The Littlewood problem and non-harmonic Fourier series

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***Abstract*—The aim of this note is to prove a lower bound of the L^1-norm of non-harmonic trigonometric polynomials of the form**

$$C\sum_{k=1}^{N}\frac{|a_k|}{k}\leq\frac{1}{T}\int_{-T/2}^{T/2}|\sum_{k=1}^{N}a_k e^{2i\pi\lambda_k t}\,\mathrm{d}t$$

where $T>1$, $C=C(T)$ is an explicit constant that depends on T only, λ_k are real numbers with $|\lambda_k-\lambda_\ell|\geq 1$ and a_k are complex numbers. This extends to the non-harmonic case the Littlewood conjecture as solved by McGehee, Pigno, Smith [6] and Konyagin [5] and was previously obtained by Nazarov [7] with a non-explicit constant C.

***Index Terms*—Hardy's Inequality, Littlewood conjecture, Besicovitch norm**

I. Introduction

The aim of this note is to give a control of coefficients of a non-harmonic (non-periodic) trigonometric polynomial $\sum c_k e^{2i\pi\lambda_k t}$ in terms of its L^1-norm or its Besicovitch $\mathcal{B}^1$-norms. Recall that when $p<+\infty$, the Besicovitch $\mathcal{B}_p$-norms are defined by

$$\|\Phi\|_{\mathcal{B}_p}^p=\lim_{T\to+\infty}\frac{1}{T}\int_{[-T/2,T/2]}|\Phi(x)|^p\,\mathrm{d}x.$$

Those norms are the right substitute to $L^p([-1/2,1/2])$-norms to investigate non-harmonic trigonometric polynomials and their limits (that is, Besikovitch-almost periodic functions).

Let us start by recalling some facts about L^2-theory where those questions have a long ranging history. First, note that the problem is trivial for harmonic trigonometric polynomials ($\lambda_j=j$) when I has length at least 1, since the exponentials are orthogonal in $L^2([-1/2,1/2])$, so the estimate of the L^2-norm is given by Parseval's relation. The same is true for non-harmonic trigonometric polynomial in the $\mathcal{B}^2$-Besicovitch norm. The question becomes more interesting for $L^2(I)$-norms with I a fixed interval and has been answered in a celebrated paper by Ingham [3]. First the frequencies $(\lambda_j)_{j\geq1}$ need to be separated, so WLOG $|\lambda_j-\lambda_k|\geq 1$ and then $|I|$ needs to be >1 if one does not impose further regularity conditions on (λ_j) and then:

Theorem I.1 (Ingham)**.** *Let $(\lambda_j)_{j=1,\ldots,N}\subset\mathbb{R}$ be such that $|\lambda_j-\lambda_k|\geq 1$ and $T>1$. Let $C(t)=\dfrac{3\pi^2}{64}$ for $1<T\leq 2$ and $C(T)=\dfrac{\pi^2(T^2-1)}{8T^3}$ when $T>2$. Then, for every $(a_j)_{j=1,\ldots,N}\subset\mathbb{C}$,*

$$C(T)\sum_{k=1}^{N}|a_k|^2\leq\frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{k=1}^{N}a_k e^{2i\pi\lambda_k t}\right|^2\,\mathrm{d}t.$$

Of course, one can then take $N\to+\infty$. In functional analysis language, this states that the exponentials $(e^{2i\pi\lambda_j})$ form a Riesz sequence. This is of course closely related to the frame property of this system and can be traced back at least to the work of Duffin and Schaeffer [1] which lead to the development of frame theory.

