Batch Entanglement Detection in Parameterized Qubit States using Classical Bandit Algorithms

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Abstract

Entanglement is a key property of quantum states that acts as a resource for a wide range of tasks in quantum computing. Entanglement detection is a key conceptual and practical challenge. Without adaptive or joint measurements, entanglement detection is constrained by no-go theorems (Lu et al., 2016), necessitating full state tomography. Batch entanglement detection refers to the problem of identifying all entangled states from amongst a set of K unknown states which finds applications in quantum information processing. We devise a method to perform batch entanglement detection by performing measurements derived from a single-parameter family of entanglement witnesses from Zhu et al. (2010), followed by a thresholding bandit algorithm on the measurement data. The proposed method can perform batch entanglement detection conclusively, when the unknown states are drawn from practically well-motivated class of two qubit states $\mathcal{F}$ that include Depolarised Bell states, Bell diagonal states etc. Our key novelty lies in drawing a connection between batch entanglement detection, and a Thresholding Bandit problem in classical Multi-Armed Bandits (MAB). The connection to the MAB problem also enables us to derive theoretical guarantees on the measurement/sample complexity of the proposed technique. We demonstrate the performance of the proposed method through numerical simulations and an experimental implementation. More broadly, this paper highlights the potential for employing classical machine learning techniques for quantum entanglement detection.

1 Introduction

Quantum information theory has redefined quantum entanglement from a descriptive property of quantum states to a fundamental non-classical resource. As the basis for applications such as quantum communication, teleportation, and information processing (Bennett et al., 1993; Buhrman et al., 2001; Horodecki et al., 2009), entanglement detection and verification is a central problem. Traditionally, this involves performing quantum measurements that yield probabilistic data, enabling techniques like full-state tomography (FST) for state reconstruction. However, this faces two challenges: theoretically, even after FST, determining entanglement remains computationally intractable, and even more so in scenarios involving many qubits; practically, real-world noise and imperfections limit the accuracy of state reconstruction. Modern multi-qubit compute systems may generate a bunch of entangled states across different sets of qubits through quantum gate operations; however, gate noise (e.g., phase flip errors, depolarization) can compromise their entanglement, requiring precise verification to ensure their reliability before use in applications such as quantum computation and communication (Hong et al., 2010). In such contexts, FST can be employed for entanglement detection. However, it comes with a high computational burden which may be unnecessary. We propose an alternative approach for simultaneous real-time verification or detection of entanglement among a given set of quantum states, dubbed *batch entanglement detection*. Instead of relying on FST, learning algorithms utilize statistical patterns to simultaneously analyze measurement data from a batch of quantum states and provide high probability guarantees on what they learn, i.e., whether or not the states are entangled.

Conventional techniques for learning quantum states include extensive research on FST (see Kueng et al. (2017); Wang et al. (2019); O'Donnell & Wright (2015a;b); Banaszek et al. (2013); Flammia et al. (2012)

and references therein and also see Guta et al. (2020); Torlai et al. (2018); Quek et al. (2018); Koutný et al. (2022); Schmale et al. (2022); França et al. (2021) for machine learning-based approaches). Measurements required for FST scale exponentially with the number of qubits. While entangled measurements enable near-optimal copy complexity for FST (O'Donnell & Wright, 2015b; Haah et al., 2017), practical implementations rely on single-copy measurements with reconstruction methods like linear inversion, maximum likelihood estimation, and maximum a posteriori estimation (Teo et al., 2011; Siddhu, 2019). These reconstructed states can be tested for entanglement using well-known criterion (some are outlined in Sec. 2.2). Alternatively, entanglement can be detected by measuring *entanglement witnesses* (Horodecki et al., 1996a; Terhal, 2000; Lewenstein et al., 2000a; Chruciski & Sarbicki, 2014), observables that detect *some* entangled states. No single witness can detect all entangled states, but in the worst case, combining information obtained from measuring different witnesses aids in state reconstruction via FST. This is explored in Zhu et al. (2010), where for bipartite qubit systems, measurement operators from a family of six witnesses are used. The proposed approach for entanglement detection involves measuring a witness and formulating a *separability criterion* based on the frequencies of measurement outcomes. A negative value of the criterion indicates entanglement; otherwise, the process is repeated with another witness. If the state remains undetected by all witnesses, a tomographic reconstruction is performed (see Sections 2.1 and 2.2 for further details).

Recently, the authors of Lumbreras et al. (2022) proposed using multi-armed bandit (MAB) frameworks for learning quantum states. The MAB algorithm repeatedly chooses from several options ("arms"), with the goal of finding the arm with the best outcome (the "best arm"). The algorithm balances between exploiting the known best options and exploring others to ensure no better option is missed (more details on MAB and policies can be found in Sec. 2.3). In Lumbreras et al. (2022), the inherent linearity in the quantum mechanical description of states is capitalized and a well-known classical learning algorithm that prescribes a sequential order of choosing measurements is employed. The MAB algorithms that are used provide guarantees on the quality of the estimate of the unknown quantum state. This MAB model in Lumbreras et al. (2022) does not directly apply to batch entanglement detection. Instead, it focuses on learning *one* entire quantum state, which may be unnecessary for entanglement detection.

Our first contribution is building on the separability criteria in Zhu et al. (2010) and using suitable MAB policies for learning the same. We find connection between batch entanglement detection and the thresholding bandit problem (TBP) (Kano et al., 2018), which is structurally different from the setting in Lumbreras et al. (2022) (see Remark 1). For a given witness, the objective is to quickly and accurately estimate the m states whose separability criteria values fall below the threshold 0, dubbed the "(m, K)-quantum Multi-Armed Bandit" framework. For a given witness, accurately estimating the separability criterion requires measurements on *numerous* copies of each of the K unknown states. FST does not provide an obvious guarantee on measurement/sample complexity for entanglement detection. Our second contribution addresses this, i.e., for a given error margin, the MAB policies identify specific 'winning' trends dictated by parameter estimates. On a high level, the trends specify *which* state to measure and provides sample complexity guarantees with high probability.

The rest of this paper is organized as follows: In Section 2, we provide a brief recap of some preliminary concepts. In Section 3, we describe the (m, K)-quantum Multi-Armed Bandit framework for entanglement detection. We define a class of parameterized two-qubit states $\mathcal{F}$ and identify measurement operators that conclusively detect entanglement in $\mathcal{F}$, detailed in Sections 3.1, 3.2, and 3.3. In Section 4, we demonstrate two TBP policies for entanglement detection. Section 5 analyzes the MAB policy performance on IBMQ backends and on an ibm-brisbane device for a family of states in $\mathcal{F}$ and details the quantum circuits used for simulation. Section 6 highlights measurement scheme limitations for entanglement detection in arbitrary states through numeric examples. Finally, Section 8 conludes the paper. Detailed proofs for the results presented in the paper can be found in Appendix A.

2 Preliminaries

Let $\mathcal{H}$ be a finite dimensional Hilbert space with dimension d. A pure quantum state is represented by a unit norm vector $|\psi\rangle \in \mathcal{H}$. Let $\mathcal{L}(\mathcal{H})$ be the space of linear operators on $\mathcal{H}$, the Frobenius inner product for any $A, B \in \mathcal{L}(\mathcal{H})$, $\langle A, B\rangle \coloneqq \mathrm{Tr}\big(A^\dagger B\big)$ where $\dagger$ represents conjugate transpose. A Hermitian operator

satisfies $H = H^\dagger$. A density operator $\rho \in \mathcal{L}(\mathcal{H})$ is Hermitian, positive semi-definite, $\rho \geq 0$, and has unit trace, $\mathrm{Tr}(\rho) = 1$; it can represents both pure and mixed states. A positive operator value measure (POVM) is collection of positive operators $\{E_i \geq 0\}$ that sum to the identity, $\sum_i E_i = I$. A POVM represents a measurement where E_i corresponds to measurement outcome i, but sometimes we compress this and just say E_i is a measurement outcome.

Let $\mathcal{H}_a$ and $\mathcal{H}_b$ be finite-dimensional Hilbert spaces with dimensions d_a and d_b, respectively, and $\mathcal{H}_{ab} \coloneqq \mathcal{H}_a \otimes \mathcal{H}_b$, where $\otimes$ represents tensor product, be a bipartite Hilbert space with dimension $d = d_a d_b$. A density operator $\rho_{ab} \in \mathcal{L}(\mathcal{H}_{ab})$ is called *separable* if it can be written as a convex combination of product states, that is,

$$\rho_{ab} = \sum_i p_i \left|\phi_a^i, \chi_b^i\right\rangle \left\langle \phi_a^i, \chi_b^i\right|, \tag{1}$$

where $p_i \geq 0$ such that $\sum_i p_i = 1$ and $\left|\phi_a^i, \chi_b^i\right\rangle \coloneqq |\phi\rangle_a^i \otimes |\chi\rangle_b^i$ is a product of two pure states. We denote the set of all separable density operators by S_{ab}. Conversely, ρ_{ab} is *entangled* if it can not be written in the form equation 1.

We discuss some preliminaries on entanglement witnesses and witness-based measurements in Section 2.1, the various separability criteria for entanglement detection in Section 2.2 and the framework and background on stochastic multi-armed problems in Section 2.3.

2.1 Entanglement Witnesses and Witness Operators Measurements

Entanglement can be detected by measuring entanglement witnesses and can be defined as follows:

Definition 1 (Entanglement Witness) *An entanglement witness, denoted as $W \in \mathcal{L}(\mathcal{H}_{ab})$, is a Hermitian operator that detects some entangled state $\rho_{ent} \in \mathcal{H}_{ab}$ such that,*

$$\langle \rho_{ent}, W\rangle = \mathrm{Tr}(\rho_{ent} W) < 0, \tag{2}$$
$$\langle \rho, W\rangle = \mathrm{Tr}(\rho W) \geq 0, \ \forall \rho \in S_{ab}. \tag{3}$$

Conceptually, a witness W defines a hyperplane that delineates a set of entangled states it can detect ($D_W = \{\rho \text{ s.t. } \mathrm{Tr}(\rho W) < 0\}$) from all other states. When comparing two arbitrary witnesses W_1 and W_2, if D_{W_1} is contained within D_{W_2}, then W_2 is considered finer than W_1. Further insights into this topology are detailed in Lewenstein et al. (2000b, Lemma 1). A witness is said to be *optimal* when no other witness is finer, suggesting that it touches the boundary of the convex set of separable states (Bengtsson & Zyczkowski, 2006).

To improve the efficacy of identifying entangled states, Zhu et al. (2010) proposes a method to construct a set of measurements called Witness Operator Measurements (WOM), which we briefly discuss here. Let us consider the rank-one projector onto a pure entangled state $|\psi\rangle \in \mathcal{H}_{ab}$ denoted by $\rho(\alpha) = |\psi\rangle\langle\psi|$, where $|\psi\rangle = \cos\alpha\,|00\rangle + \sin\alpha\,|11\rangle$. Here, the Schmidt coefficients $\cos\alpha$ and $\sin\alpha$ are arranged in non-increasing order as $1 > \cos^2\alpha \geq \sin\alpha^2 > 0$. Consequently, $\alpha \in [0, \pi/4]$ is chosen to adhere to this order.

In this paper, we consider the specific form of the witnesses from Zhu et al. (2010), namely, $W = \rho_w(\alpha) = \cos^2\alpha I - \rho(\alpha)^{\top_b}$ where $\top_b$ represents partial transpose with respect to $\mathcal{H}_b$. That is, consider a rank-one POVM $\sum_i w_i\rho_i = I$ with outcomes $w_i\rho_i$ such that $w_i > 0$ and ρ_i's are projectors onto pure states. We can construct a WOM with outcomes $w_i\rho_{iw}$ where $\rho_{iw} = \lambda_{\max} I - \rho_i^{\top_b}$.

2.2 Separability criteria for entanglement detection

Using FST techniques, briefly outlined earlier, one can do a tomographic reconstruction of the state and subsequently determine its entanglement status using well-known separability criteria (Horodecki et al., 2009). For bipartite qubit systems, the Peres-Horodecki criterion (Horodecki et al., 1996b; Peres, 1996) establishes that a density operator ρ_{ab} is separable if and only if the eigenvalues of its partial transpose $\rho_{ab}^{\top_b}$ are non-negative. This criterion remains necessary and sufficient even when $d_a = 2$ and $d_b = 3$ but is

violated in higher dimensions by a class of entangled states with non-negative partial transposition. Other criteria include the range criterion (Horodecki, 1997), the matrix realignment criterion (Rudolph, 2000), the covariance matrix (CM) criterion (Gühne et al., 2007), and additional methods discussed in Gurvits (2003); Doherty et al. (2004).

