Proportional Representation in Metric Spaces and Low-Distortion Committee Selection

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Abstract

We propose and analyze a natural new definition for when a small set R of k points in a metric space is representative of a larger set. There is a set V of points to be represented (such as documents or voters), and a set C of candidates (also documents, or candidates for office) who could represent them. Our definition states (essentially) that for any set $S \subseteq V$ of points comprising a θ fraction of V, the average distance of S to their respective best θk points in R should not be larger by more than a factor γ compared to their average distance to the best θk points among all of C. This definition is a strengthening of the notions of proportional fairness and core fairness, but — different from those notions — requires that large cohesive clusters be represented proportionally to their size.

Since there are instances for which — unless γ is polynomially large — no solutions exist, we study this notion in a resource augmentation framework, implicitly stating the constraints for a set R of size k as though its size were only k/α , for $\alpha > 1$. Furthermore, motivated by the application to elections, we mostly focus on the *ordinal* model, in which the algorithm does not learn the actual distances; instead, the algorithm learns only for each point $v \in V$ and each pair of candidates c, c' which of c, c' is closer to v. Our main result is that the EXPANDING APPROVALS RULE of Aziz and Lee is (α, γ) representative in our sense with $\gamma \approx 1 + 6.71 \cdot \frac{\alpha}{\alpha-1}$.

We also obtain three novel byproducts and corollaries from our analysis. First, we show that the EXPANDING APPROVALS RULE achieves constant proportional fairness in the ordinal model, giving the first positive result on metric proportional fairness with ordinal information. Second, we show that for the core fairness objective, the EXPANDING APPROVALS RULE achieves the same asymptotic tradeoff between resource augmentation and approximation as the recent results of Li et al., which used full knowledge of the metric. Finally, our results imply a very simple single-winner voting rule with metric distortion at most 44.

1 Introduction

Selecting representatives for a large set is a common and central problem across a wide range of application areas. As three paradigmatic applications, consider selecting a small set of documents (such as pictures or text) representing a much larger collection, selecting a committee of representatives for a large population, or selecting locations for several public facilities to serve the population of a city. Naturally, there are many ways of defining what it means for a candidate set to be "representative"; we discuss some key definitions from past work in Section 6.

A commonly accepted notion of representation is based on proportionality [Humphreys, 1911, Moulin, 2003]: subgroups of the population should be represented in the selected set proportionally to their size. Stated differently, if a cohesive subset S comprises a θ fraction of the documents/population, then (at least) roughly a θ fraction of the representative set should be similar to the members of S. In terms of documents, this implies that by examining the representative documents, a user can accurately assess the contents of the document collection. For committee elections, it states that large like-minded groups of the population

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should be suitably represented in the committee. And for the location of public facilities, it implies that dense population centers should be sufficiently served with nearby facilities. Indeed, notions of fairness or representation based on this intuition have been studied extensively, as discussed in Section 6.

We are particularly interested in the common setting in which the documents or candidates/population are embedded in a metric space in which distances capture dissimilarity. For documents, this is often the result of feature-based embeddings applied to the documents, and for the selection of public facilities, the metric is naturally derived from geographic proximity or transportation times. For voters and candidates in elections, the idea of considering all agents as embedded in a metric space, and their preferences being reflective of the distances, was first articulated as *single-peaked preferences* (where the metric space is the line, e.g., representing a one-dimensional left-to-right spectrum of opinions) [Black, 1948, Moulin, 1980], but also subsequently generalized to other metrics [Barberà et al., 1993, Merrill and Grofman, 1999].

Our first main contribution (in Section 2) is a natural and novel definition of what it means for a set R of k points to "represent" a larger set V in a metric space; our notion is a strengthening of the notion of core fairness proposed recently by Li et al. [2021].

Our second main contribution (in Section 4) is to show that a natural algorithm (a special case of the Expanding Approvals Rule of Aziz and Lee [2020]) achieves strong representativeness guarantees for the new definition. It does so even though it works in the *ordinal* model (see Section 3), in which the algorithm only learns, for each voter/document/citizen, the ranking of potential representatives by increasing distances (but not the distances themselves). As immediate corollaries of our analysis, we obtain the first algorithm with constant proportional fairness for metric costs in the ordinal model, an algorithm with ordinal information achieving — up to constants — the same parameter tradeoff for approximate core fairness as the one of Li et al. [2021] (which had access to the full metric), as well as an extremely simple single-winner voting rule with constant metric distortion; see Section 4.

Finally, in Section 5, we show that when the algorithm has access to all the distances, a slight modification of the Greedy Capture algorithm of Chen et al. [2019] provides improved constants in the representativeness guarantees.

2 The Key Fairness Concepts

Our first main contribution is a new definition of a representative set in a metric space. Our definition is a strengthening of the notion of core fairness proposed by Li et al. [2021], and — as that definition does — naturally recovers an approximate median as a special case for k = 1.

We consider settings in which a set V should be "well represented" by a subset of a set C; for examples, see Section 1. We write n = |V| and m = |C| for the sizes of the sets, and k < m for the size of the subset of C that is to be selected. For concreteness in our nomenclature, we will refer to V as *voters* and C as *candidates* throughout, although we do not exploit any specific properties of this domain.

We assume that $V \cup C$ is embedded in a (pseudo-)metric space $(V \cup C, d)$.² The goal is to pick a set $R \subseteq C$ of k representatives to ensure that each sufficiently large subset $S \subseteq V$ is "well represented", in a sense we define next. In keeping with the voting-related nomenclature, we will refer to R as a committee and to S as a coalition.

We write $d_{\text{sum}}(X,Y) = \sum_{x \in X, y \in Y} d(x,y)$ for the sum of distances between all pairs in $X \times Y$. When $X = \{x\}$ is a singleton, we write $d_{\text{sum}}(x,Y) = d_{\text{sum}}(\{x\},Y)$.

2.1 Proportional Representation

Recall that our goal is to ensure that all sufficiently large coalitions of voters are well represented. Specifically, if a coalition S comprises a θ fraction of all *voters*, at least a θ fraction of the *committee* should be approximately closest to S as a whole. To phrase this requirement cleanly, we recall the definition of the

¹V = C — the most natural case when selecting documents, and a case corresponding to peer selection in the context of elections — is of course allowed.

²Recall that a metric is a non-negative function d on pairs satisfying that d(x,x) = 0 for all x, symmetry (d(x,y) = d(y,x)) for all x,y,z, triangle inequality $(d(x,z) \le d(x,y) + d(y,z))$ for all x,y,z, and positivity (d(x,y) > 0) whenever $x \ne y$. A pseudo-metric is allowed to violate positivity, i.e., multiple points of the metric space can be at distance 0 from each other.

Hare Quota $p = \lceil n/k \rceil$, the number of voters/documents/citizens represented by any one selected candidate/document/facility. As articulated by Li et al. [2021] in their definition of (approximate) core fairness, any coalition S whose size is at least the Hare Quota should have at least one roughly satisfactory representative in R. Our generalization requires that, for any positive integer t, each member of a coalition S of size $|S| \ge t \cdot p$ should have at least t representatives in R, such that no other t candidates are much better for the coalition compared to their individual best t members in R. We now formally define what it means for the subset of the committee to be "approximately optimal" for all coalitions.

Definition 2.1 (γ -proportionally representative committee). A committee R is called γ -proportionally representative if for every coalition $S \subseteq V$ of size at least $t \cdot p$, the committee satisfies:

$$\sum_{v \in S} \min_{\substack{R'_v \subseteq R \\ |R'_v| = t}} d_{sum}(v, R'_v) \le \gamma \cdot \min_{\substack{C' \subseteq C \\ |C'| = t}} d_{sum}(S, C'). \tag{1}$$

Definition 2.1 captures a notion of approximate stability: no sufficiently large coalition S of voters could find an alternative committee for themselves of corresponding size t which they strongly prefer over their individually best size-t subcommittees of R. Note that proportional representation (Definition 2.1) is a more demanding requirement than approximate core fairness in the sense of Li et al. [2021], as defined in Definition 2.3; we will elaborate on this in more detail below. A positive feature of Definition 2.1 is that it does not require a notion of "cohesive" coalitions which must be represented. Rather, a requirement for all sufficiently large coalitions is given; however, the requirements for very spread-out coalitions are typically trivially satisfied, because such coalitions do not have attractive alternatives to deviate to.

While natural and intuitive, our definition of proportional representation is unfortunately too demanding; Li et al. [2021] already showed that there are instances for which the $(1, \Omega(\sqrt{n}))$ -core³ is empty. Since Definition 2.1 has additional constraints, we can in general not hope for $\gamma = o(\sqrt{n})$.

Therefore, as in Li et al. [2021], we relax Definition 2.1 by allowing for resource augmentation — see Jiang et al. [2020] for another example and discussion of the use of resource augmentation to deal with impossibility of proportional representation. Specifically, for a resource augmentation parameter $\alpha \geq 1$, we consider the problem of selecting a committee of size k, but require the weaker stability guarantee for a committee of (smaller) size k/α , as captured by the following definition.

Definition 2.2 $((\alpha, \gamma)$ -proportionally representative committee). For a resource augmentation parameter $\alpha \geq 1$, a committee R is called (α, γ) -proportionally representative if for every coalition $S \subseteq V$ of size at least $t \cdot \alpha \cdot p$, the committee satisfies:

$$\sum_{v \in S} \min_{\substack{R'_v \subseteq R \\ |R'_v| = t}} d_{sum}(v, R'_v) \le \gamma \cdot \min_{\substack{C' \subseteq C \\ |C'| = t}} d_{sum}(S, C'). \tag{2}$$

Definition 2.2 is a strengthening of the definition of approximate core in Li et al. [2021], defined as follows:

Definition 2.3 (Approximate Core). For a resource augmentation parameter $\alpha \geq 1$, a committee R is in the (α, β) -core if for every coalition $S \subseteq V$ of size at least $\alpha \cdot p$, the committee R satisfies:

$$\sum_{v \in S} \min_{r_v \in R} d(v, r_v) \le \beta \cdot \min_{c \in C} d_{sum}(S, c). \tag{3}$$

Thus, the definition of the (α, β) -core⁴ is obtained by requiring only the constraints for t = 1 to hold in Definition 2.2.

After augmenting resources, the definition becomes less stringent, and one may naturally inquire if approximation in the objective can be avoided, i.e., whether $\gamma = 1$ (or $\beta = 1$) can be achieved. We show that this is in general not possible; in Appendix A.1, we prove the following lower bound, which already holds for the weaker notion of approximate core in the sense of Li et al. [2021]:

³The instances described in Li et al. [2021] require $n = \Theta(k^2)$. In Appendix A.2, we give a simple class of instances showing a slightly more fine-grained lower bound of $\Omega(\min(k, n/k))$, thus giving a lower bound for the entire range of k.