When considering L^1-type norms the situation is much more delicate. The question was originally investigated by Littlewood in the 50s and concerned only (periodic) trigonometric polynomials with $\{0,1\}$ coefficients (so called idempotent trigonomatric polynomials) and it was speculated that such a polynomial had minimal L^1-norm when the frequencies (λ_j) formed an arithmetic sequence, that is by the L^1-norm of the Dirichlet kernel $\|D_N\|_1$. Littlewood actually conjectured that, at least up to a constant, $\|D_N\|_1$ provides the right lower bound of the L^1-norm of idempotent trigonomatric polynomials. The question was solved in the early 80s independently by McGehee, Pigno, Smith [6] and Konyagin [5]. Moreover, both papers proved the conjecture as a corollary of a statement for arbitrary periodic trigonometic polynomials. The result relevant for this paper is the following:

Theorem I.2 (McGehee, Pigno, Smith [6])**.** *There exists a constant C such that, for every N, if $\lambda_1,\ldots,\lambda_N$ are distinct* integers *and $(a_k)_{k=1,\ldots,N}\subset\mathbb{C}$ then*

$$C\sum_{k=1}^{N}\frac{|a_k|}{k}\leq\int_{-1/2}^{1/2}\left|\sum_{k=1}^{N}a_k e^{2i\pi\lambda_k t}\right|\mathrm{d}t.$$

In [6], the constant C is explicit, $C = \dfrac{1}{30}$ and is valid for arbitrary a_k's. When the a_k's are restricted to $\{0,1\}$, this can be improved to $C = \dfrac{4}{\pi^3}$ (*see* [9], [10]). It is still a conjecture that this theorem is valid for the best (largest) possible constant $\dfrac{4}{\pi^2}$ (which is obtained by taking $\lambda_k = k$ and $a_k = 1$, in view of standard estimates of the Dirichlet kernel).

In view of Ingham's work (and its applications), it is then natural to ask for corresponding results when the frequencies are real. A clever approximation argument allows to deduce the result from Theorem I.2 (the converse direction being obvious):

Theorem I.3 (Hudson & Leckband)**.** *For every N, if $\lambda_1, \dots, \lambda_N$ are distinct* real *numbers and $(a_k)_{k=1,\dots,N} \subset \mathbb{C}$ then*

$$C \sum_{k=1}^{N} \frac{|a_k|}{k} \leq \lim_{T\to+\infty} \frac{1}{T} \int_{[-T/2,T/2]} \left| \sum_{k=1}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t$$

where C is the constant in Theorem I.2.

The next natural question is then to see what happens for $L^1([-T/2,T/2])$-norms rather than $\mathcal{B}^1$-norms. Note that such a result implies Theorem I.3 and it turns out that the question was answered a bit earlier by Nazarov [7] and, as for Ingham, requires $T > 1$. However, the constant in [7] is not explicit. In a recent paper [4], we have shown how a modification of Nazarov's argument allows to obtain this result directly and further to obtain the best constants known today:

Theorem I.4. *Let $\lambda_1 < \lambda_2 < \cdots < \lambda_N$ be N distinct real numbers and $a_1, \dots, a_N$ be complex numbers. Then*

i) we have

$$\lim_{T\to+\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k=1}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t \geq \frac{1}{26} \sum_{k=1}^{N} \frac{|a_k|}{k+1}. \quad \text{(I.1)}$$

ii) If further $a_1, \dots, a_N$ all have modulus larger than 1, $|a_k| \geq 1$ then

$$\lim_{T\to+\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k=1}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t \geq \frac{4}{\pi^3} \ln N.$$

iii) Assume further that for $k \neq \ell \in \{1, \dots, N\}$, $|\lambda_k - \lambda_\ell| \geq 1$, then for $T \geq 72$ we have

$$\frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k=1}^{N} a_k e^{2i\pi\lambda_k t} \right| \mathrm{d}t \geq \frac{1}{122} \sum_{k=1}^{N} \frac{|a_k|}{k+1}.$$

Let us make two observations. Nazarov's result is the third statement of this theorem and, further, is valid for every $T > 1$ (which is better than our result) but with a non-explicit constant C instead of $\dfrac{1}{122}$ (our computations show that Nazarov's proof gives something of the order of $(T-1)^{-12}$). To obtain our constant, we introduce an auxilary function φ, in (II.2), that differs from the one used by Nazarov. While Nazarov's function is better adapted for small intervals, our construction allows to exploit the fact that the interval $[-T/2, T/2]$ is large. A second difference with Nazarov's proof, is that we introduce several parameters that are optimised in the last step of the proof. In order to present a simpler argument, we here avoid introducing those parameters and do not explicit constants. The constants obtained with this simpler proof are anyway far from those of the statement of Theorem I.4. To further simplify things, we also restrict our attention to (I.1), to which we devote the remaining of the paper.

