# Sampling theorems in spaces of variable bandwidth generated via Wilson basis

**Beatrice Andreolli**
Faculty of Mathematics, University of Vienna
Oskar-Morgenstern-Platz 1,
1090 Vienna, Austria
Email: beatrice.andreolli@univie.ac.at

**Karlheinz Gröchenig**
Faculty of Mathematics, University of Vienna
Oskar-Morgenstern-Platz 1,
1090 Vienna, Austria
Email: karlheinz.groechenig@univie.ac.at

***Abstract*—We propose a new sampling theorem, the complete reconstruction of a function from its samples, for the space of variable bandwidth constructed using Wilson expansions. The theorem is based on the maximal gap between consecutive points and it relates the lower sampling rate to the bandwidths that have an influence on the reconstruction on each particular interval of the signal.**

## I. Motivation

A function $f \in L^2(\mathbb{R})$ is said to be bandlimited if it belongs to the so called Paley-Wiener space

$$PW_\Omega = \{f \in L^2(\mathbb{R}) : \mathrm{supp}(\hat{f}) \subseteq [-\Omega, \Omega]\}.$$

The well-studied theory of bandlimited functions is grounded on the observation that in practical applications, signal frequencies above a finite cut-off are negligible. However, mathematically, bandlimited functions are entire functions of time and therefore represent infinite signals. Real-world signals are, by nature, time-limited, and thus they require an appropriate description that accounts for this fact. According to the uncertainty principle [14], this involves the use of functions that are not bandlimited.

As a result, the concept of variable bandwidth arises naturally and it is particularly intuitive when considering music, where the highest frequency varies with time. This motivates the allowance of different local bandwidths for different intervals of a signal in mathematical representation.

In the literature, several approaches to variable bandwidth have been proposed.

1) A model that involves applying an invertible transformation on time, known as time-warping function $\gamma$, to bandlimited signals was proposed [7], [12], [21], [23], [24]. Here, the derivative $1/\gamma'(\gamma^{-1}(x))$ of the warping function is interpreted as the local bandwidth of the resulting function of variable bandwidth at $x$.
2) Aceska and Feichtinger [3], [4] define the space of functions of variable bandwidth as a weighted modulation space determined by a so-called variable bandwidth weight, which is essentially a time-varying frequency cut-off.
3) A definition of variable bandwidth that is based on the spectral subspaces of an elliptic operator $A_p f = -\frac{d}{dx}(p(x)\frac{d}{dx})f$, where $p > 0$ is a strictly positive function, was proposed in [10], [19].

The approach that we adopt is described in [6] and it uses a discrete version of the space of Aceska and Feichtinger [3], [4], where the short-time Fourier transform is replaced by a frequency truncation of a Wilson expansion.

The Wilson basis is an orthonormal basis for $L^2(\mathbb{R})$ that was introduced by Daubechies, Jaffard, and Journé in 1991 [13]. The Wilson basis functions have the following form:

$$\psi_{n,l}(x) = \begin{cases} g(x-n), & l = 0 \\ \frac{1}{\sqrt{2}}(e^{2\pi i l x} + (-1)^{l+n} e^{-2\pi i l x}) g(x - n/2), & l \neq 0, \end{cases} \tag{I.1}$$

for $n, l \in \mathbb{Z}$ and $l \geq 0$,

where the window function $g$ can be chosen to be $\mathcal{C}^\infty$ with compact support. This basis is significant because it overcomes the limitations of the Balian-Low theorem, which states that a function $g$ that generates a Gabor orthonormal system cannot be well-localized in both time and frequency. The Wilson basis provides time-frequency localization while preserving much of the structure of a Gabor system. When a system of the form (I.1) forms a Riesz basis, is called a Wilson Riesz basis. The Wilson basis has proved to be a valuable tool in time-frequency analysis and signal processing since its introduction in 1991 [8], [9], [16]. One of its most notable applications is in the detection of gravitational waves [1], [2], [11].

