On the inverse-closedness of operator-valued matrices with polynomial off-diagonal decay

1st Lukas Köhldorfer
Acoustics Research Institute
Austrian Academy of Sciences
Vienna, Austria
lukas.koehldorfer@oeaw.ac.at

2nd Peter Balazs
Acoustics Research Institute
Austrian Academy of Sciences
Vienna, Austria
peter.balazs@oeaw.ac.at

***Abstract*—We give a self-contained proof of a recently established $\mathcal{B}(\mathcal{H})$-valued version of Jaffards Lemma. That is, we show that the Jaffard algebra of $\mathcal{B}(\mathcal{H})$-valued matrices, whose operator norms of their respective entries decay polynomially off the diagonal, is a Banach algebra which is inverse-closed in the Banach algebra $\mathcal{B}(\ell^2(X;\mathcal{H}))$ of all bounded linear operators on $\ell^2(X;\mathcal{H})$, the Bochner-space of square-summable $\mathcal{H}$-valued sequences.**

***Index Terms*—Jaffard's Lemma, Wiener's Lemma, inverse-closed, spectral invariance, polynomial off-diagonal decay, operator-valued matrices, matrix algebras.**

I. Introduction

Let $\mathcal{H}$ be a Hilbert space and $\ell^2(X;\mathcal{H})$ be the Bochner-space of $\mathcal{H}$-valued sequences $(g_l)_{l\in X}$, whose associated sequence of norms $(\|g_l\|)_{l\in X}$ is square-summable. Then $\ell^2(X;\mathcal{H})$ itself is a Hilbert space and $\mathcal{B}(\ell^2(X;\mathcal{H}))$, the space of all bounded linear operators on $\ell^2(X;\mathcal{H})$, is a C*-algebra. By the *matrix calculus* for $\mathcal{B}(\ell^2(X;\mathcal{H}))$ as explained in [22, Sec. 3.1], every operator $A \in \mathcal{B}(\ell^2(X;\mathcal{H}))$ can be uniquely identified with a $\mathcal{B}(\mathcal{H})$-valued matrix $[A_{k,l}]_{k,l\in X}$ in the sense that the action of A on $(g_l)_{l\in X} \in \ell^2(X;\mathcal{H})$ precisely corresponds to matrix multiplication of $[A_{k,l}]_{k,l\in X}$ with $(g_l)_{l\in X}$ viewed as column vector. Moreover, composition of operators from $\mathcal{B}(\ell^2(X;\mathcal{H}))$ corresponds to matrix multiplication of their respective matrix representations, and taking adjoints in $\mathcal{B}(\ell^2(X;\mathcal{H}))$ corresponds to the involution

$$([A_{k,l}]_{k,l\in X})^* = ([A^*_{k,l}]_{k,l\in X})^T, \tag{I.1}$$

where the exponent T denotes transposition. Thus $\mathcal{B}(\ell^2(X;\mathcal{H}))$ can be identified with a certain Banach algebra of $\mathcal{B}(\mathcal{H})$-valued matrices (see also [19, Thm. 5.28]).

Based on the latter point of view, we consider the *Jaffard class* $\mathcal{J}_s = \mathcal{J}_s(X;\mathcal{B}(\mathcal{H}))$ of $\mathcal{B}(\mathcal{H})$-valued matrices $A = [A_{k,l}]_{k,l\in X}$, for which there exists $C > 0$, such that

$$\|A_{k,l}\| \leq C(1+|k-l|)^{-s} \qquad (\forall k,l \in X). \tag{I.2}$$

It has been shown [21], that for sufficiently regular index sets X and a sufficiently large decay parameter $s > 0$, the Jaffard class is a unital Banach *-algebra with respect to matrix multiplication and involution as in (I.1), which is contained in $\mathcal{B}(\ell^2(X;\mathcal{H}))$ and, in fact, *inverse-closed* in $\mathcal{B}(\ell^2(X;\mathcal{H}))$, meaning that

$$A \in \mathcal{J}_s \text{ and } \exists A^{-1} \in \mathcal{B}(\ell^2(X;\mathcal{H})) \implies A^{-1} \in \mathcal{J}_s. \tag{I.3}$$

Property (I.3) can be seen as a variant of *Wiener's Lemma* on absolutely convergent Fourier series [25] for $\mathcal{B}(\mathcal{H})$-valued matrices with polynomial off-diagonal decay and is known as *Jaffard's Lemma* [16] in the scalar-valued setting. This result (compare also with [1]–[4], [17], [21]) is not only interesting from an abstract point of view, but also has an enormous potential as a powerful tool for the study of localized g-frames [18], [20], operator theory [8], the study of Fourier series of operators [9] or harmonic quantum analysis [7], [24].