II. The proof of (I.1)

By $A \lesssim B$ we mean $A \leq CB$ for some C.

Next observe that in (I.1), we may scale the sequence λ_j and thus assume that $|\lambda_j - \lambda_k| \geq 1$.

We write $|a_j| = a_j u_j$, $|u_j| = 1$. Then we define

$$\Phi(t) = \sum_{j=1}^{N} a_j e^{2i\pi\lambda_j t},\ U(t) = \sum_{j=1}^{N} \frac{u_j}{j+7} e^{2i\pi\lambda_j t}$$

$$S = \sum_{j=1}^{N} \frac{|a_j|}{j+7}. \quad \text{(II.1)}$$

Here the $+7$ is harmless and only changes numerical constants (and amounts to adding 7 zeroes at the beginning of the sequences (a_j) and 7 extra frequencies). We are going to prove the following: take $T = 2K$ a large enough even integer, then

$$S \lesssim \frac{1}{2K} \int_{[-K,K]} |\Phi(t)| \,\mathrm{d}t.$$

Our modifications, more precisely improvements on Nazarov arguments happens here and in the following step.

A. First step: an auxilary function

We now define

$$\varphi = 2\mathbf{1}_{[-K/2,K/2]} * \bigstar_k \mathbf{1}_{[-1/2,1/2]} \quad \text{(II.2)}$$

where $\bigstar_k \psi$ denotes the k-fold convolution of ψ by itself. More precisely,

$$\bigstar_2 \psi(x) = \psi(x) * \psi(x) = \int_{\mathbb{R}} \psi(t)\psi(x-t)\,\mathrm{d}t$$

is the usual convolution of ψ by itself and, for $k \geq 2$, we then define $\bigstar_{k+1}\psi = \bigstar_k\psi * \psi$.

The Fourier transform of φ is

$$\mathcal{F}[\varphi](\lambda) = 2\frac{\sin K\pi\lambda}{\pi\lambda}\left(\frac{\sin\pi\lambda}{\pi\lambda}\right)^K$$

and one easily checks that φ is even, non negative with $\|\varphi\|_\infty \leq 2$ and $\mathcal{F}[\varphi](0) = \|\varphi\|_1 = 2K$ while $|\mathcal{F}[\varphi](\lambda)| \leq \dfrac{2K}{(\pi\lambda)^K}$. Then

Lemma II.1. *For $1 \leq k \leq N$,*

$$\sum_{\substack{1\leq j\leq N\\ j\neq k}} \frac{|\mathcal{F}[\varphi](\lambda_j - \lambda_k)|}{j+7} \leq \frac{1}{2}\frac{1}{k+7}. \qquad \text{(II.3)}$$

Proof. We split the sum into two parts, E_1 where we sum of the $j's$ for which $j+7 \geq (k+7)/2$ and E_2 for the remaining terms. To estimate E_1, we note that $|\lambda_j - \lambda_k| \geq |j-k|$ and use the bound on $\mathcal{F}[\varphi]$ to get

$$\begin{aligned} E_1 &\leq \frac{2K}{\pi^K(k+7)}\sum_{\substack{1\leq j\leq N\\ j\neq k}}\frac{1}{|j-k|^K} \\ &\leq \frac{4K}{\pi^K(k+7)}\sum_{\ell=1}^{\infty}\frac{1}{\ell^K} \leq \frac{1}{4(k+7)} \end{aligned}$$

if K is large enough.

