The Minimum Eternal Vertex Cover Problem on a Subclass of Series-Parallel Graphs

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Abstract. *Eternal vertex cover* is the following two-player game between a defender and an attacker on a graph. Initially, the defender positions k guards on k vertices of the graph; the game then proceeds in turns between the defender and the attacker, with the attacker selecting an edge and the defender responding to the attack by moving some of the guards along the edges, including the attacked one. The defender wins a game on a graph G with k guards if they have a strategy such that, in every round of the game, the vertices occupied by the guards form a vertex cover of G, and the attacker wins otherwise. The *eternal vertex cover number* of a graph G is the smallest number k of guards allowing the defender to win and EVC is the problem of computing the eternal vertex cover number of the given graph.
We study this problem when restricted to the well-known class of series-parallel graphs. In particular, we prove that EVC can be solved in linear time when restricted to melon graphs, a proper subclass of series-parallel graphs. Moreover, we also conjecture that this problem is NP-hard on series-parallel graphs.

Keywords: eternal vertex cover, melon graphs, series-parallel graphs.

1 Introduction

A *vertex cover* of a graph $G = (V, E)$ is a set $S \subseteq V$ such that, for every edge in E, at least one of its endpoints is in S. It is *minimum* if it is of minimum cardinality. This minimum value is called the *vertex cover number* of G and is denoted by $vc(G)$. The VERTEX COVER problem consists in determining this number.

The notion of *eternal vertex cover*, first introduced by Klostermeyer and Mynhardt [24], exploits the above definition in the context of a two-player multi-round game, where a *defender* uses mobile guards (whose number remains the same throughout the game) placed on some vertices of G to protect the edges of G from an *attacker*. The game begins with the defender placing guards on some vertices, at most one *per* vertex. In each round of the game, the attacker chooses an edge to attack. In response, the defender moves the guards so that each guard either stays at its current location or moves to an adjacent vertex; the movement of all guards in a round is assumed to happen in parallel. If a guard crosses the attacked edge during this move, it *protects* the edge from the attack. The defender wins if the edges can be protected by any sequence of attacks. If an attacked edge cannot be protected in some round, the attacker wins. Clearly, a necessary condition to protect the graph is that the set of vertices where the guards lie at each round is a vertex cover, and this justifies the name of eternal vertex cover. The Eternal Vertex Cover problem, EVC for short, consists of determining the minimum number of guards allowing the defender to protect all the edges of G, called *eternal vertex cover number* of G and denoted by $evc(G)$. In the literature, $evc(G)$ is sometimes denoted by $\alpha_m^\infty(G)$ (see for example [24]) or by $\tau^\infty(G)$ [6].

EVC finds applications in network security, drone surveillance, and war scenarios. For example, some agents are deployed on the nodes of a network in such a way that the agents watch every connection between nodes. A malicious attack forces an agent to traverse that connection and, more in general, to reconfigure the position of the agents. The eternal vertex cover game asks whether it is possible for a set of agents to respond to any sequence of attacks. Minimizing the number of agents required for an everlasting defense and understanding a winning strategy is clearly beneficial to resource allocation.

A *series-parallel graph* can be recursively constructed by observing that a single edge is a series-parallel graph, and by composing smaller series-parallel graphs either in *series* or in *parallel*. Although this class has been introduced a long time ago [17], it still attracts the attention of researchers (see, *e.g.*, [1,4,5,12,16]). Series-parallel graphs are a well-known and studied graph class from a theoretical perspective and naturally model two-terminal networks that are constructed with the series and parallel composition.

In this paper, we study the EVC problem when restricted to series-parallel graphs: we prove that it can be solved in linear time for a proper subclass of series-parallel graphs, while we conjecture that it remains NP-hard on the whole class.

1.1 Previous Results

Since its definition, EVC has been deeply studied from a computational complexity point of view: deciding whether k guards can protect all the edges of a graph is NP-hard [19]; it remains so even for bipartite graphs [28] and for biconnected internally triangulated planar graphs, although there exists a polynomial time approximation scheme for computing the eternal vertex cover number on this class of graphs [8]. The problem can be exactly solved in $2^{\mathcal{O}(n)}$ time and is FPT parameterized by solution size [19].

On the positive side, there are a few graph classes for which the problem can be efficiently solved. Indeed, it is solvable in linear time on trees and cycles [24], maximal outerplanar graphs [9], chain and split graphs [30]. Moreover, it is solvable in quadratic time on chordal graphs [8,11] and solvable in polynomial time on co-bipartite graphs [10], cographs [30] and generalized trees [6].

Connecting the vertex cover and eternal vertex cover numbers, it holds that $vc(G) \leq evc(G) \leq 2vc(G)$ [24]. Consequently, it is interesting to understand for which graphs these two parameters are very close: in [24,25] different conditions for equality hold (graphs for which this relation holds are generally called *spartan*), while in [8] it is showed that $evc(G) \leq vc(G)+1$ for every locally connected graph G.

One of the reasons of interest for series-parallel graphs is that many combinatorial problems that are computationally hard on general graphs become polynomial-time or even linear-time solvable when restricted to the series-parallel graphs (*e.g.*, vertex cover [33], dominating set [35], coloring [7], graph isomorphism [21,26] and Hamiltonian cycle [18,20]). On the other hand, very few problems are known to be NP-hard for series-parallel graphs. These include the subgraph isomorphism [14,22,27], the bandwidth [32], the edge-disjoint paths [37], the common subgraph [2] and the list edge and list total coloring [36] problems.

1.2 Discussion of Our Results

In this work, we study EVC on the class of series-parallel graphs. Preliminarily, it is worth noting that, given any graph G, it holds $vc(G) \leq evc(G) \leq 2vc(G)$ [24] and both the bounds are attainable: as an example, consider a cycle and an odd length path, respectively. Both these graphs are, in fact, series-parallel graphs, although rather special. In particular, paths (for which vertex cover and eternal vertex cover numbers are very far) are not biconnected; on the other hand, cycles (for which $vc(G)$ and $evc(G)$ are very close) are biconnected. In Section 3, we generalize the known result for cycles to k-melon graphs ($k \geq 2$), biconnected series-parallel graphs constituted by a set of pairwise internally disjoint paths linking two vertices. In particular, we show the following.

Theorem 1. EVC *is linear-time solvable when restricted to the class of melon graphs. Moreover, for every melon graph G, it holds that $evc(G) \leq vc(G) + 1$.*

The proof of the aforementioned result is based on a case-by-case analysis classifying melon graphs according to the number of paths of even and odd lengths. For each possible input melon graph, we not only compute the eternal vertex cover number in linear time, but we also provide a minimum eternal vertex cover class and defense strategies.

One could wonder whether the biconnectivity has some influence on the difference between vertex cover and eternal vertex cover numbers. In Section 4, we give a negative answer to this question when extending our analysis to the whole class of series-parallel graphs. Proving the following result.

Theorem 2. *For any integer $k \geq 0$, there is a biconnected series-parallel graph G_k such that $evc(G_k) - vc(G_k) \geq k$, and $evc(G_k) \geq (2 - \frac{2}{k-2})vc(G_k)$.*

Recalling that VC is polynomially solvable on series-parallel graphs [33], a *naïve* 2-approximation algorithm for EVC consists in simply solving VC and doubling the vertex cover number. As a side effect, Theorem 2 implies that there are series-parallel graphs whose eternal vertex cover number is arbitrarily close to the double of the vertex cover number, so showing that the approximation ratio of 2 is almost attainable on biconnected series-parallel graphs. Nevertheless, if we wish to determine the exact value of the eternal vertex cover number, this issue does not seem to be solved in polynomial time; hence, we conclude our paper proposing the following:

Conjecture. EVC *is NP-hard on series-parallel graphs.*

Due to space restrictions, we provide only sketches of proof for some results. Nevertheless, we refer to the Appendix for the complete proofs.

2 Terminology

For a positive integer k, we denote with $[k]$ the set $\{0, \dots, k\}$. Let $G = (V, E)$ be a graph, on which we recall the following definitions. Given a vertex v of G, the *closed neighborhood of* v, $N[v]$, is the set of vertices that are adjacent to v and v itself. A *path* P is a graph whose vertex set is $\{v_0, \dots, v_\ell\}$, $\ell \geq 1$, and edge set is $\{v_i v_{i+1} \mid i \in [\ell - 1]\}$; ℓ is the *length* of P.

