FM-TS: FLOW MATCHING FOR TIME SERIES GENERATION

Anonymous authors
Paper under double-blind review

ABSTRACT

Time series generation has emerged as an essential tool for analyzing temporal data across numerous fields. While diffusion models have recently gained significant attention in generating high-quality time series, they tend to be computationally demanding and reliant on complex stochastic processes. To address these limitations, we introduce FM-TS, a rectified Flow Matching-based framework for Time Series generation, which simplifies the time series generation process by directly optimizing continuous trajectories. This approach avoids the need for iterative sampling or complex noise schedules typically required in diffusion-based models. FM-TS is more efficient in terms of training and inference. Moreover, FM-TS is highly adaptive, supporting both conditional and unconditional time series generation. Notably, through our novel inference design, the model trained in an unconditional setting can seamlessly generalize to conditional tasks without the need for retraining. Extensive benchmarking across both settings demonstrates that FM-TS consistently delivers superior performance compared to existing approaches while being more efficient in terms of training and inference. For instance, in terms of discriminative score, FM-TS achieves 0.005, 0.019, 0.011, 0.005, 0.053, and 0.106 on the Sines, Stocks, ETTh, MuJoCo, Energy, and fMRI unconditional time series datasets, respectively, significantly outperforming the second-best method which achieves 0.006, 0.067, 0.061, 0.008, 0.122, and 0.167 on the same datasets. We have achieved superior performance in solar forecasting and MuJoCo imputation tasks, significantly enhanced by our innovative t power sampling method.

1 INTRODUCTION

Time series data is fundamental to modern data analysis, serving as a cornerstone in diverse domains such as finance, healthcare, energy management, and environmental studies (Lim and Zohren, 2021; Ye et al., 2024; Dama and Sinoquet, 2021; Liang et al., 2024). However, acquiring high-quality time series data often presents significant challenges, including stringent privacy regulations, prohibitive data collection costs, and data scarcity in certain scenarios. These challenges highlight the potential benefits of synthetic time series data, which can provide a cost-effective solution for data scarcity, overcome privacy concerns, and offer flexibility in generating diverse scenarios representing a wide range of possible patterns and trends. To obtain high-quality synthetic data, there is a pressing need for advanced time series generation techniques that can produce realistic and diverse patterns, accurately reflecting real-world complexities and variations.

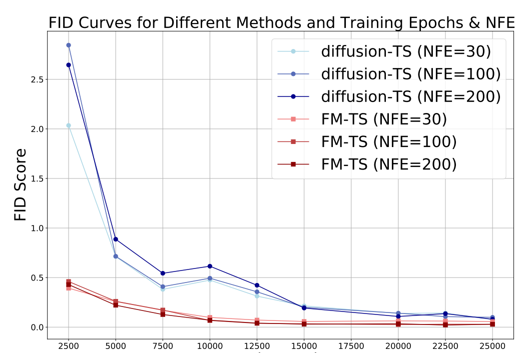


Figure 1: Comparison of FM-TS and diffusion-TS in terms of efficiency on Energy dataset under varying training epochs and number of forward evaluation steps.

Recent years have witnessed significant advancements in time series generation, ranging from VAE-based approaches (Desai et al., 2021; Xu et al., 2020) to diffusion models (Kong et al., 2021; Tashiro et al., 2021), demonstrate remarkable capabilities in capturing complex temporal dynamics.

While these studies have paved new paths for time series modeling (Coletta et al., 2023; Yoon et al., 2019a), important challenges remain in theoretical foundations and computational efficiency. Diffusion models (Ho et al., 2020; Song et al., 2020a;b) are then utilized for time series generation, yield exceptional generative quality. They offer several advantages, including their ability to capture long-range dependencies and generate diverse, high-quality samples. However, diffusion models suffer from slow generation speeds and high computational cost due to the requirement of many steps to infer (see figure 1 and (Nichol and Dhariwal, 2021)). Moreover, diffusion models struggle to preserve the long-term dependencies and intricate patterns inherent in time series data (Rasul et al., 2021).

Recently, rectified flow matching (Liu et al., 2022) has emerged as a promising generative modeling approach, because of its efficiency and capacity for scalability (Esser et al., 2024a). Rectified flow matching optimizes neural ordinary differential equation (ODE) to transport between distributions along approximately straight paths, solving a nonlinear least squares problem. This approach offers more efficient sampling than diffusion models through approximately straight paths, while providing a unified framework for generative modeling and domain transfer with theoretical guarantees on transport costs (Liu et al., 2022).

In contrast to diffusion models, rectified flow matching directly maps the latent space to the data space, whereas diffusion models must learn to denoise data based on a scheduled noise-adding process. In addition, rectified flow matching requires only a single forward pass for sampling (Liu et al., 2022), significantly enhancing both efficiency and performance. Rectified flow matching has shown superior performance in various tasks, including image generation (Kim et al., 2024; Mehta et al., 2024; Kuaishou Technology, 2024). However, it has not yet been applied to time series generation, primarily due to the unique characteristics of time series data, such as temporal dependencies and potential seasonality.

To address these challenges, we introduce FM-TS, a flow matching based framework for time series generation. Our method not only inherits the efficiency of rectified flow matching but can also generalize in both unconditional and conditional settings. The main contributions of this work are:

- FM-TS consistently outperforms existing state-of-the-art methods across a variety of time series generation datasets with notable efficiency (see Figure 1). To the best of our knowledge, this work is the first to utilize rectified flow matching to time series generation.
- For conditional time series generation, we also introduce a simple yet powerful sampling technique: t power sampling, a simple timestep shifting method (used in generation), which can boot performance of conditional generation quite a lot.
- With our novel inference design, the model trained in an unconditional setting can seamlessly generalize to conditional tasks without requiring retraining and redundant gradient-based optimization steps like (Yuan and Qiao, 2024).

The experiments on various tasks demonstrate that the proposed framework can significantly boost performance through rectified flow matching. We achieve most state-of-the-art, e.g., FM-TS can achieve context fid (lower is better) with 0.019, 0.011 on stocks, ETTh unconditional generation datasets while previous best result is 0.067, 0.061. On solar forecasting tasks, our method achieves an MSE of 213, outperforming the previous best result of 375 (Yuan and Qiao, 2024) by 43.2%.

2 Related Work

2.1 Time Series Generation

Generating realistic time series data has attracted significant attention in recent years, driven by the need for high-quality synthetic data in various domains such as finance, healthcare, and energy management (Lim and Zohren, 2021). Unconditional time series generation (Nikitin et al., 2023) is to generate time series data without specific constraints to mimic statistical properties and patterns of real data. Conditional time series generation is to Generate time series data based on specific conditions or constraints, like forecasting (Alcaraz and Strodthoff, 2022a) and imputation (Tashiro et al., 2021). Early time series generation approaches primarily utilized Generative Adversarial Networks (GANs) (Goodfellow et al., 2014). Notable works in this category include TimeGAN (Yoon et al., 2019a), which incorporates an embedding network and supervised loss to capture temporal dynamics, and RCGAN (Esteban et al., 2017), which uses a recurrent neural network architecture conditioned on auxiliary information for medical time series generation. Both TimeGAN and RCGAN are capable of conditional generation, with RCGAN specifically designed for conditional tasks, while TimeGAN can be adapted for both conditional and unconditional generation.