We define the spaces of variable bandwidth as in [6]. By letting $b : \mathbb{Z} \to \mathbb{N}$ be a bounded positive sequence such that $b(n) \leq B < \infty$, for every $n \in \mathbb{Z}$ and $\{\psi_{n,l}\}_{n\in\mathbb{Z},l\in\mathbb{N}}$ be the orthonormal Wilson basis as in (I.1), we denote with $PW_b^2(g,\mathbb{R}) \subseteq L^2(\mathbb{R})$ the Paley-Wiener-type subspace

$$PW_b^2(g,\mathbb{R}) := \Big\{ \sum_{n\in\mathbb{Z}} \sum_{l=0}^{b(n)} c_{n,l}\psi_{n,l} \in L^2(\mathbb{R}), \quad c \in \ell^2(\mathbb{Z}\times\mathbb{N}) \Big\}. \tag{I.2}$$

In this work, we study sufficient conditions for sampling for the variable bandwidth space $PW_b^2(g,\mathbb{R})$. Sampling is a fundamental concept in signal processing that involves the process of measuring a continuous signal at a finite set of discrete points in time [22]. In this regard, a set $\Lambda \subset \mathbb{R}$ is called a set of (stable) sampling if there exist constants $A, B > 0$ such that for all $f \in PW_b^2(g,\mathbb{R})$, the following sampling inequality holds:

$$A\|f\|_2^2 \leq \sum_{\lambda\in\Lambda} |f(\lambda)|^2 \leq B\|f\|_2^2. \tag{I.3}$$

The sampling inequality (I.3) indicates that all the information carried by the function is captured by the evaluation of the function at the samples. This is equivalent to requiring the reproducing kernel sequence $\{k_\lambda\}_{\lambda\in\Lambda}$ to be a frame for $PW_b^2(g,\mathbb{R})$.

Determining sufficient conditions for sampling (SCS) is a very challenging task that consists in investigating under which conditions on a sampling set $\Lambda$ a function in $PW_b^2(g,\mathbb{R})$ can be reconstructed completely. In practice, this implies that one needs to take more samples than the minimum required by the critical density to achieve a desirable level of accuracy in the reconstruction of a function. The critical density is, therefore, a theoretical concept that sets a lower bound on the number of samples required for reconstruction. The lower bound, also called lower Beurling density $D^-(\Lambda)$, has been studied in [6] by providing necessary density conditions for sampling (NDCS) (Theorem II.2 and II.3).

Section II contains the results about necessary density conditions for sampling, while in Section III the main results about sufficient conditions for sampling are presented.

## II. Necessary density conditions

Defining the usual translation and modulation operators $T_n f(t) = f(t-n)$, and $M_l f(t) = f(t)e^{2\pi i l t}$, we can rewrite the entire collection of functions that forms a Wilson basis (I.1) as

$$\psi_{n,l} = d_l(M_l + (-1)^{l+n}M_{-l})T_{\frac{n}{2}}g$$

for $(n,l) \in \mathbb{Z}^2, l \geq 0$ where $d_0 = \frac{1}{2}$ and $d_l = \frac{1}{\sqrt{2}}$, $l \geq 1$, $\psi_{2n,0} = T_n g$, $\psi_{2n+1,0} = 0$. Let

$$P_n := \sum_{l=0}^{b(n)} c_{n,l} d_l (M_l + (-1)^{l+n}M_{-l})$$

be the trigonometric polynomial of degree $b(n)$. Then, every function $f \in PW_b^2(g,\mathbb{R})$ is given by

$$f(x) = \sum_{n\in\mathbb{Z}} P_n(x) g(x - n/2). \tag{II.1}$$

For a compactly supported window $g$, the parameter $b(n)$ can be also understood as the local bandwidth of $f$ on an interval centered at $n/2$. Considering the restriction $f|_{[n/2-1/2,n/2+1/2]}$, this is approximately a trigonometric polynomial of degree $b(n)$. As a consequence, at least $b(n)+1$ samples are expected to be required within each interval $[n/2-1/2, n/2+1/2]$ to recover $f$ completely. It turns out that the real challenge is to understand the influence of the overlap of different translations of the window $g$.

In the work [6], the main results about necessary density conditions for sampling for $PW_b^2(g,\mathbb{R})$ are presented.

Firstly, one needs to prove that the variable bandwidth space defined using a Wilson basis is a reproducing kernel Hilbert space.

