Subgoal-based Demonstration Learning for Formal Theorem Proving

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Abstract

Large language models (LLMs) present a promising pathway for advancing the domain of formal theorem proving. In this paper, we aim to improve the performance of LLMs in formal theorem proving by thoroughly examining the structure and organization of demonstrative in-context examples. We introduce a subgoal-based demonstration learning framework, specifically designed to enhance the efficiency of proof search in LLMs. First, drawing upon the insights of subgoal learning from reinforcement learning and robotics, we propose the construction of distinct subgoals for each demonstration example and refine these subgoals in accordance with the pertinent theories of subgoal learning. Second, we build upon recent advances in diffusion models to predict the optimal organization, simultaneously addressing two intricate issues that persist within the domain of demonstration organization: subset selection and order determination. Our integration of subgoal-based learning has notably increased proof accuracy from 38.9% to 44.1% on the miniF2F benchmark. Furthermore, the adoption of diffusion models for demonstration organization can lead to an additional enhancement in accuracy to 45.5%, or a 5× improvement in sampling efficiency compared to previously established methods.

1. Introduction

With the ongoing investigation into the complex reasoning abilities of large language models (LLMs), there has been a surge of interest in employing them for mathematical problem solving. This has led to an expanding body of research that underscores the remarkable potential of LLMs in this domain (Lewkowycz et al., 2022; Wu et al., 2022a; Jiang et al., 2023). In this context, we pay particular attention to the problem of formal theorem proving (Polu & Sutskever, 2020). Not only due to its extensive applications in various scientific domains, such as software verification (Klein et al., 2009) and research-level mathematics (Castelvecchi et al., 2021), but also stems from its advantage of leveraging interactive proof assistants (Paulson, 2000) to automatically validate proofs generated by models, thereby eliminating the need for human proof checking. This significantly facilitates the generation of machine-verifiable proofs via LLMs.

The complexity of automated theorem proving comes from the necessity of searching through a vast space of possible logical statements and proof methods, to determine the truth-value of a given theorem. LLMs reduce the difficulty of the searching problem by factorizing the formal proof automation task into two in-context learning (§5.2) problems (Wu et al., 2022a; Jiang et al., 2023; First et al., 2023). Given a mathematical *statement*, an LLM first generates its *informal proof* as a draft. It then generates a *formal sketch* based on this draft, which is ready for an off-the-shelf prover to verify its correctness automatically.[1] In both of these steps, the quality of the demonstrative in-context examples either written by humans or generated by machines is the key to the performance of the system.

In this paper, we seek to improve the efficacy of LLMs in formal theorem proving by delving deeper into the format and the organization of these demonstrative in-context examples. We present a subgoal-based demonstration learning framework, comprising two main components. First, we restructure an *informal proof* into a *subgoal-based proof* (Figure 1(a)), drawing upon the insights of subgoal learning from reinforcement learning and robotics, where studies show that breaking down complex tasks into smaller yet more uniformed subgoals enhances the learning efficiency of the agents (Eysenbach et al., 2019; Zhang et al., 2021). To construct subgoal-based proofs that can be easily processed and handled by LLMs, we start with human-written informal proofs and then iteratively refine them through interaction with ChatGPT (OpenAI, 2022), guided by the subgoal learning theory (§2.1). Second, a recent study (Wu et al., 2022b) points out that the selection and the ordering of

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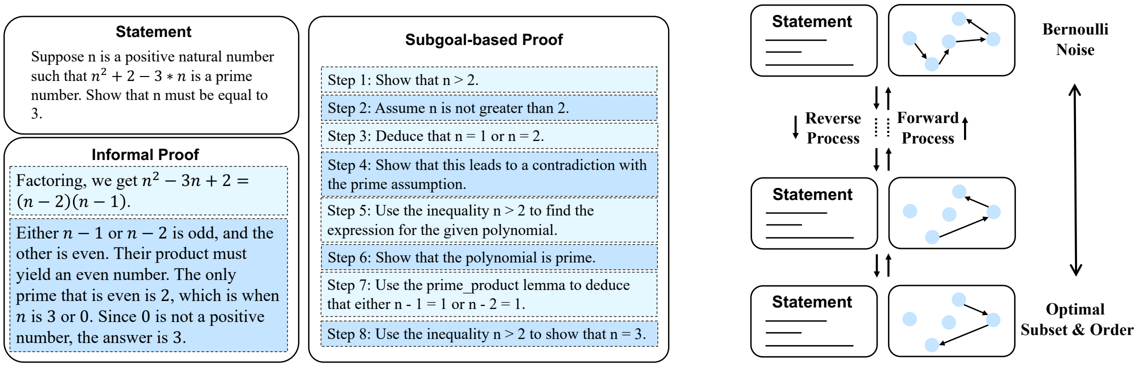
[1] In practice, the *informal proof* often serves as inline comments in the *formal sketch* to better guide the generation procedure.



Statement

Suppose n is a positive natural number such that $n^2 + 2 - 3 * n$ is a prime number. Show that n must be equal to 3.

Informal Proof

Factoring, we get $n^2 - 3n + 2 = (n-2)(n-1)$.

Either $n-1$ or $n-2$ is odd, and the other is even. Their product must yield an even number. The only prime that is even is 2, which is when n is 3 or 0. Since 0 is not a positive number, the answer is 3.

Subgoal-based Proof

Step 1: Show that n > 2.

Step 2: Assume n is not greater than 2.

Step 3: Deduce that n = 1 or n = 2.

Step 4: Show that this leads to a contradiction with the prime assumption.

Step 5: Use the inequality n > 2 to find the expression for the given polynomial.

Step 6: Show that the polynomial is prime.

Step 7: Use the prime_product lemma to deduce that either n - 1 = 1 or n - 2 = 1.

Step 8: Use the inequality n > 2 to show that n = 3.

(a) Subgoal-based Proof

(b) Demonstration Reorganization

Figure 1. **Left**: An instance of informal proof and subgoal-based proof. **Right**: Employing diffusion models to identify a more effective subset of demonstration examples, as well as the optimal order for these examples.

the in-context examples have a significant impact on performance. The lengthy formal sketches in automatic theorem proving intensify this impact, as we can only present very few cases of demonstrations. In response to that, we train a diffusion model to organize the demonstrative in-context examples for the translation process from *subgoal-based proof* to its corresponding *formal sketch* of each instance (§2.2). This approach identifies a more effective subset of demonstration examples as well as the most beneficial order of these examples (Figure 1(b)).

The proposed method significantly outperforms competing approaches in formal theorem proving tasks, achieving a pass rate of 45.5% on miniF2F dataset (Zheng et al., 2021), a 6.6% absolute and 17.0% relative improvement over the previous state-of-the-art system (Jiang et al., 2023). Furthermore, the adoption of diffusion models for demonstration selection and ordering can lead to a significant improvement in sampling efficiency, reaching previous state-of-the-art (38.5%) on miniF2F with only 20 (compared to 100) calls to the LLM.

2. Subgoal-based Demonstration Learning

Given a theorem statement x, the goal of proof synthesis is to generate a formal sketch y which can be verified by an off-the-shelf automated theorem prover (e.g., Sledgehammer) (Jiang et al., 2023). In this section, we elaborate on the subgoal-based demonstration learning framework that consists of two key components, subgoal-based proof (§2.1) and demonstration reorganization (§2.2). The *subgoal-based proof* replaces the *informal proof*, breaking down a complex problem into smaller subgoals that offer more fine-grained and uniform guidance to the LLMs. The *demonstration reorganization* takes place in the stage of generating the *formal sketch* based on the *subgoal-based proof*. This procedure is non-trivial. Given the limited context length of the LLMs, selecting relevant yet diverse demonstration examples has a significant impact on the final pass rate of these formal sketches. We denote the set of all N demonstration examples by $E = \{E_1, E_2, \cdots, E_N\}$. Each of them contains a mathematical *statement*, an *informal proof* (or a *subgoal-based proof*), and a *formal sketch*.[2]. In the remainder of this section, we first describe the iterative refinement process that produces the subgoal-based proofs given the informal proof, guided by the principles in the subgoal learning theory (Zhang et al., 2021). We then explain our solution to the demonstration reorganization. Starting from collecting arrangements that have yielded successful proofs, we use these as training data for a diffusion model, which progressively determines the most favorable reorganization during inference.

2.1. Subgoal-based Proof

The significance of LLMs to formal theorem proving is that they grant us the ability to leverage informal proofs to guide formal theorem proving, which otherwise has to be based on expensive heuristics-based brute-force search. Despite considerable progress (Lewkowycz et al., 2022; OpenAI, 2023), this approach suffers from the flawed informal proofs generated by the LLMs (Jiang et al., 2023). We propose to use subgoal-based proofs to replace the informal proofs, where the subgoals are strictly aligned with the states in the automatic provers. Following Zhang et al. (2021), we seek to obtain a *valid* sequence of subgoals which satisfies the condition that each subgoal in this sequence should be reachable from the initial state (i.e., the statement) and attain the final state (i.e., the passing state of the proof). These valid

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[2] A more comprehensive illustration of these terms is provided in Figure 3.

sequences integrate the guidance from the LLMs better with the search space of the automatic theorem provers, thereby leveraging the ability of the LLMs to the maximum extent. However, it is non-trivial to get these valid subgoal proofs as human-written subgoals often fall short of the above constraints. To address this problem, we iteratively refine the subgoal proof, in the spirit of self-play in reinforcement learning (Silver et al., 2016), making calls to both the LLM and the off-the-shelf automated theorem prover.

Subgoal Refinement. We start with manually written subgoal-based proofs, and denote these as the initial seed set $\{E_i^{(0)}\}_{i=1}^N$. This set contains subgoal-based proofs formed on the informal proofs and the statement, yet not guaranteed to be a valid sequence. We denote the sequence of subgoals in an instance as $(s_0, s_1, s_2, \cdots, s_\Delta, s_{\Delta+1})$, where Δ is the total number of subgoals and s_0 and $s_{\Delta+1}$ are two special subgoals that align with the initial and final states of the automatic prover. During the k-th iteration, we randomly select a subset of instances from the previous iteration $\{E_i^{(k-1)}\}_{i=1}^N$ as the in-context demonstration for the LLM to generate subgoals for a given instance. According to the definition, s_i is considered to be a valid subgoal if and only if it can be reached from s_0 and can reach $s_{\Delta+1}$. Therefore, for each of the subgoals, we recursively call the proof assistant to verify the validness of the most recently developed subgoal and only after Δ recursions we can obtain the new valid sequence of subgoals and add that into the next iteration as $E_i^{(k)}$. This process improves the consistency of the derived subgoals in style, thus making it easier for the LLM to learn from in the inference stage. We provide a detailed description of the algorithm in the Appendix.

2.2. Demonstration Reorganization

The demonstration examples can be lengthy in formal theorem proving. If we assume a maximum context length of 3072 tokens, only 4.79 examples on average can be included. Our experiments echo the findings by Wu et al. (2022b). These instance-based demonstration examples have a significant impact on performance. Only certain orders of carefully chosen demonstration examples lead to successful theorem proving. Consequently, identifying the optimal subset from the pool and ordering them into meaningful in-context demonstration examples is of great significance, which unfortunately is an NP-complete problem. We formulate the demonstration reorganization problem as finding the (Sub)hamiltonian graph where the nodes represent demonstration examples, and a traverse following the path corresponds to the selection and ordering of them. Building upon the recent success of applying diffusion models in addressing NP-complete problems (Graikos et al., 2022; Sun & Yang, 2023), we further treat this problem as a diffusion process on the graph. This solution has two main advantages. First, it addresses the example selection and ordering problem simultaneously. Second, the inference can be performed in parallel, which greatly reduces the time of discovering the optimal arrangement given the demonstration examples. We start by collecting successful pairs of in-context demonstration example organization and the corresponding statement x as the training data for the diffusion model. We randomly organize (select and order) the demonstration examples and query the LLM to see if it can generate the proof successfully. The passing cases will be used as the starting configuration $\boldsymbol{\psi}_0$ in the diffusion process given the statement x.

Training. The aim of employing diffusion models is to predict the optimal organization, denoted as $\boldsymbol{\psi}_0$,[3] conditioning on the theorem statement x. From the standpoint of variance inference, diffusion models adopt the following formulations to model $p_{\boldsymbol{\theta}}(\boldsymbol{\psi}_0|x)$,

$$p_{\boldsymbol{\theta}}(\boldsymbol{\psi}_0|x) \coloneqq \int p_{\boldsymbol{\theta}}(\boldsymbol{\psi}_{0:T}|x)\mathrm{d}\boldsymbol{\psi}_{1:T}, \tag{1}$$

where $\boldsymbol{\psi}_1, \cdots, \boldsymbol{\psi}_T$ serve as latent variables with the same dimensionality as $\boldsymbol{\psi}_0$. The learned reverse process progressively denoises these latent variables to reconstruct $\boldsymbol{\psi}_0$. This procedure can be formalized as follows,

$$p_{\boldsymbol{\theta}}(\boldsymbol{\psi}_{0:T}|x) = p(\boldsymbol{\psi}_T)\prod_{t=1}^{T} p_{\boldsymbol{\theta}}(\boldsymbol{\psi}_{t-1}|\boldsymbol{\psi}_t, x). \tag{2}$$

The forward process gradually corrupts $\boldsymbol{\psi}_0$ to generate noised latent variables,

$$q(\boldsymbol{\psi}_{1:T}|\boldsymbol{\psi}_0) = \prod_{t=1}^{T} q(\boldsymbol{\psi}_t|\boldsymbol{\psi}_{t-1}). \tag{3}$$

The goal of the training process is to maximize the evidence lower bound (ELBO),

$$\begin{aligned}\mathbb{E}\left[\log p_{\boldsymbol{\theta}}(\boldsymbol{\psi}_0|x)\right] &\geq \mathbb{E}_q\left[\log \frac{p_{\boldsymbol{\theta}}(\boldsymbol{\psi}_{0:T}|x)}{q_{\boldsymbol{\theta}}(\boldsymbol{\psi}_{1:T}|\boldsymbol{\psi}_0, x)}\right] \\ &= \mathbb{E}_q\left[\log p_{\boldsymbol{\theta}}(\boldsymbol{\psi}_0|\boldsymbol{\psi}_1, x) - \sum_{t>1} D_{\mathrm{KL}}[q(\boldsymbol{\psi}_{t-1}|\boldsymbol{\psi}_t, \boldsymbol{\psi}_0)\|p_{\boldsymbol{\theta}}(\boldsymbol{\psi}_{t-1}|\boldsymbol{\psi}_t, x)]\right].\end{aligned} \tag{4}$$

We employ a Graph Neural Network (GNN) for the encoding and denoising process of the graph. Following Austin et al. (2021), we adopt discrete diffusion models to model binary random variables.

Inference. During the inference stage, we obtain samples $\boldsymbol{\psi} \sim p_{\boldsymbol{\theta}}(\boldsymbol{\psi}_0|x)$ and subsequently reconstruct the order of demonstration examples from $\boldsymbol{\psi}$. We then incorporate

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[3] ψ_0 represents the sequence of demonstrations that, when fed into the prompt, results in a successful formal sketch.

examples sequentially into the LLM context, and define the output of the demonstration organization module as the sequence of examples upon reaching the LLM length constraint. More details of the implementation of the diffusion model, the implementation of GNN, and techniques used in the sampling process of ψ can be found in the Appendix.

3. Experiments

3.1. Formal Environment

Interactive Theorem Provers. Interactive Theorem Provers (ITPs), such as Isabelle (Paulson, 1994), constitute the backbone of contemporary mathematical verification systems. They facilitate the integration of mathematical definitions and theorems into a consistent logical framework, such as Higher-Order Logic or Dependent Type Theory, which is operationalized by their kernels. The kernel plays a pivotal role in the verification process, meticulously examining each theorem to ascertain its recognition by the ITP and thereby ensuring the integrity of the system. The theorem-proving process within an ITP is characterized by the articulation of the theorem in the ITP's programming language, followed by an iterative simplification into more manageable objectives or subgoals. The theorem is deemed proven once it can be distilled down to pre-established facts. The selection of Isabelle for our paper is motivated by its intuitive interface, its compatibility with a range of logical frameworks, and its comprehensive library of formalized mathematics.

Sledgehammer. Sledgehammer (Paulsson & Blanchette, 2012) serves as a powerful tool for automating reasoning within the interactive theorem prover Isabelle. It functions by transmuting the goals encapsulated in Isabelle/HOL's higher-order logic into alternative logic, such as first-order logic. These transmuted goals are then passed to off-the-shelf automated theorem provers, including E, CVC4, Z3, Vampire, and SPASS. If any of these automated theorem provers successfully derive the proof in their respective formats, Sledgehammer undertakes the task of reconstructing the proof within the Isabelle/HOL framework using certified provers, namely metis, meson, and smt. This reconstructed proof, being more interpretable to humans, significantly enhances the system's usability, thereby contributing to the efficiency and effectiveness of (interactive) theorem proving.

3.2. Dataset and Evaluation

Dataset. We evaluate our approach using the miniF2F dataset (Zheng et al., 2021), which comprises 488 formal mathematical problems derived from high-school competitions, expressed in three formal languages: Lean, HOL-Light, and Isabelle. The dataset is divided into a validation and a test set, each including 244 problems. The problems within the dataset are sourced from three distinct categories: 260 problems are extracted from the MATH dataset (Hendrycks et al., 2021), 160 problems are extracted from actual high-school mathematical competitions (AMC, AIME, and IMO), and 68 problems are crafted to mirror the difficulty level of the aforementioned competitions.

Evaluation. The task at hand entails the generation of formal sketches for problems in the miniF2F dataset. The validity of a formal sketch depends on two criteria: first, the absence of "cheating" keywords such as "sorry" and "oops" that prematurely terminate a proof before its completion; second, the capacity of the interactive theorem prover Isabelle to authenticate the corresponding formal statement with the proof. To make working with Isabelle easier, we use the Portal-to-Isabelle API, as introduced by Jiang et al. (2023). Given the absence of a training split in the miniF2F dataset, we leverage optimal organizations that yield successful proofs from the miniF2F-valid set to train the diffusion model. As proposed by Lample et al. (2022), we employ the cumulative pass rate as a measure for the results obtained from performing inference using diffusion models on the miniF2F-valid set. This involves integrating the pass rates from both the data collection stage for training and the inference stage. When it comes to other scenarios, namely conducting inference on the miniF2F-test or cases where the diffusion model is not employed, we simply provide the pass rate.

3.3. Baselines

We leverage the following baselines to substantiate the effectiveness of our proposed methodology:

Symbolic Automated Provers. We first employ Sledgehammer, a proof automation tool that is extensively utilized within the Isabelle environment. We adhere to the default configuration of Sledgehammer as provided in Isabelle2021, which encompasses a 120-second timeout and a suite of five automated theorem provers (Z3, CVC4, SPASS, Vampire, E). In alignment with Jiang et al. (2023), we employ Sledgehammer supplemented with heuristics, integrating 11 prevalent tactics (i.e., auto, simp, blast, fastforce, force, eval, presburger, sos, arith, linarith, auto simp: field simps) with Sledgehammer. If all the tactics fail or take longer than 10 seconds, the system reverts to the base Sledgehammer.