Another criterion for separability is obtained from the Witness Operator Measurements (WOMs) described in Section 2.1, which are highly efficient for entanglement detection. We review this criterion from Zhu et al. (2010) next. Specifically, let us consider two-qubit witnesses of the form:

$$\rho_w(\alpha) = \cos^2\alpha I - \left(|\psi\rangle\langle\psi|\right)^{\top_b}, \tag{4}$$

where $\left(|\psi\rangle\langle\psi|\right)^{\top_b}$ is equal to,

$$\frac{1+\cos 2\alpha}{2}|00\rangle\langle 00| + \frac{1-\cos 2\alpha}{2}|11\rangle\langle 11| + \frac{\sin 2\alpha}{2}\left(\left|\Psi^+\right\rangle\left\langle\Psi^+\right| - \left|\Psi^-\right\rangle\left\langle\Psi^-\right|\right).$$

Here, $|\psi\rangle = \cos\alpha|00\rangle + \sin\alpha|11\rangle$ such that $\alpha \in [0, \pi/4]$ and $|\Psi^\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$. We denote the projectors onto the set of eigenstates of $\rho(\alpha) = \left(|\psi\rangle\langle\psi|\right)^{\top_b}$ by $\mathcal{E} = \{|00\rangle\langle 00|, |11\rangle\langle 11|, |\Psi^+\rangle\langle\Psi^+|, |\Psi^-\rangle\langle\Psi^-|\}$. This collection of projectors $\mathcal{E}$ form a POVM and we refer to them as a Witness Basis Measurement (WBM).

Let us consider a quantum state ρ. Let $f_i := \mathrm{Tr}\{E_i\rho\}$ be the probability of obtaining outcome i when the state ρ is measured using WBM $\mathcal{E}$. The expected value of the witness $\mathrm{Tr}(\rho_w(\alpha)\rho)$ can be expressed in terms of f_i. If this expected value is less than a certain threshold (in our case, 0), we can conclude that ρ is entangled else, this test is inconclusive. When this test is inconclusive, we pick the witnesses in Table 1 sequentially. These subsequent witnesses are obtained by applying unitary transformations U_1 and U_2 on each of the qubits to change in the eigenbasis of the underlying state as shown in equation 5.

$$\rho_w(\alpha) \longrightarrow (U_1 \otimes U_2)^\dagger \rho_w(\alpha)(U_1 \otimes U_2). \tag{5}$$

Witness	U_1	U_2
1	I	I
2	I	X
3	$C^\dagger$	C
4	$C^\dagger$	XC
5	C	$C^\dagger$
6	C	$XC^\dagger$

Table 1: Changing the eigenbasis of equation 4

In Table 1, the operator C permutes between Pauli operators X, Y and Z, satisfying that $CX = YC$, $CY = ZC$ and $CZ = XC$. Expressing the eigenstates of the first witeness equation 4 in terms of Pauli operators yields three observables: $ZI + IZ$, ZZ, and $XX + YY$.

Estimates for these three observables come from measuring the first witness. Similarly, the second witness listed in Table 1 yields estimates for $ZI - IZ$, ZZ, and $XX + YY$. Thus, for a pair of witnesses, we obtain estimates for five observables by applying suitable unitary transformations, and each of the other two witness pairs provides another five expectation values. In total, we obtain estimates for 15 expectation values, providing sufficient information about the two-qubit state. This, reduction of the number of witnesses from sixteen to six offers significant practical benefits. Instead of relying solely on comparing the expected value of the witness $\rho_w(\alpha)$ against a threshold, the authors in Zhu et al. (2010) suggest adopting a more stringent criterion:

$$\min_\alpha \mathrm{Tr}\left\{\rho_{\mathrm{sep}}\left(\cos^2\alpha I - \rho_w(\alpha)\right)\right\} \geq 0, \quad \forall \rho_{\mathrm{sep}} \in S_{ab}. \tag{6}$$

which holds for all separable states and is violated by set of entangled states that *can* be detected by this family of witnesses. The above optimisation leads to the following quadratic WBM criterion,

$$S = 4f_1f_2 - (f_3 - f_4)^2 \geq 0, \quad \forall \rho_{\mathrm{sep}} \in S_{ab}. \tag{7}$$

In essence, the process of measuring the linear entanglement witnesses $\rho_w(\alpha)$ corresponds to measuring the projectors onto the eigenstate basis. The value of S equation 7 depends on the underlying WBM. Thus, for a WBM $\mathcal{E}$ and state ρ, we denote equation 7 as $S_{\mathcal{E}}(\rho)$. We note that *most* two-qubit entangled states can be detected under the six witnesses listed in Table 1.

2.3 Stochastic Multi-Armed Bandits

The stochastic Multi-Armed Bandit (MAB) framework is an archetype for many sequential decision-making problems. Within this framework, a bandit instance (problem instance) encompasses K arms (or actions) situated in an environment where stochastic rewards are yielded upon selecting an arm (termed *pulling*). Each arm $i \in [K] = \{1, 2, \ldots, K\}$ is described by a probability distribution ν_i over $\mathbb{R}$, with known support and an unknown expectation μ_i. We denote the problem instance by $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots \mu_K)$. Arm selection occurs iteratively in rounds, where during each round t, a learner (or agent) selects an arm $X_t \in [K]$ according to a specified policy. Subsequently, the learner receives a stochastic reward $Z_t \sim \nu_{X_t}$ corresponding to the selected arm. Upon receiving the reward, the learner can terminate the process or continue by updating its policy to pursue a specific objective.

In MAB literature, two objectives have been focal points of the study: (i) Maximizing the cumulative reward accumulated over multiple game rounds, necessitating a tradeoff between *exploration*(discovering arms with potentially higher rewards) and *exploitation* (repeatedly pulling the arm with the highest observed reward). (ii) The *best arm identification* (BAI) problem which focuses on pure exploration, where the learner aims to identify the arm with the highest expected reward, i.e., $i^\star = \arg\max_i \mu_i$ (known as the best arm). A BAI policy (or algorithm) consists of a *sampling rule* for arm selection, a *stopping rule* to determine the end of exploration, and a *recommendation rule* to output the best arm. The BAI problem has been explored in *fixed confidence* and *fixed budget* settings. In the fixed confidence setting, the goal is to quickly identify the best arm with a probability of at least $1 - \delta$ for a fixed error probability δ. In the fixed-budget setting, the number of arm pulls (budget) $N \in \mathbb{N}$ is fixed, and the goal is to minimize the misidentification probability of the best arm within N arm pulls. Section 2.3.1 outlines BAI, and Section 2.3.2 addresses its Good Arm Identification (GAI) variant, both in the fixed confidence setting.

2.3.1 Fixed Confidence Best Arm Identification

Consider a problem instance denoted by $\boldsymbol{\mu}$. Without loss of generality, we enumerate the arms based on their expected rewards, such that $\mu_1 > \mu_2 \geq \mu_3 \ldots \geq \mu_K$. We assume a unique best arm exists, denoted as $i^\star = 1$. The sub-optimal gaps between the arms are $\Delta_i = \mu_{i^\star} - \mu_i$. The objective is to accurately identify the best arm $i^\star$ while minimizing the number of samples used. Policies that achieve this task are classified as δ-PC policies. These policies ensure that the likelihood of the exact correctness of the outcome is at least $1 - \delta$, where δ is the *correctness threshold*. If the outcome is approximately correct with probability $1 - \delta$, then the policy is called δ-PAC.

Definition 2 (δ-PC) *Let $\hat{i}_\tau$ be the estimate of the best arm at stoppage τ. Then, an algorithm is said to be δ-PAC if it satisfies*

$$\mathbb{P}_{\boldsymbol{\mu}}(\hat{i}_\tau \neq i^\star) \leq \delta, \quad \textit{and} \quad \mathbb{P}_{\boldsymbol{\mu}}(\tau < \infty) = 1. \tag{8}$$

The primary objective is to characterize the expected stopping time $\mathbb{E}_{\boldsymbol{\mu}}[\tau]$ of the policy. Several works have attempted to provide upper and lower bounds for this objective. The successive elimination algorithm achieves a sample complexity of $\mathcal{O}(\Delta^{-2} \log(n\Delta^{-2}))$ (Even-Dar et al., 2002), while LUCB1 improves this to $\mathcal{O}(\Delta^{-2} \log \Delta^{-2})$ (Kalyanakrishnan et al., 2012). The exponential-gap elimination algorithm further reduces this to $\mathcal{O}(\Delta^{-2} \log(\log(\Delta^{-2})))$, the best-known complexity for elimination-style policies under fixed confidence (Karnin et al., 2013). These results approach the theoretical lower bound of $\mathcal{O}(\Delta^{-2})$ (Mannor & Tsitsiklis, 2004), differing only by log or log log factors. Notably, Farrell (1964) bridges this gap, proving that $\mathcal{O}(\Delta^{-2} \log \log \Delta^{-2})$ samples suffice to identify the best arm with error probability δ. Building upon this, lil'UCB Jamieson et al. (2014) uses finite-sample LIL-based concentration bounds to achieve near-optimal sample complexity, achieving order optimality in sample complexity akin to exponential-gap elimination.

Table 2: Stochastic-Quantum MAB

Attributes	**Stochastic MAB**	**Quantum MAB**
Arms	Probability distributions $(p_1, p_2, \dots p_K)$	Density operators $\{\rho_1, \rho_2, \dots, \rho_K\}$
Measurement	–	WBM $\mathcal{E}$
Measurement Data	j w.p. $p_i(j), \forall i \in [K]$	j w.p. $\text{Tr}(E_j\ \rho_i),\ \forall j \in [4], \forall i \in [K]$
Parameters to estimate	$\boldsymbol{\mu} = (\mu_1, \mu_2, \dots \mu_K)$	$\boldsymbol{S_\mathcal{E}} = (S_\mathcal{E}(\rho_1), S_\mathcal{E}(\rho_2), \dots, S_\mathcal{E}(\rho_K))$
Objective	Identify $\mathcal{G}^C = \{i \in [K] \text{ such that } \mu_i \leq \zeta\}$	Identify $\mathcal{A}_{\text{ent}} = \{i \in [K] \text{ such that } S_\mathcal{E}(\rho_i) < 0\}$

2.3.2 Fixed Confidence Good Arm Identification

Consider a problem instance $\boldsymbol{\mu}$. Alongside the acceptance error δ described in Section 2.3.1, we introduce a threshold $\zeta \in (0, 1)$ and define the set of "good" arms as $\mathcal{G} = \{i \in [K] \text{ such that } \mu_i \geq \zeta\}$. In simpler terms, the good arms are those whose means are greater than or equal to ζ. The number of good arms $|\mathcal{G}| = m$ remains unknown to the agent, leading to what we term as the (m, K)-GAI problem. Notably, the $(1, K)$-GAI reduces to the BAI problem discussed earlier. Without loss of generality, we enumerate the arms based on their expected rewards: $\mu_1 > \mu_2 \geq \ldots \geq \mu_m \geq \zeta \geq \mu_{m+1} \ldots \geq \mu_K$. Importantly, the agent is unaware of this indexing. For $i \in [K]$, $\Delta_i \coloneqq |\mu_i - \zeta|$ and $\Delta_{i,j} = \mu_i - \mu_j$. The sample complexity is expressed in terms of $\Delta = \min(\min_{i \in [K]} \Delta_i, \min_{j \in [K-1]} \Delta_{j,j+1}/2)$.

At each time instant t, the learner samples an arm $X_t \in [K]$ and receives a corresponding (random) reward $Z_t \sim \nu_{X_t}$. The agent either outputs an arm that identifies as "good" or stops when no good arms remain. We denote the stopping time of the GAI policy as τ_{stop}. Specifically, the agent outputs $\hat{X}_1, \hat{X}_2, \dots \hat{X}_{\hat{m}}$ as good arms at rounds $\tau_1, \tau_2, \dots \tau_{\hat{m}}$ respectively, where $\hat{m}$ denotes the estimate of the number of arms identified as good ones and τ_k denotes the number of rounds to identify the k^{th} good arm. The learner's objective is to accurately and rapidly identify these good arms while minimizing the number of samples used. As elaborated below, this is achieved through policies falling within the class of (λ, δ)-PAC policies.

