Ilya Sutskever, and Dario Amodei. Language models are few-shot learners. *CoRR*, abs/2005.14165, 2020. URL https://arxiv.org/abs/2005.14165.

Sébastien Bubeck, Varun Chandrasekaran, Ronen Eldan, Johannes Gehrke, Eric Horvitz, Ece Kamar, Peter Lee, Yin Tat Lee, Yuanzhi Li, Scott Lundberg, Harsha Nori, Hamid Palangi, Marco Tulio Ribeiro, and Yi Zhang. Sparks of artificial general intelligence: Early experiments with gpt-4, 2023. URL https://arxiv.org/abs/2303.12712.

Jishnu Ray Chowdhury and Cornelia Caragea. Modeling hierarchical structures with continuous recursive neural networks. *CoRR*, abs/2106.06038, 2021. URL https://arxiv.org/abs/2106.06038.

Christian Clark, Byung-Doh Oh, and William Schuler. Linear recency bias during training improves transformers' fit to reading times. In Owen Rambow, Leo Wanner, Marianna Apidianaki, Hend Al-Khalifa, Barbara Di Eugenio, and Steven Schockaert (eds.), *Proceedings of the 31st International Conference on Computational Linguistics*, pp. 7735–7747, Abu Dhabi, UAE, January 2025. Association for Computational Linguistics. URL https://aclanthology.org/2025.coling-main.517/.

Róbert Csordás, Kazuki Irie, and Jürgen Schmidhuber. The neural data router: Adaptive control flow in transformers improves systematic generalization. *CoRR*, abs/2110.07732, 2021. URL https://arxiv.org/abs/2110.07732.

Zihang Dai, Zhilin Yang, Yiming Yang, Jaime G. Carbonell, Quoc V. Le, and Ruslan Salakhutdinov. Transformer-xl: Attentive language models beyond a fixed-length context. *CoRR*, abs/1901.02860, 2019. URL http://arxiv.org/abs/1901.02860.

Soham De, Samuel L. Smith, Anushan Fernando, Aleksandar Botev, George Cristian-Muraru, Albert Gu, Ruba Haroun, Leonard Berrada, Yutian Chen, Srivatsan Srinivasan, Guillaume Desjardins, Arnaud Doucet, David Budden, Yee Whye Teh, Razvan Pascanu, Nando De Freitas, and Caglar Gulcehre. Griffin: Mixing gated linear recurrences with local attention for efficient language models, 2024.

Abhimanyu Dubey, Abhinav Jauhri, Abhinav Pandey, Abhishek Kadian, Ahmad Al-Dahle, Aiesha Letman, et al. The llama 3 herd of models, 2024. URL https://arxiv.org/abs/2407.21783.

Yann Dubois, Gautier Dagan, Dieuwke Hupkes, and Elia Bruni. Location attention for extrapolation to longer sequences. *CoRR*, abs/1911.03872, 2019. URL http://arxiv.org/abs/1911.03872.

Brian DuSell and David Chiang. Stack attention: Improving the ability of transformers to model hierarchical patterns, 2024. URL https://arxiv.org/abs/2310.01749.

Nouha Dziri, Ximing Lu, Melanie Sclar, Xiang Lorraine Li, Liwei Jiang, Bill Yuchen Lin, Peter West, Chandra Bhagavatula, Ronan Le Bras, Jena D. Hwang, Soumya Sanyal, Sean Welleck, Xiang Ren, Allyson Ettinger, Zaid Harchaoui, and Yejin Choi. Faith and fate: Limits of transformers on compositionality, 2023. URL https://arxiv.org/abs/2305.18654.

MohammadReza Ebrahimi, Sunny Panchal, and Roland Memisevic. Your context is not an array: Unveiling random access limitations in transformers, 2024. URL https://arxiv.org/abs/2408.05506.

Angela Fan, Thibaut Lavril, Edouard Grave, Armand Joulin, and Sainbayar Sukhbaatar. Accessing higher-level representations in sequential transformers with feedback memory. *CoRR*, abs/2002.09402, 2020. URL https://arxiv.org/abs/2002.09402.

Dan Friedman, Alexander Wettig, and Danqi Chen. Learning transformer programs, 2023. URL https://arxiv.org/abs/2306.01128.