Search-based Methods. In addition to the above, we incorporate baselines that utilize Monte-Carlo tree search (Silver et al., 2016) to discover the proof. This includes Thor (Jiang et al., 2022) and another version of Thor that employs an expert iteration on autoformalized data (i.e., Thor+expert iteration (Wu et al., 2022a)). Thor combines

Table 1. Pass rates on the miniF2F dataset with Isabelle. Numbers in bold denote the best performance. Numbers with a $\star$ correspond to the cumulative pass rate (Lample et al., 2022) since the evaluated statements are part of the training for diffusion models. See §3.2 for more details about cumulative pass rate. † denotes the concurrent work at the time of submission.

	valid	test
Sledgehammer	9.9%	10.4%
Sledgehammer+heuristic	18.0%	20.9%
Thor	28.3%	29.9%
Thor + expert iteration	37.3%	35.2%
DSP (540B Minerva)	42.6%	38.9%
LEGO-Prover†	52.4%	**45.5%**
Lyra†	52.8%	44.2%
Ours	48.0%*	**45.5%**

Table 2. Ablation results on the miniF2F dataset with Isabelle. Numbers with a $\star$ correspond to the cumulative pass rate.

	valid	test
Ours	48.0%*(±0.4)	45.5%(±0.6)
- subgoal & diffusion	41.4%(±0.9)	38.7%(±1.2)
- subgoal	44.3%*(±0.7)	40.6%(±0.6)
- diffusion	46.9%(±1.3)	44.1%(±0.9)

language models with automatic theorem provers to overcome the challenge of selecting beneficial premises from a vast library. Thor+expert iteration enhances a neural theorem prover by training it on theorems that have been automatically formalized.

LLM-based Method. Lastly, we incorporate three LLM-based baselines: Draft, Sketch, and Prove (DSP) (Jiang et al., 2023), LEGO-Prover (Xin et al., 2023), and Lyra (Zheng et al., 2023). DSP utilizes the $540B$ Minerva model (Lewkowycz et al., 2022) to generate informal proofs, which are then transformed into formal sketches. LEGO-Prover focuses on incrementally creating reusable theorems, streamlining the theorem proving process through the utilization of previously established results. Lyra improves the verification process by integrating error messages from external verifiers, facilitating proof optimization. Notably, all three methods, similar to our approach, employ the Sledgehammer tool, maintaining consistency in tool usage across the methodologies. For a fair comparison, we use versions of these baselines configured with 100 autoformalization attempts and without relying on ground-truth informal proofs.

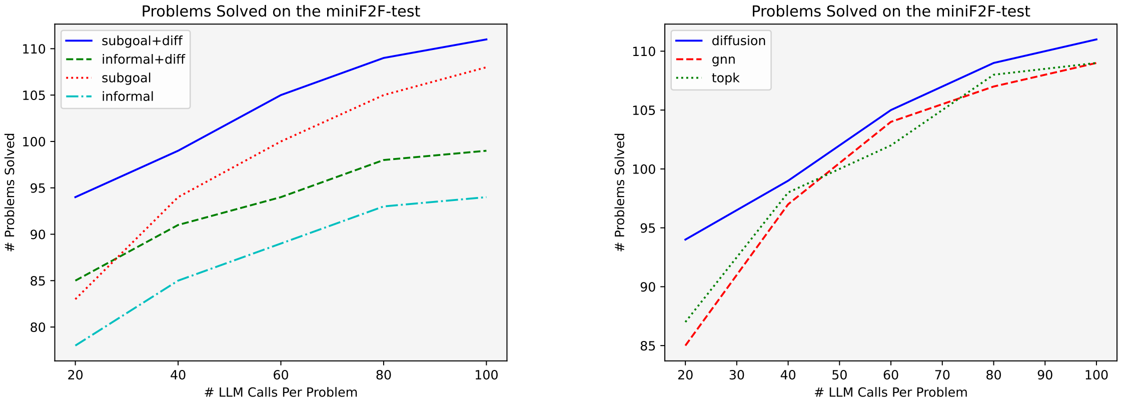
We exclude methods such as HyperTree Proof Search (HTPS) (Lample et al., 2022) and GPT-f with expert iteration (Polu et al., 2022), which are implemented using Lean (de Moura et al., 2015), a different interactive theorem prover. The disparity in tactics and automation between Lean and Isabelle renders them not directly comparable to our method.

3.4. Implementation Details

Throughout our work, we employ ChatGPT (i.e., the *gpt-3.5-turbo-0301* version) as the LLM. For the creation of the formal sketch, the temperature and max_tokens parameters of *gpt-3.5-turbo-0301* are set to 0 and 1024, respectively. For each subgoal proof, we make one formal sketch attempt, as suggested by previous literature (Jiang et al., 2023). In terms of the establishment of the subgoal-based proof, we set the number of refinement iterations to be 15, with the number of demonstration examples, denoted as N, being set to 61. For demonstration organization, we employ a randomized demonstration organization approach to generate proofs for 116 distinct statements on miniF2F-valid, which yield 137 successful proofs. We then partition the corresponding demonstration contexts into a training set and a validation set, comprising 81 and 56 instances respectively. The training of our diffusion models is conducted with a learning rate of $5e-4$, a batch size of 16, and throughout 50 epochs. We set the number of diffusion steps, represented as T, to 80. We employ an early stopping strategy on the validation set and report the performance averaged in three repetitive experiments.

3.5. Main Results

The experiment results, as shown in Table 1, yield several key observations: (1) Our method outperforms DSP,



(a) Subgoal-based Proof (b) Demonstration Reorganization

Figure 2. Number of problems solved on miniF2F-test against the number of LLM calls per problem, where the x-axis represents the number of independent problem-solving attempts. **Left**: a comparative assessment between the informal proof and subgoal-based proof under two distinct conditions: presence and absence of the diffusion model. **Right**: a comparative exploration of different in-context learning methods.

with pass rates of 48.0% on miniF2F-valid and 45.5% on miniF2F-test. This superior performance is attributable to our novel application of diffusion models for demonstration reorganization coupled with subgoal-based proof; (2) The methods Thor and Thor + expert iteration struggle due to the enormously large action space. This space significantly overshadows that of games, thereby posing challenges to the comprehensive utilization of the Monte Carlo tree search. Consequently, these methods underperform when compared to LLM-based methods; (3) DSP has pioneered the introduction of the informal proof, a critical step in the LLM-based formal theorem proving task. However, human-written informal proofs do not offer optimal compatibility with large language models. Our method, grounded in the subgoal-learning theory, is capable of inferring subgoal-based proofs that are more amenable to large language models; and (4) LEGO-Prover and Lyra, which achieves on-par performance with our model, are concurrent work with complementary focuses. LEGO-Prover focuses on the incremental creation of reusable theorems, and Lyra enhances verification via error feedback integration. The orthogonal nature presents a promising direction for integrating our approach with theirs in the future.

4. Analysis

4.1. Ablation Study

In our ablation study, we examine four variations of our model on the miniF2F dataset, as detailed in Table 2. The models include our full method (Ours), and three variants: one excluding the subgoal-based proof, denoted as "-subgoal", which utilizes informal proofs instead; one without the diffusion method, referred to as "-diffusion", employing a randomized selection for demonstrations; and one, denoted as "-subgoal & diffusion", representing the basic form of *gpt-3.5-turbo*, where both the subgoal-based proof and diffusion method are omitted. To quantify the uncertainty associated with our method, we have supplemented our results with standard deviation values. For the variants employing diffusion models ("Ours" and "-subgoal"), we trained three separate diffusion models as detailed in §3.4. For the variants without diffusion models ("-subgoal & diffusion" and "-diffusion"), the experiment was repeated three times.

Our full model achieves the highest performance on both validation and test sets. This underscores the importance of both subgoal-based proof and demonstration reorganization. The model without both components showed the lowest performance, further emphasizing the significance of these components. The models missing either the subgoal-based proof or reorganization components also show decreased performance, indicating the substantial role of each component. Additionally, it is noteworthy that the methods incorporating diffusion models exhibit relatively more stable performance, affirming the stability benefits of these models in our study.

4.2. On the Effect of Subgoal-based Proof

We further use four different variants to explore the impact of subgoal-based proof. Figure 2(a) displays the results of this experiment, where "informal" denotes the utilization of informal proofs instead of subgoal-based proof, and "diff" indicates the integration of demonstration reorganization.

Statement: Suppose n is a natural number such that n leaves a remainder of 3 when divided by 7. Show that (2n+1) leaves a remainder of 0 when divided by 7.

Subgoal-based Proof:
Step 1: Express n as $7k + 3$ for some k ∈ ℕ
Step 2: Substitute the expression for n into the expression for $2n + 1$ and simplify.
Step 3: Show that the resulting expression is divisible by 7.

Formal Sketch:

```
theorem mathd_numbertheory_370:
  fixes n :: nat
  assumes h0 : "n mod 7 = (3::nat)"
  shows "(2*n+1) mod 7 = (0::nat)"
proof -
  (* Step 1: Express n as 7k + 3 for some k ∈ ℕ. *)
  have "n = 7 * (n div 7) + (n mod 7)"
    using div_mult_mod_eq[of "n" "7"] sledgehammer
  also have "... = 7 * (n div 7) + 3”
    using h0 sledgehammer
  finally have c0: "n = 7 * (n div 7) + 3".
  (* Step 2: Substitute the expression for n into the expression for 2n + 1 and simplify. *)
  have "2*n+1 = 2*(7*(n div 7) + 3) + 1"
    using c0 sledgehammer
  also have "... = 14*(n div 7) + 7”
    sledgehammer
  also have "... = 7*(2*(n div 7) + 1)"
    sledgehammer
  finally have c1: "2*n+1 = 7*(2*(n div 7) + 1)” .
  (* Step 3: Show that the resulting expression is divisible by 7.*)
  show "(2*n+1) mod 7 = 0"
    using c1 sledgehammer
qed
```

Figure 3. A formal sketch generated by our proposed method.

The results indicate a significant difference between the approaches that incorporate subgoal-based proof (“subgoal” and “subgoal+diff”) and those that do not (“informal” and “informal+diff”). This trend remains consistent across all LLM call numbers, suggesting a noteworthy positive effect of subgoal-based proof on the overall performance of our method.

4.3. On the Effect of Demonstration Reorganization

To further investigate the effect of a diffusion model for demonstration reorganization, we draw a comparative analysis between its performance and two alternative in-context learning methodologies: the Graph Neural Network (GNN) and the Top-K. The Top-K method calculates Ada embeddings for test problem statements and in-context examples, subsequently selecting the top-k most similar examples, as measured by dot product, ensuring those with higher similarity are positioned nearer to the test problem in the prompt. The GNN is congruent with a modified version of our proposed model when the inference diffusion step is set to 1, while the efficacy of the Top-K methodology has been extensively substantiated in the literature (Liu et al., 2021). Figure 2(b) presents the empirical results, manifesting that the diffusion model’s performance increment diminishes as the number of LLM calls escalates to 100. This phenomenon stems from the fact that the module is trained on data collated from successful proofs via randomized organization sampling. Consequently, it may encounter difficulties in discerning the optimal organization for data that deviates significantly from its training dataset. Nevertheless, this limitation does not overshadow the potential of diffusion models to economize the number of LLM calls. Notably, with demonstration reorganization, our method exhibits an impressive capability of successfully deriving proofs for 94 problems (equivalently, a pass rate of 38.5%), with a mere 20 LLM calls. Remarkably, this result is comparable with that of the DSP method, which necessitates $5\times$ the number of LLM calls.

4.4. Case Study

To better comprehend the efficacy of our proposed method, we present a formal sketch of a problem that remains unproven by earlier state-of-the-art methods. As demonstrated in Figure 3, it is apparent that our strategy successfully decomposes the complex objective into three manageable subgoals, each solvable by the LLM. We provide additional comprehensive examples in the Appendix.

5. Related Work

5.1. Machine Learning for Formal Theorem Proving

Formal theorem proving has seen significant advancements through machine learning, with efforts focusing on enhancing proof search strategies (Polu & Sutskever, 2020; Polu et al., 2022; Jiang et al., 2022) and employing Large Language Models (LLMs) for autoformalization (Jiang et al., 2023). Proof search improvements are marked by the development of self-supervised strategies in Expert Iteration (Polu et al., 2022) and PACT (Han et al., 2021), alongside integrations of language models with automated provers in HyperTree Proof Search (HTPS) (Lample et al., 2022) and Thor (Jiang et al., 2022), complemented by transformer-based premise selection in Magnushammer (Mikuła et al., 2023). However, scalability remains a challenge due to the increasing complexity of theorems. In parallel, the application of LLMs for autoformalization and proof generation has been explored, with Wu et al. (2022a) and Jiang et al. (2023) demonstrating the conversion of mathematical problems into formal specifications. Baldur (First et al., 2023) goes further by producing full proofs and enhancing proving capabilities with a proof repair model. Additionally, LEGO-Prover (Xin et al., 2023) and Lyra (Zheng et al., 2023) offer unique contributions in theorem proving, focusing on the incremental development of reusable theorems and the integration of error messages from external verifiers for proof post-processing, respectively. Nevertheless, these approaches have yet fully optimized the format and the organization of demonstration examples when invoking LLMs. Our work aims to address this gap by introducing a subgoal-based demonstration learning framework that refines the use of LLMs in formal theorem proving.

5.2. In-context Learning

In-context Learning (ICL) primarily investigates two aspects: the selection and arrangement of in-context examples. For example selection, Liu et al. (2021) introduce a retrieval-based method for prompt selection, enhancing the semantic relevance over random selection. This concept was expanded by Rubin et al. (2021) who utilize a pre-trained language model for effective prompt retrieval. Additionally, Sorensen et al. (2022) propose a novel template selection method based on maximizing mutual information, circumventing the need for labeled examples or direct model access. Su et al. (2022) present a two-step framework focusing on efficiency, selecting examples from unlabeled data and retrieving task-specific examples during testing. Furthermore, Agrawal et al. (2022) develop strategies for machine translation, emphasizing the importance of example quality and domain relevance while highlighting the detrimental effects of irrelevant examples. Concerning example arrangement, Zhao et al. (2021) address the instability in few-shot learning results caused by example order, proposing a calibration method. Extending this, Lu et al. (2021) explore the sensitivity of prompt order in few-shot learning contexts. Even though previous efforts have made remarkable progress in either choosing or sequencing in-context examples, our research sets a new precedent by combining both elements. In this paper, we step out of these isolated areas of concentration, looking into an approach based on diffusion models that effectively tackles both the challenges of selection and ordering at the same time.

5.3. Subgoal Learning

Subgoal learning significantly enhances AI systems' ability to address complex tasks by introducing efficiency and structure in reinforcement learning (RL). Theoretical contributions highlight the computational benefits of subgoal rewards (Zhai et al., 2022), optimal structuring for hierarchical RL (Wen et al., 2020), option selection complexities (Jinnai et al., 2019a), and temporal abstraction integration (Fruit et al., 2017). Empirical efforts focus on subgoal exploration, planning, and curriculum learning, leveraging strategies for optimal discovery and decision-making through cover time minimization (Jinnai et al., 2019b), dynamical distance learning (Hartikainen et al., 2019), entropy maximization (Pitis et al., 2020), asymmetric self-play (OpenAI et al., 2021), and innovative planning algorithms like SoRB (Eysenbach et al., 2019), DC-MCTS (Parascandolo et al., 2020), PAIR (Li et al., 2022), and goal-oriented MCTS with hindsight experience replay (Moro et al., 2022). Curriculum learning research aims at progressively enhancing subgoal complexity to optimize learning pathways, with techniques for automatic curriculum generation and complexity scaling (Zhang et al., 2020; 2021). While there have been preliminary efforts to apply similar principles in the construction of prompts for LLMs (Khot et al., 2022), the deployment of subgoal learning theories to manage intricate tasks, such as formal theorem proving, remains largely unexplored. Our work pioneers the use of subgoal learning in this domain, with a focus on format and organization.

6. Conclusion

In this paper, we have developed a subgoal-based demonstration learning framework that significantly enhances LLMs' efficacy in formal theorem proving. Our approach combines insights from subgoal learning and diffusion models, effectively addressing the challenges of demonstration formatting and organization. As a result, we achieve a 17.0% relative improvement in proof pass rate on the miniF2F benchmark and a 5× improvement in sampling efficiency.

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Impact Statement

This work presents a novel algorithm for automated formal proof generation. Its impacts potentially span across mathematical research and software engineering: This algorithm may assist mathematicians in writing rigorous mechanized proofs, complementing their conceptual exploration; While our work is primarily focused on mathematics at present, there is potential for its application to extend to formal verification of software systems. Future research can focus on enhancing the algorithm's accuracy, broadening its application to more complex theorems, and exploring synergies with other AI techniques.

References

Agrawal, S., Zhou, C., Lewis, M., Zettlemoyer, L., and Ghazvininejad, M. In-context examples selection for machine translation. *arXiv preprint arXiv:2212.02437*, 2022.

Austin, J., Johnson, D. D., Ho, J., Tarlow, D., and van den Berg, R. Structured denoising diffusion models in discrete state-spaces. *Advances in Neural Information Processing Systems*, 34:17981–17993, 2021.

Bresson, X. and Laurent, T. An experimental study of neural networks for variable graphs, 2018. URL `https://openreview.net/forum?id=SJexcZc8G`.

Castelvecchi, D. et al. Mathematicians welcome computer-assisted proof in 'grand unification'theory. *Nature*, 595 (7865):18–19, 2021.

de Moura, L., Kong, S., Avigad, J., Van Doorn, F., and von Raumer, J. The lean theorem prover (system description). In *Automated Deduction-CADE-25: 25th International Conference on Automated Deduction, Berlin, Germany, August 1-7, 2015, Proceedings 25*, pp. 378–388. Springer, 2015.

Eysenbach, B., Salakhutdinov, R. R., and Levine, S. Search on the replay buffer: Bridging planning and reinforcement learning. *Advances in Neural Information Processing Systems*, 32, 2019.

First, E., Rabe, M. N., Ringer, T., and Brun, Y. Baldur: Whole-proof generation and repair with large language models. *arXiv preprint arXiv:2303.04910*, 2023.

Fruit, R., Pirotta, M., Lazaric, A., and Brunskill, E. Regret minimization in mdps with options without prior knowledge. *Advances in Neural Information Processing Systems*, 30, 2017.

Graikos, A., Malkin, N., Jojic, N., and Samaras, D. Diffusion models as plug-and-play priors. *arXiv preprint arXiv:2206.09012*, 2022.

Han, J. M., Rute, J., Wu, Y., Ayers, E. W., and Polu, S. Proof artifact co-training for theorem proving with language models. *arXiv preprint arXiv:2102.06203*, 2021.

Hartikainen, K., Geng, X., Haarnoja, T., and Levine, S. Dynamical distance learning for semi-supervised and unsupervised skill discovery. *arXiv preprint arXiv:1907.08225*, 2019.

Hendrycks, D., Burns, C., Kadavath, S., Arora, A., Basart, S., Tang, E., Song, D., and Steinhardt, J. Measuring mathematical problem solving with the math dataset. *arXiv preprint arXiv:2103.03874*, 2021.