In [21] the inverse-closedness of $\mathcal{J}_s$ in $\mathcal{B}(\ell^2(X;\mathcal{H}))$ is deduced from the inverse-closedness of certain weighted Schur-type algebras in $\mathcal{B}(\ell^2(X;\mathcal{H}))$ via methods from [12]. Here we give a more self-contained presentation of this fact by adapting methods from [13], [23] to the $\mathcal{B}(\mathcal{H})$-valued setting.

II. Results

A. Banach algebra properties

We consider $\mathcal{B}(\mathcal{H})$-valued matrices indexed by a *relatively separated* set $X \subset \mathbb{R}^d$, meaning that

$$\sup_{x\in\mathbb{R}^d} |X \cap (x+[0,1]^d)| < \infty. \tag{II.1}$$

For such X the following hold:

Lemma II.1. *Let $X \subset \mathbb{R}^d$ be a relatively separated set.*

(a) *[10, Lemma 1] For any $s > d$, there exists a constant $C = C(s) > 0$ such that*

$$\sup_{x\in\mathbb{R}^d} \sum_{k\in X} (1+|x-k|)^{-s} = C < \infty$$

(b) *[10, Lemma 2 (a)] For any $s > d$, there exists a constant $C = C(s) > 0$ such that for all $k,l \in X$*

$$\sum_{n\in X} (1+|k-n|)^{-s}(1+|l-n|)^{-s} \leq C(1+|k-l|)^{-s}.$$

Definition II.2. *Let $X \subset \mathbb{R}^d$ be relatively separated and $\nu_s(x) : \mathbb{R}^d \longrightarrow [0,\infty)$ the polynomial weight function given* *by $\nu_s(x) = (1+|x|)^s$, $s \geq 0$. Let $\mathcal{J}_s = \mathcal{J}_s(X;\mathcal{B}(\mathcal{H}))$ be the space of all $\mathcal{B}(\mathcal{H})$-valued matrices $A = [A_{k,l}]_{k,l\in X}$, for which*

$$\|A\|_{\mathcal{J}_s} := \sup_{k,l\in X} \|A_{k,l}\| \nu_s(k-l) \tag{II.2}$$

is finite. We call $\mathcal{J}_s$ the Jaffard class *(and later −once justified− the* Jaffard algebra*) and $\|\,.\,\|_{\mathcal{J}_s}$ the* Jaffard norm.

Since $\mathcal{J}_s$ is nothing else than the weighted Bochner space $\mathcal{J}_s = \ell^\infty_{u_s}(X\times X;\mathcal{B}(\mathcal{H}))$, where $u_s(k,l) = \nu_s(k-l)$, the Jaffard class $(\mathcal{J}_s, \|\,.\,\|_{\mathcal{J}_s})$ is a Banach space [15, Chap. 1].

Proposition II.3. *Let $s > d + r$, $r \geq 0$ and $m : \mathbb{R}^d \to [0,\infty)$ be a function for which there exists some $C > 0$, such that $m(x+y) \leq Cm(y)\nu_r(y)$ for all $x, y \in \mathbb{R}^d$. Then each $A \in \mathcal{J}_s$ defines a bounded operator on $\ell^p_m(X;\mathcal{H})$ for every $\frac{d}{s-r} < p \leq \infty$.*

Proof. For $\frac{d}{s-r} < p \leq 1$, the function $f : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}, f(x) = x^p$ is subadditive. Hence, for $g = (g_l)_{l\in X} \in \ell^p_m(X;\mathcal{H})$, we obtain

$$\begin{aligned}
&\|Ag\|^p_{\ell^p_m(X;\mathcal{H})} \\
&= \sum_{k\in X} \left\| \sum_{l\in X} A_{k,l} g_l \right\|^p m(k)^p \\
&\leq C^p \sum_{k\in X}\sum_{l\in X} \|A_{k,l}\|^p \|g_l\|^p m(l)^p (1+|k-l|)^{rp} \\
&\leq C^p \|A\|^p_{\mathcal{J}_s} \sum_{l\in X} \|g_l\|^p m(l)^p \sum_{k\in X} (1+|k-l|)^{-p(s-r)} \\
&\leq C_1 \|A\|^p_{\mathcal{J}_s} \|g\|^p_{\ell^p_m(X;\mathcal{H})},
\end{aligned}$$

where we applied [10, Lemma 2 (a)] is the last step.

The case $p = \infty$ is proven similarly.

Finally, the case $1 < p < \infty$ follows from Riesz-Thorin interpolation [15, Thm. 2.2.1]. □

Setting $r = 0$ and $m \equiv 1$ above yields the following.