Olga Golovneva, Tianlu Wang, Jason Weston, and Sainbayar Sukhbaatar. Contextual position encoding: Learning to count what's important, 2024. URL https://arxiv.org/abs/2405.18719.

Nuno M. Guerreiro, Elena Voita, and André F. T. Martins. Looking for a needle in a haystack: A comprehensive study of hallucinations in neural machine translation, 2023. URL https://arxiv.org/abs/2208.05309.

Cheng-Ping Hsieh, Simeng Sun, Samuel Kriman, Shantanu Acharya, Dima Rekesh, Fei Jia, Yang Zhang, and Boris Ginsburg. Ruler: What's the real context size of your long-context language models?, 2024. URL https://arxiv.org/abs/2404.06654.

Dieuwke Hupkes, Verna Dankers, Mathijs Mul, and Elia Bruni. The compositionality of neural networks: integrating symbolism and connectionism. *CoRR*, abs/1908.08351, 2019. URL http://arxiv.org/abs/1908.08351.

Dieuwke Hupkes, Mario Giulianelli, Verna Dankers, Mikel Artetxe, Yanai Elazar, Tiago Pimentel, Christos Christodoulopoulos, Karim Lasri, Naomi Saphra, Arabella Sinclair, Dennis Ulmer, Florian Schottmann, Khuyagbaatar Batsuren, Kaiser Sun, Koustuv Sinha, Leila Khalatbari, Maria Ryskina, Rita Frieske, Ryan Cotterell, and Zhijing Jin. A taxonomy and review of generalization research in nlp. *Nature Machine Intelligence*, 5(10):1161–1174, October 2023. ISSN 2522-5839. doi: 10.1038/s42256-023-00729-y. URL http://dx.doi.org/10.1038/s42256-023-00729-y.

Ziwei Ji, Nayeon Lee, Rita Frieske, Tiezheng Yu, Dan Su, Yan Xu, Etsuko Ishii, Ye Jin Bang, Andrea Madotto, and Pascale Fung. Survey of hallucination in natural language generation. *ACM Comput. Surv.*, 55(12), March 2023. ISSN 0360-0300. doi: 10.1145/3571730. URL https://doi.org/10.1145/3571730.

Amirhossein Kazemnejad, Inkit Padhi, Karthikeyan Natesan, Payel Das, and Siva Reddy. The impact of positional encoding on length generalization in transformers. In *Thirty-seventh Conference on Neural Information Processing Systems*, 2023. URL https://openreview.net/forum?id=Drrl2gcjzl.

Yuxuan Li and James McClelland. Representations and computations in transformers that support generalization on structured tasks. *Transactions on Machine Learning Research*, 2023. ISSN 2835-8856. URL https://openreview.net/forum?id=oFC2LAqS6Z.

Zhixuan Lin, Evgenii Nikishin, Xu He, and Aaron Courville. Forgetting transformer: Softmax attention with a forget gate. In *The Thirteenth International Conference on Learning Representations*, 2025. URL https://openreview.net/forum?id=q2Lnyegkr8.

Bingbin Liu, Jordan T. Ash, Surbhi Goel, Akshay Krishnamurthy, and Cyril Zhang. Exposing attention glitches with flip-flop language modeling, 2023a. URL https://arxiv.org/abs/2306.00946.

Bingbin Liu, Jordan T. Ash, Surbhi Goel, Akshay Krishnamurthy, and Cyril Zhang. Transformers learn shortcuts to automata. In *The Eleventh International Conference on Learning Representations*, 2023b. URL https://openreview.net/forum?id=De4FYqjFueZ.

Thang Luong, Hieu Pham, and Christopher D. Manning. Effective approaches to attention-based neural machine translation. In Lluís Màrquez, Chris Callison-Burch, and Jian Su (eds.), *Proceedings of the 2015 Conference on Empirical Methods in Natural Language Processing*, pp. 1412–1421, Lisbon, Portugal, September 2015. Association for Computational Linguistics. doi: 10.18653/v1/D15-1166. URL https://aclanthology.org/D15-1166/.

Stephen Merity, Caiming Xiong, James Bradbury, and Richard Socher. Pointer sentinel mixture models. *CoRR*, abs/1609.07843, 2016. URL http://arxiv.org/abs/1609.07843.