⁴We use γ in place of β in Li et al. [2021] to emphasize the difference in the definitions.

Proposition 2.4. For every $\alpha \geq 1$, there are instances for which the $(\alpha, 1 + 1/(2\alpha))$ -core is empty, and so, no $(\alpha, 1 + 1/(2\alpha))$ -proportional representation exists.

Furthermore, in Appendix A.2, we show that when $\alpha \to 1$, a large blowup in γ is unavoidable in general. Again, we show this lower bound result already for the approximate core:

Proposition 2.5. There are α arbitrarily close to 1 and corresponding instances for which the $(\alpha, 1/(4(\alpha - 1)))$ -core is empty, and so, no $(\alpha, 1/(4(\alpha - 1)))$ -proportional representation exists.

Note that Proposition 2.5 provides an asymptotically matching lower bound for Theorem 19 of Li et al. [2021].

While we cannot achieve $\alpha \approx 1$ or $\gamma \approx 1$ without a large blowup in the other parameter, we still seek to design (polynomial-time) algorithms for computing committees achieving a good tradeoff between α and γ ; indeed, this is the main goal of our work, studied in Section 4.

2.2 Proportional Representation, Core Fairness, and Proportionally Fair Clustering

As discussed above, proportional representation is a natural strengthening of the notion of core fairness. Another closely related concept is *proportional fairness*, introduced in the field of clustering by Chen et al. [2019], and studied further by Micha and Shah [2020]. Here, the setup is the same, and the committee R is construed as *cluster centers*:

Definition 2.6 (γ -proportional fairness). A committee R of size k is γ -proportionally fair if for every voter coalition S of size at least p and every alternate candidate c, at least one voter $v \in S$ satisfies $\min_{r \in R} d(v, r) \leq \gamma \cdot d(v, c)$.

Li et al. [2021] already pointed out that every committee in the $(1, \beta)$ -core according to their definition is also β -proportionally fair. Since $(1, \beta)$ -proportional representation implies membership in the $(1, \beta)$ -core, it is strictly stronger than β -proportional fairness.

We discuss key differences between Definitions 2.1, 2.3 and 2.6, as well as their implications. We believe that they justify Definition 2.1 as a more suitable notion of representativeness of a committee.

- 1. Regardless of the size of S and k, the definition of approximate core (and thus also proportionally fair clustering) only requires the existence of a single candidate c that is preferred in order to "satisfy" S. As a result, a large committee R could significantly distort the composition of V.
 - For example, consider k = 100, with two clusters V_1, V_2 , containing 99% and 1% of the voters, respectively. The two clusters are far from each other, and each has a large number of possible candidates, all at distance 1 from each voter in the cluster. Then, R could contain 99 candidates close to V_2 and one candidate close to V_1 , while satisfying Definition 2.3 and Definition 2.6 with $\beta = 1$. The exact same types of distances can be used to obtain similarly non-proportional outcomes for Example 1 in Li et al. [2021]. In a sense, neither Definition 2.3 nor Definition 2.6 include a notion of proportionality, and thus, they fail to achieve it.
- 2. Definition 2.6 suffers from a second weakness, which is fixed by Definition 2.3: for small k (in particular, k=1), the definition is extremely lenient. In fact, for k=1, it allows the chosen candidate to be any candidate not Pareto-dominated by another; among others, this includes any candidate ranked first by at least one voter. This is a very weak requirement for a chosen candidate being "representative". In contrast, any candidate in the $(1,\beta)$ -core (and thus any candidate who is $(1,\beta)$ -representative) must be a β -approximate median of the voters, a much more meaningful sense of being representative.
- 3. The distinction between the definitions can also be viewed through the lens of whether the utility from deviations is transferable within the deviating coalition, as pointed out by Li et al. [2021]. While proportionally fair clustering implicitly assumes non-transferable utilities, Definition 2.3 as well as Definition 2.1 correspond to transferable utilities, leading to a larger set of "deviation threats". Notice that the question of whether utility is transferable also arises in more "classical" definitions of the core (e.g., Peters [2015]).

3 The Ordinal Information Model

We have so far stated the problem of finding an (α, γ) -proportionally representative committee in a model where the metric d is known to the algorithm. This model is a natural fit for the applications of selecting representative documents or locations for facilities: the documents are typically embedded into a (often Euclidean) metric space by a known algorithm, so all distances are known; similarly, the metric space for the citizens and facilities is defined by known geography or travel times.

In contrast, for the election of a committee, parliament, or similar body, the metric space is a useful modeling tool, but is typically not known explicitly. Rather, the voters are assumed to rank the candidates by non-decreasing distance from themselves, and the algorithm has access to the rankings, but not the distances. Despite this limited information (after all, many different metrics may be consistent with the rankings), an algorithm or voting rule should select an approximately optimal solution, in our case, a proportionally representative set R exhibiting a good tradeoff between α and γ .

This framework is called the ordinal information model of the metric distortion⁵ framework [Anshelevich et al., 2018, 2021a,b, Caragiannis et al., 2022], contrasting it with the cardinal model, in which the metric is known explicitly. The worst-case loss in the objective function due to the lack of information is called (metric) distortion, and has been studied for numerous optimization problems. Most notable is the by now extensive line of work on the distortion of single-winner elections [Anshelevich et al., 2015, 2018, Munagala and Wang, 2019, Gkatzelis et al., 2020, Kizilkaya and Kempe, 2022, Gkatzelis et al., 2023]; however, more complex objectives have also been studied [Anshelevich et al., 2021b,a, Anshelevich and Zhu, 2021, Anari et al., 2023].

Viewed in this context, under the ordinal information model, our goal of selecting a committee R can be viewed as a natural extension of the metric distortion objective to multi-winner elections; a detailed comparison to other recently proposed multi-winner distortion objectives is given in Section 6.

We now define the concepts formally. The algorithm learns, for each voter $v \in V$, a ranking of candidates \succ_v . Voters rank candidates by non-decreasing distances, so $c \succ_v c'$ implies that $d(v,c) \leq d(v,c')$. We use $\pi_v(c)$ to denote the position of candidate c in v's ranking, with $\pi_v^{-1}(1)$ being v's most preferred candidate. We write $\succ_V = (\succ_v)_{v \in V}$ for the vector of all voters' rankings, and refer to it as the ranked-choice profile. An election consists of the triple (V, C, \succ_V) . We say that a pseudo-metric d is consistent with the ranked-choice profile \succ_V if it satisfies that $d(v,c) \leq d(v,c')$ whenever $c \succ_v c'$, for all v,c,c'.

An ordinal committee selection rule f receives as input the committee size k and the election (V, C, \succ_V) , and outputs a committee $R \subseteq C$ of size k.

Our notion of proportional representation for ordinal models in committee selection rules is closely related to, and intended to be a natural generalization of, the concept of metric distortion for single-winner elections. Recall that the $metric\ distortion$ of a single-winner voting rule f is the worst-case ratio (over all elections, and all metrics consistent with the election) of the total cost of the chosen winner relative to the total cost of the optimum candidate, i.e.,

$$\max_{(V,C,\succ_V)\,d\text{ consistent with }\succ_V}\frac{d_{\mathrm{sum}}(V,f(V,C,\succ_V))}{\min_{c\in C}d_{\mathrm{sum}}(V,c)}.$$

As a result, notice that in the special case when k=1 and $\alpha=1$, a committee selection rule f is γ -proportionally representative if and only if f is a single-winner voting rule with metric distortion at most γ . This is because the only coalition of size p=n is S=V, so Eq. (2) requires that for the chosen candidate $R=\{\widehat{c}\}=f(V,C,\succ_V,k)$, we have $d_{\text{sum}}(V,\widehat{c})\leq \gamma\cdot d_{\text{sum}}(V,c)$ for every candidate c; in particular, the optimal candidate $c=c^*$.

4 Our Main Result

Our second main contribution — and the key technical work in this paper — is to show that a very natural algorithm, namely, a special case of the EXPANDING APPROVALS RULE of Aziz and Lee [2020], selects an

⁵A parallel line of work (e.g., [Boutilier et al., 2015, Boutilier and Rosenschein, 2016, Procaccia and Rosenschein, 2006]) considers the same tradeoff of ordinal vs. cardinal information in a setting in which the rankings are derived from positive *utilities* rather than distances/costs.

 (α, γ) -proportionally representative committee R of size k which achieves constant α and γ ; furthermore, it does so even in the ordinal model.

The algorithm runs in iterations, parametrized by a tolerance parameter τ , which starts at 1 and increases in each round. In iteration τ , the algorithm considers, in some arbitrary order, each (remaining) voter as approving their top τ choices. As soon as at least $p = \lceil n/k \rceil$ of the remaining voters approve of a particular candidate c, this candidate is added to the committee, and those p voters are permanently removed from further consideration. We will say that c covers these voters. The fact that the algorithm processes the voters in an arbitrary order (instead of simultaneously) in each iteration achieves an (arbitrary) tie breaking implicitly, so the algorithm does not have to consider ties between multiple candidates becoming eligible for inclusion. This process continues until a committee of size k has been formed; if the algorithm would otherwise terminate with fewer candidates, it adds arbitrary candidates to ensure that the committee has size k. The algorithm is described formally as Algorithm 1.

Algorithm 1 Expanding Approvals Rule

```
Input: Election (V, C, \succ_V), Committee Size k
Output: Committee R
  Let U \leftarrow V be the set of uncovered voters.
  Let R \leftarrow \emptyset be the selected committee.
  Let N_c \leftarrow \emptyset for all c \in C.
  for \tau = 1, \ldots, m do
       for v \in V in arbitrary order do
           if v \in U then
               Let c = \pi_v^{-1}(\tau).
               if c \notin R then
                   Let N_c \leftarrow N_c \cup \{v\}, i.e., add v to the neighborhood of c.
                   if |N_c| = \lceil n/k \rceil then
                        R \leftarrow R \cup \{c\}, i.e., include c in the committee.
                       N_{c'} \leftarrow N_{c'} \setminus N_c for all c' \in C \setminus R, i.e., update the neighborhood for non-committee
  candidates.
                       U \leftarrow U \setminus N_c, update the set of uncovered voters.
                       We say that N_c has been covered by c.
  if |R| < k then add k - |R| arbitrary candidates to R.
```

Theorem 4.1. The EXPANDING APPROVALS RULE outputs a committee R of size k which is $(\alpha, \gamma(\alpha))$ -proportionally representative for all $\alpha > 1$, with $\gamma(\alpha) = 1 + \frac{7 + \sqrt{41}}{2} \cdot \frac{\alpha}{\alpha - 1} \approx 1 + 6.71 \cdot \frac{\alpha}{\alpha - 1}$.

Theorem 4.1 gives guarantees only for $\alpha > 1$, and Proposition 2.5 shows that this is unavoidable for decent approximation guarantees, unless k is very small or large. When resources are not augmented, i.e., for $\alpha = 1$, we obtain the following weaker guarantee, which is proved in Section 4.4.