2.2 Diffusion Models for Time Series

Recently, diffusion models, particularly Denoising Diffusion Probabilistic Models (DDPMs) (Ho et al., 2020), have emerged as a powerful paradigm for generative modeling across various domains. Diffusion models offer better perceptual quality compared to GANs, avoiding optimization issues in adversarial training. In the context of time series, diffusion models have shown promising results in tasks such as audio synthesis (Kong et al., 2020), time series imputation (Tashiro et al., 2021), and forecasting (Rasul et al., 2021). (Rasul et al., 2021) proposed TimeGrad, a conditional diffusion model that predicts in an autoregressive manner, guided by the hidden state of a recurrent neural network. Tashiro et al. (2021) and Alcaraz and Strodthoff (2022a) adapt diffusion models for time series imputation using self-supervised masking strategies. Shen and Kwok (2023) introduced TimeDiff, a non-autoregressive diffusion model that addresses boundary disharmony issues in time series generation. For unconditional time series generation, Lim et al. (2023) employed recurrent neural networks as the backbone for generating regular 24-time-step series using Score-based Generative Models (SGMs). Kollovieh et al. (2024) proposed a self-guiding strategy for univariate time series generation and forecasting based on structured state space models. However, these methods suffer from slow generation speeds, high computational costs, and a complex sampling schedule.

2.3 Flow matching for generation

Rectified flow matching (Liu et al., 2022) is a simple ODE method for high-quality image generation and domain transfer with minimal steps, differing from diffusion models by avoiding noise and emphasizing deterministic paths. Compared to diffusion methods, it has two main advantages, stability of training and effectiveness of inference. Rectified flow matching has shown remarkable results in video generation (Kuaishou Technology, 2024), image generation stable diffusion 3 (Esser et al., 2024b) and flux (bla, 2024), point cloud generation (Wu et al., 2023) (Kim et al., 2024), protein design (Campbell et al., 2024; Jing et al., 2024), human motion generation (Hu et al., 2023), TTS (Mehta et al., 2024; Guan et al., 2024; Guo et al., 2024). Despite the great success and effectiveness of rectified flow matching, flow matching has not yet been applied to time series generation. Witnessing the great potential of flow matching for this task, that motivates to propose FM-TS for time series generation on both unconditional and conditional settings.

3 Method

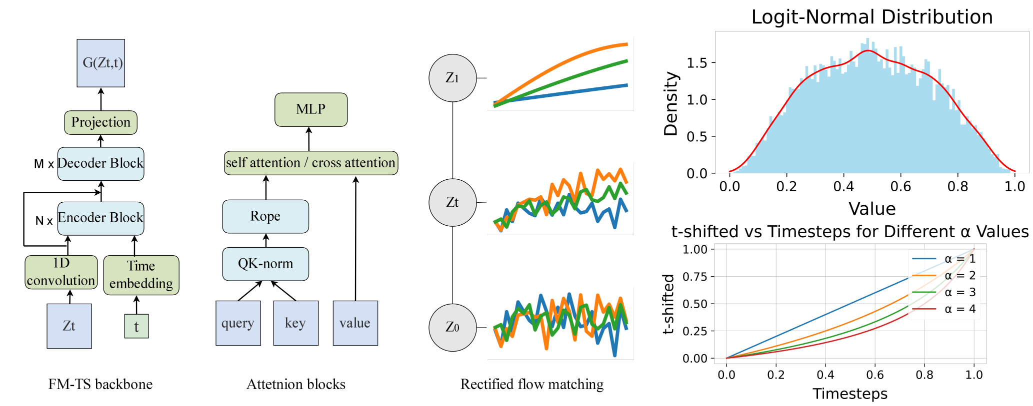
In this section, we present FM-TS, our novel framework for time series generation based on rectified flow matching. We begin by introducing the problem setting, then providing an overview of the FM-TS framework, followed by the inference pipeline of FM-TS for unconditional and conditional time series generation, respectively.

3.1 Problem Statement

Unconditional Time Series Generation Unconditional time series generation focuses on producing sequential data without any conditions, where the model learns underlying temporal patterns from a training set and generates new sequences that follow a similar distribution. Formally, the problem is defined as follows:

Let $X_{1:\ell} = (x_1, \ldots, x_\ell) \in \mathbb{R}^{\ell \times d}$ denote a time series covering ℓ time steps, where d is the dimension of observed signals.

$$\text{Input:} \quad Z_0 \sim \pi_0; \quad \text{where } Z_0 \in \mathbb{R}^{\ell \times d} \text{ and } \pi_0 \text{ is } \mathcal{N}(0, I)\,.$$
$$\text{Output:} \quad \hat{X}_{1:\ell} = G(Z_0) \in \mathbb{R}^{\ell \times d}; \quad \text{where } G \text{ transforms noise } Z_0 \text{ into the target distribution.}$$



(a) FM-TS pipeline (b) Train and inference sampling

Figure 2: **Overview of FM-TS.** (a) FM-TS pipeline. It use G as the model, which takes Z_t and t as input to generate outputs $G(Z_t, t)$ (see Eq. 3). The attention blocks in encoder/decoder blocks of G is specifically designed shown in the middle. The overall idea of learning rectified flow from Z_0 to Z_1 is illustrated in the right panel, where Z_t is a linear interpolation of Z_0 and Z_1 at timestep t. (b) The sampling strategy of training and inference. Logit-normal sampling can help the model to focus on learning the hardest part (when t is around 0.5). The t-shifting sampling in inference can generate results with better quality.

Common generative models used for G include GANs (Yoon et al., 2019b), VAEs (Desai et al., 2021), and diffusion models (Tashiro et al., 2021; Yuan and Qiao, 2024), which are capable of capturing complex temporal dependencies. During training process, G is optimized via different strategies to minimize the difference between output $\hat{X}_{1:\ell}$ and target $X_{1:\ell}$.

Conditional Time Series Generation Conditional time series generation produces sequences based on partially known data, utilizing the prior information as context. The generated sequence contains both the observed and predicted segments. Formally:

$$\begin{aligned}
\text{Input:} \quad & Z_0 \sim \pi_0, \quad y \in \mathbb{R}^{m\times d}; \quad \text{where } Z_0 \in \mathbb{R}^{\ell\times d}, \quad \pi_0 \text{ is } \mathcal{N}(0, I)\,, \\
& y \in \mathbb{R}^{m\times d} \text{ is the observed time series with length } m \text{ (where } m < \ell). \\
\text{Output:} \quad & \hat{X}_{1:\ell} = G(Z_0, y) \in \mathbb{R}^{\ell\times d}; \\
& \text{where } G \text{ transforms noise } Z_0 \text{ into the target distribution conditioned on } y.
\end{aligned}$$

Here G includes same generative models as unconditional models above.

