## Characterizations of Language Generation With Breadth

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#### Abstract

We study language generation in the limit, which was introduced by Kleinberg and Mullainathan [KM24] building on classical works of Gold [Gol67] and Angluin [Ang79]. The result of [KM24] is an algorithm for generating from any countable language collection in the limit. While their algorithm eventually generates strings from the target language *K*, it sacrifices *breadth*, *i.e.*, its ability to output all strings in *K*. The main open question of [KM24] was whether this trade-off between consistency and breadth is necessary for language generation.

Recent work by Kalavasis, Mehrotra, and Velegkas [KMV24] proposed three definitions for consistent language generation with breadth in the limit: generation with *exact* breadth, generation with *approximate* breadth, and *unambiguous* generation. Concurrent and independent work by Charikar and Pabbaraju [CP24a] introduced a different notion, called *exhaustive* generation. Both of these works explore when language generation with (different notions of) breadth is possible.

In this work, we fully characterize language generation for all these notions of breadth and their natural combinations. Building on [CP24a; KMV24], we give an unconditional lower bound for generation with exact breadth, removing a technical condition needed in [KMV24] and extending the unconditional lower bound of [CP24a] which holds for specific collections; our result shows that generation with exact breadth is characterized by Angluin's condition for identification from positive examples [Ang80]. Furthermore, we introduce a weakening of Angluin's condition and show that it tightly characterizes both generation with approximate breadth and exhaustive generation, thus showing that these two notions are equivalent. Moreover, we show that Angluin's condition further characterizes unambiguous generation in the limit as a corollary of a more general result that applies to a family of notions of breadth. We discuss the implications of our results in the statistical setting of Bousquet, Hanneke, Moran, van Handel, and Yehudayoff [BHMvY21]. Finally, we provide unconditional lower bounds for stable generators, strengthening the results of [KMV24], and we show that for stable generators all the aforementioned notions of breadth are characterized by Angluin's condition. This gives a separation for generation with approximate breadth, between stable and unstable generators.



Figure 1: Equivalences for Language Generation With Breadth in the Limit (Main Theorem 1).

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## 1 Introduction

Building on classical work on learning theory, tracing back to Gold [Gol67] and Angluin [Ang88], Kleinberg and Mullainathan [KM24] provided a formal framework for language generation. In this framework, the domain  $\mathcal{X}$  is a countable collection of strings, and there is an unknown target language K which is a subset of this domain. We know that the true language lies within a collection of possibly infinite but countably many languages  $\mathcal{L} = \{L_1, L_2, ...\}$ .<sup>1</sup> Based on this elementary setup, one can define the tasks of language identification and generation. We start our exposition with the notion of identification in the limit, that goes back to the work of Gold in the late 1960s.

## 1.1 Language Identification in the Limit

The problem of language identification in the limit from positive examples was introduced by Gold [Gol67] and further studied by Angluin [Ang79; Ang80]. For a fixed collection  $\mathcal{L}$ , an adversary and an identifier play the following game: The adversary chooses a language K from  $\mathcal{L}$  without revealing it to the identifier, and it begins *enumerating* the strings of K (potentially with repetitions)  $x_1, x_2, \ldots$  over a sequence of time steps  $n = 1, 2, 3, \ldots$ . The adversary can repeat strings in its enumeration, but the crucial point is that for every string  $x \in K$ , there must be at least one time step n at which it appears. At each time n, the identification algorithm I, given the previous examples  $x_1, x_2, \ldots, x_n$ , outputs an index  $i_n$  that corresponds to its guess for the index of the true language K. Language identification in the limit is then defined as follows.

**Definition 1** (Language Identification in the Limit [Gol67]). *Fix some K from the language collection*  $\mathcal{L} = \{L_1, L_2, ...\}$ . *The identification algorithm*  $I = (I_n)$  *identifies K in the limit if there is some*  $n^* \in \mathbb{N}$  *such that for all steps*  $n > n^*$ *, the identifier's guess*  $i_n$  *satisfies*  $i_n = i_{n-1}$  *and*  $L_{i_n} = K$ . *The language collection*  $\mathcal{L}$  *is identifiable in the limit if there is an identifier that identifies in the limit any*  $K \in \mathcal{L}$ *, for any enumeration of* K. *In this case, we say that the identifier identifies the collection*  $\mathcal{L}$  *in the limit.* 

It is important to note that the above definition imposes some stability to the algorithm: since there can be multiple appearances of *K* in the enumeration of  $\mathcal{L}$ , an algorithm identifies *K* in the limit only if it eventually *stabilizes* (*i.e.*,  $i_n = i_{n-1}$  for *n* larger than some  $n^*$ ) to a correct index (*i.e.*,  $L_{i_n} = K$ ). A natural question is which collections of languages are identifiable in the limit. Angluin [Ang80] provided a condition that characterizes such collections.

**Definition 2** (Angluin's Condition [Ang80]). *Fix a language collection*  $\mathcal{L} = \{L_1, L_2, ...\}$ . *The collection*  $\mathcal{L}$  *is said to satisfy Angluin's condition if for any index i, there is a tell-tale, i.e., a finite set of strings*  $T_i$  such that  $T_i$  is a subset of  $L_i$ , *i.e.,*  $T_i \subseteq L_i$ , and the following holds:

For all  $j \ge 1$ , if  $L_j \supseteq T_i$ , then  $L_j$  is not a proper subset of  $L_i$ .

Further, the tell-tale oracle is a primitive that, given an index *i*, outputs an enumeration of the set  $T_i$ .

It turns out that the above condition characterizes language identification in the limit.

<sup>&</sup>lt;sup>1</sup>Throughout this work, we assume *membership oracle* access to  $\mathcal{L}$  which given  $x \in \mathcal{X}$  and index *i* as input, returns  $\mathbb{1}\{x \in L_i\}$ .

**Theorem 1.1** (Characterization of Identification in the Limit [Ang80]). *The following holds for any countable collection of languages*  $\mathcal{L}$ .

- 1.  $\mathcal{L}$  is identifiable in the limit if it satisfies Angluin's condition and one has access to the tell-tale oracle.
- 2. If there is an algorithm that identifies  $\mathcal{L}$  in the limit, then Angluin's condition is true and the tell-tale oracle can be implemented.

The above tight characterization shows that language identification is information-theoretically impossible even for simple collections of languages, such as the collection of all regular languages. Crucially, access to the tell-tale oracle is necessary for identification in the limit (its existence alone is not sufficient) [Ang80, Theorem 2].

## 1.2 Language Generation in the Limit

Language generation in the limit was introduced by Kleinberg and Mullainathan [KM24] and we define it below. In this work, we define a generating algorithm  $\mathcal{G} = (\mathcal{G}_n)_{n \in \mathbb{N}}$  as a sequence of mappings, *i.e.*, for each  $n \in \mathbb{N}$ ,  $\mathcal{G}_n$  is a mapping from training sets of size n to *distributions* on the domain  $\mathfrak{X}$ . Occasionally, we may refer to generating algorithms as simply generators. As in language identification, there is a two-player game where the adversary fixes a language  $K \in \mathcal{L}$  and an enumeration of its elements. The adversary presents the enumeration sequentially to the generator, who, at each round, generates a potential example from the target language. The generator's goal is as follows:

**Definition 3** (Language Generation in the Limit [KM24]). Fix some K from the language collection  $\mathcal{L} = \{L_1, L_2, ...\}$  and a generating algorithm  $\mathcal{G} = (\mathcal{G}_n)$ . At each step n, let  $S_n \subseteq K$  be the set of all strings that the algorithm  $\mathcal{G}$  has seen so far.  $\mathcal{G}$  must output a string  $w_n \notin S_n$  (its guess for an unseen string in K). The algorithm  $\mathcal{G}$  is said to generate from K in the limit if, for all enumerations of K, there is some  $n^* \in \mathbb{N}$  such that for all steps  $n \ge n^*$ , the algorithm's guess  $w_n$  belongs to  $K \setminus S_n$  (or  $K \setminus S_n$  is empty). The collection  $\mathcal{L}$  allows for generation in the limit if there is an algorithm  $\mathcal{G}$  that, for any target  $K \in \mathcal{L}$ , generates from K in the limit.

Note that for the problem of language generation to be interesting, the languages of the collection  $\mathcal{L}$  must be of infinite cardinality. Hence, throughout this work we assume that that each language in the collection has infinite cardinality. The main result of Kleinberg and Mullainathan [KM24] is that language generation in the limit is possible for all countable collections of languages.

**Theorem 1.2** (Theorem 1 in Kleinberg and Mullainathan [KM24]). There is a generating algorithm with the property that for any countable collection of languages  $\mathcal{L} = \{L_1, L_2, ...\}$ , any target language  $K \in \mathcal{L}$ , and any enumeration of K, the algorithm generates from K in the limit.

This result is in stark contrast to negative results in language identification, mentioned in the previous section; hence, showing a strong separation between identification and generation in the limit. We proceed with the main topic of interest: *language generation with breadth*.

#### **1.3 Language Generation With Breadth in the Limit**

The main open question of Kleinberg and Mullainathan [KM24] was whether there exists a generating algorithm satisfying consistency (*i.e.*, eventually outputting elements only from the true language, which corresponds to Definition 3) and breadth (*i.e.*, eventually being able to generate from the whole range of the true language). The mathematical formulation of this question was introduced in Kalavasis, Mehrotra, and Velegkas [KMV24] and is as follows.

**Definition 4** (Language Generation With Breadth in the Limit [KMV24]). A generating algorithm  $G = (G_n)$  is said to generate with breadth in the limit for a language collection  $\mathcal{L} = \{L_1, L_2, ...\}$  if, for any  $K \in \mathcal{L}$  and enumeration of K, there is an  $n^* \ge 1$ , such that for all  $n \ge n^*$ , after seeing n elements of the enumeration,

$$\operatorname{supp}(\mathcal{G}_n) \cup S_n = K, \tag{1}$$

where  $S_n$  is the set of elements enumerated until round n.

One can also study natural relaxations of the notion of breadth. Kalavasis, Mehrotra, and Velegkas [KMV24] proposed some relaxations of Definition 4. The first relaxation allows the generating algorithm to miss (any) finitely many elements of the target language.

**Definition 5** (Language Generation with Approximate Breadth in the Limit [KMV24]). A generating algorithm  $G = (G_n)$  is said to generate with approximate breadth in the limit for a language collection  $\mathcal{L} = \{L_1, L_2, ...\}$  if, for any  $K \in \mathcal{L}$  and enumeration of K, there is an  $n^* \ge 1$ , such that for all  $n \ge n^*$ , after seeing n elements of the enumeration,

$$\operatorname{supp}(\mathcal{G}_n) \subseteq K \quad and \quad |K \setminus \operatorname{supp}(\mathcal{G}_n)| < \infty.$$
 (2)

Observe that a generating algorithm with approximate breadth avoids hallucinations (*i.e.*, outputting elements outside of *K*, see first term of (2)) but also only misses finitely many elements of the infinite language *K* (second term of (2)). Hence, any algorithm satisfying Definition 4 immediately satisfies Definition 5; see Section 1.4 for some additional motivation behind the above definition. We note that  $|K \setminus \text{supp}(\mathcal{G}_n)|$  should be finite for all  $n > n^*$  but not a fixed constant.

Charikar and Pabbaraju [CP24a], independently of and concurrently with Kalavasis, Mehrotra, and Velegkas [KMV24], came up with another notion of breadth – termed exhaustive generation<sup>2</sup> In their formulation, the generating algorithm is a sequence of mappings from sequences of the domain to *enumerations* of the domain. For any  $i, n \in \mathbb{N}$ , let  $G_n(i)$  be the *i*-th element in the enumeration of the generator that was outputted in the *n*-th round.

**Definition 6** (Exhaustive Language Generation in the Limit [CP24b]). A generating algorithm  $G = (G_n)$  is said to be an exhaustive generator in the limit for a language collection  $\mathcal{L} = \{L_1, L_2, ...\}$  if, for any  $K \in \mathcal{L}$  and enumeration of K, there is an  $n^* \ge 1$ , such that for all  $n \ge n^*$ , after seeing n elements of the enumeration,

$$\left|\bigcup_{i=1}^{\infty} \mathcal{G}_n(i) \setminus K\right| < \infty \quad and \quad S_n \cup \bigcup_{j=1}^{n-1} \mathcal{G}_j(1) \cup \bigcup_{i=1}^{\infty} \mathcal{G}_n(i) \supseteq K,$$
(3)

<sup>&</sup>lt;sup>2</sup>The definition of exhaustive generation appearing in [CP24a] is slightly different from the definition appearing in the updated version [CP24b] (which is concurrent with our work). After coordination with the authors of [CP24b], we present the updated definition in Definition 6. Later, in Section 5, we show that generation under the two definitions of exhaustive generation (from [CP24a] and [CP24b] respectively) is characterized by the same condition (see Corollary 5.5).

where  $S_n$  is the set of elements enumerated until round n.

We note that Definition 6 is (strictly) weaker than generation with exact breadth, but, seems incomparable to Definition 5. For instance, this definition allows the algorithm to hallucinate on finitely many points while approximate breadth requires the generator to be consistent.

Finally, the second relaxation of generation with breadth proposed by Kalavasis, Mehrotra, and Velegkas [KMV24] allows the generator to also hallucinate (*i.e.*, output strings outside of the target language *K*) provided it is a "better" generator for the target language than for any other language in the collection.

**Definition 7** (Unambiguous Language Generation in the Limit [KMV24]). A generating algorithm  $\mathcal{G} = (\mathcal{G}_n)$  is unambiguous in the limit for a language collection  $\mathcal{L} = \{L_1, L_2, ...\}$  if, for any  $K \in \mathcal{L}$  and enumeration of K, its support eventually becomes closer to K than to any other language  $L \neq K$  in  $\mathcal{L}$  in terms of the symmetric difference metric, i.e., there exists some  $n^* \in \mathbb{N}$  such that for all  $n \geq n^*$ , after seeing n elements of the enumeration,

$$|\operatorname{supp}(\mathcal{G}_n) \bigtriangleup K| < \min_{L \in \mathcal{L} : L \neq K} |\operatorname{supp}(\mathcal{G}_n) \bigtriangleup L|$$
, (4)

where recall that for two sets *S* and *T*,  $S \triangle T := (S \setminus T) \cup (T \setminus S)$ .

Unambiguous generation is seemingly weaker than generation with (exact) breadth and not directly comparable to generation with approximate breadth and exhaustive generation.

*Remark* 1 (Representation of the Generators). The astute reader might observe that the previous definitions allow for generating algorithms that output infinite-sized objects. However, all our generating algorithms have succinct representations and this allows for computable algorithms that sample (*i.e.*, generate) a new element, enumerate the support of all generatable elements, and, given an element, decide whether it belongs to the support (*i.e.*, whether it is part of the enumeration). On the other hand, our lower bounds are stronger, they hold for functions that might not be computable.

#### 1.4 Motivation for Generation With Approximate Breadth and Infinite Coverage

In this section, we provide further motivation behind Definition 5, generation with approximate breadth. An immediate modification of the algorithm of [KM24] can achieve *finite coverage* of the target language, for any finite number. More concretely, for any function  $f: \mathbb{N} \to \mathbb{N}$  and any countable collection of languages  $\mathcal{L}$  there exists a generating algorithm  $(\mathcal{G}_n)_{n \in \mathbb{N}}$  such that, for any target language  $K \in \mathcal{L}$  and any enumeration of K the algorithm achieves in the limit

$$\operatorname{supp}(\mathcal{G}_n) \subseteq K$$
,  $\operatorname{supp}(\mathcal{G}_n) \cap S_n = \emptyset$ , and  $|\operatorname{supp}(\mathcal{G}_n)| = f(n)$ .

where  $S_n$  is the set of elements enumerated until round n. In fact, their algorithm can achieve the stronger property of *infinite coverage* defined below.

**Definition 8** (Language Generation with Infinite Coverage in the Limit). A generating algorithm  $G = (G_n)$  is said to generate with infinite coverage in the limit for a language collection  $\mathcal{L} = \{L_1, L_2, ...\}$  if, for any  $K \in \mathcal{L}$  and enumeration of K, there is an  $n^* \ge 1$ , such that for all  $n \ge n^*$ , after seeing n elements of the enumeration (corresponding to the set  $S_n$  in round n),

$$\operatorname{supp}(\mathcal{G}_n) \subseteq K$$
,  $\operatorname{supp}(\mathcal{G}_n) \cap S_n = \emptyset$ , and  $|\operatorname{supp}(\mathcal{G}_n)| = \infty$ ,

Given the above notion of infinite coverage, a simple modification to the generating algorithm of [KM24] gives the following result.

**Proposition 1.3** (Modification of [KM24]). There is a generating algorithm with the property that for any countable collection of languages  $\mathcal{L} = \{L_1, L_2, ...\}$ , any target language  $K \in \mathcal{L}$ , and any enumeration of K, the algorithm generates with infinite coverage from K in the limit.

Thus, the aforementioned modification of the algorithm of [KM24] has the property that it does not hallucinate (*i.e.*, it does not include any elements outside of *K* in its support) and covers infinitely many (unseen) elements of the target language, but might, potentially, not cover infinitely many elements as well. Thus, a natural question is whether there exists an algorithm that does not hallucinate, can cover infinitely many elements of *K*, and also miss only *finitely* many elements of it. This is precisely the requirement of generation with approximate breadth (Definition 5).

*Proof Sketch of Proposition 1.3.* We discuss a sketch of the proof for the version of the algorithm of [KM24] that uses a subset oracle for  $\mathcal{L}$ , *i.e.*, for any  $L_i, L_j \in \mathcal{L}$  it can ask "Is  $L_i \subseteq L_j$ ?". Let us first give a high-level description of their algorithm. For large enough  $n \in \mathbb{N}$ , it creates a (potentially infinite) sequence of languages  $\mathcal{L}' = \{L_{i_1}, L_{i_2}, \ldots\} \subseteq \mathcal{L}$  such that the following hold.

- (i) For every language  $L \in \mathcal{L}'$  it holds that *L* is consistent, *i.e.*,  $S_n \subseteq L$ , where  $S_n$  is the set of elements enumerated until round *n*,
- (ii) The sequence of languages in  $\mathcal{L}'$  satisfies the inclusion:  $L_{i_1} \supseteq L_{i_2} \supseteq \ldots$ , and
- (iii)  $K \in \mathcal{L}'$ .

Then, it outputs an arbitrary string x such that  $x \notin S_n$  and  $x \in L_{i_\ell}$ , where  $i_\ell \in \mathbb{N}$  is the largest number such that  $L_{i_\ell} \in \mathcal{L}'$  and  $i_\ell \leq n$ . The immediate modification is to output a distribution  $\mathcal{G}_n$  such that  $\sup (\mathcal{G}_n) = L_{i_\ell} \setminus S_n$ . Notice that this can be done in a computable way: in order to sample from this distribution, we first sample a natural number  $\hat{n}$  (*e.g.*, from a geometric distribution on  $\mathbb{N}$ ), and then we check if  $x_{\hat{n}} \in L_{i_\ell} \setminus S_n$ .

An analogous modification can be made to the algorithm of [KM24] that only has access to a membership oracle for  $\mathcal{L}$ . For brevity, we omit the modifications to this algorithm.

#### 1.5 Summary of Our Results

Building on [CP24a; KMV24], we provide a general collection of characterizations, summarized in Figure 1. We stress that all these results are *unconditional*, in the sense that they do not rely on any particular structure of the generating algorithm, strengthening the conditional lower bounds of [KMV24]. Further, the results also hold for all countable collections of languages and not for specific families, strengthening the family-specific lower bounds of [CP24a]. Moreover, in addition to strengthening existing results, the results also establish several new lower bounds and characterizations.

We proceed with the statement of our results, which provide a clear picture of the landscape of language generation in the limit. In particular, we show that generation with exact breadth and unambiguous generation are equivalent (Definition 2). Moreover, we show that generation

with approximate breadth and exhaustive generation are both characterized by a different notion, which we call weak Angluin's condition (Definition 9), that is *strictly* weaker than Angluin's condition.

**Main Theorem 1** (Characterizations of Language Generation with Breadth). *For any countable collection of languages*  $\mathcal{L}$ *, the following hold.* 

- 1. The following are equivalent:
  - *There is an algorithm that generates from L in the limit.*
  - There is an algorithm that generates with infinite coverage from  $\mathcal{L}$  in the limit.
- 2. The following are equivalent:
  - There is an algorithm that generates with approximate breadth from  $\mathcal{L}$  in the limit.
  - There is an algorithm that generates exhaustively from  $\mathcal{L}$  in the limit.
  - *L* satisfies the weak Angluin's condition (*Definition 9*).
- 3. The following are equivalent:
  - There is an algorithm that generates with (exact) breadth from  $\mathcal{L}$  in the limit.
  - There is an algorithm that generates unambiguously from  $\mathcal{L}$  in the limit.
  - *There is an algorithm that identifies*  $\mathcal{L}$  *in the limit.*
  - *L* satisfies Angluin's condition (*Definition 2*).