A graph $G = (V, E)$ is *bipartite* if it is possible to partition the vertex set into two not empty subsets: $V = A \cup B$ so that each edge of E can only connect one vertex in A with one vertex in B; in this case, we represent G with $(A \cup B, E)$. For extended graph terminology, we refer to [15].

2.1 Eternal Vertex Cover

Given a graph $G = (V, E)$ and a subset of vertices $U \subseteq V$, we imagine each vertex of U hosting one guard, and all the edges incident to these vertices are considered *guarded*. The guards are allowed to move from one vertex to another only through an edge connecting them.

An *attack* is the selection of one edge $e \in E$ by the attacker. The defender *protects* an attacked edge if it can move a guard along that edge. Thus, it is possible only to protect guarded edges and a necessary condition for $U \subseteq V$ to be able to protect any edge from an attack is that U is a vertex cover of G.

Consider a guarded edge $e = vw$ and, without loss of generality, assume that $v \in U$. A *defense* from the attack on e is defined as a one-to-one function $\phi : U \to V$ such that e is protected, that is $\phi(v) = w$, and for each $u \in U$, $\phi(u) \in N[u]$. Given any vertex $u \in U$, we say that the guard on u *shifts to* $\phi(u)$ and, by extension, U *shifts to* U' where $U' = \phi(U) = \{\phi(u) \text{ s.t. } u \in U\}$.

The protection of an attacked edge vw with a guard on both endpoints can be easily guaranteed by shifting the guard on v to w, the guard on w to v, and every other guard stays on the same vertex. So, in the following, we implicitly assume that an attack always happens on an edge guarded by one guard, and called *single-guarded* edge. We are now ready to give the notion of eternal vertex cover.

Definition 1. *[8] Given a graph G, a family $\mathcal{U}$ of vertex covers of G all of the same cardinality is an* eternal vertex cover class *of G if the defender can protect* *any attacked edge by shifting any vertex cover of* $\mathcal{U}$ *to another vertex cover of* $\mathcal{U}$*. Each vertex cover of* $\mathcal{U}$ *is called a* configuration *for* G*. The* size *of an eternal vertex cover class* $\mathcal{U}$ *is the cardinality of any configuration of* $\mathcal{U}$*.* EVC *consists of finding the minimum size* $evc(G)$ *of an eternal vertex cover class for* G*. An eternal vertex cover class of size* $evc(G)$ *is said to be a* minimum eternal vertex cover class.

In the following, in order to determine $evc(G)$, we first provide a family $\mathcal{U}$ of vertex covers; then, for every vertex cover U of $\mathcal{U}$ and every edge e of G, we exhibit a defense function that shifts U to another vertex cover of $\mathcal{U}$ and protects e, thus showing that $\mathcal{U}$ is an eternal vertex cover class of G; finally, we show that no eternal vertex cover class of G can have size strictly smaller than $\mathcal{U}$.

2.2 Series-Parallel Graphs

Let the graphs considered from now on have two distinguished vertices, s and t, called *source* and *sink*, respectively. Let be given two vertex-disjoint graphs G_1 and G_2, with sources and sinks s_1 and t_1, s_2 and t_2, respectively. The *series composition* of G_1 and G_2 is a graph G obtained by merging t_1 with s_2, and its distinguished vertices are $s = s_1$ and $t = t_2$. The *parallel composition* of G_1 and G_2 is a graph G obtained by merging s_1 with s_2 into the distinguished vertex s and t_1 with t_2 into the distinguished vertex t. Series-parallel graphs can be constructed recursively by series and parallel compositions:

Definition 2. *[17] A* series-parallel *graph* G *is a graph with two distinguished vertices* s *and* t *that is either a single edge or can be recursively constructed by either* series *or* parallel *composition of two series-parallel graphs.*

Due to the recursive nature of series-parallel graphs, it is natural to introduce a decomposition that mimics the construction of these graphs.

Definition 3. *[34] The* SP-decomposition tree *of a series-parallel graph* G *is a rooted binary tree* T *in which each leaf corresponds to an edge of* G*, and every internal node of* T *is labeled as either a parallel or series node; starting from its edges, that are series-parallel graphs, the series-parallel subgraph associated to a subtree of* T *rooted at a node* v *is the composition indicated by the label of* v *of the two series-parallel subgraphs associated to the children of* v*;* G *is the series-parallel graph associated to the root of* T*.*

For an extended and more formal treatment of series-parallel graphs and SP-decompositions, the reader can refer *e.g.* to [16].

2.3 Melon Graphs

The main result of this paper, described and proved in Section 3, deals with a subclass of series-parallel graphs:

Definition 4. *For any integer $k \geq 1$, given k internally vertex-disjoint paths $P^{(1)}, \ldots, P^{(k)}$ whose endpoints are their distinguished vertices, a graph G is a k-*melon graph *if G can be constructed by the parallel composition of $P^{(1)}, \ldots, P^{(k)}$. A graph G is a* melon graph *if it is a k-melon graph, for some $k \geq 1$.*

In particular, paths are 1-melon graphs and cycles are 2-melon graphs. Note that for every $k \neq 2$, in every k-melon graph G, s and t are the only two vertices of G not having degree two. Melon graphs have already been studied in different research works: w.r.t. the computation of the treelength [16], for the understanding of the treewidth on hereditary graph classes [3,31] and in high-energy physics representing tensor models [13].

Let G be a k-melon graph for some $k \geq 1$. Denote with $\mathcal{P}(G)$ (or simply $\mathcal{P}$ if there is no risk for confusion) the set of paths $P^{(1)}, \ldots, P^{(k)}$ used to obtain G. A path is said to be either an *odd* or an *even path*, depending on the parity of its length. Let $\mathcal{P} = \mathcal{P}_{odd} \cup \mathcal{P}_{even}$ be the partition of $\mathcal{P}$ into odd and even paths.

Definition 5. *A k-melon graph G obtained by the paths of $\mathcal{P} = \mathcal{P}_{even} \cup \mathcal{P}_{odd}$ is an* even *(respectively* odd*) k-*melon graph *if $\mathcal{P}_{odd} = \emptyset$ (respectively $\mathcal{P}_{even} = \emptyset$), and it is* mixed *otherwise.*

In what follows, we indicate by P_e a path in $\mathcal{P}_{even}$ and by P_o a path in $\mathcal{P}_{odd}$, in order to easily have in mind its parity when confusion may arise.
Let $G = (V, E)$ and U be melon graph and a subset of V, respectively.

Let $P_e = \{v_0, \ldots, v_{2m}\} \in \mathcal{P}_{even}$, for some $m \geq 1$. We say that P_e is an *internal path* w.r.t. U if $U \cap V(P_e) = \{v_{2j} \mid j \in [m]\}$ and similarly, that P_e is an *external path* w.r.t. U if $U \cap V(P_e) = \{v_{2j+1} \mid j \in [m-1]\} \cup \{s, t\}$.

Let $P_o = \{v_0, \ldots, v_{2m+1}\} \in \mathcal{P}_{odd}$, for some $m \geq 1$. We say that P_o is an *s-path* w.r.t. U if $(U \cap V(P_o)) \setminus \{t\} = \{v_{2i+1} \mid i \in [m]\}$, while we say that P_o is an *t-path* w.r.t. U (or simply s-path, if U is clear from the context) if $(U \cap V(P_o)) \setminus \{s\} = \{v_{2i} \mid i \in [m]\}$. As an example, see Figure 1.a, that showcases internal, external s-paths and a t-path.