Corollary II.4. *If $s > d$, then*

$$\mathcal{J}_s(X) \subset \bigcap_{1\leq p\leq\infty} \mathcal{B}(\ell^p(X;\mathcal{H})).$$

Since $\mathcal{J}_s \subset \mathcal{B}(\ell^2(X;\mathcal{H}))$, the matrix calculus [22, Sec. 3.1] for the Banach algebra $\mathcal{B}(\ell^2(X;\mathcal{H}))$, mentioned in the introduction, motivates us to define a multiplication on $\mathcal{J}_s$ via matrix multiplication. For the same reason, we define an involution on $\mathcal{J}_s$ as in (I.1).

Proposition II.5. *For $s > d$, the Jaffard class $(\mathcal{J}_s, \|\,.\,\|_{\mathcal{J}_s})$ is a unital *-algebra with respect to matrix multiplication and involution as defined in (I.1). Furthermore, the involution is an isometry.*

Proof. Let $A = [A_{k,l}]_{k,l\in X}, B = [B_{k,l}]_{k,l\in X} \in \mathcal{J}_s$. For arbitrary $k, l \in X$ we have

$$\begin{aligned}
\|[A\cdot B]_{k,l}\| &\leq \sum_{n\in X} \|A_{k,n}\|\,\|B_{n,l}\| \\
&\leq \|A\|_{\mathcal{J}_s} \|B\|_{\mathcal{J}_s} \sum_{n\in X} \nu_s(k-n)^{-1}\nu_s(n-l)^{-1} \\
&\leq C\|A\|_{\mathcal{J}_s} \|B\|_{\mathcal{J}_s} \nu_s(k-l)^{-1},
\end{aligned}$$

where we applied Lemma II.1 (b) in the last step. Consequently

$$\|A\cdot B\|_{\mathcal{J}_s} \leq C\|A\|_{\mathcal{J}_s} \|B\|_{\mathcal{J}_s}, \tag{II.3}$$

which implies that the Jaffard class is an algebra. Moreover, the neutral element $\mathcal{I}_{\mathcal{J}_s} = \mathcal{I}_{\mathcal{B}(\ell^2(X;\mathcal{H}))} = \mathrm{diag}[\mathcal{I}_{\mathcal{B}(\mathcal{H})}]_{k\in X}$ is contained in $\mathcal{J}_s$, since $\|\mathcal{I}_{\mathcal{J}_s}\|_{\mathcal{J}_s} = \nu_s(0) = 1$. The involution property in $\mathcal{J}_s$ follows from the matrix calculus with respect to $\mathcal{B}(\ell^2(X;\mathcal{H}))$ via Corollary II.4. Finally, $\nu_s(-x) = \nu_s(x)$ for all $x \in \mathbb{R}^d$ implies that the involution is an isometry. □

By (II.3), the Jaffard norm is *not* a Banach algebra norm. However, we may equip the Jaffard class with the equivalent norm

$$|||A|||_{\mathcal{J}_s} := \sup_{\substack{B\in\mathcal{J}_s \\ \|B\|_{\mathcal{J}_s}=1}} \|A\cdot B\|_{\mathcal{J}_s} \qquad (A\in\mathcal{J}_s), \tag{II.4}$$

which is indeed a Banach algebra norm. As a consequence we obtain:

Corollary II.6. *[21] For any $s > d$, $(\mathcal{J}_s, |||\,.\,|||_{\mathcal{J}_s})$ is a unital Banach *-algebra.*

B. Inverse-closedness

In this section we show property (I.3), i.e., that $\mathcal{J}_s$ is inverse-closed in $\mathcal{B}(\ell^2(X;\mathcal{H}))$. While inverse-closedness in principal is an algebraic property, *Hulanicki's Lemma* [14] allows for an analytical treatment of this task. We refer the reader to [11, Prop. 2.5] for a proof of this simple but beautiful fact. Let $\sigma_{\mathcal{B}}(B)$ and $r_{\mathcal{B}}(B)$ denote the spectrum and the spectral radius of an element B from a Banach algebra $\mathcal{B}$, respectively.

Proposition II.7 (Hulanicki's Lemma). *[14] Let $\mathcal{A} \subseteq \mathcal{B}$ be a pair of unital Banach *-algebras with common identity and common involution, and suppose that $\mathcal{B}$ is symmetric, i.e. $\sigma_{\mathcal{B}}(B^*B) \subseteq [0,\infty)$ $(\forall B \in \mathcal{B})$. Then the following are equivalent:*

(i) *$\mathcal{A}$ is inverse-closed in $\mathcal{B}$.*
(ii) $r_{\mathcal{A}}(A) = r_{\mathcal{B}}(A) \qquad (\forall A = A^* \in \mathcal{A})$.
(iii) $r_{\mathcal{A}}(A) \leq r_{\mathcal{B}}(A) \qquad (\forall A = A^* \in \mathcal{A})$.