Theorem 4.2. The output R of Algorithm 1 is (1, O(n/k))-proportionally representative.

Our analysis directly implies several novel results. First, an immediate corollary of a key lemma in the proof of Theorem 4.1 is that the EXPANDING APPROVALS RULE achieves proportional fairness 5.71; this constitutes the first result achieving constant proportional fairness with metric costs in the ordinal information model:

Corollary 4.3. The Expanding Approvals Rule is a $\frac{5+\sqrt{41}}{2} \approx 5.71$ -proportionally fair clustering algorithm under the ordinal information model with metric costs.

The constant $\frac{5+\sqrt{41}}{2}$ is larger than the best proportional fairness achievable with full knowledge of the metric space, which is $1+\sqrt{2}\approx 2.41$ [Chen et al., 2019, Micha and Shah, 2020]. This is perhaps not surprising, given that in the ordinal information model, the algorithm is missing crucial information. Indeed, Appendix A.3 gives an instance in which under the ordinal information model, no deterministic algorithm

can produce a γ -proportionally fair committee for any $\gamma < 2 + \sqrt{5} \approx 4.23$; thus, there is necessarily a gap between the best proportional fairness achievable in the cardinal and ordinal models. Obtaining the best possible proportional fairness guarantees under metric costs and ordinal information is an interesting direction for future work.

A second corollary can be immediately obtained from Theorem 4.1 and Theorem 4.2: because (α, β) -representativeness implies being in the (α, β) -core, the guarantees of Theorem 4.1 and Theorem 4.2 apply verbatim to the latter. Thus, the EXPANDING APPROVALS RULE achieves approximate core fairness in the sense of Li et al. [2021] in the ordinal metric cost model; this, too, is the first positive result on core fairness in the ordinal metric cost model.

Corollary 4.4. The committee R output by Algorithm 1 is in the $(\alpha, \beta(\alpha))$ -core for all $\alpha > 1$, with $\beta(\alpha) = 1 + \frac{7+\sqrt{41}}{2} \cdot \frac{\alpha}{\alpha-1} \approx 1 + 6.71 \cdot \frac{\alpha}{\alpha-1}$. It is also in the (1, O(n/k))-core.

Note that these bounds match the information-theoretic lower bound of Proposition 2.5 and the lower bound of $\Omega(\min(k, n/k))$ for $\alpha = 1$ in Appendix A.2 up to constant factors. They mirror (with slightly worse constants) the bounds obtained by Li et al. [2021] in the model with known distances (Theorem 19).

A third corollary, also from Theorem 4.1, gives an extremely simple single-winner voting rule with constant metric distortion.

Corollary 4.5. For a given set of candidates C and voters V, consider the following voting rule: find a candidate c who is in the top τ positions of at least $\lceil n/2 \rceil$ voters, for the smallest possible τ . (Break ties arbitrarily.) Find a candidate c' who is in the top τ' positions of the remaining $\lfloor n/2 \rfloor$ voters, for the smallest possible τ' . (Again, break ties arbitrarily.) Return the one of c, c' preferred by a majority of voters.

This voting rule has metric distortion at most 44 for the single-winner election (V, C, \succ_V) .

While the distortion guarantee of 44 given by Corollary 4.5 is worse than the (optimal) metric distortion of 3 achieved by [Gkatzelis et al., 2020, Kizilkaya and Kempe, 2022], this voting rule is arguably even simpler than the rules previously known to achieve constant metric distortion (COPELAND [Anshelevich et al., 2018], PLURALITY-MATCHING [Gkatzelis et al., 2020], PLURALITY-VETO [Kizilkaya and Kempe, 2022]).

4.1 Running Time Analysis and Discussion

We briefly analyze the running time of Algorithm 1, and explain why it is a special case of the EXPANDING APPROVALS RULE, as well as its relationship to another recently proposed algorithm.

We begin with a brief running time analysis. The number of basic set operations (addition or removal of an element to/from a neighborhood, or test of membership in a set) is O(mn). This is because each voter is added and removed to/from each neighborhood at most once, and at any time, each neighborhood contains only uncovered voters.⁶ Set operations can be implemented in time $O(\log n)$ or $O(\log m)$, giving a running time of $\widetilde{O}(nm)$. Notice that because specifying all rankings requires space $\Omega(nm \log m)$, the running time is essentially linear.⁷

We next discuss why Algorithm 1 is a special case of the Expanding Approvals Rule. The Expanding Approvals Rule, as presented by Aziz and Lee [2020], follows the same pattern of increasing τ , and including a candidate when N_c (in our notation) is large enough. Because Aziz and Lee [2020] expand all N_c simultaneously, multiple candidates may become eligible for inclusion at the same time. In this case, Aziz and Lee [2020] use a fixed order on candidates to break ties. Since the order in which Algorithm 1 processes candidates in each iteration is arbitrary (and can differ from iteration to iteration), one could choose the order to emulate the ordering that would be chosen in Aziz and Lee [2020]. A second consequence of the simultaneous updating of the N_c in Aziz and Lee [2020] is that $|N_c| > \lceil n/k \rceil$ becomes possible. Aziz and Lee [2020] have all voters start with weights $w_v = 1$, and reduce the combined weights of voters in N_c by a total of $\lceil n/k \rceil$; this includes reducing the weights of voters fractionally, or selecting an arbitrary subset of size

⁶Notice that the two nested loops with an internal update for all non-committee candidates' neighborhoods initially suggest $O(m^2n)$. By keeping track of the set of all neighborhoods N_c that a given voter v has been included in, the updating can be implemented to only incur time for (v,c) pairs such that $v \in N_c$ at the time.

⁷A slight exception is when the number of voters n is superpolynomially large in m, in which case operations for the sets N_c may lead to slightly larger running time.

⁸In fact, Aziz and Lee [2020] also allow for slight variations in the total amount of weight.

 $\lceil n/k \rceil$ who are eliminated by virtue of having their weights reduced to 0. We believe that our proportional representation results hold for the EXPANDING APPROVALS RULE in full generality; however, various steps of the analysis are cleaner to formulate and argue using our approach of avoiding ties and fractional weights.

The Expanding Approvals Rule is also somewhat similar to Algorithm C of Skowron et al. [2015]. The work of Skowron et al. [2015] is concerned with selecting committees under the rules of Monroe [1995] and Chamberlin and Courant [1983]. Both of these rules define pairwise dissatisfaction values between voters and candidates, increasing in how far down in the voter's ranking the candidate appears. The goal is to select a committee minimizing the sum of dissatisfactions of all voters, where the dissatisfaction of a voter with a committee is their dissatisfaction with their favorite committee member. Algorithm C of Skowron et al. [2015] also greedily selects candidates one by one, always choosing next the candidate reducing dissatisfaction the most. Notice, however, that this is not identical to expanding approvals, as voters may see their dissatisfaction improve multiple times, and there is no hard "inclusion threshold" akin to our value of τ .

4.2 Stronger Proportional Fairness: The Key Lemma

In the analysis of Algorithm 1, a central concept is for any coalition S the set of all chosen representative candidates $r \in R$ who covered at least one voter in S. We formally define this notion; recall here that N_r is defined in Algorithm 1:

Definition 4.6 (Representatives for a Coalition). The set of representatives for the coalition $S \subseteq V$ is defined as $R[S] = \{r \in R \mid N_r \cap S \neq \emptyset\}$, i.e., $R[S] \subseteq R$ is the set of candidates $r \in R$ whose neighborhood contains at least one voter in S.

We show that the committee R returned by Algorithm 1 satisfies a somewhat stronger notion of proportional fairness, i.e., a modification of Definition 2.6. Lemma 4.7 shows that for any coalition S, the representatives R[S] are already sufficiently attractive that S will not unanimously deviate. The guarantee differs from Definition 2.6 only in the slight strengthening of replacing R with R[S].

Lemma 4.7. The committee R output by Algorithm 1 has the following stability property, with $\rho = \frac{5+\sqrt{41}}{2} \approx 5.71$: For every coalition S of size $|S| \ge p = \lceil n/k \rceil$, there exists a voter $v \in S$ with

$$\min_{r \in R[S]} d(v, r) \le \rho \cdot \min_{c \in C \setminus R} d(v, c).$$

Because $R[S] \subseteq R$, Lemma 4.7 immediately implies that Algorithm 1 outputs a $\frac{5+\sqrt{41}}{2}$ -proportionally fair clustering in the ordinal information model, which proves Corollary 4.3.

We now prove Lemma 4.7. As in the work of Micha and Shah [2020], our analysis is based on the notion of Apollonius's Circle, defined as follows:

Definition 4.8 (Apollonius's Circle). Let (\mathcal{X}, d) be an arbitrary metric space, and $x, y \in \mathcal{X}$ arbitrary points of the space. For any $\rho > 1$, let $A(x, y, \rho) = \{z \in \mathcal{X} \mid d(x, z) \cdot \rho \leq d(y, z)\}.$

 $A(x, y, \rho)$ is called the Apollonius's Circle of x and y with distance ratio ρ .

We are specifically interested in the diameter⁹ of Apollonius's Circles. The following lemma due to Micha and Shah [2020] states an upper bound on the diameter of Apollonius's Circles in arbitrary metric spaces.

Lemma 4.9 (Theorem 4 of Micha and Shah [2020]). For any metric space (\mathcal{X}, d) , points $x, y \in \mathcal{X}$ and constant $\rho > 1$, the Apollonius's Circle $A(x, y, \rho)$ has diameter at most

$$diam(A(x,y,\rho)) \le \frac{2}{\rho - 1} \cdot d(x,y).$$

Proof of Lemma 4.7. Consider a coalition S of size $|S| \ge p$ and let $c^* \in C \setminus R$ be a candidate satisfying the condition

$$\rho \cdot d(v, c^*) \le \min_{r \in R[S]} d(v, r) \qquad \text{for all } v \in S.$$
(4)

⁹Recall that the diameter of a (closed) set S of points is defined as $\operatorname{diam}(S) = \max_{x,y \in S} d(x,y)$.

We will show that under the assumption (4), ρ must satisfy $\rho \leq \frac{5+\sqrt{41}}{2}$.

We first show that R[S] is non-empty. To do so, note first that in the last iteration (i.e., for $\tau = m$), for all remaining candidates c, the neighborhood is $N_c = U$. The set of remaining candidates is also non-empty, because $c^* \notin R$. If $|U| \ge p$ at termination, the last iteration of Algorithm 1 would have added at least one more candidate. Therefore, at termination, the set of uncovered voters has size |U| . As a result, at least one voter in <math>S must have been covered by Algorithm 1, meaning that $R[S] \ne \emptyset$.