Ioffe, S. and Szegedy, C. Batch normalization: Accelerating deep network training by reducing internal covariate shift. In *International conference on machine learning*, pp. 448–456. pmlr, 2015.

Jiang, A. Q., Li, W., Tworkowski, S., Czechowski, K., Odrzygóźdź, T., Miłoś, P., Wu, Y., and Jamnik, M. Thor: Wielding hammers to integrate language models and automated theorem provers. *Advances in Neural Information Processing Systems*, 35:8360–8373, 2022.

Jiang, A. Q., Welleck, S., Zhou, J. P., Lacroix, T., Liu, J., Li, W., Jamnik, M., Lample, G., and Wu, Y. Draft, sketch, and prove: Guiding formal theorem provers with informal proofs. In *The Eleventh International Conference on Learning Representations*, 2023. URL `https://openreview.net/forum?id=SMa9EAovKMC`.

Jinnai, Y., Abel, D., Hershkowitz, D., Littman, M., and Konidaris, G. Finding options that minimize planning time. In *International Conference on Machine Learning*, pp. 3120–3129. PMLR, 2019a.

Jinnai, Y., Park, J. W., Abel, D., and Konidaris, G. Discovering options for exploration by minimizing cover time. In *International Conference on Machine Learning*, pp. 3130–3139. PMLR, 2019b.

Khot, T., Trivedi, H., Finlayson, M., Fu, Y., Richardson, K., Clark, P., and Sabharwal, A. Decomposed prompting: A modular approach for solving complex tasks. *arXiv preprint arXiv:2210.02406*, 2022.

Klein, G., Elphinstone, K., Heiser, G., Andronick, J., Cock, D., Derrin, P., Elkaduwe, D., Engelhardt, K., Kolanski, R., Norrish, M., et al. sel4: Formal verification of an os kernel. In *Proceedings of the ACM SIGOPS 22nd symposium on Operating systems principles*, pp. 207–220, 2009.

Lample, G., Lacroix, T., Lachaux, M.-A., Rodriguez, A., Hayat, A., Lavril, T., Ebner, G., and Martinet, X. Hypertree proof search for neural theorem proving. *Advances in Neural Information Processing Systems*, 35:26337–26349, 2022.

Lewkowycz, A., Andreassen, A., Dohan, D., Dyer, E., Michalewski, H., Ramasesh, V., Slone, A., Anil, C., Schlag, I., Gutman-Solo, T., et al. Solving quantitative reasoning problems with language models. *arXiv preprint arXiv:2206.14858*, 2022.

Li, A. C., Prabhudesai, M., Duggal, S., Brown, E., and Pathak, D. Your diffusion model is secretly a zero-shot classifier. *arXiv preprint arXiv:2303.16203*, 2023.

Li, Y., Gao, T., Yang, J., Xu, H., and Wu, Y. Phasic self-imitative reduction for sparse-reward goal-conditioned reinforcement learning. In *International Conference on Machine Learning*, pp. 12765–12781. PMLR, 2022.

Liu, J., Shen, D., Zhang, Y., Dolan, B., Carin, L., and Chen, W. What makes good in-context examples for gpt-3? *arXiv preprint arXiv:2101.06804*, 2021.

Lu, Y., Bartolo, M., Moore, A., Riedel, S., and Stenetorp, P. Fantastically ordered prompts and where to find them: Overcoming few-shot prompt order sensitivity. *arXiv preprint arXiv:2104.08786*, 2021.

Mikuła, M., Antoniak, S., Tworkowski, S., Jiang, A. Q., Zhou, J. P., Szegedy, C., Kuciński, Ł., Miłoś, P., and Wu, Y. Magnushammer: A transformer-based approach to premise selection. *arXiv preprint arXiv:2303.04488*, 2023.

Moro, L., Likmeta, A., Prati, E., Restelli, M., et al. Goal-directed planning via hindsight experience replay. In *International Conference on Learning Representations*, pp. 1–16, 2022.

OpenAI. Chatgpt: Optimizing language models for dialogue, 2022.

OpenAI. GPT-4 Technical Report. *arXiv e-prints*, art. arXiv:2303.08774, March 2023. doi: 10.48550/arXiv.2303.08774.

OpenAI, O., Plappert, M., Sampedro, R., Xu, T., Akkaya, I., Kosaraju, V., Welinder, P., D'Sa, R., Petron, A., Pinto, H. P. d. O., et al. Asymmetric self-play for automatic goal discovery in robotic manipulation. *arXiv preprint arXiv:2101.04882*, 2021.

Parascandolo, G., Buesing, L., Merel, J., Hasenclever, L., Aslanides, J., Hamrick, J. B., Heess, N., Neitz, A., and Weber, T. Divide-and-conquer monte carlo tree search for goal-directed planning. *arXiv preprint arXiv:2004.11410*, 2020.

Paulson, L. C. *Isabelle: A generic theorem prover*. Springer, 1994.

Paulson, L. C. Isabelle: The next 700 theorem provers. *arXiv preprint cs/9301106*, 2000.

Paulsson, L. C. and Blanchette, J. C. Three years of experience with sledgehammer, a practical link between automatic and interactive theorem provers. In *Proceedings of the 8th International Workshop on the Implementation of Logics (IWIL-2010), Yogyakarta, Indonesia. EPiC*, volume 2, 2012.

Pitis, S., Chan, H., Zhao, S., Stadie, B., and Ba, J. Maximum entropy gain exploration for long horizon multi-goal reinforcement learning. In *International Conference on Machine Learning*, pp. 7750–7761. PMLR, 2020.

Polu, S. and Sutskever, I. Generative language modeling for automated theorem proving. *arXiv preprint arXiv:2009.03393*, 2020.

Polu, S., Han, J. M., Zheng, K., Baksys, M., Babuschkin, I., and Sutskever, I. Formal mathematics statement curriculum learning. *arXiv preprint arXiv:2202.01344*, 2022.

Rubin, O., Herzig, J., and Berant, J. Learning to retrieve prompts for in-context learning. *arXiv preprint arXiv:2112.08633*, 2021.

Silver, D., Huang, A., Maddison, C. J., Guez, A., Sifre, L., Van Den Driessche, G., Schrittwieser, J., Antonoglou, I., Panneershelvam, V., Lanctot, M., et al. Mastering the game of go with deep neural networks and tree search. *nature*, 529(7587):484–489, 2016.

Sorensen, T., Robinson, J., Rytting, C. M., Shaw, A. G., Rogers, K. J., Delorey, A. P., Khalil, M., Fulda, N., and Wingate, D. An information-theoretic approach to prompt engineering without ground truth labels. *arXiv preprint arXiv:2203.11364*, 2022.

Su, H., Kasai, J., Wu, C. H., Shi, W., Wang, T., Xin, J., Zhang, R., Ostendorf, M., Zettlemoyer, L., Smith, N. A., et al. Selective annotation makes language models better few-shot learners. *arXiv preprint arXiv:2209.01975*, 2022.

Sun, Z. and Yang, Y. Difusco: Graph-based diffusion solvers for combinatorial optimization. *arXiv preprint arXiv:2302.08224*, 2023.

Vaswani, A., Shazeer, N., Parmar, N., Uszkoreit, J., Jones, L., Gomez, A. N., Kaiser, Ł., and Polosukhin, I. Attention is all you need. *Advances in neural information processing systems*, 30, 2017.

Wen, Z., Precup, D., Ibrahimi, M., Barreto, A., Van Roy, B., and Singh, S. On efficiency in hierarchical reinforcement learning. *Advances in Neural Information Processing Systems*, 33:6708–6718, 2020.

Wu, Y., Jiang, A. Q., Li, W., Rabe, M., Staats, C., Jamnik, M., and Szegedy, C. Autoformalization with large language models. *Advances in Neural Information Processing Systems*, 35:32353–32368, 2022a.

Wu, Z., Wang, Y., Ye, J., and Kong, L. Self-adaptive in-context learning. *arXiv preprint arXiv:2212.10375*, 2022b.

Xin, H., Wang, H., Zheng, C., Li, L., Liu, Z., Cao, Q., Huang, Y., Xiong, J., Shi, H., Xie, E., et al. Lego-prover: Neural theorem proving with growing libraries. *arXiv preprint arXiv:2310.00656*, 2023.

Zhai, Y., Baek, C., Zhou, Z., Jiao, J., and Ma, Y. Computational benefits of intermediate rewards for goal-reaching policy learning. *Journal of Artificial Intelligence Research*, 73:847–896, 2022.

Zhang, T., Eysenbach, B., Salakhutdinov, R., Levine, S., and Gonzalez, J. E. C-planning: An automatic curriculum for learning goal-reaching tasks. *arXiv preprint arXiv:2110.12080*, 2021.

Zhang, Y., Abbeel, P., and Pinto, L. Automatic curriculum learning through value disagreement. *Advances in Neural Information Processing Systems*, 33:7648–7659, 2020.

Zhao, Z., Wallace, E., Feng, S., Klein, D., and Singh, S. Calibrate before use: Improving few-shot performance of language models. In *International Conference on Machine Learning*, pp. 12697–12706. PMLR, 2021.

Zheng, C., Wang, H., Xie, E., Liu, Z., Sun, J., Xin, H., Shen, J., Li, Z., and Li, Y. Lyra: Orchestrating dual correction in automated theorem proving. *arXiv preprint arXiv:2309.15806*, 2023.

Zheng, K., Han, J. M., and Polu, S. Minif2f: a cross-system benchmark for formal olympiad-level mathematics. *arXiv preprint arXiv:2109.00110*, 2021.

A. More Details about Subogal-based Proof

We provide a detailed description of the subgoal refinement method (see §2.1) through Algorithm 1. In the k-th iteration, we construct demonstration examples $\{E_i^{(k)}\}_{i=1}^N$ using improved subgoal-based proofs. To construct $E_i^{(k)}$, we first extract the statement and formal sketch from $E_i^{(k-1)}$, then use an LLM to generate subgoals. Afterward, a Refine module is called to confirm the validity of the created subgoals and adjust any subgoals identified as infeasible.

A subgoal s_Δ is deemed "reachable" from s_1 if, for any s_i where $1 < i \le \Delta$, the ATP solvers can bridge the gap between s_{i-1} and s_i, under the assumption that all preceding subgoals s_j (where $j < i$) are true. This concept is crucial in cases where the proof structure necessitates the independent verification of lemma a, b from s_1, followed by the substantiation of s_2 utilizing both lemma a and b. In such scenarios, as per our definition of reachability, the verification of (s_{i-1}, s_i) inherently presumes the truth of all prior subgoals, denoted as all s_j with $j < i$, which are encapsulated within the context C in VERIF_AND_CORRECT. Specifically, the verification of (s_a, s_b) presupposes s_1 and s_a, and the verification of (s_b, s_2) presupposes s_1, s_a, and s_b. If the ITP validates the steps, the subgoal is deemed reachable; otherwise, it is considered unattainable.

In practice, the reachability of subgoals is tested using the VERIFY function in Algorithm 4, which integrates both an Interactive Theorem Prover (ITP) and a Large Language Model (LLM). It amends the formal sketch for a subgoal, assigns the LLM to complete the missing segments, and then validates these with the ITP. If the ITP affirms the steps, the subgoal is considered reachable; otherwise, it is deemed unattainable. In instances where the LLM consistently fails to generate provable subgoals for the theorem prover, the VERIFY_AND_CORRECT function will revert to returning the original subgoal pair (s_i, s_{i+1}).

We present an example to elucidate this process further (see Figures 4 to 12).[4] As shown in Figure 4, the LLM creates two subgoals for the theorem *amc12a_2003_p4*, leading to $\{s_0, s_1, s_2, s_3\}$. Refining these subgoals involves calling $\mathrm{verify\_and\_correct}(s_0, s_1)$ to improve the subgoal s_1. This is depicted in Figures 5 to 12. We first use the LLM to reconstruct the subgoal related to the first step, but this attempt fails (Figure 5). Then we break down the subgoal s_1 into three more detailed subgoals (Figure 6), each of which is then verified using the same LLM (Figures 7 to 9). Due to the unsuccessful reconstruction of the second subgoal (Figure 8), it is further broken down into two more specific subgoals (Figure 10). The last two subgoals pass the verification process successfully (Figures 11 and 12). Finally, the output of $\mathrm{verify\_and\_correct}(s_0, s_1)$, namely $S^{0\to1}$, is defined as the set that includes the steps from 1 to 4 shown in Figure 12.

Algorithm 1 Iterative Subgoal Refinement

```
Requires:   EXTRACT               extraction of statement and formal sketch
            COMPOSE               composing of a statement, formal sketch
                                  and subgoals to form a demonstration example
            INITIALIZE_SUBGOALS   generate subgoals with a LLM

function ITERATIVE_REFINEMENT($\{E_1^{(0)}, E_2^{(0)}, \cdots, E_N^{(0)}\}$)
    for $k$ in $1, 2, \ldots, K$ do
        for $i$ in $1, 2, \ldots, N$ do
            $x, y \leftarrow$ EXTRACT($E_i^{(k-1)}$)
            $s_0, s_1, \cdots, s_\Delta, s_{\Delta+1} \leftarrow$ INITIALIZE_SUBGOALS($x, y, E^{(k-1)}$)
            $S^{0\to(\Delta+1)} \leftarrow$ REFINE($(s_0, s_{\Delta+1}, \{s_1, s_2, \cdots, s_\Delta\})$)
            $E_i^{(k)} \leftarrow$ COMPOSE($x, y, S^{0\to(\Delta+1)}$)
        end for
    end for
    return $\{E_1^{(K)}, E_2^{(K)}, \cdots, E_N^{(K)}\}$
end function
```

---

[4] To simplify the illustration, we leave out redundant demonstration examples.

Algorithm 2 Refinement Algorithm

```
Requires:   VERIFY_AND_CORRECT   verify the validness of the subgoals and correct them
                                 if necessary

function REFINE($s_i$, $s_{j+1}$, $\{s_{i+1}, \cdots, s_j\}$)
    if i = j then return VERIFY_AND_CORRECT($s_i, s_{i+1}$)
    end if
    $S^{i \to i+1} \leftarrow$ REFINE($s_i, s_{i+1}, \{\}$)
    $S^{i+1 \to j+1} \leftarrow$ REFINE($s_{i+1}, s_{j+1}, \{s_{i+2}, \cdots, s_j\}$)
    return $S^{i \to i+1} \cup S^{i+1 \to j+1}$
end function
```

Algorithm 3 Verify and Correct

```
Requires:   VERIFY    verify if $s_{i+1}$ is reachable from $s_i$
            CORRECT   correct the pair of subgoals if necessary
              $M$     the maximum number of LLM calls, which is set to 10 in our experiments
              $C$     context including formal sketch and current subgoal-based proof

function VERIFY_AND_CORRECT($s_i$, $s_{i+1}$, $C$)
    budget ← $M$
    $Q \leftarrow \varnothing$        ▷ A priority queue; lower pos values indicate proximity to the formal statement
    $Q$.PUSH($(-C.pos(s_{i+1}), (s_i, s_{i+1}))$)        ▷ Push subgoals based on their distance to the formal statement
    valid_subgoals ← []
    while $Q \neq \varnothing$ and budget > 0 do
        $\_, (s, s') \leftarrow Q$.POP()        ▷ Retrieve subgoal closest to the formal statement
        budget ← budget − 1        ▷ LLM call consumed by Verify
        if VERIFY($s, s', C$) then
            valid_subgoals.APPEND($(s, s')$)
        else
            budget ← budget − 1        ▷ LLM call consumed by Correct
            subgoals ← []
            subgoals, $C$ ← CORRECT($s, s', C$)        ▷ Fix subgoal $s'$ or generate granular subgoals, then update the context
            for $(s_j, s_{j+1})$ in subgoals do
                $Q$.PUSH($(-C.pos(s_{j+1}), (s_j, s_{j+1}))$)
            end for
        end if
    end while
    if valid_subgoals ≠ ∅ then
        return valid_subgoals
    else
        return $[(s_i, s_{i+1})]$
    end if
end function
```

Algorithm 4 Verify

```
Requires:   ITP    Interactive Theorem Prover, i.e., Isabelle in our experiments
            LLM    Large Language Model, i.e., GPT-3.5-turbo in our experiments

function VERIFY($s_i$, $s_{i+1}$, C)
    llm_output ← LLM($C$.verify_prompt($s_i, s_{i+1}, C$))
    if ITP(llm_output) = pass then
        return True
    else
        return False
    end if
end function
```

Algorithm 5 Correct

```
Requires:   LLM      Large Language Model, i.e., GPT-3.5-turbo in our experiments
            PARSE    Parse the output of LLM to extract subgoals
                     which correspond to the same segment of the formal sketch as $s_{i+1}$
            UPDATE   Replace $s_{i+1}$ in C with newly generated subgoals
function CORRECT($s_i$, $s_{i+1}$, C)
    llm_output ← LLM($C$.correct_prompt($s_i, s_{i+1}, C$))
    subgoals ← PARSE(llm_output, $s_{i+1}, C$)
    C ← UPDATE(llm_output, $s_{i+1}, C$)
    return subgoals, $C$
end function
```

Your task is to transform a formal proof, written in a style suitable for a theorem-proving system, into a more explanatory and narrative style, suitable for humans. The task involves adding explanatory comments that break down the formal proof into understandable steps, providing context and insight into the underlying logical structure and reasoning.

Input:

```
theorem amc12a_2003_p24:
  fixes a b::real
  assumes "b \<le>a" and "1<b"
  shows "ln (a/b) / ln a + ln (b/a) / ln b \<le>0" (is "?L \<le> _")
proof –
  define x y where "x=ln a" and "y=ln b"
  have "y>0"
    using \<open>b>1\<close> unfolding y_def using ln_gt_zero sledgehammer
  moreover have "x \<ge>y"
    using \<open>a\<ge>b\<close> unfolding x_def y_def using assms(2) sledgehammer
  ultimately have "x>0"
    sledgehammer
  have "?L = (x-y)/x + (y-x)/y"
    apply (subst (1 2) ln_div)
    using assms unfolding x_def y_def sledgehammer
  also have "... = 2 - (y/x + x/y)"
    using \<open>x>0\<close> \<open>y>0\<close> sledgehammer
  also have "... \<le> 0"
  proof –
    have "sqrt (x/y) * sqrt (y/x) = 1"
      sledgehammer
    have "0\<le> (sqrt (x/y) - sqrt (y/x))^2"
      sledgehammer
    also have "... = x / y - 2 + y / x"
      unfolding power2_eq_square using \<open>x>0\<close> \<open>y>0\<close> \<open>sqrt (x/y) * sqrt (y/x) = 1\<close>
      sledgehammer
    finally show ?thesis
      sledgehammer
  qed
  finally show ?thesis.
qed
```

Output:

```
theorem amc12a_2003_p24:
  fixes a b::real
  assumes "b \<le>a" and "1<b"
  shows "ln (a/b) / ln a + ln (b/a) / ln b \<le>0" (is "?L \<le> _")
proof –
  (* Step 1: With the definitions $x = ln\,a$, $y = ln\,b$, and the constraints $y > 0$ and $x \geq y$, the equation simplifies into $2 - \frac{y}{x} - \frac{x}{y}$. *)
  define x y where "x=ln a" and "y=ln b"
  have "y>0"
    using \<open>b>1\<close> unfolding y_def using ln_gt_zero sledgehammer
  moreover have "x \<ge>y"
    using \<open>a\<ge>b\<close> unfolding x_def y_def using assms(2) sledgehammer
  ultimately have "x>0"
    sledgehammer
  have "?L = (x-y)/x + (y-x)/y"
    apply (subst (1 2) ln_div)
    using assms unfolding x_def y_def sledgehammer
  also have "... = 2 - (y/x + x/y)"
    using \<open>x>0\<close> \<open>y>0\<close> sledgehammer
  (* Step 2: Using the inequality $(\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}})^2 \geq 0$, it is shown that the simplified equation $2 - \frac{y}{x} - \frac{x}{y}$ is less than or equal to 0. *)
  also have "... \<le> 0"
  proof –
    have "sqrt (x/y) * sqrt (y/x) = 1"
      sledgehammer
    have "0\<le> (sqrt (x/y) - sqrt (y/x))^2"
      sledgehammer
    also have "... = x / y - 2 + y / x"
      unfolding power2_eq_square using \<open>x>0\<close> \<open>y>0\<close> \<open>sqrt (x/y) * sqrt (y/x) = 1\<close>
      sledgehammer
    finally show ?thesis
      sledgehammer
  qed
  finally show ?thesis.
qed
```

Figure 4. Illustration of the Initialize_subgoals function as denoted in Algorithm 1. ChatGPT is leveraged to generate the subgoal-based proof for a formal sketch.