In case the above conditions hold, $\mathcal{A}$ is symmetric as well.

In order to be able to employ Hulanicki's Lemma, we need some preparation.

Lemma II.8. *[13, Lemma 5.13] Let $X \subset \mathbb{R}^d$ be relatively separated, $s > d$ and $\tau_0 > 0$ be given. For any $k \in X$ and any $\tau > \tau_0$, let $M^\tau_{1,k} := \{n \in X : |k-n|_\infty \leq \lceil\tau\rceil\}$ and $M^\tau_{2,k} := \{n \in X : |k-n|_\infty > \lceil\tau\rceil\}$. Then there exists a* *constant* $C = C(X, s, \tau_0) > 0$, *such that for all* $k \in X$ *and all* $\tau > \tau_0$

$$|M^\tau_{1,k}| \leq C\tau^d \quad \textit{and} \quad \sum_{n\in M^\tau_{2,k}} \nu_s(k-n)^{-1} \leq C\tau^{d-s}. \quad \text{(II.5)}$$

Proof. In order to show the first inequality in (II.5), recall that $X \subset \mathbb{R}^d$ being relatively separated means that $\gamma := \sup_{x\in\mathbb{R}^d} |X \cap (x + [0,1]^d)|$ is finite. Since $M^\tau_{1,k}$ can be covered by $(2\lceil\tau\rceil)^d$ many translated unit cubes, we see that $|M^\tau_{1,k}| \leq (2\lceil\tau\rceil)^d\gamma$. Since $\lceil\tau\rceil \leq 1+\tau \leq (\frac{1}{\tau_0}+1)\tau$, we obtain $|M^\tau_{1,k}| \leq 2^d\gamma(\frac{1}{\tau_0}+1)^d\tau^d$.

In order to show the second inequality in (II.5) fix some arbitrary $k \in X$. Since $|z|_\infty \leq |z|$ for all $z \in \mathbb{R}^d$, it suffices to estimate the series $\sum_{n\in M^\tau_{2,k}}(1+|k-n|_\infty)^{-s}$. We may assume W.L.O.G. that X is separated, i.e. that $\inf_{x,y\in X, x\neq y}|x-y| =: \delta > 0$, since any relatively separated set $X \subset \mathbb{R}^d$ is a finite union of separated sets [6, Sec. 9.1]. Now, observe that if $n \in M^\tau_{2,k}$, i.e. $n \in X$ and $|k-n|_\infty > \lceil\tau\rceil$, then $\exists l \in S^\tau := \{l \in \mathbb{Z}^d : |l|_\infty \geq \lceil\tau\rceil\}$ such that $n = k+l+x$ for some $x \in [0,1)^d$. Thus, if for any $l \in S^\tau$ we define $X_{l,k} := X \cap (k+l+[0,1)^d)$, then we see that the family $\{X_{l,k} : l \in S^\tau\}$ covers $M^\tau_{2,k}$, hence

$$\sum_{n\in M^\tau_{2,k}} (1+|k-n|_\infty)^{-s} \leq \sum_{l\in S^\tau}\sum_{n\in X_{l,k}} (1+|k-n|_\infty)^{-s} = (*).$$

Since, by our previous observation, $n = k + l + x$ for some $l \in S^\tau$ and some $x \in [0,1)^d$, we see that $1 + |k-n|_\infty = 1 + |l+x|_\infty \geq 1 + |l|_\infty - |x|_\infty \geq |l|_\infty$. Using this together with the observation that $|X_{l,k}| \leq C'$, where the constant C' only depends on δ and the dimension d, we obtain that

$$(*) \leq C' \sum_{l\in S^\tau} |l|^{-s}_\infty =: (**).$$

Now, for each $m \in \mathbb{N}$ with $m \geq \lceil\tau\rceil$ set $S_m := \{l \in \mathbb{Z}^d : |l|_\infty = m\}$. Then $S^\tau = \bigcup_{m\geq\lceil\tau\rceil} S_m$ and each S_m consists of the integer lattice points located on the surface of a cube of side-length $2m$. Since the number of lattice points on each face of such a cube equals $(2m+1)^{d-1}$ and there are $2d$ faces per cube in total, we obtain

$$\begin{aligned}
(**) &\leq C' \sum_{m=\lceil\tau\rceil}^\infty \sum_{l\in S_m} |l|^{-s}_\infty \\
&\leq 2dC' \sum_{m=\lceil\tau\rceil}^\infty m^{-s}(2m+1)^{d-1} \\
&\leq 2dC' \sum_{m=\lceil\tau\rceil}^\infty m^{-s}(4m)^{d-1} \\
&= 2dC'4^{d-1}\left(\lceil\tau\rceil^{d-s-1} + \sum_{m=\lceil\tau\rceil+1}^\infty m^{d-s-1}\right).
\end{aligned}$$

