On the Extension and Sampling Theorem for the Coupled Fractional Fourier Transform

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***Abstract*—The fractional Fourier transform, denoted by F_θ, which is a generalization of the Fourier transform, depends on a parameter $0 \leq \theta \leq \pi/2$, so that when $\theta = 0$, F_0 is the identity transformation and when $\theta = \pi/2$, $F_{\pi/2}$ is the standard Fourier transform. The transform has been extended to higher dimensions by taking tensor products of one-dimensional transforms.**

In 2018 the author of this article introduced a novel generalization of the fractional Fourier transform to two dimensions, which is called the coupled fractional Fourier transform and is denoted by $F_{\alpha,\beta}$. This transform depends on two independent angles α and β, with $0 \leq \alpha, \beta \leq \pi/2$, so that $F_{0,0}$ is the identity transformation and $F_{\pi/2,\pi/2}$, is the two-dimensional Fourier transform. For other values of α and β, we obtain other interesting configurations of the transform. One immediate application of this transform is in time-frequency representation because of its close relationship to the Wigner distribution function.

The goal of this article is to extend the transform to a space of generalized functions and then introduce a sampling theorem for signals that are bandlimited in the domain of the transform.

The Fourier transform is an important mathematical tool used in many disciplines from mathematics to physics and engineering. In 1980 V. Namais [3]introduced the concept of Fourier transform of fractional order in which the ordinary Fourier transform is regarded as a transform of order one and the identity transform as a transform of order zero, and what is in between is called fractional Fourier transform. Moreover, he sought to maintain the additive property so that two successive applications of the transform of order one half would yield the ordinary Fourier transform.

More explicitly, the fractional Fourier transform can be viewed as a family of transformations, $\{\mathcal{F}_\alpha\}$ indexed by a parameter α, with $0 \leq \alpha \leq 1$, such that $\mathcal{F}_0$ is the identity transformation and $\mathcal{F}_1$, is the standard Fourier transformation. That is

$$\mathcal{F}_0[f] = f, \quad \mathcal{F}_1[f] = \hat{f},$$

where $\hat{f}$ is the Fourier transform of f, and, in addition, $\mathcal{F}_\alpha \mathcal{F}_\beta = \mathcal{F}_{\alpha+\beta}$. The range of the parameter does not have to be the interval $[0, 1]$ because this interval can be mapped by a simple substitution into the interval $[a, b], \ a < b$. Because of its periodicity, the fractional Fourier transform is parameterized by an angle $0 \leq \theta \leq 2\pi$, where $\mathcal{F}_0$, is the identity transformation and the conventional Fourier transform is obtained when $\theta = \pi/2$.

The fractional Fourier Transform or FrFT of a function $f(t) \in L^1(\mathbb{R})$, is defined by [1], [2], [4],

$$\mathcal{F}_\theta [f] (x) = F_\theta(x) = \int_{-\infty}^{\infty} f(t) K_\theta (x, t) \, dt, \tag{1}$$

where

$$K_\theta(x, t) = \begin{cases} c(\theta) \cdot e^{-i\left[a(\theta)(t^2+x^2) - b(\theta)xt\right]}, & \theta \neq p\pi \\ \delta(t - x), & \theta = 2p\pi \\ \delta(t + x), & \theta = (2p - 1)\pi \end{cases} \tag{2}$$

is the transformation kernel with

$$a(\theta) = \cot\theta/2, \ b(\theta) = \csc\theta, \ c(\theta) = \sqrt{\tfrac{1+i\cot\theta}{2\pi}}.$$

The fractional Fourier transform in n-variables is defined by taking the tensor product of n copies of the one-dimensional fractional Fourier transforms [4]. That is

$$\begin{aligned} & F_{\theta_1,\cdots,\theta_n}(\omega_1, \cdots, \omega_n) \\ = \ & \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} K_{\theta_1}(t_1, \omega_1) \cdots K_{\theta_n}(t_n, \omega_n) \\ \times \ & f(t_1, \cdots, t_n) dt_1 \cdots dt_n, \end{aligned}$$

where $K_{\theta_i}(t_i, \omega_i)$, $i = 1, 2, \cdots, n$, is the kernel of the one-dimensional fractional Fourier transform given by (2)