Definition 3 ((λ, δ)-PAC) *Let $\hat{m}$ denote the number of good arms identified by the agent. A (λ, δ)-PAC algorithm satisfies the following conditions:*

1. *If there are at least λ good arms, then*

$$\mathbb{P}_{\boldsymbol{\mu}} \left[\{\hat{m} < \lambda\} \cup \bigcup_{i \in \{\hat{X}_1, \hat{X}_2, \ldots \hat{X}_\lambda\}} \{\mu_i < \zeta\} \right] \leq \delta,$$

2. *If there are fewer than λ good arms,*

$$\mathbb{P}_{\boldsymbol{\mu}} [\hat{m} \geq \lambda] \leq \delta,$$

An algorithm is called δ-PAC if it is (λ, δ)-PAC for all $\lambda \in [K]$.

Just like in the BAI context (refer to Section 2.3.1), the objective in GAI is to determine the expected stopping time $\mathbb{E}_{\boldsymbol{\mu}}[\tau_{\text{stop}}]$. The GAI algorithm consists of two key components: a *sampling* rule and an *identification* rule. The former dictates the arm selection process, while the latter guides the agent in distinguishing between good and bad arms. GAI confronts a novel challenge called the exploration-exploitation dilemma of confidence. Here, exploration involves the agent pulling arms other than the empirical best arm to identify potentially 'good' arms with fewer pulls. At the same time, exploitation entails pulling the empirical best arm to increase confidence in its classification as a good arm. To address this challenge, Kano et al. (2018) proposed a hybrid algorithm for the dilemma of confidence (HDoC). In HDoC, the sampling rule is derived from the UCB algorithm for cumulative regret minimization (Auer et al., 2002), while the identification rule is based on the LUCB algorithm for BAI (Kalyanakrishnan et al., 2012) and the APT algorithm for the thresholding bandits problem (Locatelli et al., 2016). The proposed HDoC algorithm (LUCB-G) requires $\mathcal{O}\left(\Delta^{-2}\left(K \log \frac{1}{\delta} + K \log K + K \log \frac{1}{\Delta}\right)\right)$ samples. However, a drawback of the LUCB-G algorithm

is its impracticality when Δ is very small. To address this issue and achieve faster convergence in the identification phase, Tsai et al. (2024) propose utilizing confidence widths derived from the finite LIL bound, akin to the approach in the lil'UCB algorithm (Jamieson et al., 2014). They demonstrate a reduction in the required number of samples, achieving a sample complexity of $\mathcal{O}\left(\Delta^{-2}\left(K\log\frac{1}{\delta}+K\log K+K\log\log\frac{1}{\Delta}\right)\right)$. The specific connections between BAI/GAI and entanglement detection are elaborated in Section 3 and 4.

3 The Quantum MAB Framework For Entanglement Detection

In this section, we introduce the quantum Multi-Armed Bandit (MAB) framework for batch entanglement detection. First, we highlight the structural similarity between this framework and the stochastic MAB model. In stochastic MAB, pulling an arm i corresponds to sampling from a probability distribution $p_i(\cdot)$ with known support and unknown mean μ_i. When arm i is pulled, a reward j is obtained with probability (w.p.) $p_i(j)$. In each round, different arms can be pulled, yielding independent and identically distributed (i.i.d.) rewards. Analogously, in the quantum setting, each arm represents an unknown quantum state ρ. When ρ is measured, the underlying probability distribution of the rewards is determined by the measurement $\mathcal{E}$. Specifically, if a Witness Basis Measurement (WBM) $\mathcal{E}$ is chosen, measuring a state ρ with $\mathcal{E}$ will result in a reward $j\in\{1,2,3,4\}$ with probability $\mathrm{Tr}(\rho E_j)$. Once the measurement is fixed, the rewards obtained from measuring ρ are i.i.d. The subtle difference between the two models lies in the source of the rewards. In the stochastic MAB model, rewards are obtained by sampling from i.i.d. distributions, whereas in the quantum MAB model, the rewards depend on the chosen WBM.

The proposed MAB framework is as follows: A WBM $\mathcal{E}$ is chosen by the learner. Given this WBM $\mathcal{E}$, the MAB routine is applied to the (m,K) instance of quantum states (arms) with the objective of minimizing the number of measurements (arm pulls) required to detect the m entangled states. It is important to note that there is no guarantee that all the m entangled states are detectable under a given WBM. Given this, along with the fact that m is unknown necessitates the repetition of the MAB routine for all WBMs. Thus, during each MAB iteration, we do not use the measurement data (rewards) to decide the choice of the subsequent WBM.

In the Best Arm Identification (BAI) setting of stochastic MAB, the primary parameters of interest are the means of the rewards. Similarly, in the quantum analogue, $S_{\mathcal{E}}(\rho)$ is the parameter of interest. As discussed in Section 2.2, for a given state ρ and WBM $\mathcal{E}$, the value of $S_{\mathcal{E}}(\rho)$ determines whether the state is entangled. The specific problem we consider involves K arms (states), of which m are bad (entangled), and our goal is to identify these entangled states. We summarize this correspondence concisely in Table 2.

More formally, the objective of the learner is to accurately identify $\mathcal{A}_{\text{ent}}=\{i\in[K]\text{ such that }S_{\mathcal{E}}(\rho_i)<0\}$, while minimizing the number of measurements. This aligns with the goal of the (m,K)-Bad Arm identification which aims to identify all those arms $\mathcal{G}^C=\{i\in[K]\text{ such that }\mu_i\leq\zeta\}$ whose means μ_i fall below a specified threshold ζ. In essence, solving the (m,K)-Bad Arm identification is tantamount to addressing the (m,K)-quantum MAB problem. We define the (m,K)-quantum MAB setting as follows,

Definition 4 *The (m,K)-quantum Multi-Armed Bandit (MAB) setting for entanglement detection is fully characterized by the tuple $(\mathcal{A},\mathcal{E})$. Here, $\mathcal{A}$ denotes a finite action set with $|\mathcal{A}|=K$, consisting of $(K-m)$ two-qubit separable states and m two-qubit entangled states. The term $\mathcal{E}$ corresponds to a suitable Witness Basis Measurement (WBM).*

Remark 1 *The d-dimensional discrete multi-armed quantum bandit model (Lumbreras et al., 2022) is different from our formulation. The authors consider arms to be a finite set of observables and the environment, an unknown quantum state ρ. The objective is to learn the unknown quantum state ρ through an exploration-exploitation tradeoff. Given sequential oracle access to copies of ρ, each round involves selecting an observable to maximize its expectation value (reward). The information from previous rounds (history) aids in refining the action choice, thereby minimizing the regret, which is the difference between the obtained and maximal rewards. The authors also exploit the inherent linear structure in measurement outcomes and map it to the linear bandit setting. Specifically, let $\{\sigma\}_{i=1}^{d^2}$ be a set of orthogonal Hermitian matrices. The unknown environment $\rho=\sum_{i=1}^{d^2}\mathrm{Tr}(\rho\sigma_i)\sigma_i=\sum_{i=1}^{d^2}\theta_i\sigma_i$ and arm $\mathcal{O}_t=\sum_{i=1}^{d^2}\mathrm{Tr}(\mathcal{O}_t\sigma_i)\sigma_i=\sum_{i=1}^{d^2}A_{t,i}\sigma_i$.*

Then, $\mathrm{Tr}(\rho\mathcal{O}_t) = \boldsymbol{\theta}^\top \mathbf{A}_t$ *where* $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots \theta_{d^2})$ *and* $\mathbf{A}_t = (A_{t,1}, A_{t,2}, \dots A_{t,d^2})$*. In round* t*, pulling arm* $\mathcal{O}_t$ *provides a reward* $X_t = \boldsymbol{\theta}^\top \mathbf{A}_t + \eta_t$*, where* η_t *is 1-subgaussian.*

To demonstrate the functionality of MAB policies, we identify suitable WBMs for families of parameterized two-qubit states denoted by $\mathcal{F}$. We denote the first two witnesses in Table 1 as $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively. Here, $\mathcal{E}_1 \coloneqq \{|00\rangle\langle 00|, |11\rangle\langle 11|, |\Psi^+\rangle\langle\Psi^+|, |\Psi^-\rangle\langle\Psi^-|\}$ and $\mathcal{E}_2 \coloneqq \{|01\rangle\langle 01|, |10\rangle\langle 10|, |\Phi^+\rangle\langle\Phi^+|, |\Phi^-\rangle\langle\Phi^-|\}$.

3.1 Two-qubit Depolarized Bell States

For $w \in \mathbb{R}, -1/3 \leq w \leq 1$, a two-qubit **Depolarized Bell** state $\rho(w)$ is given by,

$$\rho(w) = w\,|\Upsilon\rangle\langle\Upsilon| + (1-w)\frac{I}{4}. \tag{9}$$

Here, $|\Upsilon\rangle$ represents any one of the four Bell states $|\Psi^\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$, $|\Phi^\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$. When $\Upsilon = |\Psi^-\rangle$, equation 9 is called a Werner state, and when $\Upsilon = |\Phi^+\rangle$, equation 9 is called an Isotropic state. The Peres-Horodecki criterion guarantees that $\rho(w)$ is separable when $-1/3 \leq w \leq 1/3$ and is entangled when $1/3 < w \leq 1$. Table 3 outlines the specific choices of WBM for the combination of the maximally mixed state with each of the four Bell states. When measured with these corresponding WBMs, the entangled depolarized Bell states are conclusively detected, determined by the value of $S = (w-1)^2/4 - w^2$ which is strictly positive for $-1 \leq w \leq 1/3$ and negative for $w > 1/3$.

Table 3: WBM for Depolarized Bell States

Depolarized State	**Pauli Basis**	**WBM**
$w\,\|\Phi^+\rangle\langle\Phi^+\| + (1-w)I/4$	$\big[I + \alpha(XX - YY + ZZ)\big]/4$	$\mathcal{E}_2$
$w\,\|\Psi^+\rangle\langle\Psi^+\| + (1-w)I/4$	$\big[I + \alpha(XX + YY - ZZ)\big]/4$	$\mathcal{E}_1$
$w\,\|\Psi^-\rangle\langle\Psi^-\| + (1-w)I/4$	$\big[I + \alpha(-XX - YY - ZZ)\big]/4$	$\mathcal{E}_1$
$w\,\|\Phi^-\rangle\langle\Phi^-\| + (1-w)I/4$	$\big[I + \alpha(-XX + YY + ZZ)\big]/4$	$\mathcal{E}_2$

3.2 Two-qubit Bell diagonal States

Bell diagonal states are a probabilistic mixture of the four Bell states. These states are more general than the ones in equation 9. Given parameters p_1, p_2, p_3 and p_4 such that $p_i \geq 0, \sum_i p_i = 1$, the Bell diagonal state is defined,

$$\rho_{\text{Bell}} = p_1\,|\Phi^+\rangle\langle\Phi^+| + p_2\,|\Psi^+\rangle\langle\Psi^+| + p_3\,|\Psi^-\rangle\langle\Psi^-| + p_4\,|\Phi^-\rangle\langle\Phi^-|. \tag{10}$$

The eigenvalues of $\rho_{\text{Bell}}^{\top_b}$ are calculated to be $1/2 - p_1$, $1/2 - p_2$, $1/2 - p_3$ and $1/2 - p_4$. Consequently, a Bell diagonal state is entangled if any one of these probabilities exceeds $1/2$, while the sum of the other three probabilities is less than $1/2$. Conversely, a Bell diagonal state is separable if all probabilities are less than or equal to $1/2$. Expressing equation 10 in the Pauli basis yields,

$$\rho_{\text{Bell}} = \frac{1}{4}\left[I + aXX + bYY + cZZ\right],$$

where $a = p_1 + p_2 - p_3 - p_4$, $b = -p_1 + p_2 - p_3 + p_4$ and $c = p_1 - p_2 - p_3 + p_4$.

When ρ_{Bell} is entangled, the index for which $p_i > 1/2$ determines the sign of a, b, and c, see Table 4. It is notable that the signs of a, b and c follow a similar pattern to the Pauli basis expansion of various Depolarized Bell states listed in Table 3. We observe that, for suitable combinations of a, b, and $c \in \{+1, -1\}$, the Bell diagonal state reduces to one of the Depolarized Bell states and states can be detected using the same WBMs, as in Table 3. Specifically, the value of S under the two WBMs in Table 4 is equal to $(1-p_1-p_4)^2 - 4(p_1-p_4)^2$ and $(1-p_2-p_3)^2 - 4(p_2-p_3)^2$, respectively. Depending on the probabilistic mixture, one of the two WBMs will conclusively result in $S < 0$.

