Provably Learning from Language Feedback

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Keywords: large language models, sequential decision-making, no-regret learning, bandit.

Summary

Interactively learning from observation and language feedback is an increasingly studied area driven by the emergence of large language model (LLM) agents. While impressive empirical demonstrations have been shown, so far a principled framing of these decision problems remains lacking. In this paper, we formalize the Learning from Language Feedback (LLF) problem, assert sufficient assumptions to enable learning despite latent rewards, and introduce "transfer eluder dimension" as a complexity measure to characterize the hardness of LLF problems. We show that the transfer eluder dimension captures the intuition that information in feedback changes the learning complexity of LLF. We demonstrate cases where learning from rich language feedback can be exponentially faster than learning from reward. We develop a no-regret algorithm, called LLF-UCB, that provably solves LLF problems through sequential interactions, with performance guarantees that scale with the transfer eluder dimension of the problem. Our contributions mark a first step towards designing principled agent learning from generic language feedback.

Contribution(s)

1. We formalize the interface in which agents sequentially interact while reasoning with feedback produced by an underlying hypothesis (summarized by Fig. 1) and define a verifier which evaluates the semantic consistency between candidate hypotheses and observed feedback. Through the notion of hypothesis and verifier, we give a precise definition of informative feedback and establish conditions such that LLF is feasible and can be efficiently solved.
   Context: To work with the generality of language, we rely on the concept of hypothesis testing and elimination in machine learning (De Jong et al., 1993; Lehmann & Romano, 2022) except with hypotheses that can be expressed in words.
2. We capture the learning difficulty with a new notion of complexity, which we call *transfer eluder dimension*. This complexity measure captures how efficiently language feedback can reduce uncertainty about rewards.
   Context: This complexity measure is based on eluder dimension (Russo & Van Roy, 2013) and adapted to the LLF setting.
3. We develop LLF-UCB, a provably efficient algorithm for LLF. We prove that LLF-UCB achieves a regret bound that scales gracefully with the transfer eluder dimension and time horizon T. Crucially, our analysis shows that in certain environments, LLF-UCB can be *exponentially* more efficient than learning from reward alone.
   Context: Our result marks the first formal connection between no-regret learning and language feedback.
4. We empirically validate the efficacy of LLF-UCB by implementing an approximate version that utilizes LLMs as verifiers. Our experiments on Wordle, Battleship and Minesweeper confirm that LLF-UCB and its variants consistently outperform in-context learning LLM baselines.
   Context: We compare to the ReAct (Yao et al., 2023) baseline agent.

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Abstract

Interactively learning from observation and language feedback is an increasingly studied area driven by the emergence of large language model (LLM) agents. While impressive empirical demonstrations have been shown, so far a principled framing of these decision problems remains lacking. In this paper, we formalize the Learning from Language Feedback (LLF) problem, assert sufficient assumptions to enable learning despite latent rewards, and introduce "transfer eluder dimension" as a complexity measure to characterize the hardness of LLF problems. We show that the transfer eluder dimension captures the intuition that information in feedback changes the learning complexity of LLF. We demonstrate cases where learning from rich language feedback can be exponentially faster than learning from reward. We develop a no-regret algorithm, called LLF-UCB, that provably solves LLF problems through sequential interactions, with performance guarantees that scale with the transfer eluder dimension of the problem. Our contributions mark a first step towards designing principled agent learning from generic language feedback.

1 Introduction

Large language models (LLMs) have reshaped the landscape of how machines learn and interact with the world, demonstrating remarkable capabilities across a wide range of tasks (Bommasani et al., 2021; BIG-bench authors, 2023; Anil et al., 2024; Hurst et al., 2024; Jaech et al., 2024; Guo et al., 2025; Yamada et al., 2025). Trained on large corpra of web data, these models can interact with the world through natural language, opening up new settings for sequential decision-making problems. Unlike traditional sequential decision-making approaches where agents learn from scalar reward signals (Sutton & Barto, 2018), LLM can act as agents that interpret and reason with natural language feedback such as critique (Du et al., 2023; Akyürek et al., 2023a), guidance (Fu et al., 2024; Nie et al., 2023; Wei et al., 2024; Cheng et al., 2024), or detailed explanations (Chen et al., 2023; Cheng et al., 2023).

Consider an LLM agent that produces a summary of a story, and receives feedback: "The summary is mostly accurate, but it overlooks the main character's motivation." Such feedback conveys notably richer information than a numerical score, e.g., 0.7 out of 1, as it identifies a specific flaw and suggests a direction for improvement. With LLMs' abilities to understand and respond in natural language Touvron et al. (2023), such feedback can be leveraged to drastically increase learning efficiency. This represents a fundamental shift in how AI systems can learn through continuous, rich interactions beyond rewards alone (Silver & Sutton, 2025). Despite promising empirical results in utilizing language feedback for sequential decision-making (Liu et al., 2023; Chen et al., 2024), a rigorous theoretical framework remains lacking.

We introduce a formal framework of Learning from Language Feedback (LLF), the first mathematical model of learning from language feedback in decision making. The LLF paradigm was proposed in (Cheng et al., 2023) as an interface to benchmark LLM agents' learning ability, which generalizes the classical learning-from-reward reinforcement learning setting to general in-context problem solving by replacing numerical rewards with text feedback. However, it is unclear when LLF is feasible or whether it is harder to learn than the more traditional reward-aware RL setting. Intuitively, one might think language feedback can provide more information to help learning. Indeed, people have empirically found constructive feedback to be more effective for LLM agents to learn from than conveying reward alone in words (Mu et al., 2022; Liu et al., 2024; Zhong et al., 2024). The complexity and generality of language make it difficult to formally quantify the information in language feedback. For general language feedback, can we precisely define helpful versus noisy feedback? Can we capture the complexity of LLF based on the information in feedback and does constructive feedback indeed lead to a lower problem complexity? Can we design a provably correct algorithm that learn solely from language? The goal of this paper is to provide constructive answers to all these questions.

To work with the generality of language, we rely on the concept of hypothesis testing and elimination in machine learning (De Jong et al., 1993; Lehmann & Romano, 2022) except with hypotheses that can be expressed in words. We formalize the interface in which agents sequentially interact while reasoning with feedback produced by an underlying hypothesis (summarized by Fig. 1). We also define a verifier which evaluates the semantic consistency between candidate hypotheses and observed feedback. Through the notion of hypothesis and verifier, we give a precise definition of informative feedback and establish conditions such that LLF is feasible and can be efficiently solved.

Specifically, we capture the learning difficulty with a new notion of complexity based on eluder dimension (Russo & Van Roy, 2013), which we call *transfer eluder dimension*. This complexity measure captures how efficiently language feedback can reduce uncertainty about rewards. Building on this concept, we develop LLF-UCB, a provably efficient algorithm for LLF. We prove that LLF-UCB achieves a regret bound that scales gracefully with the transfer eluder dimension and time horizon T, establishing the first formal connection between no-regret learning and language feedback. Crucially, our analysis shows that in certain environments, LLF-UCB can be *exponentially* more efficient than learning from reward alone. We empirically validate the efficacy of LLF-UCB by implementing an approximate version that utilizes LLMs as verifiers. Our experiments on Wordle, Battleship and Minesweeper confirm that LLF-UCB and its variants consistently outperform in-context learning LLM baselines. Altogether, our work contributes a first principled framework for understanding and designing learning agents guided by language.

2 Formulating Learning from Language Feedback

In this section, we give a formal mathematical model to describe the LLF process (illustrated by Fig. 1) and introduce natural assumptions to frame the learning problem so that LLF can be rigorously studied. In what follows, we first define the interaction setup. Then we introduce the notion of text hypotheses for world modeling. Finally, we define the verifier to evaluate hypothesis-feedback consistency, which later gives a measure on the informativeness of feedback. These constructions provide a basis for studying LLF's learnability and analyzing regret in the next section.

2.1 Formal Setup of LLF

Let $\mathcal{T}$ be a finite set of tokens. We denote the set of all finite token sequences by $\mathcal{T}^+ = \cup_{k\geq 1}\mathcal{T}^k \cup \{\emptyset\}$, where $\mathcal{T}^k$ denotes the set of length-K token sequences. There is a set $\mathcal{O} \subset \mathcal{T}^+$ of token sequences that we refer to as the *feedback* space. For an arbitrary set $\mathcal{X}$, we use $\Delta(\mathcal{X})$ to denote the set of all probability distributions with support on $\mathcal{X}$.

We define the problem of Learning from Language Feedback (LLF)[1] with a finite action set $\mathcal{A}$. At time step t, the agent interacts with the environment by executing an action $A_t \in \mathcal{A}$ and observing feedback $O_t \in \mathcal{O}$ sampled from a feedback distribution $f^* : \mathcal{A} \to \Delta(\mathcal{O})$; a reward $R_t = r^*(A_t)$ is incurred, based on a reward function $r^* : \mathcal{A} \to [0, 1]$, though R_t is not revealed to the agent. Here we suppose the reward is generated by a deterministic function r^*; our results can be extended to

[1]In the original formulation in (Cheng et al., 2023), a problem context is given before learning to provide background to interpret feedback. We omit writing the problem context for simplicity but equivalently *assume that the agent can interpret the feedback through the verifier* that we will introduce later.

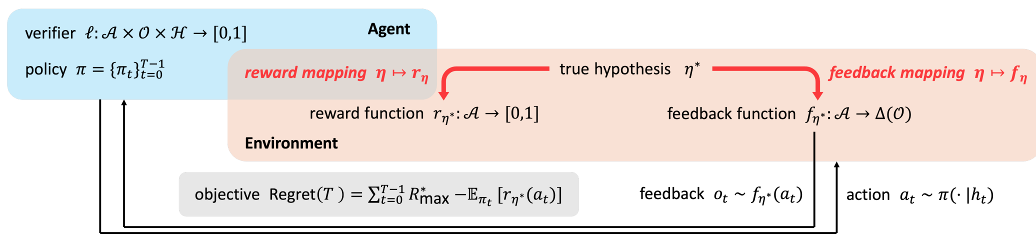


Figure 1: The LLF setup. The environment has a hypothesis η^* representable via text tokens unknown to the agent. Reward as a function of η^* is latent and used only to benchmark the agent via regret to an optimal policy. Feedback as a function of η^* is observed by the agent. Three ingredients are sufficient for no-regret learning: feedback is *unbiased* (Assumption 3), agent can interpret feedback (Assumption 2), and agent considers hypotheses $\mathcal{H}$ including η^* (precursor to Assumption 1).

stochastic rewards. A policy is a distribution on $\mathcal{A}$. We denote $\Pi = \Delta(\mathcal{A})$ and the agent's policy at time step t for sampling A_t as π_t. We measure the performance of the agent in the LLF setup by regret, which is defined as $\mathrm{Regret}(T) = \sum_{t=0}^{T-1} R^*_{\max} - \mathbb{E}_{\pi_t}[R_t]$, where T is the total number time steps, $R^*_{\max} = \max_{a \in \mathcal{A}} r^*(a)$, and the expectation is taken over feedback randomness and the algorithm's inner randomization.

This setup is similar to a bandit problem in RL, and the goal of the agent is to find actions that maximize the reward. However, unlike RL, here the agent *does not observe the rewards* $\{R_t\}$, and must learn to maximize the reward solely using natural language feedback $\{O_t\}$.

Remark 1. The setup above can be naturally extended to a contextual setting (an analogy of contextual bandit problems; please see Appendix D.2 for details), where the agent receives a context in each time step before taking an action. While the feedback in the context-less setting here may be viewed similar to a context, the main difference is that the optimal actions in the context-less setting do not change between iterations; on the other hand, in the contextual setting, the optimal actions in each time step depend on the context presented to the agent at that point.

2.2 Environment Model and Text Hypothesis

The environment in the LLF setup is defined by a feedback function $f^* : A \to \Delta(\mathcal{O})$ and a reward function $r^* : \mathcal{A} \to [0,1]$. We suppose they are "parameterized" by some text description, which we call a *hypothesis*, belonging to a (possibly exponentially large) hypothesis space $\mathcal{H} \subset \mathcal{T}^+$. One can think of a hypothesis as describing the learning problem and mechanism of generating feedback in texts such as natural language or codes. For example, in a recommendation environment, a hypothesis can be a text description of a user's interests, or in a videogame environment, a hypothesis can describe the game's code logic. A hypothesis can also represent a finite-sized numerical array (e.g., neural network weights) along with operations to decode it into reward and feedback. In short, a hypothesis is a sufficient text description of the learning problem such that the reward and the feedback functions can be fully determined.

With the hypothesis space $\mathcal{H}$, we model the feedback mechanism through a *feedback mapping* $\eta \mapsto f_\eta$ that maps each hypothesis $\eta \in \mathcal{H}$ to a *feedback function* $f_\eta : \mathcal{A} \to \Delta(\mathcal{O})$. Similarly, we model a *reward mapping* $\eta \mapsto r_\eta$ that maps a hypothesis $\eta \in \mathcal{H}$ to a *reward function* $r_\eta : \mathcal{A} \to [0,1]$. We denote by $\eta^* \in \mathcal{H}$ the true hypothesis of the environment, and use shorthand $f^* = f_{\eta^*}$ and $r^* = r_{\eta^*}$. This construction is reminiscent of classical bandit settings where the reward function is parameterized, such as the linear case $r^*(a) = \phi(a)^\top \theta^*$ for some known feature map ϕ and unknown ground-truth parameter θ^*. We generalize this by using the reward mapping $\eta \mapsto r_\eta$ as an analogue of the feature map and the hypothesis η^* as the parameter. Following the convention in the literature, we assume that the parameterization, i.e., the reward mapping $\eta \mapsto r_\eta$, is *known* to the agent, but the parameter η^* is *unknown*. See Fig. 1 for an overview.

Assumption 1. We assume that the agent has access to the reward mapping $r_\eta : \eta \mapsto r_\eta$.

In practice, the reward mapping can be implemented using an LLM to process a given hypothesis text, e.g., to tell whether an action is correct/incorrect (Zheng et al., 2023; Weng et al., 2023; Gu et al., 2024). We do not assume knowing the feedback mapping $\eta \mapsto f_\eta$, however, as precisely generating language feedback in practice is difficult.

2.3 Measuring Information in Feedback

Without any connection between feedback and reward, learning to minimize regret from feedback is impossible. Intuitively, for LLF to be feasible, language feedback must contain information that can infer the solution, like reward, action rankings, or whether an action is optimal. To study LLF learnability, we need a way to quantify this information. Since it is impossible to categorize and enumerate all possible language feedback in general (i.e., we cannot always embed language feedback into a finite-dimensional vector), we adopt a weak, implicit definition of information based on a sensing function.