In particular, in two dimensions we have

$$F_{\theta_1,\theta_2}(\omega_1, \omega_2) = \int_{\mathbb{R}} \int_{\mathbb{R}} K_{\theta_1}(t_1, \omega_1) K_{\theta_2}(t_2, \omega_2) \\ f(t_1, t_2) dt_1 dt_2,$$

In 2020 a novel two-dimensional fractional Fourier transform $F_{\alpha,\beta}$ that is not a tensor product of two one-dimensional fractional Fourier transforms was introduced [5], [6]. Unlike in the tensor product case, this transform does not depend on the two angles α and β separately but depends on the the sum and the difference of the two angles. This is the reason this transform is sometime called the coupled two-dimensional fractional Fourier transform (CFrFT).

Now we introduce the definition of the two-dimensional fractional Fourier transform.

Definition 1. *For a function $f \in L^1(\mathbb{R}^2)$ we define the two-dimensional fractional Fourier transform as*

$$F_{\alpha,\beta}(z_1, \overline{z}_1) = \int_{\mathbb{R}^2} k_{\alpha,\beta}(z_1, \overline{z}_1, z_2, \overline{z}_2) f(z_2, \overline{z}_2) dz_2$$

where $\gamma=(\alpha+\beta)/2\neq n\pi$ *and* $\delta=(\alpha-\beta)/2$. *Or*

$$F_{\alpha,\beta}\left(u,v\right)=\int_{\mathbb{R}^2}k_{\alpha,\beta}\left(x,y,u,v\right)f(x,y)dxdy, \qquad (3)$$

where $z_1=u+iv,\ z_2=x+iy$, *where*

$$\begin{aligned} & k_{\alpha,\beta}\left(z_1,\overline{z}_1,z_2,\overline{z}_2\right)\\ = & \ d(\gamma)\exp\left\{-a(\gamma)\left(x^2+y^2+u^2+v^2\right)\right.\\ + & \left. \ b(\gamma,\delta)\left(ux+vy\right)+c(\gamma,\delta)\left(vx-uy\right)\right\}, \end{aligned} \qquad (4)$$

where

$$a(\gamma)=i\frac{\cot\gamma}{2} \quad , \quad b(\gamma,\delta)=\frac{i\cos\delta}{\sin\gamma} \qquad (5)$$

$$c(\gamma,\delta)=\frac{i\sin\delta}{\sin\gamma} \quad , \quad d(\gamma)=\frac{ie^{-i\gamma}}{2\pi\sin\gamma}. \qquad (6)$$

From now on we will write a,b,c,d instead of $a(\gamma),b(\gamma,\delta),c(\gamma,\delta),d(\gamma)$ for short. We will also use the notation $\tilde{a},\tilde{b},\tilde{c},\tilde{d}$ to denote the same quantities but without the imaginary number $i=\sqrt{-1}$, that is, $a=i\tilde{a},b=i\tilde{b},..$ etc. The definition may be extended to $f\in L^2\left(\mathbb{R}^2\right)$ in the usual way. When $\alpha=\beta$, $\delta=0$, and $\gamma=\alpha$ and the two-dimensional fraction Fourier transform becomes a tensor product of two one-dimensional fractional Fourier transforms, i.e.,

$$\begin{aligned} & F_{\alpha,\alpha}\left(u,v\right)=d(\alpha)\int_{\mathbb{R}^2}\exp\left\{-a(\alpha)\right.\\ \times & \ \left(x^2+y^2+u^2+v^2\right)\\ + & \left. \ b(\alpha)\left(ux+vy\right)\right\}f(x,y)dxdy, \end{aligned}$$

where $a(\alpha)=i\cot\alpha/2,\ b(\alpha)=i\csc\alpha$, which is the standard fractional Fourier transform defined in (1). Furthermore, if $\alpha=\beta=\pi/2$, the two-dimensional fraction Fourier transforms reduces to the standard two-dimensional Fourier transform.

I. Extension of the CFrFT

The coupled fractional Fourier transform is defined for $f\in L^p(\mathbb{R}^2),\ 1\leq p\leq 2$. In this section we extend the transform to a space of generalized functions. The following lemma will be needed.