Let $r^* \in R[S]$ be the first candidate added to the committee with $N_{r^*} \cap S \neq \emptyset$, and τ the tolerance level of the algorithm at the time of addition. Let $v \in S \cap N_{r^*}$ be an arbitrary voter in S who was among the first to be covered by Algorithm 1. We show that v cannot be too much further from r^* than from c^* .

Claim 4.10.
$$d(v, r^*) \leq 2 \cdot diam(S \cup \{c^*\})$$
.

Proof. Let $R_{\leq \tau} \subseteq R$ be the set of all candidates included by Algorithm 1 for tolerance values strictly smaller than τ , and $\widehat{C} = C \setminus R_{\leq \tau}$. By definition of r^* and τ , we have that $R[S] \subseteq \widehat{C}$ and $c^* \in \widehat{C}$.

Suppose for contradiction that $d(v, r^*) > 2 \cdot \operatorname{diam}(S \cup \{c^*\})$. We will show below that every candidate $c \in \widehat{C}$ who is ranked ahead of c^* by some voter $v' \in S$ must be ranked ahead of r^* by v. Because v also ranks c^* ahead of r^* , this implies that $\pi_{v'}(c^*) < \pi_v(r^*)$ for all $v' \in S$. But then, at the end of the iteration with tolerance level $\tau - 1$, we would have had $S \subseteq N_{c^*}$, and therefore $|N_{c^*}| \ge p$. As a result, Algorithm 1 would have chosen c^* for some tolerance level strictly smaller than τ , which would have covered v. This is a contradiction to the choice of r^* as the first representative chosen in the algorithm which covers a voter in S.

To prove the missing claim, let $c \in \widehat{C}$ be a candidate, and $v' \in S$ a voter ranking c ahead of c^* . Using the triangle inequality repeatedly, along with the assumption that $d(v, r^*) > 2 \cdot \text{diam}(S \cup \{c^*\})$, we can bound

$$\begin{aligned} d(v,c) &\leq d(v,v') + d(v',c) \\ &\leq d(v,v') + d(v',c^*) \\ &\leq 2 \cdot \text{diam}(S \cup \{c^*\}) \\ &< d(v,r^*), \end{aligned}$$

which proves that v ranks c ahead of r^* .

Define $T:=\{x\in V\cup C\mid \rho\cdot d(x,c^*)\}\leq d(x,r^*)$. Because the assumption (4) implies that $\rho\cdot d(v,c^*)\leq \min_{r\in R[S]}d(v,r)$, and $r^*\in R[S]$, we obtain that $S\cup\{c^*\}\subseteq T$, so $\dim(S\cup\{c^*\})\leq \dim(T)$. Furthermore, because $T=A(c^*,r^*,\rho)$, Lemma 4.9 implies that $\dim(S\cup\{c^*\})\leq \dim(T)\leq \frac{2}{\rho-1}\cdot d(r^*,c^*)$. By triangle inequality, $d(r^*,c^*)\leq d(v,r^*)+d(v,c^*)\leq (1/\rho+1)\cdot d(v,r^*)$, again by the assumption (4).

Next, by substituting Claim 4.10, we obtain that

$$\operatorname{diam}(S \cup \{c^*\}) \leq \frac{4}{\rho - 1} \cdot \frac{\rho + 1}{\rho} \cdot \operatorname{diam}(S \cup \{c^*\}) = 4 \cdot \frac{\rho + 1}{\rho^2 - \rho} \cdot \operatorname{diam}(S \cup \{c^*\}).$$

Solving¹⁰ this inequality for ρ implies that $\rho \leq \frac{5+\sqrt{41}}{2} \approx 5.71$, completing the proof of the lemma.

4.3 Proof of Theorem 4.1 and Corollaries

Proof of Theorem 4.1. Using Lemma 4.7, we are now ready to complete the proof of Theorem 4.1. In fact, we will show that the committee R returned by Algorithm 1 satisfies an even stronger stability guarantee than the claimed (α, γ) -proportional representation. We will show that for every coalition $S \subseteq V$ of size at least $t \cdot \alpha \cdot p$, the committee satisfies:

$$\min_{\substack{R' \subseteq R \\ |R'| = t}} d_{\text{sum}}(S, R') \le \gamma \cdot \min_{\substack{C' \subseteq C \\ |C'| = t}} d_{\text{sum}}(S, C').$$
(5)

¹⁰The implication only holds when $\operatorname{diam}(S \cup \{c^*\}) > 0$. However, notice that when $\operatorname{diam}(S \cup \{c^*\}) = 0$, Claim 4.10 directly implies that $d(v, r^*) = 0 \le d(v, c^*)$, and thus $\rho \le 1$.

That is, for the coalition S, there is a subcommittee R' of size t which is almost as good as the best C. Note that the cost of S for R' obviously is an upper bound on the sum of costs for voters $v \in S$ for their individually optimal size-t subcommittees.

Let S be an arbitrary coalition of size $|S| \ge t \cdot \alpha \cdot p$. Let C^* be a set of t candidates with smallest total distance to S, i.e., $C^* \in \operatorname{argmin}_{C':|C'|=t} d_{\operatorname{sum}}(S,C')$. We will define a perfect matching between C^* and R such that for every $c \in C^*$, its match $r \in R$ is an "approximately good" alternative. Overall, this will demonstrate that R is not much worse than C^* .

Let c_1, c_2, \ldots, c_t be an enumeration of the candidates in C^* , such that the candidates in $C^* \cap R$ precede the ones in $C^* \setminus R$; apart from this requirement, the order can be arbitrary. We define the matching representatives r_1, \ldots, r_t iteratively. First, for each $c_i \in C^* \cap R$, we define $r_i = c_i$. Subsequently, for each iteration i, having already defined r_1, \ldots, r_{i-1} , we let $S_i = S \setminus \bigcup_{j=1}^{i-1} N_{r_j}$, i.e., S_i is the set of all voters in S except those covered by the first i-1 selected candidates r_j . Let $T_i \in \operatorname{argmin}_{T \subseteq S_i, |T| = p} \sum_{v \in T} d(v, c_i)$ be a subset of S_i of size $|T_i| = p$ comprising the p voters in S_i closest to c_i (ties broken arbitrarily). As $|S| \ge \lceil \alpha \cdot t \cdot p \rceil$ and each N_{r_j} has size at most p, $|S_i|$ has size at least p for each i, and thus T_i is well defined. Because $c_i \notin R$ in the current case, by Lemma 4.7, there exists a voter $v_i \in T_i$ and candidate $r_i \in R[T_i] \subseteq R[S]$ such that

$$d(v_i, r_i) \le \rho \cdot d(v_i, c_i). \tag{6}$$

Now, we confirm that the resulting assignment is indeed a matching. Observe that by definition of $R[S_i]$, we know that $S_i \cap N_{r_j} = \emptyset$ for all j < i; in particular, $r_j \notin R[S_i]$ for all j < i. This implies that r_1, \ldots, r_t are distinct from each other, and so the assignment is a matching. We also remark here that Eq. (6) holds for the i with $c_i \in R$ as well, because $r_i = c_i$ implies $d(v_i, r_i) = d(v_i, c_i)$ for those candidates.

In the rest of the proof, we will show that r_i is an approximately good alternative to c_i for all of S, showing that $d_{\text{sum}}(S, r_i) < \gamma(\alpha) \cdot d_{\text{sum}}(S, c_i)$. Fix an arbitrary index $i \in \{1, \ldots, t\}$. Let \widehat{v}_i be a voter in T_i maximizing the distance $d(v, c_i)$ to c_i over all $v \in T_i$, i.e., a voter at (or possibly tied for) the p^{th} largest distance from c_i among voters in T_i .

For any voter $v \in S$, using the triangle inequality and (6), we can bound the distance

$$d(v, r_{i}) \leq d(v, c_{i}) + d(v_{i}, c_{i}) + d(v_{i}, r_{i})$$

$$\leq d(v, c_{i}) + (1 + \rho) \cdot d(v_{i}, c_{i})$$

$$\leq d(v, c_{i}) + (1 + \rho) \cdot d(\widehat{v}_{i}, c_{i})$$

$$\leq d(v, c_{i}) + (1 + \rho) \cdot \max\{d(v, c_{i}), d(\widehat{v}_{i}, c_{i})\}.$$
(7)

Let $P_i = \{v \in S \mid d(v, c_i) < d(\widehat{v}_i, c_i)\}$ be the subset of S containing all voters v with $d(v, c_i) < d(\widehat{v}_i, c_i)$. Because T_i was chosen to be the p voters closest to c_i inside S_i , and \widehat{v}_i the voter furthest from c_i in T_i , we get that $P_i \subseteq T_i \cup \bigcup_{j=1}^{i-1} N_{r_j}$ and so $|P_i| \leq p \cdot i$. Summing the distances to r_i over all voters $v \in S$, and using (in the second step) that $d(\widehat{v}_i, c_i) \leq d(v, c_i)$ for all $v \in S \setminus P_i$, we obtain the bound

$$\sum_{v \in S} d(v, r_i) \leq \sum_{v \in S} d(v, c_i) + (1 + \rho) \cdot \left(\sum_{v \in P_i} d(\widehat{v}_i, c_i) + \sum_{v \in S \setminus P_i} d(v, c_i) \right)$$

$$\leq \sum_{v \in S} d(v, c_i) + (1 + \rho) \cdot \left(\left(\frac{|P_i|}{|S| - |P_i|} + 1 \right) \cdot \sum_{v \in S \setminus P_i} d(v, c_i) \right)$$

$$\leq \sum_{v \in S} d(v, c_i) + (1 + \rho) \cdot \left(\left(\frac{p \cdot i}{|S| - p \cdot i} + 1 \right) \cdot \sum_{v \in S \setminus P_i} d(v, c_i) \right)$$

$$\leq \sum_{v \in S} d(v, c_i) + (1 + \rho) \cdot \left(\left(\frac{1}{\alpha - 1} + 1 \right) \cdot \sum_{v \in S \setminus P_i} d(v, c_i) \right)$$

$$\leq \left(1 + (1 + \rho) \cdot \frac{\alpha}{\alpha - 1} \right) \cdot \sum_{v \in S} d(v, c_i)$$

In the penultimate step, we used that $\frac{p \cdot i}{|S| - p \cdot i} \le \frac{p \cdot i}{\alpha \cdot p \cdot i - p \cdot i} = \frac{1}{\alpha - 1}$. Finally, summing up over all indices i, we obtain

$$\sum_{i=1}^{t} d_{\text{sum}}(S, r_i) \le \left(1 + (1 + \rho) \cdot \frac{\alpha}{\alpha - 1}\right) \cdot \sum_{i=1}^{t} d_{\text{sum}}(S, c_i).$$

П

Substituting $\rho = \frac{5+\sqrt{41}}{2}$ now completes the proof.

