Frames for Source Recovery in Dynamical Systems

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***Abstract*—This paper addresses the problem of recovering constant source terms in discrete dynamical systems described by $x_{n+1} = Ax_n + w$, where x_n represents the state in a Hilbert space $\mathcal{H}$, A is a bounded linear operator, and w is a source term within a closed subspace $W \subseteq \mathcal{H}$. Using time-space sampling measurements, we establish necessary and sufficient conditions for stable recovery of w, independent of the unknown initial state x_0. This work has practical applications in areas such as environmental monitoring, where precise source identification is critical.**

I. Introduction

We consider the following discrete-time dynamical system:

$$x_{n+1} = Ax_n + w, \quad n \in \mathbb{N}, \qquad w \in W, \tag{1}$$

where $x_n \in \mathcal{H}$ is the *n-th state of the system*, and $\mathcal{H}$ is a separable Hilbert space, $A \in \mathcal{B}(\mathcal{H})$ is bounded operator o $\mathcal{H}$, $w \in W \subseteq \mathcal{H}$ is the *source* or *forcing term*, and W is a closed subspace of $\mathcal{H}$.
The goal is to find the unknown source $w \in W$ from the space-time sample measurements

$$\mathcal{D}(x_0, w) = [\langle x_n, g_j \rangle]_{n,j} \tag{2}$$

where $\langle x_n, g_j \rangle$ are obtained by inner products with vectors of a Bessel system $\mathcal{G} = \{g_j\}_{j \geq 1} \subset \mathcal{H}$. The index $n \in \mathbb{N}$ represents discrete time, while $j \in J$ stands for spatial location coded by the index j.
The model (1) may describe by environmental monitoring applications for identifying the locations and magnitude of pollution sources (see Fig. 1)[1] . For this application, the goal is to strategically place sensors across different locations to collect relevant data that allows to monitor the pollutant from the smokestacks.

This work is a follow up on the work [1] which was inspired by several other articles [2]–[6]. Other work related to source recovery and other problems in dynamical sampling can be found in [7]–[24] and the reference therein.

II. The mathematical description of problem

As part of the main problem, given a dynamical system (1), we wish to recover the source term w in a stable way from the data provided in measurements $\mathcal{D}(x_0, w)$. To describe the notion of *stable reconstruction* we need to specify some

Fig. 1: Smokestacks in an industrial zone of the city emit pollutants. Four measuring devices $\{g_1, \dots, g_4\}$ are strategically placed throughout the city to collect space-time data for recovering the pollutant intensities $w \in W$ emitted by the smokestacks.

ambient spaces $\mathcal{B}$ in which the data sits together with an appropriate norm $\|\cdot\|_{\mathcal{B}}$ in each case. This setting will allow us to describe the reconstruction operator $\mathcal{R}$ as a continuous linear mapping from the data space $\mathcal{B}$ to the Hilbert space $\mathcal{H}$ containing the source term w.

A. The measurement space

There are two cases of dynamical systems that we will wish to consider. Briefly speaking, they are as follows.

(i) In the first case, the data matrix $\mathcal{D}(x_0, w) = [\langle x_n, g_j \rangle]_{n \in [N],\, j \geq 1}$ is obtained from finitely many iterations, where $[N] = \{0, 1, 2, \dots, N-1\}$, $N \geq 1$.
(ii) In the second setting, the data matrix $\mathcal{D}(x_0, w) = [\langle x_n, g_j \rangle]_{n \geq 0,\, j \geq 1}$ stems from infinitely many time iterations.