For conditional time series generation, this can be further categorized into 2 main directions:
1) **Forecasting**: G is trained as forecasting functions that maps past observation to future predictions given $y = (x_1, x_2, ..., x_m)$.
2) **Imputation**: The model G is trained to fill in missing values at unobserved timesteps, given that y is derived from m observed timesteps within the range of 1 to ℓ.

The difference between forecasting and imputation is the position of known values. The mask $M \in \mathbb{R}^{\ell\times d}$ indicating the known/missing values which will be used in Algorithm 1.

3.2 Rectified Flow Matching for Time Series Generation.

In FM-TS, we propose to learn rectified flow as the model G for time series generation. Rectified flow (Liu et al., 2022) is a method of learning ordinary differential equation (ODE) models to transport between two empirical distributions π_0 and π_1. In our setting, π_0 is $\mathcal{N}(0, I)$, and π_1 is the target distribution, where $X_{1:\ell} \sim \pi_1$. Thus, the problem can be reformulated as: given empirical observations of two distributions $Z_0 \sim \pi_0$ and $Z_1 \sim \pi_1$, find a transport map G: $\mathbb{R}^{\ell\times d} \to \mathbb{R}^{\ell\times d}$ that can map distribution π_0 to π_1. G is designed to find the transport map between two distributions instead of pairwise mapping. After successful learning of G, we expect that $Z_1 := G(Z_0) \sim \pi_1$ when input $Z_0 \sim \pi_0$.

Algorithm 1 Inference of FM-TS for conditional generation

Input:
$\mathbf{y} \in \mathbb{R}^{l\times d}$: target time series
$\mathbf{M} \in \mathbb{R}^{l\times d}$: observation mask, where 1,0 indicates observed and missing values, respectively.
N: number of forward evaluations
G: trained flow matching model
Output:
$\hat{\mathbf{Z}}$: generated time series with condition observations.

$\hat{\mathbf{Z}} \sim \mathcal{N}(0,\mathbf{I})$, $\mathbf{Z}_0 \sim \mathcal{N}(0,\mathbf{I})$ ▷ Initialize $\hat{\mathbf{Z}}$, $\mathbf{Z}_0$
for $i \leftarrow 0$ **to** $\mathrm{N}-1$ **do**
  $t_i \leftarrow i/\mathrm{N}$
  $t_i \leftarrow t_i^k$ ▷ t_i to the power of k
  $\mathbf{Z}_0 \sim \mathcal{N}(0,\mathbf{I})$ ▷ Reinitialize noise at each step
  $\mathbf{Z}_{t_i} \leftarrow t_i\hat{\mathbf{Z}} + (1-t_i)\mathbf{Z}_0$
  $\mathbf{Z}_{t_i}[\mathbf{M}] \leftarrow t_i\mathbf{y}[\mathbf{M}] + (1-t_i)\mathbf{Z}_0[\mathbf{M}]$ ▷ Replace with observed series
  $\mathbf{v} \leftarrow G(\mathbf{Z}_{t_i}, t_i)$ ▷ Flow matching step
  $\hat{\mathbf{Z}} \leftarrow \mathbf{Z}_{t_i} + (1-t_i)\mathbf{v}$ ▷ One Euler step
end for
return $\hat{\mathbf{Z}}$

Given the empirical observations of two distributions $Z_0 \sim \pi_0$ and $Z_1 \sim \pi_1$, the rectified flow induced from (Z_0, Z_1) is an ODE on time $t \in [0,1]$,

$$\frac{dZ_t}{dt} = v(Z_t,t), where \quad t \in [0,1], Z_t \in \mathbb{R}^{\ell\times d} \tag{1}$$

where the drift force v: $\mathbb{R}^{\ell\times d} \to \mathbb{R}^{\ell\times d}$ is set to drive the flow to follow the direction $(Z_1 - Z_0)$ of the linear path between Z_0 to Z_1 as much as possible.

This can be achieved by solving a least squares regression problem:

$$\min_v \int_0^1 \mathbb{E}\left[\|Z_1 - Z_0 - v(Z_t,t)\|^2\right] dt \tag{2}$$

where Z_t is a linear interpolation between Z_0 and Z_1: $Z_t = t\cdot Z_1 + (1-t)\cdot Z_0$, where v is expected to learn with the neural network model G.

Therefore, the model G can be optimized by predicting the direction vector between $Z_1 - Z_0$ via the following loss function

$$\mathcal{L} = \mathbb{E}_{t\sim\text{Logit-Normal}}[\|(Z_1 - Z_0) - G(Z_t,t)\|^2] \tag{3}$$

where G is the model used in FM-TS to learn the drift force v. For each sample, t is randomly drawn from a Logit-Normal distribution (Esser et al., 2024b), while Z_1 is sampled from the target time series distribution π_1, and Z_0 is sampled from the standard normal distribution π_0.

The overview framework is demonstrated in Fig. 2. Here the unconditional time series generation model G can be directly trained via loss in Eq. 3 by taking Z_t and t as input to predict the drift force v between Z_0 and Z_1. Then the trained unconditional model can be directly used for conditional generation without the need for task-specific training of a conditional generation model.

3.3 Sampling Process for Inference

To generate new time series, we use a sampling process that follows the shifting of timestep schedules approach (Esser et al., 2024b). Starting from $Z_0 \sim \mathcal{N}(0,1)$, we iteratively refine it using:

$$Z_{(i+1)/N} = Z_{i/N} + (t_{i+1}^{\text{shifted}} - t_i^{\text{shifted}})\cdot G(Z_{i/N}, t_i^{\text{shifted}}) \tag{4}$$

where N is the total number of iterations, and i is iteratively updated from 0 to $N-1$, t_i^{shifted} is predefined time step at iteration i (see Eq. 5), G is the trained model.