Table 4: WBM for Bell Diagonal States

Probabilistic mixture	a	b	c	WBM
$p_1 > 0.5,\ p_2 + p_3 + p_4 < 0.5$	$+$	$-$	$+$	$\mathcal{E}_2$
$p_2 > 0.5,\ p_1 + p_3 + p_4 < 0.5$	$+$	$+$	$-$	$\mathcal{E}_1$
$p_3 > 0.5,\ p_1 + p_2 + p_4 < 0.5$	$-$	$-$	$-$	$\mathcal{E}_1$
$p_4 > 0.5,\ p_1 + p_2 + p_3 < 0.5$	$-$	$+$	$-$	$\mathcal{E}_2$

3.3 Two-qubit Amplitude Damping on Depolarized Bell States

A qubit amplitude damping channel is a source of noise in superconducting circuit-based quantum computing and thus, serves as a realistic channel model for simulating lossy processes in these systems. Mathematically, it can be obtained from an isometry J,

$$J : \mathcal{H}_a \mapsto \mathcal{H}_b \otimes \mathcal{H}_c; \quad J^\dagger J = I_a \tag{11}$$

where $\mathcal{H}_a$ denotes the Hilbert space for the channel's input, and $\mathcal{H}_b$ and $\mathcal{H}_c$ represent the Hilbert spaces for the direct and complementary channel outputs, respectively. An isometry of the form,

$$\begin{aligned} J_1 \left|0\right\rangle_a &= \left|0\right\rangle_b \left|1\right\rangle_c, \\ J_1 \left|1\right\rangle_a &= \sqrt{1-r}\left|1\right\rangle_b \left|1\right\rangle_c + \sqrt{r}\left|0\right\rangle_b \left|0\right\rangle_c, \end{aligned} \tag{12}$$

where $0 \leq r \leq 1$ defines a pair of channels, $\mathcal{B}(A) = \mathrm{Tr}_c(JAJ^\dagger)$ and $\mathcal{C}(A) = \mathrm{Tr}_b(JAJ^\dagger)$. Here, $\mathcal{B}$ is an amplitude damping channel with damping probability r for the state $\left|1\right\rangle_a$ to decay to output state $\left|0\right\rangle_b$. The isometry $J_1 = K_0 \otimes \left|0\right\rangle + K_1 \otimes \left|1\right\rangle$ where K_0 and K_1 (Kraus) damping operators such that $K_0 = [0, \sqrt{r}; 0, 0]$ and $K_1 = [1, 0; 0, \sqrt{1-r}]$. For a single qubit represented by state ρ, the amplitude damped output is given by,

$$\mathcal{B}(\rho) = K_0 \rho K_0^\dagger + K_1 \rho K_1^\dagger. \tag{13}$$

We can extend equation 13 for two qubit states with damping probabilities r and q for the first and second qubit respectively. Assuming that $r = q$, we consider Depolarized Bell states equation 9 with amplitude damping.

Proposition 5 *For any damping probability $r > 0$, a Depolarized Bell state with amplitude damping can not be expressed as a Bell diagonal state equation 10.*

This fact can be readily demonstrated through a straightforward calculation. Consider the Isotropic state $\rho(w) = w\left|\Phi^+\right\rangle\left\langle\Phi^+\right| + (1-w)\frac{I}{4}$, which can be represented by the Bell diagonal state formed with probability distribution $(p_1, p_2, p_3, p_4) = ((3w+1)/4, (1-w)/4, (1-w)/4, (1-w)/4)$. In a Bell diagonal state, the diagonal elements corresponding to $\left|00\right\rangle\left\langle 00\right|$ and $\left|11\right\rangle\left\langle 11\right|$ are identical. In the case of an amplitude damped Isotropic state, we observe that,

$$p_2 = p_3 = \frac{1-r}{4}\left(w - wr - r - 1\right).$$

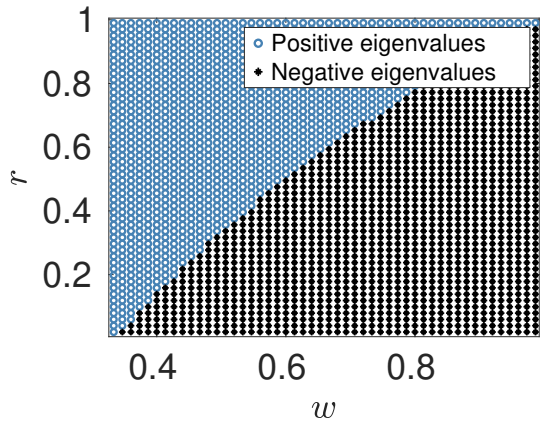
However, obtaining closed-form expressions for p_1 and p_4 when $r > 0$ is cumbersome. Specifically, the values on the diagonal corresponding to $\left|00\right\rangle\left\langle 00\right|$ and $\left|11\right\rangle\left\langle 11\right|$ is given by $w(r^2+1)/2 - (w-1)4(r+1)^2/4$ and $w(r-1)^2/2 - (w-1)(r-1)^2/4$, respectively. These expressions are equal only when $r = 0$.

Proposition 6 *For every $w \in [\frac{1}{3}, 1]$, there exists $\tilde{r} \subset [0, 1]$ such that an amplitude damped Depolarized Bell state becomes separable.*

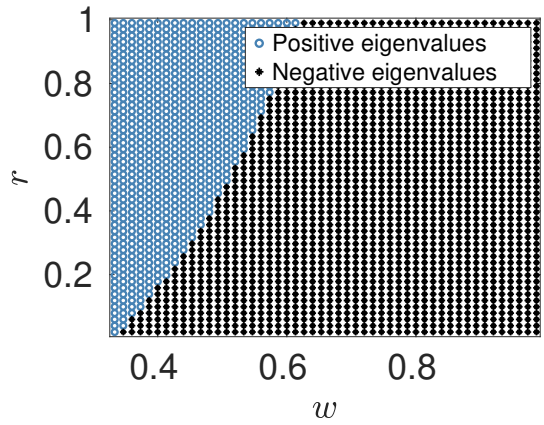
The PPT criterion asserts that a two-qubit state is entangled if and only if its partial transpose contains atleast one negative eigenvalue. For Bell states that are both amplitude damped and depolarized, we evaluate the eigenvalues and observe that one of them can exhibit either positive or negative values contingent upon the range of r. Detailed findings are presented in Table 5 and depicted graphically in Fig. 1a and Fig. 1b. Furthermore, the WBM for amplitude damped and Depolarized Bell states aligns with that of depolarized Bell states, as outlined in Table 3.

Table 5: The four eigenvalues of amplitude damped Depolarized Bell states

State with $\|\Phi\rangle^{\pm}$	State with $\|\Psi\rangle^{\pm}$	**Sign of eigenvalue**
$\frac{(w+1)(1-r^2)}{4}$	$\frac{(1-r)(1+r+w-wr)}{4}$	Always positive
$\frac{(w+1)(1-r)^2}{4}$	$\frac{(1-r)(1+r+w-wr)}{4}$	Always positive
$\frac{w(r-1)^2+(r+1)^2}{4}$	$\frac{r^2+1-w(1-r)^2+2\sqrt{w^2(1-r)^2+r^2}}{4}$	Always positive
$\frac{-r^2(w-1)+wr+(1-3w)}{4}$	$\frac{r^2+1-w(1-r)^2-2\sqrt{w^2(1-r)^2+r^2}}{4}$	Positive and Negative



(a) Range of r for eigenvalue corresponding to $\left|\Phi^{\pm}\right\rangle$



(b) Range of r for eigenvalue corresponding to $\left|\Psi^{\pm}\right\rangle$

Figure 1: A phase diagram representing the region of damping and depolarizing parameters, r and w, respectively, where the dampeddepolarized Bell state has negative or positive partial transpose.

4 Stochastic MAB policies for Entanglement Detection

We apply stochastic MAB algorithms for entanglement detection in the parameterized states $\mathcal{F}$ from Section 3. The terminology follows the alignment with classical counterparts, as outlined in Table 2. Consider a set of K unknown states, denoted by $\mathcal{A} = \{\rho_1, \rho_2, \ldots, \rho_K\} \in \mathcal{F}$. To perform measurements on the arms, the learner must know the underlying WBM. Thus, we assume knowledge of the specific forms of the arms in $\mathcal{A}$. For instance, $\mathcal{A}$ could represent the set of isotropic states detectable under the WBM $\mathcal{E}_2$, where each state is of the form $\rho_i = w_i \left|\Phi^+\right\rangle \left\langle\Phi^+\right| + (1 - w_i)\frac{I}{4}$, with w_i being unknown for all $i \in [K]$. With this assumption, we describe the MAB routine as follows: In each round $t \in \mathbb{N}$,

- The learner selects a state $\rho_i \in \mathcal{A}$.
- The learner performs a measurement $\mathcal{E}$ and obtains outcome j with probability $f_j = \mathrm{Tr}\{\rho_i E_j\}$, where $j \in \{1, 2, 3, 4\}$.
- The learner updates the values of $\hat{\boldsymbol{S}}_{\mathcal{E}}$ and identifies the entangled arm(s) or continues.

For a given WBM $\mathcal{E}$, the values of $S_{\mathcal{E}}$ are bounded in $[-1, 1]$. We use concentration inequalities applicable to 1-subgaussian[1] random variables—specifically, the law of iterated logarithm (Jamieson et al., 2014) for a finite sum of 1-subgaussian random variables:

[1]A 1-subgaussian random variable is a real, centered random variable X that satisfies $\mathbb{E}[e^{sX}] \leq e^{s^2/2}$ for any $s \in \mathbb{R}$.

Algorithm 1 Successive Elimination Algorithm

Input: $\zeta = 0$, δ, $\mathcal{A}$, WBM $\mathcal{E}$
Output: Ω
  Initialize active set $\Omega \leftarrow \mathcal{A}$
  Set initial estimates: $\hat{S}_{i,N_i(t)} = 0,\ \forall i \in \Omega$
  for $t = 1, 2, 3, \ldots$ **do**
    for $\rho_i \in \Omega$ **do**
      Perform measurement $\mathcal{E}$ on ρ_i
      Update $\hat{S}_{i,N_i(t)}$ based on outcome $j \in \{1, 2, 3, 4\}$
      Update confidence width $U\left(N_i(t), \frac{\delta}{c_\varepsilon K}\right)$ (see Lemma 7)
      Compute lower confidence bound: $\mathrm{LCB}_i(t) \leftarrow \hat{S}_{i,N_i(t)} - U\left(N_i(t), \frac{\delta}{c_\varepsilon K}\right)$
    end for
    if $\mathrm{LCB}_i(t) > 0$ for $i \in \Omega$ **then**
      Update active set: $\Omega \leftarrow \Omega - \{i\}$
    end
    if $|\Omega| = 1$ **then**
      Return Ω
    end
  end for

Lemma 7 *Let $X_1, X_2, \ldots X_t$ be* i.i.d. *sub-gaussian random variables with scale parameter $\sigma = 1$. For any $\varepsilon \in (0, 1)$, $\delta \in \left(0, \frac{\log(1+\varepsilon)}{e}\right)$, one has with probability at least $1 - c_\varepsilon \delta^{(1+\varepsilon)}$ for all $t \geq 1$,*

$$\frac{1}{t}\sum_{s=1}^{t} X_s \leq U(t, \delta), \tag{14}$$

where $U(t, \delta) = (1 + \sqrt{\varepsilon})\sqrt{\frac{2(1+\varepsilon)}{t}\log\left(\frac{\log((1+\varepsilon)t)}{\delta}\right)}$ is the confidence width and $c_\varepsilon = \frac{2+\varepsilon}{\varepsilon}\left(\frac{1}{\log(1+\varepsilon)}\right)^{1+\varepsilon}$.

Proof: Readers can refer in Jamieson et al. (2014, Lemma 1). □

In the subsequent sections, we discuss two MAB policies: successive elimination, which is applicable when there is a guarantee of one entangled arm among K arms, and the HDoC policy, designed for scenarios where there are m entangled arms among K, with m being unknown.

4.1 Successive Elimination Algorithm

Consider the set of states $\mathcal{A} = \{\rho_1, \rho_2, \ldots, \rho_K\}$ detectable under WBM $\mathcal{E}$, with the guarantee that exactly one arm in the set is entangled. The underlying problem instance $\boldsymbol{S}_\mathcal{E}$ satisfies the condition $S_\mathcal{E}(\rho_1) \geq S_\mathcal{E}(\rho_2) \geq \cdots > S_\mathcal{E}(\rho_{K-1}) > 0 > S_\mathcal{E}(\rho_K)$. To address this, we adapt the Successive Elimination algorithm (Even-Dar et al., 2002), as outlined in Algorithm 1. This modified algorithm takes as input the set $\mathcal{A}$, the threshold $\zeta = 0$, WBM $\mathcal{E}$ and the error probability δ, and it outputs the entangled state $i^\star = \arg\min_{i\in[K]} S_\mathcal{E}(\rho_i)$. Let $N_i(t)$ denote the number of times ρ_i has been measured in t rounds and $\hat{S}_{i,N_i(t)}$ is the estimate of $S_\mathcal{E}(\rho_i)$ obtained on measuring ρ_i until time t. The algorithm maintains an active set Ω and measures every state in it. In order to identify $i^\star$, the policy eliminates states states Lower Confidence Bound (LCB) exceeds the threshold and halts when only one state remains in the active set.