Your task is to complete the missing part of a Isabelle proof based on the corresponding description provided. The proof will consist of several steps, and you will need to use your knowledge of mathematical concepts and logical reasoning to fill in the gap.

Input:

```
theorem amc12a_2003_p24:
  fixes a b::real
  assumes "b \<le>a" and "1<b"
  shows "ln (a/b) / ln a + ln (b/a) / ln b \<le>0" (is "?L \<le> _")
proof –
  (* Step 1: With the definitions $x = ln\, a$, $y = ln\, b$, and the constraints $y > 0$ and $x \geq y$, the equation simplifies into $2 - \frac{y}{x} - \frac{x}{y}$. *)

  (* Step 2: Using the inequality $(\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}})^2 \geq 0$, it is shown that the simplified equation $2 - \frac{y}{x} - \frac{x}{y}$ is less than or equal to 0. *)
  also have "... \<le> 0"
  proof –
    have "sqrt (x/y) * sqrt (y/x) = 1"
      sledgehammer
    have "0\<le> (sqrt (x/y) - sqrt (y/x))^2"
      sledgehammer
    also have "... = x / y - 2 + y / x"
      unfolding power2_eq_square using \<open>x>0\<close> \<open>y>0\<close> \<open>sqrt (x/y) * sqrt (y/x) = 1\<close>
      sledgehammer
    finally show ?thesis
      sledgehammer
  qed
  finally show ?thesis.
qed
```

Output:

```
theorem amc12a_2003_p24:
  fixes a b::real
  assumes "b \<le>a" and "1<b"
  shows "ln (a/b) / ln a + ln (b/a) / ln b \<le>0" (is "?L \<le> _")
proof –
  (* Step 1: With the definitions $x = ln\, a$, $y = ln\, b$, and the constraints $y > 0$ and $x \geq y$, the equation simplifies into $2 - \frac{y}{x} - \frac{x}{y}$. *)
  define x y where "x=ln a" and "y=ln b"
  have "x > 0" and "y > 0"
    using assms by (simp_all add: ln_gt_zero)
  have "?L = 2 - y / x - x / y"
    using assms x_def y_def by (simp add: add_divide_distrib diff_divide_distrib ln_div)
  (* Step 2: Using the inequality $(\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}})^2 \geq 0$, it is shown that the simplified equation $2 - \frac{y}{x} - \frac{x}{y}$ is less than or equal to 0. *)
  also have "... \<le> 0"
  proof –
    have "sqrt (x/y) * sqrt (y/x) = 1"
      sledgehammer
    have "0\<le> (sqrt (x/y) - sqrt (y/x))^2"
      sledgehammer
    also have "... = x / y - 2 + y / x"
      unfolding power2_eq_square using \<open>x>0\<close> \<open>y>0\<close> \<open>sqrt (x/y) * sqrt (y/x) = 1\<close>
      sledgehammer
    finally show ?thesis
      sledgehammer
  qed
  finally show ?thesis.
qed
```

Figure 5. An instance of the "verify" component within the Verify_and_correct function in Algorithm 2. ChatGPT encounters a *failure* in reconstructing the proof associated with *step 1*, thereby deeming it an unsuitable subgoal.

The Isabelle Proof Assistant was unable to verify Step 1. To resolve this, consider providing more detailed information or clarity to enhance the explanation for this particular step.

Input:

```
theorem amc12a_2003_p24:
  fixes a b::real
  assumes "b \<le>a" and "1<b"
  shows "ln (a/b) / ln a + ln (b/a) / ln b \<le>0" (is "?L \<le> _")
proof –
  (* Step 1: With the definitions $x = \ln a$, $y = \ln b$, and the constraints $y > 0$ and $x \geq y$, the equation simplifies into $2 - \frac{y}{x} - \frac{x}{y}$. *)
  define x y where "x=ln a" and "y=ln b"
  have "y>0"
    using \<open>b>1\<close> unfolding y_def using ln_gt_zero sledgehammer
  moreover have "x \<ge>y"
    using \<open>a\<ge>b\<close> unfolding x_def y_def using assms(2) sledgehammer
  ultimately have "x>0"
    sledgehammer
  have "?L = (x-y)/x + (y-x)/y"
    apply (subst (1 2) ln_div)
    using assms unfolding x_def y_def sledgehammer
  also have "... = 2 - (y/x + x/y)"
    using \<open>x>0\<close> \<open>y>0\<close> sledgehammer
  (* Step 2: Using the inequality $(\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}})^2 \geq 0$, it is shown that the simplified equation $2 - \frac{y}{x} - \frac{x}{y}$ is less than or equal to 0. *)
  also have "... \<le> 0"
  proof –
    have "sqrt (x/y) * sqrt (y/x) = 1"
      sledgehammer
    have "0\<le> (sqrt (x/y) - sqrt (y/x))^2"
      sledgehammer
    also have "... = x / y - 2 + y / x"
      unfolding power2_eq_square using \<open>x>0\<close> \<open>y>0\<close> \<open>sqrt (x/y) * sqrt (y/x) = 1\<close>
      sledgehammer
    finally show ?thesis
      sledgehammer
  qed
  finally show ?thesis.
qed
```

Output:

```
theorem amc12a_2003_p24:
  fixes a b::real
  assumes "b \<le>a" and "1<b"
  shows "ln (a/b) / ln a + ln (b/a) / ln b \<le>0" (is "?L \<le> _")
proof –
  (* Step 1: Define $x = \ln a$ and $y = \ln b$. *)
  define x y where "x=ln a" and "y=ln b"
  (* Step 2: With the conditions set as $y > 0$ and $x \geq y$, we transition the equation to be framed in terms of $x$ and $y$, thereby yielding
$\frac{x-y}{x} + \frac{y-x}{y}$. *)
  have "y>0"
    using \<open>b>1\<close> unfolding y_def using ln_gt_zero sledgehammer
  moreover have "x \<ge>y"
    using \<open>a\<ge>b\<close> unfolding x_def y_def using assms(2) sledgehammer
  ultimately have "x>0"
    sledgehammer
  have "?L = (x-y)/x + (y-x)/y"
    apply (subst (1 2) ln_div)
    using assms unfolding x_def y_def sledgehammer
  (* Step 3: Simplify the expression to $2 - \frac{y}{x} - \frac{x}{y}$. *)
  also have "... = 2 - (y/x + x/y)"
    using \<open>x>0\<close> \<open>y>0\<close> sledgehammer
  (* Step 4: Using the inequality $(\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}})^2 \geq 0$, it is shown that the simplified equation $2 - \frac{y}{x} - \frac{x}{y}$ is less than or equal to 0. *)
  also have "... \<le> 0"
  proof –
    have "sqrt (x/y) * sqrt (y/x) = 1"
      sledgehammer
    have "0\<le> (sqrt (x/y) - sqrt (y/x))^2"
      sledgehammer
    also have "... = x / y - 2 + y / x"
      unfolding power2_eq_square using \<open>x>0\<close> \<open>y>0\<close> \<open>sqrt (x/y) * sqrt (y/x) = 1\<close>
      sledgehammer
    finally show ?thesis
      sledgehammer
  qed
  finally show ?thesis.
qed
```

Figure 6. An instance of the "correct" component within the Verify_and_correct function in Algorithm 2. ChatGPT works on the decomposition of the original subgoal (i.e., step 1 in the input) into a series of more granular subgoals (i.e., steps 1 - 3 in the output).

Your task is to complete the missing part of a Isabelle proof based on the corresponding description provided. The proof will consist of several steps, and you will need to use your knowledge of mathematical concepts and logical reasoning to fill in the gap.

Input:

```
theorem amc12a_2003_p24:
  fixes a b::real
  assumes "b \<le>a" and "1<b"
  shows "ln (a/b) / ln a + ln (b/a) / ln b \<le>0" (is "?L \<le> _")
proof –
  (* Step 1: Define $x = ln a$ and $y = ln b$. *)

  (* Step 2: With the conditions set as $y > 0$ and $x \geq y$, we transition the equation to be framed in terms of $x$ and $y$, thereby yielding $\frac{x-y}{x} + \frac{y-x}{y}$. *)
  have "y>0"
    using \<open>b>1\<close> unfolding y_def using ln_gt_zero sledgehammer
  moreover have "x \<ge>y"
    using \<open>a\<ge>b\<close> unfolding x_def y_def using assms(2) sledgehammer
  ultimately have "x>0"
    sledgehammer
  have "?L = (x-y)/x + (y-x)/y"
    apply (subst (1 2) ln_div)
    using assms unfolding x_def y_def sledgehammer
  (* Step 3: Simplify the expression to $2 - \frac{y}{x} - \frac{x}{y}$. *)
  also have "... = 2 - (y/x + x/y)"
    using \<open>x>0\<close> \<open>y>0\<close> sledgehammer
  (* Step 4: Using the inequality $(\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}})^2 \geq 0$, it is shown that the simplified equation $2 - \frac{y}{x} - \frac{x}{y}$ is less than or equal to 0. *)
  also have "... \<le> 0"
  proof –
    have "sqrt (x/y) * sqrt (y/x) = 1"
      sledgehammer
    have "0\<le> (sqrt (x/y) - sqrt (y/x))^2"
      sledgehammer
    also have "... = x / y - 2 + y / x"
      unfolding power2_eq_square using \<open>x>0\<close> \<open>y>0\<close> \<open>sqrt (x/y) * sqrt (y/x) = 1\<close>
      sledgehammer
    finally show ?thesis
      sledgehammer
  qed
  finally show ?thesis.
qed
```

Output:

```
theorem amc12a_2003_p24:
  fixes a b::real
  assumes "b \<le>a" and "1<b"
  shows "ln (a/b) / ln a + ln (b/a) / ln b \<le>0" (is "?L \<le> _")
proof –
  (* Step 1: Define $x = ln a$ and $y = ln b$. *)
  define x y where "x=ln a" and "y=ln b"
  (* Step 2: With the conditions set as $y > 0$ and $x \geq y$, we transition the equation to be framed in terms of $x$ and $y$, thereby yielding $\frac{x-y}{x} + \frac{y-x}{y}$. *)
  have "y>0"
    using \<open>b>1\<close> unfolding y_def using ln_gt_zero sledgehammer
  moreover have "x \<ge>y"
    using \<open>a\<ge>b\<close> unfolding x_def y_def using assms(2) sledgehammer
  ultimately have "x>0"
    sledgehammer
  have "?L = (x-y)/x + (y-x)/y"
    apply (subst (1 2) ln_div)
    using assms unfolding x_def y_def sledgehammer
  (* Step 3: Simplify the expression to $2 - \frac{y}{x} - \frac{x}{y}$. *)
  also have "... = 2 - (y/x + x/y)"
    using \<open>x>0\<close> \<open>y>0\<close> sledgehammer
  (* Step 4: Using the inequality $(\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}})^2 \geq 0$, it is shown that the simplified equation $2 - \frac{y}{x} - \frac{x}{y}$ is less than or equal to 0. *)
  also have "... \<le> 0"
  proof –
    have "sqrt (x/y) * sqrt (y/x) = 1"
      sledgehammer
    have "0\<le> (sqrt (x/y) - sqrt (y/x))^2"
      sledgehammer
    also have "... = x / y - 2 + y / x"
      unfolding power2_eq_square using \<open>x>0\<close> \<open>y>0\<close> \<open>sqrt (x/y) * sqrt (y/x) = 1\<close>
      sledgehammer
    finally show ?thesis
      sledgehammer
  qed
  finally show ?thesis.
qed
```

Figure 7. An instance of the "verify" component within the Verify_and_correct function in Algorithm 2. ChatGPT *successfully* reconstructs the proof associated with *step 1*, thus validating it as a viable subgoal.

Your task is to complete the missing part of a Isabelle proof based on the corresponding description provided. The proof will consist of several steps, and you will need to use your knowledge of mathematical concepts and logical reasoning to fill in the gap.

Input:

```
theorem amc12a_2003_p24:
  fixes a b::real
  assumes "b \<le>a" and "1<b"
  shows "ln (a/b) / ln a + ln (b/a) / ln b \<le>0" (is "?L \<le> _")
proof –
  (* Step 1: Define $x = ln\, a$ and $y = ln\, b$. *)
  define x y where "x=ln a" and "y=ln b"
  (* Step 2: With the conditions set as $y > 0$ and $x \geq y$, we transition the equation to be framed in terms of $x$ and $y$, thereby yielding
$\frac{x-y}{x} + \frac{y-x}{y}$. *)

  (* Step 3: Simplify the expression to $2 - \frac{y}{x} - \frac{x}{y}$. *)
  also have "... = 2 - (y/x + x/y)"
    using \<open>x>0\<close> \<open>y>0\<close> sledgehammer
  (* Step 4: Using the inequality $(\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}})^2 \geq 0$, it is shown that the simplified equation $2 - \frac{y}{x} - \frac{x}{y}$ is less than or equal to 0. *)
  also have "... \<le> 0"
  proof –
    have "sqrt (x/y) * sqrt (y/x) = 1"
      sledgehammer
    have "0\<le> (sqrt (x/y) - sqrt (y/x))^2"
      sledgehammer
    also have "... = x / y - 2 + y / x"
      unfolding power2_eq_square using \<open>x>0\<close> \<open>y>0\<close> \<open>sqrt (x/y) * sqrt (y/x) = 1\<close>
      sledgehammer
    finally show ?thesis
      sledgehammer
  qed
  finally show ?thesis.
qed
```

Output:

```
theorem amc12a_2003_p24:
  fixes a b::real
  assumes "b \<le>a" and "1<b"
  shows "ln (a/b) / ln a + ln (b/a) / ln b \<le>0" (is "?L \<le> _")
proof –
  (* Step 1: Define $x = ln\, a$ and $y = ln\, b$. *)
  define x y where "x=ln a" and "y=ln b"
  (* Step 2: With the conditions set as $y > 0$ and $x \geq y$, we transition the equation to be framed in terms of $x$ and $y$, thereby yielding
$\frac{x-y}{x} + \frac{y-x}{y}$. *)
  have "y>0"
    using \<open>b>1\<close> unfolding y_def using ln_gt_zero sledgehammer
  moreover have "x \<ge> y"
    using \<open>b\<le>a\<close> unfolding x_def y_def using ln_mono sledgehammer
  hence "(x - y)/x + (y - x)/y = 2 - y/x - x/y"
    using \<open>x>0\<close> \<open>y>0\<close> by (simp add: diff_divide_distrib)
  (* Step 3: Simplify the expression to $2 - \frac{y}{x} - \frac{x}{y}$. *)
  also have "... = 2 - (y/x + x/y)"
    using \<open>x>0\<close> \<open>y>0\<close> sledgehammer
  (* Step 4: Using the inequality $(\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}})^2 \geq 0$, it is shown that the simplified equation $2 - \frac{y}{x} - \frac{x}{y}$ is less than or equal to 0. *)
  also have "... \<le> 0"
  proof –
    have "sqrt (x/y) * sqrt (y/x) = 1"
      sledgehammer
    have "0\<le> (sqrt (x/y) - sqrt (y/x))^2"
      sledgehammer
    also have "... = x / y - 2 + y / x"
      unfolding power2_eq_square using \<open>x>0\<close> \<open>y>0\<close> \<open>sqrt (x/y) * sqrt (y/x) = 1\<close>
      sledgehammer
    finally show ?thesis
      sledgehammer
  qed
  finally show ?thesis.
qed
```

Figure 8. An instance of the “verify” component within the Verify_and_correct function in Algorithm 2. ChatGPT encounters a *failure* in reconstructing the proof associated with *step 2*, thereby deeming it an unsuitable subgoal.

Your task is to complete the missing part of a Isabelle proof based on the corresponding description provided. The proof will consist of several steps, and you will need to use your knowledge of mathematical concepts and logical reasoning to fill in the gap.