As a special case of Theorem 4.1 and Theorem 4.2, we obtain Corollary 4.4, simply by focusing on just the case t=1 in Theorem 4.1. In fact, by recalling that the proof of Theorem 4.1 established the somewhat stronger guarantee Eq. (5), we obtain the following Corollary 4.11, strengthening Corollary 4.4. This corollary states that no coalition of at least $\alpha \cdot p$ voters (for $\alpha > 1$) has a candidate outside R whom they strongly prefer on average to their best single candidate in R[S]:

Corollary 4.11. Let R be the committee output by the Expanding Approvals Rule. Let $\alpha > 1$, and $\beta(\alpha) = 1 + \frac{7 + \sqrt{41}}{2} \cdot \left(\frac{\alpha}{\alpha - 1}\right) \approx 1 + 6.71 \cdot \left(\frac{\alpha}{\alpha - 1}\right)$. For any coalition S of size $|S| \ge \alpha \cdot p$,

$$\min_{r \in R[S]} d_{sum}(S, r) \leq \beta(\alpha) \cdot \min_{c \in C \setminus R} d_{sum}(S, c).$$

Finally, we show how Theorem 4.1 implies Corollary 4.5, i.e., the existence of an extremely simple single-winner voting rule with constant distortion.

Proof of Corollary 4.5. Consider the committee R output by Algorithm 1 when run with k=2. By Theorem 4.1, applied with $\alpha=n/p\leq 2$, we get that $\min_{r\in R}d_{\mathrm{sum}}(V,r)\leq \gamma(2)\cdot \min_{c\in C}d_{\mathrm{sum}}(V,c)$. This implies that at least one representative in R has distortion at most $\gamma(2)\approx 14.42$ for the single-winner election. Let r_1 be the winner of the majority election between the two candidates in R, and r_2 the other candidate. Since r_1 is preferred over r_2 by at least half of the voters, Lemma 6 of Anshelevich et al. [2018] implies that $d_{\mathrm{sum}}(V,r_1)\leq 3d_{\mathrm{sum}}(V,r_2)$. This implies that r_1 has distortion at most $3\cdot\gamma(2)\leq 44$ for the single-winner election.

4.4 Proportional Representation Without Augmentation: Proof of Theorem 4.2

Let S be a coalition of size $|S| = p \cdot t$. Consider the $r_i, c_i, v_i, i = 1, ..., k$ defined in the proof of Theorem 4.1. Using the bound from Eq. (7) that $d(v, r_i) \leq d(v, c_i) + (1 + \rho) \cdot d(v_i, c_i)$ for all $v \in S$ and all i, we obtain that

$$d_{\text{sum}}(S,R) \le \sum_{v \in S} \sum_{i} (d(v,c_i) + (1+\rho) \cdot d(v_i,c_i)) = d_{\text{sum}}(S,C^*) + (1+\rho) \cdot |S| \cdot \sum_{i} d(v_i,c_i).$$
(8)

The rest of the proof will be concerned with upper-bounding $\sum_i d(v_i, c_i)$. Notice that it is possible that some/many of the v_i are the same, necessitating the following approach. For each i < k, let $u_i \in S_i \setminus T_i$ such that all of the u_i are distinct. Notice that because $|S_i| \geq (k+1-i)$ for all i, it is possible to define such u_i inductively by decreasing i, starting with i = k-1: there is always some $u \in S_i$ available which was not chosen as u_j for any j > i. Further define $u_k = v_k$. First, observe that each v appears at most twice¹¹ as a u_i . Second, because of the choice of T_i as being closest to c_i among voters in S_i , and because $v_i \in T_i$, we get that $d(u_i, c_i) \geq d(v_i, c_i)$ for all i, so $\sum_i d(v_i, c_i) \leq \sum_i d(u_i, c_i)$. Because each u_i appears at most twice, we immediately get the bound that $\sum_i d(u_i, c_i) \leq 2d_{\text{sum}}(S, C^*)$. However, to prove our desired guarantee, we need the stronger bound that $\sum_i d(u_i, c_i) \leq O(1/k) \cdot d_{\text{sum}}(S, C^*)$. The remainder of the proof will be concerned with proving this bound.

We start from the inequality $2\sum_{v\in S}\sum_i d(v,c_i) \geq \sum_{i,j} d(u_i,c_j)$, which also follows because each v appears at most twice as a u_i . We will now lower-bound the sum on the right-hand side. We begin by

The only such v is possibly $v = u_k = u_{k-1}$, and that only in the case p = 1. Otherwise, each v appears at most once.

defining the following sets of pairs of indices (i, j) with $i, j = 1, \dots, k$:

$$A = \left\{ (i,j) \mid d(u_i, c_j) \ge \frac{d(u_i, c_i)}{2} \text{ and } d(u_j, c_i) \ge \frac{d(u_j, c_j)}{2} \right\}$$

$$B = \left\{ (i,j) \mid d(u_i, c_j) \ge \frac{d(u_i, c_i)}{2} \text{ and } d(u_j, c_i) < \frac{d(u_j, c_j)}{2} \right\}$$

$$C = \left\{ (i,j) \mid d(u_i, c_j) < \frac{d(u_i, c_i)}{2} \text{ and } d(u_j, c_i) \ge \frac{d(u_j, c_j)}{2} \right\}$$

$$D = \left\{ (i,j) \mid d(u_i, c_j) < \frac{d(u_i, c_i)}{2} \text{ and } d(u_j, c_i) < \frac{d(u_j, c_j)}{2} \right\}.$$

When $d(u_i, c_j) < \frac{d(u_i, c_i)}{2}$, by triangle inequality, we have that $d(c_i, c_j) \ge d(u_i, c_i) - d(u_i, c_j) > \frac{d(u_i, c_i)}{2}$. Again by triangle inequality, this implies that for all v, we have that $d(v, c_i) + d(v, c_j) \ge d(c_i, c_j) > \frac{d(u_i, c_i)}{2}$. Similarly, when $d(u_j, c_i) < \frac{d(u_j, c_j)}{2}$, we have that $d(v, c_i) + d(v, c_j) > \frac{d(u_j, c_j)}{2}$ for all v. Now consider the four classes of pairs of indices defined above:

- 1. If $(i,j) \in A$, then $d(u_i,c_j) + d(u_j,c_i) \ge \frac{1}{2} \cdot (d(u_i,c_i) + d(u_j,c_j))$, directly from the definition of A.
- 2. If $(i,j) \in B$, then $d(v,c_i) + d(v,c_j) > \frac{d(u_j,c_j)}{2}$ for all v. In particular for $v = u_i$, we get that $d(u_j,c_j) \leq 2d(u_i,c_i) + 2d(u_i,c_j) \leq 6d(u_i,c_j) \text{ because } d(u_i,c_j) \geq \frac{d(u_i,c_i)}{2}. \text{ Because we have } d(u_i,c_j) \geq \frac{d(u_i,c_i)}{2} \text{ and } d(u_i,c_j) \geq \frac{d(u_j,c_j)}{6}, \text{ by adding the first inequality plus three times the second, we get that } d(u_i,c_j) + d(u_j,c_i) \geq d(u_i,c_j) \geq \frac{1}{8} \cdot (d(u_i,c_i) + d(u_j,c_j)).$
- 3. If $(i,j) \in C$, then a completely symmetric argument with the roles of i,j reversed gives that $d(u_i,c_j)$ + $d(u_i, c_i) \ge \frac{1}{8} \cdot (d(u_i, c_i) + d(u_i, c_i)).$
- 4. Finally, when $(i,j) \in D$, we have that $d(v,c_i) + d(v,c_j) \ge \frac{d(u_i,c_i)}{2}$ and $d(v,c_i) + d(v,c_j) \ge \frac{d(u_j,c_j)}{2}$ for all v. By averaging these two inequalities, we get that

$$d(v, c_i) + d(v, c_j) \ge \frac{1}{4} \cdot (d(u_i, c_i) + d(u_j, c_j)). \tag{9}$$

Notice that in the three first cases, we have that $d(u_i, c_j) + d(u_j, c_i) \ge \frac{1}{8} \cdot (d(u_i, c_i) + d(u_j, c_j))$, so summing up over all pairs (i, j), we obtain that

$$2\sum_{i,j} d(u_i, c_j) \ge \sum_{(i,j) \notin D} d(u_i, c_j) + d(u_j, c_i) \ge \frac{1}{8} \cdot \sum_{(i,j) \notin D} d(u_i, c_i) + d(u_j, c_j).$$
(10)

We now focus on the fourth case. Note that $(i,j) \in D$ if and only if $(j,i) \in D$. Consider an undirected graph G on $\{1,\ldots,k\}$ which contains the edge (i,j) if and only if $(i,j) \in D$. Because the maximum degree of any node in G is at most k-1, by Vizing's Theorem¹², the edges of G can be partitioned into k matchings (some of which may be empty). Let M_{ℓ} for $\ell=1,\ldots,k$ be the pairs $(i,j)\in D$ in the ℓ^{th} matching. Substituting $v = u_{\ell}$ in Eq. (9), we get for any pair¹³ $(i,j) \in M_{\ell}$ that $d(u_{\ell},c_i) + d(u_{\ell},c_j) \ge \frac{1}{4} \cdot (d(u_i,c_i) + d(u_j,c_j))$. Summing over all $\ell = 1, \ldots, k$ and all $(i,j) \in M_{\ell}$, we obtain that

$$\sum_{\ell=1}^{k} \sum_{(i,j)\in M_{\ell}} d(u_{\ell}, c_i) + d(u_{\ell}, c_j) \ge \frac{1}{4} \cdot \sum_{(i,j)\in D} d(u_i, c_i) + d(u_j, c_j).$$

On the left-hand side, observe that because each M_ℓ is a matching, and $u_\ell \neq u_{\ell'}$ for all $\ell \neq \ell'$, each term $d(u_\ell, c_i)$ can occur at most once. In other words, we get that $\sum_{\ell=1}^k \sum_{(i,j) \in M_\ell} d(u_\ell, c_i) + d(u_\ell, c_j) \leq 1$

 $^{^{12}}$ Vizing's Theorem states that any simple undirected graph can be edge-colored with at most one more color than its highest degree, ensuring no two adjacent edges share the same color.