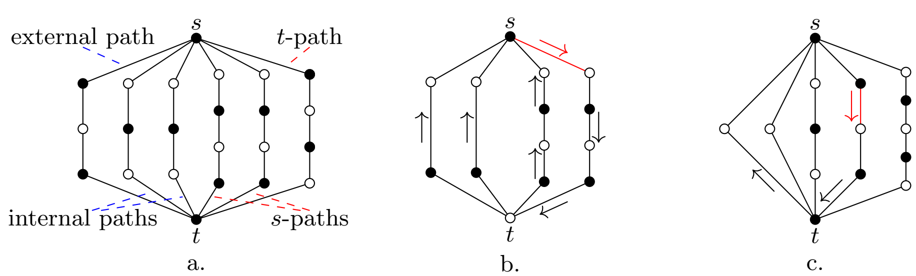

Fig. 1. a. mixed melon graph where path definitions of Subsection 2.3 are highlighted; b. and c. strategies for an odd melon graph (Lemma 1), and for an even melon graph (Lemma 2), respectively.

3 Eternal Vertex Cover on Melon Graphs

In the following, we will prove Theorem 1 separately on even, odd, and mixed melon graphs via constructive sketches of proofs. Note that it is very well-known how to solve EVC on 1- and 2-melon graphs, *i.e.*, paths and cycles [24]; hence, in the rest of this work, we only consider k-melon graphs with $k \geq 3$.

Every odd melon graph G is bipartite and the two partition classes A and B have the same size: $|A| = |B| = vc(G)$.

Lemma 1. *Let $G = (A \cup B, E)$ be an odd k-melon graph. It holds that $evc(G) = vc(G)$, and the family $\mathcal{U} = \{A, B\}$ is a minimum eternal vertex cover class of G.*

Sketch of Proof. It is possible to show that for every edge e of an odd melon graph, there exists a perfect matching M_e containing e.

To defend a configuration, say A, from the attack on e, the guards shift through the edges of M_e. Since M_e is a perfect matching between A and B, the resulting configuration is B. See also Figure 1.b. □

Given a k-even melon graph G, for each fixed $i \in [k]$, we denote with U_i the vertex set such that the path $P^{(i)}$ is an external path w.r.t. U_i and the path $P^{(j)}$ is an internal path w.r.t. U_i, for every $j \in [k]$ and $j \neq i$.

Lemma 2. *Let G be an even k-melon graph. It holds that $evc(G) = vc(G) + 1$, and the family $\mathcal{U} = \{U_i \mid i \in [k]\}$ is a minimum eternal vertex cover class of G, where the sets U_i are defined above.*

Sketch of Proof. It is possible to show that G has a unique minimum vertex cover and hence cannot be defended, so $evc(G) \geq vc(G)+1$. To defend a configuration $U_i \in \mathcal{U}$ from the attack on an edge e, all the guards except one shift along a specific cycle of G, which contains the edge e and is formed from the union of an internal and an external path of G. See also Figure 1.c. □

Given a mixed melon graph G, let $P_e \in \mathcal{P}_{even}$ and let $\mathcal{S}_o$ be any subset of $\mathcal{P}_{odd}$. We denote with $U_{P_e,\mathcal{S}_o}$ the vertex set such that:

- P_e is an external path w.r.t. $U_{P_e,\mathcal{S}_o}$;
- every path in $\mathcal{P}_{even} \setminus \{P_e\}$ is an internal path w.r.t. $U_{P_e,\mathcal{S}_o}$;
- every path in $\mathcal{S}_o$ is an s-path w.r.t. $U_{P_e,\mathcal{S}_o}$;
- every path in $\mathcal{P}_{odd} \setminus \mathcal{S}_o$ is a t-path w.r.t. $U_{P_e,\mathcal{S}_o}$.

Lemma 3. *Let G be a mixed k-melon graph; if $|\mathcal{P}_{even}| \geq 2$ and $|\mathcal{P}_{odd}| \geq 2$, then it holds that $evc(G) = vc(G) + 1$ and the family $\mathcal{U} = \{U_{P_e,\mathcal{S}_o} \mid P_e \in \mathcal{P}_{even},\ \emptyset \neq \mathcal{S}_o \subset \mathcal{P}_{odd}\}$ is a minimum eternal vertex cover class of G, where the sets $U_{P_e,\mathcal{S}_o}$ are defined above.*

Sketch of Proof. This sketch is similar to the one of Lemma 2. To defend a configuration $U_{P_e,\mathcal{S}_o} \in \mathcal{U}$ from the attack on an edge e, all guards but at most one shift along a specific cycle of G, which contains the edge e and is formed from the union of either an internal and an external path or an s- and a t-path. See also Figures 2.a and .b. □

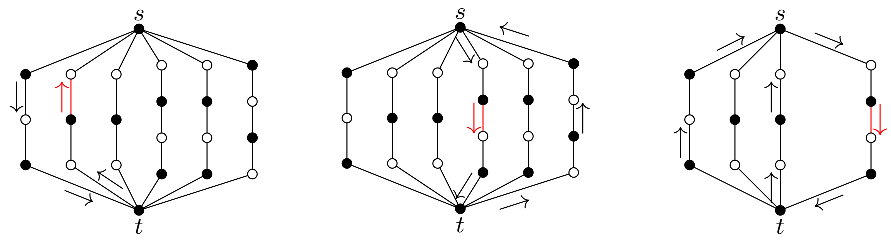

Fig. 2. a. and b. two cases of the strategy for a mixed melon graph with at least two even paths and at least two odd paths (Lemma 3); c. strategy for a mixed melon graph with at least two even paths and only one odd path (Lemma 4).

Consider now the case where $\mathcal{P}_{odd}$ contains a single path P_o. Let $x \in \{s,t\}$ and $P_e \in \mathcal{P}_{even}$. We denote with U_{x,P_e} the vertex set such that:

- P_e is an external path w.r.t. U_{x,P_e};
- every path in $\mathcal{P}_{even} \setminus \{P_e\}$ is an internal path w.r.t. U_{x,P_e};
- P_o is an x-path w.r.t. U_{x,P_e}.

Lemma 4. *Let G be a mixed k-melon graph; if $|\mathcal{P}_{odd}| = 1$, then it holds that $evc(G) = vc(G) + 1$ and the family $\mathcal{U} = \{U_{x,P_e} \mid x \in \{s,t\}, P_e \in \mathcal{P}_{even}\}$ is a minimum eternal vertex cover class of G, where the sets U_{x,P_e} are defined above.*

Sketch of Proof. This sketch is similar to the one of Lemma 2. To defend a configuration $U_{P_e,\mathcal{S}_o} \in \mathcal{U}$ from the attack on an edge e we distinguish two cases: if e belongs to a path in $\mathcal{P}_{even}$, all guards but at most one shift along a specific cycle of G which contains the edge e and is formed from the union of an internal and an external path. If e belongs to the unique path $P_o \in \mathcal{P}_{odd}$, all guards shift along three paths: P_0, the external path P_e and one internal path P'_e. See also Figure 1.c. □

Finally, consider the case where $\mathcal{P}_{even}$ contains a single path P_e, and $\mathcal{P}_{odd}$ contains at least two paths. The set U_s (resp. U_t) is a vertex set not containing t (resp. s) such that P_e is an external path and every path in $\mathcal{P}_{odd}$ is a s-path (resp. t-path) w.r.t. U_s (resp. U_t). Moreover, for any subset $\mathcal{S}_o$ of $\mathcal{P}_{odd}$, let $U_{\mathcal{S}_o}$ be the vertex set of G such that:

- P_e is an internal path,
- every path in $\mathcal{S}_o$ is a s-path
- every path in $\mathcal{P}_{odd} \setminus \mathcal{S}_o$ is a t-path w.r.t. $U_{\mathcal{S}_o}$.

Observe that U_s, U_t and every $U_{\mathcal{S}_o}$ are vertex covers of G and have all the same cardinality. Indeed, the extra guard present in the external path is compensated by the presence of exactly one guard in $\{s,t\}$. Vice versa, the second guard on the set $\{s,t\}$ is compensated by one less guard in the internal path. As an example, see Figure 3, that showcases the configurations U_s and $U_{\mathcal{S}_o}$.