Table 1: Unconditional time series Generation Benchmark with 24-length

Metric	Methods	Sines	Stocks	ETTh	MuJoCo	Energy	fMRI
Discriminative Score (Lower is Better)	**FM-TS**	**0.005±.005**	**0.019±.013**	**0.011±.015**	**0.005±.005**	**0.053±.010**	**0.106±.018**
	Diffusion-TS	0.006±.007	0.067±.015	0.061±.009	0.008±.002	0.122±.003	0.167±.023
	TimeGAN	0.011±.008	0.102±.021	0.114±.055	0.238±.068	0.236±.012	0.484±.042
	TimeVAE	0.041±.044	0.145±.120	0.209±.058	0.230±.102	0.499±.000	0.476±.044
	Diffwave	0.017±.008	0.232±.061	0.190±.008	0.203±.096	0.493±.004	0.402±.029
	DiffTime	0.013±.006	0.097±.016	0.100±.007	0.154±.045	0.445±.004	0.245±.051
	Cot-GAN	0.254±.137	0.230±.016	0.325±.099	0.426±.022	0.498±.002	0.492±.018
Predictive Score (Lower is Better)	**FM-TS**	**0.092±.000**	**0.036±.000**	**0.118±.005**	0.008±.001	**0.250±.000**	**0.099±.000**
	Diffusion-TS	0.093±.000	0.036±.000	0.119±.002	**0.007±.000**	**0.250±.000**	**0.099±.000**
	TimeGAN	0.093±.019	0.038±.001	0.124±.001	0.025±.003	0.273±.004	0.126±.002
	TimeVAE	0.093±.000	0.039±.000	0.126±.004	0.012±.002	0.292±.000	0.113±.003
	Diffwave	0.093±.000	0.047±.000	0.130±.001	0.013±.000	0.251±.000	0.101±.000
	DiffTime	0.093±.000	0.038±.001	0.121±.004	0.010±.001	0.252±.000	0.100±.000
	Cot-GAN	0.100±.000	0.047±.001	0.129±.000	0.068±.009	0.259±.000	0.185±.003
	Original	0.094±.001	0.036±.001	0.121±.005	0.007±.001	0.250±.003	0.090±.001
Context-FID Score (Lower is Better)	**FM-TS**	**0.002±.000**	**0.015±.003**	**0.024±.001**	**0.009±.000**	**0.031±.004**	0.128±.009
	Diffusion-TS	0.006±.000	0.147±.025	0.116±.010	0.013±.001	0.089±.024	**0.105±.006**
	TimeGAN	0.101±.014	0.103±.013	0.300±.013	0.563±.052	0.767±.103	1.292±.218
	TimeVAE	0.307±.060	0.215±.035	0.805±.186	0.251±.015	1.631±.142	14.449±.969
	Diffwave	0.014±.002	0.232±.032	0.873±.061	0.393±.041	1.031±.131	0.244±.018
	DiffTime	0.006±.001	0.236±.074	0.299±.044	0.188±.028	0.279±.045	0.340±.015
	Cot-GAN	1.337±.068	0.408±.086	0.980±.071	1.094±.079	1.039±.028	7.813±.550
Correlational Score (Lower is Better)	**FM-TS**	0.015±.006	0.012±.011	**0.022±.010**	**0.183±.051**	**0.650±.201**	**0.938±.039**
	Diffusion-TS	**0.015±.004**	**0.004±.001**	0.049±.008	0.193±.027	0.856±.147	1.411±.042
	TimeGAN	0.045±.010	0.063±.005	0.210±.006	0.886±.039	4.010±.104	23.502±.039
	TimeVAE	0.131±.010	0.095±.008	0.111±.020	0.388±.041	1.688±.226	17.296±.526
	Diffwave	0.022±.005	0.030±.020	0.175±.006	0.579±.018	5.001±.154	3.927±.049
	DiffTime	0.017±.004	0.006±.002	0.067±.005	0.218±.031	1.158±.095	1.501±.048
	Cot-GAN	0.049±.010	0.087±.004	0.249±.009	1.042±.007	3.164±.061	26.824±.449

The time steps t_i^{shifted} is generated following stable diffusion 3 (Esser et al., 2024b)-like time shifting sampling schedule. The time shifting aims to improve the quality of high-resolution image synthesis by ensuring that the model applies the appropriate amount of noise at each timestep, which is also beneficial for time series generation. The shifting schedule is shown as Figure 2b:

$$t_i^{\text{shifted}} = 1 - \frac{\alpha \cdot t_i}{1 + (\alpha - 1) \cdot t_i} \tag{5}$$

where $t_i = i/N$ with N total timesteps, and α is a hyperparameter. For reference, the visualization of the relationship between t_i^{shifted} and t_i under different α is shown as Fig. 2b. The larger α is, the more shifting scale is.

For conditional generation, a slightly different inference pipeline of FM-TS is illustrated in Algorithm 1, with the following major design changes relative to unconditional generation. ❶ t **power sampling with** k: We find that when $k < 1$, the sampling part can focus on the later sampling steps, which can be quite useful for conditional generation. (Algorithm 1 line 4). ❷ **Add noise at each step**: The algorithm adds the noise at each step(Algorithm 1 line 5). ❸ **One Euler Step Generation**: The algorithm uses one Euler step to generate $\hat{\mathbf{Z}}$ from Z_0 (Algorithm 1 line 9). With the above design, FM-TS effectively combines the strengths of flow matching with conditional information, enabling guided generation of time series data.

4 Experiments

4.1 Datasets

Our evaluation employs six diverse datasets: The 3 real-world datasets include Stocks [1] for measuring daily stock price data, ETTh [2] (Zhou et al., 2021) for interval electricity transformer data, and Energy [3] for UCI appliance energy prediction. The 3 simulation datasets include fMRI [4] for simulated

[1]https://finance.yahoo.com/quote/GOOG/history?p=GOOG

[2]https://github.com/zhouhaoyi/ETDataset

[3]https://archive.ics.uci.edu/ml/datasets/Appliances+energy+prediction

[4]https://www.fmrib.ox.ac.uk/datasets/netsim/

Table 2: Benchmark of Unconditional Long-term Time Series Generation

Dataset		Length	FM-TS	Diffusion-TS	TimeGAN	TimeVAE	Diffwave	DiffTime	Cot-GAN
ETTh	Discriminative (Lower Better)	64	**0.010±.004**	0.106±.048	0.227±.078	0.171±.142	0.254±.074	0.150±.003	0.296±.348
		128	**0.040±.012**	0.144±.060	0.188±.074	0.154±.087	0.274±.047	0.176±.015	0.451±.080
		256	0.081±.022	**0.060±.030**	0.444±.056	0.178±.076	0.304±.068	0.243±.005	0.461±.010
	Predictive (Lower Better)	64	**0.115±.005**	0.116±.000	0.132±.008	0.118±.004	0.133±.008	0.118±.004	0.135±.003
		128	**0.104±.013**	0.110±.003	0.153±.014	0.113±.005	0.129±.003	0.120±.008	0.126±.001
		256	**0.107±.005**	0.109±.013	0.220±.008	0.110±.027	0.132±.001	0.118±.003	0.129±.000
	Context-FID (Lower Better)	64	**0.039±.003**	0.631±.058	1.130±.102	0.827±.146	1.543±.153	1.279±.083	3.008±.277
		128	**0.128±.007**	0.787±.062	1.553±.169	1.062±.134	2.354±.170	2.554±.318	2.639±.427
		256	**0.302±.018**	0.423±.038	5.872±.208	0.826±.093	2.899±.289	3.524±.830	4.075±.894
	Correlational (Lower Better)	64	**0.027±.015**	0.082±.005	0.483±.019	0.067±.006	0.186±.008	0.094±.010	0.271±.007
		128	**0.030±.011**	0.088±.005	0.188±.006	0.054±.007	0.203±.006	0.113±.012	0.176±.006
		256	**0.025±.008**	0.064±.007	0.522±.013	0.046±.007	0.199±.003	0.135±.006	0.222±.010
Energy	Discriminative (Lower Better)	64	0.131±.046	**0.078±.021**	0.498±.001	0.499±.000	0.497±.004	0.328±.031	0.499±.001
		128	0.301±.013	**0.143±.075**	0.499±.001	0.499±.000	0.499±.001	0.396±.024	0.499±.001
		256	0.404±.070	**0.290±.123**	0.499±.000	0.499±.000	0.499±.000	0.437±.095	0.498±.004
	Predictive (Lower Better)	64	0.250±.009	**0.249±.000**	0.291±.003	0.302±.001	0.252±.001	0.252±.000	0.262±.002
		128	0.249±.001	**0.247±.001**	0.303±.002	0.318±.000	0.252±.000	0.251±.000	0.269±.002
		256	0.247±.001	**0.245±.001**	0.351±.004	0.353±.003	0.251±.000	0.251±.000	0.275±.004
	Context-FID (Lower Better)	64	**0.058±.010**	0.135±.017	1.230±.070	2.662±.087	2.697±.418	0.762±.157	1.824±.144
		128	0.100±..002	**0.087±.019**	2.535±.372	3.125±.106	5.552±.528	1.344±.131	1.822±.271
		256	**0.083±..011**	0.126±.024	5.052±.831	3.768±.998	5.572±.584	4.735±.729	2.533±.467
	Correlational (Lower Better)	64	**0.534±.110**	0.672±.035	3.668±.106	1.653±.208	6.847±.083	1.281±.218	3.319±.062
		128	0.521±.201	**0.451±.079**	4.790±.116	1.820±.329	6.663±.112	1.376±.201	3.713±.055
		256	0.391±.146	**0.361±.092**	4.487±.214	1.279±.114	5.690±.102	1.800±.138	3.739±.089