**Lemma II.1.** *The space $PW_b^2(g,\mathbb{R})$ of variable bandwidth defined in* (I.2) *is a reproducing kernel Hilbert space with reproducing kernel*

$$k(x,y) = \sum_{n\in\mathbb{Z}} \sum_{l=0}^{b(n)} \overline{\psi_{n,l}(y)}\psi_{n,l}(x).$$

To show that at each point in $\mathbb{R}$, the linear functional is bounded, the characterization of Wilson basis in [18, Corollary 8.5.4] is used.

The first result about necessary density conditions for the case of compactly supported windows $g$ states as follows.

**Theorem II.2** (NDCS - compact support)**.** *Let $g \in \mathcal{C}(\mathbb{R})$, real-valued, and even with $\operatorname{supp}(g) \subseteq [-m,m]$. Let $PW_b^2(g,\mathbb{R})$ be the space of variable bandwidth defined in* (I.2) *and let $\Lambda \subseteq \mathbb{R}$ be a set of sampling for $PW_b^2(g,\mathbb{R})$. Then, every open interval $(\alpha,\beta)$ contains at least*

$$\lceil \beta - \alpha - 2m \rceil + \sum_{n/2\in[\alpha+m,\beta-m]} b(n)$$

*points of* $\Lambda$.

From Lemma II.1, the result follows by applying an easy argument that involves counting dimensions as in [5].

A more general result holds for basis functions $g$ with unbounded support.

**Theorem II.3** (NDCS)**.** *Let* $g \in \mathcal{C}(\mathbb{R})$*, real-valued, even, and such that* $|g(x)| \le C(1+|x|)^{-1-\epsilon}$ *for* $C > 0$*,* $\epsilon > 0$*. Let* $PW_b^2(g,\mathbb{R})$ *be the space of variable bandwidth defined in* (I.2) *with* $b(n) \ge 1$ *for every* $n \in \mathbb{Z}$ *and let* $\Lambda \subseteq \mathbb{R}$ *be a set of sampling for* $PW_b^2(g,\mathbb{R})$*. Then*

$$D^-(\Lambda) := \liminf_{r\to\infty} \inf_{x\in\mathbb{R}} \frac{\#(\Lambda \cap B_r(x))}{2r} \ge 1 + \overline{b}$$

*where* $\overline{b} = \liminf_{r\to\infty} \inf_{x\in\mathbb{R}} \frac{1}{2r} \sum_{\frac{n}{2}\in B_r(x)} b(n)$.

$\overline{b}$ represents a sort of average bandwidth over an interval of length $2r$. The theorem shows that whenever the window function $g$ has good decay properties, the lower Beurling density $D^-(\Lambda)$ confirms our expectations of having at least $1+\overline{b}$ samples of $\Lambda$ contained in a ball of radius $r$.

*Remark.* We would like to point out that for windows $g$ which are compactly supported, Theorem II.2 implies Theorem II.3.

## III. Sufficient conditions for sampling

In this section, we investigate under which sufficient conditions on a sampling set a function $f \in PW_b^2(g,\mathbb{R})$ can be reconstructed completely.

The main ingredients to get sufficient conditions for sampling are Wirtinger's inequality and Bernstein's inequality.

**Lemma III.1** (Wirtinger's inequality)**.** *If* $f, f' \in L^2(a,b)$*,* $a < c < b$*, and* $f(c) = 0$*, then*

$$\int_a^b |f(x)|^2 dx \le \frac{4}{\pi^2} \max\{(b-c)^2, (c-a)^2\} \int_a^b |f'(x)|^2 dx. \tag{III.1}$$

The inequality follows from [20, p.184], by applying a change of variables.

Bernstein's inequality provides a bound for the derivative of a trigonometric polynomial.