Lemma 5. *Let G be a mixed k-melon graph; if $|\mathcal{P}_{even}| = 1$, it holds that $evc(G) = vc(G)$ and the family $\mathcal{U} = \{U_s, U_t\} \cup \{U_{\mathcal{S}_o} \mid \emptyset \neq \mathcal{S}_o \subset \mathcal{P}_{odd}\}$ is a minimum eternal vertex cover class of G, where the sets U_s, U_t and $U_{\mathcal{S}_o}$ are defined above.*

Sketch of Proof. Every configuration of $\mathcal{U}$ is a minimum vertex cover of G; therefore, to prove the claim, it is enough to show that $\mathcal{U}$ is an eternal vertex cover class of G. Let U be a configuration of $\mathcal{U}$ and e be a single-guarded edge of G. We consider two different cases, distinguishing whether U is of the form either U_x, for some $x \in \{s,t\}$, or $U_{\mathcal{S}_o}$, for some non-empty proper subset $\mathcal{S}_o$ of $\mathcal{P}_{odd}$. If $U = U_s$, for every non-empty proper subset $\mathcal{S}_o$ of $\mathcal{P}_{odd}$, it holds that $U_{\mathcal{S}_o}$ protects U_s. The case $U = U_t$ is proved in a symmetric way. If $U = U_{\mathcal{S}_o}$, for some non-empty proper subset $\mathcal{S}_o$ of $\mathcal{P}_{odd}$. Then, either U_x, with $x \in \{s,t\}$, or $U_{\mathcal{S}'_o}$, for some non-empty proper subset $\mathcal{S}'_o$ of $\mathcal{P}_{odd}$, defends $U_{\mathcal{S}_o}$ from the attack on e. □

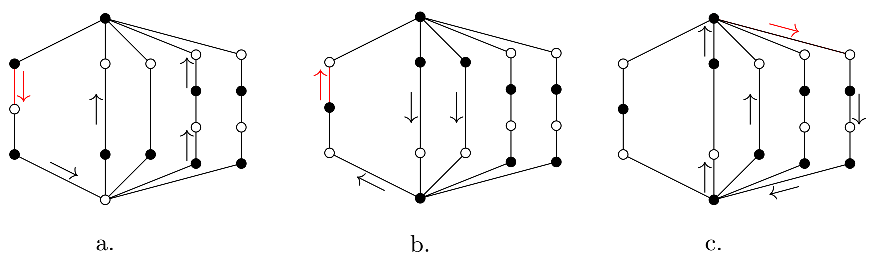

Fig. 3. A mixed melon graph with at least two odd paths and only one even path; a. strategy to defend U_s, b. and c. strategies to defend $U_{\mathcal{S}_o}$ (Lemma 5).

We used the classification of melon graphs based on the parity of the paths constituting them to completely solve the EVC problem on this graph class. We are finally ready to prove the main result of this section.

Theorem 1. EVC *is linear-time solvable when restricted to the class of melon graphs. Moreover, for every melon graph G, it holds that $evc(G) \leq vc(G) + 1$.*

Proof. We start by running a BFS on G rooted at its source s to evaluate the cardinality k of $\mathcal{P}$, and the two sets $\mathcal{P}_{even}$ and $\mathcal{P}_{odd}$. This takes $\mathcal{O}(|V| + |E|)$ time.

Recall that for $k \leq 2$, the claim is already well-known to be true. Hence, assume that $k \geq 3$. According to the cardinality of $\mathcal{P}_{even}$ and $\mathcal{P}_{odd}$, exactly one among Lemmas 1, 2, 3, 4 and 5 applies and the value of $evc(G)$ is obtained in constant time. Theselemmas also imply that $evc(G) \leq vc(G) + 1$. □

4 Toward Eternal Vertex Cover on Series-Parallel Graphs

In view of the recursive structure of series-parallel graphs, it is natural to wonder whether it is possible to extend the efficient computability given by Theorem 1 to the whole class of series-parallel graphs. The following conjecture leans towards a negative answer.

Conjecture EVC *is NP-hard on series-parallel graphs.*

This conjecture is based on many considerations, and the rest of this section is devoted to formalizing a couple of them. We show that melon graphs and series-parallel graphs behave differently w.r.t. the eternal vertex cover number and their SP-decompositions have different properties. These differences support our conjecture that computing the eternal vertex cover number on series-parallel graphs is significantly harder than computing the vertex cover number on series-parallel graphs or the eternal vertex cover number on melon graphs. The first statement supporting the conjecture is the already-stated result Theorem 2, which we are now ready to prove.

Theorem 2. *For any integer $k \geq 0$, there is a biconnected series-parallel graph G_k such that $evc(G_k) - vc(G_k) \geq k$, and $evc(G_k) \geq (2 - \frac{2}{k-2})vc(G_k)$.*

Proof. Let H_k denote the $(k+3)$-melon graph where each of the $k+3$ paths is of length 2; in other words, H_k is a complete bipartite graph $K_{2,k+3}$. Let H'_k be the series composition of H_k and of a 2-length path so that the source of H'_k coincides with the source of the 2-length path and the sink of H'_k coincides with the sink of H_k. For every $k \geq 2$, we define the biconnected series-parallel graph G_k as the parallel composition of k copies of H'_k and one copy of H_k. Let $s_1, \ldots, s_k$, s and t be the sources of the k copies of H_k inside H'_k, the source of G_k and the sink of G_k, respectively. Note that s and t have a high degree, due to the presence of H_k, which is put in parallel with the copies of H'_k. See Figure 4 for a representation of G_3.

In order to show that $evc(G_k)$ and $vc(G_k)$ fulfill the inequalities of the claim, in the following, we first exactly evaluate $vc(G_k)$, then provide a lower bound for $evc(G_k)$. Preliminarily, observe that $U = \{s_1, \ldots, s_k, s, t\}$ is the unique minimum vertex cover of G_k. Indeed, for any other vertex cover $U' \neq U$, if $U \subset U'$ then trivially $|U'| > |U|$, otherwise U' does not contain U and, for example, $s_i \notin U'$. This means that each of the $k+4$ neighbors of s_i belongs to U'. Since the neighborhoods of each s_j are disjoint, $|U'| \geq |U| + k + 3 = 2k + 5$. Even worse bounds are obtained when assuming that $s \notin U'$ or $t \notin U'$. Thus, it holds that $vc(G_k) = k + 2$.

Now, let $\mathcal{U}$ be a minimum eternal vertex cover class of G_k. Each configuration U' of $\mathcal{U}$ must necessarily contain U because, if by contradiction we supposed U' does not include U, then we would have obtained $evc(G_k) = |U'| \geq 2k + 5 > 2vc(G_k)$, which is a contradiction because $evc(G_k) \leq 2vc(G_k)$ [24]. We exploit the property that $U \subset U'$, for each $U' \in \mathcal{U}$ to provide a lower bound for $evc(G_k)$.

The informal idea is that guards on the vertices of U, which are the only vertices of G_k having high degree, require an additional guard hosted by a neighboring vertex, so that they can be replaced to still defend G_k whenever moved by the strategy. We now prove that every configuration $U' \in \mathcal{U}$ contains a vertex in $N[u]$ besides u, for each $u \in U$. If $N[u] \subseteq U'$, the claim is trivially true, so assume that there exists a neighbor v of u that is not in U'. Since U' is a configuration of an eternal vertex cover class of G_k, there exists a defense function ϕ that protects U' from the attack on uv and, in particular, $\phi(u) = v$. Since $\phi(U')$, the configuration obtained from U' after the defense, contains U then it must exist a vertex $v' \in U'$ such that $\phi(v') = u$. Thus, v' is a neighbor of u that belongs to U', which completes the proof of the claim. This means that $evc(G_k) \geq 2k + 2$.

Thanks to the previous claim and to the fact that the k sets $N[s_i]$ are pairwise disjoint, it holds that $|U'| \geq |U| + k$, that is $evc(G_k) - vc(G_k) \geq k$. Moreover, $\frac{evc(G_k)}{vc(G_k)} \geq \frac{2k+2}{k+2} = 2 - \frac{2}{k-2}$. □

It remains an open problem to determine whether there exists a biconnected series-parallel graph G such that $evc(G) = vc(G)$.