blood-oxygen-level-dependent time series, Sines [5] (Yoon et al., 2019b) generated from different frequencies, amplitudes, and phases, and Mujoco [6] from multivariate physics simulation.

These datasets offer a comprehensive range of time series characteristics, including periodic and aperiodic patterns, varying dimensionality, and different levels of feature correlation, allowing for a thorough evaluation of our method across diverse scenarios.

Following practices in time generation (Yuan and Qiao, 2024),We have 4 metrics to evaluate our method: 1) **Discriminative Score** (Yoon et al., 2019b): Measures distributional similarity between real and synthetic data. A post-hoc time series classification model (2-layer LSTM) is trained to distinguish between real and synthetic sequences. The classification error on a held-out test set is reported, with lower scores indicating higher quality synthetic data. 2) **Predictive Score** (Yoon et al., 2019b): Assesses the usefulness of synthetic data for predictive tasks. A post-hoc sequence prediction model (2-layer LSTM) is trained on synthetic data to predict next-step temporal vectors. The model is then evaluated on real data, with performance measured by mean absolute error (MAE). Lower scores indicate better preservation of predictive characteristics in synthetic data. 3) **Context-Fréchet Inception Distance (Context-FID)** (Jeha et al., 2022): Quantifies the quality of synthetic time series by computing the difference between representations that fit into the local context. This metric captures both distributional similarity and temporal dependencies. 4) **Correlational Score** (Liao et al., 2020): Evaluates the preservation of temporal dependencies by comparing cross-correlation matrices of real and synthetic data. The absolute error between these matrices is computed, with lower scores indicating better preservation of temporal structure.

4.2 Implementation details

Our FM-TS model adapts the rectified flow matching approach for time series data. The architecture is based on an encoder-decoder transformer, similar to the model in Diffusion-TS (Yuan and Qiao, 2024), but with several key enhancements: QK-RMSNorm (bla, 2024), RoPE (Su et al., 2024), Logit-Normal sampling strategy (Esser et al., 2024b), Attention register (Darcet et al., 2023; Xiao et al., 2023) and Sigmoid attention (Ramapuram et al., 2024). We set the default values of alpha to 3 and k to 0.0625 (with k specifically applied in conditional generation tasks). For more details, please see Supplementary materials.

4.3 Unconditional time series generation

[5]https://github.com/jsyoon0823/TimeGAN
[6]https://github.com/google-deepmind/dm_control

We benchmarked FM-TS against other methods for unconditional time series generation across six datasets. As shown in Table 1, FM-TS consistently outperforms other methods on most evaluation metrics. On the discriminative score, FM-TS achieves 0.005, 0.019, 0.011, 0.005, 0.053, and 0.106 on the Sines, Stocks, ETTh, MuJoCo, Energy, and fMRI datasets, respectively. In comparison, the second-best method, Diffusion-TS, achieves 0.006, 0.067, 0.061, 0.008, 0.122, and 0.167 on the same datasets. This represents a reduction in discriminative score ranging from 17% to 82%, validating FM-TS's great improvement. We attribute this superior performance to the synergy of rectified flow matching with time series-specific optimizations.

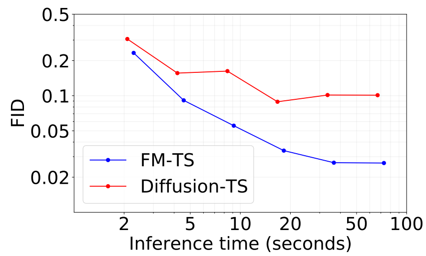


Figure 3: FID results with different N, the N list is 1, 2, 4, 8, 16, 32.

In Table 2, we extended unconditional time series generation to longer sequences (64, 128, 256) on ETTh and Energy datasets. We observe FM-TS excels on the ETTh dataset, achieving best scores in 11 out of 12 metrics (except on Discriminative score with 256-length on Energy dataset) across all lengths, with particularly strong performance in Context-FID. On the Energy dataset, FM-TS shows mixed results, outperforming in Context-FID but a little bit falling behind Diffusion-TS in others, suggesting dataset-specific characteristics may influence its effectiveness on longer sequences.

4.4 Efficiency Benchmark of FM-TS

Compared to Diffusion-TS (Yuan and Qiao, 2024), FM-TS not only delivers superior performance across various settings but also demonstrates significantly better efficiency in both training and inference. To evaluate training efficiency, we benchmarked FM-TS and Diffusion-TS across multiple training epochs on the Energy dataset. As shown in Figure 1, We observe that FM-TS consistently achieves superior FID scores compared to Diffusion-TS, with training epochs ranging from 2,500 to 25,000. Notably, FM-TS outperforms even with as few as 30 iterations ($N = 30$), whereas Diffusion-TS can not achieve even with 200 inference steps. The observed efficiency in terms of required iterations N can be attributed to the straightness property of rectified flow matching, a phenomenon extensively studied by Liu et al. (2022). To further assess inference efficiency, we compared the final models of FM-TS and Diffusion-TS, testing different numbers of iterations (N) during sampling for inference. As seen in Figure 3, FM-TS not only delivers better performance but also achieves faster inference times compared to Diffusion-TS, highlighting its efficiency advantages. This empirical evidence indicates that FM-TS is capable of facilitating more rapid and accurate time series generation.

