# Incremental maximum likelihood estimation for efficient adaptive filtering

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Abstract—Adaptive filtering is a well-known problem with a wide range of applications, including echo cancellation. Extensive research during the past few decades has led to the invention of various algorithms. However, the known computationallyefficient solutions show a tradeoff between convergence speed and accuracy. Moreover, running these algorithms involves heuristically setting various parameters that considerably affect their performances. In this paper, we propose a new algorithm which we refer to as online block maximum likelihood (OBML). OBML is a computationally-efficient online learning algorithm that employs maximum likelihood (ML) estimation every P samples. We fully characterize the expected performance of OBML and show that i) OBML is able to asymptotically recover the unknown coefficients and ii) its expected estimation error asymptotically converges to zero as  $O(\frac{1}{t})$ . We also derive an alternative version of OBML, which we refer to as incremental maximum likelihood (IML), which incrementally updates its ML estimate of the coefficients at every sample. Our simulation results verify the analytical conclusions for memoryless inputs, and also show excellent performance of both OBML and IML in an audio echo cancellation application with strongly correlated input signals.

Index Terms—Adaptive filtering, Echo cancellation, Stochastic optimization, Maximum likelihood estimation

## I. INTRODUCTION

Adaptive filtering is a fundamental problem with numerous applications in signal processing. In a generic system identification scenario, noisy observations of the response of an unknown linear system to a known input signal are used to estimate the parameters of the unknown system. During the past sixty years, there has been extensive research on developing efficient high-performance adaptive filters. The work has led to the development of various well-known algorithms, such as least mean squares (LMS), normalized least mean squares (NLMS), recursive least squares (RLS) and the affine projection algorithm (APA). Except for RLS, such methods are efficient in terms of computational complexity and memory requirement, especially if they are implemented in the frequency domain. (See for example [1]–[3].) However, the computationally-efficient solutions, such as NLMS and APA, either suffer from low convergence speed or achieve low accuracy. Various heuristic solutions have been developed to improve the performance of such methods by dynamically tuning their parameters to obtain favorable tradeoffs between convergence speed and long-term accuracy. In this paper, we propose a new, theoretically motivated algorithm with a single control parameter, whose behavior for memoryless Gaussian inputs can be explicitly characterized, and which has the same asymptotic convergence as RLS.

We also show experimental evidence that the algorithm has excellent performance in acoustic echo cancellation, an adaptive filtering application where the input signal can be modeled as having significant memory. In acoustic echo cancellation (AEC), the goal is to remove the echo generated by the transmitters (e.g. loudspeakers) from the signal captured by the receivers (e.g. microphones). To achieve this goal, EC algorithms use the (known) signal from the transmitter and the signal captured by the receiver to adaptively learn the channel coefficients describing the channel that exists between the transmitter and the receiver. (Ref to Fig. 1 for a block diagram of a two-way audio communication systems with EC.)

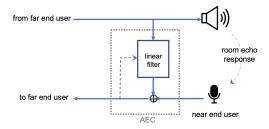


Fig. 1: Audio echo cancellation in the presence of multiple speakers and multiple microphones

#### A. Problem statement

Let  $x_t$  and  $y_t$  denote the transmitted signal and received signal at time t, respectively. Assume that  $x_t$  and  $y_t$  are related through a linear time-invariant filter modeled by  $\mathbf{w}^* \in \mathbb{R}^L$ , such that  $y_t$  is a noisy observation of a linear function of the input vector  $\mathbf{x}_t = [x_t, \dots, x_{t-L+1}] \in \mathbb{R}^L$  as

$$y_t = \mathbf{x}_t^T \mathbf{w}^* + z_t,$$

where  $(z_t)_t \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma_z^2)$  denotes the independent noise. The goal of an adaptive filter is to adaptively estimate  $\mathbf{w}^*$  from measurements  $(\mathbf{x}_{t'}, y_{t'})_{t'=1}^t$ . Assuming that the filter  $\mathbf{w}^*$  is static and does not change, ideally, at time t, the adaptive filter's goal is to minimize the following cost function:

$$\ell_t(\mathbf{w}) = \sum_{i=1}^t (\mathbf{x}_i^T \mathbf{w} - y_i)^2.$$

However, solving this optimization at every iteration is computationally infeasible. Moreover, in many application including AEC, it is not reasonable to assume that the filter coefficients