Lemma 8 *Algorithm 1 is δ-PC.*

Proof: The proof is presented in Appendix A.1.1. □

The correctness of Algorithm 1 and the copy complexity of identifying the entangled arm is presented below.

Theorem 9 *With probability at least $1-\delta$, the entangled state $i^\star = K = \arg\min_{i\in[K]} S_{\mathcal{E}}(\rho_i)$ remains in the active set Ω till termination.*

Proof: The proof is presented in Appendix A.1.2. □

Theorem 10 *With probability at least $1-\delta$, Algorithm 1 identifies the entangled state $i^\star$, requiring $\sum_{i\in[K]} \mathcal{O}\left(\Delta_i^{-2}\log\left(\frac{K\log\Delta_i^{-2}}{\delta}\right)\right)$ copies. Here, $\Delta_i = |S_{\mathcal{E}}(\rho_i)-\zeta|$ denotes the sub-optimality gap with respect to the threshold ζ.*

Proof: The proof is presented in Appendix A.1.3. □ We observe that the sample complexity derived in Theorem 10 is within a $\log(K)$ factor of the optimal bound, as demonstrated in Theorem 1 of Jamieson et al. (2014). This result follows from the concentration bound established in Lemma 7, which forms the basis for the MAB policy described in the following section.

4.2 lil'HDoC Algorithm

The lil'HDoC algorithm (Tsai et al., 2024) builds on the HDoC algorithm (Kano et al., 2018) by leveraging finite LIL concentration bounds (Lemma 7) instead of the LCB-based identification rule (Kalyanakrishnan et al., 2012). To explore among promising arms, lil'HDoC adopts the sampling rule from Kano et al. (2018), derived from the UCB algorithm for regret minimization (Auer et al., 2002). It improves sample complexity over HDoC by utilizing the LIL bound, where the $\sqrt{\log\log t/t}$ factor has a higher growth rate than the $\sqrt{\log t/t}$ factor in the LCB bound. In other words, there exists a value T such that for all $t > T$, $c_1, c_2 \in \mathbb{R}^+$,

$$c_1\sqrt{\frac{\log t}{t}} > c_2\sqrt{\frac{\log\log t}{t}}.$$

The confidence bound for HDoC grows as $\alpha(t) = \sqrt{\ln\left(\frac{4Kt^2}{\delta}\right)/2t}$. Through straightforward calculations, the smallest integer T such that the confidence bound $U\left(T, \delta/c_\varepsilon K\right)$ is greater than $\alpha(T)$ is,

$$T \geq \frac{1}{4}\log(K+1)\log\left(\max\left(\frac{1}{\delta}, 2\right)\right)c_\varepsilon^{3/2}. \tag{15}$$

Thus, if each state is measured T times initially, lil'HDoC achieves comparable identification capabilities to HDoC with $\mathcal{O}\left(\log(K+1)\log\left(\max\left(1/\delta, 2\right)\right)\right)$ copies of each state.

Consider K states such that $S_{\mathcal{E}}(\rho_1) \geq S_{\mathcal{E}}(\rho_2) \ldots > S_{\mathcal{E}}(\rho_{K-m}) > 0 > S_{\mathcal{E}}(\rho_{K-m+1}) \ldots > S_{\mathcal{E}}(\rho_K)$, with m being unknown. The algorithm takes as input, the set of states $\mathcal{A}$, threshold $\zeta = 0$, WBM $\mathcal{E}$ and the error probability δ and outputs $\mathcal{A}_{\text{ent}} = \{i \in [K] \text{ such that } S_{\mathcal{E}}(\rho_i) < 0\}$. The algorithm maintains an active set Ω and terminates when the set $\Omega = \emptyset$

To demonstrate the correctness of Algorithm 2, we first show that the algorithm is (λ, δ)-PAC for all $\lambda \in [K]$ and then characterize the copy complexity of identifying m entangled states.

Lemma 11 *Algorithm 2 is δ-PAC.*

Proof: *The proof is presented in Appendix A.2.1.* □

Theorem 12 *With probability at least $1-\delta$, the algorithm identifies all the states in $\mathcal{A}_{ent}$.*

Proof: The proof is presented in Appendix A.2.2. □

With $T = 1$ in equation 15, it can be seen from Theorem 10 that the number of samples required to identify an entangled state $\rho_i \in \mathcal{A}$ is $\mathcal{O}\left(\Delta_i^{-2}\log\left(\frac{K\log\Delta_i^{-2}}{\delta}\right)\right)$. However, in practice, T is chosen to be larger than 1, and the total sample complexity is expressed in terms of $\Delta = \min_{i\in[K]}\Delta_i$.

Algorithm 2 LIL'HDOC ALGORITHM

Input: $\zeta = 0$, δ, $\mathcal{A}$, WBM $\mathcal{E}$
Output: $\mathcal{A}_{\text{ent}}$
  Initialize active set $\Omega \leftarrow \mathcal{A}$, $\mathcal{A}_{\text{ent}} \leftarrow \emptyset$
  Set initial estimates: $\hat{S}_{i,N_i(t)} = 0, \ \forall i \in \Omega$
  for $\rho_i \in \Omega$ **do**
    Perform measurement $\mathcal{E}$ on ρ_i for T times
    $N_i(t) \leftarrow T$
    Update $\hat{S}_{i,T}$ based on outcome $j \in \{1,2,3,4\}$
  end for
  while $\Omega \neq \emptyset$ **do**
    Find $h_t = \arg\max_{i \in \mathcal{A}} \ \hat{S}_{i,N_i(t)} + \sqrt{\frac{\log t}{2N_i(t)}}$
    Perform measurement $\mathcal{E}$ on ρ_{h_t}
    $t \leftarrow t+1$
    Update $\hat{S}_{i,N_i(t)}$ based on outcome $j \in \{1,2,3,4\}$
    Update confidence width $U\left(N_i(t), \frac{\delta}{c_\varepsilon K}\right)$
    if $\hat{S}_{h_t,N_{h_t}(t)} - U\left(N_{h_t}(t), \frac{\delta}{c_\varepsilon K}\right) \geq \zeta$ **then**
      Remove h_t from Ω
    else if $\hat{S}_{h_t,N_{h_t}(t)} + U\left(N_{h_t}(t), \frac{\delta}{c_\varepsilon K}\right) < \zeta$ **then**
      Add h_t to $\mathcal{A}_{\text{ent}}$
      Remove h_t from Ω
    end
  end while

Theorem 13 *With probability $1-\delta$ and an initialization of T measurements, Algorithm 2 identifies the entangled states using $\mathcal{O}\left(\Delta^{-2}\left(K\log\frac{1}{\delta} + K\log K + K\log\log\frac{1}{\Delta}\right)\right) + \mathcal{O}\left(K\log(K+1)\log\left(\max\left(\frac{1}{\delta}, e\right)\right)\right)$ copies.*

Proof: The first term in the sample complexity is derived in Appendix A.1.3 and the second term follows from equation 15. □

5 Implementation and Simulations

This section presents an experimental workflow for detecting entangled states from an ensemble of Bell Diagonal states. Sections 5.1 and 5.2 describe the procedures for generating Bell Diagonal states (BDS) and their corresponding WBMs, respectively. The performance of the MAB policies (see Section 4) are presented through numerical findings in Sections 5.4.

5.1 Generating Bell Diagonal States

Bell Diagonal States (BDS) are constructed as convex combinations of the four Bell states equation 10, forming a geometric tetrahedron $\mathcal{T}$ and are represented by:

$$\rho_{\text{Bell}} = \sum_{j=1}^{4} p_j \left|\Upsilon\right\rangle \left\langle\Upsilon\right| = \frac{1}{4}\left[I + \sum_{j=1}^{3} t_j \sigma_j^A \otimes \sigma_j^B\right]. \tag{16}$$

Here, σ_j's are the Pauli operators and (t_1, t_2, t_3) are the coordinates within the tetrahedron $\mathcal{T}$.

The mapping $\{p_j\}_{j=1}^4 \to (t_1, t_2, t_3)$ equation 17 is implemented through the quantum circuit proposed by Pozzobom & Maziero (2019); Riedel Gårding et al. (2021) and is shown in Fig. 2.

$$\begin{aligned}
\sqrt{p_1} &= \cos(\psi) \\
\sqrt{p_2} &= \sin(\psi)\cos(\theta) \\
\sqrt{p_3} &= \sin(\psi)\sin(\theta)\cos(\varphi) \\
\sqrt{p_4} &= \sin(\psi)\sin(\theta)\sin(\varphi)
\end{aligned} \tag{17}$$

The sub-circuit G encodes the probabilities $\{p_j\}_{j=1}^4$ into canonical coordinates (ψ, θ, φ) on the unit 3-sphere, and sub-circuit B entangles the states in the Bell basis. Finally, BDS $\rho_{\text{Bell}} = \rho_{cd}$ is obtained by taking a partial trace on qubits a and b.

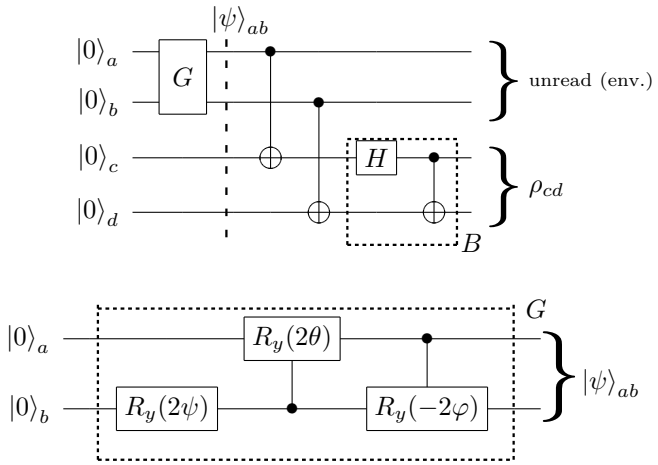


Figure 2: Four-qubit circuit for generating BDS with canonical encoder G shown below.

5.2 Implementing Witness Basis Measurements

As outlined in Table 4, BDS are detectable under WBMs $\mathcal{E}_1$ and $\mathcal{E}_2$. To measure in the Pauli-Z basis, we apply appropriate unitary transformations to $\mathcal{E}_1$ and $\mathcal{E}_2$. The corresponding transformations are realized through circuits $\text{CIRC}_{\mathcal{E}_1}$ and $\text{CIRC}_{\mathcal{E}_2}$ shown in Fig. 3 and applied to qubits c and d (see Fig. 2) before measurement.

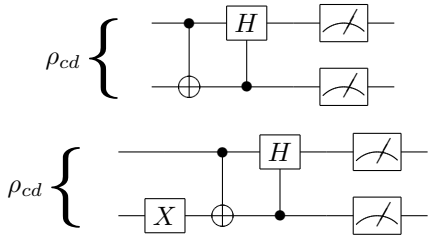


Figure 3: Circuits $\text{CIRC}_{\mathcal{E}_1}$ (top) and $\text{CIRC}_{\mathcal{E}_2}$ (bottom) perform the unitary transformations required to map $\mathcal{E}_1$ and $\mathcal{E}_2$ into the Pauli-Z basis.

5.3 Workflow for entanglement detection

We propose a workflow for detecting entanglement in BDS without assuming prior knowledge of the specific WBM. Instead, WBMs are sequentially adapted using suitable unitary transformations, as detailed in Table 1. To generate set $\mathcal{A} = \{\rho_1, \rho_2, \ldots, \rho_K\}$ of BDS, we construct K sets of probabilities for combining the

four Bell states. Specifically, m states are generated with $\max_j p_j > \frac{1}{2}$, while the remaining $K-m$ states satisfy $\max_j p_j \leq \frac{1}{2}$. These probabilities are encoded following the procedure outlined in Fig. 2, where the BDS circuit for state ρ_i is denoted as BDSi. Subsequently, one of two WBM circuits, $\text{CIRC}_{\mathcal{E}_1}$ or $\text{CIRC}_{\mathcal{E}_2}$, is appended to the respective BDS circuit. Algorithm 3 outlines this workflow for BDS and takes the following inputs: threshold $\zeta = 0$, error δ, BDS circuits $\{\text{BDS}_i\}$, WBM circuits $\text{CIRC}_{\mathcal{E}_1}$ and $\text{CIRC}_{\mathcal{E}_2}$ and the initial choice of WBM. Notably, the initial WBM selection is arbitrary, as the sequence of WBM adaptations does not rely on prior state estimation.