The proofs for all the results of the list of characterizations are provided in the upcoming Section 2. We mention that using reductions from [KMV24], the above characterizations also have implications in the statistical setting, which we discuss in Section 2.5.

*Remark* 2 (Oracle Access for Main Theorem 1). Following the phrasing of [KM24], we provide both *functions* and *algorithms* that generate in the limit. An algorithm only accesses  $\mathcal{L}$  via a membership oracle (and potentially a tell-tale oracle). When a generator uses other types of oracles (*e.g.*, subset oracle), we call it a *function*. To be more specific for the generators of Main Theorem 1, for Item (1), we design a function that uses membership and subset queries as well as an algorithm that uses only membership queries ([KM24] and Proposition 1.3). For approximate breadth in Item (2), we design a function that uses membership and subset queries (Lemma 2.4) as well as an algorithm that uses membership and tell-tale queries (Lemma 2.6). For exhaustive generation in Item (2), we design a function that uses membership, subset, and finite-difference queries (Lemma 2.9) and an algorithm that uses membership and tell-tale queries (Lemma 2.10). Finally, for Item (3), we design a function that uses membership and subset queries and an algorithm that uses membership and tell-tale queries (Lemma 2.10). Finally, for Item (3), we design a function that uses membership and subset queries and an algorithm that uses membership and tell-tale queries (Lemma 2.10). Finally, for Item (3), we design a function that uses membership and subset queries and an algorithm that uses membership and tell-tale queries (Lemma 2.10). Finally, for Item (3), we design a function that uses membership and subset queries and an algorithm that uses membership and tell-tale queries (Lemma 2.10). Finally, for Item (3), we design a function that uses membership and subset queries and an algorithm that uses membership and tell-tale queries (Lemma 2.10).

**Landscape with Stable Generators.** Prior work [KMV24] also studied the problem of generation with breadth using generating algorithms that are *stable*. Roughly speaking, this means that their support eventually stops changing and stabilizes. Under this condition, perhaps surprisingly, the landscape for generation with breadth changes and we provide new results that characterize several definitions of stable generation with breath discussed so far. In particular, we show that the

requirement of stability makes the problem of generation with approximate breadth and the problem of exhaustive generation strictly harder (see Figure 4): there exist stable generators with these properties if and only if the collection satisfies Angluin's condition for identifiability whereas before, when unstable generators were also allowed, one only required the weak Angluin's condition (Lemma 3.2). As another example of the stark change in the landscape, we also show that there exists a collection that satisfies the weak Angluin's condition (hence admits non-stable generator with approximate breadth), but for which no stable generator can achieve the much weaker requirement of infinite coverage (Theorem 3.4). For further details and additional results, we refer to Section 3.

**Characterizations for All Possible Notions of Generation.** Finally, Main Theorem 1 combined with simple observations are sufficient to characterize all possible notions of generation at a certain granularity as explained in Figure 2.

	<b>No Hallucinations</b> $ \operatorname{supp}(G)\backslash K  = 0$	<b>Finite Hallucinations</b> $  \operatorname{supp}(G) \setminus K   < \infty$	<b>Infinite Hallucinations</b> $ \operatorname{supp}(G)\setminus K  = \infty$
<b>Zero Missing Elements</b> $ K \setminus \text{supp}(G)  = 0$	Angluin's Condition (i.e., Exact Breadth)	Weak Angluin's Condition	All Countable Collections
Finite Missing Elements $ K \setminus \text{supp}(G)  < \infty$	Weak Angluin's Condition (i.e., Approximate Breadth)	Weak Angluin's Condition	All Countable Collections
<b>Infinite Present Elements</b> $ K \cap \operatorname{supp}(G)  = \infty$	All Countable Collections (i.e., Infinite Coverage)	All Countable Collections	All Countable Collections
<b>Finite Present Elements</b> $ K \cap \text{supp}(G)  < \infty$	All Countable Collections	All Countable Collections	All Countable Collections

Figure 2: *Characterizations of All Possible Notions of Generation:* This figure lists all possible notions of language generation (at a certain granularity) and the condition characterizing each notion. Rows capture the extent of breadth (*i.e.*, how many elements are missed from the target language). Columns capture the extent of hallucinations (*i.e.*, how many elements outside of the target language are included). Generation becomes easier as one moves down the rows and/or to the right along columns. To achieve any notion in the last column, it is sufficient to generate the whole domain (*i.e.*, ensure supp(G) = X) and to achieve any notion in the last two rows, it is sufficient to use the extension of [KM24]'s algorithm from Proposition 1.3.

**Roadmap.** We already proved Item 1 in Section 1.4. In Section 2, we discuss Items 2 and 3. In particular, Section 2.1 provides the unconditional characterization for generation with exact breadth, Section 2.2 provides the unconditional characterization for generation with approximate breadth, Section 2.3 gives the result for exhaustive generation, and Section 2.4 provides the unconditional characterization for unambiguous generation. This set of results also has implications for the statistical setting of [KMV24], which is introduced and discussed in Section 2.5. Finally, in Section 3, we discuss the notion of stability and present results characterizing when different notions of stable generation with breadth are achievable. Before proceeding to the formal statements and proofs of our results, we discuss related works.

### 1.6 Recent Works on Language Generation With Breadth

**Independent and Concurrent Work.** Independently and concurrently to the current paper, Charikar and Pabbaraju [CP24b] also obtained the same characterization for generation with (exact) breadth and exhaustive generation. After learning about each other's results via personal communication, we coordinated with Charikar and Pabbaraju [CP24b], and decided to give the same name to the characterizing condition (the "weak Angluin's condition").

**Comparison with [CP24b].** Compared to [CP24b], we characterize some additional notions of breadth (*e.g.*, the notions of approximate breadth and unambiguous generation appearing in [KMV24]; see Figures 1 and 2 for details) and also explore the landscape of language generation with stable generators (Section 3 and Figure 4).

**Other Results of [CP24b].** On the other hand, [CP24b] show that non-uniform generation (without breadth) is achievable for all countable collections given access to a certain oracle for  $\mathcal{L}$ , and they prove that membership oracle access to  $\mathcal{L}$  is not sufficient for uniform generation, even when  $\mathcal{L}$  consists of only two languages. Finally, they propose and study a setting of generation with feedback, reminiscent of active learning: they provide a dimension whose finiteness characterizes whether a collection  $\mathcal{L}$  admits an algorithm that generates uniformly from  $\mathcal{L}$  with feedback. In fact, these additional results were already present in [CP24a], an earlier version of [CP24b], which preceded our work.

In the remainder of this section, we draw a comparison of our work and the works of [KMV24] and [CP24a] (which is an earlier version of [CP24b]). Both prior works provide results that our work builds upon. We also note that there has been additional recent work on language generation by Li, Raman, and Tewari [LRT24], who generalize the results of Kleinberg and Mullainathan [KM24] to non-countable collections, characterize uniform generatability and introduce and study non-uniform generatability.

**Results of** [KMV24] **for Exact Breadth.** Regarding Definition 4 for consistent generation with breadth in the limit, the work of [KMV24] shows an impossibility result conditioned on the decidability of the membership oracle problem (MOP; see Definition 19): for any language collection that is not identifiable in the limit, no generating algorithm, for which the MOP is decidable, can generate with breadth. They complement this negative result with an algorithm (for which MOP is decidable) that generates with breadth for all collections of languages identifiable in the limit.

**Results of** [KMV24] **for Relaxations of Breadth.** [KMV24] provide two stronger versions of this result. First, they show that for any language collection that is not identifiable in the limit, no generating algorithm, that is stable and for which the MOP is decidable, can achieve either generation with approximate breadth in the limit (*i.e.*, Definition 5) or unambiguous generation in the limit (*i.e.*, Definition 7). For these impossibility results, the generating algorithms need to

be stable in the sense that after a finite amount of rounds, their support is stabilized and does not change. Stability is well motivated and already appears as a requirement for identification in the limit in the original definition of [Gol67] (see Definition 1).

**Other Results of** [KMV24]. We underline that all these results are corollaries of a broader collection of results that operate in a statistical setting, building on an early work of [Ang88] and [BHMvY21]; we refer to [KMV24] for further details.

**Results of** [CP24a] **for Exhaustive Generation.** Concurrently and independently of [KMV24], Charikar and Pabbaraju [CP24a] studied language generation with breadth via exhaustive generation (Definition 18).<sup>3</sup> As we have already mentioned, [CP24a] provided a collection of languages that illustrates the separation between generation with exact breadth and exhaustive generation. More concretely, they proved that there exists a generator that achieves Definition 18 for this collection, but there does not exist a generator that satisfies exact breadth (Definition 4). Furthermore, [CP24a] presented a countable collection of languages and proved that no algorithm can generate exhaustively in the limit from this collection, which immediately implies that there does not exist a generator that achieves a question asked by [KMV24]. In fact, a simple adaptation of the proof of [CP24a] can be used to show that no algorithm can achieve generation with approximate breadth for this *particular collection*.

**Comparison of Exhaustive Generation with Definitions of Breadth from** [KMV24]. By inspecting the original definition of exhaustive generation (Definition 18) and approximate breadth (Definition 5), it is immediate that Definition 18 is stronger (perhaps not strictly) than Definition 5, and seemingly incomparable with Definition 7.<sup>3</sup>

**Other Results of** [CP24a]. Finally, as mentioned before, [CP24a]'s work also presents other results, which do not fall in the intersection of [KMV24] and [CP24a] and, instead, are more closely related with the recent work of [LRT24].

**Comparing** [CP24a] **and** [KMV24]. In summary, the impossibility result of [CP24a] holds for two specific language collections but is unconditional. In contrast, the lower bounds of [KMV24] hold for all non-identifiable language collections but require some restrictions on the generators: For generation with exact breadth (Definition 4), MOP should be decidable; for generation with approximate breadth (Definition 5) or unambiguous generation (Definition 7), MOP should be decidable and the generator should be stable. Moreover, [CP24a] give a non-identifiable collection for which exhaustive generation is possible, but generation with (exact) breadth is not, thus also showing that exhaustive generation is strictly easier than identification.

**Brief Summary of Our Results and Comparison With** [CP24a; KMV24]. In this work, we give conditions that characterize when each of the above notions of breadth can be achieved, significantly strengthening some of the results of both [KMV24] and [CP24a]. Our characterizations for

<sup>&</sup>lt;sup>3</sup>As mentioned before, [CP24b], which is the updated version of [CP24a], relaxed the definition of exhaustive generation from Definition 18 to Definition 6. Concretely, the original definition (Definition 18) does not allow the generator to hallucinate, while the updated one (Definition 6) allows for finite hallucinations. The two definitions turn out to be characterized by the same condition (see Section 5). The same characterization also holds for the notion of *relaxed exhaustive generation* mentioned in Remark 4 of [CP24b], which does not require coverage of the target language.

(exact) breadth and unambiguous generation, generalize the lower bound technique in [CP24a] from the specific family they consider to all language collections via a different construction based on Angluin's condition. Further, our characterizations of approximate breadth and exhaustive generation, are based on a weaker variant of Angluin's condition that we introduce. Next, like [KMV24] and building on their machinery, we also study the implications of these characterizations in the statistical setting, where the observed strings are sampled from an underlying distribution and not chosen by an adversary. Finally, following [KMV24], we also study the landscape of generation when the generating algorithm is required to be stable and, perhaps surprisingly, find that several notions of generations are significantly harder to achieve with stable generators compared with unstable generators.

## 2 Main Results: Unconditional Characterizations

In this section, we present all the unconditional results provided in this work. Additional results for stable generators appear in Section 3.

## 2.1 Unconditional Characterization of Generation With (Exact) Breadth

In this section, we prove a strong impossibility result for language generation with exact breadth. This provides a stronger version of a result by [KMV24] for the in-the-limit setting which excluded generators for which the MOP (Definition 19) is undecidable.

**Lemma 2.1** (Impossibility for Generation with Exact Breadth). Let  $\mathcal{L}$  be a countable collection of languages that is not identifiable in the limit. Then, no algorithm can generate with breadth from  $\mathcal{L}$  in the limit.

If  $\mathcal{L}$  is identifiable in the limit, then recent work by [KMV24] shows that consistent generation with breadth is possible in the limit. This algorithm combined with Lemma 2.1 gives us the following result, which completely characterizes generation with breadth in the Gold-Angluin model.

**Theorem 2.2** (Characterization of Generation with Exact Breadth). *For any countable collection of languages*  $\mathcal{L}$ *, one of the following holds.* 

- 1. If  $\mathcal{L}$  satisfies Definition 2, then there is a generator that generates with breadth from  $\mathcal{L}$  in the limit.
- 2. If  $\mathcal{L}$  does not satisfy *Definition 2*, then no generator can generate with breadth from  $\mathcal{L}$  in the limit.

**Notation.** For any enumeration *E*, we use the notation E(i) to denote its *i*-th element, E(1:i) to denote its first *i* elements, and  $E(i:\infty)$  to denote all but the first i - 1 elements.

*Proof of Lemma* 2.1. Since  $\mathcal{L}$  is not identifiable in the limit, it does not satisfy Angluin's condition (Definition 2). Hence, there exists a language  $L^* \in \mathcal{L}$  such that the following holds:

for all finite subsets  $T \subseteq L^*$ , there exists a language  $L_T \in \mathcal{L}$ ,  $T \subseteq L_T$  and  $L_T \subsetneq L^*$ . (5)

Fix  $L^* \in \mathcal{L}$  to be any language for which this holds. Let  $E^{\infty}_*$  be an arbitrary enumeration of  $L^*$ , without repetitions. Let *K* and  $E^{\infty}_K$  respectively denote the target language and its enumeration that we will construct to show the impossibility result.

We will show that for any generating algorithm  $G = (G_n)$  there exists a choice of the target language *K* in  $\mathcal{L}$  (which may be different from  $L^*$ ) and an enumeration  $E_K^{\infty}$  of it such that if *K* is the target language and the adversary provides enumeration  $E_K^{\infty}$  to *G*, then the algorithm *G* cannot generate with breadth from *K* in the limit.

We will construct the enumeration iteratively and select *K* based on the generating algorithm. The construction of the enumeration proceeds in multiple (possibly infinite) phases. At any point  $t \in \mathbb{N}$  of the interaction, we denote by  $S_t$  the set of elements enumerated so far.

**Phase 1 of Construction.** To construct the first phase, we present the generator with the first element of the enumeration of  $L^*$ , *i.e.*,  $x_{i_1} \coloneqq E_*^{\infty}(1)$ . Let  $L_{j_1}$  be some language such that  $x_{i_1} \in L_{j_1}$  and  $L_{j_1} \subsetneq L^*$ , *i.e.*, it is a proper subset of  $L^*$ . Notice that such a language is guaranteed to exist by picking  $T = \{x_{i_1}\}$  in the violation of Angluin's condition (5).

 Subphase A (Enumerate L<sub>j1</sub> Until Generator Generates with Breadth from L<sub>j1</sub>): Consider an enumeration E<sub>1</sub><sup>∞</sup> of the language L<sub>j1</sub> that is constructed by traversing E<sub>\*</sub><sup>∞</sup> and using the elements of L<sub>j1</sub> that appear in it, in the same order as they appear, *i.e.*, for every i ∈ N it holds that E<sub>1</sub><sup>∞</sup>(i) is the *i*-th element of L<sub>j1</sub> that appears in E<sub>\*</sub><sup>∞</sup>. Notice that this is indeed a valid enumeration of L<sub>j1</sub> as L<sub>j1</sub> is a subset of L\*. At any round t of the first phase, the adversary presents the element E<sub>1</sub><sup>∞</sup>(t) to the generator.

Consider two cases: i) either there is some finite  $t_1 \in \mathbb{N}$  such that  $S_{t_1} \cup \text{supp}(\mathcal{G}_{t_1}) = L_{j_1}$  or ii) there is no such  $t_1 \in \mathbb{N}$ . In the latter case, we pick the target language  $K = L_{j_1}$  and the target enumeration  $E_K^{\infty} = E_1^{\infty}$ , and the lower bound follows. Hence, assume that we are in the former case, and let  $\hat{x}_1$  be the first element of  $E_1^{\infty}$  for which the condition holds. Note that  $S_{t_1}$  is the set of strings shown to the generating algorithm after which it starts to generate with breadth from  $L_{j_1}$ .

Let  $\hat{S}_1$  be the set of elements of  $E^{\infty}_*$  that appear before  $\hat{x}_1$  in  $E^{\infty}_*$  and have not appeared in  $S_{t_1}$ . If  $\hat{S}_1 \neq \emptyset$ , we go to Subphase B.1 and, otherwise if  $\hat{S}_1 = \emptyset$ , we go to Subphase B.2.

- Subphase B.1 (Add Any Skipped Elements): We will use the set S<sub>1</sub> to extend the construction of the target enumeration E<sup>∞</sup><sub>K</sub>. To do this, we enumerate the elements from S<sub>1</sub> in an arbitrary order and we fix the prefix of the target enumeration E<sup>∞</sup><sub>K</sub> to be (S<sub>t1</sub>, S<sub>1</sub>). Notice that this step is well-defined since we are only adding to the already constructed enumeration. Let t̂<sub>1</sub> be the total number of elements enumerated so far. Notice that t̂<sub>1</sub> = ∞ if and only if Case i) (from Subphase A) holds, in which case the lower bound already follows. Hence, assume for the continuation of the proof that t̂<sub>1</sub> < ∞. Now we terminate the first phase (without going to Subphase B.2).</li>
- Subphase B.2 (If Nothing Skipped Enumerate An Element Outside L<sub>j1</sub>): Notice that S<sub>1</sub> = Ø if and only if we did not skip any element of E<sup>∞</sup><sub>\*</sub> during the traversal in Subphase A. If we indeed did not skip elements of E<sup>∞</sup><sub>\*</sub> we continue traversing it and adding elements to E<sup>∞</sup><sub>K</sub> in the same order as we see them in E<sup>∞</sup><sub>\*</sub> until we find some element that does not belong to L<sub>j1</sub>. We also include this element in the enumeration E<sup>∞</sup><sub>K</sub>, we fix t̂<sub>1</sub> to be the number of elements enumerated so far and we terminate the first phase.

Notice that so far in our construction, we have enumerated the first  $\hat{t}_1$  elements of  $E_*^{\infty}$ .

Now we continue our construction inductively for phases  $\ell = 2, 3, ...$  Consider any  $\ell \ge 2$ . Suppose our construction continued from Phase 1 until Phase  $\ell$ . Then, Phase  $\ell + 1$  of our construction is as follows.

**Phase**  $\ell + 1$  of Construction. For the  $(\ell + 1)$ -th phase, consider the set  $E^{\infty}_{*}(1 : \hat{t}_{\ell})$  that has been enumerated so far. By construction,

$$E^{\infty}_*(1:\hat{t}_{\ell}) \not\subseteq L_{j_{\ell}}, \quad E^{\infty}_*(1:\hat{t}_{\ell}) \subseteq L^*, \text{ and } E^{\infty}_*(1:\hat{t}_{\ell}) \text{ is finite }.$$

We will now apply the violation of Angluin's condition (5) with  $T = E_*^{\infty}(1:\hat{t}_{\ell})$ . This means that there must exist some  $j_{\ell+1} \notin \{j_1, j_2, \dots, j_{\ell}\}$  such that

$$L_{j_{\ell+1}} \in \mathcal{L}$$
,  $L_{j_{\ell+1}} \subsetneq L^*$ , and  $E^{\infty}_*(1:\hat{t}_{\ell}) \subseteq L_{j_{\ell+1}}$ .

We now perform analogs of each subphase in Phase 1.

Subphase A (Enumerate L<sub>jℓ+1</sub> Until Generator Generates with Breadth from L<sub>jℓ+1</sub>): Consider an enumeration E<sup>∞</sup><sub>ℓ+1</sub> of L<sub>jℓ+1</sub> whose first t̂<sub>ℓ</sub> strings are E<sup>∞</sup><sub>\*</sub>(1 : t̂<sub>ℓ</sub>) and whose remaining strings are constructed by traversing E<sup>∞</sup><sub>\*</sub>(t̂<sub>ℓ</sub> + 1 : ∞) and selecting strings that belong to L<sub>jℓ+1</sub>, in the same order as they appear in E<sup>∞</sup><sub>\*</sub>. Notice that this is indeed a valid enumeration of L<sub>jℓ+1</sub> as L<sub>jℓ+1</sub> is a subset of L<sup>\*</sup>. At any round t of this phase, the adversary presents the element E<sup>∞</sup><sub>ℓ+1</sub>(t + t̂<sub>ℓ</sub>) to the generator.

Consider two cases: i) either there is some finite  $t_{\ell+1} \ge \hat{t}_{\ell} + 1$  such that  $S_{t_{\ell+1}} \cup \text{supp}(\mathcal{G}_{t_{\ell+1}}) = L_{j_{\ell+1}}$  or ii) there is no such  $t_{\ell+1} \in \mathbb{N}$ . In the latter case, we pick the target language  $K = L_{j_{\ell+1}}$  and the enumeration  $E_K^{\infty} = E_{\ell+1}^{\infty}$ , and the lower bound follows. Hence, assume that we are in the former case, and let  $\hat{x}_{\ell+1}$  be the first element of  $E_{\ell+1}^{\infty}$  for which the condition holds. Note that  $S_{t_{\ell+1}}$  is the set of strings shown to the generating algorithm after which it starts to generate with breadth from  $L_{j_{\ell+1}}$ .