Input:

```
theorem amc12a_2003_p24:
  fixes a b::real
  assumes "b \<le>a" and "1<b"
  shows "ln (a/b) / ln a + ln (b/a) / ln b \<le>0" (is "?L \<le> _")
proof -
  (* Step 1: Define $x = \ln a$ and $y = \ln b$. *)
  define x y where "x=ln a" and "y=ln b"
  (* Step 2: With the conditions set as $y > 0$ and $x \geq y$, we transition the equation to be framed in terms of $x$ and $y$, thereby yielding
$\frac{x-y}{x} + \frac{y-x}{y}$. *)
  have "y>0"
    using \<open>b>1\<close> unfolding y_def using ln_gt_zero sledgehammer
  moreover have "x \<ge>y"
    using \<open>a\<ge>b\<close> unfolding x_def y_def using assms(2) sledgehammer
  ultimately have "x>0"
    sledgehammer
  have "?L = (x-y)/x + (y-x)/y"
    apply (subst (1 2) ln_div)
    using assms unfolding x_def y_def sledgehammer
  (* Step 3: Simplify the expression to $2 - \frac{y}{x} - \frac{x}{y}$. *)

  (* Step 4: Using the inequality $(\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}})^2 \geq 0$, it is shown that the simplified equation $2 - \frac{y}{x} - \frac{x}{y}$ is less than or equal to 0. *)
  also have "... \<le> 0"
  proof -
    have "sqrt (x/y) * sqrt (y/x) = 1"
      sledgehammer
    have "0\<le> (sqrt (x/y) - sqrt (y/x))^2"
      sledgehammer
    also have "... = x / y - 2 + y / x"
      unfolding power2_eq_square using \<open>x>0\<close> \<open>y>0\<close> \<open>sqrt (x/y) * sqrt (y/x) = 1\<close>
      sledgehammer
    finally show ?thesis
      sledgehammer
  qed
  finally show ?thesis.
qed
```

Output:

```
theorem amc12a_2003_p24:
  fixes a b::real
  assumes "b \<le>a" and "1<b"
  shows "ln (a/b) / ln a + ln (b/a) / ln b \<le>0" (is "?L \<le> _")
proof -
  (* Step 1: Define $x = \ln a$ and $y = \ln b$. *)
  define x y where "x=ln a" and "y=ln b"
  (* Step 2: With the conditions set as $y > 0$ and $x \geq y$, we transition the equation to be framed in terms of $x$ and $y$, thereby yielding
$\frac{x-y}{x} + \frac{y-x}{y}$. *)
  have "y>0"
    using \<open>b>1\<close> unfolding y_def using ln_gt_zero sledgehammer
  moreover have "x \<ge>y"
    using \<open>a\<ge>b\<close> unfolding x_def y_def using assms(2) sledgehammer
  ultimately have "x>0"
    sledgehammer
  have "?L = (x-y)/x + (y-x)/y"
    apply (subst (1 2) ln_div)
    using assms unfolding x_def y_def sledgehammer
  (* Step 3: Simplify the expression to $2 - \frac{y}{x} - \frac{x}{y}$. *)
  also have "... = 2 - (y/x + x/y)"
    using \<open>x>0\<close> \<open>y>0\<close> sledgehammer
  (* Step 4: Using the inequality $(\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}})^2 \geq 0$, it is shown that the simplified equation $2 - \frac{y}{x} - \frac{x}{y}$ is less than or equal to 0. *)
  also have "... \<le> 0"
  proof -
    have "sqrt (x/y) * sqrt (y/x) = 1"
      sledgehammer
    have "0\<le> (sqrt (x/y) - sqrt (y/x))^2"
      sledgehammer
    also have "... = x / y - 2 + y / x"
      unfolding power2_eq_square using \<open>x>0\<close> \<open>y>0\<close> \<open>sqrt (x/y) * sqrt (y/x) = 1\<close>
      sledgehammer
    finally show ?thesis
      sledgehammer
  qed
  finally show ?thesis.
qed
```

Figure 9. An instance of the "verify" component within the Verify_and_correct function in Algorithm 2. ChatGPT *successfully* reconstructs the proof associated with *step 3*, thus validating it as a viable subgoal.

The Isabelle Proof Assistant was unable to verify Step 2. To resolve this, consider providing more detailed information or clarity to enhance the explanation for this particular step.

Input:

```
theorem amc12a_2003_p24:
  fixes a b::real
  assumes "b \<le>a" and "1<b"
  shows "ln (a/b) / ln a + ln (b/a) / ln b \<le>0" (is "?L \<le> _")
proof –
  (* Step 1: Define $x = ln\, a$ and $y = ln\, b$. *)
  define x y where "x=ln a" and "y=ln b"
  (* Step 2: With the conditions set as $y > 0$ and $x \geq y$, we transition the equation to be framed in terms of $x$ and $y$, thereby yielding
$\frac{x-y}{x} + \frac{y-x}{y}$. *)
  have "y>0"
    using \<open>b>1\<close> unfolding y_def using ln_gt_zero sledgehammer
  moreover have "x \<ge>y"
    using \<open>a\<ge>b\<close> unfolding x_def y_def using assms(2) sledgehammer
  ultimately have "x>0"
    sledgehammer
  have "?L = (x-y)/x + (y-x)/y"
    apply (subst (1 2) ln_div)
    using assms unfolding x_def y_def sledgehammer
  (* Step 3: Simplify the expression to $2 - \frac{y}{x} - \frac{x}{y}$. *)
  also have "... = 2 - (y/x + x/y)"
    using \<open>x>0\<close> \<open>y>0\<close> sledgehammer
  (* Step 4: Using the inequality $(\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}})^2 \geq 0$, it is shown that the simplified equation $2 - \frac{y}{x} - \frac{x}{y}$ is less than or equal to 0. *)
  also have "... \<le> 0"
  proof –
    have "sqrt (x/y) * sqrt (y/x) = 1"
      sledgehammer
    have "0\<le> (sqrt (x/y) - sqrt (y/x))^2"
      sledgehammer
    also have "... = x / y - 2 + y / x"
      unfolding power2_eq_square using \<open>x>0\<close> \<open>y>0\<close> \<open>sqrt (x/y) * sqrt (y/x) = 1\<close>
      sledgehammer
    finally show ?thesis
      sledgehammer
  qed
  finally show ?thesis.
qed
```

Output:

```
theorem amc12a_2003_p24:
  fixes a b::real
  assumes "b \<le>a" and "1<b"
  shows "ln (a/b) / ln a + ln (b/a) / ln b \<le>0" (is "?L \<le> _")
proof –
  (* Step 1: Define $x = ln\, a$ and $y = ln\, b$. *)
  define x y where "x=ln a" and "y=ln b"
  (* Step 2: Show that $y > 0$ and $x \geq y$. *)
  have "y>0"
    using \<open>b>1\<close> unfolding y_def using ln_gt_zero sledgehammer
  moreover have "x \<ge>y"
    using \<open>a\<ge>b\<close> unfolding x_def y_def using assms(2) sledgehammer
  ultimately have "x>0"
    sledgehammer
  (* Step 3: Use the definitions of $x$ and $y$ to rewrite the expression as $\frac{x-y}{x} + \frac{y-x}{y}$. *)
  have "?L = (x-y)/x + (y-x)/y"
    apply (subst (1 2) ln_div)
    using assms unfolding x_def y_def sledgehammer
  (* Step 4: Simplify the expression to $2 - \frac{y}{x} - \frac{x}{y}$. *)
  also have "... = 2 - (y/x + x/y)"
    using \<open>x>0\<close> \<open>y>0\<close> sledgehammer
  (* Step 5: Using the inequality $(\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}})^2 \geq 0$, it is shown that the simplified equation $2 - \frac{y}{x} - \frac{x}{y}$ is less than or equal to 0. *)
  also have "... \<le> 0"
  proof –
    have "sqrt (x/y) * sqrt (y/x) = 1"
      sledgehammer
    have "0\<le> (sqrt (x/y) - sqrt (y/x))^2"
      sledgehammer
    also have "... = x / y - 2 + y / x"
      unfolding power2_eq_square using \<open>x>0\<close> \<open>y>0\<close> \<open>sqrt (x/y) * sqrt (y/x) = 1\<close>
      sledgehammer
    finally show ?thesis
      sledgehammer
  qed
  finally show ?thesis.
qed
```

Figure 10. An instance of the “correct” component within the Verify_and_correct function in Algorithm 2. ChatGPT works on the decomposition of the original subgoal (specifically, step 2 in the input) into a series of more granular subgoals (specifically, steps 2 - 3 in the output).

Your task is to complete the missing part of a Isabelle proof based on the corresponding description provided. The proof will consist of several steps, and you will need to use your knowledge of mathematical concepts and logical reasoning to fill in the gap.

Input:

```
theorem amc12a_2003_p24:
  fixes a b::real
  assumes "b \<le>a" and "1<b"
  shows "ln (a/b) / ln a + ln (b/a) / ln b \<le>0" (is "?L \<le> _")
proof -
  (* Step 1: Define $x = \ln a$ and $y = \ln b$. *)
  define x y where "x=ln a" and "y=ln b"
  (* Step 2: Show that $y > 0$ and $x \geq y$. *)

  (* Step 3: Use the definitions of $x$ and $y$ to rewrite the expression as $\frac{x-y}{x} + \frac{y-x}{y}$. *)
  have "?L = (x-y)/x + (y-x)/y"
    apply (subst (1 2) ln_div)
    using assms unfolding x_def y_def sledgehammer
  (* Step 4: Simplify the expression to $2 - \frac{y}{x} - \frac{x}{y}$. *)
  also have "... = 2 - (y/x + x/y)"
    using \<open>x>0\<close> \<open>y>0\<close> sledgehammer
  (* Step 5: Using the inequality $(\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}})^2 \geq 0$, it is shown that the simplified equation $2 - \frac{y}{x} - \frac{x}{y}$ is less than or equal to 0. *)
  also have "... \<le> 0"
  proof -
    have "sqrt (x/y) * sqrt (y/x) = 1"
      sledgehammer
    have "0\<le> (sqrt (x/y) - sqrt (y/x))^2"
      sledgehammer
    also have "... = x / y - 2 + y / x"
      unfolding power2_eq_square using \<open>x>0\<close> \<open>y>0\<close> \<open>sqrt (x/y) * sqrt (y/x) = 1\<close>
      sledgehammer
    finally show ?thesis
      sledgehammer
  qed
  finally show ?thesis.
qed
```

Output:

```
theorem amc12a_2003_p24:
  fixes a b::real
  assumes "b \<le>a" and "1<b"
  shows "ln (a/b) / ln a + ln (b/a) / ln b \<le>0" (is "?L \<le> _")
proof -
  (* Step 1: Define $x = \ln a$ and $y = \ln b$. *)
  define x y where "x=ln a" and "y=ln b"
  (* Step 2: Show that $y > 0$ and $x \geq y$. *)
  have "y>0"
    using \<open>b>1\<close> unfolding y_def using ln_gt_zero sledgehammer
  moreover have "x \<ge>y"
    using \<open>a\<ge>b\<close> unfolding x_def y_def using assms(2) sledgehammer
  ultimately have "x>0"
    sledgehammer
  (* Step 3: Use the definitions of $x$ and $y$ to rewrite the expression as $\frac{x-y}{x} + \frac{y-x}{y}$. *)
  have "?L = (x-y)/x + (y-x)/y"
    apply (subst (1 2) ln_div)
    using assms unfolding x_def y_def sledgehammer
  (* Step 4: Simplify the expression to $2 - \frac{y}{x} - \frac{x}{y}$. *)
  also have "... = 2 - (y/x + x/y)"
    using \<open>x>0\<close> \<open>y>0\<close> sledgehammer
  (* Step 5: Using the inequality $(\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}})^2 \geq 0$, it is shown that the simplified equation $2 - \frac{y}{x} - \frac{x}{y}$ is less than or equal to 0. *)
  also have "... \<le> 0"
  proof -
    have "sqrt (x/y) * sqrt (y/x) = 1"
      sledgehammer
    have "0\<le> (sqrt (x/y) - sqrt (y/x))^2"
      sledgehammer
    also have "... = x / y - 2 + y / x"
      unfolding power2_eq_square using \<open>x>0\<close> \<open>y>0\<close> \<open>sqrt (x/y) * sqrt (y/x) = 1\<close>
      sledgehammer
    finally show ?thesis
      sledgehammer
  qed
  finally show ?thesis.
qed
```

Figure 11. An instance of the "verify" component within the Verify_and_correct function in Algorithm 2. ChatGPT *successfully* reconstructs the proof associated with *step 2*, thus validating it as a viable subgoal.

Your task is to complete the missing part of a Isabelle proof based on the corresponding description provided. The proof will consist of several steps, and you will need to use your knowledge of mathematical concepts and logical reasoning to fill in the gap.

Input:

```
theorem amc12a_2003_p24:
  fixes a b::real
  assumes "b \<le>a" and "1<b"
  shows "ln (a/b) / ln a + ln (b/a) / ln b \<le>0" (is "?L \<le> _")
proof –
  (* Step 1: Define $x = \ln a$ and $y = \ln b$. *)
  define x y where "x=ln a" and "y=ln b"
  (* Step 2: Show that $y > 0$ and $x \geq y$. *)
  have "y>0"
    using \<open>b>1\<close> unfolding y_def using ln_gt_zero sledgehammer
  moreover have "x \<ge>y"
    using \<open>a\<ge>b\<close> unfolding x_def y_def using assms(2) sledgehammer
  ultimately have "x>0"
    sledgehammer
  (* Step 3: Use the definitions of x and y to rewrite the expression as $\frac{x-y}{x} + \frac{y-x}{y}$. *)

  (* Step 4: Simplify the expression to $2 - \frac{y}{x} - \frac{x}{y}$. *)
  also have "... = 2 - (y/x + x/y)"
    using \<open>x>0\<close> \<open>y>0\<close> sledgehammer
  (* Step 5: Using the inequality $(\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}})^2 \geq 0$, it is shown that the simplified equation $2 - \frac{y}{x} - \frac{x}{y}$ is less than or equal to 0. *)
  also have "... \<le> 0"
  proof –
    have "sqrt (x/y) * sqrt (y/x) = 1"
      sledgehammer
    have "0\<le> (sqrt (x/y) - sqrt (y/x))^2"
      sledgehammer
    also have "... = x / y - 2 + y / x"
      unfolding power2_eq_square using \<open>x>0\<close> \<open>y>0\<close> \<open>sqrt (x/y) * sqrt (y/x) = 1\<close>
      sledgehammer
    finally show ?thesis
      sledgehammer
  qed
  finally show ?thesis.
qed
```

Output:

```
theorem amc12a_2003_p24:
  fixes a b::real
  assumes "b \<le>a" and "1<b"
  shows "ln (a/b) / ln a + ln (b/a) / ln b \<le>0" (is "?L \<le> _")
proof –
  (* Step 1: Define $x = \ln a$ and $y = \ln b$. *)
  define x y where "x=ln a" and "y=ln b"
  (* Step 2: Show that $y > 0$ and $x \geq y$. *)
  have "y>0"
    using \<open>b>1\<close> unfolding y_def using ln_gt_zero sledgehammer
  moreover have "x \<ge>y"
    using \<open>a\<ge>b\<close> unfolding x_def y_def using assms(2) sledgehammer
  ultimately have "x>0"
    sledgehammer
  (* Step 3: Use the definitions of x and y to rewrite the expression as $\frac{x-y}{x} + \frac{y-x}{y}$. *)
  have "?L = (x-y)/x + (y-x)/y"
    apply (subst (1 2) ln_div)
    using assms unfolding x_def y_def sledgehammer
  (* Step 4: Simplify the expression to $2 - \frac{y}{x} - \frac{x}{y}$. *)
  also have "... = 2 - (y/x + x/y)"
    using \<open>x>0\<close> \<open>y>0\<close> sledgehammer
  (* Step 5: Using the inequality $(\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}})^2 \geq 0$, it is shown that the simplified equation $2 - \frac{y}{x} - \frac{x}{y}$ is less than or equal to 0. *)
  also have "... \<le> 0"
  proof –
    have "sqrt (x/y) * sqrt (y/x) = 1"
      sledgehammer
    have "0\<le> (sqrt (x/y) - sqrt (y/x))^2"
      sledgehammer
    also have "... = x / y - 2 + y / x"
      unfolding power2_eq_square using \<open>x>0\<close> \<open>y>0\<close> \<open>sqrt (x/y) * sqrt (y/x) = 1\<close>
      sledgehammer
    finally show ?thesis
      sledgehammer
  qed
  finally show ?thesis.
qed
```

Figure 12. An instance of the "verify" component within the Verify_and_correct function in Algorithm 2. ChatGPT *successfully* reconstructs the proof associated with *step 3*, thus validating it as a viable subgoal.

B. More Details about Demonstration Reorganization

B.1. Parameterization

In alignment with Austin et al. (2021), we adopt discrete diffusion models to model binary random variables. Explicitly, the forward process is given by:

$$q(\boldsymbol{\psi}_t|\boldsymbol{\psi}_{t-1}) = \mathrm{Cat}\left(\boldsymbol{\psi}_t; \mathbf{p} = \delta(\boldsymbol{\psi}_{t-1})\boldsymbol{Q}_t\right), \tag{5}$$

where $\delta(\boldsymbol{\psi})$ symbolizes the one-hot encoding of $\boldsymbol{\psi}$, $\boldsymbol{Q}_t = \begin{bmatrix} (1-\beta_t) & \beta_t \\ \beta_t & (1-\beta_t) \end{bmatrix}$ denotes the transition matrix, β_t corresponds to the corruption ratio and satisfies that $\prod_{t=1}^{T}(1-\beta_t) \approx 0$. The marginal at step t and the posterior at step $t-1$ can be articulated as:

$$\begin{aligned} q(\boldsymbol{\psi}_t|\boldsymbol{\psi}_0) &= \mathrm{Cat}\left(\boldsymbol{\psi}_t; \mathbf{p} = \delta(\boldsymbol{\psi}_0)\overline{\boldsymbol{Q}}_t\right), \\ q(\boldsymbol{\psi}_{t-1}|\boldsymbol{\psi}_t, \boldsymbol{\psi}_0) &= \mathrm{Cat}\left(\boldsymbol{\psi}_{t-1}; \mathbf{p} = \frac{\delta(\boldsymbol{\psi}_t)\boldsymbol{Q}_t^\top \odot \delta(\boldsymbol{\psi}_0)\overline{\boldsymbol{Q}}_{t-1}}{\delta(\boldsymbol{\psi}_0)\overline{\boldsymbol{Q}}_t\delta(\boldsymbol{\psi}_t)^\top}\right), \end{aligned} \tag{6}$$

where $\overline{\boldsymbol{Q}}_t = \boldsymbol{Q}_1\boldsymbol{Q}_2\ldots\boldsymbol{Q}_t$. In consonance with Austin et al. (2021), we employ a denoising neural network which is tasked with the prediction of $p(\boldsymbol{\psi_0}|\boldsymbol{\psi_t})$, thereby enabling the parameterization of the reverse process:

$$p_{\boldsymbol{\theta}}(\boldsymbol{\psi}_{t-1}|\boldsymbol{\psi}_t, x) \propto \sum_{\boldsymbol{\psi}} q(\boldsymbol{\psi}_{t-1}|\boldsymbol{\psi}_t, \boldsymbol{\psi}_0)p_{\boldsymbol{\theta}}(\boldsymbol{\psi}_0|\boldsymbol{\psi}_t, x). \tag{7}$$

B.2. Implementation of GNN

Our work employs a modified version of GNN, a model that exhibits anisotropic characteristics and is enhanced by edge gating methodologies (Bresson & Laurent, 2018; Sun & Yang, 2023). We define $\mathbf{t}$ as sinusoidal representations (Vaswani et al., 2017) associated with the denoising timestep t. Consider $\boldsymbol{h}_i^\ell$ and $\boldsymbol{e}_{ij}^\ell$ as the features of node i and edge ij at a specific layer ℓ, respectively. During the transition between layers, these features disseminate via an anisotropic message propagation paradigm as follows:

$$\begin{aligned} \hat{\boldsymbol{e}}_{ij}^{\ell+1} &= \boldsymbol{P}^\ell\boldsymbol{e}_{ij}^\ell + \boldsymbol{Q}^\ell\boldsymbol{h}_i^\ell + \boldsymbol{R}^\ell\boldsymbol{h}_j^\ell, \\ \boldsymbol{e}_{ij}^{\ell+1} &= \boldsymbol{e}_{ij}^\ell + \mathrm{MLP}_e(\mathrm{BN}(\hat{\boldsymbol{e}}_{ij}^{\ell+1})) + \mathrm{MLP}_t(\mathbf{t}), \\ \boldsymbol{h}_i^{\ell+1} &= \boldsymbol{h}_i^\ell + \mathrm{ReLU}(\mathrm{BN}(\boldsymbol{U}^\ell\boldsymbol{h}_i^\ell + \mathrm{SUM}_{j\in\mathcal{N}_i}(\sigma(\hat{\boldsymbol{e}}_{ij}^{\ell+1}) \odot \boldsymbol{V}^\ell\boldsymbol{h}_j^\ell))), \end{aligned} \tag{8}$$

where $\boldsymbol{P}^\ell, \boldsymbol{Q}^\ell, \boldsymbol{R}^\ell, \boldsymbol{U}^\ell, \boldsymbol{V}^\ell \in \mathbb{R}^{d\times d}$ denote layer-specific learnable parameters with d denoting the dimension of hidden state. BN signifies the Batch Normalization operation (Ioffe & Szegedy, 2015), while SUM represents sum pooling. $\odot$ designates the Hadamard product, and $\mathcal{N}_i$ encapsulates the set of neighboring nodes of node i. Lastly, a two-layer multi-layer perceptron is denoted by $\mathrm{MLP}(\cdot)$.