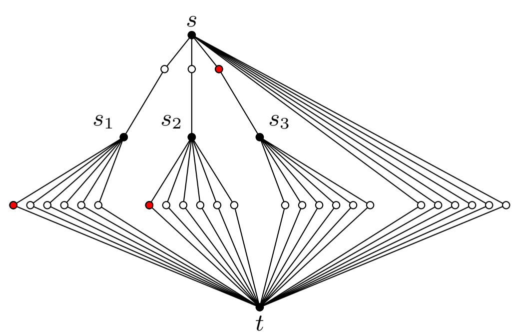

Fig. 4. The figure shows the series-parallel graph G_3 described in the proof of Theorem 2. The black vertices represent its unique minimum vertex cover U. The red vertices are an example of the position of guards to be added to U in order to get an eternal vertex cover configuration U.

We propose a graph parameter that is well-defined on series-parallel graphs, which allows us to characterize melon graphs showing that they have a much simpler structure than general series-parallel graphs. For a series-parallel graph G, we define the parameter $alt(G)$ as the maximum number of alternations between parallel and series nodes or *vice versa* in any path connecting the root and a leaf in any SP-decomposition of G. This parameter is clearly unbounded for the class of series-parallel graphs. The following result shows that melon graphs can be characterized as series-parallel graphs with alt at most 1.

Lemma 6. *For every melon graph G, $alt(G) \leq 1$. Conversely, for every series-parallel graph G with $alt(G) \leq 1$, either G is a k-melon graph or is a path with possibly multiple edges.*

Sketch of Proof. Let G be a k-melon graph, for some $k \geq 1$. The claim follows from the fact that every path P connecting the root and a leaf in any SP-decomposition of G starts with a non-empty sequence of parallel nodes and continues with a sequence of series nodes, and so P contains at most one alternation: $alt(G) \leq 1$.

Let G be a series-parallel graph with $alt(G) \leq 1$ and fix any SP-decomposition T of G. If the root of T is a parallel node, then G is constituted by a set of parallel paths between two vertices, that is, G is a melon graph. If the root of T is a series node, then G is a series of melon graphs in which the length of every path is one, *i.e.*, a set of multiple edges. □

Algorithmic techniques exploiting results on sub-structures, like divide and conquer or dynamic programming, look to be very natural on series-parallel graphs due to their recursive nature. Nevertheless, they are not immediately applicable for EVC: while $alt \leq 1$ for melon graphs guarantees a very limited number of cases, for the general case, it is impractical to relate $evc(G)$ to $evc(G_1)$ and $evc(G_2)$.

The reason is that the defense strategies for the EVC problem are, in general, not local, that is, the defense against an attack may require that every guard of a given configuration to shift to a neighbor. The idea is that combining the local information about G_1 and G_2 graphs and elaborating such information to a global solution for G is far from trivial.

5 Conclusions

This paper focuses on EVC restricted to series-parallel graphs. The problem is known to be NP-hard in general. This paper fits in the research direction of understanding the structural and complexity properties of this problem when restricted to graph classes.

We have shown that EVC can be solved in linear time for melon graphs: series-parallel graphs that are parallel composition of paths. This result is based on a case analysis of the structure of the input melon graph and generalizes the solution for cycles. Moreover, we have conjectured that this problem stays NP-hard on the whole class of series-parallel graphs. We have argued in favor of this conjecture exploiting the structural differences between melon and series-parallel graph based on the (eternal) vertex cover number and the SP-decomposition tree.

To further expand this work, we plan to consider the EVC on outerplanar graphs, *i.e.*, planar graphs that have a plane drawing with all vertices on the outer face. This class is interesting because, on the one hand, it is a subclass of series-parallel graphs and contains the maximal outerplanar graphs for which this problem is linear-time solvable [9]; on the other hand, the parameter *alt* is unbounded for outerplanar graphs.

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A Appendix

This section contains complete proof of lemmas and theorems which have been partially cut from the main body of the paper due to space restrictions.

A.1 Proof of Lemma 1

In order to prove Lemma 1, we exploit a result from [29], for which we need some additional definitions.

A *matching* M of G is a subset of vertex-disjoint edges of G. Moreover, if G is bipartite and $V = A \cup B$, a matching M is *perfect* if $|M| = \min\{|A|, |B|\}$; clearly, if $|A| = |B|$, every vertex is adjacent to some edge of a perfect matching.

Given an odd path P of length ℓ, we can recognize on it a maximum matching of cardinality $(\ell + 1)/2$ and a maximal matching of cardinality $(\ell - 1)/2$; the first one is perfect, and hence we call it *odd-perfect*, while the second leaves the two endpoints of the path out of the matching, and so we denote it as *odd-imperfect*. It is easy to see that every edge of P belongs to exactly one of these two matchings.

In support of our goal of building constructive proofs, we say that a bipartite graph G is *elementary* if it is connected and every edge belongs to some perfect matching of G [23]. The following result connects elementary graphs and their eternal vertex cover number.

Lemma 7. *[29] Let G be an elementary graph, then $evc(G) = vc(G) = |V(G)|/2$.*

We exploit the previous lemma to prove our results on odd melon graphs. Preliminarily, observe that every odd melon graph G is bipartite, so for the rest of this subsection, we assume that $G = (A \cup B, E)$. Since every path has an odd length, then one between s and t belongs to A while the other belongs to B; without loss of generality, we assume $s \in A$ and $t \in B$.

Lemma 8. *Every odd melon graph is elementary.*

Proof. Every melon graph is connected by definition, so it remains to prove that any edge e of G belongs to a perfect matching M_e that we construct as follows.

For each path of $\mathcal{P}(G)$, consider its odd-perfect and odd-imperfect matchings. Without loss of generality, let $e \in P^{(1)}$ (otherwise we can rename the paths in $\mathcal{P}(G)$). If e belongs to the odd-perfect matching of $P^{(1)}$ (see the red edge in Figure 1.a), then put in M_e all the edges of this odd-perfect matching (including e) and all the edges lying in the odd-imperfect matchings of all the other paths. If, vice versa, e belongs to the odd-imperfect matching of $P^{(1)}$ (see the red edge in Figure 1.b), then put in M_e all the edges of the odd-perfect matching of $P^{(2)}$ and all the edges lying on the odd-imperfect matchings of all the other paths (including e).

M_e contains e and is a perfect matching indeed, due to the alternating nature of M_e, for every vertex v of G, there exists exactly one edge of M_e that contains v. □

Note that each odd melon is bipartite, and it holds that $|A| = |B|$ because, for any path $P \in \mathcal{P}$, $|A \cap P| = |B \cap P|$. Moreover, A and B are two vertex covers of G. This observation is exploited to prove the following result.

Lemma 1. *Let $G=(A \cup B, E)$ be an odd k-melon graph. It holds that $evc(G) = vc(G)$, and the family $\mathcal{U}=\{A, B\}$ is a minimum eternal vertex cover class of G.*

Proof. Consider an edge e of G. Since G is elementary by Lemma 8, there exists a perfect matching M_e of G that contains e, and M_e can be found following the proof of Lemma 8.

Whenever attacked, the edge e can always be protected. Indeed, suppose first that the guards are positioned on the vertices of A; then, to protect e, every guard shifts through its incident edge in M_e, *i.e.*, for each $a \in A$, $\phi(a) = b$, where ab is the unique edge of M_e incident to a. The case in which the guards are positioned on the vertices of B is done symmetrically. See Figure 1.b. □

A.2 Proof of Lemma 2

Let G be an even melon graph. Although it is easy to see that G is bipartite, we can not exploit a strategy similar to the proof of Lemma 1 because for an even k-melon graph it holds that the two bipartitions have the same cardinality if and only if $k = 2$.

Lemma 9. *Let G be an even 2-melon graph with paths P and P', source s and sink t. Moreover, let U be a set of vertices such that P is internal and P' is external w.r.t. U, and let U' be a set of vertices such that P' is internal and P is external w.r.t. U'. Then it is possible to defend G from an attack on any single-guarded edge by shifting U to U' and vice versa.*

Proof. Let $e = zw$ be an edge of G. Intuitively, to protect e, we move the guards to turn P into an external path and P' into an internal path following the direction of the forced shift of the guard on e. Let $e = zw$ be an edge of G. Since U is a vertex cover and e is single-guarded, it is not restrictive to assume that $z \in U$ and $w \notin U$. Call $u_0, \ldots, u_{2m}$ the vertices of P and $v_0, \ldots, v_{2m'}$ the vertices of P', for some $m, m' \geq 1$, and let $u_0 = v_0 = t$ and $u_{2m} = v_{2m'} = s$. Then, to protect e, we move the guards to turn P into an external path and P' into an internal path following the forced shift of the guard from z to w.