Algorithm 3 Workflow for Entanglement Detection in BDS

Input: $\zeta = 0$, δ, $\{\text{BDS}_i\}$, $\text{CIRC}_{\mathcal{E}}$, WBM choice = 1
Output: A_{ent}, Stopping time τ
  Run Algorithm 2 on $\{\text{BDS}_i\}$ with circuit $\text{CIRC}_{\mathcal{E}_1}$ on K states
  Return entangled states $|A_{\text{ent},1}| = \tilde{m}$ and stopping time τ_1.
  if $(\tilde{m} = K)$ **then**
    $A_{\text{ent},2} \leftarrow \emptyset$, $\tau_2 \leftarrow 0$.
  else if $\tilde{m} < K$ **then**
    Run Algorithm 2 on $\{\text{BDS}_i\}$ with circuit $\text{CIRC}_{\mathcal{E}_2}$ on $K - \tilde{m}$ states
    Return entangled states $A_{\text{ent},2}$ and stopping time τ_2.
  end
  $A_{\text{ent}} \leftarrow A_{\text{ent},1} + A_{\text{ent},2}$, $\tau \leftarrow \tau_1 + \tau_2$

The learner does not initially know under which WBM the BDS are detectable. Consequently, at least one iteration of Algorithm 2 must be executed. In the first iteration, the algorithm processes circuits corresponding to K states with WBM $\mathcal{E}_1$ (or $\mathcal{E}_2$) and identifies a subset of entangled states, $\tilde{m}$, where $0 \leq \tilde{m} \leq K$. In the second iteration, Algorithm 2 is applied to the $K - \tilde{m}$ states that remain undetected by using circuits with WBM $\mathcal{E}_2$ (or $\mathcal{E}_1$) as inputs. Let us define $\Delta_1 \coloneqq \min |\boldsymbol{S}_{\mathcal{E}_1}|$, $\Delta_2 \coloneqq \min |\boldsymbol{S}_{\mathcal{E}_2}|$ and $\Delta_{\min} = \min\{\Delta_1, \Delta_2\}$, then

Corollary 14 *With probability* $1 - \delta$ *and* $T = 1$*, Algorithm 3 identifies entangled BDS using* $2\mathcal{O}\left(\Delta_{min}^{-2}\left(K \log \frac{1}{\delta} + K \log K + K \log\log \frac{1}{\Delta_{min}}\right)\right)$ *copies.*

5.4 Qiskit Experiment

The workflow presented in Algorithm 3 is simulated on IBM's Qiskit. The implementation is available in Bharati (2025). We present numerical results on the achievable copy complexity for entanglement detection in BDS. The experimental setup is given as follows:

- *Simulation Environments:* The workflow is executed across three computational setups: (i) **AerSimulator** for idealized, noiseless simulations, (ii) **FakeBrisbane** backend to simulate noisy quantum environments, and (iii) **ibm-brisbane** for real quantum processing unit (QPU) computations.
- *Problem Instance*: We consider $K = 5$ states of which $m = 3$ are entangled. The probabilities are suitably generated and the true corresponding parameters under $\mathcal{E}_1$ and $\mathcal{E}_2$ are,

$$\boldsymbol{S}_{\mathcal{E}_1} = [0.6306, -0.2688, 0.5232, 0.1796, 0.0695]$$

$$\boldsymbol{S}_{\mathcal{E}_2} = [-0.0749, 0.5963, -0.1735, 0.2801, 0.3768]$$

  .
- Each state was measured 10^6 times on backends (i, ii) and 10^5 times on (iii). Algorithm 3 was run 20 times on (i, ii) and 5 times on (iii) for $\delta \in (0, 1)$. We plot the average number of copies measured until stoppage on the y-axis and $\log(1/\delta)$ on the x-axis, as shown in Fig.4. Here, we note that the large standard deviation for the trend in backend (iii) arises due to the limited number of experiment iterations, constrained by available compute resources.

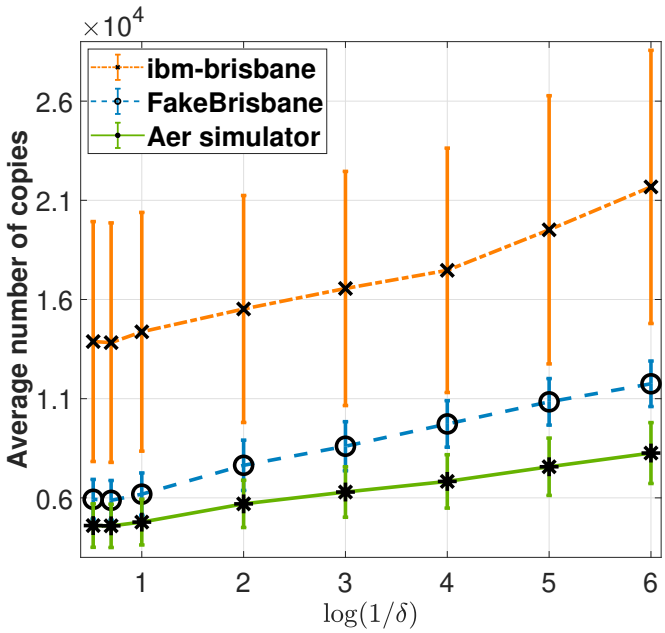


Figure 4: Copy complexity for entanglement detection in BDS v/s $\log(1/\delta)$

From Corollary 14, we observe that the factor $\log(1/\delta)$ has a multiplicative effect on the sample complexity, while the average copy complexity is primarily determined by $\Delta_{\min}$. The values of $S_{\mathcal{E}}$ are governed by the four frequencies f_1, f_2, f_3, and f_4, as defined in equation 7. While the true values of the f_i's are calculated using $\mathrm{Tr}\{\rho_{\mathrm{Bell}} E_i\}$, the values of f_i obtained from register countsbased on simulations performed on different backendsdiffer from the true values upto $\mathcal{O}(10^{-2})$. Due to measurement noise and decoherence, the goalpost for $S_{\mathcal{E}}$ varies across different backends and these differences influence $\Delta_{\min}$. One option is to mitigate the measurement noise (see details in the appendix, Sec. A.3)

6 Entanglement Detection in Arbitrary Quantum States

This section outlines a routine for detecting entanglement in arbitrary two-qubit quantum states. Specifically, we consider K arbitrary states, one of which is entangled, and describe an MAB routine along with numerical results.

6.1 Numerical Experiment

The workflow outlined in Algorithm 4 is implemented in MATLAB. The algorithm takes the following inputs: a threshold ζ, an error parameter δ, a set of K states $\mathcal{A}$ (with the promise that one state is entangled), and a permutation of $\{1, 2, 3, 4, 5, 6\}$ that specifies the order in which the WBMs should be adapted. As this is a promise problem, the algorithm terminates as soon as it identifies an entangled state, without needing to measure with all six WBMs. The different modules in the software are described below:

- *Generating arbitrary quantum states*: To generate random density matrices, we follow the method described in Zyczkowski & Sommers (2001). Specifically, we start by generating a complex matrix $A \in \mathbb{C}^{4\times 4}$, where the real and imaginary parts of each element are independently sampled from a normal distribution. We then compute the density matrix ρ by normalizing $AA^\dagger$, resulting in $\rho = AA^\dagger/\mathrm{Tr}(AA^\dagger)$. This procedure ensures that ρ is a valid density matrix.
- *Experiment Setup*: In this experiment, we generate 1000 distinct instances of $K = 5$ full rank arbitrary states, ensuring that each instance contains exactly one entangled state. These instances are validated using the PPT criterion to confirm their validity.

Algorithm 4 Entanglement detection for arbitrary states

Input: $\zeta = 0$, δ, $\mathcal{A} \leftarrow \{\rho_1, \rho_2 \dots \rho_K\}$, WBM Order P
Output: A_{ent}
  flag $\leftarrow 1$, $I \leftarrow 1$
  while flag **do**
    With $\mathcal{E} \leftarrow \mathcal{E}_{P(I)}$, run Algorithm 2 for K arms
    Return entangled arm $A_{\text{ent}}^{(I)}$
    if $|A_{\text{ent}}^{(I)}| = 1$ **then**
      flag $\leftarrow 0$
    else
      $I \leftarrow I + 1$
    end
  end while
  $A_{\text{ent}} \leftarrow A_{\text{ent}}^{(I)}$

- We test the efficacy of using the single parameter family of witnesses equation 4 to detect entanglement in arbitrary states. For $\delta \in (0, 1)$, we report the detection ratio which is the fraction of times the entangled state is accurately identified by the MAB policy. This result is shown in Fig. 5. We observe that the detection ratio diminishes with larger error margins.

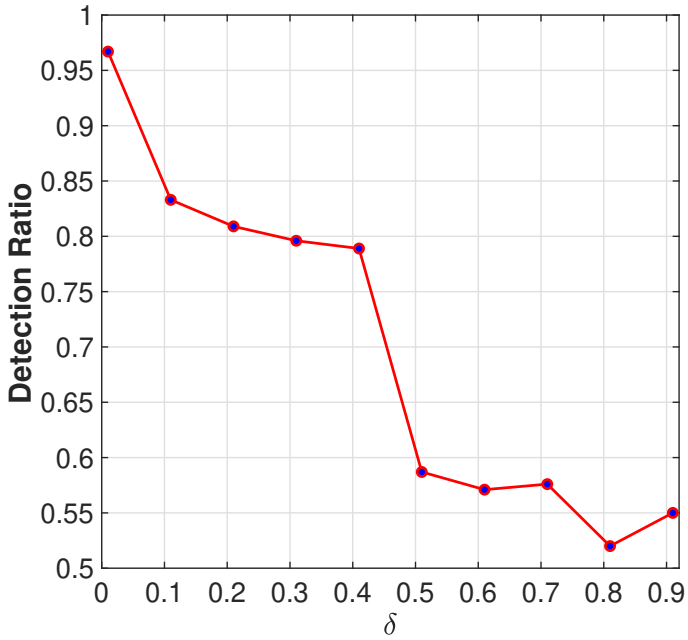


Figure 5: Entanglement Detection ratio v/s δ for arbitrary quantum states

- For a random order of WBM, we analyze how many measurements from the witness family are required to detect a single valid entangled state among a set of K states. For $\delta \in (0, 1)$, we present the frequency distribution of the number of WBMs used, displayed as a cumulative histogram in Fig. 6. For significantly larger values of δ, the lower detection ratios indicate that the algorithm terminates upon identifying the wrong state, preventing further adaptation and primarily (around 85%) relying on up to three witnesses.

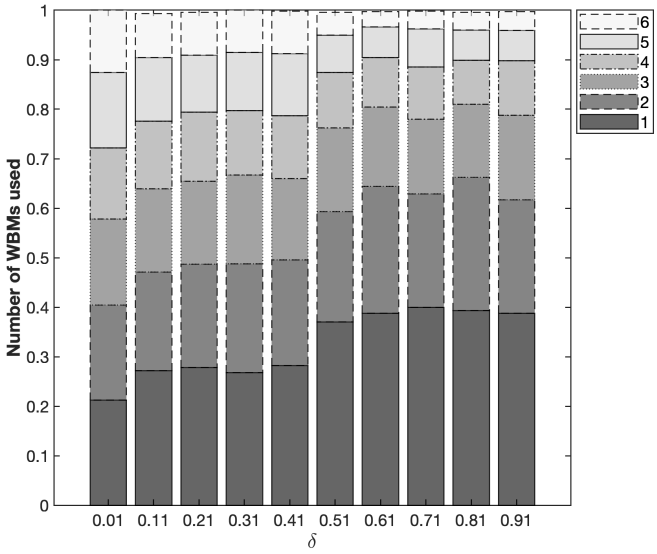


Figure 6: The cumulative histograms compare between the number of WBMs used to detect one valid entangled state across different values of δ.

