Personlalized PageRank centrality from a Fixed Point Theory perspective

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PageRank constitutes one of the most influential and successful centrality measures in the study of complex networks, notable for its transition from theoretical formulation to widespread real-world applications. Originally introduced by Page and Brin in the late 1990s to improve the ranking of web pages in Google's search engine [1], the algorithm has since demonstrated remarkable versatility, with applications extending far beyond its original scope. Contemporary uses of PageRank encompass diverse domains, including the assessment of user influence in online social platforms, the analysis of protein–protein interaction and metabolic networks in biology, the evaluation of systemic risk in financial systems, the measurement of scientific impact in citation networks, and the modeling of public transportation systems, among numerous others. At its core, PageRank on a complex network $\mathscr{G} = (V,E)$ can be interpreted as the stationary distribution of a random walk. In each iteration, the walker selects uniformly at random one of the outgoing links of the current node, but with a certain probability the process instead teleports to a randomly chosen node in the network [2, 3]. Two fundamental parameters govern this process: (i) the damping factor, which specifies the probability of following a network link rather than teleporting, and (ii) the personalization vector, which determines the distribution of nodes selected during teleportation. Several studies have analyzed how variations in these parameters affect the resulting PageRank distribution [2–5], but in this poster we will follow an approach based on classic fixed point theory.

Formally, given a network $\mathscr{G} = (V,E)$ with N nodes and a fixed damping factor $\lambda \in (0,1)$, PageRank can be understood as a functional operator $PR_\lambda : \Delta_N^+ \longrightarrow \Delta_N^+$ such that for a given personalization vector $\mathbf{v}$ in the N-dimensional simplex $\in \Delta_N^+$ (i.e the set of all normalized N-dimensional vectors with positive coordinates), the image $PR_\lambda(\mathbf{v})$ corresponds to the PageRank vector of $\mathscr{G}$ under damping factor λ and teleportation distribution $\mathbf{v}$ [6]. This functional framework motivates the use of concepts from Functional Analysis to investigate the operator PR_λ, including the study of fixed points, spectral properties, and operator stability. The main contribution of this presentation is the characterization of the existence and uniqueness of fixed points of PR_λ, together with methods for their explicit computation. Specifically, we analytically demonstrate that if $\mathscr{G}$ is a strongly connected network, then PR_λ admits a unique fixed point, which can be analytically determined. Conversely, when $\mathscr{G}$ is not strongly connected, existence of fixed points is still guaranteed, while uniqueness depends on the structure of the strongly connected components (sinks). A fixed point of PR_λ corresponds to a personalization vector $\mathbf{v}$ such that $PR_\lambda(\mathbf{v}) = \mathbf{v}$; in other words, the teleportation vector coincides with the resulting PageRank. The computation of such fixed points relies on an iterative feedback-PageRank scheme, inspired by the classical Power Method. Notably, classical fixed-point theorems (e.g., Banach's Contraction Principle) do not apply in this setting, as PR_λ constitutes a non-contractive operator with spectral radius equal to one. All theoretical results are illustrated by considering several synthetic examples, as for example, the one included in Figure 1.

Beyond their theoretical significance, these results hold practical implications for dynamic and temporal networks, where the underlying topology evolves over time. In such contexts, the personalization vector itself may vary dynamically, raising the natural question of how to select teleportation distributions that reflect intrinsic structural properties of the network.

For example, in web graphs, one could adopt a feedback mechanism in which teleportation probabilities at a given time step are determined by the PageRank distribution from the previous time step. The steady state of such a process, if it exists, corresponds precisely to the fixed point of the operator PR_{λ}. Hence, by adopting a functional operator perspective, this work establishes a rigorous analytical framework for PageRank in complex networks, elucidating both the mathematical structure of the operator and its potential for modeling adaptive and temporally evolving systems.

References

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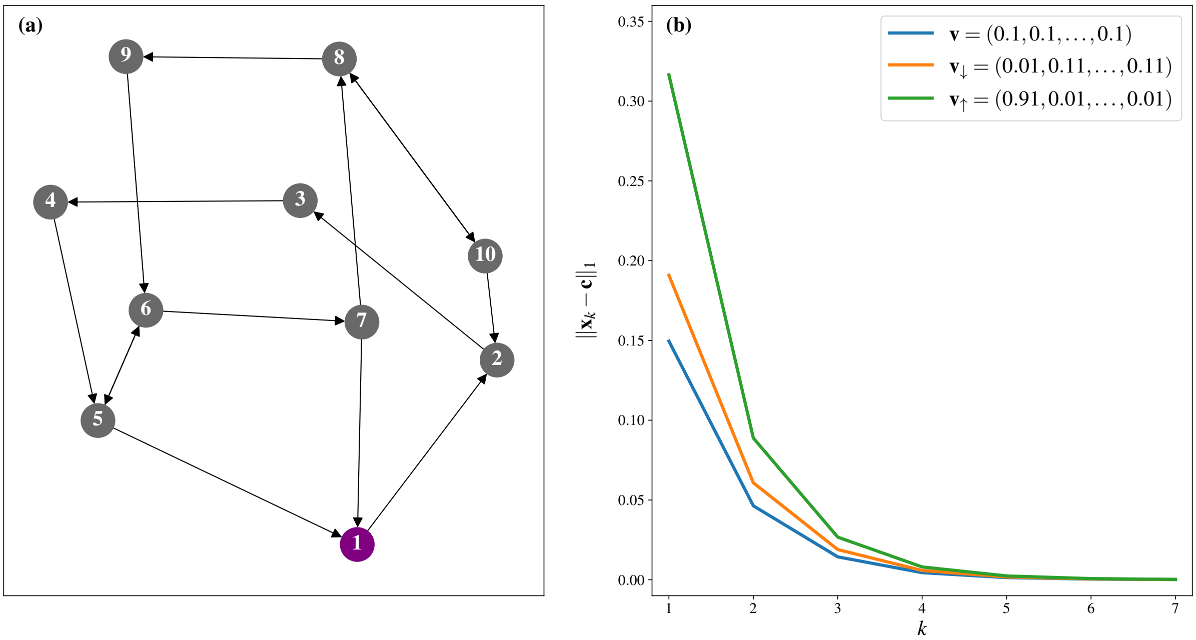

Figure 1: **Title.** Numerical simulation on the evolution of iterative feedback-PageRank of a 10-noded systhetic comoplex network and its path to the steady state for several personalization vector strategies.