**Lemma III.2** (Bernstein's inequality)**.** *Let* $P$ *be a trigonometric polynomial of degree* $n$ *i.e.*

$$P(x) = \sum_{|k|\le n} c_k e^{2\pi i k x}.$$

*Then*

$$\|P'\|_2 \le 2\pi n \|P\|_2. \tag{III.2}$$

Define the set of points $\Lambda$ as follows

$$\Lambda := \{x_{k/2,j} = k/2 + \mu_{k/2,j} : k \in \mathbb{Z},\ \mu_{k/2,j} \in [0,1/2)\} \tag{III.3}$$

where the sampling points in $\Lambda$ are ordered by magnitude,

$$\ldots < x_{k/2,j} < x_{k/2,j+1} < \ldots$$
$$\ldots < x_{(k+1)/2,j} < x_{(k+1)/2,j+1} < \ldots$$

and $\lim_{k\to\pm\infty} x_{k/2} = \pm\infty$. The sampling density is measured by the maximal gap between the sample, i.e.

$$\delta_{k/2} = \sup_{j=1,\ldots j_{\max(k/2)}} (x_{k/2,j+1} - x_{k/2,j}),$$

where the number $j_{\max(k/2)}$ depends on the specific interval $[k/2, k/2+1/2)$.

We present our new result that provides sufficient conditions for sampling for $PW_b^2(g,\mathbb{R})$.

**Theorem III.3** (SCS)**.** *Let* $g \in \mathcal{C}^1(\mathbb{R})$ *be real-valued and even, with* $\operatorname{supp}(g) \subseteq [-\frac{1}{2}-\epsilon, \frac{1}{2}+\epsilon]$ *and* $0 < \epsilon < 1/2$*.*
*Let* $PW_b^2(g,\mathbb{R})$ *be the space of variable bandwidth defined in* (I.2) *and let* $\Lambda \subseteq \mathbb{R}$ *be as in* (III.3)*. If for every* $k \in \mathbb{Z}$

$$32\left(\delta^2_{k/2-1} + \delta^2_{k/2-1/2} + \delta^2_{k/2} + \delta^2_{k/2+1/2}\right) \cdot \left(b^2(k)\|g\|^2_\infty + \frac{\|g'\|^2_\infty}{\pi^2}\right) < 1 \tag{III.4}$$

*then* $f \in PW_b^2(g,\mathbb{R})$ *can be reconstructed completely from the samples in* $\Lambda$*.*

The first part of the proof uses a technique that is similar to the one in the preprint [6]. The proof is a modification of the method proposed in [17] and then called the adaptive weights method in [15]. The idea of the proof consists in designing an approximation operator that uses only the samples $f(x_{k/2,j})$. We find an approximation of $f$ by a step function on the projection onto $PW_b^2(g,\mathbb{R})$ with $P$. We define

$$y_{k/2,j} = \frac{1}{2}(x_{k/2,j} + x_{k/2,j+1}),$$

where $y_{k/2,0} = k/2$ and $y_{k/2,j_{\max(k/2)}} = k/2 + 1/2$, to be the midpoints between the samples and set

$$\chi_{k/2,j} := \chi_{\left[y_{k/2,j-1}, y_{k/2,j}\right)}.$$

The resulting approximation operator is

$$Af = P\left(\sum_{k\in\mathbb{Z}} \sum_{j=1}^{j_{\max(k/2)}} f(x_{k/2,j})\chi_{k/2,j}\right),$$

where $P$ is the projection onto $PW_b^2(g,\mathbb{R})$. The idea of the proof consists in finding a constant $\gamma < 1$ such that $\|f - Af\|_2 \le \gamma \|f\|_2$. A Neumann series expansion of $A^{-1}$ will then make the reconstruction of $f$ possible. Wirtinger's inequality (III.1) is the essential tool for the first part of the proof and it allows to bound the $L^2$-norm of the distance between the function $f$ and its evaluation at the sampling points in $\Lambda$. The definition of a Wilson system in terms of a trigonometric polynomial (II.1) and Bernstein's inequality (III.2) are then fundamental to prove the second part. The theorem presents a sufficient condition involving the maximal gap between consecutive points of the sampling set $\Lambda$. The method deeply relies on the properties of the window function $g$ and this is visible from the fact that condition (III.4) depends on the $\|g\|_\infty$ and $\|g'\|_\infty$. Moreover, the requirement of $g$ being compactly supported is crucial. In fact, since $\mathrm{supp}(g) \subseteq [-1/2-\epsilon, 1/2+\epsilon]$, there are only four elements of the Wilson basis that play a role in the expansion of each sample $f(x_{k/2,j})$ in the interval $[k/2, k/2+1/2)$

$$f(x_{k/2,j}) = \sum_{n=k-1}^{k+2} P_n(x_{k/2,j}) g(x_{k/2,j} - n/2).$$

As a consequence, the maximal gaps of four intervals arise in condition (III.4).