Lemma 1. *To simplify the notation, let* $k=k_{\alpha,\beta}(x,y;u,v)$ *be the kernel of the coupled fractional Fourier transform given by Eq. (4), (5) and (6). We have*

$$\begin{aligned} \frac{\partial^{m+n}k}{\partial u^m\partial v^n} &= k\left[(-2au)^m+P_m(x,y;u)\right]\\ &\times\left[(-2av)^n+\tilde{P}_n(x,y;v)\right], \end{aligned}$$

where $P_m(x,y;u)$ *and* $\tilde{P}_n(x,y;v)$ *are polynomials of degree* m *and* n *in* x *and* y *and of degree* $m-1$ *in* u *and* $n-1$ *in* v, *respectively.*

Proof. It is easy to see that

$$\frac{\partial k}{\partial u}=kH, \quad \frac{\partial k}{\partial v}=k\tilde{H},$$

where

$$H=H(x,y;u)=-2au+\eta(x,y),\ \eta(x,y)=bx-cy,$$

and

$$\tilde{H}=-2av+\tilde{\eta}(x,y), \quad \tilde{\eta}(x,y)=by+cx.$$

Hence

$$\begin{aligned} \frac{\partial^2k}{\partial u^2}-2ak+kH^2 &= k\left(4a^2u^2+\eta^2-4au\eta-2a\right)\\ &= k\left[(-2au)^2+P_2(x,y;u)\right], \end{aligned}$$

where $P_2(x,y;u)=\eta^2-4au\eta-2a$ is a polynomial of degree 2 in x and y and of degree 1 in u. In general,

$$\frac{\partial^nk}{\partial u^n}=k\left[(-2au)^n+P_n(x,y;u)\right],$$

where $P_n(x,y;u)$ is a polynomial of degree n in x and y and of degree $n-1$ in u.

We prove it by induction. By differentiating the last equation with respect to u , we have

$$\begin{aligned} & \frac{\partial^{n+1}k}{\partial u^{n+1}}=\frac{\partial}{\partial u}k\left[(-2au)^n+P_n(x,y;u)\right]\\ = & \ k\left[n(-2a)^nu^{n-1}+\frac{\partial}{\partial u}P_n(x,y;u)\right]\\ + & \ \left[(-2au)^n+P_n(x,y;u)\right]Hk\\ = & \ k\left[n(-2a)^nu^{n-1}+\frac{\partial}{\partial u}P_n(x,y;u)\right.\\ + & \left. \ \left\{(-2au)^n+P_n(x,y;u)\right\}(-2au+\eta(x,y))\right]\\ = & \ k\left[(-2au)^{n+1}+P_{n+1}(x,y;u)\right], \end{aligned}$$

where

$$\begin{aligned} & P_{n+1}(x,y;u)=n(-2a)^nu^{n-1}+\frac{\partial}{\partial u}P_n(x,y;u)\\ + & \ (-2au)^n\eta+(-2au)P_n(x,y;u)+P_n(x,y;u)\eta \end{aligned}$$

is a polynomial of degree $n+1$ in x and y and of degree n in u because $P_n(x,y;u)$ is of degree n in x and y and $n-1$ in u. Similarly,

$$\frac{\partial^nk}{\partial v^n}=k\left[(-2av)^n+\tilde{P}_n(x,y;v)\right],$$

where $\tilde{P}_n(x,y;u)$ is a polynomial of degree n in x and y and of degree $n-1$ in v. Since

$$\frac{\partial^2k}{\partial u\partial v}=kH\tilde{H},$$

it can be easily shown as above that

$$\begin{aligned} \frac{\partial^{m+n}k}{\partial u^m\partial v^n} &= k\left[(-2au)^m+P_m(x,y;u)\right]\\ &\times\left[(-2av)^n+\tilde{P}_n(x,y;v)\right], \end{aligned}$$

where $P_m(x,y;u)$ and $\tilde{P}_n(x,y;v)$ are polynomials of degree m and n in x and y and of degree $m-1$ in u and $n-1$ in v, respectively. □

Now we extend the transform to a space of generalized functions. Let $\mathcal{E}(\mathbb{R}^2)$ be the testing-function space of all infinitely differentiable functions on $\mathbb{R}^2$, and $\mathcal{E}^*$ be its dual space which is the space of all generalized functions with compact support. It is known that $\mathcal{E}^*$ is a subspace of the space $\mathfrak{D}^*$ of Schwartz distributions. Since the kernel of the coupled fractional Fourier transform $k(x,y;u,v)$ is in the space $\mathcal{E}(\mathbb{R}^2)$, we have the following definition