Table 6: Examples of arbitrary pure entangled states detected by the family of witnesses equation 4

Pure entangled states $\|\psi_1\rangle, \|\psi_2\rangle$ and $\|\psi_3\rangle$	Values under $(S_{\mathcal{E}_i})_{i=1}^{6}$
$[0.2687 + 0.0375i; 0.2406 + 0.4090i; 0.0502 + 0.6162i; 0.2413 + 0.5107i]$	$(-0.1851, 0.3160, 0.1598, -0.0058, 0.2177, -0.1947)$
$[0.0565 + 0.3355i; 0.0508 + 0.0686i; 0.4885 + 0.5191i; 0.5689 + 0.2125i]$	$(0.1562, -0.0280, -0.1135, 0.1832, -0.0779, 0.1373)$
$[0.1953 + 0.4438i; 0.4958 + 0.4009i; 0.0069 + 0.3495i; 0.0322 + 0.4848i]$	$(-0.1851, 0.3160, 0.1598, -0.0058, 0.2177, -0.1947)$

In this experiment, we encountered edge cases, i.e., instances of pure states ρ with value of $S_{\mathcal{E}}(\rho) = 0$. For such edge cases, the algorithm took a significantly long time to converge and, despite this, incorrectly estimated the value of $S_{\mathcal{E}}(\rho)$. Consequently, we adjusted the threshold to -1×10^{-3} and imposed a cutoff on the sample complexity at 1×10^{12} to better reflect the real-time performance of this policy. This experiment can be extended to the scenario where there are m entangled states. However, since m is unknown and the states may be detectable under any of the WBMs, the routine would necessitate measuring under all WBMs to reliably identify the entangled states.

6.2 Numeric Examples

We present an example of a PPT-verified entangled state that yields positive values for $S_{\mathcal{E}}(\rho)$ under all six WBMs. Consider the pure entangled states and their corresponding $\boldsymbol{S}_{\mathcal{E}}$ values, as shown in Table 6. The state $\rho = \sum_{i=1}^{3} p_i \left|\psi_i\right\rangle \left\langle\psi_i\right|$, where $\left|\psi_i\right\rangle$ are defined in Table 6, and $(p_i)_{i=1}^{3} = (0.2936, 0.0655, 0.6409)$, has a negative eigenvalue of -0.029 after applying the partial transpose, thus confirming it as a PPT-verified entangled state. However, the values of $(S_{\mathcal{E}}) = (0.0732, 0.1727, 0.1257, 0.1139, 0.0736, 0.0296)$ under the six WBMs are all non-negative. This indicates that the state cannot be detected by the witness family described in equation 4.

We derive an observation on the nature of such states, focusing specifically on the eigenstate $\left|\lambda\right\rangle_{\max} = [0.3773 - 0.1445i, 0.4768 - 0.3244i, 0.4598 + 0.0809i, 0.5351]$, which corresponds to the largest eigenvalue of ρ. This eigenstate has a Schmidt coefficient close to, but not exactly equal to 1, suggesting that it lies near the boundary of separable states while still remaining entangled. The pure state $\left|\lambda\right\rangle_{\max} \left\langle\lambda\right|_{\max}$ produces the following values for $(S_{\mathcal{E}}) = (0.0380, 0.1269, 0.0401, 0.1054, 0.0221, 0.0074)$. This highlights that both pure and mixed entangled states can yield inconclusive results when measured using this specific witness family.

In these cases, it is crucial to measure all six witnesses a sufficient number of times to accurately obtain the expected values of the corresponding observables. Additionally, performing FST can assist in determining the entanglement of these states through other separability criteria.

7 Discussions

7.1 Does the MAB routine optimize WBM ordering?

As outlined in Dai et al. (2014), there are WBM optimization strategies that prescribe an optimal WBM ordering for efficiently detecting whether a *single* arbitrary two-qubit quantum state is entangled. One such adaptive strategy uses the maximum-likelihood maximum-entropy (MLME) estimate of the unknown state, based on causal measurement data. Using this estimate, the subsequent WBM is identified to be the one minimizing the quadratic separability criterion. This leads to partial estimation of the quantum state.

In the context of batch entanglement detection, where an unknown number m of entangled states out of a set of K states may be detectable under different witnesses, implementing the WBM adaptive strategies from Dai et al. (2014) would be both time-consuming and complex. This is because each of the K states may require a unique permutation of the WBM ordering. Furthermore, the goal of the proposed MAB framework is to *minimize* the number of measurements needed for detecting entanglement in a given set of quantum states under a specific WBM. Notably, this framework *does not* optimize the WBM ordering across multiple MAB runs.

The closest comparison is with Fig. 6, which depicts the cumulative frequency of WBMs used. This aligns with the cumulative percentage of states identified under the WBM family, as seen in schemes 1A and 4A of the recently reported incomplete state estimation techniques (see (Dai et al., 2014, Fig. 1)). However, this approach does not address the batch entanglement detection problem. The WBM adaptation scheme A in Dai et al. (2014) successfully detects 98% of random pure states but only 33% of full-rank mixed states. We specifically analyze the latter category, generating multiple instances of K states to quantify the number of WBMs required to detect a single entangled state, presenting results for varying δ. Notably, Dai et al. (2014) lacks numerical insights into the sample complexity and convergence rate of its proposed schemes.

8 Future Work And Conclusion

Batch entanglement detection, as discussed in this paper, is particularly useful for verifying the integrity of a batch of practically relevant entangled states, before use in applications like secure multi-channel quantum communication. We established a novel correspondence between the problem of batch entanglement detection and the Thresholding Bandit problem in stochastic Multi-Armed Bandits. We proposed the (m, K)-quantum Multi-Armed Bandit framework for entanglement detection. Focus of this framework is on identifying m entangled states out of K states, where m is potentially unknown. We apply this framework to two-qubit states using two key ingredients: a specialized set of six measurements for two-qubit states called Witness Basis Measurements (WBM) $\mathcal{E}$ and a separability criterion $S_{\mathcal{E}}$, which is based on the data obtained from these measurements and serves as the parameter that needs to be estimated. We present theoretical guarantees and numerical simulations to demonstrate how this parameter can be estimated quickly and accurately using MAB policies. First, we show that entangled states belonging to a class of parameterised two-qubit states $\mathcal{F}$ can be detected by measuring a subset of the six WBMs. With the knowledge of the WBM, we show that we can directly apply some suitable MAB policies. Second, for the same parameterised states, we present a routine for entanglement detection when the WBM is not known by enabling arbitrary sequential adaptation of the WBMs. We extend this to arbitrary two qubit quantum states and provide numerical results on the efficacy of using these measurements for detecting entanglement.

An exciting avenue for future research lies in identifying WBMs for higher-dimensional bipartite systems. The minimalistic tomographic scheme proposed in Zhu et al. (2010) significantly reduces the number of required witnesses for two-qutrits from 81 to just 11, demonstrating the potential for more efficient entanglement detection. Meanwhile, recent advancements in data-driven machine learning, particularly the use of SVMs to construct linear entanglement witnesses from local measurements (Greenwood et al., 2023), open new

possibilities for tackling the (m, K)-quantum MAB problem. By leveraging these techniques, one could optimize the number of witnesses needed to reliably detect all m states.

Entanglement detection can be reframed as a membership problem, where a state belongs to a set if it exhibits a specific propertysuch as entanglement. This perspective aligns with the partition identification problem (Juneja & Krishnasamy, 2019), where the objective is to determine the partition to which a data point belongs based on a hyperplane structure. Extending this framework to the (m, K)-quantum MAB problem could pave the way for groundbreaking approaches to adaptive entanglement detection.

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A Appendix

The following lemma is useful for some calculations.

Lemma 15 *For $t \geq 1, c > 0, \varepsilon \in (0,1), 0 < w \leq 1$,*

$$\frac{1}{t}\log\left(\frac{\log\left((1+\varepsilon)t\right)}{w}\right) \geq c \implies t \leq \frac{1}{c}\log\left(\frac{2\log\left(\frac{(1+\varepsilon)}{cw}\right)}{w}\right). \tag{18}$$

A.1 Proof for Section 4.1

A.1.1 Proof of Lemma 8

Proof: Let $\mathcal{B}$ denote the "good" event that at any time $t > 0$ and for all arms $i \in [K]$, the true value $S_{\mathcal{E}}(\rho_i)$ is well concentrated around its estimate $\hat{S}_{i,N_i(t)}$.

$$\mathcal{B} \coloneqq \bigcup_{i=1}^{K} \bigcup_{t=1}^{\infty} \left\{ |\hat{S}_{i,N_i(t)} - S_i| \leq U\left(N_i(t), \frac{\delta}{c_\varepsilon K}\right) \right\}$$

From Lemma 7 and by applying the union bound, we get that

$$\mathbb{P}\left[\mathcal{B}\right] \geq 1 - c_\varepsilon K \left(\frac{\delta}{c_\varepsilon K}\right)^{1+\varepsilon} \geq 1 - \delta \tag{19}$$

where Eq. 19 holds because $\varepsilon \in (0, 1)$ and $c_\varepsilon \geq 1$. □

A.1.2 Proof of Theorem 9

Proof: Recall that the threshold $\zeta = 0$ and problem instance $\boldsymbol{S}_{\mathcal{E}}$ is such that $S_{\mathcal{E}}(\rho_1) \geq S_{\mathcal{E}}(\rho_2) \geq S_{\mathcal{E}}(\rho_3) \ldots > S_{\mathcal{E}}(\rho_{K-1}) > 0 > S_{\mathcal{E}}(\rho_K)$. Let us consider the case that the event $\mathcal{B}$ described in Lemma 8 holds. As outlined in Algorithm 1, the arm $i^\star$ will be dropped from the active set Ω if $\mathrm{LCB}_{i^\star}(t) > 0$. That is,

$$\begin{aligned}
&\hat{S}_{i^\star,N_{i^\star}(t)} - U\left(N_{i^\star}(t), \frac{\delta}{c_\varepsilon K}\right) > 0 \\
&\hat{S}_{i^\star,N_{i^\star}(t)} - |\hat{S}_{i^\star,N^\star_i(t)} - S_{i^\star}| > 0 \\
&\implies S_{i^\star} > 0
\end{aligned}$$

This contradicts the assumption about the problem instance $\boldsymbol{S}$ because $S_{i^\star} = S_{\mathcal{E}}(\rho_K) < 0$ and so, the arm $i^\star$ will not be dropped from the active set Ω as long as event $\mathcal{B}$ holds. □

A.1.3 Proof of Theorem 10

Proof: Let us consider the case where $\mathcal{B}$ holds. By the elimination rule of Algorithm 1, an arm i is removed from the active set Ω if $\mathrm{LCB}_i(t) > 0$. We have that,

$$\begin{aligned}
&\hat{S}_{i,N_i(t)} - U\left(N_i(t), \frac{\delta}{c_\varepsilon K}\right) \geq \zeta \\
&\hat{S}_{i,N_i(t)} - S_i + \Delta_i \geq U\left(N_i(t), \frac{\delta}{c_\varepsilon K}\right) \\
&\implies \Delta_i \geq 2U\left(N_i(t), \frac{\delta}{c_\varepsilon K}\right)
\end{aligned} \tag{20}$$

Let us denote N_i to be the number of samples of arm i, that is, $N_i = \inf\{t : U\left(N_i(t), \frac{\delta}{c_\varepsilon K}\right) \leq \frac{\Delta_i}{2}\}$. The minimum value of N_i can be obtained by solving,

$$\begin{aligned}
&U\left(N_i, \frac{\delta}{c_\varepsilon K}\right) = \frac{\Delta_i}{2} \\
&(1+\sqrt{\varepsilon})\sqrt{\frac{2(1+\varepsilon)}{N_i}\log\left(\frac{\log\left((1+\varepsilon)N_i\right)}{\delta/c_\varepsilon K}\right)} = \frac{\Delta_i}{2} \\
&\frac{1}{N_i}\log\left(\frac{\log\left((1+\varepsilon)N_i\right)}{\delta/c_\varepsilon K}\right) = \frac{\Delta_i^2}{8(1+\varepsilon)(1+\sqrt{\varepsilon})^2}
\end{aligned} \tag{21}$$

From Lemma 15, we get that,

$$N_i = \frac{8(1+\varepsilon)(1+\sqrt{\varepsilon})^2}{\Delta_i^2} \log\left(\frac{2c_\varepsilon K \log\left(\frac{8c_\varepsilon(1+\varepsilon)^2(1+\sqrt{\varepsilon})^2}{\delta}\frac{K}{\Delta_i^2}\right)}{\delta}\right) \tag{22}$$

Thus, the total number of samples required to identify the arm $i^\star$ with a probability of at least $1-\delta$ is $N \leq \sum_{i=1}^{K} N_i$. □