The same methodology can be employed when substituting a Wilson orthonormal basis with a Wilson Riesz basis $\{\tilde{\psi}_{n,l}\}_{n\in\mathbb{Z}, l=0,\dots,b(n)}$ with lower Riesz bound $C > 0$. In this case, we have more flexibility in the choice of the window but the sufficient condition needs to be adjusted to take care of the Riesz bound. The next result is a sufficient condition for sampling theorem for the space of variable bandwidth

$$\widetilde{PW}_b^2(g,\mathbb{R}) = \Big\{ \sum_{k\in\mathbb{Z}} \sum_{l=0}^{b(n)} c_{n,l} \tilde{\psi}_{n,l} \in L^2(\mathbb{R}), c \in \ell^2 \Big\} \tag{III.5}$$

constructed using a Wilson Riesz basis.

**Theorem III.4** (SCS - Wilson Riesz basis)**.** *Let $g \in \mathcal{C}^1(\mathbb{R})$ with $\mathrm{supp}(g) \subseteq [-\frac{1}{2}-\epsilon, \frac{1}{2}+\epsilon]$ and $0 < \epsilon < 1/2$.*
*Assume $g$ satisfies the following conditions:*

*(a)* $g(x) = g(-x)$, $\quad \forall x \in \mathbb{R}$.
*(b)* $g(x) = 1$, *for* $x \in [-\frac{1}{2}+\epsilon, \frac{1}{2}-\epsilon]$.
*(c)* $g(1-x) = 1 - g(x)$, *for* $x \in [\frac{1}{2}-\epsilon, \frac{1}{2}+\epsilon]$.
*(d)* *$g$ is decreasing in* $[\frac{1}{2}-\epsilon, \frac{1}{2}+\epsilon]$.

*Let $\widetilde{PW}_b^2(g,\mathbb{R})$ be the space of variable bandwidth defined in* (III.5) *and let $\Lambda \subseteq \mathbb{R}$ be as in* (III.3)*. If for every $k \in \mathbb{Z}$*

$$\frac{8}{C}\Big(\delta_{k/2-1}^2 + 4\delta_{k/2-1/2}^2 + 4\delta_{k/2}^2 + \delta_{k/2+1/2}^2\Big) \cdot \cdot \left(b^2(k) + \frac{\|g'\|_\infty^2}{\pi^2}\right) < 1 \tag{III.6}$$

*then $f \in \widetilde{PW}_b^2(g,\mathbb{R})$ can be reconstructed completely from the samples in $\Lambda$.*

As for Theorem III.3, there are only four elements of the Wilson basis that are involved in the expansion of each sample $f(x_{k/2,j})$. The influence of these four elements is translated into the impact of the neighboring bandwidths on each interval expressed by (III.6). Indeed, the bandwidths of the two central windows have a stronger influence on the sampling rate, which is motivated by the constant 4 in (III.6), while the windows at the boundaries have a lighter weight in determining the density of the space. Therefore, the condition (III.6) provides a better understanding of the spaces of variable bandwidth and it gives information about the role played by the overlap of different translations of $g$.

## Acknowledgment

B. Andreolli was supported by the Austrian Science Fund (FWF) projects P31887-N32 and P33217. K. Gröchenig was supported by the Austrian Science Fund (FWF) project P31887-N32.

## References

[1] Abbott, B. P., et al. Observation of gravitational waves from a binary black hole merger. *Phys. Rev. Lett. 116*, 6 (2016), 061102, 16. Authors include B. C. Barish, K. S. Thorne and R. Weiss.

[2] Abbott, B. P., et al. Observing gravitational-wave transient GW150914 with minimal assumptions. *Phys. Rev. D93*, 12 (2016), 122004.

[3] Aceska, R., and Feichtinger, H. G. Functions of variable bandwidth via time-frequency analysis tools. *J. Math. Anal. Appl. 382*, 1 (2011), 275–289.

[4] Aceska, R., and Feichtinger, H. G. Reproducing kernels and variable bandwidth. *J. Funct. Spaces Appl.* (2012), Art. ID 469341, 12.