Let  $\widehat{S}_{\ell+1}$  be the set of strings of  $E_*^{\infty}$  that appear before  $\widehat{x}_{\ell+1}$  in  $E_*^{\infty}$  and have not appeared in the enumeration  $S_{t_{\ell+1}}$ . If  $\widehat{S}_{\ell+1} \neq \emptyset$ , we go to Subphase B.1 and, otherwise if  $\widehat{S}_{\ell+1} = \emptyset$ , we go to Subphase B.2.

- Subphase B.1 (Add Any Skipped Elements): We will use the set S<sub>ℓ+1</sub> to extend the construction of the target enumeration E<sub>K</sub><sup>∞</sup>. To do this, we enumerate the elements from S<sub>ℓ+1</sub> in an arbitrary order and we fix the prefix of the target enumeration E<sub>K</sub><sup>∞</sup> to be (S<sub>t<sub>ℓ+1</sub></sub>, S<sub>ℓ+1</sub>). Notice that this step is well-defined since we are only adding to the already constructed enumeration. Let t<sub>ℓ+1</sub> be the set of elements enumerated so far. Notice that t<sub>ℓ+1</sub> = ∞ if and only if Case i) (from Subphase A) holds, in which case the lower bound already follows. Hence, assume for the continuation of the proof that t<sub>ℓ+1</sub> < ∞. Now we terminate the (ℓ + 1)-th phase without going to Subphase B.2.</li>
- Subphase B.2 (If Nothing Skipped Enumerate An Element Outside L<sub>jℓ+1</sub>): Notice that \$\hfills\$\vec{f}\_{ℓ+1} = Ø\$ if and only if we did not skip any element of E<sup>∞</sup><sub>\*</sub> during the traversal in Subphase A. If we indeed did not skip elements of E<sup>∞</sup><sub>\*</sub> we continue traversing it and adding elements to E<sup>∞</sup><sub>K</sub> in the same order as we see them in E<sup>∞</sup><sub>\*</sub> until we find some element that does not belong to L<sub>jℓ+1</sub>. We also include this element in the enumeration E<sup>∞</sup><sub>K</sub>, we set t<sub>ℓ+1</sub> to be the number of elements enumerated so far and we terminate Phase ℓ + 1.



Figure 3: Illustration of the Construction in the Proof of Lemma 2.1. Fix any enumeration  $a, b, c, d, e, f, g, \ldots$  of the language  $L^*$ , depicted in the first row. The enumeration of K is initially empty in the construction and this is depicted in the second row. To begin the construction, we apply the contrapositive to Angluin's condition with  $T = \{a\}$  (*i.e.*, with the set highlighted in blue in the first row). This results in a language  $L_1$  that contains T and is a subset of  $L^*$ . For this illustration, suppose that the enumeration of  $L_1$  is as presented in the fourth row. The elements shared between  $L_1$  and  $L^*$  are highlighted in red in the third row. From the fourth row, we can see that the strings in  $L_1$ 's enumeration,  $E_1^*$ , follow the same relative order as in  $E_*^{\infty}$ . Further, note that c, d, and f are skipped from the enumeration since they do not belong to  $L_1$  (*i.e.*, they are not highlighted in red). Now, the algorithm in the proof is trained on the enumeration  $E_1^{\infty}$  (Subphase A), and we consider two cases: Case (i): Assume that after seeing element *e*, the algorithm starts generating with breadth. Then we update  $E_K^{\infty}$  by adding all elements of  $E_1^{\infty}$  until *e* and then add all the elements that we skipped from  $E_*^{\infty}$ ; this is shown in the fifth row where we added *c* and d. This scenario corresponds to Subphase B.1 in the proof since at least one element from the enumeration of  $E_*^{\infty}$  was skipped during Subphase A. Next, we again apply the contrapositive to Angluin's condition. This time, we set  $T = \{a, b, e, c, d\}$  (denoted in blue in the sixth row), and, then repeat the process. Case (ii): Assume that the algorithm generates with breadth after seeing b. Then, we update  $E_K^{\infty}$  by adding a, b and then the first element that is not in  $L_1$ , i.e., c. This is depicted in the seventh row. This scenario corresponds to Subphase B.2 in the proof since no strings from  $E_*^{\infty}$  were skipped during Subphase A. Next, we again apply the contrapositive to Angluin's condition. This time, we use  $T = \{a, b, c\}$  (denoted in blue in the last row) and repeat the process.

Notice that so far we have enumerated the first  $\hat{t}_{\ell+1} > \hat{t}_{\ell} + 1$  elements of  $E_*^{\infty}$ .

**Inductive Argument.** As explained, we continue the construction of the target enumeration inductively. If there is some phase  $\ell$  such that Case ii) (in Subphase A) is activated, then the lower bound follows. Let us now assume that Case ii) is not activated for any phase  $\ell \in \mathbb{N}$ . Then, we have constructed an enumeration of  $L^*$  (by construction of the sets  $S_{t_\ell}$  and  $\widehat{S}_\ell$  for each  $\ell \in \mathbb{N}$ ) such that  $S_t \cup \text{supp}(\mathcal{G}_t) \neq L^*$  for infinitely many  $t \in \mathbb{N}$ . Now, the lower bound follows by setting the target language  $K = L^*$  and the target enumeration to the one we have constructed inductively over all phases.

### 2.2 Unconditional Characterization of Generation With Approximate Breadth

In this section, we show that the following *strict* weakening of Angluin's condition characterizes language generation with approximate breadth (recall Definition 5).

**Definition 9** (Weakening of Angluin's Condition). *Fix a language collection*  $\mathcal{L} = \{L_1, L_2, ...\}$ . *The collection*  $\mathcal{L}$  *is said to satisfy the weak Angluin's condition if for any index i, there is a tell-tale, i.e., a finite set of strings*  $T_i$  *such that*  $T_i$  *is a subset of*  $L_i$ *, i.e.,*  $T_i \subseteq L_i$ *, and the following holds:* 

For all  $j \ge 1$  such that  $L_j \supseteq T_i$ , one of the following holds.

- *Either* L<sub>i</sub> *is not a proper subset of* L<sub>i</sub>; *or*
- $L_i$  is a proper subset and misses finitely many elements of  $L_i$ , i.e.,  $|L_i \setminus L_j| < \infty$ .

Further, the tell-tale oracle is a primitive that, given an index *i*, outputs an enumeration of the set  $T_i$ .

In Remark 3, we give a collection  $\mathcal{L}$ , taken from [CP24a], which witnesses that the above modification of Angluin's condition is a *strict* weakening of Definition 2.

*Remark* 3 (Separation Between Definition 2 and Definition 9 [CP24a]). We highlight that there is a separation between the collections of languages that satisfy Definition 2 and Definition 9, which is taken from [CP24a]. Let  $\mathcal{X} = \mathbb{N}$ ,  $L_i = \mathbb{N} \setminus \{i\}$ , and  $\mathcal{L} = \{\mathbb{N}, L_1, L_2, ...\}$ . Then,  $\mathcal{L}$  does not satisfy Definition 2 but satisfies Definition 9. Thus, Definition 9 is a strictly weaker condition than Definition 2.

We note that [KMV24] showed that if the MOP is decidable for the generator and the generator is stable (see Section 3 for a discussion on stability and Definition 11 for a formal definition), then Angluin's original condition characterizes language generation with approximate breadth. Hence, the following result shows a generator that is (1) unstable or (2) has an undecidable MOP can generate a strictly larger set of language collections with approximate breadth than stable generators with decidable MOP.

**Theorem 2.3** (Characterization of Generation with Approximate Breadth). *For any countable collection of languages*  $\mathcal{L}$ *, one of the following holds.* 

- 1. If  $\mathcal{L}$  satisfies *Definition* 9, then there is a generator that generates with approximate breadth from  $\mathcal{L}$  in the limit.
- 2. If  $\mathcal{L}$  does not satisfy *Definition* 9, then no generator can generate with approximate breadth from  $\mathcal{L}$  in the limit.

We will prove the result in two parts. First, we show that if  $\mathcal{L}$  satisfies Definition 9 then there exists some algorithm which generates from  $\mathcal{L}$  with consistency and approximate breadth in the limit. For this result, we consider two algorithms: the first one that has access to a "subset oracle" for  $\mathcal{L}$  (which can answer queries of the form "Is  $L_i \subseteq L_j$ ?") (Lemma 2.4) and the second one only has access to a membership oracle for  $\mathcal{L}$  (which can answer queries of the form "Is  $L_i \subseteq L_j$ ?") (Lemma 2.4) and the second one only has access to a membership oracle for  $\mathcal{L}$  (which can answer queries of the form "given a string w and i, is  $w \in L_i$ ?") and the tell-tale oracle from Definition 9 (Lemma 2.6). Interestingly, the former algorithm is (essentially) the one proposed by [KM24]. Subsequently, we will show that if  $\mathcal{L}$  does not satisfy Definition 9, then no algorithm can generate from  $\mathcal{L}$  with consistency and approximate breadth in the limit.

**Lemma 2.4** (Function for Generation with Approximate Breadth). Let  $\mathcal{L}$  be a countable collection of languages that satisfies Definition 9. Then, there exists a generating algorithm that, given access to a membership oracle for  $\mathcal{L}$  and a subset oracle for  $\mathcal{L}$  (that given indices *i*, *j* outputs Yes if  $L_i \subseteq L_j$  and No otherwise), generates from  $\mathcal{L}$  with approximate breadth in the limit.

This proof is inspired by the proof of Theorem B.2 in [KMV24], the difference is that, instead of using Angluin's condition (Definition 2), we use its weakening (Definition 9).

*Proof of Lemma* 2.4. The algorithm  $\mathcal{A}$  is illustrated below. This algorithm follows the steps of the generation algorithm of [KM24] (see Steps 1 to 5). The only change is in its last Step 6 where it generates a random sample from the set of interest.

for  $t \in \{1, 2, ...\}$  do:

- 1. Observe element  $x_t$  and let  $S_t$  be the set of all elements observed so far.
- 2. Construct a version space  $V_t$  consisting of all languages in  $\mathcal{L}_{<t}$  consistent with  $S_t$ , *i.e.*,

$$V_t := \left\{ L_j \colon 1 \le j \le t \,, \ L_j \supseteq S_t \right\} \,.$$

# Define a language  $L_i \in V_t$  to be critical if  $L_i$  is the smallest-index language in  $V_t$  or  $L_i$  is a subset of all languages preceding it in  $V_t$ , i.e.,  $L_i \subseteq L_j$  for all  $1 \le j < i$ .

- 3. If  $V_t = \emptyset$ , **output** an arbitrary element of  $\mathfrak{X}$  and **go** to the next iteration.
- 4. Construct the set  $C_t \subseteq V_t$  of all critical languages.

# To construct the set of critical languages  $C_t$  the algorithm needs access to the subset oracle.

- 5. Let  $L_i$  be the largest-indexed language in the set of critical languages  $C_t$ .
- 6. **output** a sample from any distribution whose support is  $L_i \setminus S_t$ . This can be done in a computable fashion by first sampling a natural number *n* from (*e.g.*, the geometric distribution on  $\mathbb{N}$ ) and then outputting the *n*-th string from  $L_i \setminus S_t$ .

Let *z* be the first index such that  $K = L_z$ . The proposed algorithm generates with approximate breadth from *K* when after some finite time  $t^*$ , and for  $t > t^*$ , the last language in the set of critical languages  $C_t$ ,  $L_i = L_i(t)$ , satisfies that

$$L_i \subseteq K$$
 and  $|K \setminus L_i| < \infty$ .

This condition is implied by the following two conditions.

- (A) *K* is eventually included in set of critical languages  $C_t$  and is never removed after that.
- (B) Eventually all the languages  $L_j$  with j > z that are in  $C_t$  satisfy  $L_j \subseteq K$  and  $|K \setminus L_j| < \infty$ .

Result (4.3) of [KM24] shows that there is a finite time  $t_A$  after which Condition (A) holds. We will show that there is also a finite time  $t_B$  after which Condition (B) holds. This shows that, for any  $t \ge \max{\{t_A, t_B\}}$ ,  $\mathcal{A}$  generates with approximate breadth from K.

**Condition (B) holds after a finite time.** Since  $\mathcal{L}$  satisfies the weakening of Angluin's condition (Definition 9),  $K = L_z$  has a finite tell-tale set  $T_z$ , such that, any language  $L \in \mathcal{L}$  containing the tell-take  $T_z$  satisfies one of the following:

- Either *L* is not a proper subset of *K*;
- Or *L* is a proper subset of *K* and satisfies  $|K \setminus L| < \infty$ .

(Recall that  $T_z$  is not known to us; our proof will not need this.) Fix any j > z and any time  $t_B \ge t_A$  after which K is guaranteed to be a critical language and after which  $S_t \supseteq T_z$  (which happens at a finite time since  $T_z$  is finite and, so, all elements of  $T_z$  appear in the enumeration of K at some finite time). Our goal is to show that for any  $t \ge t_B$ , and any j > z for which  $L_j$  is in  $C_t$ , it holds that

$$L_i \subseteq K$$
 and  $|K \setminus L_i| < \infty$ .

By the definition of critical languages and the fact that  $L_j$  appears after  $K = L_z$  in the set of critical languages (as j > z), it follows that  $L_j \subseteq K$ . Hence, it remains to show that  $|K \setminus L_j| < \infty$ . To see this, observe that since  $L_j \in C_t$  and  $C_t \subseteq V_t$ ,  $L_j$  is in the version space  $V_t$  and, hence, by the definition of  $V_t$ ,  $L_j \supseteq S_t$ . Therefore, in particular,  $L_j \supseteq T_z$  (as  $S_t \supseteq T_z$ ). Now, Definition 9 combined with the observation that  $L_j \subseteq K$  implies that  $|K \setminus L_j| < \infty$  as required.

Building on the result of Kalavasis, Mehrotra, and Velegkas [KMV24] (Corollary B.2 in their paper), the previous result shows that the function<sup>4</sup> of Kleinberg and Mullainathan [KM24] with access to a subset query oracle achieves the "best-of-three" worlds for generation, without requiring any prior information about  $\mathcal{L}$ , only subset and membership oracle access.

**Corollary 2.5.** Let  $\mathcal{L}$  be a countable collection of languages. Exactly one of the following holds for the subset-oracle-based function of Kleinberg and Mullainathan [KM24].

- If *L* satisfies Angluin's condition, the function generates with exact breadth in the limit.
- If *L* does not satisfy Angluin's condition but satisfies the weak Angluin's condition, the function generates with approximate breadth in the limit.
- If *L* does not satisfy the weak Angluin's condition, the function generates with infinite coverage in the limit.

<sup>&</sup>lt;sup>4</sup>To be precise, the function is that of [KM24] together with a process to sample from a language given membership access to it; see *e.g.*, Step 6 in the Algorithm of Lemma 2.4.

Next, we give an algorithm that generates with approximate breadth without requiring access to a subset oracle.

**Lemma 2.6** (Algorithm for Generation with Approximate Breadth). Let  $\mathcal{L}$  be a countable collection of languages that satisfies *Definition 9*. Then, there exists a generating algorithm that, given access to a membership oracle for  $\mathcal{L}$  and the tell-tale oracle from *Definition 9*, generates from  $\mathcal{L}$  with approximate breadth in the limit.

*Proof of Lemma* 2.6. Let  $S_n$  be the set of elements the adversary has enumerated up to round  $n \in N$ . For every  $i, n \in \mathbb{N}$ , let  $T_n^i$  be the first n elements enumerated from the tell-tale oracle when called on language  $L_i$ . Let also  $x_1, x_2, ...$ , be an enumeration of the domain  $\mathfrak{X}$ . Our proof is reminiscent of Angluin's approach [Ang80], and the generating algorithm requires only one extra step, namely removing the elements  $x_1, ..., x_n$  from the support of the outputted distribution. However, due to the relaxed condition we are using, our analysis is more technically involved.

For every round  $n \in \mathbb{N}$ , the generating algorithm constructs the sets  $T_n^i$  using the tell-tale oracle for all languages  $L_i$  with  $1 \le i \le n$ . Let  $g_n \in \mathbb{N}$ ,  $1 \le g_n \le n$ , be the smallest number (if any) such that  $S_n \subseteq L_{g_n}$  and  $T_n^{g_n} \subseteq S_n$ . If no such number exists, let  $\mathcal{G}_n$  be some arbitrary distribution. Otherwise, let  $\mathcal{G}_n$  be a distribution with supp $(\mathcal{G}_n) = L_{g_n} \setminus (S_n \cup \{x_1, \ldots, x_n\})$ .<sup>5</sup>

Fix a canonical enumeration  $x_1, x_2, \ldots$  of  $\mathfrak{X}$ .

for  $n \in \{1, 2, ...\}$  do:

- 1. Let  $S_n$  be the set of all elements observed so far.
- 2. Create the list  $\mathcal{L}_{< n} = \{L_1, ..., L_n\}.$
- 3. For each language  $L_i$  in  $\mathcal{L}_{\leq n}$ , let  $T^i = \text{TellTaleOracle}(L_i), i \in [n]$ .
- 4. Truncate the outputs of the oracle and keep only their first *n* elements

$$T_n^i = (T^i(1), \ldots, T^i(n)), \ i \in [n].$$

5. Find smallest index  $g_n \in \{1, ..., n\}$  such that  $S_n \subseteq L_{g_n}$  and  $T_n^{g_n} \subseteq S_n$ .

# This is the minimum indexed language in  $\mathcal{L}_{\leq n}$  that is consistent and its truncated tell-tale is contained in the observed elements.

- 6. If no such  $g_n$  exists, **output** an arbitrary point from  $\mathfrak{X}$  and **go** to the next iteration.
- 7. Otherwise, define a distribution  $\mathcal{G}_n$  with supp $(\mathcal{G}_n) = L_{g_n} \setminus (S_n \cup \{x_1, \dots, x_n\})$ .

# The intuition for removing the first *n* elements  $x_1, \ldots, x_n$  of the canonical enumeration of  $\mathfrak{X}$  is as follows. A bad scenario for our algorithm is that there exists some language  $L_{g_n}$  in the enumeration of  $\mathfrak{L}$  before  $L_z = K$  such Step 5 will be stuck on  $L_{g_n}$ . Then we can guarantee that  $|L_{g_n} \setminus K| < \infty$ . Since this set is finite, by removing parts of the enumeration of  $\mathfrak{X}$  of increasing

<sup>&</sup>lt;sup>5</sup>One can sample from this distribution in a computable fashion.

but finite size, we will eventually remove  $|L_{g_n} \setminus K|$ , and obtain a sampler that (i) is consistent and (ii) misses only finitely many elements from K.

8. **Output** a sample from the distribution  $G_n$ .

We will show that this algorithm generates with approximate breadth in the limit. Let *K* be the target language and  $z \in \mathbb{N}$  be the smallest number such that  $L_z = K$ . We consider two cases.

**Case A** (z = 1):  $S_n \subseteq L_1, \forall n \in \mathbb{N}$  and since the tell-tale set  $T^1$  of  $L_1$  is finite and the adversary presents a complete presentation of K, it holds that  $T_n^1 \subseteq S_n$  for sufficiently large n. Thus, in the limit, it holds that  $g_n = 1$ , thus supp $(G_n) = L_1 \setminus (S_n \cup \{x_1, \ldots, x_n\})$ , and the proof is concluded by noting that supp $(G_n) \subseteq K$  and  $|S_n \cup \{x_1, \ldots, x_n\}| < \infty$ , for all sufficiently large n.

**Case B** (z > 1): We now move on to the case z > 1. Then, for every language  $L_i$ ,  $1 \le i \le z - 1$ , that precedes  $L_z$ , exactly one of the following holds:

- (i) either there exists some  $x_{j_i} \in L_z$  but  $x_{j_i} \notin L_i$ , or
- (ii)  $L_z \subsetneq L_i$ .

If Case (i) holds, then there exists some  $n_i \in \mathbb{N}$  such that  $S_{n_i} \not\subseteq L_i$ . Thus, since there are finitely many languages before z for which Case (i) holds, after finitely many  $n \in \mathbb{N}$  all of them will have been contradicted by  $S_n$ . Thus, we consider some  $n_0 \in \mathbb{N}$  large enough so that for all  $n \ge n_0$  every language  $L_i$ ,  $1 \le i \le z - 1$ , for which  $S_n \subseteq L_i$  satisfies  $L_z \subsetneq L_i$ .

Let  $\mathfrak{I} = \{i_1, \ldots, i_\ell\}$  be the set of the indices for which the previous holds. For every  $j \in \mathfrak{I}$ , and for all  $j' \in \mathbb{N}$  for which the tell-tale set of  $L_j$  is a subset of  $L_{j'}$ , *i.e.*,  $T^j \subseteq L_{j'}$ , one of the following two cases hold by the definition of the weak Angluin's condition: (a) either  $L_{j'}$  is not a proper subset of  $L_j$  or (b)  $|L_j \setminus L_{j'}| < \infty$ .