13 This holds for all $(i,j) \in D$, not just those in M_{ℓ} . But we will use it only for $(i,j) \in M_{\ell}$.

 $\sum_{i,j} d(u_i,c_j)$, where the right-hand sum is not restricted to D. In summary, we have shown that

$$\sum_{i,j} d(u_i, c_j) \ge \frac{1}{4} \cdot \sum_{(i,j) \in D} d(u_i, c_i) + d(u_j, c_j). \tag{11}$$

Adding twice Eq. (10) and Eq. (11), we get that

$$5 \cdot \sum_{i,j} d(u_i, c_j) \ge \frac{1}{4} \cdot \sum_{(i,j)} d(u_i, c_i) + d(u_j, c_j) = \frac{k}{2} \cdot \sum_i d(u_i, c_i) \ge \frac{k}{2} \cdot \sum_i d(v_i, c_i),$$

so $\sum_i d(v_i, c_i) \leq \frac{10}{k} \cdot \sum_{i,j} d(u_i, c_j) \leq \frac{20}{k} \cdot d_{\text{sum}}(S, C^*)$. Substituting this bound into Eq. (8), we finally obtain

$$d_{\operatorname{sum}}(S,R) \leq d_{\operatorname{sum}}(S,C^*) + (1+\rho) \cdot |S| \cdot \frac{20}{k} \cdot d_{\operatorname{sum}}(S,C^*) = O(n/k) \cdot d_{\operatorname{sum}}(S,C^*).$$

This completes the proof that the output of Algorithm 1 is (1, O(n/k))-proportionally representative. \square

5 Proportional Representation with Known Distances

When the metric space is known, i.e., in the cardinal information model, we provide an algorithm with better approximation constants. In particular, we establish that a slight modification of the GREEDY CAPTURE algorithm of Chen et al. [2019] provides good (α, γ) -representativeness guarantees.

GREEDY CAPTURE works as follows: it continuously grows balls around each candidate, at the same rate. When a ball around a candidate c contains $p = \lceil n/k \rceil$ voters who have not been removed yet, c is included in the committee, and the voters are removed. Even once c is included in the committee, the ball around c continues growing, and any additional voter included in c's ball is immediately removed as well. In the nomenclature of our Algorithm 1, this means that candidate c can "cover" more than p voters under GREEDY CAPTURE.

We modify the algorithm by preventing balls of included candidates from growing. That is, as soon as cis included in the committee, and the voters in the ball are removed, c and its ball are removed from further consideration. The full algorithm is given as Algorithm 2.

```
Algorithm 2 Truncated Greedy Capture
Input: (V, C, k), and metric d on V \cup C
Output: Committee R
   Let U \leftarrow V be the initial set of uncovered voters.
   Let R \leftarrow \emptyset be the selected committee.
   Let \delta = 0 be the ball radius.
   Set N_c \leftarrow \emptyset for each candidate c \in C.
   while U \neq \emptyset do
       Continuously increase \delta.
       for v \in V in arbitrary order do
           if v \in U then
               for c \in C \setminus R in arbitrary order do
                    if d(v,c) \leq \delta then
                        Set N_c \leftarrow N_c \cup \{v\}.
                        if |N_c| = \lceil n/k \rceil then
                            R \leftarrow R \cup \{c\}, i.e., include c in the committee.
                            N_{c'} \leftarrow N_{c'} \setminus N_c for all c' \in C \setminus R, i.e., update the neighborhood for non-committee
   candidates.
                            U \leftarrow U \setminus N_c, i.e., update the set of uncovered voters.
```

We say N_c has been covered by c.

if |R| < k then add k - |R| arbitrary candidates to R.

Theorem 5.1. When all distances are known to the algorithm, Truncated Greedy Capture yields a committee R which is simultaneously $(\alpha, \gamma(\alpha))$ -proportionally representative for all $\alpha > 1$, with $\gamma(\alpha) = 1 + (2 + \sqrt{2}) \cdot \frac{\alpha}{\alpha - 1} \approx 1 + 3.42 \cdot \frac{\alpha}{\alpha - 1}$.

In order to prove Theorem 5.1, we first prove that Algorithm 2 satisfies the analogue of Lemma 4.7, but with improved constants. Recall that for any coalition S of voters, $R[S] \subseteq R$ denotes the set of candidates $r \in R$ who had at least one voter from S in their neighborhood when r was added to R.

Lemma 5.2. The committee R output by Algorithm 2 satisfies the following stability property, with $\rho = 1 + \sqrt{2} \approx 2.42$. For any coalition S of size $|S| \ge p = \lceil n/k \rceil$, there exists a voter $v \in S$ with

$$\min_{r \in R[S]} d(v, r) \le \rho \cdot \min_{c \in C \setminus R} d(v, c).$$

The proof of this lemma is very similar to the proof of Theorem 1 of Chen et al. [2019]. The fact that balls around included candidates c continue to grow in the original GREEDY CAPTURE algorithm turns out to be unimportant for its theoretical guarantees (though likely beneficial in practice). By stopping the growth, we obtain the stronger guarantee of the existence of an $r \in R[S]$ (rather than $r \in R$). For completeness, we give a self-contained proof of Lemma 5.2.

Proof of Lemma 5.2. Let R be the output of Algorithm 2. Let $S \subseteq V$ be any coalition of voters of size $|S| \ge p$. Let $c \in C \setminus R$ be an arbitrary candidate. We will show the existence of a candidate $r \in R[S]$ with $d(v,r) \le \rho \cdot d(v,c)$.

Let $\delta = \max_{v \in S} d(v, c)$, which implies that $S \subseteq B(c, \delta)$. There must exist some candidate $r \in R[S]$ such that $B(r, \delta) \cap S \neq \emptyset$; otherwise, when Algorithm 2 considered ball radius δ , it would have included c (as all voters in S would have been uncovered and in N_c at that time). Notice that the slight strengthening of our result compared to Theorem 1 of Chen et al. [2019] is that they only guarantee $r \in R$, whereas we guarantee (and need) $r \in R[S]$.

Fix a candidate r with $B(r, \delta) \cap S \neq \emptyset$ (whose existence we just proved). Let $v \in B(r, \delta)$ be arbitrary, and v^* be a voter with $d(v^*, c) = \delta$. Using the triangle inequality, along with the bounds that $d(v^*, c) = \delta$ and $d(v, r) \leq \delta$, we now show that at least one of v, v^* prefers c over r by at most a small amount:

$$\min\left(\frac{d(v,r)}{d(v,c)}, \frac{d(v^*,r)}{d(v^*,c)}\right) \le \min\left(\frac{d(v,r)}{d(v,c)}, \frac{d(v^*,c) + d(v,c) + d(v,r)}{d(v^*,c)}\right)$$

$$\le \min\left(\frac{\delta}{d(v,c)}, 2 + \frac{d(v,c)}{\delta}\right)$$

$$\le \max_{z\ge 0} \min\left\{z, 2 + \frac{1}{z}\right\}$$

$$= 1 + \sqrt{2}.$$

With the voter for the given set S being either v or v^* , this completes the proof.

Proof of Theorem 5.1. In the proof of Theorem 4.1, the only properties of Algorithm 1 that were used were:

- Lemma 4.7.
- The disjointness of the neighborhoods N_r for $r \in R$.
- The fact that each neighborhood N_r for $r \in R$ had size $|N_r| = p$.

Because Algorithm 2 also satisfies the second and third properties, we can substitute the improved bound of Lemma 5.2 into the proof and adjust the constants; this completes the proof of Theorem 5.1. \Box

Finally, we briefly analyze the running time of Algorithm 2. While the algorithm is stated as running in continuous time, the points δ at which a ball includes another voter can be easily pre-computed (they comprise exactly the distances d(v,c) for all pairs (v,c)) and sorted in increasing order. All checks can then be performed by simple operations or set membership tests, and the remaining analysis is exactly as for Algorithm 1, giving a running time of $\widetilde{O}(nm)$, i.e., essentially linear unless $n \gg m$.

6 Related Work

6.1 Multi-Winner Metric Distortion

Several papers have proposed extensions of the notion of (metric) distortion to multi-winner (or committee) elections. Goel et al. [2018] consider the cost of a set R to be the sum of all distances between V and the members of R. Because this sum decomposes, it is minimized by choosing the k candidates individually closest to V (in terms of sums of distances); as a result, the set R tends to be as "homogeneous" as possible. For example, if V is partitioned into two clusters of 51% and 49%, and there are enough candidates near each cluster, then the optimal solution would choose all candidates close to the cluster comprising 51% of V. We note that Goel et al. [2018] did not propose this objective as a natural objective for evaluating how representative R is; rather, the goal was to show that many results relating (and bounding) distortion and fairness naturally generalize from single-winner to multi-winner elections.

An alternative notion was proposed by Caragiannis et al. [2022]. Their definition is parametrized by $q \leq k$: each individual $v \in V$ evaluates the cost of R as the cost of the q^{th} closest representative in R; the objective to minimize is then the sum of these costs. When q=1, the objective coincides with (uncapacitated) k-median; in contrast, for q=k, a committee has low cost for v only if all of its members are close to v. Thus, in the regime of large q, the definition suffers from the same drawback as that of Goel et al. [2018]: it rewards committees R (almost) all of whose members are close to the largest cluster within V. Caragiannis et al. [2022] show that while the objective can be well approximated with ordinal information in the "homogeneous" case q > k/2, the distortion is unbounded for q < k/3; though no results are shown under resource augmentation.

This notion of q-cost was used as a definition of fairness or representativeness in the context of sortition in subsequent work by Ebadian et al. [2022b] and Ebadian and Micha [2023]. In Ebadian et al. [2022b] and part of Ebadian and Micha [2023], this notion is used primarily as an analysis tool: the analysis focused on uniformly random selection of individuals, and the representativeness/fairness of the resulting panels, i.e., no knowledge of the metric whatsoever (not even ordinal) was assumed. This is because the random selection procedure itself was a primary focus of the work. However, in addition, Ebadian and Micha [2023] were also interested in the question of how much more representativeness of panels could be achieved with full knowledge of the underlying metric; we discuss those results and their relationship with our work in Section 6.3.

6.2 The Core and Proportionality in Social Choice

There has been a large amount of prior work relating notions of fairness, multi-winner elections, distortion, and the core. For example, Ebadian et al. [2022a] study the *utilitarian distortion* of randomized single-winner voting rules. They use fairness both as a tool to derive novel low-distortion randomized voting rules, and — under a different notion of proportional fairness — as an optimization goal in its own right. They show that α -approximate proportional fairness in their definition implies membership in the α -core.

In general cooperative games, the notion of core captures a stability desideratum: that the outcome be stable against deviations by any subgroup of players seeking better utilities. The exact coalitions and available "outside options" of utilities give rise to specific core solution concepts. This concept is further extended to inverse utility games, where the objective is to reduce the cost incurred by the players [Tamir, 1993, Chen et al., 2019]. The stability found in the core of cost-sharing games closely mirrors the notion of proportional representation of points in a metric space. In a vein similar to proportional representation, the core in cost-sharing games guarantees that large fractions of players have low cost under the chosen solution, deterring them from deviating to alternatives.

In social choice, core definitions often emphasize proportionality, suggesting that a θ fraction of the population should "control" an equal fraction of the outcome [Moulin, 2003]. In the context of electing a committee, various notions of core stability require that every sufficiently cohesive and large voter group should sufficiently "approve" of the committee. Early research by Dummett [1984] aimed to ensure proportionality for solid coalitions (PSC) on ranked ballots. These coalitions consist of voter groups whose set of top candidates are the same (for corresponding set sizes), though their ordering of these top candidates may differ.