For the second sum, note that $\lambda_k - \lambda_j \geq k - j \geq (k+7)/2$. The bound on $\mathcal{F}[\varphi]$ then gives

$$\begin{aligned} E_2 &\leq \sum_{j+7<(k+7)/2}|\mathcal{F}[\varphi](\lambda_j-\lambda_k)| \\ &\leq k\frac{K2^K}{\big(\pi(k+7)\big)^K} \end{aligned}$$

since there are at most k terms in this sum. It remains to notice that

$$E_2 \leq \frac{k}{k+7}\frac{K2^{K+1}}{\pi^K 7^{K-2}}\frac{1}{k+7} \leq \frac{1}{4(k+7)}$$

if K is large enough. □

From this, we deduce the following:

Corollary II.2. *For $k = 1, \ldots, N$*

$$\left|\frac{1}{2K}\int_{[-K,K]} U(t)e^{2i\pi\lambda_k t}\varphi(t)\,\mathrm{d}t - \frac{u_k}{k+7}\right| \leq \frac{1}{4K}\frac{1}{k+7}. \qquad \text{(II.4)}$$

Proof. Indeed, we write the integral as

$$\begin{aligned} &= \sum_{j=1}^{N}\frac{u_j}{j+7}\frac{1}{2K}\int_{[-K,K]} e^{-2i\pi\lambda_j t}e^{2i\pi\lambda_k t}\varphi(t)\,\mathrm{d}t \\ &= \sum_{j=1}^{N}\frac{u_j}{j+7}\frac{1}{2K}\mathcal{F}[\varphi](\lambda_j-\lambda_k). \end{aligned}$$

Separating the term $j = k$ and estimating the remaing ones with lemma II.1 gives the result. □

It remains to multiply (II.4) by $|a_k|$, and then to sum over k on both sides to obtain

Proposition II.3. *If K is large enough*

$$S \leq \frac{5}{4}\left|\frac{1}{2K}\int_{[-K,K]} U(t)\Phi(t)\varphi(t)\,\mathrm{d}t\right|.$$

Here we need an explicit constant ($5/4$ is an upper bound of $4K/(4K-1)$ when K is large enough).

B. Modification of U into a uniformly bounded V

First, up to adding a few zero terms at the end of the sequence (a_j), we can assume that $N+7 = 2^{n+1}-1$. We will now decompose U into dyadic blocs: write $\mathcal{D}_j = \{k \in \mathbb{N} : 2^j \leq k < 2^{j+1}\}$ and

$$f_j(t) = \sum_{r+7\in\mathcal{D}_j}\frac{u_r}{r+7}e^{-2i\pi\lambda_r t}$$

so that $U = \sum_{j=3}^{n} f_j$. We will now modify U into an other sum V that is near to U but uniformly bounded. We start by estimating the norms of the f_j's:

Lemma II.4. *With the above notation, we have*

1) $\|f_j\|_{L^2([-K,K])} \leq 2^{-\frac{j}{2}}\sqrt{2K+1}$;
2) $\|f_j\|_\infty \leq 1$.

Proof. This is a standard fact. Set $v_r = \dfrac{u_r}{r+7}$, then

$$\|f_j\|^2_{L^2([-K,K])} = \sum_{r,s\in\mathcal{D}_j-7} v_r\overline{v_s}\int_{-K}^{K} e^{2i\pi(\lambda_s-\lambda_r)t}\,\mathrm{d}t.$$

One then isolates the term $r = s$ in the sum and estimate the remaining ones using Hilbert's Inequality [8]. □

For a function $F \in L^2(I_K)$ and $s \in \mathbb{Z}$, we write its Fourier coefficients

$$c_s^K(F) = \frac{1}{2K}\int_{-K}^{K} F(t)e^{-i\pi\frac{st}{K}}\,\mathrm{d}t.$$

To each $|f_j| \in L^2(I_K)$ we associate $h_j \in L^2(I_K)$ defined via its Fourier series as

$$h_j(t) = c_0^K(|f_j|) + 2\sum_{s=1}^{\infty} c_s^K(|f_j|)e^{-i\pi\frac{st}{K}}.$$

Lemma II.5. *For $3 \leq j \leq n$, the following two properties hold*

1) $\mathrm{Re}(h_j) = |f_j| \geq 0$ *and* $\Re(h_j) \in L^\infty([-K,K])$.