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stay constant. Therefore, in designing adaptive filters, the goal is to find an algorithm that adaptively estimates w\*, such that at every iteration it updates its current estimate by only using the latest P observations, where P is a small number (typically smaller than 10). At time t, the adaptive filter receives as input: i)  $\mathbf{x}_t$ : input vector at time t (e.g. the loudspeaker signal vector in AEC), ii)  $y_t$ : the output value at time t (e.g., the microphone signal in AEC), iii)  $(\mathbf{x}_{t'}, y_{t'})$ ,  $t' = t - 1, \dots, t - P + 1$  and iv)  $\mathbf{w}_t$ , and outputs an updated estimate of the filter coefficients  $\mathbf{w}_{t+1}$ . The performance of such a adaptive filter can be evaluated in terms of its i) convergence rate, i.e., how fast it can learn the filter coefficients, ii) accuracy, i.e., how well it can learn the coefficients in steady state (i.e., after enough time has elapsed) and iii) computational complexity and memory requirements. The ultimate goal in designing adaptive filters is to have a computationally-efficient fast-converging method that is also asymptotically accurate  $(\|\mathbf{w}_t - \mathbf{w}^*\| \to 0$ , as  $t\to\infty$ ).

#### B. Related work

Adaptive filtering is a well-studied problem with various well-known solutions, such as least mean squares (LMS), normalized LMS (NLMS), proportionate NLMS (PNLMS) [4], [5], affine projection algorithm (APA) [6], and recursive least squares (RLS) [7]. These algorithms show a trade-off between the performance criteria mentioned earlier. Except for RLS, these algorithm all have reasonable computational complexities that are linear in the number of unknown coefficients, L. However, they show a sharp trade-off between convergence speed and accuracy: To achieve an accurate estimate of the filter coefficients, one typically needs to set the parameters of these algorithms in a way that leads to very slow convergence. On the other extreme, the RLS algorithm is extremely efficient in learning filter coefficients, both in terms of accuracy and convergence speed. However, RLS is not a practical method in many applications as both its computational complexity and memory requirements are quadratic in L.

Another aspect of these algorithms is that they all have parameters, such as step size, regularization parameter, and memory order (*P*), that strongly affect their performance and stability. Adaptive tuning of these parameters has been the subject of various studies and multiple heuristic methods for parameter optimization have been proposed throughout the years. For instance, in [8], the authors propose variable step-size APA and NLMS. Optimizing the step size of NLMS is studied in [9]–[11]. Joint optimization of the step size and the regularization parameter of NLMS algorithm is studied in [12]. Various methods for dynamically setting the order and the step size of APA algorithm is studied in [13]–[17].

Adaptive filtering, as known in signal processing community, is closely related to stochastic optimization and online learning problems that are widely studied by statisticians and more recently by computer scientists. The famous LMS algorithm or block LMS are essentially variants of stochastic gradient descent (SGD) and batch SGD, respectively. The difference between LMS and SGD is that in SGD input vectors are typically selected at random from a dataset, while in

LMS the input vectors  $(\mathbf{x}_t)_t$  are closely related as  $\mathbf{x}_{t+1} = [x_{t+1}, x_t, \dots, x_{t-L+2}]$  and  $\mathbf{x}_t = [x_t, \dots, x_{t-L+1}]$  overlap in L-1 entries  $(x_t, \dots, x_{t-L+2})$ . This makes theoretical analysis of LMS and other adaptive filtering methods highly complex. For SGD with step size vanishing at the appropriate rate and independent observations, it is known that it asymptotically recovers the solution at the optimal rate [18].

# C. Our contributions

In this paper, we propose two adaptive algorithms which we refer to as online block maximum likelihood (OBML) and incremental maximum likelihood (IML). They are motivated by a Bayesian model inspired by an analysis of the convergence behavior of APA. Both algorithms are use updates that are similar in form to APA with fixed memory order  $P \geq 1$ , but are dynamically controlled by a single parameter playing the role of regularization and stepsize. The algorithms include an explicit formula for how the single control parameter should be set to optimize performance. The control parameter should ideally be set based on knowledge of the current squared error (aka misalignment) of the filter, which can only be estimated in practice. While OBML updates the filter coefficients once after every P samples, IML updates the filter incrementally after each received sample. When the true misalignment is provided as an input to the algorithms, we refer to them as genie-aided OBML (GA-OBML) and genie-aided IML (GA-IML) respectively.