We introduce the notion of a *verifier* to formalize information the agent can extract from feedback. The verifier represents a mechanism that assesses whether a hypothesis is consistent with observed feedback given to an action; for example, a verifier implemented by an LLM may rule out hypotheses that are semantically incompatible with feedback observations.

Assumption 2 (Verifier)**.** We assume that there is a verifier, which defines a loss $\ell : \mathcal{A} \times \mathcal{O} \times \mathcal{H} \to [0, 1]$, and the agent has access to the verifer through ℓ. For any action $a \in \mathcal{A}$, feedback $o \in \mathcal{O}$ and hypothesis $\eta \in \mathcal{H}$, the value $\ell(a, o, \eta)$ quantifies how well η aligns with the feedback on action a. If η is consistent with o on action a, then $\ell(a, o, \eta) = 0$; otherwise, it returns a non-zero penalty.

A concrete example may help ground this abstract assumption. Suppose the agent chooses an action a corresponding to a text summary of a story, and receives feedback o in the form of text critique, such as: "The summary is mostly accurate, but it misses an important detail about the main character's motivation." Suppose each hypothesis $\eta \in \mathcal{H}$ corresponds to a set of rubrics to judge summaries. A verifier must output a score $\ell(a, o, \eta)$. If a rubric η implies that summaries should capture the main character's motivation, then $\ell(a, o, \eta) = 0$, indicating consistency. Otherwise, the loss value is positive. Such a verifier can be implemented by prompting an LLM to assess whether the feedback o is consistent with applying rubric η to the summary a.

The set of feedback-consistent hypotheses naturally captures information in the feedback. Ideally, feedback generated from $f_\eta(\cdot)$ should be self-consistent, i.e., $\mathbb{E}_{O\sim f_\eta(a)}[\ell(a, O, \eta)] = 0$ for all $a \in \mathcal{A}$ and $\eta \in \mathcal{H}$. However, in practice, both the feedback and the verifier may be noisy or imperfect and there may be some $a \in \mathcal{A}$ such that $\mathbb{E}_{O\sim f^*(a)}[\ell(a, O, \eta^*)] > 0$. To accommodate this potential noise while preserving learnability, we adopt a weaker assumption than self-consistency: although the feedback may be noisy, it is *unbiased* such that each hypothesis minimizes the expected verifier loss under its induced distribution.

Assumption 3 (Unbiased Feedback)**.** For all $a \in \mathcal{A}$ and $\eta \in \mathcal{H}$, $\eta \in \min_{\eta' \in \mathcal{H}} \mathbb{E}_{O\sim f_\eta(a)}[\ell(a, O, \eta')]$.

The notion of verifier can be used to formalize *semantic equivalence* among hypotheses. In natural language, many token sequences share the same underlying semantic meaning. For LLF, such distinctions are not meaningful and should not affect the learning outcome. This invariance can be captured by the verifier introduced above. We deem hypotheses as equivalent whenever they induce identical loss functions across all inputs. We use this to define the geometry of the hypothesis space.

Definition 1 (Hypothesis Equivalence)**.** We define the distance between two hypotheses $\eta, \eta' \in \mathcal{H}$ as $d_{\mathcal{H}}(\eta, \eta') := \sup_{a\in\mathcal{A}, o\in\mathcal{O}} |\ell(a, o, \eta) - \ell(a, o, \eta')|$. If $d_{\mathcal{H}}(\eta, \eta') = 0$, we say η and η' are *equivalent*.

This definition provides a criteria to determine the equivalence of hypotheses, as two hypotheses with zero distance are indistinguishable from the agent's perspective. In applications involving LLM-generated feedback, the loss function ℓ can be designed to reflect semantic similarity, e.g., by assigning similar values to outputs that are paraphrases of one another, based on token-level matching, embedding-based metrics, or LLM-prompted judgments (Wang & Yu, 2023; Chuang et al., 2022; Asai & Hajishirzi, 2020; Bubeck et al., 2023).

Remark 2. Readers familiar with reinforcement learning from human feedback (RLHF) or AI feedback (RLAIF) may wonder if such a loss structure is necessary. Indeed, one may alternatively define a scoring function $g : \mathcal{A} \times \mathcal{O} \to [0, 1]$ that directly evaluates an action-feedback pair and impose some relationships between the scoring function and the underlying reward. This construction is a special case to our framework, which we discuss in detail in Section 3.3.

3 Learnability and Provable Algorithm

Compared to numerical reward signals, feedback can potentially carry more information. In LLF, to interpret this feedback and guide learning, the agent is equipped with: *1)* The verifier loss function ℓ and *2)* The reward mapping $\eta \mapsto r_\eta$. This structure reflects a central feature of LLF: the agent must reason over the hypothesis space $\mathcal{H}$ via the verifier to minimize regret defined by the hidden rewards.

But can an agent learn to maximize reward despite not observing it? For instance, if feedback does not convey useful information for problem solving, it is unrealistic to expect any learning to happen. On the other hand, if feedback directly reveals the optimal action, then the problem can be solved in two steps. Naturally, one would expect the learnability and complexity of LLF problems to depend on the information that feedback conveys. The goal of this section is to give natural structures and assumptions to the LLF setup that characterizes the difficulty of the learning problem.

3.1 Transfer Eluder Dimension

To quantify information in the feedback, we utilize the verifier, introduced in Section 2.3, to propose a new complexity measure called *transfer eluder dimension* based on the eluder dimension (Russo & Van Roy, 2013). At a high level, transfer eluder dimension characterizes how effectively information in the feedback reduces uncertainty about the unknown reward function. When it is small, a single piece of feedback carries a lot of information about the reward, which enables LLF to be much more efficient than learning from reward.

Definition 2. Define $\ell_\eta^{min}(a) := \min_{\eta'} \mathbb{E}_{O \sim f_\eta(a)}[\ell(a, O, \eta')]$. Given a verifier loss ℓ, an action $a \in \mathcal{A}$ is ϵ*-transfer dependent* on actions $\{a_1, \ldots, a_n\} \subset \mathcal{A}$ with respect to $\mathcal{H}$ if any pair of hypotheses $\eta, \eta' \in \mathcal{H}$ satisfying $\sum_{i=1}^{n} \left(\mathbb{E}_{o \sim f_{\eta'}(a_i)}[\ell(a_i, o, \eta)] - \ell_{\eta'}^{\min}(a_i) \right) \leq \epsilon^2$, also satisfies $|r_\eta(a) - r_{\eta'}(a)| \leq \epsilon$. Further, a is ϵ*-transfer independent* of $\{a_1, \ldots, a_n\}$ with respect to $\mathcal{H}$ if a is not ϵ-transfer dependent on $\{a_1, \ldots, a_n\}$.

Intuitively, this definition says that an action a is transfer independent of $\{a_1, \ldots, a_n\}$ if two hypotheses that give similar feedback according to the verifier at $\{a_1, \ldots, a_n\}$ can differ significantly in their reward predictions at a. This differs from the original definition of eluder dimension (Definition 4), which measures discrepancies in both the history and new observation using reward. Our goal is accurate reward prediction, not feedback recovery. This intuition motivates the definition of the transfer eluder dimension.

Definition 3 (Transfer eluder dimension)**.** The ϵ*-transfer eluder dimension* $\dim_{TE}(\mathcal{H}, \ell, \epsilon)$ of $\mathcal{H}$ with respect to the verifier loss ℓ is the length d of the longest sequence of elements in $\mathcal{A}$ such that, for some $\epsilon' \geq \epsilon$, every action element is ϵ'-transfer independent of its predecessors.

Unlike the eluder dimension, transfer eluder dimension measures dependence based on two quantities: the verifier loss and the reward function. This extension allows us to capture information in the feedback relevant to reward learning. Later in Section 3.4, we will present a provable algorithm that attains a sublinear regret rate in LLF in terms of the transfer eluder dimension.

3.2 Example Forms of Feedback

We discuss several example forms of feedback and compute the corresponding transfer eluder dimensions. The nature of feedback critically affects learning efficiency: uninformative feedback (e.g., random text) leads to infinite transfer eluder dimension, while some feedback can provide more information than reward and accelerate learning. For example, in a constraint satisfaction problem, feedback that reveals satisfied constraints can shrink the set of potentially true hypotheses. In the toy example below, reward-only learning requires exponential time (2^L), whereas the transfer eluder dimension is 1, so LLF gives an exponential speed up.

Example 1 (Bitwise feedback on 0-1 string)**.** Consider an action set $\mathcal{A} = \{0,1\}^L$. The space of hypotheses $\mathcal{H}$ contains all possible length-L 0-1 strings. Each hypothesis η contains a particular fixed target string $s(\eta)$ and the corresponding text instruction to provide reward and feedback about the target. The reward function r_η corresponding to a hypothesis η is such that $r(a) = 1$ if $a = s(\eta)$ and $r(a) = 0$ otherwise. In other words, rewards are sparse and every suboptimal arm incurs a regret of 1. Feedback to an action $a = (a_1, \ldots, a_L)$ is bitwise, which tells in words the correctness of each bit in the 0-1 string (i.e. whether $a_i = s_i$ for $s(\eta) = (s_1, \ldots, s_L)$. Equivalently, we can abstract the feedback as $f_\eta(a) = (\mathbb{1}\{a_i = s_i\})_{i=1}^L$ and define the loss function $\ell(a, o, \eta) = \frac{1}{L}\sum_{i=1}^L \mathbb{1}\{o_i \neq \mathbb{1}\{a_i = s_i\}\}$ to measure the discrepancy between the feedback and the correctness indicated by hypothesis η. For any $\epsilon < \frac{1}{L}$, the transfer eluder dimension $\dim_{TE}(\mathcal{H}, \ell, \epsilon) = 1$, as for any action a', the expected loss $\mathbb{E}_{O\sim f_{\eta'}(a')}[\ell(a', O, \eta)] < \frac{1}{L}$ iff $\eta = \eta'$.

We can also use feedback to reveal information e.g. about the optimality of selected actions, improving directions, or explanation of mistakes. Below we use an example to illustrate how different forms of feedback can drastically change the problem complexity.

Example 2 (Reasoning steps)**.** Consider a math reasoning problem where one tries to construct a hidden sequence of L-step reasoning $a^* = (s_1^*, \ldots, s_L^*)$, where each $s_i \in \mathcal{S} \subset \mathcal{T}^+$ is a token sequence that represents a correct reasoning at step i, and $\mathcal{S}$ is a finite set of token sequences that represent possible reasoning steps. The action set $\mathcal{A} = \cup_{k=1}^L (\mathcal{T}^+)^k$ consists of all possible reasoning of L steps. Each hypothesis represents a full solution to the problem and rubrics to critique partial answers with. Reward is 1 if all steps are correct and 0 otherwise. Below we show the transfer eluder dimension with $\epsilon < \frac{1}{2L}$ for different feedback (see Appendix B.4 for the exact forms of verifiers and proofs). We consider four feedback types, which corresponds to the reward, hindsight-negative, hindsight-positive, and future-positive feedback, respectively, in the LLF's feedback taxonomy proposed in (Cheng et al., 2023). Directly learning from rewards incurs exponential complexity, as the agent must enumerate all possible sequences. Feedback that identifies the first mistake enables stage-wise decomposition and yields exponential improvement in L, though each stage still requires brute-force search. If the feedback is more constructive, showing not only where the fist mistake is but also how to correct for it, the problem complexity does not depend on $|\mathcal{S}|$. Finally, if the feedback tells the answer right away, the complexity becomes constant, as the agent can learn the solution immediately after one try.

Feedback	$\dim_{TE}(\mathcal{H}, \ell, \epsilon)$
1. (reward) binary indicator of whether all steps are correct	$O(\|\mathcal{S}\|^L)$
2. (explanation) index of the first incorrect step	$O(\|\mathcal{S}\|L)$
3. (suggestion) give correction for the first mistake	$O(L)$
4. (demonstration) all the correct steps	$O(1)$

3.3 Comparison to Learning from Reward

We have shown examples where the transfer eluder dimension is bounded and decreases as the feedback provides more information than reward. Here we prove the generality of this observation. Below we show that if the feedback contains reward information, then the transfer eluder dimension of LLF is no larger than the traditional eluder dimension of RL in Definition 4.

Algorithm 1 LLF via Upper Confidence Bound (LLF-UCB)

Input $\mathcal{A}$, $\mathcal{O}$, T, reward mapping $\eta \mapsto r_\eta$, verifier loss $\ell : \mathcal{A} \times \mathcal{O} \times \mathcal{H} \to [0, 1]$
Initialize $t = 0$, $A_0 \sim \mathrm{Unif}(\mathcal{A})$
for $t = 0, 1, \ldots, T$ **do**
  observe O_t
  define $\mathcal{H}_t \coloneqq \{\eta \in \mathcal{H} : \frac{1}{t} \sum_i \ell(A_i, O_i, \eta) - \min_{\eta' \in \mathcal{H}} \frac{1}{t} \sum_i \ell(A_i, O_i, \eta') \leq \epsilon_t\}$
  $(\pi_p, \eta_p) \leftarrow \arg\min_{\pi \in \Pi} \max_{\eta \in \mathcal{H}_t} [r_\eta(\pi_\eta) - r_\eta(\pi)]$
  if $r_{\eta_p}(\pi_{\eta_p}) - r_{\eta_p}(\pi_p) = 0$ **then**
    $A_t \sim \pi_p(\cdot)$ // Stopping criterion
  else
    $(\pi_o, \eta_o) \leftarrow \arg\max_{\pi \in \Pi} \max_{\eta \in \mathcal{H}_t} r_\eta(\pi)$ // UCB policy
    $A_t \sim \pi_o(\cdot)$
  end if
end for

Definition 4 (Eluder Dimension)**.** An action $a \in \mathcal{A}$ is ϵ-dependent on actions $\{a_1, \ldots, a_n\} \subset \mathcal{A}$ with respect to a reward class $\mathcal{R}$ if any $r, r' \in \mathcal{R}$ satisfying $\sum_{i=1}^{n} (r(a_i) - r'(a_i))^2 \leq \epsilon^2$, also satisfies $|r(a) - r'(a)| \leq \epsilon$. Further, a is ϵ-independent of $\{a_1, \ldots, a_n\}$ if it is not ϵ-dependent on $\{a_1, \ldots, a_n\}$. The *ϵ-eluder dimension* $\dim_E(\mathcal{R}, \epsilon)$ of $\mathcal{R}$ is the length d of the longest sequence of elements in $\mathcal{A}$ such that, for some $\epsilon' \geq \epsilon$, every action element is ϵ'-independent of its predecessors.