Consider j' = z and any  $j \in \mathcal{I}$ . Since, by construction,  $L_z \subsetneq L_j$ , the previous argument shows that either **(I)**  $T^j \not\subseteq L_z$  or **(II)**  $|L_j \setminus L_z| < \infty$ .

If *j* falls into Case (**I**) then for large enough *n* it holds that  $T_n^j \not\subseteq L_z$ , thus  $T_n^j \not\subseteq S_n$ , and due to the way we have defined  $g_n, g_n \neq j$ .<sup>6</sup> Thus, we let  $\mathcal{I}'$  be the set of indices  $j \in \mathbb{N}, 1 \leq j \leq z - 1$ , such that  $T^j \subseteq L_z$  and  $L_z \subsetneq L_j$  and, hence, since we fall into Case (**II**) the previous argument implies that  $|L_j \setminus L_z| < \infty$  for each  $j \in \mathcal{I}'$ .

We consider again two cases: if  $\mathcal{I}' = \emptyset$ , then for large enough *n* it holds that  $g_n = z$ . Hence, the correctness follows from the previous arguments.

We now handle the more complicated case  $\mathfrak{I}' \neq \emptyset$ . Let  $j^*$  be the first element of  $\mathfrak{I}'$ . For large enough n, the choice of  $g_n$  will stabilize to  $j^*$ . To see this, notice that  $S_n \subseteq L_{j^*}$  for all  $n \in$  $\mathbb{N}$ ,  $T_n^{j^*} = T^{j^*}$  for sufficiently large n (since  $T^{j^*}$  is finite), and since  $T^{j^*} \subseteq L_z$  (and the adversary presents a complete presentation of  $L_z$ ), for large enough n it holds that  $T_n^{j^*} \subseteq S_n$ . Thus, indeed for all sufficiently large n it holds that  $g_n = j^*$ . By definition of  $\mathfrak{I}'$ , it holds that  $|L_{j^*} \setminus L_z| < \infty$ . Let  $x_{\ell_{j^*}}$  be the largest element of the enumeration of  $\mathfrak{X}$  for which  $x_{\ell_{j^*}} \in L_{j^*}$  but  $x_{\ell_{j^*}} \notin L_z$  (this always exists as  $j^* \in \mathfrak{I}'$  and, hence,  $L_z \subsetneq L_{j^*}$  and  $|L_{j^*} \setminus L_z| < \infty$ .). For  $n \ge \ell_{j^*}$  it holds that

<sup>&</sup>lt;sup>6</sup>Observe that if we had assumed the stronger Definition 2 (Angluin's condition), then this step implies that we can identify  $L_z$  in the limit, since only Case (I) is valid. This is exactly how the tell-tale-based algorithm of [Ang80] works.

 $L_{j^*} \setminus \{x_1, \ldots, x_n\} \subseteq L_z$ . This shows that, indeed,  $\operatorname{supp}(\mathcal{G}_n) \subseteq K$ , for large enough n, since we set  $\operatorname{supp}(\mathcal{G}_n) = L_{j^*} \setminus (S_n \cup \{x_1, \ldots, x_n\})$ . Moreover, since  $L_z \subsetneq L_{j^*}$ , and  $|\{x_1, \ldots, x_n\}| < \infty$ , it holds that  $|L_z \setminus (L_{j^*} \setminus \{x_1, \ldots, x_n\})| < \infty$ , for all  $n \in \mathbb{N}$ . Hence, the generator generates with approximate breadth from K in the limit.

*Remark* 4. The generating algorithm that achieves approximate breadth in the limit for languages that satisfy the weak version of Angluin's condition has the property that the Membership Oracle Problem is decidable. Hence, by the results of [KMV24], it cannot be stable, and, indeed, it is not since its support changes at each iteration.

Finally, we state the impossibility result for language generation with approximate breadth, which together with the previous algorithm imply Theorem 2.3.

**Lemma 2.7** (Impossibility for Generation with Approximate Breadth). Let  $\mathcal{L}$  be a countable collection of languages that does not satisfy *Definition 9*. Then, there is no generating algorithm that generates with approximate breadth from  $\mathcal{L}$  in the limit.

The proof of Lemma 2.7 follows using a similar construction as in the proof of Lemma 2.1. We prove the lower bound (Lemma 2.7) using a generalization of Lemma 2.1 which holds for any notion of breadth satisfying a certain uniqueness criterion. We defer this generalization and its implications to Section 4, and provide a sketch of the approach below.

*Proof Sketch of Lemma* 2.7. The proof idea for Lemma 2.7 is as follows. We perform the construction as in Section 2.1 (see *e.g.*, Figure 3). In contrast to the exact breadth case, we now use the contrapositive of the weak Angluin's condition. Concretely, the contrapositive to the weak Angluin's criterion implies that there exists a language  $L^* \in \mathcal{L}$  such that the following hold:

 $\forall$  finite  $T \subseteq L^*$ ,  $\exists L_T \in \mathcal{L}$ , such that  $T \subseteq L_T$ ,  $L_T \subsetneq L^*$ , and  $|L^* \setminus L_T| = \infty$ .

We will use this language  $L^*$  and proceed with the construction without change. At each phase *i*, we use some language  $L_{j_i}$  as in the proof of Lemma 2.1. There are two cases. First, the easy case is when the algorithm never generates with approximate breadth from  $L_{j_i}$ . Then we set  $K = L_{j_i}$  and we are done. Hence, assume that for infinitely many phases, the algorithm generates with approximate breadth from the corresponding languages. Then, we will set  $K = L^*$ . This is because (i) each language  $L_{j_i}$  misses infinitely many elements from  $L^*$  (by the contrapositive of the weak Angluin's condition) and (ii) there is a time step  $t_i$  where the generating algorithm generates with approximate breadth from  $L_{j_i}$  for all  $i \in \mathbb{N}$ . Combining (i) and (ii), we get that the algorithm infinitely often misses infinitely many elements from  $L^*$ , giving the desired lower bound.

### 2.3 Unconditional Characterization of Exhaustive Generation

Next we show that Definition 9 also characterizes exhaustive generation (Definition 6). This shows that exhaustive generation in the limit is indeed equivalent to generation with approximate breadth in the limit. In fact, we will show that if  $\mathcal{L}$  does not satisfy Definition 9 then generation under a notion of breadth weaker than Definition 6 is not possible. Moreover, we will show an algorithm that achieves the stronger variant of Definition 6 which requires zero hallucinations in

the enumeration after some finite *n* (instead of finitely many). In particular, this result shows that the two versions of exhaustive generation considered by [CP24a; CP24b] are characterized by the same condition.

**Theorem 2.8** (Characterization of Exhaustive Generation). For any countable collection of languages  $\mathcal{L}$ , one of the following holds.

1. If  $\mathcal{L}$  satisfies *Definition* 9, then there is a generator that generates exhaustively from  $\mathcal{L}$  in the limit.

2. If  $\mathcal{L}$  does not satisfy *Definition 9*, then no generator can generate exhaustively from  $\mathcal{L}$  in the limit.

As in the previous section, we will prove this result in two parts. The first part of the theorem follows immediately from a modification of the algorithm for generation with approximate breadth. We refer to the upcoming Lemma 2.9 for details. As before, we give two algorithms: the first one has access to certain additional oracles and the second one requires access to the tell-tale oracle in Definition 9. Subsequently, we prove the second part of the theorem in Lemma 2.11.

We first give a function that achieves exhaustive generation.

**Lemma 2.9** (Function for Exhaustive Generation). Let  $\mathcal{L}$  be a countable collection of languages that satisfies Definition 9. Then, there exists a generating algorithm that, given access to a membership oracle for  $\mathcal{L}$ , a subset oracle for  $\mathcal{L}$  (that given indices *i*, *j* outputs Yes if  $L_i \subseteq L_j$  and No otherwise) and a finite difference oracle for  $\mathcal{L}$  (that given indices *i*, *j* with  $L_i \subset L_j$  outputs Yes if  $|L_j \setminus L_i| < \infty$  and No otherwise), exhaustively generates from  $\mathcal{L}$  (and is consistent with the target language) in the limit.

The generation in the above result satisfies a property stronger than Definition 6:

*Remark* 5. In addition to achieving exhaustive generation, the generator is consistent with the target language and, hence, does not have *any* hallucinations.

The generator in Lemma 2.9 is as follows.

Fix the following: a special character  $x_0 \notin X$  and a canonical enumeration  $x_1, x_2, ...$  of X. Initialize  $\ell_0 = 0$ . for  $t \in \{1, 2, ...\}$  do:

- 1. Observe element  $x_t$  and let  $S_t$  be the set of all elements observed so far.
- 2. Construct a version space  $V_t$  consisting of all languages in  $\mathcal{L}_{\leq t}$  consistent with  $S_t$ , *i.e.*,

$$V_t := \left\{ L_j \colon 1 \le j \le t \,, \ L_j \supseteq S_t \right\} \,.$$

3. If  $V_t = \emptyset$ , **output** an arbitrary element of  $\mathfrak{X}$  and **go** to the next iteration.

# Define a language  $L_i \in V_t$  to be critical if  $L_i$  is the smallest-indexed language in  $V_t$  or  $L_i$  is a subset of all languages preceding it in  $V_t$ , i.e.,  $L_i \subseteq L_j$  for all  $1 \le j < i$ .

4. Construct the set  $C_t = \{L_{i_1^t} \supseteq L_{i_2^t} \supseteq \cdots \supseteq L_{i_t^t}\} \subseteq V_t$  of critical languages for some  $j \leq t$ .

# To construct the set of critical languages  $C_t$  the algorithm needs access to the subset oracle.

5. Find the smallest indexed language L = L(t) in  $C_t$  such that  $|L \setminus L_{i_j^t}| < \infty$ . Create the set  $C'_t$  by removing all the languages in  $C_t$  before L.

# To perform this filtering, the algorithm needs access to the finite difference oracle.

- 6. If  $C'_t = \emptyset$ , **output** an arbitrary element of  $\mathfrak{X}$  and **go** to the next iteration.
- 7. Let  $L_i = L_{i(t)}$  be the minimum indexed language in the set of filtered critical languages  $C'_t$ .
- 8. If  $i(t) \neq i(t-1)$ , set  $\ell_t = 0$ ; else  $\ell_t = \ell_{t-1} + 1$ .
- 9. **output** the enumeration of  $L_i \setminus \{x_0, \ldots, x_{\ell_t}\}$  induced by the canonical enumeration of  $\mathfrak{X}$  fixed at the start.

*Proof of Lemma 2.9.* We will show that the above function exhaustively generates and is consistent with the true language in the limit. Let *K* be the target language and  $z \in \mathbb{N}$  be the smallest number such that  $L_z = K$ . We will use the case analysis of Lemma 2.6. Fix some symbol  $x_0 \notin \mathcal{X}$ .

**Case A** (z = 1): Since z = 1, the true language is the first critical language and is never filtered from  $C'_t$ . Moreover, the counters  $\ell_t$  will never be reset (in Step 8) and, in fact, satisfy  $\ell_t = t$ . Hence, for each  $t \in \mathbb{N}$ , the algorithm  $\mathcal{G}_t$  enumerates the set  $K \setminus (S_t \cup \{x_0, \ldots, x_t\})$  induced by the canonical enumeration of  $\mathcal{X}$ . It follows that, for each removed  $x_i$ , there is some t where it is the first element of the output enumeration. Further, the output enumeration is always consistent with K. Hence, the resulting generator exhaustively generates K. In fact, it has the stronger property that it never hallucinates.

**Case B** (z > 1): Consider the languages before  $L_z$  in the enumeration of  $\mathcal{L}$ . There are two cases: For any *i* < *z*, either there exists an element that belongs to  $L_z$  but not  $L_i$  or  $L_z \subseteq L_i$ . If the first case holds, then eventually the distinguishing element will appear in the enumeration of K and make L<sub>i</sub> inconsistent. Hence, let us assume that for all i < z, we only care about indices i for which  $L_i \supseteq L_z$ . We claim that eventually the index of Step 5 stabilizes in the limit. In particular, we will show that it stabilizes to the smallest index  $i^*$  such that  $L_{i^*} \supseteq L_z$  and  $|L_{i^*} \setminus L_z| < \infty$ ; note that if there is no language  $L_i \supseteq L_z$ , then  $i^*$  must be z. Before proving this claim, we show that it implies the result. Let  $1 \le i^* \le z$  be the index that Step 5 eventually stabilizes on. We know that  $L_{i^*} \supseteq K$  (by our earlier argument that any index  $1 \le i \le z$  not satisfying this property is eliminated after a finite time) and  $|L_{i^*} \setminus K| < \infty$  (by construction). We now show how to exhaustively generate K in the limit, this corresponds to Steps 8 and 9 of the above function. To see this, observe that as  $|L_{i^*} \setminus K| < 1$  $\infty$ , after a finite number of steps  $L_{i^*} \setminus \{x_0, \ldots, x_{\ell_t}\} \subseteq K$  (and, hence, the algorithm eventually stops hallucinating). Further, since at step t (for large enough t), we output the enumeration of  $L_{i^*} \setminus \{x_0, \ldots, x_{\ell_t}\}$  induced by the canonical enumeration of  $\mathcal{X}$ , it follows, for each removed  $x_i$ , there is some t where it is the first element of the output enumeration. Hence, the resulting generator exhaustively generates K. In fact, it has the stronger property that it eventually stops making any hallucinations.

*Proof of the claim.* It remains to prove our claim that the index of Step 5 stabilizes in the limit. Since  $\mathcal{L}$  satisfies the weak Angluin's condition, then *K* has a finite tell-tale set  $T_K$ . We condition on

the following events: (A) *K* is a critical language, and (B)  $S_t \supset T_K$ . Condition (A) is satisfied for any  $t \ge z$  and (B) is satisfied after a finite time since  $T_K$  is finite and all its elements appear at a finite point in the enumeration of *K*. Conditioned on these events the critical list  $C_t$  is of the form

$$L_{i_1^t} \supseteq L_{i_2^t} \supseteq \cdots \supseteq K \supseteq L_{j_1^t} \supseteq \cdots$$

First, observe that there are finitely many languages before *K* in this list: this is because *K* appears at a finite point in this list. Next, we claim that conditioned on the above events the indices  $i_1^t, i_2^t, \ldots$  of the languages appearing *before K* in the list never change. The proof is via induction.

- *Base Case:* First, consider the first index  $i_1^t$ . It is defined as the smallest index language consistent with  $S_t$ . Moreover, due to the structure above it has the property that  $L_{i_1^t} \supseteq K$  and, hence, it never becomes inconsistent with  $S_{t'}$  for  $t' \ge t$ . Therefore, the index  $i_1^t$  never changes in subsequent steps.
- *Induction Step:* Next, we complete the induction argument, suppose indices  $i_1^t, i_2^t, \ldots, i_r^t$  never change in subsequent steps, then we claim that the index  $i_{r+1}^t$  (if it exists) also never changes in subsequent steps. This is because  $i_{r+1}^t$  is defined as the smallest indexed language that is (1) consistent with  $S_t$  and (2) has the property that  $L_{i_{r+1}^t} \subseteq L_{i_r^t}$ . The former always holds for all subsequent  $t' \ge t$  since  $L_{i_{r+1}^t} \supseteq S_t \supseteq T_K$  and the latter holds for all subsequent  $t' \ge t$  since  $i_r^t$  never changes.

Now we are ready to prove that the index i(t) selected in Step 5 stabilizes. Recall that i(t) is the smallest index satisfying that (1)  $L_{i(t)}$  appears before K in the critical list and (2)  $|L_{i(t)} \setminus L_{i_j^t}| = |L_{i(t)} \setminus K| + |K \setminus L_{i_j^t}| < \infty$ . Observe that  $|L_{i(t)} \setminus L_{i_j^t}| = |L_{i(t)} \setminus K| + |K \setminus L_{i_j^t}|$  and, by construction,  $|K \setminus L_{i_j^t}| < \infty$  and, therefore, Condition (2) is equivalent to  $|L_{i(t)} \setminus K| < \infty$ . Fix any t satisfying Conditions A and B above and the corresponding i(t). For all subsequent  $t' \ge t$ ,  $L_{i(t)}$  continues to appear before K in the critical list since we proved that all indices before K in the critical list stabilize. Further,  $|L_{i(t)} \setminus K| < \infty$  since it is independent of t'. Therefore, i(t) = i(t') since i(t) satisfies both properties that determine i(t'). It follows that for  $t' \ge t$ , the index selected in Step 5 never changes.

Moreover, a small adaptation of the proof of Lemma 2.6 gives a generator that generates exhaustively (Definition 6) in the limit provided one has access to the tell-tale oracle from Definition 9.

**Lemma 2.10** (Algorithm for Exhaustive Generation). Let  $\mathcal{L}$  be a countable collection of languages that satisfies Definition 9. Then, there exists a generating algorithm that, given access to a membership oracle for  $\mathcal{L}$  and the tell-tale oracle from Definition 9, exhaustively generates from  $\mathcal{L}$  in the limit.

*Proof of Lemma* 2.10. The argument in the proof of Lemma 2.6 shows that the choice of the index  $g_n$  stabilizes in the limit. Moreover,  $K \subseteq L_{g_n}$  and  $|L_{g_n} \setminus K| < \infty$ . To achieve exhaustive generation, the only modification needed is that we keep track of another index  $\ell_n$  which is initialized at 0, increases by 1 in every round, and every time the choice of  $g_n$  changes, we reset  $\ell_n = 0$ . The enumeration we output is  $L_{g_n} \setminus \{x_0, \ldots, x_{\ell_n}\}$ , where we use the notational convention that  $x_0$  is some special element that does not appear in  $\mathcal{X}$ . Moreover, the sequence in which the element appears in the enumeration is the natural order induced by (some canonical) enumeration of  $\mathcal{X}$ .

Assume that *n* is large enough so that  $g_n$  has stabilized. It is easy to see two things: for every element  $\hat{x}$  of  $L_{g_n}$ , there exists some finite round  $\hat{n} \in \mathbb{N}$  such that  $\hat{x}$  is the first element in the enumeration we have outputted. Moreover, since  $L_z \subseteq L_{g_n}$  and  $|L_{g_n} \setminus L_z| < \infty$ , after some finite  $n \in \mathbb{N}$  it holds that  $L_{g_n} \setminus \{x_0, \ldots, x_{\ell_n}\} \subseteq L_z$ . Moreover, every time an element  $x_i$  is omitted from the enumeration we output, there has been some prior iteration where it has been the first element in the enumeration. These arguments show that the modified generator is an exhaustive generator for  $\mathcal{L}$ .

Finally, we state the impossibility result for exhaustive language generation, which together with the previous algorithm imply Theorem 2.8.

**Lemma 2.11** (Impossibility for Exhaustive Generation). Let  $\mathcal{L}$  be a countable collection of languages that does not satisfy *Definition 9*. Then, there is no generating algorithm that exhaustively generates from  $\mathcal{L}$  in the limit.

The second part of the theorem follows by building on the construction in the proof of Lemma 2.1. We defer the formal proof to Appendix B. Also see Section 5, where we show that the construction in the proof of Lemma 2.1 implies an impossibility result for any notion of breath satisfying certain criterion (Definition 17) and that exhaustive generation satisfies this criterion (Observation 5.2).

### 2.4 Unconditional Characterization of Unambiguous Generation

In this section, we characterize the language collections for which unambiguous generation in the limit is possible. We start with an impossibility result. Lemma 2.12 is a stronger version of a result by [KMV24] for unambiguous generation in the limit which showed that generators that are stable and for which the MOP is decidable cannot generate unambiguously. In contrast, our result below holds for *all* generators.

**Lemma 2.12** (Impossibility for Unambiguous Generation). Let  $\mathcal{L}$  be a countable collection of languages that is not identifiable in the limit. Then, no algorithm can unambiguously generate from  $\mathcal{L}$  in the limit.

The proof of Lemma 2.12 follows from the construction in the proof of Lemma 2.1. We defer the formal proof to Appendix A. Also see Section 4, where we show that the construction in the proof of Lemma 2.1 implies an impossibility result for any notion of breath satisfying a uniqueness criterion (Definition 15) and that unambiguous generation satisfies this uniqueness criterion (Observation 4.2). To be more precise, the uniqueness criterion for unambiguous generation is that, roughly speaking, for any pair  $L \neq L'$  of different languages in the class  $\mathcal{L}$ , any algorithm that unambiguously generates from L, *cannot* unambiguously generate from L' at the same time. This is immediate from the definition of unambiguous generation since the algorithm should be strictly closer (in symmetric difference) to L than any other  $L' \in \mathcal{L}$ . Hence, in Section 4, we show how to obtain unconditional lower bounds for all notions of breadth that satisfy such uniqueness criteria, and, as an application, we prove Lemma 2.12. Complementing Lemma 2.12, if  $\mathcal{L}$  is identifiable in the limit, then [KMV24] shows that consistent generation with breadth is possible in the limit and, hence, unambiguous generation is also possible in the limit. Hence, we get the following result which completely characterizes generation with breadth in the Gold-Angluin model. **Theorem 2.13** (Characterization of Unambiguous Generation). *For any countable collection of languages*  $\mathcal{L}$ *, one of the following holds.* 

- 1. If  $\mathcal{L}$  satisfies *Definition 2*, then there is a generator that unambiguously generates from  $\mathcal{L}$  in the *limit*.
- 2. If  $\mathcal{L}$  does not satisfy Definition 2, then no generator can unambiguously generate from  $\mathcal{L}$  in the limit.