With respect to OBML and IML, we have the following key results:

- 1) Asymptotic optimality of GA-OBML: Assuming that i) the transmitted signal is independent and identically distributed (i.i.d.) Gaussian and ii) the input vectors  $(\mathbf{x}_t)_t$  are independent, we *fully* characterize the expected estimation error of GA-OBML and show that asymptotically its expected estimation error, i.e.,  $\mathrm{E}[\|\mathbf{w}_t \mathbf{w}^*\|^2]$  converges to zero as  $O(\frac{1}{t})$ . Comparing the performance with the performance of offline least squares, we show that the achieved convergence rate is in fact optimal.
- Fast convergence of GA-OBML: under the same assumptions as above, we show that GA-OBML initially converges exponentially when the initial misalignment is high.
- 3) Effectiveness of GA-IML: In empirical AEC simulations, we show that IML has generally similar performance to OBML, and outperforms it for small values of P>1.
- 4) Genie not required: In empirical simulation results in an acoustic echo cancellation setting, we show that it is sufficient to use estimate misalignment in the control loop; that is, OBML and IML have similar performance to GA-OBML and GA-IML, respectively.

In summary, we show that IML and OBML are theoretically-justified ML-based methods, without any parameter tuning, that achieves state-of-the-art performance with fast initial convergence (similar to APA), asymptotic accuracy (similar to LMS with vanishing step size).

#### D. Notation

Throughput the paper bold letters indicate vectors. Matrices are denoted by upper-case letters, such as A and X. For an  $m \times n$  matrix A, let  $\sigma_{\min}(A)$  and  $\sigma_{\max}(A)$  denote the minimum and maximum singular values of A. Sets are denoted by calligraphic letters, such as  $\mathcal X$  and  $\mathcal Y$ .

### E. Organization of this paper

In Section II, we analyze the convergence behavior of regularized APA. Inspired by the analysis, in Section III, we propose the genie-aided OBML method, which an online MLE-based method for estimating the coefficients. GA-OBML is a block update algorithm that updates its estimate every *P* samples. In Section IV, we derive an alternative MLE-based method that incrementally updates its estimate at every time step. We will show that although IML is harder to analyze compared to OBML, it is closely related and empirically performs better in some scenarios of interest. Section V presents our simulation results that show the effectiveness of OBML and IML methods in AEC. The proof of our main theoretical result is presented in Section VI. Section VII concludes the paper.

#### II. ON CONVERGENCE BEHAVIOR OF APA

The affine projection algorithm (APA) is an algorithm proposed to address the slow convergence problem of LMS and NLMS algorithms. Unlike LMS and NLMS, APA employs P latest observation vectors, for some P > 1. Let  $X_t = [\mathbf{x}_t, \ldots, \mathbf{x}_{t-P+1}]$ , and  $\mathbf{y}_t = [y_t, \ldots, y_{t-P+1}]^T$ . Standard APA algorithm is designed as follows: Given  $(X_t, \mathbf{y}_t)$  and  $\mathbf{w}_t$ , the coefficients are updated such that the new vector is closest vector in  $\mathbb{R}^L$ , which satisfies the last P measurements exactly, i.e.,  $X_t^T \mathbf{w} = \mathbf{y}_t$ . That is, it sets  $\mathbf{w}_{t+1} = \arg\min_{\mathbf{w}: X_t^T \mathbf{w} = \mathbf{y}_t} \|\mathbf{w} - \mathbf{w}_t\|^2$ , or

$$\mathbf{w}_{t+1} = \mathbf{w}_t + X_t (X_t^T X_t)^{-1} (\mathbf{y}_t - X_t^T \mathbf{w}_t).$$

APA in its standard form shows instability issues that are caused by the inversion involved in  $(X_t^T X_t)^{-1}$ . To address this issue, regularized APA is defined as

$$\mathbf{w}_{t+1} = \mathbf{w}_t + \mu X_t (X_t^T X_t + \delta I_P)^{-1} (\mathbf{y}_t - X_t^T \mathbf{w}_t),$$

where  $\delta$  and  $\mu$  denote the regularization parameter and step size, respectively. Regularized APA can be motivated analytically as follows. First, note that, because the measurements are noisy, even the true coefficient vector  $\mathbf{w}^*$  does not satisfy  $X_t^T\mathbf{w} = \mathbf{y}_t$ . The noise variance determines the typical distance to be expected between  $X_t^T\mathbf{w}$  and  $\mathbf{y}_t$ . Taking noise into account, we could tweak the APA optimization as follows:

min 
$$\|\mathbf{w} - \mathbf{w}_t\|^2$$
  
s.t.  $\|X_t^T \mathbf{w} - \mathbf{y}_t\|^2 \le P\sigma_z^2$ . (1)