First, by using the verifier, we define the statement "feedback to contain reward information".

Definition 5 (Feedback containing reward information)**.** The feedback function f_η is *reward-informative* of r_η with respect to the verifier ℓ if there is $C_F > 0$ such that $\forall \eta' \in \mathcal{H}$, $a \in \mathcal{A}$, $|r_\eta(a) - r_{\eta'}(a)|^2 \leq C_F \mathbb{E}_{o \sim f_\eta(a)}[\ell(a, o, \eta') - \ell_\eta^{min}(a)]$. We say an LLF problem is *reward-informative* if (f^*, r^*, ℓ) satisfies the above condition.

This assumption states that the verifier can distinguish hypotheses based on feedback to the same extent as their reward differences. In other words, if two hypotheses differ in their corresponding rewards, then from the verifier can tell they are different. Therefore, standard RL problems are a special case of reward-informative LLF problems.

An reward-informative example is when the unobserved reward is a function of the feedback. Concretely, suppose $r_\eta(a) = \mathbb{E}_{o \sim f_\eta(a)}[g(a, o)]$ for some known $g : \mathcal{A} \times \mathcal{O} \to [0, 1]$. Note that the reward mapping $\eta \mapsto r_\eta$ is known, but the reward function itself is still hidden from the agent (since η^* is unknown). Consider $\ell(a, o, \eta) \coloneqq (g(a, o) - r_\eta(a))^2 = (g(a, o) - \mathbb{E}_{o' \sim f_\eta(a)}[g(a, o')])^2$. Then one can verify that $\eta \in \arg\min_{\eta' \in \mathcal{H}} \mathbb{E}_{o \sim f_\eta(a)}[\ell(a, o, \eta')]$ and show that this feedback-verifier pair is reward-informative. (see Appendix B.3). In addition to this example, one can check that the forms of feedback used in Section 3.2 are reward-informative too. Note that reward-informative feedback can also contain information other than reward as shown in Section 3.2.

With this definition in place, we show that if feedback contains reward information, the transfer eluder dimension is no larger than the eluder dimension for the reward class induced by $\mathcal{H}$.

Proposition 1. *For reward-informative LLF problems with C_F as in Definition 5, it holds that $\dim_{TE}(\mathcal{H}, C_F \ell, \epsilon) \leq \dim_E(\mathcal{R}_\mathcal{H}, \epsilon)$, where $\mathcal{R}_\mathcal{H} = \{r_\eta : \eta \in \mathcal{H}\}$ is the effective reward class of $\mathcal{H}$.*

Proposition 1 implies that reward-informative LLF problems are no harder than their reward-only counterparts, such as those solved by the standard UCB algorithm over the reward class $\mathcal{R}_\mathcal{H}$ using reward extracted from the language feedback by some LLM.

3.4 LLF-UCB Algorithm

To validate our characterization of learnability based on the transfer eluder dimension, we design a simple UCB-style algorithm, LLF-UCB, outlined in Algorithm 1. LLF-UCB uses feedback to guide exploration using the optimism principle (Auer et al., 2002). As a concrete instantiation of how our conceptual framework can inform algorithmic design, LLF-UCB serves as a sanity check that LLF problems with finite transfer eluder dimensions can indeed be solved provably efficiently, with a regret guarantee that depends sublinearly on the transfer eluder dimension.

Theorem 1. *Under Assumption 1 and Assumption 2, for all $T \in \mathbb{N}$, the regret of LLF-UCB satisfies*

$$\mathrm{Regret}(T) \leq \widetilde{O}\left(T^{3/4}\left(\log N(\mathcal{H}, \epsilon_T^{\mathcal{H}}, d_{\mathcal{H}})\right)^{1/4}\sqrt{\dim_{TE}(\mathcal{H}, \ell, \epsilon_T^{\mathcal{H}})}\right),$$

where $N(\mathcal{H}, \epsilon_T^{\mathcal{H}}, d_{\mathcal{H}})$ denotes the $\epsilon_t^{\mathcal{H}}$-covering number of $\mathcal{H}$ based on the pseudo-metric $d_{\mathcal{H}}$, $\dim_{TE}(\mathcal{H}, \ell, \epsilon_T^{\mathcal{H}})$ denotes the $\epsilon_T^{\mathcal{H}}$-transfer eluder dimension of $\mathcal{H}$, and $\epsilon_T^{\mathcal{H}} = \max\left\{\frac{1}{T^2}, \min_{a\in\mathcal{A}} \inf\{|r_\eta(a) - r^*(a)| : \eta \in \mathcal{H}, \eta \neq \eta^*\}\right\}$.

While the order $\widetilde{O}(T^{3/4})$ on the time horizon T may appear suboptimal compared to classical $\widetilde{O}(\sqrt{T})$ optimal rates for bandit learning with direct reward feedback, this slower rate is in fact a principled consequence of our minimal assumptions. Specifically, our analysis makes no structural assumptions on the verifier loss ℓ beyond boundedness. If we have more structural knowledge of ℓ, say, that it is α-strongly convex, then the bound can be tightened to match the optimal order $\widetilde{O}(\sqrt{T})$. We defer a detailed treatment of these improvements to Appendix A.2, provide a sketch of the general argument in Theorem 1 in Appendix A.1, and include complete technical details in Appendix A.2.

We now describe the main components of LLF-UCB. Given a hypothesis $\eta \in \mathcal{H}$, let π_η denote its optimal policy. At each step t, the algorithm maintains a confidence set $\mathcal{H}_t$ consisting of hypotheses that remain approximately consistent with observed actions and feedback, as measured by cumulative verifier loss. The algorithm then identifies a hypothesis η_o that achieve maximal optimal reward, and follows an optimal policy π_o under this hypothesis. An additional design in LLF-UCB compared to standard UCB is a stopping criterion. It checks for a consensus optimal action among all hypotheses in the confidence set. If the minimax regret $\min_{\pi\in\Pi} \max_{\eta\in\bar{\mathcal{H}}} r_\eta(\pi_\eta) - r_\eta(\pi) = 0$, then the minimizer policy only selects actions that are simultaneously optimal for all candidate hypotheses (see Lemma 5).

As discussed in Section 3.3, feedback in a trivial LLF problem can directly reveal the optimal action but nothing about the reward. If this is the case, the stopping criteria ensures that the algorithm will not over-explore when it is certain that some action is optimal. Directly querying LLM for an action by prompting with the interaction history (with the lowest temperature) would be similar to drawing actions from π_η where η is randomly sampled from $\arg\min_{\eta'\in\mathcal{H}} \sum_i \ell(A_i, O_i, \eta')$. In the classical RL setting, such a greedy algorithm does not explore and therefore does not always have low-regret. Since RL is a special case of reward-informative LLF, we conjecture that this greedy algorithm also does not have regret guarantees for general LLF. We will compare this baseline in all of our experiments and confirm that LLF-UCB reliably outperforms this baseline.

4 Related Work

While using LLMs for general problem solving has been studied for a long time (Xie et al., 2022; Guo et al., 2024; Akyürek et al., 2023b), relatively fewer prior works studied the use of LLMs for sequential decision-making. There are roughly two routes to improving the agent's performance with language feedback. One is to directly deploy LLMs as agents in decision-making problems by incorporating feedback into subsequent prompts or an external memory buffer (Yao et al., 2023; Brooks et al., 2023; Shinn et al., 2023; Wang et al., 2024; Krishnamurthy et al., 2024; Nie et al., 2024; Xi et al., 2025). Another route is to process this feedback and use it to finetune a model's weights (Chen et al., 2024; Scheurer et al., 2022; Raparthy et al., 2023; Lee et al., 2023; Qu et al., 2025). This approach requires a considerable amount of offline interaction data. More recent work has investigated more sophisticated methods to improve exploration with LLMs, such as directly learning exploration behavior through supervised fine-tuning (Nie et al., 2024), preference-based learning (Tajwar et al., 2025), or reinforcement learning (Schmied et al., 2025), or prompting LLMs

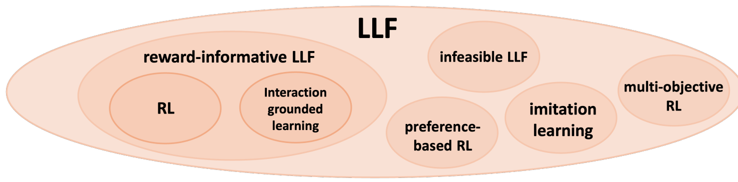
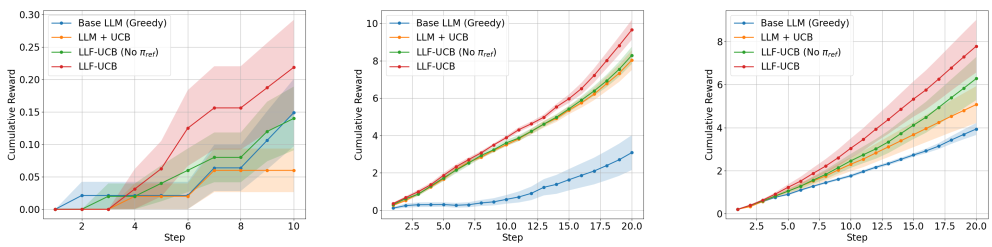


Figure 2: **LLF and its relationship to existing paradigms**. LLF covers many existing paradigms: (1) reinforcement learning (RL): agent learning from a scalar reward signal, (2) interaction-guided learning (IGL) (Xie et al., 2021): agent observes a generic feedback vector that can decode a latent reward signal, (3) reward-informative LLF: agent observes language feedback that can be translated into a scalar reward signal (Xie et al., 2024), (4) multi-objective RL: extension of RL to problems with multiple objectives, combined via a utility function, (5) preference-based RL: feedback provides a comparison between two actions, (6) imitation learning: feedback provides an expert demonstration.



(a) Wordle (50 scenarios) (b) Battleship (20 scenarios) (c) Minesweeper (20 scenarios)

Figure 3: We show the cumulative reward that the agent is able to obtain during a fixed number of interactions with the environment. Shaded area represents the standard error of cumulative reward across different scenarios.

to mimic a perfect Bayesian learner (Arumugam & Griffiths, 2025). However, up to date, these results have been empirical.

We aim to bridge this gap by introducing a formal framework and guarantees for learning from language feedback. Our framework is closely related to multi-armed bandits (Lai & Robbins, 1985) and contextual bandits (Langford & Zhang, 2007). The class of algorithms that achieve diminishing long-term average reward are termed "no-regret algorithms" (Auer et al., 2002; Thompson, 1933; Russo et al., 2018). One widely adopted strategy relies on the "optimism in the face of uncertainty" principle. Our algorithm design follows the same spirit as UCB (Auer et al., 2002). A key difference is that our algorithm does not observe rewards at all, but instead rely on decoding information in the feedback through a verifier loss to construct the confidence set. A recent line of work utilizes UCB-like heuristics for LLM agents, but they either consider hypotheses as code that specifies an MDP (Tang et al., 2024), and/or assume that the agent observes the ground-truth numerical reward (Tang et al., 2024; N et al., 2024; Nie et al., 2024).

Beyond scalar rewards, many learning settings offer richer forms of feedback. Prior work has explored bandits with side observations (Wang et al., 2003; Kocák et al., 2014), partial monitoring (Bartók et al., 2014), and preference-based feedback (Fürnkranz et al., 2012). To characterize sample complexity in reward-aware RL, (Russo & Van Roy, 2013) introduces the eluder dimension. Our work extends this notion beyond reward learning (see Fig. 2), opening a new avenue to understanding agent learning in the era of generative AI.

5 Discussion

We develop a formal foundation for learning from language feedback (LLF), a setting where agents must learn from language feedback rather than scalar rewards. We introduce the transfer eluder dimension as a complexity measure that quantifies how feedback affects the efficiency of learning.

When feedback is informative, we show that LLF can achieve exponential efficiency gain compared to traditional reward-based learning. To demonstrate the practicality of this framework, we propose LLF-UCB, a no-regret algorithm with performance guarantees in terms of the transfer eluder dimension.

5.1 Empirical Studies

In addition to theory, we also validate a practical approximation of Algorithm 1 in experiments using three LLF problems (Wordle, Battleship and Minesweeper) constructed from the benchmark Tajwar et al. (2025). Please see Appendix E for details. We consider the following LLF agents.

Greedy is the ReAct (Yao et al., 2023) agent that generates a hypothesis and returns its action.

UCB uses an LLM to generate N hypotheses (thoughts), the best actions under each hypothesis, and M additional exploratory actions. The agent evaluates all the generated actions on all the hypotheses using an LLM, forming an $N \times (N + M)$ matrix. The agent then select the hypothesis with the highest score and perform the corresponding best action. If there are ties, the first generated action among ties is chosen.

LLF-UCB adds the stopping criterion in Algorithm 1 to the UCB agent. After computing the hypothesis-action score matrix, the agent first checks whether a *consensus action* a exists—i.e., an action that achieves the highest score across all hypotheses. If true, then that action is returned. Otherwise, the UCB procedure above is used, except with a different tie-breaking rule. If multiple hypotheses yield the same highest score, we normalize the score by subtracting the average score of exploratory actions. We have found tie-breaking to play a nontrivial role in LLMs, as LLMs favor certain hypotheses and actions, unlike traditional UCB where ties can be broken arbitrarily. After normalization, we select the hypothesis with the highest normalized score. If a tie still remains, we select the first generated action among ties.

Results We plot the cumulative reward as a function of the number of environment interaction steps on WORDLE, BATTLESHIP, and MINESWEEPER in Figure 4. We see that for all three environments, the base LLM, where we only greedily choose the first action, performs worse generally. In environments where information-gathering is more necessary, such as in BATTLESHIP or in MINESWEEPER, agents designed to conduct strategic explorations tend to outperform the greedy base LLM by a large margin. Our LLF-UCB agents consistently outperform both the greedy baseline and barebone UCB agents. In particular, on BATTLESHIP and MINESWEEPER, LLF-UCB achieves a significant performance improvement over the baselines. We leave further analysis to Appendix E.