In particular, assume that e is either an edge of P and $z = u_{2j}$ and $w = u_{2j+1}$ for some $j \in [m-1]$, or an edge of P' and $z = v_{2j+1}$ and $w = v_{2j}$ for some $0 \leq j \in [m'-1]$. Then, to protect e, we use the following defense function ϕ:

- $\phi(u_{2i}) = u_{2i+1}$, for every $i \in [m-1]$;
- $\phi(v_{2i+1}) = v_{2i}$, for every $i \in [m'-1]$;
- $\phi(s) = s$.

It is clear that $\phi(z) = w$ and $\phi(C) = C'$. Due to symmetry, a similar defense function defends the attack of e when it is either an edge of P and $z = u_{2j}$ and $w = u_{2j-1}$ for some $0 \neq j \in [m]$, or an edge of P' and $z = v_{2j-1}$ and $w = v_{2j}$ for some $0 \neq j \in [m'-1]$. □

Given a k-even melon graph G, for each fixed $i \in [k]$, we denote with U_i the vertex set such that the path $P^{(i)}$ is an external path w.r.t. U_i and the path $P^{(j)}$ is an internal path w.r.t. U_i, for every $j \in [k]$ and $j \neq i$. In the following result, we exploit Lemma 9 to defend any even k-melon with $k \geq 3$ with its guards on the vertices of U_i by considering the even 2-melon graph induced by $P^{(i)}$ and one of the internal paths w.r.t. U_i.

Lemma 2. *Let G be an even k-melon graph. It holds that $evc(G) = vc(G) + 1$, and the family $\mathcal{U} = \{U_i \mid i \in [k]\}$ is a minimum eternal vertex cover class of G, where the sets U_i are defined above.*

Proof. First, observe that, fixed any $i \in [k]$, the set U_i is a vertex cover of G with $vc(G)+1$ elements. Indeed, due to the alternating nature of the definition, every edge of G contains exactly one vertex of U_i with the exception of the two edges of the external path $P^{(i)}$ which are incident to s and t, whose both endpoints are vertices of U_i.

Consider now the set U of vertices of G such that every path $P \in \mathcal{P}$ is internal w.r.t. U. Clearly, since no external paths are in U, it holds that $|U_i| = |U| + 1$, for every $i \in [k]$. Moreover, U is a vertex cover and it is of minimum cardinality because every edge is incident to exactly one vertex in U. Finally, U is the unique minimum vertex cover of G, and so it cannot be a configuration of a minimum eternal vertex cover class. It follows that $evc(G)$ is at least $vc(G) + 1$. Then, proving that $\mathcal{U}$ is an eternal vertex cover class of G also shows that $\mathcal{U}$ is minimum.

Let U_i be any configuration of $\mathcal{U}$ and let e be the attacked edge of G. Let $P^{(j)}$ be the path which contains e. If $j = i$ (see Figure 1.c), let $P^{(k)}$ be any internal path of $\mathcal{P}$ w.r.t. U_i and let G' be the subgraph of G induced by the vertices of $P^{(i)}$ and $P^{(k)}$. If $j \neq i$ (see Figure 5.a), let G' be the subgraph of G induced by the vertices of $P^{(i)}$ and $P^{(j)}$. Observe that G' is an even 2-melon graph; calling ϕ' the defense function of Lemma 9 to defend G' from the attack on e, to protect e in G we define the defense function ϕ as follows: $\phi(v) = \phi'(v)$ if v is a vertex of G' and $\phi(v) = v$ otherwise. It is easy to see that ϕ protects e. $\square$

A.3 Proof of Lemma 3

Lemma 10. *Let G be an odd 2-melon graph with paths P and P', source s and sink t. Moreover, let U be a set of vertices such that P is an s-path and P' is a t-path w.r.t. U and let U' be a set of vertices such that P is a t-path and P' is an s-path w.r.t. U. Then it is possible to defend G from an attack on any single-guarded edge by shifting U to U' and vice versa.*

Proof. Let $e = zw$ be an edge of G. Intuitively, to protect e, we move the guards to turn P into a t-path and P' into a s-path following the direction of the forced shift of the guard on e. Let $e = zw$ be an edge of G. Since C is a vertex cover and e is single-guarded, it is not restrictive to assume that $z \in C$ and $w \notin C$. Call $u_0, \ldots, u_{2m+1}$ the vertices of P and $v_0, \ldots, v_{2m'+1}$ the vertices of P', for

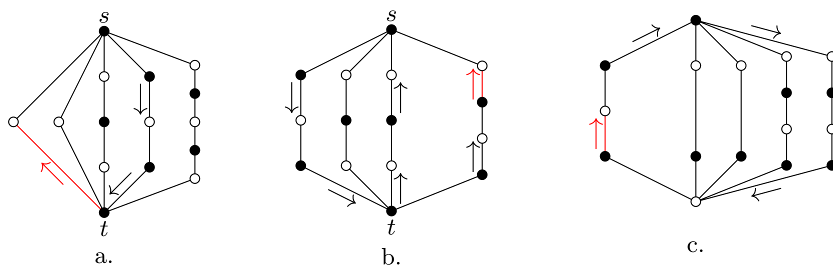

Fig. 5. a. strategy for an even melon graph (Lemma 2); b. strategy for a mixed melon graph with at least two even paths and only one odd path (Lemma 4); c. strategy for a mixed melon graph with at least two odd paths and only one even path to defend U_s (Lemma 5).

some $m, m' \geq 0$, and let $u_0 = v_0 = t$ and $u_{2m} = v_{2m'} = s$. Then, to defend from the attack on e, we move the guards to turn P into a t-path and P' into an s-path following the forced shift of the guard from z to w.

In particular, first, assume that e is either an edge of P and $z = u_{2j+1}$ and $w = u_{2j+2}$ for some $j \in [m-1]$, or an edge of P' and $z = v_{2j+2}$ and $w = v_{2j+1}$ for some $0 \neq j \in [m']$. Then, to defend from the attack on e, we use the following defense function ϕ:

- $\phi(u_{2i+1}) = u_{2i+2}$, for every $i \in [m-1]$;
- $\phi(v_{2i+2}) = v_{2i+1}$, for every $0 \neq i \in [m'-1]$, ;
- $\phi(s) = s$ and $\phi(t) = t$.

It is clear that $\phi(z) = w$ and $\phi(C) = C'$.

Now, assume that e is either an edge of P and $z = u_{2j+1}$ and $w = u_{2j}$ for some $0 \neq j \in [m]$, or an edge of P' and $z = v_{2j}$ and $w = v_{2j+1}$ for some $j \in [m']$. Then, to defend from the attack on e, we use the following defense function ϕ:

- $\phi(u_{2i+1}) = u_{2i}$, for every $i \in [m]$;
- $\phi(v_{2i}) = v_{2i+1}$, for every $i \in [m']$.

Similarly, it is clear that $\phi(z) = w$ and $\phi(C) = C'$. □

Let $P_e \in \mathcal{P}_{even}$ and let $\mathcal{S}_o$ be any subset of $\mathcal{P}_{odd}$. We denote with $U_{P_e,\mathcal{S}_o}$ the vertex set such that:

- P_e is an external path w.r.t. $U_{P_e,\mathcal{S}_o}$;
- every path in $\mathcal{P}_{even} \setminus \{P_e\}$ is an internal path w.r.t. $U_{P_e,\mathcal{S}_o}$;
- every path in $\mathcal{S}_o$ is an s-path w.r.t. $U_{P_e,\mathcal{S}_o}$;
- every path in $\mathcal{P}_{odd} \setminus \mathcal{S}_o$ is a t-path w.r.t. $U_{P_e,\mathcal{S}_o}$.