[5] Aldroubi, A., and Gröchenig, K. Beurling-Landau-type theorems for non-uniform sampling in shift invariant spline spaces. *J. Fourier Anal. Appl. 6*, 1 (2000), 93–103.

[6] Andreolli, B., and Gröchenig, K. Variable bandwidth via Wilson bases. *arXiv preprint arXiv: 2305.17290* (2023).

[7] Brueller, N., Peterfreund, N., and Porat, M. Non-stationary signals: optimal sampling and instantaneous bandwidth estimation. In *Proceedings of the IEEE-SP Int. Symp. on Time-Frequency and Time-Scale Analysis* (1998), p. 113–115.

[8] BÖLCSKEI, H., FEICHTINGER, H. G., GRÖCHENIG, K., AND HLAWATSCH, F. Discrete-time multi-window Wilson expansions: Pseudo frames, filter banks, and lapped transforms. In *Proc. IEEE-SP Int. Symp. on Time-Frequency and Time-Scale Analysis*. 1996, pp. 525–528.

[9] BÖLCSKEI, H., GRÖCHENIG, K., HLAWATSCH, F., AND FEICHTINGER, H. G. Oversampled Wilson expansions. *IEEE Signal Proc. Lett. 4* (1997), 106–108.

[10] CELIZ, M. J., GRÖCHENIG, K., AND KLOTZ, A. Spectral subspaces of Sturm-Liouville operators and variable bandwidth. *arXiv preprint arXiv:2304.07811* (2023).

[11] CHASSANDE-MOTTIN, E., JAFFARD, S., AND MEYER, Y. Des ondelettes pour détecter les ondes gravitationnelles. *Gaz. Math.*, 148 (2016), 61–64.

[12] CLARK, J., PALMER, M., AND LAWRENCE, P. A transformation method for the reconstruction of functions from nonuniformly spaced samples. *IEEE Transactions on Acoustics, Speech, and Signal Processing 33*, 5 (1985), 1151–1165.

[13] DAUBECHIES, I., JAFFARD, S., AND JOURNÉ, J.-L. A simple Wilson orthonormal basis with exponential decay. *SIAM J. Math. Anal. 22*, 2 (1991), 554–573.

[14] DONOHO, D. L., AND STARK, P. B. Uncertainty principles and signal recovery. *SIAM J. Appl. Math. 49*, 3 (1989), 906–931.

[15] FEICHTINGER, H. G., AND GRÖCHENIG, K. Theory and practice of irregular sampling. In *Wavelets: mathematics and applications*, Stud. Adv. Math. CRC, Boca Raton, FL, 1994, pp. 305–363.

[16] FEICHTINGER, H. G., GRÖCHENIG, K., AND WALNUT, D. Wilson bases and modulation spaces. *Math. Nachr. 155* (1992), 7–17.

[17] GRÖCHENIG, K. Reconstruction algorithms in irregular sampling. *Math. Comp. 59*, 199 (1992), 181–194.

[18] GRÖCHENIG, K. *Foundations of time-frequency analysis*. Applied and Numerical Harmonic Analysis. Birkhäuser Boston, Inc., Boston, MA, 2001.

[19] GRÖCHENIG, K., AND KLOTZ, A. What is variable bandwidth? *Comm. Pure Appl. Math. 70*, 11 (2017), 2039–2083.

[20] HARDY, G. H., LITTLEWOOD, J. E., AND PÓLYA, G. *Inequalities*. Cambridge, at the University Press,, 1952. 2d ed.

[21] HORIUCHI, K. Sampling principle for continuous signals with time-varying bands. *Information and Control 13* (1968), 53–61.

[22] OLEVSKII, A. M., AND ULANOVSKII, A. *Functions with disconnected spectrum*, vol. 65 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2016. Sampling, interpolation, translates.

[23] SHAVELIS, R., AND GREITANS, M. Signal sampling according to time-varying bandwidth. In *2012 Proceedings of the 20th European Signal Processing Conference (EUSIPCO)* (2012), pp. 1164–1168.

[24] WEI, D., AND OPPENHEIM, A. V. Sampling based on local bandwidth. In *2007 Conference Record of the Forty-First Asilomar Conference on Signals, Systems and Computers* (2007), pp. 1103–1107.