Which Latent-Confounded Causal Structures imply Inequality Constraints?

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The essential premise of causal inference is that data generating processing can be described by causal relationships between variables. [1–5] Understanding the implications of such causal relationships enables not only adjudication between competing causal hypothesis but even the quantitative effect estimation of counterfactual interventions. Graphical models vary in terms of the strictness of their assumptions. The strongest assumption is typically causal sufficiency, which is the belief that the observed data arose from a self-contained process not involving any latent variables. In contrast to general causal structures involving latent confounding, *latent-free* causal structures have two exceptional features: Firstly, all counterfactual do-conditionals are point-identifiable in the absence of latent variables. Secondly, every distribution which is *Markov* relative to a latent-free structure — i.e., the distribution exhibits corresponding conditional independence for every d-separation relation between visible nodes in the structure — is *compatible* with the structure, i.e., admits a structural equation model realization.

For causal structures involving latent variables, however, these properties need not hold. We focus on the second property: For some causal structures — including but not limited to all latent-free structures — Markovianity implies compatibility. For other causal structures, Markovianity is insufficient, and all observable statistics compatible with such a structure must satisfy inequality constraints. We provide a rigorous and extensive classification of latent-involving causal structures relative to whether or not the presence of the latent variables contribute additional constraints that

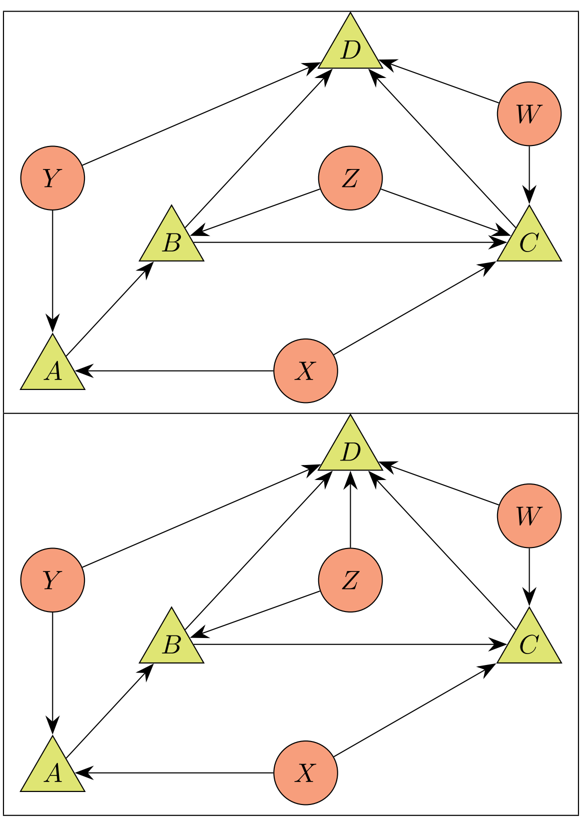

Figure 1: Two examples of causal structures involving latent nodes (reddish circles) for which we we able to establish that they each implies inequality constraints on their observed nodes (yellowish triangles).

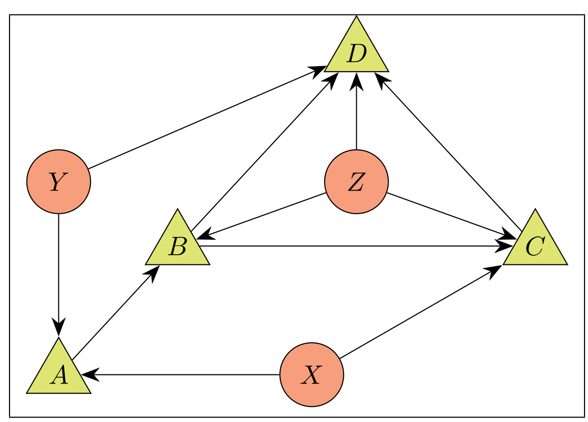

Figure 2: One of only three latent-involving causal structures with four observed nodes which we failed to classify. We strongly suspect this structure implies inequality constraints.

manifest as inequalities.

Our work illuminates the many obstacles which have prevented prior work from performing such a classification for causal structures involving more than three observable variables. Our progress is a result of combining superficially-unrelated methods from across disparate literature [6–13]. We especially build on the work of Robin Evans, by working within the modern framework of mDAGs [10], exploiting ideas such as e-separation [8, 12], and by leveraging the recently-proven result that a latent-involving structure always implies inequality constraints unless the structure is observationally equivalent to a latent-free one [13]. We also derive novel theorems that allow us to exploit certain methods more computationally efficiently than would otherwise be possible.

Some example results collected in our work are that a causal structure (hereafter, "DAG") must imply inequality constraints whenever:

- There is a pair of d-inseparable visible nodes in the DAG are not directly connected. [7]
- There is a set of visible nodes in the DAG each pair of which is d-inseparable but such that the whole set does not share a common ancestor.
- The Partial Ancestral Graph (PAG) associated with the DAG contains a bidirected edge. [13]
- There is a *support* which is simultaneously Markov with respect to a DAG but not actually compatible with the DAG.

We also show that the technique of analyzing (in)compatible supports [11] ultimately subsumes all other results. We developed an open-source Python implementation of the support analysis technique and utilized it to great effect for this classification task, highlighting the plethora of DAGs recognized as manifesting inequality constraints that would have been overlooked by traditional methods.

This classification task required us to push forward on two distinct research frontiers: Not only have we made headway with regards to recognizing the *presence* of inequality constraints, we also advanced the start of the art with regards to certifying the *absence* of inequality constraints. We adapt and modernize prior results of theoretical physicists [9] to develop an algorithm capable of showing that a latent-involving DAG is observationally equivalent to a latent-free one. It has been conjectured that this algorithm successfully picks out *all* DAGs not impying inequality constraints, and our work provides further evidence in favor of this conjecture.

In practice, we unambiguously classify 99.9989% of all possible DAGs involving latent confounding. We would like to call the attention of the community to three causal structures which we strongly suspect imply inequality constraints despite being unable to prove as such by the techniques available to us.

In conclusion, our research significantly advances the methodology of causal hypothesis testing by introducing a sophisticated framework for recognizing the implications of inequality constraints with a high degree of precision. Dropping the causal sufficiency assumption necessitates accounting for the complex interplay of observed and latent variables, but doing so allows data scientists to improve the reliability of causal claims. The deeper theoretical understanding of causal implications afforded by our work paves the way for more accurate inference in data obscured by unobserved factors. This can have practical ramifications in domains such as epidemiology, economics, and social sciences.

This work has additional foundational implications, in that it clarifies how to certify the *existence* of some inequality without actually *constructing* said inequality. Our techniques, rather, witness an implicit inequality by constructing a *distribution* which is incompatible with the given causal structure despite being Markov relative to it. Collecting such incompatible distributions, and appreciating the graphical features which make them incompatible, may inspire future research into inequality derivation or even causal discovery sensitive to inequality constraints, either explicit or implicit.

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