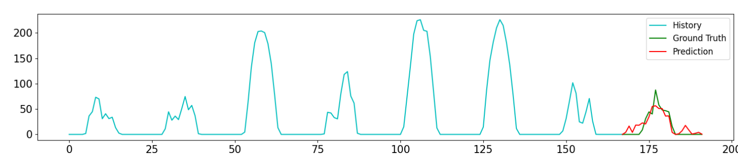


Figure 4: An example of solar forecasting results.

4.5 Conditional time series generation

After validating FM-TS on unconditional time series generation, we further assessed its generalizability for conditional time series generation. Instead of retraining the model, we employed the specialized inference algorithm, detailed in Algorithm 1, to incorporate observed information into inference for conditional setting. As stated in Section 3.1, conditional time series generation includes two primary tasks: forecasting and imputation. To demonstrate the effectiveness of FM-TS, following the practice in (Alcaraz and Strodthoff, 2022a) and (Tashiro et al., 2021), we benchmarked it on Solar and Mujoco datasets.

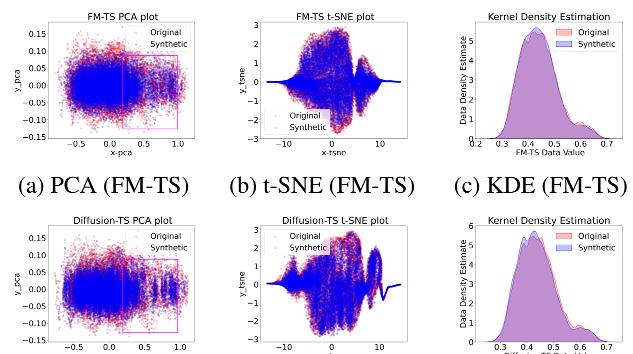
Table 3 presents the forecasting performance on the Solar dataset. Given a sequence length of 168, FM-TS achieved a superior mean-squared-error of 2.18e2 when predicting the next 24 time points, significantly outperforming the second-best model, Diffusion-TS, which scored 3.75e2. This highlights the substantial improvement in prediction accuracy and sequence alignment with FM-TS in forecasting. In Fig. 4, we presented an example of the forecasting results by FM-TS and target, where FM-TS successfully captures the incoming peak region in the future time series. Additionally, we evaluated FM-TS on the imputation task (following setting of (Alcaraz and Strodthoff, 2022b)) using the MuJoCo dataset in Table 3, where it consistently outperformed other methods under varying missing data ratios. Despite most competing methods being specifically designed for conditional time series generation, FM-TS demonstrated superior performance across multiple scenarios. The Mean Squared Error (MSE) for missing rate 70% condition has decreased from 0.00027 of Diffusion-TS to 0.00007, representing a substantial 74.1% reduction.

Table 3: Time Series Forecasting and Imputation Results

Model	Solar Forecasting	Mujoco Imputation	
	168 → 24	Missing(70 %)	Missing(80 %)
GP-copula	9.8e2	–	–
TransMAF	9.30e2	–	–
TLAE	6.8e2	–	–
RNN GRU-D	–	11.34	14.21
ODE-RNN	–	9.86	12.09
NeuralCDE	–	8.35	10.71
Latent-ODE	–	3.00	2.95
NAOMI	–	1.46	2.32
NRTSI	–	0.63	1.22
CSDI	9.0e2	0.24	0.61
SSSD	5.03e2	0.59	1.00
Diffusion-TS	3.75e2	0.00027	0.00032
FM-TS	**2.13e2**	**0.00007**	**0.00014**

4.6 Visualization comparison of FM-TS

To offer a more direct comparison between generated and target sequences, we followed the practices outlined in (Yuan and Qiao, 2024), mapping both generated and target sequences into an embedding space using PCA (Shlens, 2014) and t-SNE (Van der Maaten and Hinton, 2008). In Fig. 5a, 5d, 5b, 5e, we present a comparison of PCA and t-SNE visualizations between sequences generated by FM-TS and Diffusion-TS, as well as the corresponding target sequences. It is evident that the embeddings from FM-TS show greater consistency with the target sequences in both visualizations, highlighting the superior performance of FM-TS. We further analyzed the results using kernel density estimation (KDE) (Chen, 2017), shown in Fig. 5c and 5f. The KDE for FM-TS aligns more closely with the target sequences, especially on the right slope, where Diffusion-TS exhibits noticeable fluctuations, further validating FM-TS's superior accuracy.



(a) PCA (FM-TS) (b) t-SNE (FM-TS) (c) KDE (FM-TS)

(d) PCA (Diffusion-TS) (e) t-SNE (Diffusion-TS) (f) KDE (Diffusion-TS)

Figure 5: Embedding visualization comparison of generated sequences by FM-TS and Diffusion-TS methods relative to the target sequences using PCA, t-SNE, and Kernel Density Estimation. Here red indicates the target sequences, where blue indicates the generated sequences.

4.7 Ablation Study

In this section, we will study the key components in FM-TS framework to understand their contributions.

Logit-Normal distribution for training t sampling In Table 4, we compared the performances on energy dataset on the 4 metrics with uniform distribution and our default logit-normal distribution for training. It is clear that logit-normal distribution shown in Fig. 2b is essential for training a stable and accurate model. That validates our assumption that distribution can encourage model to learn the hardest information.

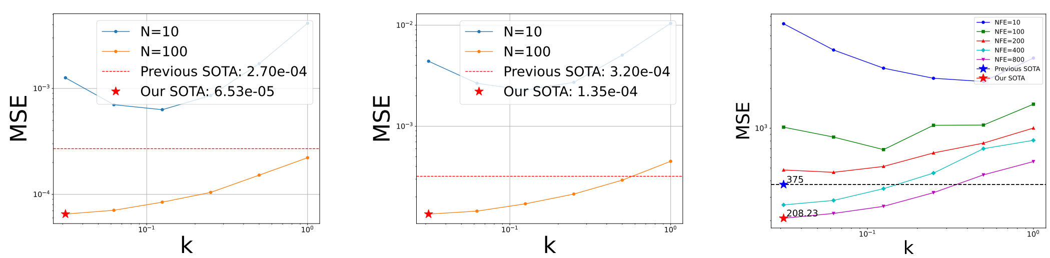
Table 4: Training Sampling Strategy Comparison

Method	FID	CS	DS	PS
FM-TS	0.028	0.721	0.058	0.250
uniform sampling	0.029	0.676	0.056	0.250

Number of iterations N

The number of iteration steps is a critical factor in balancing performance and efficiency. As shown in Fig. 3, performance begins to saturate when $N = 32$. Based on a comparison with Diffusion-TS, we identify $N = 32$ as the optimal point for achieving a balance between performance and computational efficiency. That validates our assumption FM-TS is both accurate and efficient compared to that of Diffusion-TS.

t power sampling factor k for conditional generation In Algorithm 1, we proposed to use power sampling factor k to control conditional time series generation. In Fig. 6, we compared different k for generation under different number of inference iterations N. When N becomes larger, which indicates the inference becomes more stable, we found that a small k can lead to better performance. The effectiveness of conditional generation can be significantly improved by focusing on later sampling steps in the diffusion process. Setting $k < 1$ in t^k, where $t \in (0, 1]$, enables more effective conditioning. For instance, with $t = 0.25$ and $k = 0.5$, $t^k = 0.5$ represents a later time step than t, bringing generated samples closer to the target distribution.