Subsequently, more research focused on approval ballots instead of ranked ballots; here, each voter specifies for each candidate whether they approve of the candidate or not; equivalently, the voter specifies the set of all approved candidates. Aziz et al. [2017] introduced the concept of Justified Representation (JR). Justified Representation (JR) requires that cohesive groups be well-represented in the committee as follows: if a set of θn voters is cohesive in the sense of having at least θk candidates whom they all approve, then at least θk of those unanimously approved candidates must be selected. Unfortunately, a committee achieving JR may not exist. Subsequent work of Sánchez-Fernández et al. [2017], Peters et al. [2021], Skowron [2021] further refined this concept, in part with an eye towards ensuring existence of a committee satisfying the definition. They consider various alternate and subtly differing definitions of cohesiveness; see also the survey by Lackner and Skowron [2023].

In studying axiomatic approaches for proportional representation, it has been observed repeatedly (see the in-depth discussion of this strand of literature in Brill and Peters [2023]) that various cohesiveness definitions for coalitions — both for approval and ranked ballots — are very demanding. As a result, only few coalitions are cohesive, imposing few requirements on a selected committee. In response, Brill and Peters [2023] propose new definitions for both approval and ranked ballots. Their PJR+ notion requires the following. For any coalition S which, based on proportionality, is "entitled" to ℓ committee members, if the union of committee members approved by at least one voter in the coalition has size smaller than ℓ , then the committee is not allowed to omit any candidate who is unanimously approved by the coalition. The rank-PJR+ notion for ranked ballots requires this to hold for the "approval ballots" obtained by truncating all ranked ballots at position r, for all r. This constitutes a more demanding notion of proportional representation under ranked choice ballots, compared to PSC [Dummett, 1984]. Brill and Peters [2023] show that while Single-Transferable Vote fails rank-PJR+, the Expanding Approvals Rule satisfies it.

General notions of stability are also studied by Cheng et al. [2020], Jiang et al. [2020]. Cheng et al. [2020] again consider deviation threats in which coalitions S may deviate to a set of size $\frac{|S|}{n} \cdot k$ if there exists such a set that they prefer over the chosen committee. They consider both approval and ranked ballots: for approval ballots, voters prefer the committee that contains more candidates whom the voter approves, while for ranked ballots, voters prefer the committee with the single highest-ranked candidate according to the voter's order. The main focus of Cheng et al. [2020] is on lotteries over committees that achieve stability in this sense (since no deterministic stable committee may exist). Jiang et al. [2020] consider more general monotone preferences of voters over committees. They consider approximate stability, similar to our resource augmentation: they require that no coalition S have a deviation threat of size $\frac{|S|}{cn} \cdot k$ for $c \ge 1$ that they prefer over the chosen committee. The main contribution of Jiang et al. [2020] is to show that for a constant c, there always exists an approximately stable committee in this sense, so long as the voters' preferences are monotone. They prove this by starting from a lottery over committees, which is then derandomized.

6.3 Clustering, Fairness and Proportionality

Representing a large point set is also a — sometimes explicit, sometimes implicit — goal of clustering points in a metric space. A review of this very large literature is beyond the scope of our work. Of the common objective functions (most notably, k-center, k-means, and k-median), k-median [Arya et al., 2001] is closest to our objective here, since it minimizes the sum of distances of points to the respective closest of the selected k cluster centers. In the basic definition, like proportionally fair clustering, it suffers from the fact that a large and dense cluster can be "served" by just one selected representative.

Capacitated versions address this issue by limiting the number of points "served" by a cluster center (see, e.g., Byrka et al. [2015]). Our algorithms for both ordinal and cardinal models similarly assign to each representative c a subset of size p from V that c "covers". To appreciate the difference between the objectives, suppose that roughly p points are at a large distance M from all others points which are within a unit circle. If a representative is at distance $(1-\epsilon)M$ from these remote points, a k-median solution might include it since the objective reduction by ϵM would outweigh any decisions made within the unit circle. In contrast, our proportional representation objective might exclude it since the relative improvement would be marginal. Thus, the proportional representation objective is much less sensitive to few outliers. It should also be noted that both objectives require resource augmentation for positive results [Byrka et al., 2015]. While the objectives look quite different, it would be interesting to explore a link between them and to determine if one implies non-trivial guarantees for the other.

Notions of fairness have also been incorporated into clustering problems more explicitly. These can be roughly divided into fairness notions towards individuals and towards coalitions. A notion of fairness towards individuals was proposed by Jung et al. [2020], Chakrabarti et al. [2022]: they consider a committee of size k to be α -fair if for each individual i, the committee contains a member that is no more than α times as far from i compared to i's n/kth closest candidate. Fairness in this sense may come at a significant cost for the overall utility function (such as k-median), and Mahabadi and Vakilian [2020], Vakilian and Yalçımer [2022] give algorithms with improved tradeoffs between the objectives of fairness and overall clustering quality.

An alternative notion of individual fairness was considered by Chakrabarti et al. [2022]. Their notion somewhat resembles envy-freeness: each point has a self-declared set of "peers", and would like its distance to its closest center to be larger by at most a factor α than that of its peers; the latter could be the minimum or average distance of peers to their closest center.

Yet another notion was considered in the work of Ebadian and Micha [2023]. Their work was motivated by Sortition: choosing random citizen panels such that each citizen has (approximately) uniform probability of being part of the panel, and the panel overall is representative in the sense of multi-winner elections discussed above: i.e., for each individual, the panel contains q members approximately closest to the individual. In order to obtain a good tradeoff between individual inclusion probabilities and approximation to the core, Ebadian and Micha [2023] consider another modified version of the Greedy Capture algorithm. Like our Truncated Greedy Capture algorithm, their version freezes a ball when representatives are chosen; however, this happens only when a ball contains q times the Hare quota many points, and at this point, the q representatives from the ball are chosen uniformly at random. Interestingly, the approximation to the core is shown to be the same $\frac{5+\sqrt{41}}{2}$ which appears in our analysis of the Expanding Approvals Rule for Proportionally Fair Clustering. The technical steps leading to this bound (i.e., specific triangle inequalities and combinations) are similar, even though the algorithms and overall analysis are different.

An alternative approach, more in line with our definition, is to require fairness (or lack of deviation incentives) for sufficiently large groups. Perhaps the most prominent examples are the notions of proportional fairness [Chen et al., 2019, Micha and Shah, 2020] and approximate core [Li et al., 2021], both discussed in depth in Section 2.2.

Simultaneously and independently of our work, Aziz et al. [2023] also propose a notion of proportional representation. Their motivation is quite similar to ours: they also observe that neither proportionally fair clustering nor the approximate core achieve the goal (stated in those works) of representing clusters proportionally to their size. They formulate a basic axiom called UNANIMOUS PROPORTIONALITY (UP), which requires that if there is a coalition S comprising a θ fraction of the population all of whose members are in the exact same location, then a θ fraction of the committee should be made up of points closest to S. This is a clean axiomatization of the issue we discuss in Section 2.2; indeed, our notion of proportional representation also leads to satisfying the UP axiom. Aziz et al. [2023] then propose a definition they term PROPORTIONALLY REPRESENTATIVE FAIRNESS (PRF). This notion requires the following, for each set S. Let y be the diameter of S, and ℓ the number of candidates who are at distance at most y from all members of S. Let θ the fraction of the population that S comprises. Then, the committee must contain at least $\min(\theta k, \ell)$ candidates who are at distance at most y from all members of S. Aziz et al. [2023] then propose an algorithm for selecting a committee satisfying their definition. The algorithm is essentially an EXPANDING APPROVALS RULE with known distances; that is, balls around candidates are grown, and when a ball contains n/k voters, the candidate is included, and the voters are "removed" (by having their weight decreased). As with the algorithms we consider, proportionality is ensured by having the balls around included candidates not grow any further. Notice that the algorithms are designed with knowledge of the metric space. The notions of proportional representation of our work and Aziz et al. [2023] are obviously similar in their goals and approaches, although we are not aware of a formal reduction. It should be noted that while our notion is a strict strengthening of the notion of core fairness proposed by Li et al., 2021, this is not the case for the notion of Aziz et al. [2023].

In further simultaneous and independent work, Kellerhals and Peters [2023] aim to draw connections between the plethora of different fairness/proportionality notions alluded to above. Among others, they show that proportional fairness and individual fairness imply each other (up to losses of constant factors). More importantly, they define metric notions of ranked fairness. Recall from Section 6.2 that notions of justified representation (JR, EJR, PJR, PJR+) can be extended to ranked (rather than approval) ballots by considering all the cutoffs r, and requiring the conditions to hold for each implied approval ballot. As

an alternative, Kellerhals and Peters [2023] propose notions of rank-JR, rank-PJR, rank-PJR+ based on distance cutoffs: they consider agents as approving all candidates within distance r, and require that the corresponding justified representation axioms hold for the implied approval ballots. They show that various natural algorithms (such as GREEDY CAPTURE and a distance-based EXPANDING APPROVALS RULE) satisfy corresponding axioms. Furthermore, they show implications between these new axioms and proportional and individual fairness (as well as the proportional representation notion of Aziz et al. [2023]), obtaining results on the representation guarantees for the outputs of these algorithms for various notions. In several cases, the obtained guarantees match the best known results.

7 Concluding Remarks and Open Questions

We presented a natural new definition of when a committee is "representative" of a larger set of points in a metric space. This definition is applicable for documents (where the metric is typically known) or for the election of a committee or other representative political body. In the latter case, typically, only ordinal information about the metric is available, and our definition gives a natural objective for distortion-based analysis of multi-winner elections. Our main result is that the EXPANDING APPROVALS RULE of Aziz and Lee [2020] achieves constant representativeness with resource augmentation by a constant factor, even with just ordinal information; we also showed that some resource augmentation in the analysis is unavoidable, as for large committees, there are examples for which a polynomial lower bound on representativeness is unavoidable.

Aziz and Lee [2020] showed that the EXPANDING APPROVALS RULE satisfies several desirable axiomatic properties for multi-winner elections, in addition to being a natural rule in its own right. We believe that our result thus adds to the evidence for EXPANDING APPROVALS RULE being a potentially useful rule to be used in practice.

Theorems 4.1 and 5.1 give bounds for general metric spaces. When the metric is known to be Euclidean, by using the stronger bound of Theorem 5 of Micha and Shah [2020] in place of Lemma 4.9 (Theorem 4 of Micha and Shah [2020]), the constant in Lemma 4.7 can be improved, which directly carries through to an improved constant in Theorems 4.1 and 5.1.