Solving the Lagrangian form of the above optimization  $\min_{\mathbf{w}}[\|\mathbf{w} - \mathbf{w}_t\|^2 + \lambda \|X_t^T \mathbf{w} - \mathbf{y}_t\|^2]$ , it follows that

$$\mathbf{w}_{t+1} = (I + \lambda X_t X_t^T)^{-1} (\mathbf{w}_t + \lambda X_t \mathbf{y}_t). \tag{2}$$

Employing Woodbury matrix identity, we can simplify this update rule:

$$\mathbf{w}_{t+1} = (I + \lambda X_t X_t^T)^{-1} (\mathbf{w}_t + \lambda X_t \mathbf{y}_t)$$

$$= (I - \lambda X_t (I_P + \lambda X_t^T X_t)^{-1} X_t^T) (\mathbf{w}_t + \lambda X_t \mathbf{y}_t)$$

$$= \mathbf{w}_t + X_t (\delta I_P + X_t^T X_t)^{-1} (\mathbf{y}_t - X_t^T \mathbf{w}_t), \tag{3}$$

where  $\delta = 1/\lambda$ .

To understand the convergence behavior of APA, we employ the update rule in (2). Recall that under our model,

$$\mathbf{y}_t = X_t^T \mathbf{w}^* + Z_t,$$

where  $Z_t = [z + t, ..., z_{t-P+1}]$ . Subtracting  $\mathbf{w}^*$  from both side of (2), it follows that

$$\mathbf{w}_{t+1} - \mathbf{w}^* = \left(I - X_t (\delta I_P + X_t^T X_t)^{-1} X_t^T\right) (\mathbf{w}_t - \mathbf{w}^*) + X_t (\delta I_P + X_t^T X_t)^{-1} Z_t.$$
(4)

Assume that  $\mathbf{w}_0 = \mathbf{0}_L$ , and define

$$P_t = I_L - X_t (\delta I_P + X_t^T X_t)^{-1} X_t^T.$$

it follows that

$$\mathbf{w}_{t+1} = \mathbf{w}^* - \prod_{t'=1}^t P_{t'} \mathbf{w}^* + \sum_{t'=1}^t \prod_{t''=t'+1}^t P_{t''} X_{t'} (\delta I_P + X_{t'}^T X_{t'})^{-1} Z_{t'}.$$
 (5)

This shows that at time t+1 conditioned on the input signal,  $\mathbf{w}_{t+1}$  has a Gaussian distribution with mean

$$\mathrm{E}[\mathbf{w}_{t+1}] = \mathbf{w}^* - \prod_{t'=1}^t P_{t'} \mathbf{w}^*.$$

Note that  $P_t$  has L-P eigenvalues that are equal to one. The corresponding eigenvectors represent the directions that are orthogonal to the space spanned by the columns of  $X_t$ . For directions that are not orthogonal to the space spanned by the columns of  $X_t$ , the corresponding eigenvalues  $\nu_i$  are strictly smaller than 1, and in fact can be expressed as  $\nu_i = \delta/\left(\delta+S_{ii}^2\right)$  where  $\{S_{ii}\}$  are the non-zero singular values of  $X_t$ . This implies that the estimation bias along the directions that are observed in the input data vanishes quickly, as long as the regularization parameter  $\delta$  is not larger than the typical singular values of  $X_t$ .

## III. ONLINE MLE ESTIMATION FOR AEC

Our analysis of the previous section suggests that  $\mathbf{w}_t$ , the APA's estimate of  $\mathbf{w}^*$  at iteration t, is a Gaussian random vector with a mean quickly converging to  $\mathbf{w}^*$ . In this section, first inspired by the analysis of the previous section, we derive a new AEC algorithm that employs MLE. For i.i.d. Gaussian sources, we fully characterize the expected temporal performance of the proposed method and show that it converges to the optimal coefficient at a rate that is asymptotically optimal.