Definition 2. *Let $f \in \mathcal{E}^*$. We define the coupled fractional Fourier transform of f as*

$$F(u,v) = F_{\alpha,\beta}(u,v) = \langle f(x,y), k_{\alpha,\beta}(x,y;u,v)\rangle.$$

Theorem 1. *Let $f \in \mathcal{E}^*$. Then its coupled fractional Fourier transform $F(u,v)$ is in the space $\mathcal{E}(\mathbb{R}^2)$ and satisfies*

$$\left|\frac{\partial^{m+n}F}{\partial u^m \partial v^n}\right| \le C|u|^m|v|^n$$

for sufficiently large u and v. That is F is a $C^\infty(\mathbb{R}^2)$ function in both u and v and does not grow faster than a polynomial as $|u|$ and $|v|$ go to infinity, i.e, F is a tempered function.

Proof. By taking the derivatives of F with respect to u and v, we have by Lemma1

$$\begin{aligned}
&\frac{\partial^{m+n}F}{\partial u^m \partial v^n} = \langle f, \frac{\partial^{m+n}k}{\partial u^m \partial v^n}\rangle \\
= \;& \langle f, k\left[(-2au)^m + P_m(x,y;u)\right]\left[(-2av)^n \right.\\
+ \;& \left.\tilde{P}_n(x,y;u)\right]\rangle \\
= \;& \langle f, (-2au)^m(-2av)^n k\left[1 + \frac{\tilde{P}_n}{(-2av)^n}\right.\\
+ \;& \left.\frac{P_m}{(-2au)^m} + \frac{P_m\tilde{P}_n}{(-2au)^m(-2av)^n}\right]\rangle \\
= \;& (-2au)^m(-2av)^n\langle f, k\left[1 + G(x,y,;u,v)\right]\rangle \\
= \;& (-2au)^m(-2av)^n\left\{\langle f,k\rangle + \langle f, kG\rangle\right\}
\end{aligned}$$

where

$$G(x,y;u,v) = \frac{\tilde{P}_n}{(-2av)^n} + \frac{P_m}{(-2au)^m} + \frac{P_m\tilde{P}_n}{(-2au)^m(-2av)^n}.$$

Since G is a polynomial in x and y it is a multiplier of the space $\mathcal{E}(\mathbb{R}^2)$ and hence $\langle f, kG\rangle$ is well defined and consequently so is $\frac{\partial^{m+n}F}{\partial u^m \partial v^n}$. Finally, since $|G(x,y;u,v)|$ can be made arbitrary small for u and v sufficiently large, it would follow that

$$\left|\frac{\partial^{m+n}F}{\partial u^m \partial v^n}\right| \le C|u|^m|v|^n,$$

for some constant C. □

Remark: The coupled fractional Fourier transform maps the space $\mathcal{E}^*$ into a subspace of the space of tempered distribution. Next we derive the inversion formula for the fractional Fourier transform of the generalized function f.

Theorem 2. *Let $\mathcal{F}_{\alpha,\beta}[f](u,v) = F_{\alpha,\beta}(u,v)$ be the coupled FrFT of a generalized function f with compact support. Then*

$$f(x,y) = \lim_{r\to\infty}\lim_{R\to\infty}\int_{-r}^{r}\int_{-R}^{R} F_{\alpha,\beta}(u,v)k_{-\alpha,-\beta}(x,y;u,v)\,dudv$$

where the limits are taken in the space $\mathcal{S}^*$ of tempered distributions.