In our experiments, we define $\boldsymbol{h}_i^0 = \boldsymbol{W}[\mathrm{Ada}(x); \mathrm{Ada}(E_i^{(K)})]$ where $\boldsymbol{W} \in \mathbb{R}^{d\times 3072}$ is a learnable parameter. $\mathrm{Ada}(x), \mathrm{Ada}(E_i^{(K)}) \in \mathbb{R}^{1536\times 1}$ denote the ada embeddings [5] of the statement x and the i-th demonstration example, respectively. The operator $[\cdot;\cdot]$ denotes the concatenation operation between two vectors. $\boldsymbol{e}_{ij}^0$ are initialized as sinusoidal features of the edges.

B.3. Sampling Process

A straightforward strategy for creating a demonstration organization is by directly sampling $\boldsymbol{\psi} \sim p_{\boldsymbol{\theta}}(\boldsymbol{\psi}_0|x)$. However, this strategy introduces two key challenges: (1) A cycle in $\boldsymbol{\psi}$ might be present, indicating that at least one demonstration example is selected multiple times; (2) $\boldsymbol{\psi}$ could include multiple separate sub-graphs, making it difficult to define the relative position between two demonstration examples from two different sub-graphs. Taking a cue from treating diffusion models as discriminative approaches (Li et al., 2023), we start by randomly creating 200 potential solutions. Using the diffusion model's ability to provide conditional density estimates, we rate these 200 potential solutions and select the one with the highest score to build the final demonstration organization. We then reconstruct the sequence of demonstration examples from $\boldsymbol{\psi}$, adding examples one by one into the LLM context until we hit the length limit of the LLM.

---

[5] `https://platform.openai.com/docs/guides/embeddings`

B.4. Hyperparameters and Hardware Setup

In the course of our experiment, we employ a 3-layer Anisotropic Graph Neural Network with a hidden state dimensionality set to 256. We sweep the learning rate from $[1e-4, 2e-4, 5e-4, 7e-4]$ and sweep batch size from $[4, 8, 16, 32]$. The processes of training and inference for the diffusion models are conducted on an NVIDIA RTX 3090 GPU.

C. More In-depth Analysis

C.1. Impact of Prompt Wording

To systematically investigate the influence of varying prompt wordings on the effectiveness of our proposed method, we further conduct an experiment on the miniF2F-test set. The specific prompt is as follows:

```
Assume the role of a mathematician proficient in Isabelle.
When provided with both informal and formal statements of a problem,
your responsibility is to formulate a formal proof that Isabelle
can verify.
```

Table 3. Comparison of pass rates with different prompt wordings over 100 autoformalization attempts on the miniF2F-test set.

Prompt Type	**pass@100**
Original (Proposed)	45.5%
Alternative Prompt	41.0%

As can be observed in Table 3, the original prompt yielded slightly better results compared to the alternative prompt. This suggests that the phrasing of the prompt can have an influence on the performance, and our original choice was more effective in achieving higher pass rates.

C.2. Sensitivity to Example Ordering

To assess the robustness and adaptability of our model, we explored the sensitivity of the model to the ordering of examples. Specifically, we maintained the examples derived from our diffusion model but altered their sequence. The experiment involves randomizing the sequence of examples after they were organized using the diffusion model. We then evaluated the model's ability to solve problems under this modified setup. The evaluation was conducted over 20 autoformalization attempts on the miniF2F-test set.

Table 4. Impact of example ordering.

Method	**# of Problems Solved**
Ours (Ordered)	94
Shuffled Order	39

The results in Table 4 demonstrate that the ordering of examples significantly impacts the model's performance. It is evident that the structured order provided by the diffusion model is important, as disrupting this order resulted in a substantial decline in the number of problems solved.

C.3. Impact of Demonstration Selection and Presentation Order

We conducted experiments comparing our method, which meticulously selects and orders demonstrations, to an alternative approach where the demonstrations are shuffled.

The results in Table 5 underscore the significance of both the selection and the order in which demonstrations are presented. It is evident that a methodical approach to selecting and ordering demonstrations yields superior results compared to a scenario where demonstrations are presented in a shuffled manner.

Table 5. Comparison of our method vs. shuffled demonstrations

Method	# of Problems Solved
Ours	94
Shuffled	39

C.4. Impact of Increasing Demonstrations

We conduct experiments to elucidate the impact of varying the number of demonstrations on the performance of our model.

Table 6. Performance analysis with varying numbers of demonstrations.

Number of Demonstrations	# of Problems Solved
Ours	94
1 example	16
2 examples	29
3 examples	34

The empirical results in Table 6 underscore the collective significance of all demonstrations in enhancing the model's performance. It is evident that the incorporation of multiple demonstrations is pivotal, substantiating the efficacy of our diffusion-based model.

C.5. Impact of Randomly Generated Problem Names

The incorporation of original problem names could potentially facilitate models like ChatGPT in associating them to readily available solutions online. This scenario posits a risk of primarily assessing the model's capability to translate human proof sketches to subgoals and formal proofs, rather than its proficiency in innovating novel subgoals. To mitigate this concern and uphold the rigor of our evaluation, we initiated experiments substituting original problem names with random identifiers and analyzed the ensuing impact on performance. The evaluation was conducted over 20 autoformalization attempts on the miniF2F-test set.

Table 7. Performance comparison with original and random problem names on the miniF2F-test set.

Method	Original	Random
Ours	38.5%	39.8%
Top-K	35.7%	34.8%

The results in Table 7 indicate that our method maintains robust performance, even manifesting a marginal enhancement when random names are deployed. This steadfastness in performance, regardless of the naming conventions adopted, accentuates the robustness and adaptability of our approach across diverse experimental conditions.

C.6. Impact of Proof Sketch Quality

To comprehensively assess the robustness of our method in relation to the quality of informal proof sketches, we conducted an additional experiment that involved the generation of informal proofs using gpt-3.5-turbo-0613 for problems from the miniF2F-valid dataset. Utilizing the methodology outlined in §2.1, we constructed subgoal-based proofs for these informal proofs. For this purpose, a set of 61 problems was carefully chosen to serve as demonstration samples, aligning with the experimental setup specified in §3.4. The results of this experiment are shown in Table 8.

The results from Table 8 demonstrate that our method maintains a notable degree of robustness against variations in the quality of informal proof sketches. This robustness indicates the adaptability and dependability of our approach when confronted with different levels of proof sketch quality, affirming its practical applicability in diverse scenarios.

Table 8. Performance comparison with varying proof sketch quality.

Method	valid	test
Ours (DSP Informal Proofs)	48.0% (±0.4)	45.5% (±0.6)
Ours (GPT-3.5 Informal Proofs)	47.7% (±0.5)	45.0% (±0.7)

C.7. Evaluation of Optimal Organization Search

To further evaluate the efficiency of our method, we conducted experiments involving 200 randomized organization sampling attempts for each statement within the miniF2F-valid and miniF2F-test datasets. These efforts aimed to determine the potential of finding an optimal organization for each statement. The results of this extensive search are displayed in Table 9.

Table 9. Performance comparison with optimal organization search.

Method	valid	test
Ours	48.0%	45.5%
Optimal Organization Search	48.4%	44.3%

These results corroborate previous observations (Jiang et al., 2023) that the advantages of extensive organization searches tend to level off beyond a certain number of attempts (in this case, 100). These results highlight the inherent challenges of the miniF2F dataset and emphasize the effectiveness of our diffusion model.

C.8. Analysis of Sledgehammer Usage

To provide a comprehensive understanding of how our method and DSP (Jiang et al., 2023) utilize the Sledgehammer tool, we have conducted a detailed analysis. This analysis was carried out on a machine with 64 CPU cores, focusing on the average number of Sledgehammer calls and their execution times for each solved statement. The results are shown in Table 10, which illustrates the average number of Sledgehammer calls and their corresponding durations (in seconds) for each method.

Table 10. Average number of Sledgehammer calls and durations.

Method	Calls		Duration (seconds)	
	valid	test	valid	test
DSP	2.33	2.39	3.29	2.98
Ours	2.88	3.22	4.16	4.94

The results in Table 10 indicate that our method exhibits a slight increase in both the frequency of Sledgehammer calls and their execution times in comparison with DSP. Specifically, this increase is primarily observed in statements that our method can solve but DSP cannot. For these statements, the number of Sledgehammer calls on the miniF2F-valid and miniF2F-test are 3.21 and 5.38, respectively. This suggests that the need for Sledgehammer becomes more important as the problem's complexity increases.

D. Discussions about Correlation Between Input Statements and Organization

We further investigate the relationship between input statements and the demonstration organization generated by diffusion models within the miniF2F dataset. We aimed to determine whether the diffusion model tends to generate a generic organization broadly applicable across various statements or if it tailors unique organizations to individual statements.

The empirical results suggest that there is not a "one-size-fits-all" optimal demonstration organization. Specifically, even the most adaptable organization we identified could successfully prove only **3** distinct statements. Furthermore, we only observed **4** instances of such adaptable organizations on the miniF2F-test. This highlights the tailored nature of each organization to its corresponding statement. The limited scope of even the most adaptable organization indicates the challenge in identifying the optimal demonstration organization for each unique statement. This complexity further emphasizes the effectiveness of our model in efficiently searching for and determining the most suitable demonstration organization for individual statements.

E. Discussions about the Train/Test Data Leakage

The potential for train/test leakage, stemming from the use of original problem names from the miniF2F dataset, was brought to our attention. In response, we undertook a thorough analysis. Our findings indicated that about 32.0% of problems could be identifiable by ChatGPT based on their names, and approximately 40% of informal statements were susceptible to leakage. To further elucidate this, a comprehensive manual search was conducted on the test set's informal statements and their corresponding proofs. This search revealed that around 40% had been available online before September 1, 2021, with platforms like Mathway and Mathematics Stack Exchange being significant contributors. Despite several miniF2F problems lacking Isabelle solutions, we revised the potentially leaked statements. Experiments with randomized identifiers further confirmed that our method's performance is not contingent on data leakage (see Appendix C.5).

F. Additional Examples

We provide additional cases in Figure 13 to 18 to demonstrate the efficacy of our method.[6]

Figures 13 to 15 display examples where our method effectively guides the proof process. For instance, in Figure 13, the method leverages demonstrations involving division and modulus operations, successfully navigating towards a clear proof path. Similarly, in Figure 14, it utilizes demonstrations based on squaring and square root operations, accurately predicting potential subgoals. Figure 15 continues this trend, demonstrating our method's consistent ability to discern viable subgoals, thus facilitating the construction of proofs.

Additionally, Figures 16 to 18 display the same statements as Figures 13 to 15 but with the demonstration examples reordered. This reorganization illustrates the impact of demonstration order on the generation of proofs, revealing how changes in the order can lead to unexpected errors. The comparison between these two groups highlights the significance of demonstration organization in influencing final performance.

---

[6] In Figures 13 to 18, identical statements are guided by a consistent set of demonstration examples, but in a different order. Some less relevant demonstration examples have been omitted due to space constraints.

As a mathematician familiar with Isabelle, your task is to provide a formal proof in response to a given problem statement.
Your proof should be structured and clearly written, meeting the following criteria:
- It can be verified by Isabelle.
- Each step of the proof should be explained in detail using comments enclosed in "(*" and "*)".
- The explanation for each step should be clear and concise, avoiding any unnecessary or apologetic language.
Please ensure that your proof is well-organized and easy to follow, with each step building upon the previous one.

Problem:
Let n be an integer and p be a prime number. Show that p divides n if and only if n^2 is congruent to 0 modulo p.

Proof:

```
theorem numbertheory_prmdvsneqnsqmodpeq0:
  fixes n :: int
    and p :: nat
  assumes "prime p"
  shows "p dvd n \<longleftrightarrow> (n^2) mod p = 0"
  (* Step 1: Assume that $p$ divides $n$. Show that $n^2$ is congruent to 0 modulo $p$.
    Step 2: Assume that $n^2$ is congruent to 0 modulo $p$. Show that $p$ divides $n$. *)
proof -
  (* Step 1: Assume that $p$ divides $n$. Show that $n^2$ is congruent to 0 modulo $p$.*)
  assume "p dvd n"
  then show "(n^2) mod p = 0" sledgehammer
  (* Step 2: Assume that $n^2$ is congruent to 0 modulo $p$. Show that $p$ divides $n$. *)
  next assume "(n^2) mod p = 0"
  then show "p dvd n" sledgehammer
qed
```

Problem:
Prove that for any natural number n, 3 divides $n^3 + 2n$.

Proof:

```
theorem induction_divisibility_3divnto3m2n:
  fixes n::nat
  shows "3 dvd n^3 + 2 * n"
  (* Step 1: Prove the base case $n = 0$.
    Step 2: Assume the statement is true for some $n$ and prove it for $n + 1$.
    Step 3: Expand $(n + 1)^3+2(n + 1)$ and simplify.
    Step 4: Use the assumption that 3 divides $n^3 + 2n$ to show that 3 divides $3n^2 + 3n + 3$.
    Step 5: Combine steps 3 and 4 to show that 3 divides $(n + 1)^3+2(n + 1)$ .
    Step 6: Conclude that the statement is true for all natural numbers $n$. *)
proof (induct n)
  (* Step 1: Prove the base case $n = 0$.*)
  case 0
    then show ?case sledgehammer
  (* Step 2: Assume the statement is true for some $n$ and prove it for $n + 1$.*)
  next case (Suc n)
  have ?case when "n=0"
    using that sledgehammer
  moreover have ?case when "n>0"
  proof -
  (* Step 3: Expand $(n + 1)^3+2(n + 1)$ and simplify.*)
    have "(n + 1)^3 + 2 * (n + 1) = (n^3+3*n^2+3*n+1) + (2*n+2)"
      unfolding power3_eq_cube power2_eq_square sledgehammer
    also have "... = n^3+2*n + 3*n^2+3*n+3" sledgehammer
    finally have "(n+1)^3+2*(n+1) = n^3+2*n + 3*n^2+3*n+3".
  (* Step 4: Use the assumption that 3 divides $n^3 + 2n$ to show that 3 divides $3n^2 + 3n + 3$.*)
    moreover have "3 dvd n^3+2*n" using Suc(1) sledgehammer
    moreover have "3 dvd 3*n^2+3*n+3" sledgehammer
  (* Step 5: Combine steps 3 and 4 to show that 3 divides $(n + 1)^3+2(n + 1)$ .*)
    ultimately have "3 dvd (n+1)^3+2*(n+1)" sledgehammer
  (* Step 6: Conclude that the statement is true for all natural numbers $n$. *)
    then show ?thesis sledgehammer
  qed
  ultimately show ?case sledgehammer
qed
```

Problem:
Suppose n is a natural number such that n leaves a remainder of 3 when divided by 7. Show that $(2n + 1)$ leaves a remainder of 0 when divided by 7.

Proof:

```
theorem mathd_numbertheory_370:
  fixes n :: nat
  assumes h0 : "n mod 7 = (3::nat)"
  shows "(2*n+1) mod 7 = (0::nat)"
```

Prompt

ChatGPT Output

```
  (* Step 1: Express $n$ as $7k + 3$ for some k∊ℕ$.
    Step 2: Substitute the expression for $n$ into the expression for $2n + 1$ and simplify.
    Step 3: Show that the resulting expression is divisible by 7. *)
proof -
  (* Step 1: Express $n$ as $7k + 3$ for some k∊ℕ$.*)
  have "n = 7 * (n div 7) + (n mod 7)" using div_mult_mod_eq[of "n" "7"] sledgehammer
  also have "... = 7 * (n div 7) + 3" using h0 sledgehammer
  finally have c0: "n = 7 * (n div 7) + 3".
  (* Step 2: Substitute the expression for $n$ into the expression for $2n + 1$ and simplify.*)
  have "2*n+1 = 2*(7*(n div 7) + 3) + 1" using c0  sledgehammer
  also have "... = 14*(n div 7) + 7" sledgehammer
  also have "... = 7*(2*(n div 7) + 1)" sledgehammer
  finally have c1: "2*n+1 = 7*(2*(n div 7) + 1)".
  (* Step 3: Show that the resulting expression is divisible by 7. *)
  show "(2*n+1) mod 7 = 0" using c1 sledgehammer
qed
```

Figure 13. A case from miniF2F-valid. Our method identifies viable subgoals and leverages demonstration examples with division and modulus operations to complete the proof.

As a mathematician familiar with Isabelle, your task is to provide a formal proof in response to a given problem statement.
Your proof should be structured and clearly written, meeting the following criteria:
- It can be verified by Isabelle.
- Each step of the proof should be explained in detail using comments enclosed in "(*" and "*)".
- The explanation for each step should be clear and concise, avoiding any unnecessary or apologetic language.

Please ensure that your proof is well-organized and easy to follow, with each step building upon the previous one.

Problem:
Suppose that x is a positive real number. Show that $2-\sqrt{2} \geq 2 - x - \frac{1}{2x}$.

Proof:

```
theorem algebra_amgm_faxinrrp2msqrt2geq2mxm1div2x:
  fixes x :: real
  assumes "x > 0"
  shows "2 - sqrt 2 \<ge> 2 - x - 1/ (2 * x)"
  (* Step 1: Show that $\sqrt{2} \leq x + \frac{1}{2x}$.
    Step 2: Rearrange the inequality to get $2x^2 + 1 - 2x\sqrt{2} \geq 0$.
    Step 3: Factor the expression to get $(\sqrt{2}x - 1)^2 \geq 0$.
    Step 4: Conclude that the inequality is true. *)
proof -
  (* Step 1: Show that $\sqrt{2} \leq x + \frac{1}{2x}$. *)
  have c0: "2 * x > 0" using assms sledgehammer
  have "sqrt 2 \<le> x + 1 / (2 * x)"
  proof -
    define y where "y = sqrt 2"
    have c1: "2 = y * y"
    proof -
      have "2 = (sqrt 2) * (sqrt 2)" sledgehammer
      then have "... = y * y" using \<open>y = sqrt 2\<close> sledgehammer
      then show ?thesis sledgehammer
    qed
    have "(2 * x) * x + 1 - (2 * x) * (sqrt 2) = (y * y * x * x) + 1 - (2 * x) * y" using c1 y_def sledgehammer
    also have "... = (y*x) * (y*x) - 2 * (y*x) + 1" sledgehammer
    also have "... = (y*x - 1) * (y*x - 1)" sledgehammer
    also have "... \<ge> 0" sledgehammer
    ultimately have c2: "(2 * x) * x + 1 - (2 * x) * (sqrt 2) \<ge> 0" sledgehammer
    have "(2*x) * (x + 1/(2*x) - sqrt 2) = (2 * x) * x + (2 * x) * (1/(2*x)) - (2*x) * sqrt 2" sledgehammer
    also have "... = (2 * x) * x + 1 - (2*x) * sqrt 2" using c0 sledgehammer
    (* Step 2: Rearrange the inequality to get $2x^2 + 1 - 2x\sqrt{2} \geq 0$. *)
    also have "... \<ge> 0" using c2 sledgehammer
    (* Step 3: Factor the expression to get $(\sqrt{2}x - 1)^2 \geq 0$. *)
    ultimately have "(2*x) * (x + 1/(2*x) - sqrt 2) \<ge> 0" sledgehammer
    hence "x + 1/(2*x) - sqrt 2 \<ge> 0" using mult_pos_pos sledgehammer
    (* Step 4: Conclude that the inequality is true. *)
    then show ?thesis sledgehammer
  qed
  then show ?thesis sledgehammer
qed
```

Problem:
Suppose that x and y are two real numbers such that $x+y=13$ and $xy=24$. Find $\sqrt{x^2+y^2}$ and show that it is equal to 11.