Since $\lceil\tau\rceil^{d-s-1} \leq \lceil\tau\rceil^{d-s}$ and $m^{d-s-1} \leq \int_{m-1}^m x^{d-s-1}\,dx$, we obtain in total that

$$\begin{aligned}
\sum_{n\in M^\tau_{2,k}} (1+|k-n|)^{-s} &\leq C''\left(\lceil\tau\rceil^{d-s} + \int_{\lceil\tau\rceil}^\infty x^{d-s-1}\,dx\right) \\
&= C''\left(1+\frac{1}{s-d}\right)\lceil\tau\rceil^{d-s},
\end{aligned}$$

which yields the desired inequality. $\square$

Next we show that $\mathcal{J}_s$ is continuously embedded in $\mathcal{B}(\ell^2(X;\mathcal{H}))$.

Lemma II.9. *Let* $s > d$. *Then there exists* $C > 0$, *such that*

$$\|A\|_{\mathcal{B}(\ell^2(X;\mathcal{H}))} \leq C\|A\|_{\mathcal{J}_s} \qquad (\forall A \in \mathcal{J}_s). \quad \text{(II.6)}$$

Proof. The statement is proved essentially as [13, Lemma 5.2]:

Let $A \in \mathcal{J}_s$ be arbitrary. Since $(\mathcal{J}_s, |||\,.\,|||_{\mathcal{J}_s})$ is a unital Banach *-algebra by Corollary II.6, we have by Gelfand's formula for the spectral radius that $r_{\mathcal{J}_s}(A) = \lim_{n\to\infty}|||A^n|||^{\frac{1}{n}}_{\mathcal{J}_s}$ for any $A \in \mathcal{J}_s$. Furthermore, since $\mathcal{J}_s \subset \mathcal{B}(\ell^2(X;\mathcal{H}))$ by Corollary II.4, a simple argument (see e.g. [11, Lemma 2.4]) shows that $r_{\mathcal{B}(\ell^2(X;\mathcal{H}))}(A) \leq r_{\mathcal{J}_s}(A)$ for all $A \in \mathcal{J}_s$. Thus

$$\begin{aligned}
\|A\|^2_{\mathcal{B}(\ell^2(X;\mathcal{H}))} &= \|A^*A\|_{\mathcal{B}(\ell^2(X;\mathcal{H}))} \\
&= r_{\mathcal{B}(\ell^2(X;\mathcal{H}))}(A^*A) \\
&\leq r_{\mathcal{J}_s}(A^*A) \\
&= \lim_{n\to\infty} |||(A^*A)^n|||^{\frac{1}{n}}_{\mathcal{J}_s} \\
&\leq |||A^*A|||_{\mathcal{J}_s} \\
&\leq |||A^*|||_{\mathcal{J}_s}|||A|||_{\mathcal{J}_s} \\
&\leq C\|A\|^2_{\mathcal{J}_s},
\end{aligned}$$

for all $A \in \mathcal{J}_s$, where we used (II.4) in the last line. $\square$

Lemma II.10. *Let* $A = [A_{k,l}]_{k,l\in X}$ *be a* $\mathcal{B}(\mathcal{H})$*-valued matrix and* $1 \leq p \leq \infty$. *If* A *defines an element in* $\mathcal{B}(\ell^p(X;\mathcal{H}))$, *then the following hold:*

(a) *For* $1 \leq p < \infty$,

$$\sup_{l\in X}\sup_{\|f\|_{\mathcal{H}}=1}\left(\sum_{k\in X}\|A_{k,l}f\|^p_{\mathcal{H}}\right)^{\frac{1}{p}} \leq \|A\|_{\mathcal{B}(\ell^p(X;\mathcal{H}))}.$$