2) $\|h_j\|_{L^2([-K,K])} \lesssim 2^{-\frac{j}{2}}\sqrt{2K+1}$.

Proof. First, as $|f_j|$ is real valued, $c_0^K(|f_j|)$ is also real, and $\overline{c_s^K(|f_j|)} = c_{-s}^K(|f_j|)$ for every $s \geq 1$. A direct computation then shows that $\mathrm{Re}(h_j) = |f_j|$ while Parseval shows that $\|h_j\|_2 \leqslant \sqrt{2}\|f_j\|_2$.

Finally, as f_j is a trigonometric polynomial, it is a bounded function thus so is $\mathrm{Re}(h_j)$. □

We now take a parameter $0 < \eta \leq 1$ that will be a small number ($\eta = 10^{-3}$ would do) and define a sequence $(F_j)_{j=3,\dots,n}$ inductively through

$$F_3 = f_3 \quad \text{and} \quad F_{j+1} = F_j e^{-\eta h_{j+1}} + f_{j+1}.$$

Lemma II.6. *For $3 \leq j \leq n$, $\|F_j\|_\infty \lesssim \eta^{-1}$.*

Proof. Set $E := \sup_{0<x\leq 1} \dfrac{x}{1-e^{-\eta x}} = \dfrac{1}{1-e^{-\eta}}$ and note that $\eta^{-1} \leq E \leq 2\eta^{-1}$.

By definition of E, if $C \leq E$ and $0 \leq x \leq 1$, then $Ce^{-\eta x} + x \leq Ee^{-\eta x} + x \leq E$.

We can now prove by induction over j that $|F_j| \leq E$ from which the lemma follows. First, when $j = m$, from Lemma II.4 we get $\|F_3\|_\infty = \|f_3\|_\infty \leq 1 \leq E$.

Assume now that $\|F_j\|_\infty \leq E$, then

$$\begin{aligned} |F_{j+1}(t)| &= |F_j(t)e^{-\eta h_{j+1}(t)} + f_{j+1}(t)| \\ &\leq |F_j(t)|e^{-\eta \Re\left(h_{j+1}(t)\right)} + |f_{j+1}(t)| \\ &= |F_j(t)|e^{-\eta |f_{j+1}(t)|} + |f_{j+1}(t)|. \end{aligned}$$

As $|f_{j+1}(t)| \leq 1$ and $|F_j(t)| \leq E$, we get $|F_{j+1}(t)| \leq E$ as claimed. □

Lemma II.7. *For $3 \leq k \leq n$ and $j = 3,\dots,k$ let $g_{j,k} = e^{-\eta H_{j,k}}$ with*

$$H_{j,k} = \begin{cases} h_{j+1} + \ldots + h_k & \text{when } j < k \\ 0 & \text{when } j = k \end{cases}.$$

Then $F_k = \displaystyle\sum_{j=3}^{k} f_j g_{j,k}$. Moreover

$$\|H_{j,k}\|_{L^2([-K,K])} \lesssim 2^{-\frac{j}{2}}\sqrt{2K+1}.$$

Proof. The first part is easily obtained by induction on k. For the second, the triangular inequality and Lemma II.5 give a sum of a geometric series. □

The next lemma is a direct consequence of [6, Lemma, p 614]:

Lemma II.8. *Let $3 \leq k \leq n$ and $3 \leq j \leq k$, then*

1) *the negative Fourier coefficients of $g_{j,k}(t) - 1$ vanish so that its Fourier series writes*

$$g_{j,k}(t) - 1 = \sum_{s\geqslant 0} c_s^K(g_{j,k} - 1)e^{i\pi\frac{st}{K}};$$

2) *$\|g_{j,k} - 1\|_{L^2([-K,K])} \lesssim \eta 2^{-\frac{j}{2}}\sqrt{2K+1}$ so that $|c_s^k(g_{j,k} - 1)| \lesssim 2^{-\frac{j}{2}}\sqrt{2K+1}$.*