In the first case, all data measurements sit in the space $\mathcal{B}(\ell^2, \mathbb{C}^N)$, which can be described as the family of all infinite matrices $D = [d_{ij}]$ with (finitely many) N rows

[1]image generated using DALL.E

$r_1, \ldots, r_N$, where each row $r_i = (d_{i1}, d_{i2}, \ldots) \in \ell^2$. This space $\mathcal{B}(\ell^2, \mathbb{C}^N)$ is endowed with the norm

$$\|D\|_{\ell^2 \to \mathbb{C}^N} = \sum_{i=1}^{N} \left(\sum_{j=1}^{\infty} |d_{ij}|^2 \right)^{1/2}, \quad \text{for } D \in \mathcal{B}(\ell^2, \mathbb{C}^N). \tag{3}$$

For the second case of infinitely many time iterations, we will use the space $\mathcal{B}^s(\ell^2, \ell^\infty)$ which is a closed subspace of $\mathcal{B}(\ell^2, \ell^\infty)$. The latter is the family of all infinite matrices for which the norm

$$\|D\|_{\ell^2 \to \ell^\infty} = \sup_{i \geq 1} \left(\sum_{j=1}^{\infty} |d_{ij}|^2 \right)^{1/2}, \quad \text{for } D \in \mathcal{B}(\ell^2, \ell^\infty), \tag{4}$$

is finite. The former space $\mathcal{B}^s(\ell^2, \ell^\infty)$ is the closed subspace consisting of matrices whose rows form a Cauchy sequence in ℓ^2. More explicitly, we provide the following definition.

Definition 1. *The space $\mathcal{B}^s(\ell^2, \ell^\infty)$ is the set of matrices $\{D = [d_{i,j}] : i \geq 1, j \geq 1\}$ such that each row r_i of D belongs to ℓ^2, and there exists a $t \in \ell^2$ such that $\lim_{i \to \infty} \|r_i - t\|_{\ell^2} = 0$. The norm $\|D\|_{\ell^2 \to \ell^\infty}$ is defined as $\sup_{i \geq 1} \|r_i\|_{\ell^2}$.*

Note that due to the equivalence of norms in $\mathbb{C}^N$,we may replace $\sum_{i=1}^{N}$ by $\sup_{1 \leq i \leq N}$ in (3), and so $\mathcal{B}^s(\ell^2, \mathbb{C}^N) = \mathcal{B}(\ell^2, \mathbb{C}^N)$. A detailed description of these spaces, in particular, an equivalent description of $\mathcal{B}^s(\ell^2, \ell^\infty)$, is available in [25]. Throughout the general description of the spaces $\mathcal{B}(\ell^2, \mathbb{C}^N)$ and $\mathcal{B}(\ell^2, \ell^\infty)$, we use the index i, commencing from the initial value 1, for the rows of matrices involved in the discussion. However, when analyzing dynamical systems, we adopt a different indexing scheme, mostly denoted by n and starting at 0.

B. Stable recovery

Consider a dynamical system of the form (1) with measurements $\mathcal{D}(x_0, w)$ given by sampling through a Bessel sequence $\mathcal{G} = \{g_j\}_{j \geq 1}$ in $\mathcal{H}$ as in (2).

(i) If there are finitely many time iterations, we say that the source term $w \in W \subseteq \mathcal{H}$ can be recovered from the data $\mathcal{D}(x_0, w)$ in a stable way if there exists a bounded linear operator $\mathcal{R} : \mathcal{B}(\ell^2, \mathbb{C}^N) \to \mathcal{H}$ such that

$$\mathcal{R}\big(\mathcal{D}(x_0, w)\big) = w$$

for all $x_0 \in \mathcal{H}$ and all $w \in W$.

(ii) If we have infinitely many time iterations, we say that the source term $w \in W \subseteq \mathcal{H}$ can be recovered from the data $\mathcal{D}(x_0, w)$ in a stable way if there exists a bounded linear operator $\mathcal{R} : \mathcal{B}^s(\ell^2, \ell^\infty) \to \mathcal{H}$ such that

$$\mathcal{R}\big(\mathcal{D}(x_0, w)\big) = w$$

for all $x_0 \in \mathcal{H}$ and all $w \in W$.

The differences between the measurement spaces $\mathcal{B}(\ell^2, \mathbb{C}^N)$ and $\mathcal{B}^s(\ell^2, \ell^\infty)$, and consequently the emerging reconstruction operators $\mathcal{R}$, are profound and is discussed in depth in [25].