(b)

$$\sup_{k,l\in X}\|A_{k,l}\| \leq \|A\|_{\mathcal{B}(\ell^p(X;\mathcal{H}))}.$$

Proof. (a) For $l \in X$ and $1 \leq p \leq \infty$, define $P^p_l : \ell^p(X;\mathcal{H}) \longrightarrow \ell^p(X;\mathcal{H})$, $P^p_l(f_k)_{k\in X} = (\delta_{k,l}f_k)_{k\in X}$. Then for arbitrary but fixed $l \in X$ and $1 \leq p < \infty$ we see that

$$\begin{aligned}
\|A\|^p_{\mathcal{B}(\ell^p(X;\mathcal{H}))} &= \sup_{\|f\|_{\ell^p(X;\mathcal{H})}=1}\sum_{k\in X}\left\|\sum_{l\in X}A_{k,l}f_l\right\|^p \\
&\geq \sup_{\substack{f=P^p_l f \\ \|f\|_{\ell^p(X;\mathcal{H})}=1}}\sum_{k\in X}\|A_{k,l}f_l\|^p \\
&= \sup_{\|f\|_{\mathcal{H}}=1}\sum_{k\in X}\|A_{k,l}f\|^p.
\end{aligned}$$

Taking the p-th root and supremum over all $l \in X$ yields (a).

(b) For $1 \le p < \infty$, we have by (a) that

$$\|A_{k,l}\| \le \sup_{\|f\|_{\mathcal{H}}=1} \left(\sum_{k \in X} \|A_{k,l} f\|^p \right)^{\frac{1}{p}} \le \|A\|_{\mathcal{B}(\ell^p(X;\mathcal{H}))}$$

for all $k, l \in X$. Taking the supremum over all $k, l \in X$ yields the claim. The case $p = \infty$ is omitted. $\square$

After these preparatory results, we are able to prove the decisive ingredient for proving our main theorem.

Lemma II.11. *Let $s > d$ and $\gamma = 1 - \frac{d}{s} > 0$. Then there exists a positive constant C, such that*

$$\|A^2\|_{\mathcal{J}_s} \le C \|A\|_{\mathcal{J}_s}^{2-\gamma} \|A\|_{\mathcal{B}(\ell^2(X;\mathcal{H}))}^{\gamma} \qquad (\forall A \in \mathcal{J}_s). \quad \text{(II.7)}$$

Proof. We abbreviate $\mathcal{B} = \mathcal{B}(\ell^2(X;\mathcal{H}))$. Let $A \in \mathcal{J}_s$ be arbitrary and assume W.L.O.G. that $A \neq 0$. Hölder's inequality on $\mathbb{R}^2$ with respect to the exponent s (and its conjugated Hölder exponent) implies that

$$\nu_s(k-l) < 2^s \left(\nu_s(k-n) + \nu_s(n-l)\right) \qquad (\forall k, l, n \in X).$$

Thus, for arbitrary $k, l \in X$ we can estimate

$$\begin{aligned} &\|[A^2]_{k,l}\| \nu_s(k-l) \\ &\le \sum_{n \in X} \|A_{k,n}\| \|A_{n,l}\| \nu_s(k-l) \\ &< 2^s \sum_{n \in X} \|A_{k,n}\| \|A_{n,l}\| \left(\nu_s(k-n) + \nu_s(n-l)\right) \\ &\le 2^s \|A\|_{\mathcal{J}_s} \left(\sum_{n \in X} \|A_{n,l}\| + \sum_{n \in X} \|A_{k,n}\| \right). \end{aligned}$$

Recall from Proposition II.9, that there exists $C_1 > 0$, such that $\|A\|_{\mathcal{B}} \le C_1 \|A\|_{\mathcal{J}_s}$. In particular, for $\theta > 0$ (to be chosen later), there exists $\tau_0 > 0$, such that

$$\tau := \|A\|_{\mathcal{J}_s}^{\theta} \|A\|_{\mathcal{B}}^{-\theta} \ge C_1^{-\theta} > \tau_0 > 0. \quad \text{(II.8)}$$

Hence we are in the setting of Lemma II.8 and may estimate

$$\begin{aligned} &\sum_{n \in X} \|A_{n,l}\| \\ &\le \sum_{n \in M_{1,l}^{\tau}} \|A_{n,l}\| + \sum_{n \in M_{2,l}^{\tau}} \|A_{n,l}\| \\ &\le |M_{1,l}^{\tau}| \|A\|_{\mathcal{B}} + \|A\|_{\mathcal{J}_s} \sum_{n \in M_{2,l}^{\tau}} \nu_s(n-l)^{-1} \\ &\le C_2 \Big(\tau^d \|A\|_{\mathcal{B}} + \|A\|_{\mathcal{J}_s} \tau^{d-s} \Big) \\ &= C_2 \Big(\|A\|_{\mathcal{J}_s}^{d\theta} \|A\|_{\mathcal{B}}^{1-d\theta} + \|A\|_{\mathcal{J}_s}^{1+(d-s)\theta} \|A\|_{\mathcal{B}}^{(-(d-s)\theta)} \Big), \end{aligned}$$

where we applied Lemma II.10 (b) in the second estimate, and $C_2 > 0$ denotes the constant arising in Lemma II.8. Analogous reasoning also yields

$$\begin{aligned} &\sum_{n \in X} \|A_{k,n}\| \\ &\le C_2 \Big(\|A\|_{\mathcal{J}_s}^{d\theta} \|A\|_{\mathcal{B}}^{1-d\theta} + \|A\|_{\mathcal{J}_s}^{1+(d-s)\theta} \|A\|_{\mathcal{B}}^{(-(d-s)\theta)} \Big). \end{aligned}$$