5.2 Limitations and Open Questions

One might wonder if the transfer eluder dimension forms a lower bound for LLF. The answer, however, is negative, as some LLF problems are trivially solvable despite having infinite transfer eluder dimension. For example, suppose rewards are arbitrary but feedback always reveals an optimal action. The transfer eluder dimension is unbounded in this case, yet the learning problem is easy.

The difference between this and the earlier demonstration case in Example 2 is that latter's reward class are constrained to be binary and the optimal action is unique, which keeps the transfer eluder dimension finite. We highlight that this argument assumes worst-case verifier behavior, while LLMs in practice impose inductive biases on how feedback is interpreted. Empirically, we find that when explicitly presented with an optimal action, LLMs tend to trust and act on it, bypassing further learning to infer full rewards. LLF-UCB captures this using the early stopping criterion (line 8), whereas näive reward-driven UCB fails.

This counterexample points to a gap in our current understanding: the true complexity of LLF may lie between worst-case reward identification and optimal behavior learning. A promising direction is to adapt DEC (Foster et al., 2024) to the LLF setting. However, the existing algorithm there is not directly implementable using LLMs. Closing this gap by developing a complexity measure that both lower-bounds regret and informs practical algorithm design remains an important open question.

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Supplementary Materials

The following content was not necessarily subject to peer review.

A Regret Analysis

A.1 Proof Sketch

We sketch the regret analysis in four main steps. The full proof is presented in Appendix A.2.

Step 1: Define confidence sets For each hypothesis $\eta \in \mathcal{H}$, we define $\mathcal{L}_t(\eta) = \sum_{i=0}^{t-1}\left(\mathbb{E}_{O\sim f_{\eta^*}(A_i)}[\ell(A_i,O,\eta)] - \ell_{\eta^*}^{\min}(A_i)\right)$ to be the cumulative population prediction error and $L_t(\eta) = \sum_{i=0}^{t-1}\ell(A_i,O_i,\eta) = \sum_{i=0}^{t-1}\ell_i(\eta)$ to be the cumulative empirical verifier loss. We define confidence sets $\mathcal{H}_t = \{\eta \in \mathcal{H} : L_t(\eta) \le \min_{\eta'\in\mathcal{H}} L_t(\eta') + \beta_t\}$ where β_t is a confidence parameter.

Step 2: Regret decomposition We let the width of a subset $\mathcal{V} \subseteq \mathcal{H}$ at an action $a \in \mathcal{A}$ be $w_{\mathcal{V}}(a) = \sup_{\overline{\eta}\in\mathcal{V}} |r_{\overline{\eta}}(a) - r^*(a)|$. Then, we can decompose the regret in terms of version space widths: $\mathrm{Regret}(T,\eta^*) \le \sum_{t=0}^{T-1}\mathbb{E}\left[w_{\mathcal{V}_t}(A_t)\cdot \mathbb{1}\{\eta^*\in\mathcal{V}_t\} + \mathbb{1}\{\eta^*\notin\mathcal{V}_t\}\right]$.

Step 3: Bounding the sum of widths via transfer eluder dimension The key step is to show that if the width $w_{\mathcal{H}_t}(A_t) > \epsilon$ for some $\epsilon > 0$, then A_t must be ϵ-dependent on only $O(\beta_t/\epsilon^2)$ disjoint historical action sequences, where β_t is the confidence parameter. By the definition of the transfer eluder dimension $d_{TE} = \dim_{TE}(\mathcal{H},\ell,\epsilon)$, in any sequence of N actions, there must be some action that is ϵ-dependent on at least $\Omega(N/d)$ previous ones. Combining these facts forces the number of large-width version spaces $\sum_{t=0}^{T-1}\mathbb{1}\{w_{\mathcal{H}_t}(A_t) > \epsilon\}$ to be bounded by $O(\beta_T d/\epsilon^2)$. Rearranging terms and choosing a suitable sequence of ϵ gives that with high probability, $\sum_{t=0}^{T-1} w_{\mathcal{V}_t}(A_t) \le O(d_{TE} + 2\sqrt{3 d_{TE}\beta_T T})$. Note that when the stopping criteria is triggered, the per-step regret of all following steps become zero, and so the regret of LLF-UCB is always bounded above by that without the stopping criteria.

Step 4: Prove high-probability confidence set concentration It remains to define suitable β_t's and show that $\eta^* \in \mathcal{V}_t$ for all $t \in \mathbb{N}$ with high probability. Depending on what structural assumptions are known for the verifier loss ℓ, we determine the rate of decay of β_t. If we only make the minimal assumption that ℓ is bounded, then $\beta_T = \widetilde{O}(\sqrt{T})$. Putting everything together proves Theorem 1.

A.2 Full Analysis

We first define the version spaces used in the algorithm. As shorthand notations, define

$$\mathcal{L}_t(\eta) = \sum_{i=0}^{t-1}\left(\mathbb{E}_{O\sim f_{\eta^*}(A_i)}[\ell(A_i,O,\eta)] - \ell_{\eta^*}^{\min}(A_i)\right)$$

to be the cumulative population prediction error and

$$L_t(\eta) = \sum_{i=0}^{t-1}\ell(A_i,O_i,\eta) = \sum_{i=0}^{t-1}\ell_i(\eta)$$

to be the cumulative empirical verifier loss. A small value of $L_t(\eta)$ means η is close to consistent with observed feedback. Let $\mathcal{V}_t \subseteq \mathcal{H}$ be the version space of all hypotheses still plausible after t rounds of interactions. Concretely,

$$\mathcal{V}_t = \{\eta \in \mathcal{H} : L_t(\eta) \le \min_{\eta'\in\mathcal{H}} L_t(\eta') + \beta_t\}, \tag{1}$$

where $\beta_t > 0$ is an appropriately chosen confidence parameter so that we do not throw away the true hypothesis η^* due to noise.

A useful approach to bounding the regret is to decompose it in terms of version spaces. Define the width of a subset $\mathcal{V} \subseteq \mathcal{H}$ at an action $a \in \mathcal{A}$ by

$$w_{\mathcal{V}}(a) = \sup_{\overline{\eta} \in \mathcal{V}} |r_{\overline{\eta}}(a) - r^*(a)| \,.$$

Proposition 2 (Regret decomposition)**.** *Fix any sequence $\{\mathcal{V}_t : t \in \mathbb{N}\}$, where $\mathcal{V}_t \subseteq \mathcal{H}$ is measurable with respect to $\sigma(H_t)$. Then for any $T \in \mathbb{N}$,*

$$\mathrm{Regret}(T, \eta^*) \leq \sum_{t=0}^{T-1} \mathbb{E}\left[w_{\mathcal{V}_t}(A_t) \cdot \mathbb{1}\{\eta^* \in \mathcal{V}_t\} + \mathbb{1}\{\eta^* \notin \mathcal{V}_t\}\right].$$

Proof. Define the upper bound $U_t(a) = \sup\{r_\eta(a) : \eta \in \mathcal{V}_t\}$. Let $a^* \in \arg\max_{a \in \mathcal{A}} r^*(a)$. When $\eta^* \in \mathcal{V}_t$, the bound $r^*(a) \leq U_t(a)$ hold for all actions. This implies

$$\begin{aligned} r^*(\eta^*) - r^*(A_t) &\leq (U_t(a^*) - r^*(A_t)) \cdot \mathbb{1}\{\eta^* \in \mathcal{V}_t\} + \mathbb{1}\{\eta^* \notin \mathcal{V}_t\} \\ &\leq w_{\mathcal{V}_t}(A_t) \cdot \mathbb{1}\{\eta^* \in \mathcal{V}_t\} + \mathbb{1}\{\eta^* \notin \mathcal{V}_t\} + [U_t(a^*) - U_t(A_t)] \cdot \mathbb{1}\{\eta^* \in \mathcal{V}_t\} \end{aligned}$$

Since the algorithm selects an action A_t that maximizes $U_t(a)$, the conclusion follows by taking the expectation and summing over all $t = 0, \ldots, T-1$. $\square$

If the version spaces $\mathcal{V}_t$ are constructed to contain η^* with high probability, this proposition reduces upper bounding the regret to bounding the expected sum of widths $\sum_{t=0}^{T-1} \mathbb{E}[w_{\mathcal{V}_t}(A_t)]$.

We first introduce a class of Martingale exponential inequalities that will be useful throughout our analysis, including bounding the sum of widths and proving the high-confidence events $\eta^* \in \mathcal{V}_t$. Consider random variables $(X_t | t \in \mathbb{N})$ adapted to the filtration $(\mathcal{F}_t | t \in \mathbb{N})$. Assume $\mathbb{E}[\exp(\lambda X_t)]$ is finite for all λ and $\mathbb{E}[X_t | \mathcal{F}_{t-1}] = 0$. We assume that there is a uniform upper bound on the cumulant generating function (i.e., log moment generating function) for the conditional distribution of X_t.

Lemma 1 (Cumulant generating function)**.** *If there is a sequence of convex functions $\{\psi_t : [0, \infty) \to \mathbb{R}\}_{t=0}^{\infty}$ with $\psi_t(0) = 0$ such that, for all $t \in \mathbb{N}$ and all $\lambda \in [0, \infty)$,*

$$\log \mathbb{E}\left[e^{\lambda |X_t|} | \mathcal{F}_{t-1}\right] \leq \psi_t(\lambda),$$

then for all $\delta \in (0,1)$ and $T \in \mathbb{N}$, with probability $1 - \delta$,

$$\left|\sum_{t=0}^{T-1} X_t\right| \leq \inf_{\lambda \in [0,\infty)} \left\{\frac{\sum_{t=0}^{T-1} \psi_t(\lambda) + \log(2/\delta)}{\lambda}\right\}.$$

Proof. Let $S_T = \sum_{t=0}^{T-1} X_t$. By Markov's inequality, for all $u \in \mathbb{R}$ and $\lambda \in [0, \infty)$,

$$\begin{aligned} \mathbb{P}(S_T \geq u) = \mathbb{P}\left(e^{\lambda S_T} \geq e^{\lambda u}\right) &\leq \frac{\mathbb{E}[e^{\lambda S_T}]}{e^{\lambda u}} = \frac{\mathbb{E}[\mathbb{E}[e^{\lambda S_T} | \mathcal{F}_{T-1}]]}{e^{\lambda u}} = \frac{\mathbb{E}[e^{\lambda \sum_{t=0}^{T-2} X_t} \mathbb{E}[e^{\lambda X_{T-1}} | \mathcal{F}_{T-1}]]}{e^{\lambda u}} \\ &\leq \frac{\mathbb{E}[e^{\lambda \sum_{t=0}^{T-2} X_t}] \exp(\psi_{T-1}(\lambda))}{e^{\lambda u}} \leq \cdots \leq \frac{\exp(\sum_{t=0}^{T-1} \psi_t(\lambda))}{e^{\lambda u}}. \end{aligned}$$

This gives

$$\mathbb{P}(S_T \geq u) \leq \exp\left(-\lambda u + \sum_{t=0}^{T-1} \psi_t(\lambda)\right)$$

for all $\lambda \in [0, \infty)$. Applying the same argument to $-X_t$, we have

$$\mathbb{P}(S_T \leq -u) = \mathbb{P}(-S_T \geq u) \leq \exp\left(-\lambda u + \sum_{t=0}^{T-1} \psi_t(\lambda)\right).$$

Solving for u to achieve a $\delta/2$ probability for each side, and taking the infimum over $\lambda \in [0, \infty)$, we have with probability at least $1 - \delta$,

$$S_T \leq \inf_{\lambda \in [0,\infty)} \left\{ \frac{\sum_{t=0}^{T-1} \psi_t(\lambda) + \log(2/\delta)}{\lambda} \right\}.$$

$\square$

We now proceed to bounding the sum of widths $\sum_{t=0}^{T-1} \mathbb{E}[w_{\mathcal{V}_t}(A_t)]$ when the event $\eta^* \in \mathcal{V}_t$ holds. As a first step, we show that there cannot be many version spaces $\mathcal{V}_t$ with a large width. For all $t \in \mathbb{N}$ and $\eta, \eta' \in \mathcal{H}$, we define the martingale difference

$$Z_t(\eta, \eta') = \mathbb{E}_{O \sim f_{\eta^*}(A_t)} \left[\ell(A_t, O, \eta) - \ell(A_t, O, \eta') | \mathcal{G}_{t-1}\right] - \left(\ell(A_t, O_t, \eta) - \ell(A_t, O_t, \eta')\right).$$

Notice that Z_t have expectation zero and constitutes a martingale difference sequence adapted to the filtration $(\mathcal{G}_t | t \in \mathbb{N})$ where $\mathcal{G}_t$ is the σ-algebra generated by all observations $\{(a_0, o_1), \ldots, (a_t, o_t)\}$ up to time t.

Proposition 3. *If the conditions in Lemma 1 holds for $(Z_t | t \in \mathbb{N})$ adapted to $(\mathcal{G}_t | t \in \mathbb{N})$ with cumulative generating function bound $(\psi_t | t \in \mathbb{N})$, $(\beta_t \geq 0 | t \in \mathbb{N})$ in (1) is a nondecreasing sequence such that for all $t \in \mathbb{N}$, $\beta_t \geq \inf_{\lambda \in [0,\infty)} \left\{ \frac{\sum_{i=0}^{t-1} \psi_i(\lambda) + \log(10t^2/3\delta)}{\lambda} \right\}$, then for all $\delta \in (0, 1)$, with probability at least $1 - \delta$,*

$$\sum_{t=0}^{T-1} \mathbb{1}\{w_{\mathcal{V}_t}(A_t) > \epsilon\} \cdot \mathbb{1}\{\eta^* \in \mathcal{V}_t\} \leq \left(\frac{3\beta_T}{\epsilon^2} + 1\right) \dim_{TE}(\mathcal{H}, \ell, \epsilon)$$

for all $T \in \mathbb{N}$ and $\epsilon > 0$.