III. Main Results

A. Reconstruction Conditions

The main results give necessary and sufficient conditions on the set $\mathcal{G} = \{g_j\}_{j \geq 1}$ for a stable reconstruction of $w \in W$ in two cases. The first case is when $W = \mathcal{H}$, and the second case is when $W \subsetneq \mathcal{H}$. The proof of the results can be found in [25].

The first result is academic and concerns the situation where $W = \mathcal{H}$. From the point of view of smokestacks, it states that there is a smokestack at each location in space. Thus, it is neither the most interesting from the point of view of applications nor is the mathematical result unexpected. In some sense, we expect to have a sampling device (modeled by g_j) at every location in space j and that the set of vectors $\{g_j\}_{j \geq 1}$ must form a frame. However, this problem gave us insight into the more interesting problem in which $W \subsetneq \mathcal{H}$ described later.

Theorem 1. *Let $\mathcal{H}$ be a separable Hilbert space, and let $\mathcal{G} = \{g_j\}_{j \geq 1}$ be a Bessel sequence in $\mathcal{H}$. Consider the dynamical system (1), with an arbitrary initial state $x_0 \in \mathcal{H}$. Then the source term $w \in \mathcal{H}$ can be recovered from the measurements $\mathcal{D}(x_0, w) = [\langle x_n, g_j \rangle]_{n \in [N],\, j \geq 1}$ in a stable way for some $1 \leq N < \infty$ if and only if $\mathcal{G} = \{g_j\}_{j \geq 1}$ is a frame for $\mathcal{H}$.*

The theorem above states that we only need finitely many times in the space-time sampling scheme to recover the source w, albeit we need infinitely many spatial devices at all spatial locations $\mathcal{G} = \{g_j\}_{j \geq 1}$. The case where $W \subsetneq \mathcal{H}$ is much more interesting. For example, $\mathcal{H}$ may be infinite-dimensional while $\dim W = 4$, modeling four smokestacks located in various locations. Intuitively, we do not need to place measurement devices at all spatial locations. In fact, we seek to find a minimum number of locations that require a device. We think that well-placed measuring devices of the order of 4 should be sufficient. The mathematical statement of how many and where to place them is stated in the following theorem.

Theorem 2. *Let $\mathcal{H}$ be a separable Hilbert space, let W be a closed subspace of $\mathcal{H}$, and let $\mathcal{G} = \{g_j\}_{j \in J}$ be a Bessel sequence in $\mathcal{H}$. Consider the dynamical system (1) with an arbitrary initial state $x_0 \in \mathcal{H}$, and with $\|A\| < 1$. Then each source term $w \in W$ of the system can be recovered from the measurements $\mathcal{D}(x_0, w) = [\langle x_n, g_j \rangle]_{n \geq 0, j \in J}$ in a stable way if and only if $\{P_W (I - A^*)^{-1} g_j\}_{j \in J}$ is a frame for W.*

In this case, unlike the previous theorem, we need infinitely many time samples in general. However, the cardinality $\#|\mathcal{G}|$ of $\mathcal{G} = \{g_j\}_{j \in J}$ and $\dim W$ should be of the same order. Notice also that the vectors g_j need not be in W. In fact, it is not $\mathcal{G} = \{g_j\}_{j \in J}$ that must form a frame for W, but $\{P_W (I - A^*)^{-1} g_j\}_{j \in J}$ that forms a frame for W. In some sense, this result tells us how many devices we need and where to place them in order to recover $w \in W$. When $W = \mathcal{H}$, it is possible to use this theorem to recover Theorem 1 using frame theory.

IV. Acknowledgment

The research of Akram Aldroubi and Le Gong is supported in part by grant DMS-2208030. Javad Mashreghi was supported by grants from the Fulbright Foundation and the Canada Research Chair program.

This conference paper is a concise summary of a full-length manuscript that has been submitted for publication and is currently under review.

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