Proof:

```
theorem mathd_algebra_510:
  fixes x y :: real
  assumes h0 : "x+y=13"
    and h1 : "x*y=24"
  shows "sqrt (x^2 + y^2) = 11"
  (* Step 1: Use the identity $x^2 + y^2 = (x+y)^2 - 2xy$ to find $x^2 + y^2$ in terms of x and y.
    Step 2: Substitute the given values of x and y to find $x^2 + y^2$ .
    Step 3: Show that $x^2 + y^2 = 121$.
    Step 4: Take the square root of both sides to find $\sqrt{x^2 + y^2}$.
    Step 5: Show that $\sqrt{x^2 + y^2} = 11$. *)
proof -
  (* Step 1: Use the identity $x^2 + y^2 = (x+y)^2 - 2xy$ to find $x^2 + y^2$ in terms of x and y. *)
  have c0: "x^2 + y^2 = (x+y)^2 - 2*x*y" sledgehammer
  (* Step 2: Substitute the given values of x and y to find $x^2 + y^2$ . *)
  also have "... = (13)^2 - 2*24" using h0 h1 sledgehammer
  (* Step 3: Show that $x^2 + y^2 = 121$. *)
  also have "... = 121" sledgehammer
  finally have c1: "x^2 + y^2 = 121" sledgehammer
  (* Step 4: Take the square root of both sides to find $\sqrt{x^2 + y^2}$. *)
  then show ?thesis sledgehammer
  (* Step 5: Show that $\sqrt{x^2 + y^2} = 11$. *)
qed
```

Problem:
Suppose that x and y are two real numbers such that $\frac{x+y}{2} = 7$ and $\sqrt{xy} = \sqrt{19}$. Find $x^2 + y^2$.

Proof:

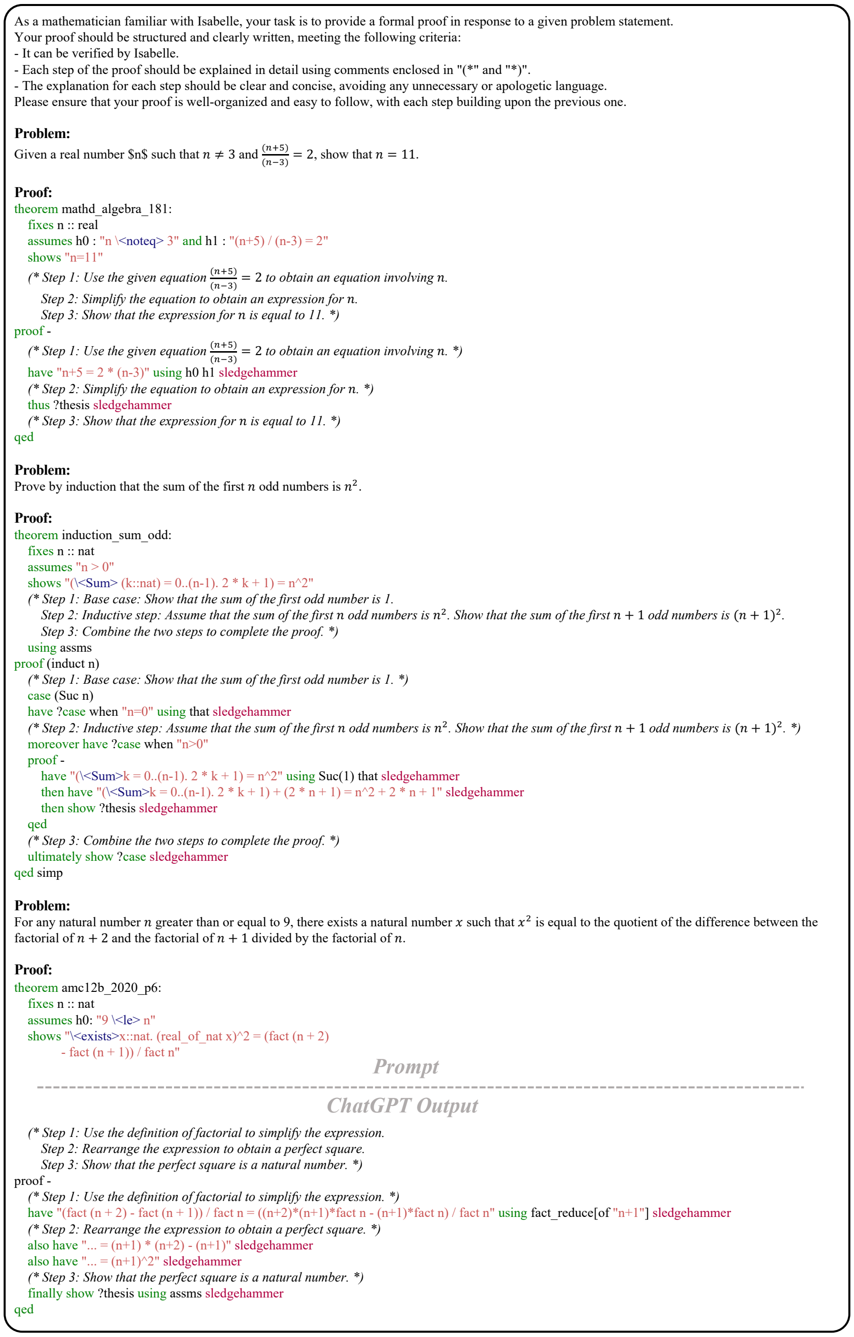
```
theorem mathd_algebra_332:
  fixes x y ::real
  assumes "(x + y) / 2 = 7"
    and "sqrt (x * y) = sqrt 19"
  shows "x^2 + y^2 = 158"
```

Prompt

ChatGPT Output

```
  (* Step 1: Use the identity $(x+y)^2 = x^2 + 2xy + y^2$ to find $x^2 + y^2$ in terms of $x + y$ and xy.
    Step 2: Substitute the given values of $x + y$ and xy to find $x^2 + y^2$.
    Step 3: Show that $x^2 + y^2 = 158$. *)
proof -
  (* Step 1: Use the identity $(x+y)^2 = x^2 + 2xy + y^2$ to find $x^2 + y^2$ in terms of $x + y$ and xy. *)
  have c0: "x^2 + y^2 = (x+y)^2 - 2*x*y" sledgehammer
  (* Step 2: Substitute the given values of $x + y$ and xy to find $x^2 + y^2$. *)
  also have "... = (2*7)^2 - 2*(sqrt 19)^2" using assms sledgehammer
  (* Step 3: Show that $x^2 + y^2 = 158$. *)
  also have "... = 158" sledgehammer
  finally show "x^2 + y^2 = 158" sledgehammer
qed
```

Figure 14. A case from miniF2F-test. Our method predicts viable subgoals and utilizes demonstration examples involving squaring and square root operations to finalize the proof.



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- The explanation for each step should be clear and concise, avoiding any unnecessary or apologetic language.

Please ensure that your proof is well-organized and easy to follow, with each step building upon the previous one.

Problem:
Given a real number n such that $n \neq 3$ and $\frac{(n+5)}{(n-3)} = 2$, show that $n = 11$.

Proof:

```
theorem mathd_algebra_181:
  fixes n :: real
  assumes h0 : "n \<noteq> 3" and h1 : "(n+5) / (n-3) = 2"
  shows "n=11"
  (* Step 1: Use the given equation $\frac{(n+5)}{(n-3)} = 2$ to obtain an equation involving $n$.
     Step 2: Simplify the equation to obtain an expression for $n$.
     Step 3: Show that the expression for $n$ is equal to 11. *)
proof -
  (* Step 1: Use the given equation $\frac{(n+5)}{(n-3)} = 2$ to obtain an equation involving $n$. *)
  have "n+5 = 2 * (n-3)" using h0 h1 sledgehammer
  (* Step 2: Simplify the equation to obtain an expression for $n$. *)
  thus ?thesis sledgehammer
  (* Step 3: Show that the expression for $n$ is equal to 11. *)
qed
```

Problem:
Prove by induction that the sum of the first n odd numbers is n^2.

Proof:

```
theorem induction_sum_odd:
  fixes n :: nat
  assumes "n > 0"
  shows "(\<Sum> (k::nat) = 0..(n-1). 2 * k + 1) = n^2"
  (* Step 1: Base case: Show that the sum of the first odd number is 1.
     Step 2: Inductive step: Assume that the sum of the first $n$ odd numbers is $n^2$. Show that the sum of the first $n+1$ odd numbers is $(n+1)^2$.
     Step 3: Combine the two steps to complete the proof. *)
  using assms
proof (induct n)
  (* Step 1: Base case: Show that the sum of the first odd number is 1. *)
  case (Suc n)
  have ?case when "n=0" using that sledgehammer
  (* Step 2: Inductive step: Assume that the sum of the first $n$ odd numbers is $n^2$. Show that the sum of the first $n+1$ odd numbers is $(n+1)^2$. *)
  moreover have ?case when "n>0"
  proof -
    have "(\<Sum>k = 0..(n-1). 2 * k + 1) = n^2" using Suc(1) that sledgehammer
    then have "(\<Sum>k = 0..(n-1). 2 * k + 1) + (2 * n + 1) = n^2 + 2 * n + 1" sledgehammer
    then show ?thesis sledgehammer
  qed
  (* Step 3: Combine the two steps to complete the proof. *)
  ultimately show ?case sledgehammer
qed simp
```

Problem:
For any natural number n greater than or equal to 9, there exists a natural number x such that x^2 is equal to the quotient of the difference between the factorial of $n+2$ and the factorial of $n+1$ divided by the factorial of n.

Proof:

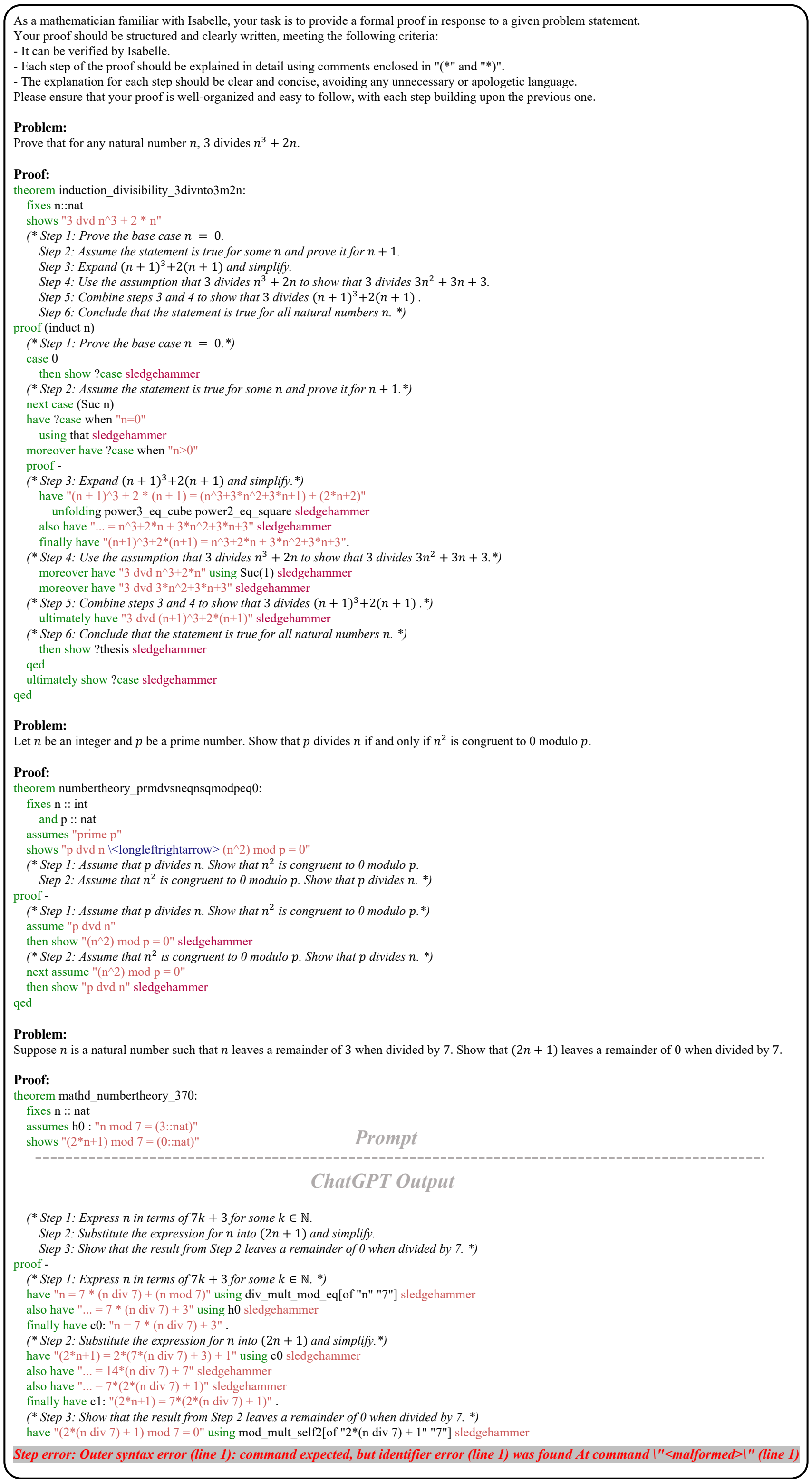
```
theorem amc12b_2020_p6:
  fixes n :: nat
  assumes h0: "9 \<le> n"
  shows "\<exists>x::nat. (real_of_nat x)^2 = (fact (n + 2)
         - fact (n + 1)) / fact n"
```

Prompt

ChatGPT Output

```
  (* Step 1: Use the definition of factorial to simplify the expression.
     Step 2: Rearrange the expression to obtain a perfect square.
     Step 3: Show that the perfect square is a natural number. *)
proof -
  (* Step 1: Use the definition of factorial to simplify the expression. *)
  have "(fact (n + 2) - fact (n + 1)) / fact n = ((n+2)*(n+1)*fact n - (n+1)*fact n) / fact n" using fact_reduce[of "n+1"] sledgehammer
  (* Step 2: Rearrange the expression to obtain a perfect square. *)
  also have "... = (n+1) * (n+2) - (n+1)" sledgehammer
  also have "... = (n+1)^2" sledgehammer
  (* Step 3: Show that the perfect square is a natural number. *)
  finally show ?thesis using assms sledgehammer
qed
```

Figure 15. A case from miniF2F-test. Our method recognizes viable subgoals and successfully finishes the proof.



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- The explanation for each step should be clear and concise, avoiding any unnecessary or apologetic language.

Please ensure that your proof is well-organized and easy to follow, with each step building upon the previous one.

Problem:
Prove that for any natural number n, 3 divides $n^3 + 2n$.

Proof:

```
theorem induction_divisibility_3divnto3m2n:
  fixes n::nat
  shows "3 dvd n^3 + 2 * n"
  (* Step 1: Prove the base case $n = 0$.
    Step 2: Assume the statement is true for some $n$ and prove it for $n + 1$.
    Step 3: Expand $(n + 1)^3 + 2(n + 1)$ and simplify.
    Step 4: Use the assumption that 3 divides $n^3 + 2n$ to show that 3 divides $3n^2 + 3n + 3$.
    Step 5: Combine steps 3 and 4 to show that 3 divides $(n + 1)^3 + 2(n + 1)$ .
    Step 6: Conclude that the statement is true for all natural numbers $n$. *)
proof (induct n)
  (* Step 1: Prove the base case $n = 0$.*)
  case 0
    then show ?case sledgehammer
  (* Step 2: Assume the statement is true for some $n$ and prove it for $n + 1$.*)
  next case (Suc n)
  have ?case when "n=0"
    using that sledgehammer
  moreover have ?case when "n>0"
  proof -
  (* Step 3: Expand $(n + 1)^3 + 2(n + 1)$ and simplify.*)
    have "(n + 1)^3 + 2 * (n + 1) = (n^3+3*n^2+3*n+1) + (2*n+2)"
      unfolding power3_eq_cube power2_eq_square sledgehammer
    also have "... = n^3+2*n + 3*n^2+3*n+3" sledgehammer
    finally have "(n+1)^3+2*(n+1) = n^3+2*n + 3*n^2+3*n+3".
  (* Step 4: Use the assumption that 3 divides $n^3 + 2n$ to show that 3 divides $3n^2 + 3n + 3$.*)
    moreover have "3 dvd n^3+2*n" using Suc(1) sledgehammer
    moreover have "3 dvd 3*n^2+3*n+3" sledgehammer
  (* Step 5: Combine steps 3 and 4 to show that 3 divides $(n + 1)^3 + 2(n + 1)$ .*)
    ultimately have "3 dvd (n+1)^3+2*(n+1)" sledgehammer
  (* Step 6: Conclude that the statement is true for all natural numbers $n$. *)
    then show ?thesis sledgehammer
  qed
  ultimately show ?case sledgehammer
qed
```

Problem:
Let n be an integer and p be a prime number. Show that p divides n if and only if n^2 is congruent to 0 modulo p.

Proof:

```
theorem numbertheory_prmdvsneqnsqmodpeq0:
  fixes n :: int
    and p :: nat
  assumes "prime p"
  shows "p dvd n \<longleftrightarrow> (n^2) mod p = 0"
  (* Step 1: Assume that $p$ divides $n$. Show that $n^2$ is congruent to 0 modulo $p$.
    Step 2: Assume that $n^2$ is congruent to 0 modulo $p$. Show that $p$ divides $n$. *)
proof -
  (* Step 1: Assume that $p$ divides $n$. Show that $n^2$ is congruent to 0 modulo $p$.*)
  assume "p dvd n"
  then show "(n^2) mod p = 0" sledgehammer
  (* Step 2: Assume that $n^2$ is congruent to 0 modulo $p$. Show that $p$ divides $n$. *)
  next assume "(n^2) mod p = 0"
  then show "p dvd n" sledgehammer
qed
```

Problem:
Suppose n is a natural number such that n leaves a remainder of 3 when divided by 7. Show that $(2n + 1)$ leaves a remainder of 0 when divided by 7.