Proof. We first show that if $w_{\mathcal{V}_t}(A_t) > \epsilon$ and $\eta^* \in \mathcal{V}_t$ then with high probability, A_t is ϵ-dependent on fewer than $O(\beta_t/\epsilon^2)$ disjoint subsequences of $(A_0, A_1, \ldots, A_{t-1})$. To see this, note that if $w_{\mathcal{V}_t}(A_t) > \epsilon$ and $\eta^* \in \mathcal{V}_t$, there exists $\overline{\eta} \in \mathcal{V}_t$ such that $|r_{\overline{\eta}}(A_t) - r_{\eta^*}(A_t)| > \epsilon$. By definition, if A_t is ϵ-dependent on a subsequence $(A_{i_1}, \ldots, A_{i_k})$ of $(A_0, \ldots, A_{t-1})$, then

$$\sum_{j=1}^{k} \left(\mathbb{E}_{O \sim f_{\eta^*}(A_{i_j})}[\ell(A_{i_j}, O, \overline{\eta})] - \ell_{\eta^*}^{\min}(A_{i_j}) \right) > \epsilon^2.$$

It follows that if A_t is ϵ-dependent on K disjoint subsequences of $(A_0, \ldots, A_{t-1})$ then

$$\sum_{i=0}^{t-1} \left(\mathbb{E}_{O \sim f_{\eta^*}(A_i)}[\ell(A_i, O, \overline{\eta})] - \ell_{\eta^*}^{\min}(A_i) \right) > K\epsilon^2.$$

Then

$$\begin{aligned}
&\sum_{i=0}^{t-1}\left(\mathbb{E}_{O\sim f_{\eta^*}(A_i)}[\ell(A_i,O,\overline{\eta})]-\ell_{\eta^*}^{\min}(A_i)\right)\\
&=\sum_{i=0}^{t-1}\mathbb{E}_{O\sim f_{\eta^*}(A_i)}\left[\ell(A_i,O,\overline{\eta})-\ell(A_i,O,\eta^*)\right]\\
&=\left[\sum_{i=0}^{t-1}\ell(A_i,O_i,\eta^*)-\min_{\eta'\in\mathcal{H}}\sum_{i=0}^{t-1}\ell(A_i,O_i,\eta')\right]-\left[\sum_{i=0}^{t-1}\ell(A_i,O_i,\overline{\eta})-\min_{\eta'\in\mathcal{H}}\sum_{i=0}^{t-1}\ell(A_i,O_i,\eta')\right]\\
&\quad+\left[\sum_{i=0}^{t-1}\left[\ell(A_i,O_i,\overline{\eta})-\ell(A_i,O_i,\eta^*)\right]-\sum_{i=0}^{t-1}\mathbb{E}_{O\sim f_{\eta^*}(A_i)}\left[\ell(A_i,O,\overline{\eta})-\ell(A_i,O,\eta^*)\right]\right]\\
&\le\left|\sum_{i=0}^{t-1}\ell(A_i,O_i,\eta^*)-\min_{\eta'\in\mathcal{H}}\sum_{i=0}^{t-1}\ell(A_i,O_i,\eta')\right|+\left|\sum_{i=0}^{t-1}\ell(A_i,O_i,\overline{\eta})-\min_{\eta'\in\mathcal{H}}\sum_{i=0}^{t-1}\ell(A_i,O_i,\eta')\right|\\
&\quad+\left[\sum_{i=0}^{t-1}\left[\ell(A_i,O_i,\overline{\eta})-\ell(A_i,O_i,\eta^*)\right]-\sum_{i=0}^{t-1}\mathbb{E}_{O\sim f_{\eta^*}(A_i)}\left[\ell(A_i,O,\overline{\eta})-\ell(A_i,O,\eta^*)\right]\right]\\
&\le 2\beta_t+\sum_{i=0}^{t-1}\left[\ell(A_i,O_i,\overline{\eta})-\ell(A_i,O_i,\eta^*)\right]-\sum_{i=0}^{t-1}\mathbb{E}_{O\sim f_{\eta^*}(A_i)}\left[\ell(A_i,O,\overline{\eta})-\ell(A_i,O,\eta^*)\right]\\
&=2\beta_t-\sum_{i=0}^{t-1}Z_i(\overline{\eta},\eta^*).
\end{aligned}$$

Using Lemma 1,

$$\mathbb{P}\left(\left|\sum_{i=0}^{t-1}Z_i(\overline{\eta},\eta^*)\right|>\inf_{\lambda\in[0,\infty)}\left\{\frac{\sum_{i=0}^{t-1}\psi_i(\lambda)+\log(2/\delta)}{\lambda}\right\}\right)\le\delta.$$

We choose a sequence $\{\delta_t\}_{t\in\mathbb{N}_{>0}}$ where $\delta_t=\frac{3\delta}{5t^2}$, and so $\sum_{t=1}^{\infty}\delta_t<\delta$. Using a union bound over all $t\in\mathbb{N}_{>0}$, we have that with probability at least $1-\delta$, for all $t\in\mathbb{N}$,

$$\left|\sum_{i=0}^{t-1}Z_i(\overline{\eta},\eta^*)\right|\le\inf_{\lambda\in[0,\infty)}\left\{\frac{\sum_{i=0}^{t-1}\psi_i(\lambda)+\log(10t^2/3\delta)}{\lambda}\right\}\le\beta_t.$$

Since $\{\beta_t\}_{t\in\mathbb{N}}$ is nondecreasing in t, we have that with probability at least $1-\delta$, $K\epsilon^2\le 3\beta_T$. It follows that with probability at least $1-\delta$, $K\le 3\beta_T/\epsilon^2$.

Next, we show that in any action sequence $(a_1,\dots,a_\tau)$, there is some element a_j that is ϵ-dependent on at least $\tau/d-1$ disjoint subsequences of $(a_1,\dots,a_{j-1})$, where $d=\dim_{TE}(\mathcal{H},\ell,\epsilon)$. To show this, for an integer K satisfying $Kd+1\le\tau\le Kd+d$, we will construct K disjoint subsequences $B_1,\dots,B_K$. First let $B_i=(a_i)$ for $i=1,\dots,K$. If a_{K+1} is ϵ-dependent on each subsequence $B_1,\dots,B_K$, our claim is established. Otherwise, select one subsequence for which a_{K+1} is ϵ-independent to and append a_{K+1} to it. Repeat this process for elements with indices $j>K+1$ until a_j is ϵ-dependent on each subsequence or $j=\tau$. In the latter scenario $\sum|B_i|\ge Kd$, and since each element of a subsequence B_i is ϵ-independent of its predecessors, $|B_i|=d$. In this case, a_τ must be ϵ-dependent on each subsequence, by the definition of $\dim_{TE}(\mathcal{H},\ell,\epsilon)$.

Now consider taking $(A_1,\dots,A_\tau)$ to be the subsequence $(A_{t_1},\dots,A_{t_\tau})$ of $(A_1,\dots,A_T)$ consisting of elements A_t for which $w_{\mathcal{V}_t}(A_t)>\epsilon$. As we have established, each A_{t_j} is ϵ-dependent on fewer than $3\beta_T/\epsilon^2$ disjoint subsequences of $(A_1,\dots,A_{j-1})$ with probability at least $1-\delta$. Combining this with the fact we have established that there is some a_j that is ϵ-dependent on at least $\tau/d-1$ disjoint subsequences of $(a_1,\dots,a_{j-1})$, we have $\tau/d-1\le 3\beta_T/\epsilon^2$. It follows that $\tau\le\left(3\beta_T/\epsilon^2+1\right)d$ with probability at least $1-\delta$, as desired. $\square$

We are now ready to bound the sum of widths $\sum_{t=0}^{T-1} \mathbb{E}[w_{\mathcal{V}_t}(A_t)]$ when the event $\eta^* \in \mathcal{V}_t$ holds. Consider the $\epsilon_T^{\mathcal{H}}$-transfer eluder dimension of $\mathcal{H}$, where

$$\epsilon_t^{\mathcal{H}} = \max\left\{\frac{1}{t^2}, \min_{a\in\mathcal{A}} \inf\{|r_\eta(a) - r^*(a)| : \eta \in \mathcal{H}, \eta \neq \eta^*\}\right\}.$$

Lemma 2. *If the conditions in Lemma 1 holds for $(Z_t|t \in \mathbb{N})$ adapted to $(\mathcal{G}_t|t \in \mathbb{N})$ with cumulative generating function bound $(\psi_t|t \in \mathbb{N})$, $(\beta_t \geq 0|t \in \mathbb{N})$ in (1) is a nondecreasing sequence such that for all $t \in \mathbb{N}$, $\beta_t \geq \inf_{\lambda\in[0,\infty)} \left\{\frac{\sum_{i=0}^{t-1}\psi_i(\lambda)+\log(10t^2/3\delta)}{\lambda}\right\}$, then for all $\delta \in (0,1)$, with probability at least $1-\delta$,*

$$\sum_{t=0}^{T-1} w_{\mathcal{V}_t}(A_t) \cdot \mathbb{1}\{\eta^* \in \mathcal{V}_t\} \leq \frac{1}{T} + \min\left\{\dim_{TE}(\mathcal{H},\ell,\epsilon_T^{\mathcal{H}}), T\right\} + 2\sqrt{3\dim_{TE}(\mathcal{H},\ell,\epsilon_T^{\mathcal{H}})\beta_T T}$$

for all $T \in \mathbb{N}$.

Proof. Let $d_T = \dim_{TE}(\mathcal{H},\ell,\epsilon_T^{\mathcal{H}})$ and $w_t = w_{\mathcal{V}_t}(A_t)$. Reorder the sequence $(w_1,\ldots,w_T) \to (w_{i_1},\ldots,w_{i_T})$ where $w_{i_1} \geq w_{i_2} \geq \cdots \geq w_{i_T}$. We have

$$\begin{aligned}
&\sum_{t=0}^{T-1} w_{\mathcal{V}_t}(A_t) \cdot \mathbb{1}\{\eta^* \in \mathcal{V}_t\} \\
&= \sum_{t=0}^{T-1} w_{i_t} \cdot \mathbb{1}\{\eta^* \in \mathcal{V}_{i_t}\} \\
&= \sum_{t=0}^{T-1} w_{i_t} \cdot \mathbb{1}\{\eta^* \in \mathcal{V}_{i_t}\} \cdot \mathbb{1}\{w_{i_t} > \epsilon_T^{\mathcal{H}}\} + \sum_{t=0}^{T-1} w_{i_t} \cdot \mathbb{1}\{\eta^* \in \mathcal{V}_{i_t}\} \cdot \mathbb{1}\{w_{i_t} \leq \epsilon_T^{\mathcal{H}}\} \\
&\leq \frac{1}{T} + \sum_{t=0}^{T-1} w_{i_t} \cdot \mathbb{1}\{\eta^* \in \mathcal{V}_{i_t}\} \cdot \mathbb{1}\{w_{i_t} > \epsilon_T^{\mathcal{H}}\}.
\end{aligned}$$

The last inequality follows since either $\epsilon_T^{\mathcal{H}} = 1/T^2$ and $\sum_{t=0}^{T-1} \epsilon_T^{\mathcal{H}} = 1/T$ or $\epsilon_T^{\mathcal{H}}$ is set below the smallest possible width and hence $\mathbb{1}\{w_{i_t} \leq \epsilon_T^{\mathcal{H}}\}$ never occurs. We have that $w_{i_t} \leq 1$. Also, $w_{i_t} > \epsilon \iff \sum_{k=0}^{T-1} \mathbb{1}\{w_{\mathcal{V}_k}(a_k) > \epsilon\} \geq t$. By Proposition 3, this can only occur if $t < \left(3\beta_T/\epsilon^2 + 1\right)\dim_{TE}(\mathcal{H},\ell,\epsilon)$ with probability at least $1-\delta$. For $\epsilon \geq \epsilon_T^{\mathcal{H}}$, since $\dim_{TE}(\mathcal{H},\ell,\epsilon')$ is nonincreasing in ϵ', $\dim_{TE}(\mathcal{H},\ell,\epsilon) \leq \dim_{TE}(\mathcal{H},\ell,\epsilon_T^{\mathcal{H}}) = d_T$. Therefore, when $w_{i_t} > \epsilon \geq \epsilon_T^{\mathcal{H}}$, $t \leq \left(3\beta_T/\epsilon^2 + 1\right) d_T$ which implies $\epsilon \leq \sqrt{\frac{3\beta_T d_T}{t-d_T}}$. This shows that if $w_{i_t} > \epsilon_T^{\mathcal{H}}$, then $w_{i_t} \leq \min\{1, \sqrt{\frac{3\beta_T d_T}{t-d_T}}\}$. Thus,

$$\begin{aligned}
\sum_{t=0}^{T-1} w_{i_t} \cdot \mathbb{1}\{\eta^* \in \mathcal{V}_{i_t}\} \cdot \mathbb{1}\{w_{i_t} > \epsilon_T^{\mathcal{H}}\} &\leq d_T + \sum_{t=d_T+1}^{T-1} \sqrt{\frac{3\beta_T d_T}{t-d_T}} \\
&\leq d_T + \sqrt{3\beta_T d_T} \int_{t=1}^{T-1} \frac{1}{\sqrt{t}} dt \\
&= d_T + 2\sqrt{3\beta_T d_T T}.
\end{aligned}$$

Since the sum of widths is always bounded by T, this implies with probability $1-\delta$,

$$\begin{aligned}
&\sum_{t=0}^{T-1} w_{\mathcal{V}_t}(a_t) \cdot \mathbb{1}\{\eta^* \in \mathcal{V}_t\} \\
&\leq \min\left\{T, \frac{1}{T} + \dim_{TE}(\mathcal{H},\ell,\epsilon_T^{\mathcal{H}}) + 2\sqrt{3\dim_{TE}(\mathcal{H},\ell,\epsilon_T^{\mathcal{H}})\beta_T T}\right\} \\
&\leq \frac{1}{T} + \min\left\{\dim_{TE}(\mathcal{H},\ell,\epsilon_T^{\mathcal{H}}), T\right\} + 2\sqrt{3\dim_{TE}(\mathcal{H},\ell,\epsilon_T^{\mathcal{H}})\beta_T T}.
\end{aligned}$$

$\square$

So far, we have only considered LLF-UCB without the stopping criteria. We remark that when the stopping criteria is triggered, the per-step regret of all following steps become zero, and so the regret of the full LLF-UCB is always bounded above by that without the stopping criteria. Combining this observation with Lemma 2 and Proposition 2, we arrive at the following abstract regret bound in terms of the version space confidence parameter β_T.