#### 2.5 Implications to Language Generation in the Statistical Setting

Our results have direct implications to the statistical setting that [KMV24] considered. In this setting, there is a countable language collection  $\mathcal{L}$ , a "valid" distribution  $\mathcal{P}$  supported on a language  $K \in \mathcal{L}$ , and the generating algorithm takes as input string drawn i.i.d. from  $\mathcal{P}$ . For every different notion of breadth considered in Section 1.3, one can define an error function for the generating algorithm ( $\mathcal{G}_n$ ) $_{n \in \mathbb{N}}$  as

$$\operatorname{er}(\mathcal{G}_n) = \mathbb{1}\{\neg P(\mathcal{G}_n)\}, \qquad (6)$$

where  $P(\cdot)$  is a predicate defined based on the underlying notion of breadth and its value is True if the breadth property is achieved by  $G_n$  and False, otherwise.

Given this definition (6), [KMV24] define the error rate for generation with breadth via the universal rates framework of Bousquet, Hanneke, Moran, van Handel, and Yehudayoff [BHMvY21].

**Definition 10** (Error Rate [BHMvY21]). Let  $\mathcal{L}$  be a countable collection of languages, er be an error function defined in Equation (6), and  $R : \mathbb{N} \to [0, 1]$  be a rate function such that  $\lim_{n\to\infty} R(n) = 0$ . We say that rate  $R(\cdot)$  is achievable for  $\mathcal{L}$  if there exists a generating algorithm  $G = (G_n)$  such that

$$\forall \mathcal{P} \in \operatorname{Val}(\mathcal{L}) \exists C, c > 0 \quad such that \quad \mathbb{E}\left[\operatorname{er}(\mathcal{G}_n)\right] \leq C \cdot R(c \cdot n) \quad \forall n \in \mathbb{N},$$

where  $\operatorname{Val}(\mathcal{L})$  the set of all valid distributions with respect to  $\mathcal{L}$ . Conversely, we say that no rate faster than  $R(\cdot)$  is achievable for  $\mathcal{L}$  if for any generating algorithm  $\mathcal{G} = (\mathcal{G}_n)$  there exists a valid distribution  $\mathcal{P}$  and  $c, \mathcal{C} > 0$  such that  $\mathbb{E}[\operatorname{er}(\mathcal{G}_n)] \geq C \cdot R(c \cdot n)$ , for infinitely many  $n \in \mathbb{N}$ . We say that no rate is achievable for  $\mathcal{L}$  if for any generating algorithm  $\mathcal{G} = (\mathcal{G}_n)$  there exists a valid distribution  $\mathcal{P}$  such that  $\limsup_{n\to\infty} \mathbb{E}[\operatorname{er}(\mathcal{G}_n)] > 0.$ 

[KMV24] proved bounds in this statistical setting for language identification, generation with exact breadth for algorithms for which the MOP is decidable,<sup>7</sup> and generation with approximate breadth for algorithms that are stable in the limit,<sup>8</sup> and for which the MOP is decidable. To get these results, [KMV24] showed connections between the online setting considered in the previous sections and the statistical setting. Using the new results in this work, and the results of [KMV24], we can get characterizations for the statistical rates under these two notions of breadth removing the requirement for decidability of the MOP oracle and stability of the generating algorithm.

<sup>&</sup>lt;sup>7</sup>Recall this is a mild technical condition that requires that the generating algorithm can answer queries about whether a string x is in its support.

<sup>&</sup>lt;sup>8</sup>Roughly speaking, stability means that after finitely many steps, the support of the distribution outputted by the generating algorithm does not change. For the formal definition, see Definition 11.

**Theorem 2.14** (Rates for Generation with Exact Breadth). For any non-trivial collection of languages  $\mathcal{L}$  no rate faster than  $e^{-n}$  is achievable for generation with exact breadth. Moreover, For any collection that is identifiable in the limit, there exists an algorithm that achieves generation with exact breadth at rate  $e^{-n}$ . Conversely, for any non-identifiable collection, no rate is achievable for generation with exact breadth.

For the non-triviality requirement, we refer the interested reader to [KMV24]. The  $e^{-n}$  lower bound and upper bound follow immediately from their results. The lower bound for no rates achievable follows from the approach of [KMV24] (with a few modifications in their construction) and Lemma 2.1. For brevity, we only sketch the modifications here:

- [KMV24] make use of a construction of [Ang88] which connects the adversarial setting "inthe-limit" to the statistical setting "in-the-limit" (Theorem 5.6 in their paper) for language identification. A similar result can be shown for generation with exact breadth.
- [KMV24] make use of majority votes over learners that identify the target language. In Lemma 5.8 they use the voting scheme, (a modification of) Angluin's result [Ang88], and the Borel-Cantelli lemma to show that no rate is achievable for language identification, for collections that do not satisfy Angluin's criterion (Definition 2). The same approach can be used to derive the lower bound for generation with exact breadth, by using a slightly different majority voting scheme. At a very high level, following [KMV24]<sup>9</sup> we split the dataset into different batches and train the generating algorithm, and we can show that for large enough *n*, a *c*-fraction of these generators satisfies the generation with exact breadth property (for, *e.g.*, *c* > 2/3). In order to combine their outputs, we define an (implicit) distribution as follows: we keep sampling from all the batches until a *c*-fraction of them outputs the same element. It is not hard to see that (i) this process terminates in finite time, <sup>10</sup> (ii) only elements of *K* have positive probability of being outputted.

A similar result can be obtained for language generation with approximate breadth, using the criterion from Definition 9.

**Theorem 2.15** (Rates for Generation with Approximate Breadth). For any non-trivial collection of languages  $\mathcal{L}$  no rate faster than  $e^{-n}$  is achievable for generation with approximate breadth. For any collection that satisfies Definition 9, there exists an algorithm that achieves generation with approximate breadth at rate  $e^{-n}$ . Conversely, for any collection that does not Definition 9, no rate is achievable for generation with exact breadth.

The above pair of results provides statistical rates for language generation with exact and approximate breadth. Obtaining statistical rates for unambiguous generation is an interesting direction.

## 3 The Role of Stability in Language Identification and Generation

In his original work, Gold [Gol67] defined language identification in the limit by requiring that the guess of the learner stabilizes to some index  $i \in \mathbb{N}$  that corresponds to an occurrence of the

<sup>&</sup>lt;sup>9</sup>The same approach has been used extensively in the universal rates literature, starting from [BHMvY21].

<sup>&</sup>lt;sup>10</sup>One small complication is that if a *c*-fraction does not satisfy the desired property, the algorithm might not terminate. To fix that, in every step we either terminate with probability 1/2 or we do the sampling strategy we described with probability 1/2. If we terminate, we run the algorithm from [KM24] to generate a valid string from *K*.

target language. Interestingly, we can show that stability for language identification in the limit comes without loss of generality: if there exists an algorithm that, in the limit, oscillates between different guesses of the target language, then it can be converted to an algorithm which, in the limit, identifies the same index of the target language (see, *e.g.*, Lemma 5.4 from [KMV24]).<sup>11</sup> Hence, it is natural to ask whether generation in the limit can be achieved using stable generating algorithms.

**Definition 11** (Stable Generating Algorithm [KMV24]). A generating algorithm  $(G_n)$  is stable for a language collection  $\mathcal{L}$  if for any target language  $K \in \mathcal{L}$  and for any enumeration of K, there is some finite  $n^* \in \mathbb{N}$  such that for all  $n, n' \ge n^*$ , it holds that  $\operatorname{supp}(G_n) = \operatorname{supp}(G_{n'})$ .

[KMV24] showed that for every collection  $\mathcal{L}$  that does not satisfy Angluin's condition (Definition 2), no generating algorithm that (1) is stable and (2) for which the MOP (Definition 19) is decidable, can achieve generation with approximate breadth in the limit (Theorem C.1 in [KMV24]). Recall that Theorem 2.3 shows that, for every collection that satisfies the weak Angluin's condition (Definition 9), there exists an (unstable) generating algorithm for which the MOP is decidable and achieves approximate breadth in the limit. Moreover, Definition 9 is strictly weaker than Definition 2. Thus, these results already show that the stability requirement makes the problem of generation with approximate breadth strictly more challenging, for all natural algorithms for which the MOP is decidable.

## 3.1 Characterization of Stable Generation With Approximate Breadth

In this section, we characterize stable generation with approximate breadth. The main result is that achieving generation with approximate breadth becomes significantly harder if one insists on having a stable generator. Recall that if one does not require the generator to be stable, then generation with approximate breadth is possible if and only if the language collection satisfies the weak Angluin's condition (Definition 9). The main result of this section states that if stability is required, then generation with approximate breadth is possible if and only if the language collection is identifiable (*i.e.*, if and only if it satisfies Angluin's condition; Definition 2), a much stronger criterion compared to the weak Angluin's condition.

**Theorem 3.1** (Characterization of Generation With Approximate Breadth For Stable Generators). *For any countable collection of languages*  $\mathcal{L}$ *, one of the following holds.* 

- 1. If  $\mathcal{L}$  satisfies *Definition 2*, then there is a stable generator that generates with approximate breadth from  $\mathcal{L}$  in the limit.
- 2. If  $\mathcal{L}$  does not satisfy *Definition* 2, then no stable generator can generate with approximate breadth from  $\mathcal{L}$  in the limit.

To get this result, we first need to give a lower bound for language collections that do not satisfy Angluin's condition. This is provided in the next lemma.

**Lemma 3.2.** Let  $\mathcal{L}$  be a countable collection of languages that is not identifiable in the limit. Then, no stable algorithm can generate from  $\mathcal{L}$  with approximate breadth in the limit.

<sup>&</sup>lt;sup>11</sup>We suspect that this result was known in prior work, but we could not find a better reference for it.

Now, the characterization of Theorem 3.1 follows since if  $\mathcal{L}$  is identifiable in the limit, then there is a generator that generates with exact breadth from  $\mathcal{L}$  in the limit and, hence, by definition, also generates with approximate breadth from  $\mathcal{L}$  in the limit and is stable.

In the rest of the section, we give an overview of the proof of Lemma 3.2. The proof of Lemma 3.2 uses a construction very similar to the construction in the proof of Lemma 2.1. We defer the complete construction to Appendix C and just present the implication of the construction which is sufficient to prove Lemma 3.2.

**Lemma 3.3.** Let  $\mathcal{L}$  be a countable collection of languages that is not identifiable in the limit. Let  $\mathcal{G} = (\mathcal{G}_n)$  be a stable generating algorithm. If  $\mathcal{G}$  generates with approximate breadth from  $\mathcal{L}$  in the limit, then there is a language  $L^* \in \mathcal{L}$ , an enumeration  $E^*$  of  $L^*$ , a sequence of distinct languages  $L_1, L_2, \dots \in \mathcal{L}$ , and a strictly increasing sequence  $t(1), t(2), \dots \in \mathbb{N}$ , such that the following holds.

- For each  $i \in \mathbb{N}$ ,  $L_{t(i)}$  is a proper subset of  $L^*$ , i.e.,  $L_{t(i)} \subsetneq L^*$ ; and
- Given strings from  $E^*$  as input, for each  $i \in \mathbb{N}$ ,  $\mathcal{G}_{t(i)}$  generates with approximate breadth from  $L_{t(i)}$ .

Recall that in the proof of Lemma 2.1 we (implicitly) showed the same result except the notion of breadth was "(exact) breadth" instead of "approximate breadth." To gain some intuition, note that in the case of exact breadth, the above result already gives us a contradiction to the fact that G generates with exact breadth from  $\mathcal{L}$  in the limit: indeed,  $t_1, t_2, \ldots$  gives us infinitely many points at which G generates with breadth from a language different from  $L^*$  and, hence, by definition, does *not* generate with breadth from  $L^*$ . This contradiction must imply that no stable generator can generate with breadth from any non-identifiable collection.

The contradiction with approximate breadth is less clear since, for a fixed *i*, generator  $G_{t(i)}$  can generate with approximate breadth from both  $L_{t(i)}$  and  $L^*$ . Indeed, if the generator is unstable (*i.e.*, it can change its support infinitely often), then there is no contradiction – and generation with approximate breadth is possible for certain non-identifiable collections (Theorem 2.3). Hence, to obtain a contradiction, we need to leverage the stability of the generator.

*Proof of Lemma 3.2.* Consider the construction in Lemma 3.3. Let  $K = L^*$  and suppose that the adversary follows the enumeration  $E^*$ .

Let  $C_B, C_S \colon \mathbb{N} \to \mathbb{N}$  be two counters: for each t,  $C_B(t)$  counts the number of values  $1 \le i \le t$ for which  $\mathcal{G}_i$  does *not* generate with approximate breadth from  $L^*$  and  $C_S(t)$  counts the number of values  $2 \le i \le t$  for which  $\operatorname{supp}(\mathcal{G}_i) \ne \operatorname{supp}(\mathcal{G}_{i-1})$ . In other words,  $C_B(t)$  is the number of times  $\mathcal{G}$  does not generate with approximate breadth from  $L^*$  in the first *t*-steps and  $C_S(t)$  is the number of times  $\mathcal{G}$  changes its support in the first *t*-steps.

Toward a contradiction suppose that G is stable and generates with approximate breadth from K in the limit (when given the enumeration  $E^*$ ). This, by definition, implies that

$$\lim_{t \to \infty} C_B(t) < \infty \quad \text{and} \quad \lim_{t \to \infty} C_S(t) < \infty.$$
(7)

The former implies that there are only finitely many values of  $i \in \mathbb{N}$  such that  $\mathcal{G}_{t(i)}$  does not generate with approximate breadth from  $L_{t(i)}$  (where t(i) and  $L_{t(i)}$  are from Lemma 3.3). In other words, there are infinitely many values, say,  $\tau(1) < \tau(2) < \cdots \in \mathbb{N}$ , such that, for each i,  $\mathcal{G}_{\tau(i)}$ generates with approximate breadth from  $L^*$ . Moreover, Lemma 3.3 says that, for each  $i \in \mathbb{N}$ ,  $\mathcal{G}_{\tau(i)}$  generates with approximate breadth from  $L_{\tau(i)}$ . Since  $\mathcal{G}_{\tau(i)}$  generates with approximate breadth from both  $L^*$  and  $L_{\tau(i)}$  and  $L_{\tau(i)} \subsetneq L^*$ , it follows that: for each  $i \in \mathbb{N}$ ,

$$L_{\tau(i)} = \operatorname{supp}(\mathcal{G}_{\tau(i)}) \cup R \quad \text{where} \quad R \subseteq L^* \setminus \operatorname{supp}(\mathcal{G}_{\tau(i)}).$$
(8)

Fix any *i*. Let

$$\mathfrak{s}(i) \coloneqq \left| L^* \setminus \operatorname{supp}(\mathcal{G}_{\tau(i)}) \right| \, .$$

Since  $\mathcal{G}_{\tau(i)}$  generates with approximate breadth from  $L^*$ ,  $s(i) < \infty$ . We claim that

$$\operatorname{supp}(\mathcal{G}_{\tau(i)}) \neq \operatorname{supp}(\mathcal{G}_{\tau(i+j)}) \quad \text{for some} \quad 1 \le j \le S(i) \coloneqq 2^{S(i)} + 1.$$
(9)

**Proof of Equation (9).** To see this, toward a contradiction, suppose that

$$\operatorname{supp}(\mathcal{G}_{\tau(i)}) = \operatorname{supp}(\mathcal{G}_{\tau(i+1)}) = \cdots = \operatorname{supp}(\mathcal{G}_{\tau_{i+S(i)}}).$$

This combined with Equation (8) implies that, for each  $1 \le j \le S(i)$ ,  $L_{\tau(i+j)} = \operatorname{supp}(\mathcal{G}_{\tau(i)}) \cup R_j$ for some finite set  $R_j \subseteq L^* \setminus \operatorname{supp}(\mathcal{G}_{\tau(i)})$ . Since all of  $L_1, L_2, \ldots$  are different, it must hold that all of  $R_1, R_2, \ldots, R_{S(i)}$  are different. This is a contradiction since each  $R_i$  is a subset of  $R_i \subseteq L^* \setminus$  $\operatorname{supp}(\mathcal{G}_{\tau(i)})$  and there are only  $S(i) - 1 = 2^{s(i)}$  such subsets.

**Completing the Proof of Lemma 3.2.** Equation (9) shows that, for each  $i \in \mathbb{N}$ , starting from the  $\tau(i)$ -th step, the support of the generator changes after finitely many steps. Since  $\tau_1, \tau_2, \ldots, \in \mathbb{N}$  is a strictly increasing and infinite sequence, this implies that the support of the generator changes infinitely often as it is provided more and more examples and, hence,  $\lim_{t\to\infty} C_S(t) = \infty$  which contradicts the fact that  $\mathcal{G}$  is stable (7). Hence, our assumption that  $\mathcal{G}$  is stable and generates with approximate breadth from  $\mathcal{L}$  in the limit must be false. Therefore, no stable generator can generate with approximate breadth from any non-identifiable collection.

#### 3.2 A Collection for Which No Stable Generator Has Infinite Coverage

The next result shows that there is a language collection  $\mathcal{L}$  for which there exists an algorithm that achieves approximate breadth in the limit, but no stable algorithm can achieve the (strictly) weaker notion of generating with infinite coverage in the limit. The collection  $\mathcal{L}$  is due to [CP24a], who observed that a trivial generating algorithm that does not get *any* input generates from  $\mathcal{L}$  exhaustively in the limit. Since exhaustive generation implies, by definition, generation with approximate breadth, we only need to prove the impossibility result for generation with infinite coverage by stable generators.

We first provide the collection and then state the result.

**Example 1** ([CP24a]). Let  $\mathcal{X} = \mathbb{N}$ ,  $L_{\infty} = \mathbb{N}$ , for every  $i \in \mathbb{N}$  let  $L_i = \mathbb{N} \setminus \{i\}$ , and let  $\mathcal{L} = \{L_{\infty}, L_1, L_2, \ldots\}$ . Notice that every pair of languages  $L_i, L_j \in \mathcal{L}$  differ in at most two elements, so it follows that  $\mathcal{L}$  satisfies Definition 9. To see that it does not satisfy Angluin's condition (Definition 2), consider the language  $L_{\infty}$ . Then, for every finite subset  $T \subseteq L_{\infty}$  there is some language  $L_T$  such that  $T \subseteq L_T$  and  $L_T \subsetneq L_{\infty}$ .

We continue with the statement of the theorem.

**Theorem 3.4.** There exists a countable collection of languages  $\mathcal{L}$  that satisfies the weak Angluin's condition (*Definition 9*), and for which no stable generating algorithm can achieve generation with infinite coverage in the limit (*Definition 8*).

*Proof.* Consider the collection defined in Example 1. Since it satisfies the weak Angluin's condition (Definition 9), by Theorem 2.3, it follows that there exists an algorithm that achieves generation with approximate breadth in the limit.<sup>12</sup> Assume towards contradiction that there exists a stable generating algorithm  $\mathcal{G} = (\mathcal{G}_n)_{n \in \mathbb{N}}$  that achieves generation with infinite coverage in the limit. We will pick a target language and an enumeration of it that witnesses the lower bound based on the given algorithm  $\mathcal{G}$ . We denote the target language by K and the target enumeration by  $E_K^{\infty}$ . Like in the previous proofs, for any enumeration E, we use the notation E(i) to denote its *i*-th element, E(1:i) to denote its first *i* elements, and  $E(i:\infty)$  to denote all but the first i - 1 elements.

As in the previous proofs of the impossibility results, we consider several phases for our construction. First, we start with the enumeration  $E_{\mathbb{N}}^{\infty} = (1, 2, 3, ...)$ . Notice that this is a valid enumeration for  $L_{\infty}$ . We consider two cases: (I) either there is some  $n \in \mathbb{N}$  such that  $|\operatorname{supp}(\mathcal{G}_n)| = \infty$ , or (II) if there is no such n the lower bound follows immediately by picking  $K = \mathbb{N}$  and the hard enumeration  $E_K^{\infty} = E_{\mathbb{N}}^{\infty}$ . For the continuation of the proof, assume that the former case holds and let  $n_1$  denote the first timestep for which this holds. Notice that up to that point we have enumerated  $(1, \ldots, n_1)$ . Let  $\hat{n}_1 \in \mathbb{N}$  be the smallest number strictly greater than  $n_1$  that is in the support of  $\mathcal{G}_{n_1}$ . Notice that such a number must exist because  $|\operatorname{supp}(\mathcal{G}_{n_1})| = \infty$ .