Assume that at time t we have access to i)  $\mathbf{w}_t \sim \mathcal{N}(\mathbf{w}^*, M_t)$ , and ii)  $(X_t, \mathbf{y}_t)$ , where  $X_t \in \mathbb{R}^{L \times P}$  and  $\mathbf{y}_t \in \mathbb{R}^P$ , and  $\mathbf{y}_t = X_t^T \mathbf{w}^* + Z_t$ . Further, assume that  $\mathbf{w}_t$  is independent of  $(X_t, \mathbf{y}_t)$ . Given  $\mathbf{w}_t$  and  $(X_t, \mathbf{y}_t)$ , one can ask

the following question: what is the ML estimate of  $\mathbf{w}^*$ ? Given the distributions of  $\mathbf{w}_t$  and  $(X_t, \mathbf{y}_t)$ , the ML estimate  $\mathbf{w}^*$  is the solution of the following optimization:

$$\mathbf{w}_{t+1}^{(\mathrm{ML})} = \underset{\mathbf{w} \in \mathbb{R}^L}{\mathrm{arg} \min}[(\mathbf{w}_t - \mathbf{w})^T M_t^{-1}(\mathbf{w}_t - \mathbf{w}) + \frac{1}{\sigma_z^2} \|\mathbf{y}_t - X_t^T \mathbf{w}\|^2].$$

Therefore,

$$\mathbf{w}_{t+1}^{(\text{ML})} = (M_t^{-1} + \frac{1}{\sigma_z^2} X_t X_t^T)^{-1} (\frac{1}{\sigma_z^2} X_t \mathbf{y}_t + M_t^{-1} \mathbf{w}_t).$$
 (6)

Note that

$$\mathbf{w}_{t+1}^{(\mathrm{ML})} = (\sigma_{z}^{2} M_{t}^{-1} + X_{t} X_{t}^{T})^{-1} \Big( X_{t} (\mathbf{y}_{t} - X_{t}^{T} \mathbf{w}_{t} + X_{t}^{T} \mathbf{w}_{t})$$

$$+ \sigma_{z}^{2} M_{t}^{-1} \mathbf{w}_{t} \Big)$$

$$= (\sigma_{z}^{2} M_{t}^{-1} + X_{t} X_{t}^{T})^{-1} \Big( X_{t} (\mathbf{y}_{t} - X_{t}^{T} \mathbf{w}_{t})$$

$$+ (X_{t} X_{t}^{T} + \sigma_{z}^{2} M_{t}^{-1}) \mathbf{w}_{t} \Big)$$

$$= \mathbf{w}_{t} + (\sigma_{z}^{2} M_{t}^{-1} + X_{t} X_{t}^{T})^{-1} X_{t} (\mathbf{y}_{t} - X_{t}^{T} \mathbf{w}_{t}).$$
 (7)

To simplify the update rule, assume that  $M_t = m_t I_L$ . Using the identity  $(I+AB)^{-1}A = A(I+BA)^{-1}$ ,  $(\sigma_z^2 M_t^{-1} + X_t X_t^T)^{-1} X_t = X_t \left(c^{-1}I + X_t^T X_t\right)^{-1}$ , where,  $c_t$ , the confidence parameter, is defined as

$$c_t = \frac{m_t}{\sigma_z^2}.$$

Then,  $\mathbf{w}_{t+1}^{(\mathrm{ML})}$  can be written as  $\mathbf{w}_{t+1}^{(\mathrm{ML})} = \mathbf{w}_t + X_t (\frac{1}{c_t} I_P + X_t^T X_t)^{-1} (\mathbf{y}_t - X_t^T \mathbf{w}_t)$ . This expression provides the MLE of  $w^*$  given the assumption  $\mathbf{w}_t \sim \mathcal{N}(\mathbf{w}^*, m_t I)$  with known parameter  $m_t$ . Moreoever, if we know that the covariance of  $\mathbf{w}_t$  is of the form  $m_t I$ , but  $m_t$  is unknown, we can derive the MLE of  $m_t$  as the solution to the optimization

$$m_t^{(\mathrm{ML})} = \operatorname*{arg\,max}_{m \in \mathbb{R}^+} \left( -\frac{L}{2} \log m - \frac{1}{2m} \|\mathbf{w}_t - \mathbf{w}^*\|^2 \right),$$

which leads to

$$m_t^{(\mathrm{ML})} = \frac{1}{L} \|\mathbf{w}_t - \mathbf{w}^*\|^2.$$

Putting the pieces together, we derive the update rule which we refer to as genie-aided online block maximum likelihood (GA-OBML) update rule:

$$\mathbf{w}_{t+1} = \mathbf{w}_t + X_t \left(\frac{1}{c_t} I_P + X_t^T X_t\right)^{-1} (\mathbf{y}_t - X_t^T \mathbf{w}_t), \quad (8)$$

with confidence parameter defined as

$$c_t = \frac{\|\mathbf{w}_t - \mathbf{w}^*\|^2}{\sigma_z^2 L}.$$
 (9)

To compute the confidence parameter at time t, one needs to have access to the misalignment at time t, i.e.,  $\|\mathbf{w}_t - \mathbf{w}^*\|^2$ , which is obviously not available in practice. That is why we refer to this algorithm as "genie-aided". However, as explained in Section V, this quantity can be estimated reasonably well in practice. In particular, the "delay and extrapolation" approach [19] enables us to estimate the misalignment with only marginal loss of performance compared with the genie-aided version, especially for P > 2. Pseudo-code for the GA-OBML

algorithm is provided in Algorithm 1.