We now recall the definition of U and define V:

$$U(t) = \sum_{j=3}^{n} f_j \text{ and } V = F_n = \sum_{j=3}^{n} f_j g_{j,n}.$$

Proposition II.9. *There is an η such that, for K large enough and for $1 \leq k \leq N$ we have*

$$\left|\frac{1}{2K}\int_{[-K,K]} \big(U(t) - V(t)\big)e^{2i\pi\lambda_k t}\varphi(t)\,\mathrm{d}t\right| \leq \frac{1}{2}\frac{1}{k+7}.$$

Proof. To simplify notation, we will simply write $g_j = g_{j,n}$. We fix $k \in [1, N]$ and denote by ℓ the unique integer such that $k + 7 \in \mathcal{N}_\ell$. We want to bound $R_1 + R_2$ where

$$R_1 = \frac{1}{2K}\int_{[-K,K]} \sum_{3\leq j\leq \ell-2} f_j(g_j - 1)e^{2i\pi\lambda_k t}\varphi(t)\,\mathrm{d}t,$$

and

$$R_2 = \frac{1}{2K}\int_{[-K,K]} \sum_{\ell-1\leq j\leq n} f_j(g_j - 1)e^{2i\pi\lambda_k t}\varphi(t)\,\mathrm{d}t.$$

Let us first bound R_1. For this, notice that if $s \in \mathbb{Z}$, then

$$\begin{aligned} &\int_{[-K,K]} f_j(t)\varphi(t)e^{2i\pi\lambda_k t}e^{-2i\pi\frac{s}{K}t}\,\mathrm{d}t \\ &= \int_{[-K,K]} \sum_{r+7\in\mathcal{N}_j} \frac{u_r}{r+7}\varphi(t)e^{-2i\pi(\lambda_r - \lambda_k + \frac{s}{K})}\,\mathrm{d}t \\ &= \sum_{r+7\in\mathcal{N}_j} \frac{u_r}{r+7}\mathcal{F}[\varphi]\left(\lambda_r - \lambda_k + \frac{s}{K}\right). \end{aligned}$$

It follows that $\displaystyle\int_{[-K,K]} f_j(g_j - 1)e^{2i\pi\lambda_k t}\varphi(t)\,\mathrm{d}t =$

$$\begin{aligned} &\int_{[-K,K]} f_j(t)\varphi(t)e^{2i\pi\lambda_k t}\sum_{s\geq 0} c_s^K(g_j - 1)e^{2i\pi\frac{st}{K}}\,\mathrm{d}t \\ &= \sum_{s=0}^{+\infty} c_s^K(g_j - 1)\int_{[-K,K]} f_j(t)\varphi(t)e^{2i\pi\lambda_k t + \frac{2i\pi st}{K}}\,\mathrm{d}t \\ &= \sum_{s=0}^{+\infty} c_s^K(g_j - 1)\times \\ &\qquad \times \sum_{r+M\in\mathcal{N}_j} \frac{u_r}{r+7}\mathcal{F}[\varphi]\left(\lambda_r - \lambda_k - \frac{s}{K}\right). \end{aligned}$$

Inverting both sums and injecting the result in the definition of R_1, we obtain

$$R_1 = \frac{1}{2K}\sum_{3\leq j\leq \ell-2}\sum_{r+7\in\mathcal{N}_j}\frac{u_r}{r+M}\times$$
$$\times\sum_{s=0}^{\infty}c_s^K(g_j-1)\mathcal{F}[\varphi]\left(\lambda_r-\lambda_k-\frac{s}{K}\right).$$

Now set $c_s(r)=c_s^K(g_j-1)$ when $r+7\in\mathcal{N}_j$ and use parity of $\mathcal{F}[\varphi]$ to rewrite

$$R_1 = \frac{1}{2K}\sum_{8\leq r+7<2^{\ell-1}}\frac{u_r}{r+7}$$
$$\times\sum_{s=0}^{\infty}c_s(r)\mathcal{F}[\varphi]\left(\lambda_k-\lambda_r+\frac{s}{K}\right).$$