(a) MSE with changing N and k on solar dataset imputation tasks, with missing ratio 0.7

(b) MSE with changing N and k on solar dataset imputation tasks, with missing ratio 0.8

(c) MSE with changing N and k on Mujoco dataset forecasting tasks

Figure 6: Conditional generation with different k

5 Conclusion

We introduced FM-TS, a novel time series generation framework based on rectified flow matching. FM-TS achieves efficient one-pass generation while maintaining high-quality output. Experimental results demonstrate FM-TS's superior speed in training and inference, consistently outperforming state-of-the-art methods across various datasets and tasks in both conditional and unconditional generation. A key innovation of FM-TS is the novel t power sampling technique, which significantly enhances performance in conditional generation tasks. By using t^k with $k < 1$, the model focuses on later steps in the generation process, allowing for more effective incorporation of conditional information. This adaptive sampling strategy proves particularly beneficial in tasks like forecasting and imputation, where FM-TS demonstrates substantial improvements over existing methods.

References

Bryan Lim and Stefan Zohren. Time-series forecasting with deep learning: a survey. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 379 (2194):20200209, February 2021. ISSN 1471-2962. doi: 10.1098/rsta.2020.0209. URL http://dx.doi.org/10.1098/rsta.2020.0209.

Jiexia Ye, Weiqi Zhang, Ke Yi, Yongzi Yu, Ziyue Li, Jia Li, and Fugee Tsung. A survey of time series foundation models: Generalizing time series representation with large language model, 2024. URL https://arxiv.org/abs/2405.02358.

Fatoumata Dama and Christine Sinoquet. Time series analysis and modeling to forecast: a survey, 2021. URL https://arxiv.org/abs/2104.00164.

Yuxuan Liang, Haomin Wen, Yuqi Nie, Yushan Jiang, Ming Jin, Dongjin Song, Shirui Pan, and Qingsong Wen. Foundation models for time series analysis: A tutorial and survey. In *Proceedings of the 30th ACM SIGKDD Conference on Knowledge Discovery and Data Mining*, volume 619 of *KDD '24*, page 6555–6565. ACM, August 2024. doi: 10.1145/3637528.3671451. URL http://dx.doi.org/10.1145/3637528.3671451.

Abhyuday Desai, Cynthia Freeman, Zuhui Wang, and Ian Beaver. Timevae: A variational auto-encoder for multivariate time series generation, 2021. URL https://arxiv.org/abs/2111.08095.

Tianlin Xu, Li K. Wenliang, Michael Munn, and Beatrice Acciaio. Cot-gan: Generating sequential data via causal optimal transport, 2020. URL https://arxiv.org/abs/2006.08571.

Zhifeng Kong, Wei Ping, Jiaji Huang, Kexin Zhao, and Bryan Catanzaro. Diffwave: A versatile diffusion model for audio synthesis, 2021. URL https://arxiv.org/abs/2009.09761.

Yusuke Tashiro, Jiaming Song, Yang Song, and Stefano Ermon. Csdi: Conditional score-based diffusion models for probabilistic time series imputation. *Advances in Neural Information Processing Systems*, 34:24804–24816, 2021.

Andrea Coletta, Sriram Gopalakrishan, Daniel Borrajo, and Svitlana Vyetrenko. On the constrained time-series generation problem, 2023. URL https://arxiv.org/abs/2307.01717.

Jinsung Yoon, Daniel Jarrett, and Mihaela van der Schaar. Time-series generative adversarial networks. In H. Wallach, H. Larochelle, A. Beygelzimer, F. d'Alché-Buc, E. Fox, and R. Garnett, editors, *Advances in Neural Information Processing Systems*, volume 32. Curran Associates, Inc., 2019a. URL https://proceedings.neurips.cc/paper_files/paper/2019/file/c9efe5f26cd17ba6216bbe2a7d26d490-Paper.pdf.

Jonathan Ho, Ajay Jain, and Pieter Abbeel. Denoising diffusion probabilistic models. *arXiv preprint arxiv:2006.11239*, 2020.

Yang Song, Jascha Sohl-Dickstein, Diederik P Kingma, Abhishek Kumar, Stefano Ermon, and Ben Poole. Score-based generative modeling through stochastic differential equations. *arXiv preprint arXiv:2011.13456*, 2020a.

Jiaming Song, Chenlin Meng, and Stefano Ermon. Denoising diffusion implicit models. *arXiv preprint arXiv:2010.02502*, 2020b.

Alexander Quinn Nichol and Prafulla Dhariwal. Improved denoising diffusion probabilistic models. In *International conference on machine learning*, pages 8162–8171. PMLR, 2021.

Kashif Rasul, Calvin Seward, Ingmar Schuster, and Roland Vollgraf. Autoregressive denoising diffusion models for multivariate probabilistic time series forecasting. In *International Conference on Machine Learning*, pages 8857–8868. PMLR, 2021.

Xingchao Liu, Chengyue Gong, and Qiang Liu. Flow straight and fast: Learning to generate and transfer data with rectified flow. *arXiv preprint arXiv:2209.03003*, 2022.

Patrick Esser, Sumith Kulal, Andreas Blattmann, Rahim Entezari, Jonas Müller, Harry Saini, Yam Levi, Dominik Lorenz, Axel Sauer, Frederic Boesel, Dustin Podell, Tim Dockhorn, Zion English, Kyle Lacey, Alex Goodwin, Yannik Marek, and Robin Rombach. Scaling rectified flow transformers for high-resolution image synthesis, 2024a. URL https://arxiv.org/abs/2403.03206.

Sungwon Kim, Kevin Shih, Joao Felipe Santos, Evelina Bakhturina, Mikyas Desta, Rafael Valle, Sungroh Yoon, Bryan Catanzaro, et al. P-flow: a fast and data-efficient zero-shot tts through speech prompting. *Advances in Neural Information Processing Systems*, 36, 2024.

Shivam Mehta, Ruibo Tu, Jonas Beskow, Éva Székely, and Gustav Eje Henter. Matcha-TTS: A fast TTS architecture with conditional flow matching. In *Proc. ICASSP*, 2024.

Kuaishou Technology. Klingai. https://klingai.kuaishou.com/, 2024. URL https://klingai.kuaishou.com/. Accessed: DATE.

Xinyu Yuan and Yan Qiao. Diffusion-TS: Interpretable diffusion for general time series generation. In *The Twelfth International Conference on Learning Representations*, 2024. URL https://openreview.net/forum?id=4h1apFjO99.