# Algorithm 1 GA-OBML algorithm

Some intuitive insight into the GA-OBML algorithm can be obtained by considering different ranges of the confidence parameter  $c_t$ . During initial steps,  $\|\mathbf{w}_t - \mathbf{w}^*\|^2$  is large and therefore  $c_t$  is large as well. In this case, GA-OBML behaves more like standard APA with aggressive step size  $\mu = 1$  and small regularization  $\delta$ . As time proceeds and the misalignment error becomes small, the confidence parameter  $c_t$  becomes small as well. In that case, GA-OBML becomes more conservative and its update rule becomes similar to block LMS with a very small step size  $\mu = c_t$ . That is, we have approximately  $\mathbf{w}_{t+1} = \mathbf{w}_t + c_t X_t \left( \mathbf{y}_t - X_t^T \mathbf{w}_t \right)$ . Therefore, in summary, by setting the confidence parameter as (9), GA-OBML is able to perform a soft transition between APA and block-LMS, depending on the current ratio of misalignment and noise.

To derive the GA-OBML update rule we made several assumptions, such as unbiasedness of our estimates, which, at best, are approximately true. However, our next theorem, which is our main theoretical result, fully characterizes the expected misalignment performance of GA-OBML and shows that it indeed is able to asymptotically recover the unknown coefficients  $\mathbf{w}^*$ .

**Theorem 1.** Assume that  $\mathbf{y}_t = X_t^T \mathbf{w}^* + Z_t$ , where  $X_t \in \mathbb{R}^{L \times P}$  and  $Z_t \in \mathbb{R}^P$  are independent and the entries of  $X_t$  and  $Z_t$  are i.i.d.  $\mathcal{N}(0, \sigma_x^2)$  and  $\mathcal{N}(0, \sigma_z^2)$ , respectively. Let  $\mathbf{w}_{t+1} = \mathbf{w}_t + X_t(\frac{1}{c_t}I_P + X_t^T X_t)^{-1}(\mathbf{y}_t - X_t^T \mathbf{w}_t)$ , where  $\mathbf{w}_t \in \mathbb{R}^L$  is a fixed vector and parameter  $c_t$  is defined as (9). Let  $r_t = \|\mathbf{w}_t - \mathbf{w}^*\|^2$  and, assume L large enough that  $\sqrt{L} - \sqrt{\log L} > \sqrt{P}$ . Then,

$$E[r_{t+1}] \le \left(1 - (1 - \gamma_L) \frac{P}{L} \frac{\sigma_x^2 r_t}{\sigma_z^2 + \sigma_x^2 r_t}\right) r_t$$

where  $\gamma_L = O(\sqrt{\frac{\log L}{L}})$  is an absolute constant depending only on P and L.

The proof of Theorem 1 is provided in Section VI.

Theorem 1 shows that after every update the expected error at time t+1 ( $\mathrm{E}[r_{t+1}]$ ) is smaller than the actual error at time t ( $r_t$ ) multiplied by a factor that is smaller than one but depends on  $r_t$ . To further understand the implications of Theorem 1, let  $a_t = r_t \sigma_x^2/\sigma_z^2$  denote the normalized misalignment error,

and define function  $f: \mathbb{R} \to \mathbb{R}$  as

$$f(a) = (1 - \eta \frac{a}{1+a})a,$$

where  $\eta \in (0,1)$  is defined as

$$\eta = (1 - \gamma_L) \frac{P}{L}.$$

Using these definitions, Theorem 1 states that

$$E[a_{t+1}] \le f(a_t), \tag{10}$$

for any  $t = 0, 1, \dots$  Since f is a concave function, taking the expected value of both side of (10) and using the Jensen's inequality, it follows that

$$E[a_{t+1}] \le f(E[a_t])$$

$$= (1 - (1 - \gamma_L) (\frac{P}{L}) \frac{E[a_t]}{1 + E[a_t]}) E[a_t].$$
 (11)

Thus if we repeatedly apply GA-OBML, the sequence defined by  $a_{t+1} = f(a_t)$  is an upper bound on the expected value of the normalized misalignment after t iterations, and s = tP samples. This expected error reduces monotonically, and as we will see, asymptotically, achieves the performance achieved by optimal offline algorithms.