Using $|c_s(r)|\lesssim\sqrt{2K+1}$ we get

$$|R_1|\lesssim\sqrt{2K+1}\sum_{8\leq r+7<2^{\ell-1}}\frac{1}{r+7}E_r \qquad \text{(II.5)}$$

with

$$E_r=\sum_{s=0}^{\infty}\frac{1}{2K}\left|\mathcal{F}[\varphi]\left(\lambda_k-\lambda_r+\frac{s}{K}\right)\right|.$$

To bound E_r, one first notices that the choice of ℓ and of the cutoff in the sum implies that

$$\lambda_k-\lambda_r-1\geq k-r-1\geq\frac{1}{8}(k+7).$$

One then uses the bound on $\mathcal{F}[\varphi]$ and compare the sum in E_r with an integral to get

$$E_r\lesssim\frac{8^K}{\pi^K}\frac{1}{(k+7)^{K-1}}.$$

Then

$$|R_1| \lesssim \frac{8^K}{\pi^K}\frac{\sqrt{2K+1}}{(k+7)^{K-1}}\sum_{8\leq r+7\leq 2^{\ell-1}}\frac{1}{r+7}$$
$$\lesssim \frac{8^K}{\pi^K}\frac{\sqrt{2K+1}}{(k+7)^{K-2}}$$

since the sum has at most $2^{\ell-1}\lesssim k+M$ terms ≤ 1. We thus obtain, for K large enough, $R_1=o(k^{-1})$.

As for R_2, using the fact that φ is bounded and Cauchy-Schwarz we get that $|R_2|$ is

$$\lesssim\sum_{\ell-1\leq j\leq n}\frac{1}{2K}\|f_j\|_{L^2([-K,K])}\|g_j-1\|_{L^2([-K,K])}$$
$$\lesssim\frac{2K+1}{2K}\eta\sum_{\ell-1\leq j\leq n}2^{-j}\lesssim\frac{\eta}{k+7}$$

since we have chosen ℓ so that $2^{-\ell-1}\leq\dfrac{1}{k+7}$.

It follows that

$$|R_1|+|R_2|\lesssim\big(\eta+o(1)\big)\frac{1}{k+7}\leq\frac{1}{2}\frac{1}{k+7}$$

if η is a small enough number (that depends only on the constant in the bound of R_2) and K is large enough. □

Corollary II.10.

$$\left|\frac{1}{2K}\int_{[-K,K]}\big(U(t)-V(t)\big)\Phi(t)\varphi(t)\,\mathrm{d}t\right|\leq\frac{1}{2}S$$

Proof. As $\Phi(t)=\sum_{k=1}^N a_k e^{2i\pi\lambda_k t}$, it suffices to use the triangular inequality and Proposition II.9. □

C. End of the proof

The end of the proof consists in applying first Proposition II.3

$$S\leq\frac{5}{4}\left|\frac{1}{2K}\int_{[-K,K]}U(t)\Phi(t)\varphi(t)\,\mathrm{d}t\right|.$$

Then, applying Corollary II.10 we get

$$S \leq \frac{5}{4}\left|\frac{1}{2K}\int_{[-K,K]}\big(U(t)-V(t)\big)\Phi(t)\varphi(t)\,\mathrm{d}t\right|$$
$$+\frac{5}{4}\left|\frac{1}{2K}\int_{[-K,K]}V(t)\Phi(t)\varphi(t)\,\mathrm{d}t\right|$$
$$\leq \frac{5}{8}S+\frac{5}{4}\|V\|_\infty\|\varphi\|_\infty\frac{1}{2K}\int_{[-K,K]}|\Phi(t)|\,\mathrm{d}t$$

This gives the desired bound

$$S\lesssim\frac{1}{2K}\int_{[-K,K]}|\Phi(t)|\,\mathrm{d}t.$$

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