Altogether, we obtain

$$\begin{aligned} &\|[A^2]_{k,l}\| \nu_s(k-l) \\ &\le 2^{s+1} C_2 \Big(\|A\|_{\mathcal{J}_s}^{1+d\theta} \|A\|_{\mathcal{B}}^{1-d\theta} + \|A\|_{\mathcal{J}_s}^{2+(d-s)\theta} \|A\|_{\mathcal{B}}^{(-(d-s)\theta)} \Big) \end{aligned}$$

for all $k, l \in X$. Now, we choose $\theta = \frac{1}{s} > 0$, which yields $1 + (d-s)\theta = \frac{d}{s} = d\theta$, and therefore

$$\|[A^2]_{k,l}\| \nu_s(k-l) \le 2^{s+2} C_2 \|A\|_{\mathcal{J}_s}^{2-\gamma} \|A\|_{\mathcal{B}}^{\gamma}.$$

Taking the supremum over all $k, l \in X$ yields the claim. $\square$

Now we are finally able to prove the main result of this section. The main contribution to its proof is Lemma II.11. In fact, having Lemma II.11 available at our hands, the proof of the subsequent theorem can be established exactly as the proof of [13, Theorem 5.15]. For completeness reason we provide the details.

Theorem II.12. *[21, Cor. 4.5] For every $s > d$, the Jaffard algebra $(\mathcal{J}_s, |||\,.\,|||_{\mathcal{J}_s})$ is inverse-closed in $\mathcal{B}(\ell^2(X;\mathcal{H}))$. In particular, $(\mathcal{J}_s, |||\,.\,|||_{\mathcal{J}_s})$ is a symmetric Banach algebra whenever $s > d$.*

Proof. Since $\mathcal{B} := \mathcal{B}(\ell^2(X;\mathcal{H}))$ is a symmetric Banach *-algebra, we can deduce both the inverse-closedness of $\mathcal{J}_s$ in $\mathcal{B}$ and the symmetry of $\mathcal{J}_s$ from Hulanicki's Lemma II.7, once we have verified the inequality of spectral radii

$$r_{\mathcal{J}_s}(A) \le r_{\mathcal{B}}(A) \qquad (\forall A = A^* \in \mathcal{J}_s). \quad \text{(II.9)}$$

We establish the verification of (II.9) via *Brandenburg's trick* [5], which relies on an estimate of the kind (II.7).

By norm equivalence (II.4) there exist positive constants K_1, K_2 such that

$$K_1 |||A^n|||_{\mathcal{J}_s} \le \|A^n\|_{\mathcal{J}_s} \le K_2 |||A^n|||_{\mathcal{J}_s}$$

for all $A \in \mathcal{J}_s$ and all $n \in \mathbb{N}$. Taking n-th roots and letting $n \to \infty$ implies via Gelfand's formula that

$$r_{\mathcal{J}_s}(A) = \lim_{n\to\infty} |||A^n|||_{\mathcal{J}_s}^{\frac{1}{n}} = \lim_{n\to\infty} \|A^n\|_{\mathcal{J}_s}^{\frac{1}{n}} \qquad (\forall A \in \mathcal{J}_s).$$

Now we combine the latter observation with Lemma II.11 and obtain that

$$\begin{aligned} r_{\mathcal{J}_s}(A) &= \lim_{n\to\infty} \|A^{2n}\|_{\mathcal{J}_s}^{\frac{1}{2n}} \\ &\le \lim_{n\to\infty} C^{\frac{1}{2n}} \Big(\|A^n\|_{\mathcal{J}_s}^{\frac{1}{n}} \Big)^{\frac{2-\gamma}{2}} \Big(\|A^n\|_{\mathcal{B}}^{\frac{1}{n}} \Big)^{\frac{\gamma}{2}} \\ &= r_{\mathcal{J}_s}(A)^{\frac{2-\gamma}{2}} r_{\mathcal{B}}(A)^{\frac{\gamma}{2}} \end{aligned}$$

holds for all $A \in \mathcal{J}_s$. Rearranging the latter yields

$$r_{\mathcal{J}_s}(A)^{\frac{\gamma}{2}} \le r_{\mathcal{B}}(A)^{\frac{\gamma}{2}} \qquad (\forall A \in \mathcal{J}_s).$$

Since $\gamma > 0$, this implies (II.9). $\square$

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REFERENCES

[1] A. G. Baskakov. Wiener's theorem and the asymptotic estimates of the elements of inverse matrices. *Functional Analysis and Its Applications*, 24(3):222–224, July 1990.