Proof. We need to show that

$$\langle f(x,y), \phi(x,y)\rangle = \lim_{r\to\infty}\lim_{R\to\infty}\langle\int_{-r}^{r}\int_{-R}^{R} F_{\alpha,\beta}(u,v)$$

$$k_{-\alpha,-\beta}(x,y,u,v)\,dudv, \phi(x,y)\rangle\,,$$

for all $\phi \in \mathcal{S}(\mathbb{R}^2)$, where $\mathcal{S}(\mathbb{R}^2)$ is the Schwartz space of all infinitely differentiable functions with rapid decay . We have

$$\begin{aligned}
&\langle\int_{-r}^{r}\int_{-R}^{R} F_{\alpha,\beta}(u,v)k_{-\alpha,-\beta}(x,y,u,v)\,dudv, \phi(x,y)\rangle \\
= \;& \int_{\mathbb{R}^2}\phi(x,y)\,dxdy\int_{-r}^{r}\int_{-R}^{R} F_{\alpha,\beta}(u,v)k_{-\alpha,-\beta}(x,y,u,v)\,dudv \\
= \;& \int_{\mathbb{R}^2}\phi(x,y)\,dxdy\int_{-r}^{r}\int_{-R}^{R}\langle f(w,z), k_{\alpha,\beta}(w,z,u,v)\rangle \\
\times \;& k_{-\alpha,-\beta}(x,y,u,v)\,dudv \\
= \;& \langle f(w,z), \int_{\mathbb{R}^2}\phi(x,y)\,dxdy\int_{-r}^{r}\int_{-R}^{R} k_{\alpha,\beta}(w,z,u,v) \\
\times \;& k_{-\alpha,-\beta}(x,y,u,v)\,dudv\rangle
\end{aligned}$$

Changing the order of integration is possible because ϕ is infinitely differentiable with rapid decay and the integrant is a continuous function of x,y and u,v. Therefore,

$$\begin{aligned}
&\lim_{r\to\infty}\lim_{R\to\infty}\langle\int_{-r}^{r}\int_{-R}^{R} F_{\alpha,\beta}(u,v) \\
\times \;& k_{-\alpha,-\beta}(x,y;u,v)\,dudv, \phi(x,y)\rangle \\
= \;& \lim_{r\to\infty}\lim_{R\to\infty}\langle f(w,z), \int_{R^2}\phi(x,y)\,dxdy\int_{-r}^{r}\int_{-R}^{R} k_{\alpha,\beta}(w,z;u,v) \\
\times \;& k_{-\alpha,-\beta}(x,y;u,v)\,dudv\rangle \\
= \;& \langle f(w,z), \lim_{r\to\infty}\lim_{R\to\infty}\int_{\mathbb{R}^2}\phi(x,y)\,dxdy\int_{-r}^{r}\int_{-R}^{R} k_{\alpha,\beta}(w,z;u,v) \\
\times \;& k_{-\alpha,-\beta}(x,y;u,v)\,dudv\rangle \\
= \;& \langle f(w,z), \int_{\mathbb{R}^2}\phi(x,y)\,dxdy\int_{\mathbb{R}^2} k_{\alpha,\beta}(w,z;u,v) \\
\times \;& k_{-\alpha,-\beta}(x,y;u,v)\,dudv\rangle \\
= \;& \langle f(w,z), \int_{\mathbb{R}^2}\phi(x,y)\,\delta(x,y;w,z)dxdy\rangle = \langle f(w,z), \phi(w,z)\rangle.
\end{aligned}$$

□

II. Sampling Theorem

Here we state the sampling theorem for the coupled fractional Fourier transform without proof because the long proof will exceed the page limitation, but the proof will be published somewhere else.

Theorem 3. *Let f be bandlimited to $\Omega = [-r,r]\times[-R,R]$, in the FrFT domain. Then f can be reconstructed from its samples via the formula*

$$\tilde{f}(x,y) = \sum_{m,n=-\infty}^{\infty}\tilde{f}(x_{m,n},y_{m,n})\frac{\sin(m\pi - rw_1)\sin(n\pi - Rw_2)}{(m\pi - rw_1)(n\pi - Rw_2)}, \tag{7}$$

where $\tilde{f}(x,y) = e^{-a(x^2+y^2)} f(x,y)$, $\gamma = (\alpha+\beta)/2$, *and* $\delta = (\alpha-\beta)/2$,

$$w_2 = \tilde{b}y + \tilde{c}x \quad \text{and } w_1 = \tilde{b}x - \tilde{c}y, \tag{8}$$

and

$$x_{m,n} = \pi \sin\gamma \left(\frac{m}{r}\cos\delta + \frac{n}{R}\sin\delta \right),$$

and

$$y_{m,n} = \pi \sin\gamma \left(\frac{n}{R}\cos\delta - \frac{m}{r}\sin\delta \right).$$

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