Selecting controlled edges in edge-based pinning control synchronization

Keywords: Synchronization, Adaptive network, Pinning control, Edge centrality, Fiedler value

Extended Abstract

Network synchronization is a fundamental and pivotal issue in the field of network science, which has garnered widespread attention across various disciplines, including physics, biology, mathematics, and engineering. In recent years, an edge-based adaptive synchronization strategy was proposed [1]. The goal is to improve synchronization through adaptively increasing the edge weights. However, selecting controlled edges remains a challenge, as relying on classical edge centrality metrics fails to achieve precise selection.

Consider an N-sized network dynamics given by [1]

$$\begin{cases} \dot{\xi}_i(t) = f(\xi_i(t)) - \sigma \sum_{j=1}^{N} l_{ij}\xi_j(t), & i = 1,2,...,N, \\ \dot{l}_{ij}(t) = \dot{l}_{ji}(t) = -\alpha(\xi_i - \xi_j)^T(\xi_i - \xi_j), & (i,j) \in \bar{E}, \end{cases} \tag{1}$$

where $\xi_i(t) \in R^d$ is the state of the ith node, $f \in C[R^d, R^d]$ a smooth nonlinear function, σ the constant coupling strength, $L = (l_{ij})$ the Laplacian matrix, α a positive constant representing the control gain and $\bar{E}$ the controlled edge set. Suppose that the function f satisfies the QUAD condition, which is a commonly adopted mild assumption. The synchronized state is steady if there exists a positive constant ω such that

$$[I_N \otimes (U - aI_n)] - \sigma\lambda_2(L^{\omega}_{\bar{E}})(I_N \otimes I_d) < 0, \tag{2}$$

where $a > 0$ and U is a diagonal matrix related to the QUAD condition. The weighted Laplacian matrix $L^{\omega}\bar{E}$ is derived after weighting the edge set $\bar{E}$ with ω. Taking into account the influence of the network topology, for the edge set $\bar{E}$, higher value of $\lambda_2(L^{\omega}_{\bar{E}})$ (known as the Fiedler value [2]) in Eq. (2) indicate better control efficiency.

We have presented some conclusions to assist in selecting controlled edges, based on papers published in 2023, 2024, and 2025.

(1) We have quantified the dynamical contribution value of each edge, defined as the sensitivity of the Fiedler value to different edge weights. Higher sensitivity implies greater contribution value, indicating superior performance in control processes. This value is eventually expressed via the Fiedler vector as $I_{e_{ij}} = (x_i - x_j)^2$, where x_p denotes the p-th component of $x_2(L)$ (eigenvector of the Fiedler value). Interestingly, the "Pareto principle" has been observed in the dynamical contribution of edges, with only a small portion of edges having a significant impact on dynamics, as seen in Fig. 1. This implies that controlling a few critical edges suffices to achieve control objectives.

(2) From a graph generation perspective, we demonstrate that cycle structures—a recently emphasized edge combination [3]—enhance network synchronization. This confirms that cycles are beneficial controlled structures. Furthermore, we have proposed a ranking index I_{c_i}=$\sum_{(p,q)\in c_i}(x_p - x_q)^2$ (accumulated contribution value of edges in a cycle) to measure the importance of a cycle. The index guides the edge-based pinning control. The synchronization process achieved by controlling the cycles c_1, c_3, and c_6 are depicted in Fig. 2. It is found

that the importance of c_1, c_3, and c_6 decreases progressively, which aligns with the sequence in which synchronization was achieved.

References

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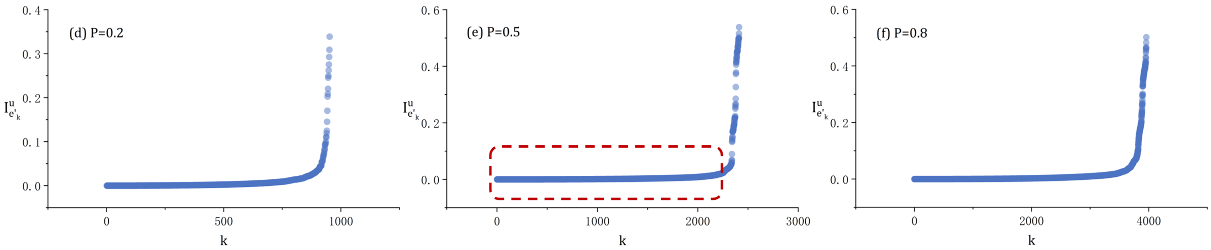

Figure 1: **"Pareto principle".** Distribution of dynamical contribution values of all edges across varying probabilities p in random networks with 100 nodes, where each dot represents the dynamical contribution value of edge e_k numbered in ascending order.

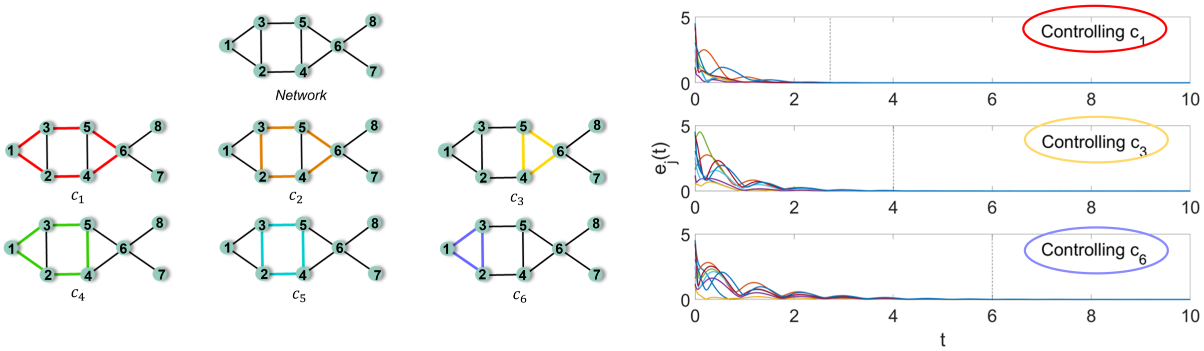

Figure 2: **Control synchronization.** (a) A sample network with 6 cycles. Cycles are ranked as $c_1, c_2, ..., c_6$. (b) The norm of error $e_j(t) = ||\xi_j(t) - s(t)||_2, 1 \leq j \leq 8$ versus time t by respectively controlling cycles c_1, c_3 and c_6 in the network, where $\xi_j(t)$ and $s(t)$ are the node state and the synchronization state respectively. The average error is obtained after 50 simulations by using initial state value range of $[-25, 25]$.