We now extend the target enumeration  $E_K^{\infty}(1:\hat{n}_1-1) = (1,2,\ldots,\hat{n}_1-1)$ . Notice that this is well-defined since we only add elements to the already constructed enumeration. We continue building the target enumeration by skipping the element  $\hat{n}_1$  and including the element  $\hat{n}_1 + 1$  to it, *i.e.*, the  $\hat{n}_1$ -th element of the constructed enumeration is  $\hat{n}_1 + 1$ . We continue adding consecutive elements to the enumeration  $E_K^{\infty}$  until the first timestep  $n > \hat{n}_1 + 1$  such that  $\operatorname{supp}(\mathcal{G}_n) \neq \operatorname{supp}(\mathcal{G}_{n_1})$ and  $|\operatorname{supp}(\mathcal{G}_n)| = \infty$ . Notice that if no such n exists the lower bound already follows by picking the target language  $K = L_{\hat{n}_1}$  and the constructed target enumeration. This is because in every timestep either  $\operatorname{supp}(\mathcal{G}_n) = \operatorname{supp}(\mathcal{G}_{n_1})$  (and therefore  $\operatorname{supp}(\mathcal{G}_n) \not\subseteq K$  because  $\hat{n}_1 \in \operatorname{supp}(\mathcal{G}_n)$ ) or  $|\operatorname{supp}(\mathcal{G}_n)| < \infty$ , hence the algorithm does not achieve generation with infinite coverage in the limit. For the continuation of the proof, let  $n_2$  denote the first timestep for which  $\operatorname{supp}(\mathcal{G}_{n_2}) \neq$  $\operatorname{supp}(\mathcal{G}_{n_1})$  and  $|\operatorname{supp}(\mathcal{G}_{n_2})| = \infty$ . We then add the element  $\hat{n}_1$  to the constructed prefix of the enumeration  $E_K^{\infty}$  and terminate the first phase.

Notice that at the end of the first phase we have enumerated all the elements  $\{1, 2, ..., n_2 - 1\}$  and the support of the generating algorithm has changed at least once or we have the desired lower bound. We continue inductively in exactly the same way until **(I)** either some phase cannot be terminated in which case the lower bound follows because the property of infinite coverage in the limit is not achieved or **(II)** we construct infinitely many phases which witness infinitely many changes in the support of the generating algorithm, hence showing it cannot be stable. This concludes the proof.

<sup>&</sup>lt;sup>12</sup>As we explained, this also follows from the work of [CP24a].

#### 3.3 Sufficient Condition for Stable Generation With Infinite Coverage

In this section, we provide a sufficient condition on the language collection  $\mathcal{L}$  that guarantees the existence of a stable generating algorithm that generates with infinite coverage in the limit. In particular, we can show that if a collection has finite closure dimension [LRT24], then there exists a stable generating algorithm that achieves infinite coverage in the limit. First, we give the definition of the closure dimension [LRT24], which is inspired by a result of [KM24] on *uniform generation*<sup>13</sup> from finite sets of languages.

**Definition 12** (Closure Dimension [LRT24]). *The closure dimension of*  $\mathcal{L}$ *, denoted by*  $d(\mathcal{L})$ *, is the largest natural number*  $\ell \in \mathbb{N}$  *for which there exist distinct*  $x_1, \ldots, x_\ell \in \mathfrak{X}$  *such that* 

ı.

$$V(x_1,\ldots,x_\ell) := \{L \in \mathcal{L} : \{x_1,\ldots,x_\ell\} \subseteq L\} \neq \emptyset \quad and \quad \left| \bigcap_{L \in V(x_1,\ldots,x_\ell)} L \right| < \infty.$$

*If for every*  $\ell \in \mathbb{N}$  *there exists a set of distinct elements that satisfies this condition we say that*  $d(\mathcal{L}) = \infty$ *.* 

In general the closure dimension can be  $\infty$ , but due to a result of [KM24], we know that all collections of languages with finitely many languages have finite closure dimension. In order to design an algorithm that achieves stable infinite coverage for any collection  $\mathcal{L}$  that has a finite closure dimension, we will make use of a stronger oracle for  $\mathcal{L}$  than just the membership oracle to it. Namely, we define the *version space intersection* (VSI) membership oracle as follows.

**Definition 13** (Membership Oracle to Version Space Intersection (VSI)). *The membership oracle to VSI is a primitive that, given a set of distinct elements*  $x_1, ..., x_n \in X$  *and a target element*  $x \in X$ *, returns* 

$$\mathbb{1}\left\{x\in\cap_{L\in V(x_1,\ldots,x_n)}L\right\}\,.$$

We remark that for finite collections  $\mathcal{L}$  this oracle can be computed just with membership oracle to  $\mathcal{L}$ , but for countable collections this oracle might not be computable.

**Proposition 3.5** (Adaptation of Lemma 3.2 in [LRT24]). Let  $\mathcal{L}$  be a collection of languages with  $d(\mathcal{L}) < \infty$  (*Definition 12*). There exists a stable (*Definition 11*) generating algorithm  $\mathcal{G} = (\mathcal{G}_n)$  for  $\mathcal{L}$  that, given the value of  $d(\mathcal{L})$ , achieves infinite coverage (*Definition 8*) using access to a VSI membership oracle for  $\mathcal{L}$ , after taking as input  $d(\mathcal{L}) + 1$  distinct elements.

In particular, since the closure dimension of any finite collection of languages is finite [KM24], for any finite collection of languages, there exists a stable generating algorithm that achieves infinite coverage. It is not hard to see that for such collections, the VSI oracle can be implemented using only membership oracle to languages in  $\mathcal{L}$ .

**Corollary 3.6** (Stable Generation for Finite Collections). *For every finite collection of languages*  $\mathcal{L}$ *, the following hold:* 

1. There exists a stable generating algorithm that achieves generation with exact breadth in the limit, using only membership oracle access to  $\mathcal{L}$ .

<sup>&</sup>lt;sup>13</sup>The exact definition of uniform generation is not important for our work. At a high level, this condition asks whether there exists some  $d \in \mathbb{N}$  such that after the generator observes d different strings from *any* target language of  $\mathcal{L}$ , then it can generate unseen strings that belong to K.

2. There exists a stable generating algorithm that achieves generation with infinite coverage after taking as input  $d(\mathcal{L}) + 1$  distinct strings, using only membership oracle access to  $\mathcal{L}$ .

Moreover, for finite collections, a stronger property is possible: the results of [KMV24] (see Proposition 3.9 in their work) show that for finite collections there exists a stable generating algorithm that achieves exact breadth in the limit (and, hence, also infinite coverage), but there might not be an upper bound on the elements needed to achieve this property.<sup>14</sup>

Finally, we prove Proposition 3.5.

I.

*Proof of Proposition 3.5.* Our proof is inspired by the Lemma 3.2 from [LRT24]. The only modification is that now the algorithm stops using new elements beyond the  $d(\mathcal{L}) + 1$  elements required to achieve infinite coverage. Moreover, we discuss the type of access to  $\mathcal{L}$  needed that is sufficient to achieve this property, which was not the focus of Li, Raman, and Tewari [LRT24]. Let  $K \in \mathcal{L}$  be any target language and  $x_1, \ldots, x_{d(\mathcal{L})+1} \in K$  be any  $d(\mathcal{L}) + 1$  distinct elements of the target language. First, notice that since  $x_1, \ldots, x_{d(\mathcal{L})+1} \in K$ ,  $V(x_1, \ldots, x_{d(\mathcal{L})+1}) \neq \emptyset$ , as  $K \in V(x_1, \ldots, x_{d(\mathcal{L})+1})$ . By the definition of the closure dimension (Definition 12) and since  $|K| = \infty$  (recall that language generation is not meaningful with finite languages and, hence, throughout this work, we consider all languages are infinite),

$$\left| \bigcap_{L \in V(x_1, \dots, x_{d(\mathcal{L})+1})} L \right| = \infty \quad \text{and} \quad \bigcap_{L \in V(x_1, \dots, x_{d(\mathcal{L})+1})} L \subseteq K.$$

Thus, the generating algorithm can stabilize its support to be  $T := \bigcap_{L \in V(x_1,...,x_{d(\mathcal{L})+1})} L$  and never change it from this point on during the interaction with the adversary. Notice that given access to a VSI membership oracle for  $\mathcal{L}$  the learner can indeed sample from a distribution supported on T as follows: first sample a natural number  $\hat{n}$  (*e.g.*, from a geometric distribution on  $\mathbb{N}$ ) and then query the VSI membership oracle with the set of elements  $x_1, \ldots, x_{d(\mathcal{L})+1}$  and the target element  $x_{\hat{n}}$ .<sup>15</sup> Repeat the process until the oracle returns Yes. Notice that this process terminates with probability 1, and the support of the induced distribution is exactly T.

As a final note on our discussion on stability, it is worth pointing out that there are collections that do not satisfy the weak Angluin's condition, nevertheless there is a stable generating algorithm that achieves infinite coverage after observing one example from the target language. The example is due to Charikar and Pabbaraju [CP24a].

**Example 2** (Stable Infinite Coverage  $\implies$  Weak Angluin's Condition). Define the domain  $\mathcal{X}$  and the language collection  $\mathcal{L}$  as follows

$$\mathfrak{X} = \mathbb{Z}$$
 and  $\mathcal{L} = \{L_{\infty} \coloneqq \mathbb{Z}, L_a \coloneqq \{a + i, i \in \mathbb{N}\} : a \in \mathbb{Z}\}$ ,

<sup>&</sup>lt;sup>14</sup>To be precise, Proposition 3.9 in [KMV24] gives an algorithm to identify finite collections in the limit. This algorithm immediately gives an algorithm for generation with exact breadth: once we know an index *z* such that  $K = L_z$ , we can sample a natural number (from, *e.g.*, an exponential distribution on  $\mathbb{N}$ ) and output the *i*-th element of  $L_z$ . The latter, in turn, can be found using the membership oracle to  $L_z$ .

<sup>&</sup>lt;sup>15</sup>To be formal, we need to use a different enumeration of the strings of  $\mathfrak{X}$  and the strings that define the target version space. We overload the notation for simplicity.

where  $\mathbb{Z}$  is the set of integer numbers. Notice that both  $\mathfrak{X}$  and  $\mathfrak{L}$  are countable, and each  $L \in \mathfrak{L}$  is also countable. Consider the language  $L_{\infty}$  and any finite  $T \subseteq L_{\infty}$ . Let  $i_T$  be the smallest element of the subset T. Then,  $T \subseteq L_{i_T}, L_{i_T} \subsetneq L_{\infty}$ , and  $|L_{\infty} \setminus L_{i_T}| = \infty$ . Hence, this collection does not satisfy the weak Angluin's condition. Consider the generating algorithm  $\mathcal{G}$  which in every round n outputs a distribution with supp  $(\mathcal{G}_n) = \mathbb{N} \setminus S_1$ , where  $S_1$  is the input in round 1. It is not hard to see that for any target language K, this generating algorithm achieves infinite coverage, and is, by definition, stable.

#### 3.4 Generation With Increasing Coverage: A Strengthening of Stability

A key observation in [KM24] is that their generator's support can decrease when it sees new strings from the target *K* and, in fact, for many language collections the number of valid strings omitted from its support can grow without bound, which is an extreme form of *mode collapse*. In this light, one can view stability as a property that avoids such extreme mode collapse: any stable generator can only change its support finitely many times. A natural question is whether we can achieve something stronger than stability and, yet, more tractable than breadth. To capture this phenomenon, we introduce the following notion of *generation with strictly increasing coverage*.

**Definition 14** (Generation with Strictly Increasing Coverage). Let  $\mathcal{L}$  be a countable collection of languages. A generating algorithm  $\mathcal{G} = (\mathcal{G}_n)$  is said to have strictly increasing coverage for  $\mathcal{L}$  in the limit if, for any  $K \in \mathcal{L}$  and enumeration of K, there is an  $n^* \geq 1$  such that for all  $n \geq n^*$ , after seeing n elements of the enumeration, the following hold

- supp  $(\mathcal{G}_n) \subseteq$  supp  $(\mathcal{G}_{n+1})$ , and
- *either* supp  $(\mathcal{G}_n) = K$  or there exists some n' > n such that supp  $(\mathcal{G}_n) \subsetneq \text{supp} (\mathcal{G}_{n'})$ .

Intuitively, if a generator satisfies this property of strictly increasing coverage, then, at a high level, one may gather that it learns something new about the target language each time it sees a new string from it.

To gain intuition about when increasing coverage is achievable, let us consider two extremes. On the one hand, it is not hard to see that achieving approximate breadth along with strictly increasing coverage is significantly harder than achieving approximate breadth along: This is because if a generator has approximate breadth, then after seeing sufficiently many strings from K, its support only misses a finite number of strings from K and, then, if it further has strictly increasing coverage, its support eventually becomes equal to K implying exact breadth which is only achievable for collections satisfying Angluin's condition (Lemma 2.1). On the other hand, if one is not required to have infinite coverage<sup>16</sup> (a requirement already weaker than any notion of breadth), then it is easy to achieve strictly increasing coverage: consider the generator G in Proposition 1.3, which achieves infinite coverage for any collection  $\mathcal{L}$ , and post-process the algorithm to have a support of size at most t on round t. Since eventually G's support has infinitely many elements (as it achieves infinite coverage), it follows that the support of the above post-processed

<sup>&</sup>lt;sup>16</sup>For the subsequent discussion, we use the equivalent version of the definition of infinite coverage (Definition 8) which allows the support of the generator to contain strings from the set  $S_n$ , which is the set of all strings enumerated so far.

variant increases infinitely many times, implying that the post-processed variant achieves strictly increasing coverage.

Thus, the most interesting question is whether there is a generator that achieves infinite coverage – a property between breadth and consistent generation – while also having strictly increasing coverage. Our next result shows that there are collections for which this is indeed possible. The collection we use to show this result does not satisfy the weak Angluin's condition, so one cannot achieve even the weakest notion of breadth (namely, approximate breadth or equivalently exhaustive generation) for this collection.

**Proposition 3.7.** There exists a countable collection of languages  $\mathcal{L}$  that does not satisfy the weak Angluin's condition (*Definition 9*) and for which there exists a generating algorithm  $\mathcal{G} = (\mathcal{G}_n)$  that can achieve infinite coverage (*Definition 8*) and has strictly increasing coverage in the limit (*Definition 14*).

*Proof.* Consider the collection of arithmetic progressions used in Example 2. As we discussed, this collection does not satisfy the weak Angluin's condition. Let  $S_n$  be the set of elements enumerated up to round n and let  $\hat{t}_n$  denote the smallest element of  $S_n$ . Then, it is immediate that the generating algorithm that outputs a distribution supported on  $\{\hat{t}_n, \hat{t}_n + 1, \ldots\}$  achieves infinite coverage and has strictly increasing coverage in the limit.

We remark that the generating strategy in the above result uses information about the structure of  $\mathcal{L}$ , and not just membership access to it.

## 3.5 Landscape of Language Generation With Stable Algorithms

Our results on stable generation under various notions of breadth can be summarized as follows.

- 1. For stable generators, language generation with (exact) breadth in the limit is characterized by Angluin's condition. On the one hand, if  $\mathcal{L}$  satisfies Definition 2, then there is a stable algorithm that generates with (exact) breadth in the limit due to [KMV24]. On the other hand, if  $\mathcal{L}$  does not satisfy Angluin's condition, then no generator can generate with (exact) breadth in the limit (and so stable algorithms are excluded too) (Lemma 2.1).
- 2. For stable generators, unambiguous generation in the limit is also characterized by Angluin's condition. If  $\mathcal{L}$  satisfies Definition 2, then there is a stable algorithm that generates unambiguously in the limit due to [KMV24]. If  $\mathcal{L}$  does not satisfy Angluin's condition, then, in this work, we provide an unconditional lower bound for unambiguous generation in the limit (Lemma 2.12).
- 3. For stable generators, generation with approximate breadth in the limit is characterized by Angluin's condition. The algorithm follows from the exact breadth case. The lower bound is given in Lemma 3.2.
- 4. Since exhaustive generation is (i) implied by generation with (exact) breadth, and (ii) implies approximate breadth, it is also characterized by Angluin's dimension for stable generating algorithms.

- 5. There is a collection (see Example 1 and Theorem 3.4) of languages that satisfies the weak Angluin's condition (hence there exists a non-stable generator that achieves approximate breadth for this collection), but for which the strictly weaker requirement of generation with infinite coverage is not possible by a stable generator. Conversely, there is a collection (see Example 2) that does not satisfy the weak Angluin's condition but for which there exists a stable generator that achieves infinite coverage.
- 6. For every collection that has a finite closure dimension, there exists a stable generating algorithm that achieves infinite coverage, given access to the membership oracle to VSI (Definition 13).



(a) Unconditional Characterizations



Figure 4: *Comparison of Generation in the Limit With and Without Requiring Stability.* Each containment illustrated by a border is *strict, i.e.*, for each border there is a language collection that satisfies the outer containment but not the inner containment. Concretely, in the figure on the left, there are (1) language collections that do not satisfy the Weak Angluin's Condition (Definition 9) (see Example 2), (2) language collections that satisfy the Weak Angluin's Condition, but not Angluin's condition (See Example 1), and (3) there are language collections which satisfy Angluin's Condition (Definition 2) (*e.g.*, all finite collections). The figure on the right depicts the characterization for stable generators. In addition to what is depicted there, there are (1) language collections that satisfy the weak Angluin's condition statisfy the weak Angluin's condition and for which infinite coverage is not achievable (see Theorem 3.4) and (2) language collections for which infinite coverage is achievable but that do not satisfy the weak Angluin's Condition (Definition 9) (see Example 2). We note that (1) and (2) *are* not depicted in the right figure.

## 4 Extension to Any Notion of Breadth Satisfying Uniqueness

The goal of this section is to introduce an abstraction of the notions of breadth discussed in this manuscript and show that it is possible to extend the proof of Lemma 2.1 for these more general notions and get stronger results. As implications of this generalization, we will prove Lemma 2.7 and Lemma 2.12 in Appendix A. To this end, we present an unconditional lower bound for language generation with breadth that applies to *any* notion of language generation with breadth which satisfies the following uniqueness criterion.
**Definition 15** (Uniqueness Criterion). Consider any notion B of language generation with breadth. We say that B satisfies the uniqueness criterion with respect to a language collection  $\mathcal{L}$  if for any pair of distinct languages  $L, L' \in \mathcal{L}$ , no generator can satisfy B for both L and L' simultaneously, i.e., if a generating algorithm generates with breadth from L, then it cannot generate with breadth from L' and vice versa.

If B satisfies the uniqueness criterion for all language collections  $\mathcal{L}$ , then we simply say that B satisfies the uniqueness criterion.

To gain some intuition of this criterion, we consider a few notions of generation with breadth that we discussed in Section 1.3.

1. (Generation With Exact Breadth): First, consider the exact breadth (Definition 4). It satisfies the uniqueness criterion with respect to any language collection  $\mathcal{L}$ : this is because if a generator  $\mathcal{G}$  generates a language L with breadth, *i.e.*, supp $(\mathcal{G}) = L$ , then it necessarily does *not* generate any other language  $L' \neq L$  with breadth.

**Observation 4.1.** *Generation with (exact) breadth (Definition 4) satisfies the uniqueness criterion.* 

2. (Unambiguous Generation): Next, consider unambiguous generation (Definition 7). It also satisfies the uniqueness criterion with respect to any language collection. To see this, consider any distinct languages  $L \neq L'$ . Suppose a generator *G* unambiguously generates from *L*. This implies that

$$|\operatorname{supp}(\mathcal{G}) \triangle L| < \min_{L'' \in \mathcal{L}, \ L'' \neq L} \left| \operatorname{supp}(\mathcal{G}) \triangle L'' \right|.$$

However, setting L'' = L' implies that  $|\operatorname{supp}(\mathcal{G}) \triangle L| < |\operatorname{supp}(\mathcal{G}) \triangle L'|$  which shows that  $\mathcal{G}$  does not unambiguously generate from L'. This proves the following result.

**Observation 4.2.** *Unambiguous generation* (*Definition 7*) *satisfies the uniqueness criterion.* 

3. (Generation With Approximate Breadth): Finally, consider generation with approximate breadth (Definition 5). In general, it does not satisfy the uniqueness criterion. To see this, consider a language collection  $\mathcal{L}$  consisting of two languages  $L_1 \subseteq L_2$  that differ on finitely many elements: a generator whose support is  $L_1$  generates with approximate breadth from both  $L_1$  and  $L_2$  simultaneously.

**Observation 4.3.** *There are language collections*  $\mathcal{L}$  *for which generation with approximate breadth does not satisfy the uniqueness criterion.* 

It is not too hard to see that this is the only reason why uniqueness might be violated for generation with approximate breadth.

**Observation 4.4.** Consider any language collection  $\mathcal{L}$  satisfying that, for any pair of distinct languages  $L, L' \in \mathcal{L}$  with  $L \subseteq L', L'$  and L differ in infinitely many elements (i.e.,  $|L' \setminus L'| = \infty$ ). Generation with approximate breadth satisfies the uniqueness criterion with respect to  $\mathcal{L}$ .

We will use this observation in the next section to complete the proof of the characterization of generation with approximate breadth by a weakening of Angluin's criterion.