Lemma 3. *Let G be a mixed k-melon graph; if $|\mathcal{P}_{even}| \geq 2$ and $|\mathcal{P}_{odd}| \geq 2$, then it holds that $evc(G) = vc(G) + 1$ and the family $\mathcal{U} = \{U_{P_e,\mathcal{S}_o} \mid P_e \in \mathcal{P}_{even},\ \emptyset \neq \mathcal{S}_o \subset \mathcal{P}_{odd}\}$ is a minimum eternal vertex cover class of G, where the sets $U_{P_e,\mathcal{S}_o}$ are defined above.*

Proof. For every path set $\mathcal{S}_o$ such that $\emptyset \neq \mathcal{S}_o \subset \mathcal{P}_{odd}$, consider the set $U_{\mathcal{S}_o}$ of vertices of G such that all the even paths are internal, the odd paths in $\mathcal{S}_o$ are s-paths and the remaining odd paths are t-paths. In other words, $U_{\mathcal{S}_o}$ differs from any $U_{P_e,\mathcal{S}_o}$ only in P_e that is not external anymore, so $|U_{P_e,\mathcal{S}_o}| = |U_{\mathcal{S}_o}| + 1$, for every $P_e \in \mathcal{P}_{even}$. Moreover, let the family of the sets $U_{\mathcal{S}_o}$ be the collection of all minimum vertex covers of G; this is not an eternal vertex cover class of G because it is not possible to defend from an attack on any edge that belongs to a path in $\mathcal{P}_{even}$. It follows that $evc(G)$ is at least $vc(G) + 1$ and hence proving that $\mathcal{U}$ is an eternal vertex cover class of G also shows that $\mathcal{U}$ is minimum.

Let $U_{P_e,\mathcal{S}_o}$ be a configuration of $\mathcal{U}$ and let e be an attacked single-guarded edge of G. If e is an edge of a path $P_e \in \mathcal{P}_{even}$, let G' be the subgraph of G induced by the vertices of the paths in $\mathcal{P}_{even}$. The definition of $\mathcal{U}$ implies that G' contains at least an internal and at least an external path w.r.t. $U_{P_e,\mathcal{S}_o} \cap V(G')$, and we call ϕ' the defense function obtained from Lemma 9 when applied to the even melon graph G' to protect it from the attack on e. Then, to protect G from the attack on edge e we define the defense function ϕ as follows: $\phi(v) = \phi'(v)$ if v is a vertex of G' and $\phi(v) = v$ otherwise. It is easy to see that ϕ protects e. See Figure 2.a.

Suppose now that e is an edge of a path $P_o \in \mathcal{P}_{odd}$. Let P'_o be another path of $\mathcal{P}_{odd}$ such that $P_o \in \mathcal{S}_o$ if and only if $P'_o \in \mathcal{P}_{odd} \setminus \mathcal{S}_o$ and consider the odd 2-melon graph G' induced by the vertices of the paths of P_o and P'_o. We call ϕ' the defense function obtained from Lemma 10 when applied to the odd melon graph G' to protect it from the attack on e. To protect G from the attack on edge e we define the defense function ϕ as follows: $\phi(v) = \phi'(v)$ if v is a vertex of G' and $\phi(v) = v$ otherwise. It is easy to see that ϕ protects e. See Figure 2.b. $\square$

A.4 Proof of Lemma 4

Consider now the case where $\mathcal{P}_{odd}$ contains a single path P_o. Let $x \in \{s, t\}$ and $P_e \in \mathcal{P}_{even}$. We denote with U_{x,P_e} the vertex set such that:

- P_e is an external path w.r.t. U_{x,P_e};
- every path in $\mathcal{P}_{even} \setminus \{P_e\}$ is an internal path w.r.t. U_{x,P_e};
- P_o is an x-path w.r.t. U_{x,P_e}.

Lemma 4. *Let G be a mixed k-melon graph; if $|\mathcal{P}_{odd}| = 1$, then it holds that $evc(G) = vc(G) + 1$ and the family $\mathcal{U} = \{U_{x,P_e} \mid x \in \{s, t\}, P_e \in \mathcal{P}_{even}\}$ is a minimum eternal vertex cover class of G, where the sets U_{x,P_e} are defined above.*

Proof. The graph G has two minimum vertex covers U_x, $x \in \{s, t\}$: U_x is the set of vertices of G such that all even paths are internal paths and P_o is a x-path. In other words, U_x differs from any U_{x,P_e} only in P_e that is no longer external, so $|U_{x,P_e}| = |U_x| + 1$, for every $P_e \in \mathcal{P}_{even}$.

Let U_{x,P_e} be a configuration of $\mathcal{U}$. Due to the symmetry of G, it is not restrictive to assume that $x = s$. Let e be an attacked edge.

If e is an edge of a path in $\mathcal{P}_{even}$, let G' be the subgraph of G induced by the vertices of the paths in $\mathcal{P}_{even}$. The definition of $\mathcal{U}$ implies that G' contains at least an internal and at least an external path, and we call ϕ' the defense function obtained from Lemma 2 when applied to the even melon graph G' when protecting from the attack on e. To protect G, we define the defense function ϕ as follows: $\phi(v) = \phi'(v)$ if v is a vertex of G' and $\phi(v) = v$ otherwise. It is easy to see that ϕ protects e.

Suppose instead that $e = zw$ is an edge of the unique path $P_o \in \mathcal{P}_{odd}$ and, without loss of generality, let $z \in U_{s,P_e}$ and $w \notin U_{s,P_e}$. Call $v_0, \ldots, v_{2m+1}$ the vertices of P_o, for some $m \geq 0$, $v_0 = t$ and $v_{2m+1} = s$; since P_o is an s-path, then $z = v_{2j+1}$ for some $j \leq m$. We distinguish two cases according to whether $w = v_{2j+2}$ or $w = v_{2j}$, that is, whether the guard on z must be moved in the direction of s or of t in order to protect e. Let P'_e be any even path of G different from P_e. Say that the path P_e has $\{u_0, \ldots, u_{2m_e}\}$ as vertices, with $v_0 = t$ and $v_{2m_e} = s$, for some $m_e \geq 0$, and that the path P'_e has $\{x_0, \ldots, x_{2m'_e}\}$ as vertices, with $x_0 = t$ and $x_{2m'_e} = s$, for some $m'_e \geq 0$.

If $w = v_{2j+2}$ (and hence $j < m$), we have that U_{t,P'_e} protects U_{e,P_e} from the attack on e, all the guards on P_o and P'_e shift in the direction of s (P_o becoming a t-pathand P'_e becoming an external path), and all the guards on P_e but the one on s shift in the direction of t (P_e becoming a s-path. In particular, the defense function ϕ is defined as follows:

- $\phi(v_{2i+1}) = v_{2i+2}$, for every $i \in [m]$;
- $\phi(u_{2i+1}) = u_{2i}$, for every $i \in [m_e]$;
- $\phi(x_{2i}) = x_{2i+1}$, for every $i \in [m'_e - 1]$;
- $\phi(u) = u$, for every vertex u, that is not part of neither P_o, P_e nor P'_e.

It is clear that $\phi(z) = w$ and $\phi(U_{s,P_e}) = U_{t,P'_e}$. See Figure 5.b.

If, instead, $w = v_{2j}$, for some $j \leq m$, we have that U_{t,P'_e} protects U_{s,P_e} from the attack on e, all the guards on P_o shift in the direction of t (and P_o becoming a t-path), and all the guards on P'_e and on P_e shift in the direction of s (P'_e becoming an external path while P_e becoming an internal path). In particular, the defense function ϕ is defined as follows:

- $\phi(v_{2i+1}) = v_{2i}$, for every $i \in [m]$;
- $\phi(u_{2i+1}) = u_{2i+2}$, for every $i \in [m_e - 1]$;
- $\phi(x_{2i}) = x_{2i+1}$, for every $i \in [m'_e - 1]$;
- $\phi(u) = u$, for every vertex u, that is not part of neither P_o, P_e nor P'_e.