To show the effectiveness of the proposed GA-OBML method, we next compare its performance, characterized in Theorem 1, with a lower bound on the performance achieved by a general class of offline algorithms that estimate  $\mathbf{w}^*$  at time s using the entire history of observations. Let  $\mathbf{y}_s = X_s^T \mathbf{w}^* + Z_s$ , where  $X_s \in \mathbb{R}^{L \times s}$  and  $Z_s \in \mathbb{R}^s$ . We define  $\mathbf{w}_s^{(\delta)}$ , a generalized regularized least squares estimator, defined as

$$\mathbf{w}_{c}^{(\delta)} = \mathbf{w}_{0} + \Delta \mathbf{w},$$

with

$$\Delta \mathbf{w} = \arg\min_{\mathbf{w}} [\|X_s^T \mathbf{w}_o + X_s^T \mathbf{w} - \mathbf{y}_s\|^2 + \delta \|\mathbf{w}\|^2].$$

Solving for  $\Delta w$ , it is straightforward to see that

$$\mathbf{w}_s^{(\delta)} = \mathbf{w}_0 + X_s(\delta I_s + X_s^T X_s)^{-1} (\mathbf{y}_s - X_s^T \mathbf{w}_0).$$

Here,  $\mathbf{w}_0$  plays the role of a rough estimate of the desired parameters  $(\mathbf{w}^*)$  available from a different set of data. Note that setting  $\delta=0$  and assuming that  $s\geq L$ ,  $\mathbf{w}_s^{(0)}=X_s(\delta I_s+X_s^TX_s)^{-1}\mathbf{y}_s$  denotes the ordinary least squares estimate of  $\mathbf{w}^*$ . In that special case,  $\mathbf{w}_s^{(0)}$  is an unbiased estimator under our assumption that  $Z_s$  is Gaussian. With  $\delta>0$ , the estimator is no longer unbiased, but if  $\delta$  is chosen correctly, the mean squared error can be somewhat lower than that of the unbiased estimate. Theorem 2 below provides a lower bound on the mean squared error achieved by  $\mathbf{w}_s^{(\delta)}$  and shows that if the initial error  $a_0=\|\mathbf{w}_0-\mathbf{w}^*\|^2\sigma_x^2/\sigma_z^2$  is known, the best regularization factor parameter is

$$\delta = c_0^{-1} = \frac{L\sigma_x^2}{a_0}.$$

(Note also that  $\mathbf{w}^{(c_0^{-1})}$  happens to be the ML estimate of  $\mathbf{w}^*$ , under a Bayesian model with  $\mathbf{w}_0 \sim \mathcal{N}(\mathbf{0}_L, \frac{a_0}{L}I_L)$ .)

**Theorem 2.** Assume that  $\mathbf{y}_s = X_s^T \mathbf{w}^* + Z_s$ , where  $X_s \in \mathbb{R}^{L \times s}$  and  $Z_s \in \mathbb{R}^s$  are independent and the entries of  $X_s$  and  $Z_s$  are i.i.d.  $\mathcal{N}(0, \sigma_x^2)$  and  $\mathcal{N}(0, \sigma_z^2)$ , respectively. Let  $\mathbf{w}_s^{(\delta)} = \mathbf{w}_0 + X_s(\delta I_s + X_s^T X_s)^{-1}(\mathbf{y}_s - X_s^T \mathbf{w}_0)$ , where  $\mathbf{w}_0 \in \mathbb{R}^L$  is a fixed vector and  $\delta \geq 0$ . Let  $a_s^{(\delta)} = \|\mathbf{w}_s^{(\delta)} - \mathbf{w}^*\|^2 \sigma_x^2 / \sigma_z^2$ . Then, if s < L,

$$\mathrm{E}[a_s^{(\delta)}] \ge \mathrm{E}[a_s^{(c_0^{-1})}] \ge \left(1 - \frac{s}{L} \frac{a_0}{1 + a_0}\right) a_0,$$

and if  $s \geq L$ ,

$$E[a_s^{(\delta)}] \ge E[a_s^{(c_0^{-1})}] \ge \frac{La_0}{sa_0 + L}.$$

Theorem 2 provides a bound on any regularized least squares method for offline estimation of  $\mathbf{w}^*$  using s samples. (In the case of  $\delta = 0$ , s < L, the matrix inverse in the definition of  $\mathbf{w}^{(\delta)_s}$  should be interpreted as a pseudo-inverse.)