Proof:

```
theorem mathd_numbertheory_370:
  fixes n :: nat
  assumes h0 : "n mod 7 = (3::nat)"
  shows "(2*n+1) mod 7 = (0::nat)"
```

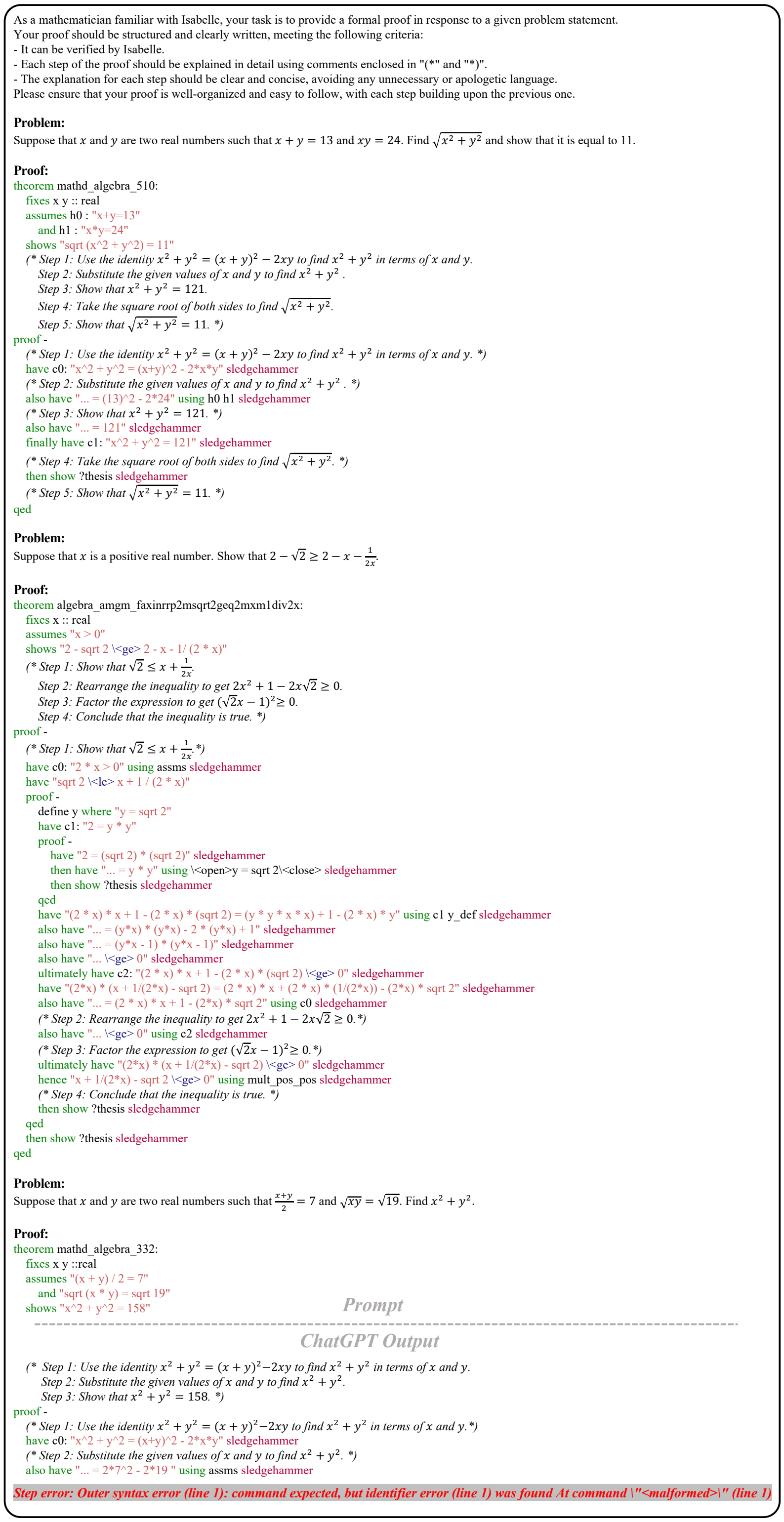
Prompt

ChatGPT Output

```
  (* Step 1: Express $n$ in terms of $7k + 3$ for some $k \in \mathbb{N}$.
    Step 2: Substitute the expression for $n$ into $(2n + 1)$ and simplify.
    Step 3: Show that the result from Step 2 leaves a remainder of 0 when divided by 7. *)
proof -
  (* Step 1: Express $n$ in terms of $7k + 3$ for some $k \in \mathbb{N}$. *)
  have "n = 7 * (n div 7) + (n mod 7)" using div_mult_mod_eq[of "n" "7"] sledgehammer
  also have "... = 7 * (n div 7) + 3" using h0 sledgehammer
  finally have c0: "n = 7 * (n div 7) + 3" .
  (* Step 2: Substitute the expression for $n$ into $(2n + 1)$ and simplify.*)
  have "(2*n+1) = 2*(7*(n div 7) + 3) + 1" using c0 sledgehammer
  also have "... = 14*(n div 7) + 7" sledgehammer
  also have "... = 7*(2*(n div 7) + 1)" sledgehammer
  finally have c1: "(2*n+1) = 7*(2*(n div 7) + 1)" .
  (* Step 3: Show that the result from Step 2 leaves a remainder of 0 when divided by 7. *)
  have "(2*(n div 7) + 1) mod 7 = 0" using mod_mult_self2[of "2*(n div 7) + 1" "7"] sledgehammer
```

Step error: Outer syntax error (line 1): command expected, but identifier error (line 1) was found At command \"<malformed>\" (line 1)

Figure 16. A case from miniF2F-valid. The shuffled demonstration organization results in the incorrect application of the "mod_mult_self2" theorem.



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- The explanation for each step should be clear and concise, avoiding any unnecessary or apologetic language.

Please ensure that your proof is well-organized and easy to follow, with each step building upon the previous one.

Problem:
Suppose that x and y are two real numbers such that $x + y = 13$ and $xy = 24$. Find $\sqrt{x^2 + y^2}$ and show that it is equal to 11.

Proof:

```
theorem mathd_algebra_510:
  fixes x y :: real
  assumes h0 : "x+y=13"
    and h1 : "x*y=24"
  shows "sqrt (x^2 + y^2) = 11"
  (* Step 1: Use the identity $x^2 + y^2 = (x + y)^2 - 2xy$ to find $x^2 + y^2$ in terms of $x$ and $y$.
    Step 2: Substitute the given values of $x$ and $y$ to find $x^2 + y^2$ .
    Step 3: Show that $x^2 + y^2 = 121$.
    Step 4: Take the square root of both sides to find $\sqrt{x^2 + y^2}$.
    Step 5: Show that $\sqrt{x^2 + y^2} = 11$. *)
proof -
  (* Step 1: Use the identity $x^2 + y^2 = (x + y)^2 - 2xy$ to find $x^2 + y^2$ in terms of $x$ and $y$. *)
  have c0: "x^2 + y^2 = (x+y)^2 - 2*x*y" sledgehammer
  (* Step 2: Substitute the given values of $x$ and $y$ to find $x^2 + y^2$ . *)
  also have "... = (13)^2 - 2*24" using h0 h1 sledgehammer
  (* Step 3: Show that $x^2 + y^2 = 121$. *)
  also have "... = 121" sledgehammer
  finally have c1: "x^2 + y^2 = 121" sledgehammer
  (* Step 4: Take the square root of both sides to find $\sqrt{x^2 + y^2}$. *)
  then show ?thesis sledgehammer
  (* Step 5: Show that $\sqrt{x^2 + y^2} = 11$. *)
qed
```

Problem:
Suppose that x is a positive real number. Show that $2 - \sqrt{2} \geq 2 - x - \frac{1}{2x}$.

Proof:

```
theorem algebra_amgm_faxinrrp2msqrt2geq2mxm1div2x:
  fixes x :: real
  assumes "x > 0"
  shows "2 - sqrt 2 \<ge> 2 - x - 1/ (2 * x)"
  (* Step 1: Show that $\sqrt{2} \leq x + \frac{1}{2x}$.
    Step 2: Rearrange the inequality to get $2x^2 + 1 - 2x\sqrt{2} \geq 0$.
    Step 3: Factor the expression to get $(\sqrt{2}x - 1)^2 \geq 0$.
    Step 4: Conclude that the inequality is true. *)
proof -
  (* Step 1: Show that $\sqrt{2} \leq x + \frac{1}{2x}$. *)
  have c0: "2 * x > 0" using assms sledgehammer
  have "sqrt 2 \<le> x + 1 / (2 * x)"
  proof -
    define y where "y = sqrt 2"
    have c1: "2 = y * y"
    proof -
      have "2 = (sqrt 2) * (sqrt 2)" sledgehammer
      then have "... = y * y" using \<open>y = sqrt 2\<close> sledgehammer
      then show ?thesis sledgehammer
    qed
    have "(2 * x) * x + 1 - (2 * x) * (sqrt 2) = (y * y * x * x) + 1 - (2 * x) * y" using c1 y_def sledgehammer
    also have "... = (y*x) * (y*x) - 2 * (y*x) + 1" sledgehammer
    also have "... = (y*x - 1) * (y*x - 1)" sledgehammer
    also have "... \<ge> 0" sledgehammer
    ultimately have c2: "(2 * x) * x + 1 - (2 * x) * (sqrt 2) \<ge> 0" sledgehammer
    have "(2*x) * (x + 1/(2*x) - sqrt 2) = (2 * x) * x + (2 * x) * (1/(2*x)) - (2*x) * sqrt 2" sledgehammer
    also have "... = (2 * x) * x + 1 - (2*x) * sqrt 2" using c0 sledgehammer
    (* Step 2: Rearrange the inequality to get $2x^2 + 1 - 2x\sqrt{2} \geq 0$. *)
    also have "... \<ge> 0" using c2 sledgehammer
    (* Step 3: Factor the expression to get $(\sqrt{2}x - 1)^2 \geq 0$. *)
    ultimately have "(2*x) * (x + 1/(2*x) - sqrt 2) \<ge> 0" sledgehammer
    hence "x + 1/(2*x) - sqrt 2 \<ge> 0" using mult_pos_pos sledgehammer
    (* Step 4: Conclude that the inequality is true. *)
    then show ?thesis sledgehammer
  qed
  then show ?thesis sledgehammer
qed
```

Problem:
Suppose that x and y are two real numbers such that $\frac{x+y}{2} = 7$ and $\sqrt{xy} = \sqrt{19}$. Find $x^2 + y^2$.

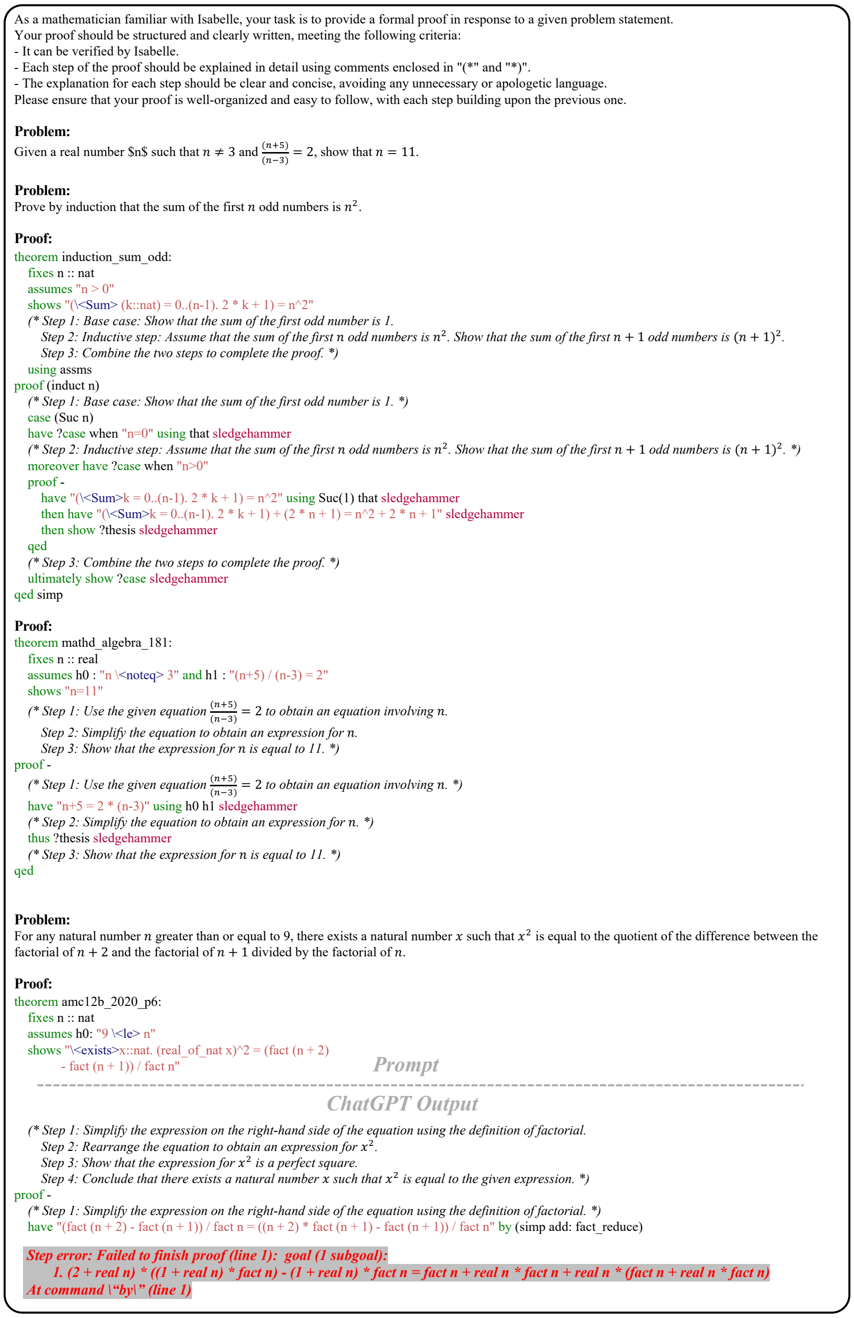
Proof:

```
theorem mathd_algebra_332:
  fixes x y ::real
  assumes "(x + y) / 2 = 7"
    and "sqrt (x * y) = sqrt 19"
  shows "x^2 + y^2 = 158"
```

```
  (* Step 1: Use the identity $x^2 + y^2 = (x + y)^2 - 2xy$ to find $x^2 + y^2$ in terms of $x$ and $y$.
    Step 2: Substitute the given values of $x$ and $y$ to find $x^2 + y^2$.
    Step 3: Show that $x^2 + y^2 = 158$. *)
proof -
  (* Step 1: Use the identity $x^2 + y^2 = (x + y)^2 - 2xy$ to find $x^2 + y^2$ in terms of $x$ and $y$.*)
  have c0: "x^2 + y^2 = (x+y)^2 - 2*x*y" sledgehammer
  (* Step 2: Substitute the given values of $x$ and $y$ to find $x^2 + y^2$. *)
  also have "... = 2*7^2 - 2*19 " using assms sledgehammer
```

Step error: Outer syntax error (line 1): command expected, but identifier error (line 1) was found At command \"<malformed>\" (line 1)

Figure 17. A case from miniF2F-test. The shuffled demonstration organization leads to an unsuitable conjecture, "... = $27^2 - 219$," which cannot be proved using the Sledgehammer.



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Problem:
Given a real number \$n\$ such that $n \neq 3$ and $\frac{(n+5)}{(n-3)} = 2$, show that $n = 11$.

Problem:
Prove by induction that the sum of the first n odd numbers is n^2.

Proof:

```
theorem induction_sum_odd:
  fixes n :: nat
  assumes "n > 0"
  shows "(\<Sum> (k::nat) = 0..(n-1). 2 * k + 1) = n^2"
  (* Step 1: Base case: Show that the sum of the first odd number is 1.
     Step 2: Inductive step: Assume that the sum of the first n odd numbers is n^2. Show that the sum of the first n + 1 odd numbers is (n + 1)^2.
     Step 3: Combine the two steps to complete the proof. *)
  using assms
proof (induct n)
  (* Step 1: Base case: Show that the sum of the first odd number is 1. *)
  case (Suc n)
  have ?case when "n=0" using that sledgehammer
  (* Step 2: Inductive step: Assume that the sum of the first n odd numbers is n^2. Show that the sum of the first n + 1 odd numbers is (n + 1)^2. *)
  moreover have ?case when "n>0"
  proof -
    have "(\<Sum>k = 0..(n-1). 2 * k + 1) = n^2" using Suc(1) that sledgehammer
    then have "(\<Sum>k = 0..(n-1). 2 * k + 1) + (2 * n + 1) = n^2 + 2 * n + 1" sledgehammer
    then show ?thesis sledgehammer
  qed
  (* Step 3: Combine the two steps to complete the proof. *)
  ultimately show ?case sledgehammer
qed simp
```

Proof:

```
theorem mathd_algebra_181:
  fixes n :: real
  assumes h0 : "n \<noteq> 3" and h1 : "(n+5) / (n-3) = 2"
  shows "n=11"
  (* Step 1: Use the given equation (n+5)/(n-3) = 2 to obtain an equation involving n.
     Step 2: Simplify the equation to obtain an expression for n.
     Step 3: Show that the expression for n is equal to 11. *)
proof -
  (* Step 1: Use the given equation (n+5)/(n-3) = 2 to obtain an equation involving n. *)
  have "n+5 = 2 * (n-3)" using h0 h1 sledgehammer
  (* Step 2: Simplify the equation to obtain an expression for n. *)
  thus ?thesis sledgehammer
  (* Step 3: Show that the expression for n is equal to 11. *)
qed
```

Problem:
For any natural number n greater than or equal to 9, there exists a natural number x such that x^2 is equal to the quotient of the difference between the factorial of $n + 2$ and the factorial of $n + 1$ divided by the factorial of n.

Proof:

```
theorem amc12b_2020_p6:
  fixes n :: nat
  assumes h0: "9 \<le> n"
  shows "\<exists>x::nat. (real_of_nat x)^2 = (fact (n + 2)
        - fact (n + 1)) / fact n"
```

Prompt

ChatGPT Output

```
  (* Step 1: Simplify the expression on the right-hand side of the equation using the definition of factorial.
     Step 2: Rearrange the equation to obtain an expression for x^2.
     Step 3: Show that the expression for x^2 is a perfect square.
     Step 4: Conclude that there exists a natural number x such that x^2 is equal to the given expression. *)
proof -
  (* Step 1: Simplify the expression on the right-hand side of the equation using the definition of factorial. *)
  have "(fact (n + 2) - fact (n + 1)) / fact n = ((n + 2) * fact (n + 1) - fact (n + 1)) / fact n" by (simp add: fact_reduce)
```

```
Step error: Failed to finish proof (line 1):  goal (1 subgoal):
    1. (2 + real n) * ((1 + real n) * fact n) - (1 + real n) * fact n = fact n + real n * fact n + real n * (fact n + real n * fact n)
At command \“by\” (line 1)
```

Figure 18. A case from miniF2F-test. The shuffling of the optimal demonstration organization leads to a complex and unresolved equation, which further results in a failure to properly simplify the factorial expression.