William Merrill, Ashish Sabharwal, and Noah A. Smith. Saturated transformers are constant-depth threshold circuits. *Transactions of the Association for Computational Linguistics*, 10:843–856, 2022. doi: 10.1162/tacl_a_00493. URL https://aclanthology.org/2022.tacl-1.49/.

William Merrill, Jackson Petty, and Ashish Sabharwal. The illusion of state in state-space models, 2024. URL https://arxiv.org/abs/2404.08819.

Iman Mirzadeh, Keivan Alizadeh, Hooman Shahrokhi, Oncel Tuzel, Samy Bengio, and Mehrdad Farajtabar. Gsm-symbolic: Understanding the limitations of mathematical reasoning in large language models, 2024. URL https://arxiv.org/abs/2410.05229.

Catherine Olsson, Nelson Elhage, Neel Nanda, Nicholas Joseph, Nova DasSarma, Tom Henighan, Ben Mann, Amanda Askell, Yuntao Bai, Anna Chen, Tom Conerly, Dawn Drain, Deep Ganguli, Zac Hatfield-Dodds, Danny Hernandez, Scott Johnston, Andy Jones, Jackson Kernion, Liane Lovitt, Kamal Ndousse, Dario Amodei, Tom Brown, Jack Clark, Jared Kaplan, Sam McCandlish, and Chris Olah. In-context learning and induction heads, 2022. URL https://arxiv.org/abs/2209.11895.

Mattia Opper and N. Siddharth. Banyan: Improved representation learning with explicit structure, 2025. URL https://arxiv.org/abs/2407.17771.

Mattia Opper, Victor Prokhorov, and N. Siddharth. Strae: Autoencoding for pre-trained embeddings using explicit structure, 2023. URL https://arxiv.org/abs/2305.05588.

Antonio Orvieto, Samuel L Smith, Albert Gu, Anushan Fernando, Caglar Gulcehre, Razvan Pascanu, and Soham De. Resurrecting recurrent neural networks for long sequences, 2023.

Bo Peng, Eric Alcaide, Quentin Anthony, Alon Albalak, Samuel Arcadinho, Huanqi Cao, Xin Cheng, Michael Chung, Matteo Grella, Kranthi Kiran GV, Xuzheng He, Haowen Hou, Przemyslaw Kazienko, Jan Kocon, Jiaming Kong, Bartlomiej Koptyra, Hayden Lau, Krishna Sri Ipsit Mantri, Ferdinand Mom, Atsushi Saito, Xiangru Tang, Bolun Wang, Johan S. Wind, Stansilaw Wozniak, Ruichong Zhang, Zhenyuan Zhang, Qihang Zhao, Peng Zhou, Jian Zhu, and Rui-Jie Zhu. Rwkv: Reinventing rnns for the transformer era, 2023.

Ofir Press, Noah A. Smith, and Mike Lewis. Train short, test long: Attention with linear biases enables input length extrapolation. *CoRR*, abs/2108.12409, 2021. URL https://arxiv.org/abs/2108.12409.

Alec Radford, Jeff Wu, Rewon Child, David Luan, Dario Amodei, and Ilya Sutskever. Language models are unsupervised multitask learners. 2019. URL https://api.semanticscholar.org/CorpusID:160025533.

Colin Raffel, Noam Shazeer, Adam Roberts, Katherine Lee, Sharan Narang, Michael Matena, Yanqi Zhou, Wei Li, and Peter J. Liu. Exploring the limits of transfer learning with a unified text-to-text transformer. *CoRR*, abs/1910.10683, 2019. URL http://arxiv.org/abs/1910.10683.

Jishnu Ray Chowdhury and Cornelia Caragea. Monotonic location attention for length generalization. In Andreas Krause, Emma Brunskill, Kyunghyun Cho, Barbara Engelhardt, Sivan Sabato, and Jonathan Scarlett (eds.), *Proceedings of the 40th International Conference on Machine Learning*, volume 202 of *Proceedings of Machine Learning Research*, pp. 28792–28808. PMLR, 23–29 Jul 2023. URL https://proceedings.mlr.press/v202/ray-chowdhury23b.html.