A.2 Proof for Section 4.2

A.2.1 Proof of Lemma 11

Proof: Firstly, we show that Algorithm 2 is (λ, δ)-PAC for arbitrary $\lambda \in [K]$. In the case where there are arms greater than or equal to λ, we show that $\mathbb{P}\left[\{\hat{m} < \lambda\} \cup \bigcup_{i \in \mathcal{A}_{\text{ent}}} \{S_i < \zeta\}\right] \leq \delta$ where $\hat{m}$ is the number of good arms identified by the agent. Since we are now considering the case when $m \geq \lambda$, the event $\{\hat{m} < \lambda\}$ implies that at least one good arm $j \in [m]$ is identified as a bad arm by the agent. That is, for some $j \in [m]$ and $t \in \mathbb{N}$, the upper confidence bound $\hat{S}_{j,N_j(t)} + U\left(N_j(t), \frac{\delta}{c_\varepsilon K}\right) < \zeta$. Thus, we have that,

$$\begin{aligned}
\mathbb{P}\left[\hat{m} < \lambda\right] &\leq \sum_{j\in[m]} \mathbb{P}\left[\bigcup_{t\in\mathbb{N}} \{\hat{S}_{j,N_j(t)} + U\left(N_j(t), \frac{\delta}{c_\varepsilon K}\right) < \zeta\}\right] \\
&\leq \sum_{j\in[m]} c_\varepsilon \left(\frac{\delta}{c_\varepsilon K}\right)^{1+\varepsilon} \qquad \text{(By Lemma 7)} \\
&\leq m c_\varepsilon \left(\frac{\delta}{c_\varepsilon K}\right)
\end{aligned} \tag{23}$$

The event $\bigcup_{i\in\{\hat{X}_1,\hat{X}_2,\ldots\hat{X}_\lambda\}}\{\mu_i < \zeta\}$ considers all those outcomes where a bad arm is identified to be a good one. Thus, for some bad arm $j \in \{\hat{X}_1, \hat{X}_2, \ldots \hat{X}_{\hat{m}}\}$ such that $j \in [K] \setminus [m]$, we have,

$$\begin{aligned}
&\mathbb{P}\left[\bigcup_{i\in\{\hat{X}_1,\hat{X}_2,\ldots\hat{X}_\lambda\}} \{S_i < \zeta\}\right] \\
&\leq \sum_{j\in[K]\setminus[m]} \mathbb{P}\left[\bigcup_{t\in\mathbb{N}} \{\hat{S}_{j,N_j(t)} - U\left(N_j(t), \frac{\delta}{c_\varepsilon K}\right) > \zeta\}\right] \\
&\leq (K-m) c_\varepsilon \left(\frac{\delta}{c_\varepsilon K}\right)
\end{aligned} \tag{24}$$

Thus, putting Eq. 23 and Eq. 24 together, we get that $\mathbb{P}\left[\{\hat{m} < \lambda\} \cup \bigcup_{i\in\{\hat{X}_1,\hat{X}_2,\ldots\hat{X}_{\hat{m}}\}}\{\mu_i < \zeta\}\right] \leq \delta$. Next, we consider the case when the number of good arms m is less than λ and show that $\mathbb{P}\left[\hat{m} \geq \lambda\right] \leq \delta$. Since there are at most λ good arms, the event $\{\hat{m} > \lambda\}$ implies that one of the output arms $j \in \{\hat{X}_1, \hat{X}_2, \ldots \hat{X}_\lambda\}$ is such that there exists some index j such that $\hat{X}_j$ is a bad arm. Thus, we have that,

$$\begin{aligned}
\mathbb{P}\left[\hat{m} \geq \lambda\right] &\leq \sum_{j\in[K]\setminus[m]} \mathbb{P}[j \in \{\hat{X}_1, \hat{X}_2, \ldots \hat{X}_\lambda\}] \\
&\leq (K-m) c_\varepsilon \left(\frac{\delta}{c_\varepsilon K}\right)^{1+\varepsilon} \\
&\leq \frac{K-m}{K} c_\varepsilon \left(\frac{\delta}{c_\varepsilon}\right) \\
&\leq \delta
\end{aligned} \tag{25}$$

We see that the algorithm is (λ, δ)-PAC for all such $\lambda \in [K]$, thereby giving us that the algorithm is δ-PAC. □

A.2.2 Proof of Theorem 12

Proof: Recall that the threshold $\zeta = 0$ and problem instance $\boldsymbol{S}_{\mathcal{E}}$ is such that $S_{\mathcal{E}}(\rho_1) \geq S_{\mathcal{E}}(\rho_2) \ldots > S_{\mathcal{E}}(\rho_{K-m}) > 0 > S_{\mathcal{E}}(\rho_{K-m+1}) \ldots > S_{\mathcal{E}}(\rho_K)$, with m being unknown. Let us consider the case that the event $\mathcal{B}$ described in Lemma 8 holds. As outlined in Algorithm 2, an arm i will be dropped if $\mathrm{LCB}_i(t) > 0$. That is,

$$\begin{aligned} &\hat{S}_{i,N_i(t)} - U\left(N_i(t), \frac{\delta}{c_\varepsilon K}\right) > 0 \\ &\hat{S}_{i,N_i(t)} - |\hat{S}_{i,N_i(t)} - S_i| > 0 \\ &\implies S_i > 0 \end{aligned}$$

Thus, as long as event $\mathcal{B}$ holds, all the arms that have $S_{\mathcal{E}} < 0$ will not dropped. Thus the lil'HDoC algorithm identifies all the arms correctly. □

A.3 Integrating Error Mitigation in MAB Algorithms for Batch Entanglement Detection

In the MAB-based workflow for entanglement detection described in Section 5, one state is measured at every time instant as dictated by the sampling rule, and the statisticsnamely, the estimates of f_1, f_2, f_3, and $f_4$are updated as new measurement outcomes are obtained. These estimates are susceptible to measurement errors, particularly readout errors, which induce inaccuracies in the measurement counts. To improve the accuracy of the estimates, we characterize such errors and wish to mitigate them (Qiskit Community, 2024). To this end, we carry out a preliminary investigation by incorporating a procedure for (a) error mitigation and (b) including error mitigation in the MAB routine, and study the impact of error mitigation on the overall copy complexity of batch entanglement detection.

A.3.1 Procedure for Error Mitigation

In Fig. 3, we apply a unitary transformation to WBM $\mathcal{E}_1$ and $\mathcal{E}_2$ to measure the state of the system ρ in the computational (Pauli Z) basis. Consequently, we obtain expectation values of the diagonal Pauli operators ZZ, ZI, and IZ. The estimates of f_1, f_2, f_3, and f_4 are linear combinations of these expectation values.

$$\begin{aligned} f_1 &= 0.25\,(1 + \langle IZ\rangle_\rho + \langle ZI\rangle_\rho + \langle ZZ\rangle_\rho) \\ f_2 &= 0.25\,(1 - \langle IZ\rangle_\rho + \langle ZI\rangle_\rho - \langle ZZ\rangle_\rho) \\ f_3 &= 0.25\,(1 + \langle IZ\rangle_\rho - \langle ZI\rangle_\rho - \langle ZZ\rangle_\rho) \\ f_4 &= 0.25\,(1 - \langle IZ\rangle_\rho - \langle ZI\rangle_\rho + \langle ZZ\rangle_\rho)\,. \end{aligned} \tag{26}$$

Thus, it is essential to obtain precise expectation values for the diagonal Pauli operators to improve the accuracy of our estimates. To do this, we use a LocalReadOut scheme from IBM's Qiskit Experiments library (Bravyi et al., 2021). In this scheme we characterize the readout errors of physical qubits on the **FakeBrisbane** backend. These errors are assumed to be local in the sense they are independent across qubits. Readout error mitigation uses a mitigator object (matrix) computed from an assignment matrix A, where each element $A_{i,j}$ represents the probability of observing outcome i when the true outcome is j. By applying this mitigator to unmitigated measurement counts, we refine our estimates by obtaining more accurate expectation values for ZZ, ZI, and IZ.

A.3.2 How and where does it fit in the MAB Routine?

In each round of the MAB policy, based on an Upper Confidence Bound (UCB) score, the sampling rule selects a quantum state to measure. Since only a single-shot measurement is performed per round, the error mitigation procedure described in Section A.3.1 is applied after a state has been measured several times. To illustrate this process, consider a specific round $t = F$, where state ρ_1 has previously been measured $T^\star$

times. The unmitigated measurement *counts* for the four possibrle outcomes are denoted as $F_1^{\text{um}}, F_2^{\text{um}}, F_3^{\text{um}}$, and F_4^{um}. The empirical frequencies of these outcomes are given by,

$$\hat{f}_i^{\text{um}}(F) = \frac{F_i^{\text{um}}}{T^\star}, \quad i \in [4]. \tag{27}$$

At this point, we invoke the error mitigation routine, supplying it with the unmitigated counts $\{F_i^{\text{um}}\}$ as input. The mitigation routine corrects for readout errors and returns mitigated expectation values of the diagonal Pauli observables, yielding mitigated estimates $\hat{f}_i^{\text{m}}(F)$. With post-processing adjustments to correct for decimal rounding errors, the corresponding mitigated measurement counts,

$$F_i^{\text{m}} = \hat{f}_i^{\text{m}}(F) \times T^\star, \quad i \in [4]. \tag{28}$$

We propose a **nested mitigative process** where the MAB algorithm invokes the error mitigation routine once every F measurement shots per state and uses the mitigated values in subsequent shots. For instance, at $t = F$, the routine produces mitigated estimates $\hat{f}_i^{\text{m}}(F)$ from which we obtain mitigated counts. Future measurement outcomes update on these mitigated counts. At $t = 2F$, the routine takes input these new counts and outputs a new set of mitigated estimates $\hat{f}_i^{\text{m}}(2F)$. This creates a nested-mitigation cycle, where each round of mitigation refines the previous one.

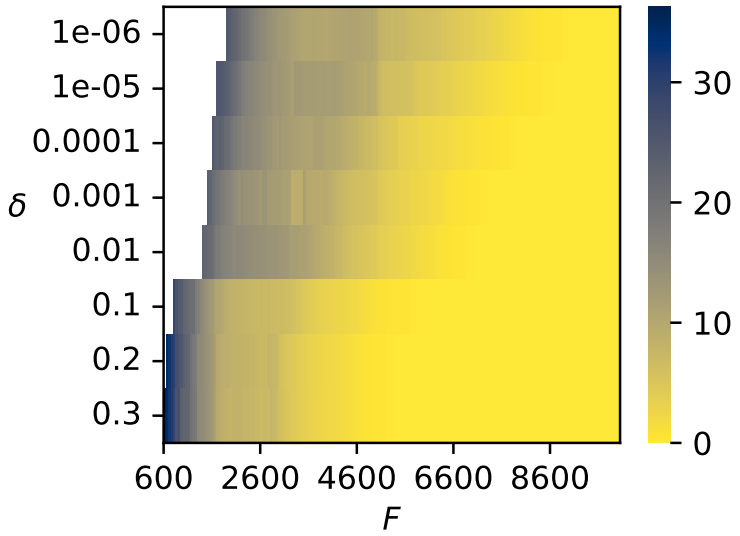

Figure 7: Heatmap of percentage error mitigation on FakeBrisbane backend for $\delta \in (0, 1)$ and various mitigation frequencies

We conduct an empirical study to assess the impact of error mitigation on the average copy complexity of the MAB algorithm. Mitigation is invoked once every F rounds, where F ranges from 50 to 10,000 in steps of 50. Here, smaller F values correspond to high-frequency mitigation and larger values indicate lower-frequency mitigation. For the problem instance described in Section 5, with $\delta \in (0, 1)$ and range of F, we execute Algorithm 3 on FakeBrisbane, averaging the copy complexity at stoppage over 20 runs. The percentage of error mitigation is quantified as the relative reduction in copy complexity compared to the case without mitigation. To ensure the algorithm correctly identifies the entangled states, we employ an error indicator that verifies whether its error remains within the prescribed threshold δ. Using this framework, we generate the heatmap in Fig. 7, which visualizes the percentage reduction in copy complexity due to error mitigation. Notably, the white regions indicate cases where the algorithm converged in finite time but failed to correctly identify the entangled states.

We observe and report the following inferences from Fig. 7. First, the effect of mitigation is δ-dependent. For larger values of δ, the mitigation effect starts only as early as ($F = 600$) and stabilizes faster ($F \sim 4000$).

In contrast, for smaller values of δ, the effect of mitigation is prominent only mid-range and stabilizes at $F \sim 7000$. Second, for $F < 600$ and smaller values of δ, the algorithm fails to detect the correct set of states under the prescribed δ. This can be attributed to over-mitigation which could potentially lead to random fluctuations in the estimates. Third, the observed stabilization zone (yellow) across values of δ suggests a critical threshold for F beyond which reducing mitigation frequency (increasing the value of F) no longer reduces errors. It remains an open question to fully understand and optimize for the use of error-mitigation and integrate them with MAB strategies.