Theorem 2. *If it holds that for some $\delta \in (0,1)$, with probability at least $1-\delta$, $\eta^* \in \mathcal{V}_t$ for all t, then for all $T \in \mathbb{N}$,*

$$\mathrm{Regret}(T) \le 1 + \frac{1}{T} + \min\{\dim_{TE}(\mathcal{H}, \ell, \epsilon_T^{\mathcal{H}}), T\} + 2\sqrt{3\dim_{TE}(\mathcal{H}, \ell, \epsilon_T^{\mathcal{H}})\beta_T T}.$$

The dominant term in the regret bound is

$$2\sqrt{3\dim_{TE}(\mathcal{H}, \ell, \epsilon_T^{\mathcal{H}})\beta_T T}.$$

For our main theorem, it remains to design suitable version spaces $\mathcal{V}_t$ and show that they contain the true hypothesis η^* with high probability. Crucially, the rate at which the confidence parameters β_t of these version spaces shrink depends on concentration properties of the verifier loss function ℓ. Note that for the general LLF framework, we have assumed only that ℓ is a bounded function taking values in $[0,1]$. If we have more structural assumptions on the verifier loss ℓ, for example, that ℓ is α-strongly convex, then we may arrive at a tighter regret bound up to order $\sqrt{T}$ by taking β_T to be of constant order.

A.3 Version Space Construction for General Bounded Loss

Consider the most general case with minimal assumptions on the loss function, namely, that it is bounded between $[0,1]$ for all inputs. Then we prove the following high-probability event:

Lemma 3 (High-probability event)**.** *For all $\delta > 0$, $\eta, \eta' \in \mathcal{H}$,*

$$\mathbb{P}\left(\mathcal{L}_T(\eta') \ge \mathcal{L}_T(\eta) + L_T(\eta') - L_T(\eta) - \sqrt{2T\log\left(\frac{10T^2}{3\delta}\right)}, \quad \forall T \in \mathbb{N}\right) \ge 1-\delta.$$

Proof. For each $t = 1, \dots, T$, define the Martingale difference sequence

$$X_t = \mathbb{E}_{O\sim f_{\eta^*}(A_t)}\left[\ell(A_t, O, \eta) - \ell(A_t, O, \eta')\right] - \left(\ell(A_t, O_t, \eta) - \ell(A_t, O_t, \eta')\right).$$

$$\begin{aligned}
&\mathcal{L}_T(\eta') - \mathcal{L}_T(\eta) - (L_T(\eta') - L_T(\eta)) \\
&= \sum_{t=0}^{T-1}\left(\mathbb{E}_{O\sim f_{\eta^*}(A_t)}[\ell(A_t, O, \eta)] - \mathbb{E}_{O\sim f_{\eta^*}(A_t)}[\ell(A_t, O, \eta')]\right) - \sum_{t=0}^{T-1}\left(\ell(A_t, O_t, \eta) - \ell(A_t, O_t, \eta')\right) \\
&= \sum_{t=0}^{T-1}\mathbb{E}_{O\sim f_{\eta^*}(A_t)}\left[\ell(A_t, O, \eta) - \ell(A_t, O, \eta')\right] - \sum_{t=0}^{T-1}\left(\ell(A_t, O_t, \eta) - \ell(A_t, O_t, \eta')\right) \\
&= \sum_{t=0}^{T-1} X_t.
\end{aligned}$$

Notice that X_t have expectation zero and constitutes a Martingale difference sequence adapted to the filtration $\{\mathcal{G}_t\}_{t\ge 1}$ where $\mathcal{G}_t$ is the σ-algebra generated by all observations $\{(A_0, O_1), \dots, (A_t, O_t)\}$ up to time t. Since feedback losses $\ell(a, o, \eta)$ are uniformly bounded between $[0,1]$, we have that

$X_t \in [-2, 2]$ with probability 1. Using Lemma 1 with $\psi_t(\lambda) = \lambda^2/2$ and taking the infimum over λ, we get

$$\mathbb{P}\left(\left|\sum_{t=0}^{T-1} X_t\right| > \sqrt{2T\log(2/\delta)}\right) \leq \delta.$$

We choose a sequence $\{\delta_T\}_{T\in\mathbb{N}_{>0}}$ where $\delta_T = \frac{3\delta}{5T^2}$ such that $\sum_{T=1}^{\infty} \delta_T < \delta$. Using a union bound over all $T \in \mathbb{N}_{\geq 0}$, we have that with probability at least $1-\delta$,

$$|\mathcal{L}_T(\eta') - \mathcal{L}_T(\eta) - (L_T(\eta') - L_T(\eta))| \leq \sqrt{2T\log\left(\frac{2}{\delta_T}\right)} = \sqrt{2T\log\left(\frac{10T^2}{3\delta}\right)} \quad \forall T \in \mathbb{N}.$$

□

Since η^* is the true hypothesis, by Assumption 3, it minimizes the population loss $\mathcal{L}_T(\eta)$ for all $T \in \mathbb{N}$. That is, for all $\eta \in \mathcal{H}$,

$$\mathcal{L}_T(\eta^*) \leq \mathcal{L}_T(\eta) \quad \forall T \in \mathbb{N}.$$

Suppose $m = |\mathcal{H}| < \infty$. By Lemma 3, for any $\eta \in \mathcal{H}$, with probability at least $1 - \delta/m$, for all $T \in \mathbb{N}$,

$$L_T(\eta^*) - L_T(\eta) \leq \mathcal{L}_T(\eta^*) - \mathcal{L}_T(\eta) + \sqrt{2T\log\left(\frac{10T^2}{3\delta}\right)} \leq \sqrt{2T\log\left(\frac{10mT^2}{3\delta}\right)}.$$

Using a union bound over $\mathcal{H}$, with probability at least $1-\delta$, the true hypothesis η^* is contained in the version space

$$\mathcal{V}_T = \left\{\eta \in \mathcal{H} : L_T(\eta) \leq \min_{\eta' \in \mathcal{H}} L_T(\eta') + \sqrt{2T\log\left(\frac{10|\mathcal{H}|T^2}{3\delta}\right)}\right\}$$

for all $T \in \mathbb{N}$. To extend this to a space of infinite hypotheses, we measure the set $\mathcal{H}$ by some discretization scale α. Recall that we define distances in the hypothesis space in terms of the loss function ℓ:

$$d_{\mathcal{H}}(\eta, \eta') = \sup_{a\in\mathcal{A}, o\in\mathcal{O}} |\ell(a, o, \eta) - \ell(a, o, \eta')|.$$

Lemma 4. *$d_{\mathcal{H}}(\cdot, \cdot)$ is a pseudometric on $\mathcal{H}$.*

Proof. We check the axioms for a pseudometric.

- nonnegativity: $d_{\mathcal{H}}(\eta, \eta) = 0$ and $d_{\mathcal{H}}(\eta, \eta') \geq 0$ for all $\eta, \eta' \in \mathcal{H}$.
- symmetry: $d_{\mathcal{H}}(\eta, \eta') = d_{\mathcal{H}}(\eta', \eta)$.
- triangle inequality: for each $a \in \mathcal{A}$ and $o \in \mathcal{O}$, $|\ell(a, o, \eta) - \ell(a, o, \eta'')| \leq |\ell(a, o, \eta) - \ell(a, o, \eta')| + |\ell(a, o, \eta') - \ell(a, o, \eta'')|$. Taking the supremum over $\mathcal{A}$ and $\mathcal{O}$ yields the desired property.

□

Let $N(\mathcal{H}, \alpha, d_{\mathcal{H}})$ denote the α-covering number of $\mathcal{H}$ in the pseudometric $d_{\mathcal{H}}$, and let

$$\beta_t^*(\mathcal{H}, \delta, \alpha) := \sqrt{2t\log\left(\frac{10N(\mathcal{H}, \alpha, d_{\mathcal{H}})t^2}{3\delta}\right)} + 2\alpha t. \tag{2}$$

Proposition 4. *For $\delta > 0$, $\alpha > 0$, and $T \in \mathbb{N}$, define*

$$\mathcal{V}_T := \left\{ \eta \in \mathcal{H} : L_T(\eta) \leq \min_{\eta' \in \mathcal{H}} L_T(\eta') + \beta_T^* \right\}$$

Then it holds that

$$\mathbb{P}\left(\eta^* \in \bigcap_{T=1}^{\infty} \mathcal{V}_T \right) \geq 1 - \delta.$$

Proof. Let $\mathcal{H}^\alpha \subseteq \mathcal{H}$ be an α-cover of $\mathcal{H}$ in the pseudometric $d_\mathcal{H}$, in the sense that for any $\eta \in \mathcal{H}$, there is an $\eta^\alpha \in \mathcal{H}^\alpha$ such that $d_\mathcal{H}(\eta, \eta^\alpha) \leq \alpha$. By a union bound over $\mathcal{H}^\alpha$, with probability at least $1 - \delta$,

$$\begin{aligned}
&(\mathcal{L}_T(\eta^\alpha) - L_T(\eta^\alpha)) - (\mathcal{L}_T(\eta^*) - L_T(\eta^*)) \leq \sqrt{2T \log\left(\frac{10|\mathcal{H}^\alpha|T^2}{3\delta}\right)} \\
\implies &(\mathcal{L}_T(\eta) - L_T(\eta)) - (\mathcal{L}_T(\eta^*) - L_T(\eta^*)) \leq \sqrt{2T \log\left(\frac{10|\mathcal{H}^\alpha|T^2}{3\delta}\right)} \\
&\qquad + \underbrace{(\mathcal{L}_T(\eta) - L_T(\eta)) - (\mathcal{L}_T(\eta^\alpha) - L_T(\eta^\alpha))}_{\text{discretization error}}.
\end{aligned}$$

The discretization error can be expanded and bounded as

$$\sum_{t=0}^{T-1} \left[\mathbb{E}_{O \sim f_{\eta^*}(A_t)} \left[\ell(A_t, O, \eta) - \ell(A_t, O, \eta^\alpha) \right] - \ell(A_t, O_t, \eta) + \ell(A_t, O_t, \eta^\alpha) \right] \leq 2\alpha T.$$

Since η^* is a minimizer of $\mathcal{L}_T(\cdot)$, we have that with probability at least $1 - \delta$,

$$L_T(\eta^*) - L_T(\eta) \leq \sqrt{2T \log\left(\frac{10|\mathcal{H}^\alpha|T^2}{3\delta}\right)} + 2\alpha T.$$

Taking the infimum over the size of α covers implies

$$L_T(\eta^*) - L_T(\eta) \leq \sqrt{2T \log\left(\frac{10N(\mathcal{H}, \alpha, d_\mathcal{H})T^2}{3\delta}\right)} + 2\alpha T.$$

$\square$

Taking $\delta = \frac{1}{T}$ and plugging $\beta_T = \beta_T^*(\mathcal{H}, \delta, \epsilon_T^\mathcal{H})$ into the abstract regret bound in Theorem 2 proves the following main theorem.

Theorem 1. *For all $T \in \mathbb{N}$,*

$$\begin{aligned}
\mathrm{Regret}(T) \leq 1 + \frac{1}{T} &+ \min\{\dim_{TE}(\mathcal{H}, \ell, \epsilon_T^\mathcal{H}), T\} \\
&+ 2\sqrt{3\sqrt{2} \log\left(\frac{10N(\mathcal{H}, \alpha, d_\mathcal{H})T^2}{3\delta}\right)^{1/2} \dim_{TE}(\mathcal{H}, \ell, \epsilon_T^\mathcal{H}) T^{3/2} + 6 \dim_{TE}(\mathcal{H}, \ell, \epsilon_T^\mathcal{H})}.
\end{aligned}$$

Proof.

$$
\begin{aligned}
&\mathrm{Regret}(T)\\
&\leq 1+\frac{1}{T}+\min\{\dim_{TE}(\mathcal{H},\ell,\epsilon_T^{\mathcal{H}}),T\}+2\sqrt{3\dim_{TE}(\mathcal{H},\ell,\epsilon_T^{\mathcal{H}})\beta_T^*(\mathcal{H},\delta,\epsilon_T^{\mathcal{H}})T}\\
&=1+\frac{1}{T}+\min\{\dim_{TE}(\mathcal{H},\ell,\epsilon_T^{\mathcal{H}}),T\}+\\
&\quad+2\sqrt{3\dim_{TE}(\mathcal{H},\ell,\epsilon_T^{\mathcal{H}})\left(\sqrt{2T\log\left(\frac{10N(\mathcal{H},\epsilon_T^{\mathcal{H}},d_{\mathcal{H}})T^2}{3\delta}\right)}+2\epsilon_T^{\mathcal{H}}T\right)T}\\
&=1+\frac{1}{T}+\min\{\dim_{TE}(\mathcal{H},\ell,\epsilon_T^{\mathcal{H}}),T\}+\\
&\quad+2\sqrt{3\sqrt{2}\log\left(\frac{10N(\mathcal{H},\alpha,d_{\mathcal{H}})T^2}{3\delta}\right)^{1/2}\dim_{TE}(\mathcal{H},\ell,\epsilon_T^{\mathcal{H}})T^{3/2}+6\epsilon_T^{\mathcal{H}}\dim_{TE}(\mathcal{H},\ell,\epsilon_T^{\mathcal{H}})T^2}\\
&\leq 1+\frac{1}{T}+\min\{\dim_{TE}(\mathcal{H},\ell,\epsilon_T^{\mathcal{H}}),T\}+\\
&\quad+2\sqrt{3\sqrt{2}\log\left(\frac{10N(\mathcal{H},\alpha,d_{\mathcal{H}})T^2}{3\delta}\right)^{1/2}\dim_{TE}(\mathcal{H},\ell,\epsilon_T^{\mathcal{H}})T^{3/2}+6\dim_{TE}(\mathcal{H},\ell,\epsilon_T^{\mathcal{H}})},
\end{aligned}
$$

where he last inequality follows since $\epsilon_T^{\mathcal{H}}\leq 1/T^2$ by definition. □

The leading term in the regret bound is of order

$$T^{3/4}\left(\log N(\mathcal{H},\epsilon_T^{\mathcal{H}},d_{\mathcal{H}})\right)^{1/4}\sqrt{\dim_{TE}(\mathcal{H},\ell,\epsilon_T^{\mathcal{H}})}.$$

Remark 3. As noted earlier on, while the order $\widetilde{O}(T^{3/4})$ on the time horizon T may appear suboptimal compared to classical $\widetilde{O}(\sqrt{T})$ optimal rates for bandit learning with direct reward feedback, this slower rate is in fact a principled consequence of our minimal assumptions. Specifically, our analysis makes no structural assumptions on the verifier loss ℓ beyond boundedness. If we have more structural knowledge of ℓ, say, that it is α-strongly convex, then the bound can be tightened to match the optimal order $\widetilde{O}(\sqrt{T})$. A notable instance is when ℓ is a squared loss. A refined analysis on the drift of conditional mean losses allows us to choose the confidence parameters β_T for the version spaces to be of order $\widetilde{O}(\log(1/\delta))$, which results in the tight $\widetilde{O}(\sqrt{T})$ regret rate.