Alexander Nikitin, Letizia Iannucci, and Samuel Kaski. Tsgm: A flexible framework for generative modeling of synthetic time series. *arXiv preprint arXiv:2305.11567*, 2023.

Juan Miguel Lopez Alcaraz and Nils Strodthoff. Diffusion-based time series imputation and forecasting with structured state space models. *arXiv preprint arXiv:2208.09399*, 2022a.

Ian Goodfellow, Jean Pouget-Abadie, Mehdi Mirza, Bing Xu, David Warde-Farley, Sherjil Ozair, Aaron Courville, and Yoshua Bengio. Generative adversarial nets. *Advances in neural information processing systems*, 27, 2014.

Cristóbal Esteban, Stephanie L. Hyland, and Gunnar Rätsch. Real-valued (medical) time series generation with recurrent conditional gans, 2017. URL https://arxiv.org/abs/1706.02633.

Zhifeng Kong, Wei Ping, Jiaji Huang, Kexin Zhao, and Bryan Catanzaro. Diffwave: A versatile diffusion model for audio synthesis. *arXiv preprint arXiv:2009.09761*, 2020.

Lifeng Shen and James Kwok. Non-autoregressive conditional diffusion models for time series prediction. In *International Conference on Machine Learning*, pages 31016–31029. PMLR, 2023.

Haksoo Lim, Minjung Kim, Sewon Park, and Noseong Park. Regular time-series generation using sgm, 2023. URL https://arxiv.org/abs/2301.08518.

Marcel Kollovieh, Abdul Fatir Ansari, Michael Bohlke-Schneider, Jasper Zschiegner, Hao Wang, and Yuyang Bernie Wang. Predict, refine, synthesize: Self-guiding diffusion models for probabilistic time series forecasting. *Advances in Neural Information Processing Systems*, 36, 2024.

Patrick Esser, Sumith Kulal, Andreas Blattmann, Rahim Entezari, Jonas Müller, Harry Saini, Yam Levi, Dominik Lorenz, Axel Sauer, Frederic Boesel, Dustin Podell, Tim Dockhorn, Zion English, Kyle Lacey, Alex Goodwin, Yannik Marek, and Robin Rombach. Scaling rectified flow transformers for high-resolution image synthesis, 2024b. URL https://arxiv.org/abs/2403.03206.

Black forest labs - artificial intelligence solutions. https://blackforestlabs.ai/, 2024. Accessed: 2024-09-25.

Lemeng Wu, Dilin Wang, Chengyue Gong, Xingchao Liu, Yunyang Xiong, Rakesh Ranjan, Raghuraman Krishnamoorthi, Vikas Chandra, and Qiang Liu. Fast point cloud generation with straight flows. In *Proceedings of the IEEE/CVF conference on computer vision and pattern recognition*, pages 9445–9454, 2023.

Andrew Campbell, Jason Yim, Regina Barzilay, Tom Rainforth, and Tommi Jaakkola. Generative flows on discrete state-spaces: Enabling multimodal flows with applications to protein co-design. *arXiv preprint arXiv:2402.04997*, 2024.

Bowen Jing, Bonnie Berger, and Tommi Jaakkola. Alphafold meets flow matching for generating protein ensembles. *arXiv preprint arXiv:2402.04845*, 2024.

Vincent Tao Hu, Wenzhe Yin, Pingchuan Ma, Yunlu Chen, Basura Fernando, Yuki M Asano, Efstratios Gavves, Pascal Mettes, Bjorn Ommer, and Cees GM Snoek. Motion flow matching for human motion synthesis and editing. *arXiv preprint arXiv:2312.08895*, 2023.

Wenhao Guan, Qi Su, Haodong Zhou, Shiyu Miao, Xingjia Xie, Lin Li, and Qingyang Hong. Reflow-tts: A rectified flow model for high-fidelity text-to-speech. In *ICASSP 2024 - 2024 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, pages 10501–10505, 2024. doi: 10.1109/ICASSP48485.2024.10447822.

Yiwei Guo, Chenpeng Du, Ziyang Ma, Xie Chen, and Kai Yu. Voiceflow: Efficient text-to-speech with rectified flow matching. In *ICASSP 2024 - 2024 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, pages 11121–11125, 2024. doi: 10.1109/ICASSP48485.2024.10445948.

Jinsung Yoon, Daniel Jarrett, and Mihaela Van der Schaar. Time-series generative adversarial networks. *Advances in neural information processing systems*, 32, 2019b.

Haoyi Zhou, Shanghang Zhang, Jieqi Peng, Shuai Zhang, Jianxin Li, Hui Xiong, and Wancai Zhang. Informer: Beyond efficient transformer for long sequence time-series forecasting. In *The Thirty-Fifth AAAI Conference on Artificial Intelligence, AAAI 2021, Virtual Conference*, volume 35, pages 11106–11115. AAAI Press, 2021.

Paul Jeha, Michael Bohlke-Schneider, Pedro Mercado, Shubham Kapoor, Rajbir Singh Nirwan, Valentin Flunkert, Jan Gasthaus, and Tim Januschowski. Psa-gan: Progressive self attention gans for synthetic time series. In *The Tenth International Conference on Learning Representations*, 2022.

Shujian Liao, Hao Ni, Lukasz Szpruch, Magnus Wiese, Marc Sabate-Vidales, and Baoren Xiao. Conditional sig-wasserstein gans for time series generation. *arXiv preprint arXiv:2006.05421*, 2020.

Jianlin Su, Murtadha Ahmed, Yu Lu, Shengfeng Pan, Wen Bo, and Yunfeng Liu. Roformer: Enhanced transformer with rotary position embedding. *Neurocomputing*, 568:127063, 2024.

Timothée Darcet, Maxime Oquab, Julien Mairal, and Piotr Bojanowski. Vision transformers need registers. *arXiv preprint arXiv:2309.16588*, 2023.

Guangxuan Xiao, Yuandong Tian, Beidi Chen, Song Han, and Mike Lewis. Efficient streaming language models with attention sinks. *arXiv*, 2023.

Jason Ramapuram, Federico Danieli, Eeshan Dhekane, Floris Weers, Dan Busbridge, Pierre Ablin, Tatiana Likhomanenko, Jagrit Digani, Zijin Gu, Amitis Shidani, and Russ Webb. Theory, analysis, and best practices for sigmoid self-attention, 2024. URL https://arxiv.org/abs/2409.04431.

Juan Lopez Alcaraz and Nils Strodthoff. Diffusion-based time series imputation and forecasting with structured state space models. *Transactions on Machine Learning Research*, 2022b. ISSN 2835-8856. URL https://openreview.net/forum?id=hHiIbk7ApW.

Jonathon Shlens. A tutorial on principal component analysis, 2014. URL https://arxiv.org/abs/1404.1100.

Laurens Van der Maaten and Geoffrey Hinton. Visualizing data using t-sne. *Journal of machine learning research*, 9(11), 2008.

Yen-Chi Chen. A tutorial on kernel density estimation and recent advances, 2017. URL https://arxiv.org/abs/1704.03924.