Anian Ruoss, Grégoire Delétang, Tim Genewein, Jordi Grau-Moya, Róbert Csordás, Mehdi Bennani, Shane Legg, and Joel Veness. Randomized positional encodings boost length generalization of transformers, 2023. URL https://arxiv.org/abs/2305.16843.

Noam Shazeer. GLU variants improve transformer. *CoRR*, abs/2002.05202, 2020. URL https://arxiv.org/abs/2002.05202.

Yikang Shen, Zhouhan Lin, Chin wei Huang, and Aaron Courville. Neural language modeling by jointly learning syntax and lexicon. In *International Conference on Learning Representations*, 2018. URL https://openreview.net/forum?id=rkgOLb-0W.

Sania Sinha, Tanawan Premsri, and Parisa Kordjamshidi. A survey on compositional learning of ai models: Theoretical and experimental practices, 2024. URL https://arxiv.org/abs/2406.08787.

Paul Smolensky, Roland Fernandez, Zhenghao Herbert Zhou, Mattia Opper, and Jianfeng Gao. Mechanisms of symbol processing for in-context learning in transformer networks, 2024. URL https://arxiv.org/abs/2410.17498.

Paul Soulos, Henry Conklin, Mattia Opper, Paul Smolensky, Jianfeng Gao, and Roland Fernandez. Compositional generalization across distributional shifts with sparse tree operations, 2024. URL https://arxiv.org/abs/2412.14076.

Jianlin Su, Yu Lu, Shengfeng Pan, Bo Wen, and Yunfeng Liu. Roformer: Enhanced transformer with rotary position embedding. *CoRR*, abs/2104.09864, 2021. URL https://arxiv.org/abs/2104.09864.

Iulia Turc, Ming-Wei Chang, Kenton Lee, and Kristina Toutanova. Well-read students learn better: The impact of student initialization on knowledge distillation. *CoRR*, abs/1908.08962, 2019. URL http://arxiv.org/abs/1908.08962.

Ashish Vaswani, Noam Shazeer, Niki Parmar, Jakob Uszkoreit, Llion Jones, Aidan N. Gomez, Lukasz Kaiser, and Illia Polosukhin. Attention is all you need. *CoRR*, abs/1706.03762, 2017. URL http://arxiv.org/abs/1706.03762.

Petar Veličković, Christos Perivolaropoulos, Federico Barbero, and Razvan Pascanu. softmax is not enough (for sharp out-of-distribution), 2024. URL https://arxiv.org/abs/2410.01104.

Gail Weiss, Yoav Goldberg, and Eran Yahav. Thinking like transformers, 2021. URL https://arxiv.org/abs/2106.06981.

Tianzhu Ye, Li Dong, Yuqing Xia, Yutao Sun, Yi Zhu, Gao Huang, and Furu Wei. Differential transformer, 2024. URL https://arxiv.org/abs/2410.05258.

Biao Zhang and Rico Sennrich. Root mean square layer normalization. *CoRR*, abs/1910.07467, 2019. URL http://arxiv.org/abs/1910.07467.

Hattie Zhou, Arwen Bradley, Etai Littwin, Noam Razin, Omid Saremi, Josh Susskind, Samy Bengio, and Preetum Nakkiran. What algorithms can transformers learn? a study in length generalization, 2023. URL https://arxiv.org/abs/2310.16028.

Chen Zhu, Wei Ping, Chaowei Xiao, Mohammad Shoeybi, Tom Goldstein, Anima Anandkumar, and Bryan Catanzaro. Long-short transformer: Efficient transformers for language and vision. *CoRR*, abs/2107.02192, 2021. URL https://arxiv.org/abs/2107.02192.

A Hyperparameters:

For synthetic tasks we train for one full pass through the data, for copy and induct this required circa 100k steps as our training set consisted of 4 million examples, while for flip-flop language modeling this corresponded to 20k. We used batch size 128 for the former tasks and batch size 64 for FFL to accomodate its larger sequence length (512).

For RoPE we set the theta to 500k following Dubey et al. (2024). For CoPE we set npos max to 64 following the original authors Golovneva et al. (2024). All models are trained using the AdamW optimiser with a low dropout of 0.01 applied to the attention weights and feedforward hidden representation.