B Proofs for Supporting Lemmas and Propositions

B.1 Proof for Proposition 1

Proof. Let $\tilde{\ell}=C_F\ell$. Let $d_{TE}=\dim_{TE}(\mathcal{H},\tilde{\ell},\epsilon)$ be the shorthand for the ϵ-transfer eluder dimension of $\mathcal{H}$ with respect to $\tilde{\ell}$. Then, there exists a length d_{TE} sequence of elements in $\mathcal{A}$ such that for some $\tilde{\epsilon}\geq\epsilon$, every action element is $\tilde{\epsilon}$-transfer independent of its predecessors. We denote such a sequence as $(a_0,\ldots,a_{d_{TE}-1})$. By definition of the transfer eluder dimension, for any $k\in\{0,\ldots,d_{TE}-2\}$, there exists a pair of hypotheses $\eta,\eta'\in\mathcal{H}$ satisfying

$$\sum_{i=0}^{k}\left(\mathbb{E}_{o\sim f_{\eta'}}(a_i)[\tilde{\ell}(a_i,o,\eta)]-\tilde{\ell}_{\eta'}^{\min}(a_i)\right)\leq\tilde{\epsilon}^2$$

but $|r_\eta(a_{k+1}) - r_{\eta'}(a_{k+1})| > \tilde{\epsilon}$. Using the definition for reward-discriminative verifiers,

$$\sum_{i=0}^{k} (r_\eta(a_i) - r_{\eta'}(a_i))^2 \le C_F \sum_{i=0}^{k} \left(\mathbb{E}_{o\sim f_{\eta'}}(a_i)[\ell(a_i, o, \eta)] - \ell_{\eta'}^{\min}(a_i) \right)$$
$$= \sum_{i=0}^{k} \left(\mathbb{E}_{o\sim f_{\eta'}}(a_i)[\tilde{\ell}(a_i, o, \eta)] - \tilde{\ell}_{\eta'}^{\min}(a_i) \right) \le \tilde{\epsilon}^2.$$

By the definition of the (regular) eluder dimension, every action in the sequence $(a_0, \ldots, a_{d_{TE}-1})$ is ϵ-independent of its predecessors. Therefore, $d_{TE} \le \dim_E(\mathcal{R}, \epsilon)$ since the latter is the length of the longest sequence of independent actions. We may conclude that $\dim_E(\mathcal{R}, \epsilon) \ge \dim_{TE}(\mathcal{H}, C_F \ell, \epsilon)$.

□

B.2 Proof for Lemma 5

Lemma 5. *Consider some $\bar{\mathcal{H}}$. Suppose $\min_{\pi\in\Pi} \max_{\eta\in\bar{\mathcal{H}}} r_\eta(\pi_\eta) - r_\eta(\pi) = 0$. Let $\hat{\pi}$ be a minimizer. Let $\mathcal{A}_\eta^*$ denote the set of optimal actions with respect to r_η. Then supp$(\hat{\pi}) \subseteq \mathcal{A}_\eta^*$, for all $\eta \in \bar{\mathcal{H}}$.*

Proof. We prove by contradiction. Suppose $\hat{\pi}$ takes some action a' outside of $\mathcal{A}_\eta^*$ for some $\eta \in \bar{\mathcal{H}}$ with probability p'. Let $\pi' = \hat{\pi} - p'\mathbb{1}[a = a'] + p'\text{Unif}[a \in \mathcal{A}_\eta^*]$. Then it follows $r_\eta(\pi') > r_\eta(\hat{\pi})$, which is a contradiction. Therefore, supp$(\hat{\pi}) \subseteq \mathcal{A}_\eta^*$, for all $\eta \in \bar{\mathcal{H}}$. □

B.3 Proof of the Reward-Informative Feedback Example

Suppose $r_\eta(a) = \mathbb{E}_{o\sim f_\eta(a)}[g(a, o)]$ for some known $g : \mathcal{A} \times \mathcal{O} \to [0, 1]$. Note that the reward mapping $\eta \mapsto r_\eta$ is known, but the reward function itself is still hidden from the agent (since η^* is unknown). We define $\ell(a, o, \eta) \coloneqq (g(a, o) - r_\eta(a))^2 = (g(a, o) - \mathbb{E}_{o'\sim f_\eta(a)}[g(a, o')])^2$, which gives

$$\mathbb{E}_{o\sim f_\eta(a)}[\ell(a, o, \eta')] = \mathbb{E}_{o\sim f_\eta(a)} \left[(g(a, o) - \mathbb{E}_{o'\sim f_{\eta'}(a)}[g(a, o')])^2 \right].$$

One can easily verify that $\eta \in \arg\min_{\eta'\in\mathcal{H}} \mathbb{E}_{o\sim f_\eta(a)}[\ell(a, o, \eta')]$. With this definition, we have that

$$\begin{aligned}
|r_\eta(a) - r_{\eta'}(a)|^2 &= (\mathbb{E}_{o\sim f_\eta(a)}[g(a, o)] - \mathbb{E}_{o\sim f_{\eta'}(a)}[g(a, o)])^2 \\
&= (\mathbb{E}_{o\sim f_\eta(a)}[g(a, o) - \mathbb{E}_{o'\sim f_{\eta'}(a)}[g(a, o')]])^2 \\
&\le \mathbb{E}_{o\sim f_\eta(a)}[(g(a, o) - \mathbb{E}_{o'\sim f_{\eta'}(a)}[g(a, o')])^2] \\
&= \mathbb{E}_{o\sim f_\eta(a)}[\ell(a, o, \eta')]
\end{aligned}$$

This shows the feedback is reward-informative.

B.4 Proof of Reasoning Example

binary indicator of whether all steps are correct This problem is equivalent to a bandit problem with $|\mathcal{S}|^L$ arms. Here $f_\eta(a) = r(a)$, so the transfer eluder dimension reduces to the standard eluder dimension, which is bounded by the size of the action space.

index of the first incorrect step Here we prove for $\epsilon < 1/2L$. Given the rubric of η^*, partition the action space into L sets, where $\mathcal{A}_l = \{(s_1, \ldots, s_L) | s_1, \ldots, s_{l-1}$ are correct and s_l is incorrect$\}$ for $l = 1, \ldots, L$, where $\mathcal{A}_0$ denotes sequences where s_1 is incorrect. By this definition, we have $\mathcal{A}_i \bigcap \mathcal{A}_j = \emptyset$, for $i \ne j$, and $\mathcal{A}^* \bigcup (\bigcup_{l=1}^{L} \mathcal{A}_l) = \mathcal{A}$, where $\mathcal{A}^* = \{a^*\}$

Suppose we have an independent action sequence $(a_1, \ldots, a_K)$ in the sense of Definition 3 where each action is ϵ-independent of their predecessors. We show it can have no more than $|\mathcal{S}|$ actions from each $\mathcal{A}_l$ for $l \in [1, L]$. By definition of the feedback, for $a \in \mathcal{A}_l$, $f^*_\eta(a) = l$. Suppose we have more than $|\mathcal{S}|$ actions from $\mathcal{A}_l$. It implies that a token must be used twice at the lth position. Say it's s_l and it's shared by $a^1, a^2 \in \mathcal{A}_l$. Then we show a^2 is ϵ-dependent on a^1 when $\epsilon < 1/L$. For $\eta \in \mathcal{H}$, satisfying $\mathbb{E}_{o\sim f^*(a^0)}[|o - f_\eta(a^0)|^2/L^2] = |l - f_\eta(a^0)|^2/L^2 \le \epsilon^2$, we have $l - L\epsilon \le f_\eta(a^0) \le l + L\epsilon$. Since $\epsilon < 1/2L$ and $f_\eta(a^0)$ is an integer, this implies $f_\eta(a^0) = l$. That is, for such an η satisfying the constraint given by a^0, s_l is incorrect. This implies $f_\eta(a^1) \le l$. Therefore, $r_\eta(a^0) = r_\eta(a^1) = 0$.

Therefore, the length of independent action sequences is bounded by $|\mathcal{S}|L + |\mathcal{A}^*| = |\mathcal{S}|L + 1$.

give correction for the first mistake In this case, the feedback not only returns the index of the first incorrect step l, but also reveals the correct reasoning action s^*_l. Let $a^*_\eta = (s_1(\eta), \dots, s_L(\eta))$ denote the L reasoning steps based on the hypothesis η. The reward function of any action a and hypothesis η is $r_\eta(a) = \mathbb{I}\{a^*_\eta = a\}$. For an action $a = (s_1, \dots, s_L)$ and feedback $o := (l, s_l(\eta))$ generated based on $f_\eta(a)$, we have $s_j = s_j(\eta)$ for all $j < l$ and $s_l \ne s_l(\eta)$. Now, given any feedback $o := (l, s^*_l)$, we define the following loss $\ell(a, o, \eta) = \frac{1}{L}\left(\sum_{j=1}^{l-1} \mathbb{I}\{s_j(\eta) = s_j\} + \mathbb{I}\{s_l(\eta) = s^*_l\}\right)$. This verifer loss evaluates whether η and η' have the same first l reasoning steps.

For $\epsilon < 1$, suppose an action sequence $(a_1, \dots, a_K)$ where each action is ϵ-independent of their predecessors. If action a is ϵ-independent, there exists η, η' such that $\sum_{i=1}^K \mathbb{E}_{o_i \sim f_{\eta'}(a)}[l(a_i, o_i, \eta)] \le \epsilon$ and $|r_\eta(a) - r_{\eta'}(a)| > \epsilon$. By definition of the feedback and loss, we know η,η' have the same initial $\max_i l_i$ reasoning steps. However, we know that $r_\eta(a) \ne r_{\eta'}(a)$ indicating at least one index $l > \max_i l_i$ where $s_l \in \{s_l(\eta), s_l(\eta')\}$ and $s_l(\eta) \ne s_l(\eta')$, resulting in feedback $o = (l, s_l(\eta'))$ for a. Thus, the sequence of indices in feedback $o_1, o_2, \dots$ is monotonic. As we have L reasoning steps, for any pair η, η', the sequence length is bounded by L.

demonstration Here, the feedback directly demonstrates correct reasoning sequence $a^* = (s^*_1, \dots, s^*_L)$ and is independent of the agent's action sequence. For action $a = (s_1, \dots, s_L)$ and hypothesis η, we define the loss as $\ell(a, o, \eta) = \mathbb{I}\{o = a^*_\eta\}$. Therefore, for any η, η' and $\epsilon < 1$, if a satisfies: $\mathbb{E}_{o\sim f_{\eta'}(a)}\ell(a, o, \eta) \le \epsilon$, we have $a^*_\eta = a^*_{\eta'}$, implying $r_\eta(a) = r_{\eta'}(a)$ for all $a \in |\mathcal{S}|^L$ and a transfer Eluder dimension of 1.

C LLF and its relationship to existing paradigms

In this section, we describe the relation of LLF with existing paradigms of learning from feedback, as alluded to in Fig. 2 in more detail. In all discussed paradigms, we focus our comparison on how different forms of feedback are subsumed within LLF, while other environment parameters are loosely assumed to be included in the LLF agent's hypothesis space. LLF covers the following learning paradigms commonly discussed in the literature:

Reinforcement learning (RL) In RL, upon seeing an environment state $x_t \in \mathcal{X}$, the agent chooses an action $a_t \in \mathcal{A}$ and observes a scalar reward feedback $r_t \in \mathbb{R}$. The rewards and states observed by the agent at any decision step t, can depend on the past observed states and actions. In LLF, the agent's hypothesis $\eta \in \mathcal{H}$ returns a reward function $r_\eta : \mathcal{A} \times \mathcal{X} \to [0, 1]$, while the feedback function is exactly the same: $f_\eta = r_\eta$. Hence, RL is trivially subsumed by LLF.

Interaction-guided Learning (IGL) (Xie et al., 2021) In IGL, the environment generates a latent scalar reward $r(x, a) \in [0, 1]$ but only reveals a rich feedback vector $y \in \mathcal{Y}$. To enable learning, IGL framework assumes reward decodability, i.e., the existence of a decoder $\psi \in \Psi$, such that $\psi : \mathcal{Y} \times \mathcal{A} \to [0, 1]$, capable of extracting reward estimates for the agent. LLF naturally accommodates this by modeling both the latent reward r_η and the feedback mapping f_η (hence the feedback y), allowing the agent to reason about the consistency between the decoded rewards and the observed feedback vectors without needing to identify the true decoder ψ^* or the true feedback function f^*.

Reward-informative LLF Reward-informative LLF, defined formally in Definition 5, subsumes the special case where the latent reward function is itself a function of the observed feedback (Xie et al., 2024). This framework generalizes both RL and IGL, capturing scenarios where feedback is rich and structured (e.g., language) but ultimately reflects reward. As discussed in Section 3.3, this class of LLF problems can be no harder than the reward-only setting and may even improve sample efficiency by leveraging structure in the feedback to recover the reward signal more effectively.

Multi-objective RL (MORL) MORL extends the standard RL framework to environments that return vector-valued rewards rather than a single scalar. The central challenge in MORL is balancing trade-offs across multiple objectives, often handled via scalarization methods (see single-policy learning approaches in (Roijers et al., 2013; Zhang & Golovin, 2020)) or Pareto front exploration (Mossalam et al., 2016). In LLF, this is naturally captured by allowing the agent's hypothesis to represent vector-valued reward functions. Furthermore, the verifier loss $\ell : \mathcal{A} \times \mathcal{O} \times \mathcal{H}$ can be extended accordingly. Since the reward vector may be under-determined with respect to the underlying utility function, we treat MORL as distinct from reward-informative LLF (Definition 5), which assumes informativeness of feedback with respect to scalar reward.

Preference-based RL In PbRL, the environment does not reveal scalar reward feedback. Instead, the agent receives pairwise preferences over actions (or trajectories), e.g., that action a is preferred over action a'. These comparisons may be between actions selected by the agent or between one agent-chosen action and a reference provided by the environment. LLF captures this setting by modeling the feedback function f_η as a binary comparator over pairs of actions such that $f_\eta(a, a') \in \{0, 1\}$ indicates the binary preference. The underlying reward model can be implicitly defined in the hypothesis η such that it induces such preferences. Thus, this preference based structure fits within LLF.