It is clear that $\phi(z) = w$ and $\phi(U_{s,P_e}) = U_{t,P'_e}$. This completes the proof. See Figure 2.c. □

A.5 Proof of Lemma 5

Consider the case where $\mathcal{P}_{even}$ contains a single path P_e, and $\mathcal{P}_{odd}$ contains at least two paths. The set U_s (resp. U_t) is a vertex set not containing t (resp. s) such that P_e is an external path and every path in $\mathcal{P}_{odd}$ is a s-path (resp. t-path) w.r.t. U_s (resp. U_t). Moreover, for any subset $\mathcal{S}_o$ of $\mathcal{P}_{odd}$, let $U_{\mathcal{S}_o}$ be the vertex set of G such that:

– P_e is an internal path,
– every path in $\mathcal{S}_o$ is a s-path
– every path in $\mathcal{P}_{odd} \setminus \mathcal{S}_o$ is a t-path w.r.t. $U_{\mathcal{S}_o}$.

Observe that U_s, U_t and every $U_{\mathcal{S}_o}$ are vertex covers of G and have all the same cardinality. Indeed, the extra guard present in the external path is compensated by the presence of exactly one guard in $\{s, t\}$. Vice-versa, the second guard on the set $\{s, t\}$ is compensated by one less guard in the internal path. Figure 3 showcases the configurations U_s and $U_{\mathcal{S}_o}$.

Lemma 5. *Let G be a mixed k-melon graph; if $|\mathcal{P}_{even}| = 1$, it holds that $evc(G) = vc(G)$ and the family $\mathcal{U} = \{U_s, U_t\} \cup \{U_{\mathcal{S}_o} \mid \emptyset \neq \mathcal{S}_o \subset \mathcal{P}_{odd}\}$ is a minimum eternal vertex cover class of G, where the sets U_s, U_t and $U_{\mathcal{S}_o}$ are defined above.*

Proof. Every configuration of $\mathcal{U}$ is a minimum vertex cover of G; therefore, to prove the claim, it is enough to show that $\mathcal{U}$ is an eternal vertex cover class of G. Let U be a configuration of $\mathcal{U}$ and e be a single-guarded edge of G. We consider two different cases, distinguishing whether U is of the form either U_x, for some $x \in \{s, t\}$, or $U_{\mathcal{S}_o}$, for some non-empty proper subset $\mathcal{S}_o$ of $\mathcal{P}_{odd}$.

Case 1: U is of the form U_x, for some $x \in \{s, t\}$. Thanks to the symmetry of G, it is not restrictive to assume $U = U_s$. For every non-empty proper subset $\mathcal{S}_o$ of $\mathcal{P}_{odd}$, it holds that $U_{\mathcal{S}_o}$ protects U_s.

To prove this claim, it is not restrictive to assume that the attacked edge $e = zw$ is such that $z \in U_s$ and $w \notin U_s$. We analyze different cases according to the position of e in G. Let path P_e be a sequence of vertices $u_0, \ldots, u_{2m}$, for some $m \geq 1$ such that $u_0 = t$, $u_{2m} = s$. Moreover, let P_o be any path in $\mathcal{P}_{odd}$ and recall that P_o is a s-path. Let P_o be a sequence of vertices $v_0, \ldots, v_{2m_o+1}$, for some $m_o \geq 0$ such that $v_0 = t$, $v_{2m_o+1} = s$.

Assume first that e is an edge of the unique path $P_e \in \mathcal{P}_{even}$, then $z = u_{2j+1}$ for some $j \in [m-1]$. Informally, a defending strategy consists of the guards shifting along the cycle formed by P_e and any other odd path around it, say P_o. Formally, the edge can be attacked in order to move the guard on z either in the direction of s or of t. Suppose first that $w = u_{2j+2}$. To defend G from this attack, we define a defense function ϕ as follows:

– $\phi(u_{2i+1}) = u_{2i+2}$, for every $i \in [m-1]$;
– $\phi(v_{2i+1}) = v_{2i}$, for every $i \in [m_o]$;
– $\phi(v) = v$, for every x, that is not part of neither P_e nor P_o.

It is clear that $\phi(z) = w$ and $\phi(U_s) = U_{\{P_o\}}$. See Figure 5.c.

Suppose now that $w = u_{2j}$, for some $j \in [m-1]$. To defend G from this attack, we define a defense function ϕ as follows:

– $\phi(u_{2i+1}) = u_{2i}$, for every $i \in [m-1]$, ;
– $\phi(v_{2i+1}) = v_{2i+2}$, for every $i \in [m_o - 1]$, ;
– $\phi(v) = v$, for every v, that is not part of neither P_e nor P_o.

It is easy to see that $\phi(z) = w$ and $\phi(U_s) = U_{\{P_o\}}$. See Figure 3.a.

Assume now that e is an edge of some path $P_o \in \mathcal{P}_{odd}$. It is easy to see that by exploiting one of the two defense functions defined above, we obtain to shift U_s to $U_{\{P_o\}}$ and successfully defend from the attack on e.

Case 2: U is of the form $U_{\mathcal{S}_o}$, for some non-empty proper subset $\mathcal{S}_o$ of $\mathcal{P}_{odd}$. Then, either U_x, with $x \in \{s, t\}$, or $U_{\mathcal{S}'_o}$, for some non-empty proper subset $\mathcal{S}'_o$ of $\mathcal{P}_{odd}$, defends $U_{\mathcal{S}_o}$ from the attack on e. To prove this claim, assume again that $e = zw$ with $z \in U_{\mathcal{S}_o}$ and $w \notin U_{\mathcal{S}_o}$. We analyze different cases according to the position of e in G.

First, suppose that e is an edge of the unique path $P_e \in \mathcal{P}_{even}$. Thanks to the symmetry of G, we can assume $z = u_{2j}$ and $w = u_{2j+1}$, for some $j \in [m-1]$. To defend G from this attack, we define a defense function ϕ as follows:

- $\phi(u_{2i}) = u_{2i+1}$, for every $i \in [m-1]$;
- $\phi(x) = x_t$, for every $x \neq t$ of $U_{\mathcal{S}_o}$ that is part of a t-path, where x_t is the neighbor of x in the path from x to s that does not contain t;
- $\phi(x) = x$, for every x of $U_{\mathcal{S}_o}$ that is part of an s-path.

It is easy to see that $\phi(z) = w$ and $\phi(U_{\mathcal{S}_o}) = U_s$. See Figure 3.b.

Now, assume that e is an edge of some path $P_o \in \mathcal{P}_{odd}$. Let P'_o be another path of $\mathcal{P}_{odd}$ such that P'_o is a t-path if P_o is an s-path and an s-path otherwise. Let G' be the subgraph of G induced by the vertices of the paths P_o and P'_o. To defend from the attack on e we define a defense function as follows: $\phi(v) = \phi'(v)$ if v is a vertex of G' and $\phi(v) = v$ otherwise, where ϕ' is the defense function obtained from Lemma 10 when applied to the odd 2-melon graph G' when defending from the attack on e. This completes the case analysis and the proof. □

A.6 Proof of Lemma 6

Lemma 6. *For every melon graph G, $alt(G) \leq 1$. Conversely, for every series-parallel graph G with $alt(G) \leq 1$, either G is a k-melon graph or is a path with possibly multiple edges.*

Proof. First, let G be a k-melon graph, for some $k \geq 1$, and let us prove by induction on k that $alt(G) \leq 1$. If $k = 1$, then G is either a single edge or a path, which is obtained recursively by the series composition of two 1-melon graphs. Thus, all non-leaf vertices of any SP-decomposition of G are series vertices and then $alt(G) = 0$.

Suppose now $k \geq 2$. Then G can only be obtained recursively by the parallel composition of two x- and y-melon graphs with $x, y \geq 1$ and $x + y = k$. Thus, every path P connecting the root and a leaf in any SP-decomposition of G starts with a non-empty sequence of parallel nodes and continues with a sequence of series nodes and so P contains at most one alternation: $alt(G) \leq 1$.

Now, let G be a series-parallel graph with $alt(G) \leq 1$ and fix any SP-decomposition T of G. If G is a single edge, then the statement trivially holds, so from now on we assume that G has at least two edges. It is well known that the type of the root is the same in every SP-decomposition of G: indeed, the root is a series node if G contains a cut-vertex and is a parallel node otherwise. If the root of T is a parallel node, then G is constituted by a set of parallel paths between two vertices, that is, G is a melon graph. If the root of T is a series node, then G is a series of melon graphs in which the length of every path is one, *i.e.*, a set of multiple edges. □