[2] A. G. Baskakov. Estimates for the entries of inverse matrices and the spectral analysis of linear operators. *Izvestiya: Mathematics*, 61(6):1113, dec 1997.

[3] A. G. Baskakov and I. A. Krishtal. Memory estimation of inverse operators. *Journal of Functional Analysis*, 267(8):2551–2605, 2014.

[4] A. G. Baskakov and I. A. Krishtal. Spectral properties of an operator polynomial with coefficients in a Banach algebra. In *Frames and harmonic analysis. AMS special session on frames, wavelets and Gabor systems and special session on frames, harmonic analysis, and operator theory, North Dakota State University, Fargo, ND, USA, April 16–17, 2016. Proceedings*, pages 93–114. Providence, RI: American Mathematical Society (AMS), 2018.

[5] L. Brandenburg. On identifying the maximal ideals in Banach algebras. *Journal of Mathematical Analysis and Applications*, 50(3):489–510, 1975.

[6] O. Christensen. *An Introduction to Frames and Riesz Bases*. Birkhäuser, 2016.

[7] M. A. de Gosson. *Quantum Harmonic Analysis. An Introduction.* De Gruyter, Berlin, Boston, 2021.

[8] R. G. Douglas. *Banach Algebra Techniques in Operator Theory*. Springer, 1998.

[9] H. G. Feichtinger and W. Kozek. *Quantization of TF lattice-invariant operators on elementary LCA groups*, pages 233–266. Birkhäuser Boston, Boston, MA, 1998.

[10] K. Gröchenig. Localization of Frames, Banach Frames, and the Invertibility of the Frame Operator. *J. Fourier Anal. Appl.*, 10(2):105–132, 2004.

[11] K. Gröchenig. *Wiener's lemma: Theme and variations. An introduction to spectral invariance and its applications.*, chapter 5, pages 175 – 234. Applied and Numerical Harmonic Analysis. Birkhäuser, 2010.

[12] K. Gröchenig and M. Leinert. Symmetry and inverse-closedness of matrix algebras and functional calculus for infinite matrices. *Transactions of the American Mathematical Society*, 358(6):2695–2711, 2006.

[13] J. Holböck. Localized frames and applications. Master's thesis, University of Vienna, 2022.

[14] A. Hulanicki. On the spectrum of convolution operators on groups with polynomial growth. *Invent. Math.*, 17:135–142, 1972.

[15] T. Hytönen, J. van Neerven, M. Veraar, and L. Weis. *Analysis in Banach Spaces, Volume I: Martingales and Littlewood-Paley Theory*. Springer, 12 2016.

[16] S. Jaffard. Propriétés des matrices "bien localisées" preé de leur diagonale et qualques applications. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 7(5):461–476, 1990.

[17] I. Krishtal. Wiener's lemma: Pictures at an exhibition. *Revista de la Unión Matemática Argentina*, 52(2):61–79, 01 2011.

[18] I. A. Krishtal. Wiener's lemma and memory localization. *Journal of Fourier Analysis and Applications*, 17(4):674–690, August 2011.

[19] L. Köhldorfer. Fusion Frames and Operators. Master's thesis, University of Vienna, 2021.

[20] L. Köhldorfer and P. Balazs. Intrinsically localized g-frames. *In preparation*, 2025.

[21] L. Köhldorfer and P. Balazs. Wiener pairs of Banach algebras of operator-valued matrices. *Submitted*, 2025.

[22] L. Köhldorfer, P. Balazs, P. Casazza, S. Heineken, C. Hollomey, P. Morillas, and M. Shamsabadhi. *A Survey of Fusion Frames in Hilbert Spaces*, chapter 21. Springer Nature Switzerland, 2023.

[23] Q. Sun. Wiener's lemma for infinite matrices. *Trans. Amer. Math. Soc.*, 359(7):3099–3123, 2007.

[24] R. Werner. Quantum harmonic analysis on phase space. *Journal of Mathematical Physics*, 25(5):1404–1411, 05 1984.

[25] N. Wiener. Tauberian theorems. *Annals of Mathematics*, 33(1):1–100, 1932.