Imitation learning (IL) In IL, the agent learns from demonstrations of expert behavior rather than explicit feedback or rewards. To make a closer comparison with LLF, we can consider the interactive imitation learning setting, where the agent observes expert actions (corrections) for the all environment observations. IL can be modeled within the LLF framework by considering expert actions as a form of feedback $f^*_\eta = a^*$. Any hypothesis $\eta \in \mathcal{H}$ considered by the LLF agent can evaluate a verifier loss which corresponds to the discrepancy between the optimal action of the hypothesis a^*_η and expert action a^*. IL is thus a special case of LLF where the feedback space is the action space itself, and consistency between the agent's output and expert-labeled actions is the verifier loss.

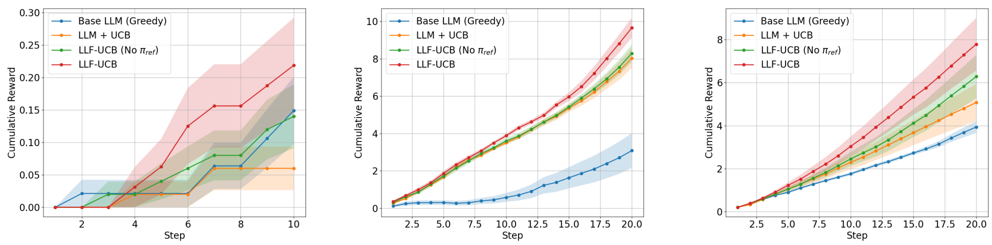
D Extensions

D.1 Special Case of Reward-Agnostic Feedback

Text feedback may contain information beyond what is relevant to the reward. In particular, one could imagine a special case, where feedback does not reveal much about the reward, but still provides enough to identify an optimal action over time. One simple example is when the feedback directly reveals the optimal action, regardless of the action chosen. In this case, the transfer eluder dimension as defined could be arbitrarily large, but ideally an efficient LLF agent should choose the optimal action in the following steps instead of trying to identify the mean reward for each action.

D.2 Extension to Contextual Bandits

Our formulation can be modified slightly to accommodate learning with a context. In a contextual problem, a Markov process X_t independently takes values in a set $\mathcal{X}$ that the agent views as contexts. We may define the full set of actions to be the set of context-action pairs $\mathcal{A} := \{(x, a) : x \in \mathcal{X}, a \in \mathcal{A}(x)\}$, where $\mathcal{A}(x)$ is the set of available actions under the context x. Instead of having a fixed action space $\mathcal{A}$ across time, consider time-varying action sets



(a) Wordle (50 scenarios) (b) Battleship (20 scenarios) (c) Minesweeper (20 scenarios)

Figure 4: We show the cumulative reward that the agent is able to obtain during a fixed number of interactions with the environment. Shaded area represents the standard error of cumulative reward across different scenarios. The battleship result looks different here because we fixed a bug on how we sample random actions to construct $\pi_{\texttt{ref}}$ in the experiments with the main paper submission.

$\mathcal{A}_t := \{(X_t, a) : a \in \mathcal{A}(X_t)\}$. At each time t, an action $a_t \in \mathcal{A}_t$ will be selected. In accordance, the policy $\pi = \{\pi_t | t \in \mathbb{N}\}$ is now a sequence of functions indexed by time, each mapping the history $H_t = (\mathcal{A}_0, A_0, R_0, \ldots, \mathcal{A}_{t-1}, A_{t-1}, R_{t-1}, \mathcal{A}_t)$ to a distribution over $\mathcal{A}$ with support $\mathcal{A}_t$. Our analysis for the context-free setting directly carries over.

E Experiment Details

In this section, we present the details of the implementation of our proposed provable agent in three environments that require the LLM agent to learn from language feedback. In particular, we use the following three gym environments proposed in Tajwar et al. (2025).

Wordle In each scenario, the environment selects a secret 5-letter word from a predefined dictionary. The agent can attempt to guess the word, receiving feedback after each guess indicating correct letters and their positions. In our experiment, we used 50 scenarios to evaluate our algorithm. To better illustrate Example 2 in Sec 3.2, we modify the feedback from the original environment to only contain information about the first incorrect character. For example, if the target word is "totem", and the agent's guess is "apple", the feedback is "The first letter 'a' is incorrect." Considering that this feedback provides less information than the typical wordle feedback, we allow the agents to make 10 attempts before termination.

Battleship Battleship is a 2D grid environment where three hidden ships must be located and sunk within 20 turns. The agent fires at one cell per turn, receiving hit/miss feedback and ship type (Carrier, Battleship, Destroyer). Success requires strategic exploration to find ships and exploitation to sink them efficiently. We use 20 scenarios (maps of ship layout) to evaluate our agent. For this game, we offer a per-step reward, such as "a ship was hit but not sunk" would correspond to 0.5 points. This point system is only used for evaluation purposes to showcase the agent's ability to explore. We do not communicate any numerical reward information to the agent.

Minesweeper Minesweeper is a 2D grid puzzle with hidden mines. At each turn, the agent reveals one cell, aiming to uncover all safe cells within 20 turns without hitting a mine. Revealed cells show the number of adjacent mines, and a '0' triggers automatic reveals of surrounding safe cells. Success depends on sequential reasoning and updating hypotheses based on observed clues. The agent receives a 0.2 reward for choosing a square that does not have a mine, and a 1.0 reward for fully solving the game. Invalid moves incur a -0.2 penalty.

E.1 LLF-UCB with Parallel Thought Sampling

First, we define three types of LLM calls used throughout our algorithm implementation: generating hypotheses and candidate actions, constructing a reference policy, and evaluating actions under different hypotheses. Given observation o, and a number of actions to sample N,

1. `propose(o, N)`: At each step, we invoke `propose(o, N)` to prompt the LLM to generate N diverse hypotheses candidates $\{h_1, \dots, h_N\}$ and their corresponding actions $\mathcal{A} = \{a_1, \dots, a_N\}$ given the current observation o. Specifically, we use chain-of-thought style prompting to generate the action. We view the reasoning of that action as the hypothesis. The collection of hypotheses are used to approximate the constraint in Algorithm 1.
2. $\pi_{\texttt{ref}}$`(o)`: To define the reference policy π_{ref}, we prompt the LLM to produce M exploratory or unconventional actions $\mathcal{A}' = \{a_{N+1}, \dots, a_{N+M}\}$ that are valid yet intentionally deviate from typical behavior. The prompt encourages the model to generate creative, non-obvious alternatives.
3. `evaluate(a, h)`: Given all actions and hypotheses, This function evaluates an action a under a given hypothesis h, returning a score in the range $[0, 1]$ quantifying how well the action aligns with the proposed reasoning. Note, we do not use thoughts ("random thought") that produced the exploration actions.

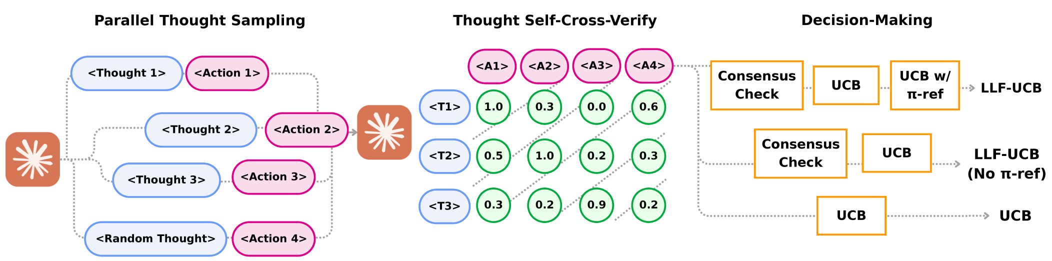


Figure 5: **Algorithm Diagram.** Note that we do not use a ground-truth verifier during the self-cross-check process. The agent proposes actions and uses different actions' chain-of-thought to conduct cross-check. Our proposed algorithm is an inference-time algorithm with a self-judge.

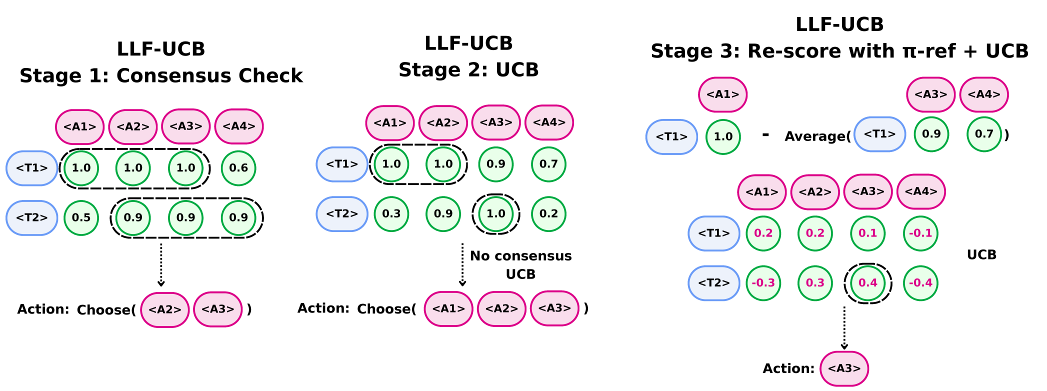


Figure 6: **LLF-UCB Algorithm.** We show that the LLF-UCB algorithm has three steps. The consensus check is first performed to see each hypothesis' highest scoring actions overlap. If such overlap does not exist, a UCB-style hypothesis elimination process is then carried out – only hypotheses with the highest scoring actions are kept. Without π_{ref}, LLF-UCB will do tie-breaking. However, if we introduce a uniform policy π_{ref}, then we can re-calculate the score of each action by subtracting over the average – in this example, we were given A3 and A4 as random actions.

We consider the following agents for comparison. We also implement two variants of the LLF-UCB agents, with slightly different procedures on how the action is chosen.

Greedy This agent generates one hypothesis and one action, and returns that action immediately. This the ReAct-style baseline.

UCB We first ask an LLM to generate N candidate hypotheses and their corresponding actions, as well as M exploratory actions from the reference policy. Then the agent evaluates all of the actions under all of the hypotheses, forming a matrix of $N \times (N + M)$, where we evaluate each hypothesis to all proposed actions and exploratory actions. The agent then select the hypothesis with the highest score and perform the corresponding best action. If there are ties, the first generated action among ties is chosen.

LLF-UCB We first ask an LLM to generate N candidate hypotheses and their corresponding actions, as well as M exploratory actions from the reference policy. Like UCB agent, the agent evaluates all of the actions under all of the hypotheses, forming a matrix of $N \times (N + M)$. Then, to select an action, following Lemma 5, our agent first checks whether a *consensus action* a exists—i.e., an action that achieves the highest score across all hypotheses. Specifically, if for a given action a such that, $\forall h, \forall a_i$, we have $\texttt{evaluate}(h, a) \geq \texttt{evaluate}(h, a_i)$, then a is identified as a consensus action and selected immediately. If no such consensus action exists, we first calculate the score for each hypothesis based on the best action, i.e. $\texttt{score}(h_i) = \max_j \texttt{score}(h_i, a_j)$, and then only keep the hypotheses with the highest score. If multiple hypotheses yield the same highest score, different from the UCB agents which break ties randomly, here we apply a tie-breaking procedure by normalizing scores using the exploratory actions. To break the tie, we subtract the average score over the M exploratory actions from each score: $\overline{\text{score}}(h_i, a_j) \leftarrow \text{score}(h_i, a_j) - \mathbb{E}_{a \sim \pi_{\text{ref}}}[\text{score}(h_i, a)]$. After normalization, we select the hypothesis with the highest normalized score. If a tie still remains, we randomly sample one of the top-scoring hypotheses. The final action is then selected as the highest-scoring action under the chosen hypothesis, with ties again resolved via random sampling.

LLF-UCB (No π_{ref}) We run a variant of our LLF-UCB algorithm without π_{ref}, meaning that we do not perform the final subtraction step to compute $\overline{\text{score}}(h_i, a_j)$. This is direct an approximation of the theoretical algorithm in Algorithm 1, whereas **LLB-UCB** above adds a tie-breaking rule based on π_{ref} which Algorithm 1 does not cover.

E.2 Empirical Results

We plot the cumulative reward as a function of the number of environment interaction steps on WORDLE, BATTLESHIP, and MINESWEEPER in Figure 4. We see that for all three environments, the base LLM, where we only greedily choose the first action, performs worse generally. In environments where information-gathering is more necessary, such as in BATTLESHIP or in MINESWEEPER, agents designed to conduct strategic explorations tend to outperform the greedy base LLM by a large margin.

As shown, our LLF-UCB agents consistently outperform both the greedy baseline and barebone UCB agents. In particular, on BATTLESHIP and MINESWEEPER, LLF-UCB achieves a significant performance improvement over the baselines. Although the theoretical version of our algorithm does not use π_{ref}, we found that across these three environments, performing an explicit score normalization is beneficial. This normalization computes the score for each action as the gap between the score for such action and averaged score of random actions. The gap encodes the implicit directive of choosing actions that have the largest gain over random actions, using the LLM's ability to self-verify.

E.3 Prompt Templates

Propose Action Prompt

Given the information above, please propose some hypotheses and act according to those hypotheses.
You can propose at most {num_actions} hypotheses.
Please propose a reasonable number of hypotheses – each hypothesis represents what you think.
Please provide your actions in the following format:
Action 1: <think>...</think> <answer>action 1</answer>
...
Action {num_actions}: <think>...</think> <answer>your {num_actions}th action</answer>

Propose Exploration Action Prompt (π_{ref})

Given the information above, please propose {num_actions} completely different and unexpected actions. These should be valid in the environment but should explore unusual or creative approaches.
Try to think outside the box and propose actions that might not be immediately obvious or conventional.
Here are the actions you have already proposed:
{actions}
Please avoid proposing the same actions.
Please provide your actions in the following format:
Action 1: <think>...</think> <answer>your first random/exploratory action</answer>
...
Action {num_actions}: <think>...</think> <answer>your {num_actions}th random/exploratory action</answer>

Evaluate Hypothesis

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{task description}
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Now you have a new task. You are a given a hypothesis (thought/instruction) and actions.
You need to evaluate how good or bad the action is given the hypothesis.

Hypothesis:
<think>
{hypothesis}
</think>

Rate all the actions indiviually based on whether the action is aligned with the hypothesis.

Action {action_idx}: <action>{action}</action>

Make sure the score you assign is between 0 and 1. Please provide your scores in the
following format:

Action 1 for the Hypothesis:
<think> ... </think>
<score>...</score>
...
Action {num_actions} for the Hypothesis:
<think> ... </think>
<score>...</score>
```