Having developed some intuition about the uniqueness criterion, we are ready to state the main result in this section: An unconditional lower bound for language generation with breadth for any notion of breadth that satisfies the uniqueness criterion (for all language collections).

**Definition 16.** Consider any notion B of language generation with breadth. We will say that an algorithm generates with B-breadth from  $\mathcal{L}$  in the limit, if it can generate with breadth with respect to notion B in the limit.

We have the following result, whose proof appears in Appendix A.

**Theorem 4.5** (Impossibility for Any Notion of Breadth Satisfying Uniqueness). Let *B* be any notion of generation that satisfies the uniqueness criterion. Let  $\mathcal{L}$  be a countable collection of languages that is not identifiable in the limit. Then, no algorithm can generate with B-breadth from  $\mathcal{L}$  in the limit.

# 5 Extension to Any Notion of Breadth Satisfying Finite Non-Uniqueness

This section presents a relaxation of the notion of uniqueness introduced in the previous section. We show that it is possible to extend the proof of Lemma 2.7 to this notion. As implications of this generalization, we prove that the two notions of exhaustive generation proposed by [CP24a] and [CP24b] respectively are characterized by the weak Angluin's condition. We begin with the definition of the relaxation of uniqueness.

**Definition 17** (Finite Non-Uniqueness Criterion). Consider any notion B of language generation with breadth. We say that B satisfies the finite non-uniqueness criterion with respect to a language collection  $\mathcal{L}$  if for any pair of distinct languages  $L, L' \in \mathcal{L}$ , a generator can satisfy B for both L and L' simultaneously if and only if they differ on a finite number of elements, i.e.,  $|L \triangle L'| < \infty$ . If B satisfies the finite non-uniqueness criterion for all language collections  $\mathcal{L}$ , then we simply say that B satisfies the finite non-uniqueness criterion.

This is a strict relaxation of the uniqueness condition introduced in the previous section. Hence, in particular, all notions of breadth satisfying the uniqueness condition also satisfy the finite non-uniqueness condition. To gain further intuition, let us consider some notions of breadth that did not satisfy the uniqueness condition and check whether they satisfy the finite-non-uniqueness condition.

1. (Generation With Approximate Breadth): Consider generation with approximate breadth (Definition 5). We saw in the last section that it does not satisfy the uniqueness criterion (Observation 4.3). However, it does satisfy finite non-uniqueness: To see this, consider any pair of languages *L* and *L'* that differ in infinitely many elements, *i.e.*,  $|L \triangle L'| = \infty$ . Now, if a generator *G* generates a language *L* with approximate breadth, *i.e.*,  $\supp(G) \subseteq L$  and  $|L \setminus \supp(G)| < \infty$ , then it necessarily does *not* generate *L'* with approximate breadth since if it did then it must imply that

$$|L \triangle L'| = |L \setminus L'| + |L' \setminus L| \quad \stackrel{\operatorname{supp}(\mathcal{G}) \subseteq L, L'}{\leq} \quad |L \setminus \operatorname{supp}(\mathcal{G})| + |L' \setminus \operatorname{supp}(\mathcal{G})| < \infty.$$

which contradicts the fact that  $|L \triangle L'| = \infty$ .

**Observation 5.1.** *Generation with approximate breadth* (*Definition 5*) *satisfies the finite non-uniqueness criterion.* 

2. (Exhaustive Generation): Next, we turn to exhaustive generation. Recall that in the formulation of exhaustive generation, the generating algorithm is a sequence of mappings from sequences of the domain to *enumerations* of the domain. Let  $\mathcal{G}(1 : \infty)$  be the set containing all the items  $\mathcal{G}$  enumerates. We claim that exhaustive generation satisfies finite non-uniqueness. To see this, consider any pair of languages L and L' that differ in infinitely many elements, *i.e.*,  $|L \triangle L'| = \infty$ . Now, if a generator  $\mathcal{G}$  generates exhaustively generates both L and L', then, by definition,

$$|L \setminus \mathcal{G}(1:\infty)|, \quad |L' \setminus \mathcal{G}(1:\infty)|, \quad |\mathcal{G}(1:\infty) \setminus L|, \quad |\mathcal{G}(1:\infty) \setminus L'| < \infty.$$
(10)

This contradicts the fact that  $|L \triangle L'| = \infty$  since

$$\begin{aligned} |L \triangle L'| &= |L \setminus L'| + |L' \setminus L| \\ &\leq \left( \left| \mathcal{G}(1:\infty) \triangle L' \right| + |L \setminus \mathcal{G}(1:\infty)| \right) + \left( \left| \mathcal{G}(1:\infty) \triangle L \right| + |L' \setminus \mathcal{G}(1:\infty)| \right) \\ &\leq 3 \cdot \left( |L \setminus \mathcal{G}(1:\infty)| + |\mathcal{G}(1:\infty) \setminus L| + |L' \setminus \mathcal{G}(1:\infty)| + |\mathcal{G}(1:\infty) \setminus L'| \right) \\ &\stackrel{(10)}{<} \infty. \end{aligned}$$

**Observation 5.2.** *Exhaustive generation (Definition 6) satisfies the finite non-uniqueness criterion.* 

3. (A Variant of Exhaustive Generation from [CP24a]): Next, we consider the first version of exhaustive generation, which appeared in [CP24a].

**Definition 18** (Variant of Exhaustive Language Generation in the Limit [CP24a]). A generating algorithm  $\mathcal{G} = (\mathcal{G}_n)$  is said to be an exhaustive generator in the limit for a language collection  $\mathcal{L} = \{L_1, L_2, ...\}$  if, for any  $K \in \mathcal{L}$  and enumeration of K, there is an  $n^* \ge 1$ , such that for all  $n \ge n^*$ , after seeing n elements of the enumeration,

$$\bigcup_{i=1}^{\infty} \mathcal{G}_n(i) \subseteq K \quad and \quad S_n \cup \bigcup_{j=1}^{n-1} \mathcal{G}_j(1) \cup \bigcup_{i=1}^{\infty} \mathcal{G}_n(i) \supseteq K, \quad (11)$$

where  $S_n$  is the set of elements enumerated until round n.

This notion is strictly stronger than the second version of exhaustive generation in Definition 6 and, hence, also satisfies the finite non-uniqueness criterion.

**Observation 5.3.** *Definition 6* (*i.e.*, the variant of exhaustive generation from [CP24a]) satisfies the finite non-uniqueness criterion.

*Remark* 6 (Comparison of Definition 18 and Generation with Approximate Breadth). We remark that this variant of exhaustive generation is weaker (perhaps not strictly) than generation with approximate breadth (Definition 5). In particular, an algorithm that satisfies the above definition also satisfies Definition 5. To see this, note that exhaustive generation implies that after some finite time, the generator misses only finitely many elements of the target language (*i.e.*, the set  $S_n \cup \bigcup_{j=1}^{n-1} G_j(1)$  has finitely many elements). Hence, lower bounds for generation with approximate breadth imply lower bounds for exhaustive generation.

Having developed some intuition about the finite non-uniqueness criterion, we are ready to state the main result of this section: An unconditional lower bound for language generation with breadth for any notion of breadth that satisfies the finite non-uniqueness criterion (for all language collections).

**Theorem 5.4** (Impossibility for Any Notion of Breadth Satisfying Finite Non-Uniqueness). Let *B* be any notion of generation that satisfies the finite non-uniqueness criterion. Let  $\mathcal{L}$  be any countable collection of languages that does not satisfy the weak Angluin's condition (Definition 9). Then, no algorithm can generate with B-breadth from  $\mathcal{L}$  in the limit.

An immediate implication of Theorem 5.4 and observations from earlier in this section is that no generator can achieve generation with approximate breadth, exhaustive generation, or the variant of exhaustive generation from [CP24a] for any language collection that does not satisfy the weak Angluin's condition (Definition 9). This combined with algorithms presented earlier immediately implies the following equivalence result.

**Corollary 5.5** (Equivalence of Approximate Breadth, Exhaustive Generation and Its Variant). *Let*  $\mathcal{L}$  *be any countable collection of languages. The following are equivalent.* 

- *L* satisfies the weak Angluin's condition (*Definition 9*).
- There is an algorithm that generates with approximate breadth from  $\mathcal{L}$  in the limit.
- There is an algorithm that exhaustively generates (*Definition 6*)  $\mathcal{L}$  in the limit.
- There is an algorithm that generates according to *Definition 18* (i.e., the variant of exhaustive generation from [CP24a]) for  $\mathcal{L}$  in the limit.

The proofs of Theorem 5.4 and Corollary 5.5 appear in Appendix B.

# 6 Conclusion

In this section, we summarize the results of this manuscript. We show the following characterizations that significantly strengthen the results of [CP24a; KMV24] (see Sections 1.5 and 1.6 for further discussion).

• Generation with *infinite coverage* (Definition 8) (but no breadth) is achievable for all countable collections by a small modification of the algorithm of [KM24] (Lemma 2.4, also see Lemma 2.6).

- Generation with *exact breadth* in the limit is characterized by Angluin's condition. If Definition 2 (Angluin's condition) holds for *L*, then *L* is identifiable in the limit, and there is a generating algorithm that generates with breadth from *L* in the limit. Otherwise, no algorithm can generate with breadth from *L* in the limit (Theorem 2.2).
- Generation with *approximate breadth* in the limit is characterized by the weaker variant of Angluin's condition (Definition 9). If Definition 9 holds for  $\mathcal{L}$ , then there is a generating algorithm that generates with approximate breadth from  $\mathcal{L}$  in the limit. Otherwise, no algorithm can generate with approximate breadth from  $\mathcal{L}$  in the limit (Theorem 2.3).
- *Exhaustive generation* in the limit is equivalent to generation with approximate breadth in the limit (Theorem 2.8).
- Unambiguous generation in the limit is characterized by Angluin's condition (Theorem 2.13). If Definition 2 holds for *L*, then *L* is identifiable in the limit, and there is a generating algorithm that unambiguously generates from *L* in the limit. Otherwise, no algorithm can unambiguously generate from *L* in the limit.

Moreover, we derive additional results for stable generators, a natural property of generators derived from the work of Gold [Gol67]. For the family of stable generators for which the MOP is also decidable, [KMV24] show that Angluin's condition characterizes all the above notions of breadth, *i.e.*, Definitions 4 to 7. We strengthen this result by removing the requirement of MOP's decidability in [KMV24]'s result. In particular, we show that the family of stable generators (which may or may not have a decidable MOP), can achieve language generation with breadth from collection  $\mathcal{L}$  – for any notion of breadth (Definitions 4 to 7) – if and only if  $\mathcal{L}$  satisfies Angluin's condition (see the right plot of Figure 4). Interestingly, we show that there are collections which satisfy the weak Angluin's condition, nevertheless there does not exist a stable generator that achieves infinite coverage – a notion of breadth that is achievable for all countable collections by non-stable learners. This demonstrates that the landscape of generation looks significantly different for stable generators than for unstable generators.

We believe the above results provide a clear picture of the landscape of language generation in the limit.

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### A The Proof of Theorem 4.5 and Implications

The proof of Theorem 4.5 is an extension of the proof of Lemma 2.1.

Because the proofs of Theorem 4.5 and Lemma 2.1 are very similar, we highlight the changes in red.

*Proof of Theorem* 4.5. As before, for any enumeration *E*, we use the notation *E*(*i*) to denote its *i*-th element, E(1:i) to denote its first *i* elements, and  $E(i:\infty)$  to denote all but the first *i* – 1 elements. Since  $\mathcal{L}$  is not identifiable in the limit, it does not satisfy Angluin's condition (Definition 2). Hence, there exists a language  $L^* \in \mathcal{L}$  such that the following holds:

for all finite subsets  $T \subseteq L^*$ , there exists a language  $L_T \in \mathcal{L}$ ,  $T \subseteq L_T$  and  $L_T \subsetneq L^*$ . (12)

Fix  $L^* \in \mathcal{L}$  to be any language for which this holds. Let  $E^{\infty}_*$  be an arbitrary enumeration of  $L^*$ , without repetitions. Let *K* and  $E^{\infty}_K$  respectively denote the target language and its enumeration that we will construct to show the impossibility result.

We will show that for any generating algorithm  $G = (G_n)$  there exists a choice of the target language K in  $\mathcal{L}$  (which may be different from  $L^*$ ) and an enumeration of it such that if K is the target language and the adversary provides enumeration  $E_K^{\infty}$  to G, then the algorithm G cannot generate with breadth in the limit.

We will construct the enumeration iteratively and select *K* based on the generating algorithm. The construction of the enumeration proceeds in multiple (possibly infinite) phases. At any point  $t \in \mathbb{N}$  of the interaction, we denote by  $S_t$  the set of elements enumerated so far.

**Phase 1 of Construction.** To construct the first phase, we present the generator with the first element of the enumeration of  $L^*$ , *i.e.*,  $x_{i_1} \coloneqq E_*^{\infty}(1)$ . Let  $L_{j_1}$  be some language such that  $x_{i_1} \in L_{j_1}$  and  $L_{j_1} \subsetneq L^*$ , *i.e.*, it is a proper subset of  $L^*$ . Notice that such a language is guaranteed to exist by picking  $T = \{x_{i_1}\}$  in the violation of Angluin's condition (12).

• Subphase A (Enumerate  $L_{j_1}$  Until Generator Generates with Breadth from  $L_{j_1}$ ): Consider an enumeration  $E_1^{\infty}$  of the language  $L_{j_1}$  that is constructed by traversing  $E_*^{\infty}$  and using the elements of  $L_{j_1}$  that appear in it, in the same order as they appear, *i.e.*, for every  $i \in \mathbb{N}$  it holds that  $E_1^{\infty}(i)$  is the *i*-th element of  $L_{j_1}$  that appears in  $E_*^{\infty}$ . Notice that this is indeed a valid enumeration of  $L_{j_1}$  as  $L_{j_1}$  is a subset of  $L^*$ . At any round *t* of the first phase, the adversary presents the element  $E_1^{\infty}(t)$  to the generator.

Consider two cases: i) either there is some finite  $t_1 \in \mathbb{N}$  such that  $\mathcal{G}_{t_1}$  generates with *B*breadth from  $L_{j_1}$  or ii) there is no such  $t_1 \in \mathbb{N}$ . In the latter case, we pick the target language  $K = L_{j_1}$  and the target enumeration  $E_K^{\infty} = E_1^{\infty}$ , and the lower bound follows since we have found a pair of *K* and  $E_K^{\infty}$  for which the generator never achieves *B*-breadth. Hence, assume that we are in the former case, and let  $\hat{x}_1$  be the first element of  $E_1^{\infty}$  for which the condition holds. Note that, at this point,  $\mathcal{G}_{t_1}$  does *not* generate with *B*-breadth from  $L^*$  since *B* satisfies the uniqueness criterion and  $L_{j_1} \neq L^*$ . Further, note that  $S_{t_1}$  is the set of strings shown to the generating algorithm after which it starts to generate with breadth from  $L_{j_1}$ .

Let  $\hat{S}_1$  be the set of elements of  $E^{\infty}_*$  that appear before  $\hat{x}_1$  in  $E^{\infty}_*$  and have not appeared in  $S_{t_1}$ . If  $\hat{S}_1 \neq \emptyset$ , we go to Subphase B.1 and, otherwise if  $\hat{S}_1 = \emptyset$ , we go to Subphase B.2.

- Subphase B.1 (Add Any Skipped Elements): We will use the set  $\hat{S}_1$  to extend the construction of the target enumeration  $E_K^{\infty}$ . To do this, we enumerate the elements from  $\hat{S}_1$  in an arbitrary order and we fix the prefix of the target enumeration  $E_K^{\infty}$  to be  $(S_{t_1}, \hat{S}_1)$ . Notice that this step is well-defined since we are only adding to the already constructed enumeration. Let  $\hat{t}_1$  be the total number of elements enumerated so far. Notice that  $\hat{t}_1 = \infty$  if and only if Case i) (from Subphase A) holds, in which case the lower bound already follows. Hence, assume for the continuation of the proof that  $\hat{t}_1 < \infty$ . Now we terminate the first phase (without going to Subphase B.2).
- Subphase B.2 (If Nothing Skipped Enumerate An Element Outside L<sub>j1</sub>): Notice that S<sub>1</sub> = Ø if and only if we did not skip any element of E<sup>∞</sup><sub>\*</sub> during the traversal in Subphase A. If we indeed did not skip elements of E<sup>∞</sup><sub>\*</sub> we continue traversing it and adding elements to E<sup>∞</sup><sub>K</sub> in the same order as we see them in E<sup>∞</sup><sub>\*</sub> until we find some element that does not belong to L<sub>j1</sub>. We also include this element in the enumeration E<sup>∞</sup><sub>K</sub>, we fix t̂<sub>1</sub> to be the number of elements enumerated so far and we terminate the first phase.

Notice that so far in our construction, we have enumerated the first  $\hat{t}_1$  elements of  $E_*^{\infty}$ .

Now we continue our construction inductively for phases  $\ell = 2, 3, ...$  Consider any  $\ell \ge 2$ . Suppose our construction continued from Phase 1 until Phase  $\ell$ . Then, Phase  $\ell + 1$  of our construction is as follows.

**Phase**  $\ell + 1$  of Construction. For the  $(\ell + 1)$ -th phase, consider the set  $E^{\infty}_{*}(1 : \hat{t}_{\ell})$  that has been enumerated so far. By construction,

$$E^{\infty}_*(1:\hat{t}_{\ell}) \not\subseteq L_{i_{\ell}}, \quad E^{\infty}_*(1:\hat{t}_{\ell}) \subseteq L^*, \text{ and } E^{\infty}_*(1:\hat{t}_{\ell}) \text{ is finite }.$$

We will now apply the violation of Angluin's condition (12) with  $T = E_*^{\infty}(1:\hat{t}_{\ell})$ . This means that there must exist some  $j_{\ell+1} \notin \{j_1, j_2, \dots, j_{\ell}\}$  such that

 $L_{j_{\ell+1}} \in \mathcal{L}$ ,  $L_{j_{\ell+1}} \subsetneq L^*$ , and  $E^{\infty}_*(1:\hat{t}_{\ell}) \subseteq L_{j_{\ell+1}}$ .

We now perform analogs of each subphase in Phase 1.

Subphase A (Enumerate L<sub>jℓ+1</sub> Until Generator Generates with Breadth from L<sub>jℓ+1</sub>): Consider an enumeration E<sup>∞</sup><sub>ℓ+1</sub> of L<sub>jℓ+1</sub> whose first t<sub>ℓ</sub> strings are E<sup>∞</sup><sub>\*</sub>(1 : t<sub>ℓ</sub>) and whose remaining strings are constructed by traversing E<sup>∞</sup><sub>\*</sub>(t<sub>ℓ</sub> + 1 : ∞) and selecting strings that belong to L<sub>jℓ+1</sub>, in the same order as they appear in E<sup>∞</sup><sub>\*</sub>. Notice that this is indeed a valid enumeration of L<sub>jℓ+1</sub> as L<sub>jℓ+1</sub> is a subset of L\*. At any round t of this phase, the adversary presents the element E<sup>∞</sup><sub>ℓ+1</sub>(t + t<sub>ℓ</sub>) to the generator.

Consider two cases: i) either there is some finite  $t_{\ell+1} \ge \hat{t}_{\ell} + 1$  such that  $\mathcal{G}_{t_{\ell+1}}$  generates with *B*-breadth from  $L_{j_{\ell+1}}$  or ii) there is no such  $t_{\ell+1} \in \mathbb{N}$ . In the latter case, we pick the target language  $K = L_{j_{\ell+1}}$  and the enumeration  $E_K^{\infty} = E_{\ell+1}^{\infty}$ , and the lower bound follows since we have found a pair of K and  $E_K^{\infty}$  for which the generator never achieves *B*-breadth. Hence, assume that we are in the former case, and let  $\hat{x}_{\ell+1}$  be the first element of  $E_{\ell+1}^{\infty}$  for which the condition holds. Note that, at this point,  $\mathcal{G}_{t_{\ell+1}}$  goes *not* generate with *B*-breadth from  $L^*$  since *B* satisfies the uniqueness criterion and  $L_{j_{\ell+1}} \neq L^*$ . Further, note that  $S_{t_{\ell+1}}$  is the set of strings shown to the generating algorithm after which it starts to generate with breadth from  $L_{j_{\ell+1}}$ . Let  $\widehat{S}_{\ell+1}$  be the set of strings of  $E_*^{\infty}$  that appear before  $\widehat{x}_{\ell+1}$  in  $E_*^{\infty}$  and have not appeared in the enumeration  $S_{t_{\ell+1}}$ . If  $\widehat{S}_{\ell+1} \neq \emptyset$ , we go to Subphase B.1 and, otherwise if  $\widehat{S}_{\ell+1} = \emptyset$ , we go to Subphase B.2.