In general, the constants in our upper and lower bounds do not match. Closing the gaps for all of the notions of representation studied here would be of interest, both in the model with known metric space and with ordinal information. Another possible direction of interest is to understand under what natural conditions about the metric space a constant factor in representativeness can be achieved without resource augmentation.

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Lower-Bound Examples

Lower Bound for large α

In this section, we prove Proposition 2.4, showing that for every $\alpha > 1$, there are instances (V, c, d, k) for which the $(\alpha, 1 + 1/(2\alpha))$ -core is empty.

Proof of Proposition 2.4. Given any $\alpha > 1$, let $q = \lceil 2\alpha \rceil$. The committee size is k = 2q - 1. In our instance, the voters and candidates are the same, i.e., V = C, and there are n = 2k of them; thus, the Hare Quota is p=2. The voters/candidates form two clusters B_1, B_2 , each of size k. Within each cluster, each pair has distance 1, and between clusters, the distance is large.

Let R be any committee of size k. For one of the two clusters (w.l.o.g. B_1), the committee R selects at most q-1 candidates. Let S be a coalition of q voters from $B_1 \setminus R$; notice that such a coalition exists, because B_1 contains 2q-1 voters, of whom at most q-1 are in R. Because $q \geq 2\alpha$, the coalition S is of size at least $\alpha \cdot p$.

Because each $v \in S$ is at distance at least 1 from its closest representative in R, the total cost of S is at least q. By including any one member of S instead, the cost for that member would be reduced to 0, so the total cost would be q-1. As a result, we obtain that the cost ratio is at least $1+\frac{1}{q-1}>1+\frac{1}{2\alpha}$, so whenever $\beta \leq 1 + \frac{1}{2\alpha}$, we have that $\sum_{v \in S} \min_{r \in R} d(v, r) > \beta \cdot d_{\text{sum}}(S, c)$ for some $c \in C \setminus R$. In particular, R cannot be in the $(\alpha, 1 + 1/(2\alpha))$ -core, and because the argument applies to arbitrary R, we have shown that the $(\alpha, 1 + 1/(2\alpha))$ -core is empty.

Lower Bound for small α

In this section, we prove Proposition 2.5, showing that for α arbitrarily close to 1, there are instances for which the $(\alpha, 1/(4(\alpha-1)))$ -core is empty.

Proof of Proposition 2.5. Consider $\alpha \in (1,3/2)$ such that $\frac{1}{\alpha-1}$ is an integer. Let the committee size be $k=\frac{1}{\alpha-1}$, and consider an instance consisting of k+1 clusters B_i , each containing one candidate and k-1voters. Thus, the total number of voters is $n = k^2 - 1$, and the Hare Quota is $p = \lceil n/k \rceil = k$.

The distances between all voters and the candidate within a cluster are 0, whereas the distances between any pair of candidate/voter from different clusters is 1.14

Any committee R of size k must omit the candidate in at least one of the clusters B_i . Consider the coalition S comprising all of the k-1 voters in B_i , plus $\alpha \cdot p - (k-1) = (k+1) - (k-1) = 2$ voters from some other B_i .

Because the candidate from B_i is not in the committee, each of the voters in B_i incurs cost at least 1, so the total cost of B_i is at least k-1. On the other hand, the candidate from B_i would have been an alternative leading to cost at most 2, since the voters in B_i would incur cost 0, while the voters in B_j would incur cost 1. Thus, $\sum_{v \in S} \min_{r \in R} d(v, r) > \beta \cdot d_{\text{sum}}(S, c)$ for some $c \in C \setminus R$, whenever $\beta < \frac{k-1}{2} = \frac{2-\alpha}{2(\alpha-1)}$. Again, this implies that R cannot be in the approximate core, and it applies to arbitrary R. Because $\alpha < 3/2$, we get that $\frac{2-\alpha}{2(\alpha-1)} > \frac{1}{4(\alpha-1)}$, implying that the $(\alpha, 1/(4(\alpha-1)))$ -core is empty.

By modifying the proof of Proposition 2.5 slightly, we also obtain a slightly more refined bound on when the $(1,\beta)$ -core is empty compared to Li et al. [2021], encompassing the whole range of k. Specifically, we will show that for every n, k and $\beta < \frac{1}{16} \cdot \min(k, n/k)$, there are instances for which the $(1, \beta)$ -core is empty. For $k = \sqrt{n}$, this recovers (up to small constants, lost only for mathematical convenience) Theorem 7 of Li et al. [2021], but extends it to arbitrary values of k.

Take n, k as given. If k > n/4, then the claimed lower bound holds trivially (because the given bound on β is smaller than 1), so we assume $k \leq n/4$. Again, we consider an instance with k+1 clusters B_i , each containing one candidate. The number of voters in cluster i is $b_i \in \{\lfloor n/(k+1)\rfloor, \lceil n/(k+1)\rceil\}$, with the b_i chosen such that $\sum_{i=1}^{k+1} b_i = n$. As in the proof of Proposition 2.5, distances within clusters are 0, and between clusters, the distances are 1. The Hare Quota is $p = \lceil n/k \rceil$.

¹⁴These distances can be implemented in a (k+1)-dimensional Euclidean space, by embedding cluster i at distance $1/\sqrt{2}$ on the ith coordinate axis; though this is not essential, as we allow abstract metric spaces as well. Also, notice that we could easily modify the example to be a metric instead of pseudo-metric, by replacing the distances of 0 with some small ϵ .

Again, any committee R of k candidates must omit at least one cluster B_i , and we consider the coalition S of the $b_i \ge \lfloor n/(k+1) \rfloor$ voters in B_i , plus $p-b_i$ voters from some other cluster B_j . The total cost of S for the committee R is at least b_i , while it would be at most $p-b_i$ if the candidate from B_i had been included. Thus, $\sum_{v \in S} \min_{r \in R} d(v, r) > \beta \cdot d_{\text{sum}}(S, c)$ for any $c \in B_i$ and $\beta < \frac{b_i}{n-b_i}$. We now lower-bound this ratio:

$$\frac{b_i}{p - b_i} \ge \frac{\lfloor n/(k+1) \rfloor}{\lceil n/k \rceil - \lfloor n/(k+1) \rfloor}$$

$$\ge \frac{n/(k+1) - 1}{n/k + 2 - n/(k+1)}$$

$$= \frac{n - (k+1)}{n/k + 2(k+1)}$$

$$\stackrel{k \le n/4}{\ge} \frac{1}{2} \cdot \frac{n}{n/k + 2(k+1)}$$

$$\ge \frac{1}{4} \cdot \min\left(\frac{n}{n/k}, \frac{n}{2(k+1)}\right)$$

$$> \frac{1}{16} \cdot \min(k, n/k).$$

Thus, we have shown that for the given instance, the $(1,(1/16) \cdot \min(k,n/k))$ -core is empty.

A.3 Separation between Cardinal and Ordinal Proportionally Fair Clustering

In Corollary 4.3, we showed the existence of a deterministic 5.71-proportionally fair clustering algorithm under ordinal information. On the other hand, Chen et al. [2019], Micha and Shah [2020] gave a deterministic $1+\sqrt{2}\approx 2.41$ -proportionally fair clustering algorithm when the metric space is known to the algorithm. Here, we show a separation between proportionally fair clustering in the ordinal and cardinal models, by giving an instance under ordinal information for which no deterministic algorithm can be better than $2+\sqrt{5}\approx 4.23$ -proportionally fair.

We consider a committee selection problem with n=6 voters, m=6 candidates, and committee size k=3. The voters are $V=\{v_1,v_2,v_3,v_1',v_2',v_3'\}$, and the candidates are $C=\{c_1,c_2,c_3,c_1',c_2',c_3'\}$. The rankings of the voters are as follows:

$$v_1 : c_1 \succ c_2 \succ c_3 \succ c_1' \succ c_2' \succ c_3'$$

$$v_2 : c_2 \succ c_3 \succ c_1 \succ c_1' \succ c_2' \succ c_3'$$

$$v_3 : c_3 \succ c_1 \succ c_2 \succ c_3'$$

$$v_4 : c_1' \succ c_2' \succ c_3' \succ c_1 \succ c_2 \succ c_3$$

$$v_2 : c_2' \succ c_3' \succ c_1' \succ c_2 \succ c_3$$

$$v_3 : c_3' \succ c_1' \succ c_2' \succ c_3'$$

$$v_3' : c_3' \succ c_1' \succ c_2' \succ c_3$$

Because the committee size is k=3, either at most one candidate is chosen from $\{c_1, c_2, c_3\}$ or at most one candidate is chosen from $\{c'_1, c'_2, c'_3\}$. Without loss of generality, assume that at most one candidate is chosen from $\{c_1, c_2, c_3\}$.

The metric space we define will have two clusters, one comprising $c_1, c_2, c_3, v_1, v_2, v_3$, and the other $c'_1, c'_2, c'_3, v'_1, v'_2, v'_3$. The two clusters are far from each other.

If no candidate from $\{c_1, c_2, c_3\}$ is chosen, then the voter coalition $S = \{v_1, v_2\}$ will have extremely high cost (the distance between the clusters), and strongly prefer to deviate to any candidate in $\{c_1, c_2, c_3\}$. So for the rest of the analysis, we assume that exactly one candidate from $\{c_1, c_2, c_3\}$ is chosen. We assume that the chosen candidate is c_1 —the construction below can be altered straightforwardly simply by switching the names of candidates if another candidate were chosen.

Consider the following distances between the voters and candidates, with $\delta = \frac{\sqrt{5}-1}{2}$, and ϵ some arbitrarily small number (used solely to avoid ties):

	c_1	c_2	c_3
$v_1(c_1 \succ c_2 \succ c_3)$	3	$3 + \epsilon$	$3+2\epsilon$
$v_2(c_2 \succ c_3 \succ c_1)$	$3+2\delta$	1	$1 + \epsilon$
$v_3(c_3 \succ c_1 \succ c_2)$	$2+\delta$	$2 + \delta + \epsilon$	δ

The distances between c'_1, c'_2, c'_3 and v'_1, v'_2, v'_3 are all 1, and the distances between the two clusters are 100. It can be verified that all of these distances together satisfy the triangle inequality (the only interesting cases occur between c_1, c_2, c_3 and v_2, v_3), and are consistent with the rankings.

Now consider the coalition $\{v_2,v_3\}$. For each of them, the best candidate in the committee is c_1 , with respective costs $3+2\delta$ and $2+\delta$. By deviating to c_3 , their new respective costs would be $1+\epsilon$ and δ . The resulting cost improvement would be $\min(\frac{3+2\delta}{1+\epsilon},\frac{2+\delta}{\delta}) \geq 3+2\delta-O(\epsilon)$. Thus, as $\epsilon \to 0$, we see that no proportional fairness guarantee of $\gamma < 3+2\delta = \sqrt{5}+2\approx 4.236$ can be possible information-theoretically.