- Subphase B.1 (Add Any Skipped Elements): We will use S<sub>ℓ+1</sub> to extend the construction of the target enumeration E<sub>K</sub><sup>∞</sup>. To do this, we enumerate the elements from S<sub>ℓ+1</sub> in an arbitrary order and we fix the prefix of the target enumeration E<sub>K</sub><sup>∞</sup> to be (S<sub>t<sub>ℓ+1</sub></sub>, S<sub>ℓ+1</sub>). Notice that this step is well-defined since we are only adding to the already constructed enumeration. Let t<sub>ℓ+1</sub> be the set of elements enumerated so far. Notice that t<sub>ℓ+1</sub> = ∞ if and only if Case i) (from Subphase A) holds, in which case the lower bound already follows. Hence, assume for the continuation of the proof that t<sub>ℓ+1</sub> < ∞. Now we terminate the (ℓ + 1)-th phase without going to Subphase B.2.</li>
- Subphase B.2 (If Nothing Skipped Enumerate An Element Outside L<sub>jℓ+1</sub>): Notice that *Ŝℓ+1* = Ø if and only if we did not skip any element of E<sup>∞</sup><sub>\*</sub> during the traversal in Subphase A. If we indeed did not skip elements of E<sup>∞</sup><sub>\*</sub> we continue traversing it and adding elements to E<sup>∞</sup><sub>K</sub> in the same order as we see them in E<sup>∞</sup><sub>\*</sub> until we find some element that does not belong to L<sub>jℓ+1</sub>. We also include this element in the enumeration E<sup>∞</sup><sub>K</sub>, we set *t̂ℓ+1* to be the number of elements enumerated so far and we terminate Phase ℓ + 1.

Notice that so far we have enumerated the first  $\hat{t}_{\ell+1} > \hat{t}_{\ell} + 1$  elements of  $E_*^{\infty}$ .

**Inductive Argument.** As explained, we continue the construction of the target enumeration inductively. If there is some phase  $\ell$  such that Case ii) (in Subphase A) is activated, then the lower bound follows. Let us now assume that Case ii) is not activated for any phase  $\ell \in \mathbb{N}$ . Then, we have constructed an enumeration of  $L^*$  (by construction of the sets  $S_{t_\ell}$  and  $\widehat{S}_\ell$  for each  $\ell \in \mathbb{N}$ ) such that  $\mathcal{G}_t$  does not generate with *B*-breadth form  $L^*$  for infinitely many  $t \in \mathbb{N}$ . Now, the lower bound follows by setting the target language  $K = L^*$  and the target enumeration to the one we have constructed inductively over all phases.

#### A.1 The Proof of Lemma 2.12 (Impossibility for Unambiguous Generation)

In this section, we prove Lemma 2.12, which we restate below.

**Lemma 2.12** (Impossibility for Unambiguous Generation). Let  $\mathcal{L}$  be a countable collection of languages that is not identifiable in the limit. Then, no algorithm can unambiguously generate from  $\mathcal{L}$  in the limit.

*Proof.* This is a corollary of Theorem 4.5 since unambiguous generation satisfies the uniqueness criterion as shown in Observation 4.2.

#### **B** The Proof of Theorem 5.4 and Implications

In this section, we prove Theorem 5.4.

*Proof of Theorem 5.4.* The proof of this lower bound uses the construction in the proof of Theorem 4.5 with one change: now the language  $L_T$  (introduced at the start of the proof) is the language determined by the contrapositive to the weak Angluin's criterion (Definition 9) and not the contrapositive to the (usual) Angluin's criterion (Definition 2). Concretely, the contrapositive to the weak Angluin's criterion implies that there exists a language  $L^* \in \mathcal{L}$  such that the following holds:

 $\forall T \subseteq L^*$ ,  $\exists L_T \in \mathcal{L}$ , such that  $T \subseteq L_T$ ,  $L_T \subsetneq L^*$ , and  $|L^* \setminus L_T| = \infty$ . (13)

We will use this language  $L^*$  and proceed with the construction without change.

Having completed the construction, we proceed to the proof. The only place in which the proof uses a property of the criterion for breadth is when it invokes the uniqueness criterion with respect to the pair of languages  $L_T$  and  $L^*$  (once in Subphase A of each phase). Here, T is the set  $E^{\infty}_*(1)$  in the first phase and  $E^{\infty}_*(1:\hat{t}_{\ell})$  in the  $\ell$ -th phase. Now, we cannot directly invoke the uniqueness criterion since, in general, there are pairs of languages L, L' for which generation with approximate breadth does not satisfy the uniqueness criterion (Observation 4.3). However, since  $|L^* \setminus L_T| = \infty$  and since the notion of breadth B satisfies the finite non-uniqueness criterion, we can conclude that no generator can generate with B-breadth from both  $L^*$  and  $L_T$  simultaneously, as desired. Hence, we can use the finite non-uniqueness criterion in analyzing each phase of the construction and the result follows as in the proof of Theorem 4.5.

#### B.1 The Proof of Lemma 2.7 (Impossibility for Generation With Approximate Breadth)

In this section, we prove Lemma 2.7, which we restate below.

**Lemma 2.7** (Impossibility for Generation with Approximate Breadth). Let  $\mathcal{L}$  be a countable collection of languages that does not satisfy *Definition 9*. Then, there is no generating algorithm that generates with approximate breadth from  $\mathcal{L}$  in the limit.

*Proof of Lemma* 2.7. This is a corollary of Theorem 5.4 since unambiguous generation satisfies the uniqueness criterion as shown in Observation 5.1.

#### B.2 The Proof of Lemma 2.11 (Impossibility for Exhaustive Generation)

In this section, we prove Lemma 2.11, which we restate below.

**Lemma 2.11** (Impossibility for Exhaustive Generation). Let  $\mathcal{L}$  be a countable collection of languages that does not satisfy Definition 9. Then, there is no generating algorithm that exhaustively generates from  $\mathcal{L}$  in the limit.

*Proof of Lemma 2.11.* This is a corollary of Theorem 5.4 since unambiguous generation satisfies the uniqueness criterion as shown in Observation 5.2.  $\Box$ 

# **B.3** The Proof of Corollary 5.5 (Equivalence of Approximate Breadth, Exhaustive Generation and Its Variant)

In this section, we prove Corollary 5.5, which we restate below.

**Corollary 5.5** (Equivalence of Approximate Breadth, Exhaustive Generation and Its Variant). *Let*  $\mathcal{L}$  *be any countable collection of languages. The following are equivalent.* 

- *L* satisfies the weak Angluin's condition (*Definition 9*).
- There is an algorithm that generates with approximate breadth from  $\mathcal{L}$  in the limit.
- There is an algorithm that exhaustively generates (Definition 6)  $\mathcal{L}$  in the limit.
- There is an algorithm that generates according to *Definition 18* (i.e., the variant of exhaustive generation from [CP24a]) for  $\mathcal{L}$  in the limit.

*Proof of Corollary* 5.5. First, suppose  $\mathcal{L}$  does not satisfy weak Angluin's condition (Definition 9). Then, since approximate breadth, exhaustive generation, or Definition 18 satisfy finite non-uniqueness (Observations 5.1 to 5.3), Theorem 5.4 implies that no generator can achieve any of these notions of generation for  $\mathcal{L}$  in the limit.

Next, suppose that  $\mathcal{L}$  does satisfy weak Angluin's condition (Definition 9). Now, Lemma 2.6 and Lemma 2.9 give algorithms that, in the limit, satisfy the definition of generation with approximate breadth and exhaustive generation for  $\mathcal{L}$ . Further, recall that apart from exhaustive generation, the generator in Lemma 2.9 has the additional property that it does not hallucinate (see Remark 5). Hence, it also satisfies Definition 18.

## C Proof Omitted From Section 3

In this section, we prove Lemma 3.3, which we restate below.

**Lemma 3.3.** Let  $\mathcal{L}$  be a countable collection of languages that is not identifiable in the limit. Let  $\mathcal{G} = (\mathcal{G}_n)$  be a stable generating algorithm. If  $\mathcal{G}$  generates with approximate breadth from  $\mathcal{L}$  in the limit, then there is a language  $L^* \in \mathcal{L}$ , an enumeration  $E^*$  of  $L^*$ , a sequence of distinct languages  $L_1, L_2, \dots \in \mathcal{L}$ , and a strictly increasing sequence  $t(1), t(2), \dots \in \mathbb{N}$ , such that the following holds.

- For each  $i \in \mathbb{N}$ ,  $L_{t(i)}$  is a proper subset of  $L^*$ , i.e.,  $L_{t(i)} \subsetneq L^*$ ; and
- *Given strings from*  $E^*$  *as input, for each*  $i \in \mathbb{N}$ *,*  $\mathcal{G}_{t(i)}$  *generates with approximate breadth from*  $L_{t(i)}$ *.*

The proof of Lemma 3.3 uses a very similar construction to Lemma 2.1: Since  $\mathcal{L}$  is non-identifiable in the limit, it must violate Angluin's condition. The proof selects  $L^*$  to be a language that witnesses the failure of Angluin's condition. Then, it carefully constructs an enumeration, and during this process constructs the sequence of languages  $L_1, L_2, \ldots$  from the contrapositive to Angluin's condition by setting the (potential) tell-tale sets *T* to be prefixes of the enumeration being constructed.

For the ease of the reader, we highlight changes in the present construction compared to the one in the proof of Lemma 2.1 in red color.

*Proof of Lemma* 3.3. As before, for any enumeration *E*, we use the notation *E*(*i*) to denote its *i*-th element, E(1:i) to denote its first *i* elements, and  $E(i:\infty)$  to denote all but the first *i* – 1 elements. Since  $\mathcal{L}$  is not identifiable in the limit, it does not satisfy Angluin's condition (Definition 2). Hence, there exists a language  $L^* \in \mathcal{L}$  such that the following holds:

for all finite subsets  $T \subseteq L^*$ , there exists a language  $L_T \in \mathcal{L}$ ,  $T \subseteq L_T$  and  $L_T \subsetneq L^*$ . (14)

Fix  $L^* \in \mathcal{L}$  to be any language for which this holds. Let  $E_*^{\infty}$  be an arbitrary enumeration of  $L^*$ , without repetitions. Let  $\mathcal{G}_n$  be any stable generating algorithm that generates with approximate breadth from  $\mathcal{L}$ . The construction of the enumeration  $E^*$  depends on the generator  $\mathcal{G}$ . It proceeds in multiple (possibly infinite) phases. At any point  $t \in \mathbb{N}$  of the interaction, we denote by  $S_t$  the set of elements enumerated so far.

**Phase 1 of Construction.** To construct the first phase, we present the generator with the first element of the enumeration of  $L^*$ , *i.e.*,  $x_{i_1} := E_*^{\infty}(1)$ . Let  $L_{j_1}$  be some language such that  $x_{i_1} \in L_{j_1}$  and  $L_{j_1} \subsetneq L^*$ , *i.e.*, it is a proper subset of  $L^*$ . Notice that such a language is guaranteed to exist by picking  $T = \{x_{i_1}\}$  in the violation of Angluin's condition (14).

Subphase A (Enumerate L<sub>j1</sub> Until Generator Generates with Approximate Breadth from L<sub>j1</sub>): Consider an enumeration E<sub>1</sub><sup>∞</sup> of the language L<sub>j1</sub> that is constructed by traversing E<sub>\*</sub><sup>∞</sup> and using the elements of L<sub>j1</sub> that appear in it, in the same order as they appear, *i.e.*, for every *i* ∈ N it holds that E<sub>1</sub><sup>∞</sup>(*i*) is the *i*-th element of L<sub>j1</sub> that appears in E<sub>\*</sub><sup>∞</sup>. Notice that this is indeed a valid enumeration of L<sub>j1</sub> as L<sub>j1</sub> is a subset of L\*. At any round *t* of the first phase, the adversary presents the element E<sub>1</sub><sup>∞</sup>(*t*) to the generator.

Consider two cases: i) either there is some finite  $t_1 \in \mathbb{N}$  such that  $\mathcal{G}_{t_1}$  generates with approximate breadth from  $L_{j_1}$  (*i.e.*,  $|L_{j_1} \setminus \text{supp}(\mathcal{G}_{t_1})| < \infty$ ) or ii) there is no such  $t_1 \in \mathbb{N}$ . In the latter case, we pick the target language  $K = L_{j_1}$  and the target enumeration  $E_K^{\infty} = E_1^{\infty}$ , and we get a contradiction to the fact that  $\mathcal{G}$  generates with approximate breadth from  $\mathcal{L}$  in the limit. Hence, we must be in the former case, and let  $\hat{x}_1$  be the first element of  $E_1^{\infty}$  for which the condition holds. Note that  $S_{t_1}$  is the set of strings shown to the generating algorithm after which it starts to generate with breadth from  $L_{j_1}$ .

Let  $\widehat{S}_1$  be the set of elements of  $E_*^{\infty}$  that appear before  $\widehat{x}_1$  in  $E_*^{\infty}$  and have not appeared in  $S_{t_1}$ . If  $\widehat{S}_1 \neq \emptyset$ , we go to Subphase B.1 and, otherwise if  $\widehat{S}_1 = \emptyset$ , we go to Subphase B.2.

- Subphase B.1 (Add Any Skipped Elements): We will use the set S<sub>1</sub> to extend the construction of the enumeration E\*. To do this, we enumerate the elements from S<sub>1</sub> in an arbitrary order and we fix the prefix of the enumeration E\* to be (S<sub>t1</sub>, S<sub>1</sub>). Notice that this step is well-defined since we are only adding to the already constructed enumeration. Let t<sub>1</sub> be the total number of elements enumerated so far. Notice that t<sub>1</sub> = ∞ if and only if Case i) (from Subphase A) holds, which we saw was impossible. Hence, t<sub>1</sub> < ∞. Now we terminate the first phase (without going to Subphase B.2).</li>
- Subphase B.2 (If Nothing Skipped Enumerate An Element Outside L<sub>j1</sub>): Notice that S<sub>1</sub> = Ø if and only if we did not skip any element of E<sup>∞</sup><sub>\*</sub> during the traversal in Subphase A. If we indeed did not skip elements of E<sup>∞</sup><sub>\*</sub> we continue traversing it and adding elements to E<sup>\*</sup> in

the same order as we see them in  $E_*^{\infty}$  until we find some element that does not belong to  $L_{j_1}$ . We also include this element in the enumeration  $E^*$ , we fix  $\hat{t}_1$  to be the number of elements enumerated so far and we terminate the first phase.

Notice that so far in our construction, we have enumerated the first  $\hat{t}_1$  elements of  $E_*^{\infty}$ .

Now we continue our construction inductively for phases  $\ell = 2, 3, ...$  Consider any  $\ell \ge 2$ . Suppose our construction continued from Phase 1 until Phase  $\ell$ . Then, Phase  $\ell + 1$  of our construction is as follows.

**Phase**  $\ell + 1$  of Construction. For the  $(\ell + 1)$ -th phase, consider the set  $E^{\infty}_{*}(1 : \hat{t}_{\ell})$  that has been enumerated so far. By construction,

$$E^{\infty}_*(1:\hat{t}_\ell) \not\subseteq L_{j_\ell}$$
,  $E^{\infty}_*(1:\hat{t}_\ell) \subseteq L^*$ , and  $E^{\infty}_*(1:\hat{t}_\ell)$  is finite.

We will now apply the violation of Angluin's condition (5) with  $T = E_*^{\infty}(1:\hat{t}_{\ell})$ . This means that there must exist some  $j_{\ell+1} \notin \{j_1, j_2, \dots, j_{\ell}\}$  such that

$$L_{j_{\ell+1}} \in \mathcal{L}$$
,  $L_{j_{\ell+1}} \subsetneq L^*$ , and  $E^{\infty}_*(1:\hat{t}_{\ell}) \subseteq L_{j_{\ell+1}}$ .

We now perform analogs of each subphase in Phase 1.

Subphase A (Enumerate L<sub>jℓ+1</sub> Until Generator Generates with Approximate Breadth from L<sub>jℓ+1</sub>): Consider an enumeration E<sup>∞</sup><sub>ℓ+1</sub> of L<sub>jℓ+1</sub> whose first t<sub>ℓ</sub> strings are E<sup>∞</sup><sub>\*</sub>(1 : t<sub>ℓ</sub>) and whose remaining strings are constructed by traversing E<sup>∞</sup><sub>\*</sub>(t<sub>ℓ</sub> + 1 : ∞) and selecting strings that belong to L<sub>jℓ+1</sub>, in the same order as they appear in E<sup>∞</sup><sub>\*</sub>. Notice that this is indeed a valid enumeration of L<sub>jℓ+1</sub> as L<sub>jℓ+1</sub> is a subset of L<sup>\*</sup>. At any round t of this phase, the adversary presents the element E<sup>∞</sup><sub>ℓ+1</sub>(t + t<sub>ℓ</sub>) to the generator.

Consider two cases: i) either there is some finite  $t_{\ell+1} \ge \hat{t}_{\ell} + 1$  such that  $\mathcal{G}_{t_{\ell+1}}$  generates with approximate breadth from  $L_{j_{\ell+1}}$  (*i.e.*,  $|L_{j_{\ell+1}} \setminus \text{supp}(\mathcal{G}_{t_{\ell+1}})| < \infty$ ) or ii) there is no such  $t_{\ell+1} \in \mathbb{N}$ . In the latter case, we pick the target language  $K = L_{j_{\ell+1}}$  and the enumeration  $E_K^{\infty} = E_{\ell+1}^{\infty}$ , and we get a contradiction to the fact that  $\mathcal{G}$  generates with approximate breadth from  $\mathcal{L}$  in the limit. Hence, assume that we are in the former case, and let  $\hat{x}_{\ell+1}$  be the first element of  $E_{\ell+1}^{\infty}$  for which the condition holds. Note that  $S_{t_{\ell+1}}$  is the set of strings shown to the generating algorithm after which it starts to generate with breadth from  $L_{j_{\ell+1}}$ .

Let  $\widehat{S}_{\ell+1}$  be the set of strings of  $E_*^{\infty}$  that appear before  $\widehat{x}_{\ell+1}$  in  $E_*^{\infty}$  and have not appeared in the enumeration  $S_{t_{\ell+1}}$ . If  $\widehat{S}_{\ell+1} \neq \emptyset$ , we go to Subphase B.1 and, otherwise if  $\widehat{S}_{\ell+1} = \emptyset$ , we go to Subphase B.2.

Subphase B.1 (Add Any Skipped Elements): We will use the set S<sub>ℓ+1</sub> to extend the construction of the enumeration E\*. To do this, we enumerate the elements from S<sub>ℓ+1</sub> in an arbitrary order and we fix the prefix of the enumeration E\* to be (S<sub>t<sub>ℓ+1</sub></sub>, S<sub>ℓ+1</sub>). Notice that this step is well-defined since we are only adding to the already constructed enumeration. Let t<sub>ℓ+1</sub> be the set of elements enumerated so far. Notice that t<sub>ℓ+1</sub> = ∞ if and only if Case i) (from Subphase A) holds, which we saw was impossible. Hence, t<sub>ℓ+1</sub> < ∞. Now we terminate the (ℓ + 1)-th phase without going to Subphase B.2.</li>

 Subphase B.2 (If Nothing Skipped Enumerate An Element Outside L<sub>jℓ+1</sub>): Notice that *Ŝ*<sub>ℓ+1</sub> = Ø if and only if we did not skip any element of E<sup>∞</sup><sub>\*</sub> during the traversal in Subphase A. If we indeed did not skip elements of E<sup>∞</sup><sub>\*</sub> we continue traversing it and adding elements to E<sup>\*</sup> in the same order as we see them in E<sup>∞</sup><sub>\*</sub> until we find some element that does not belong to L<sub>jℓ+1</sub>. We also include this element in the enumeration E<sup>∞</sup><sub>K</sub>, we set t̂<sub>ℓ+1</sub> to be the number of elements enumerated so far and we terminate Phase ℓ + 1.

Notice that so far we have enumerated the first  $\hat{t}_{\ell+1} > \hat{t}_{\ell} + 1$  elements of  $E_*^{\infty}$ .

**Inductive Argument.** As explained, we continue the construction of the target enumeration inductively. If there is some phase  $\ell$  such that Case ii) (in Subphase A) is activated, then we get a contradiction to the fact that G generates with approximate breadth from  $\mathcal{L}$  in the limit. Hence, Case ii) must never be activated for any phase  $\ell \in \mathbb{N}$ . Then, we have constructed an enumeration of  $L^*$  (by construction of the sets  $S_{t_{\ell}}$  and  $\hat{S}_{\ell}$  for each  $\ell \in \mathbb{N}$ ), a sequence of distinct languages  $L_{j_1}, L_{j_2}, \ldots$  (each satisfying  $L_{j_i} \subsetneq L^*$ ), and a strictly increasing sequence of numbers  $t_1, t_2, \cdots \in \mathbb{N}$ , such that, for each i, the generator  $G_{t_i}$ , generates with approximate breadth from  $L_{j_1}$ .

#### D Membership Oracle Problem

In this section, we define the Membership Oracle Problem (MOP), which is required for the impossibility results of [KMV24], but not required for the characterizations in our work. For more details, we refer to Definitions 5 and 6 in [KMV24].

**Definition 19** (Membership Oracle Problem [KMV24]). *Given a generator* G, *the membership oracle problem for* G, *denoted as* MOP(G), *is defined as follows: given the description of* G *and a string* x, *output Yes if*  $x \in \text{supp}(G)$  *and output No otherwise.*