The bound has two phases - an initial fast convergence from  $a_0$  to  $a_0/(1+a_0)\approx 1$ , after s=L samples, followed by a gradual improvement with  $a_s\approx L/s$ , as  $s\to\infty$ . Below, we examine the implications of Theorems 1 and 2 in these two regimes and show that despite being an on-line algorithm with fixed memory P, GA-OBML performs nearly as well as offline methods, for memoryless inputs.

Next we compare the upper bound on GA-OBML performance (the sequence  $a_t$  generated as  $a_{t+1} = f(a_t)$ ) with the lower bound  $a_s$  on performance of offline regularized least squares.

#### A. Initial steps

Assume that we initialize GA-OBML algorithm at  $\mathbf{w}_0 = \mathbf{0}_L$ , which corresponds to  $a_0 = \|\mathbf{w}^*\|^2 \sigma_x^2 / \sigma_z^2$ . If the filter  $\mathbf{w}^*$  is such that initial residual echo dominates the noise, we have  $a_0 \gg 1$ , which implies that  $\frac{a_t}{1+a_t} \approx 1$  and therefore, during initial steps, we expect GA-OBML to show an almost linear convergence with  $a_{t+1} \leq (1 - \frac{P}{L})a_t$  or  $a_t \leq (1 - \frac{P}{L})^t a_0 \approx e^{-(P/L)t} a_0 = e^{-s/L} a_0$ , where we have assumed  $P \ll L$ . GA-OBML thus takes approximately  $L \log a_0$  samples to achieve  $a_s \approx 1$ , just a logarithmic factor longer than the L samples required by the offline estimation.

Note that in this analysis, for memoryless input, the convergence speed (in terms of the number of samples) does not depend on P. The motivation for studying memory order P>1 is that increasing P improves convergence rate for correlated sources observed, such as those observed in audio echo cancellation applications. Later we show empirically that for audio input signals the convergence rate of GA-OBML improves with P.

# B. Asymptotic optimality of GA-OBML

Next we compare the asymptotic convergence behavior of GA-OBML with our lower bound on offline regularized least squares methods.

From Theorem 2, for  $s \ge L$ , the normalized expected error of offline algorithms is at least  $a_s = La_0/(sa_0 + L)$ , or  $a_s \approx L/s$ .

For GA-OBML, Lemma 2 proved in Appendix A shows that the upper bound sequence  $a_{t+1} = a_t (1 - (1 - \gamma_L)(\frac{P}{L}) \frac{a_t}{1 + a_t})$ , for large enough t, satisfies  $a_t \approx (1 - \gamma_L)^{-1} L/(Pt)$ . Recalling s = tP, we have  $a_s \approx (1 - \gamma_L)^{-1} L/s$ . This implies that GA-OBML achieves asymptotically the same performance as OLS, up to a factor  $(1 - \gamma_L)^{-1}$  that approaches 1 for large L.

Fig. 2 shows the temporal behavior of upper and lower bounds for  $a_s$  derived for GA-OBML and offline methods, respectively, for various initial error levels  $a_0$ . The performances of the two methods coincide for  $s \ll L$  and differ only by the factor  $1 - \gamma_L$ , for  $s \gg L$ . At the transition between the two regimes  $(s \approx L)$ , a gap between on-line and offline methods is apparent.

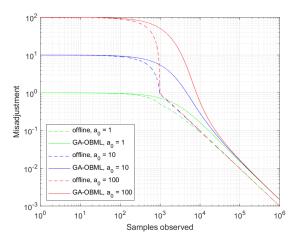


Fig. 2: Comparing bounds on misalignment of GA-OBML to bounds on misalignment of offline methods for three different initial values. (Here, L=1000 and P=4.)

# C. Connections with variants of APA and NLMS with dynamic control

It has long been recognized that by dynamically controlling the parameters of adaptive filtering algorithms, such as NLMS and APA, it is possible to achieve fast initial convergence while also continuing to slowly improve over time. The literature is most mature in the case of NLMS, which coincides with APA for P=1.

Most literature on dynamic control of NLMS has focused on step size - using large step size for fast initial convergence and later decreasing the step size for gradual reduction in misalignment. A method proposed in [19] (VR-NLMS) defines a regularized form of NLMS, i.e.  $\mathbf{w}_{t+1} = \mathbf{w}_t + \frac{1}{\|\mathbf{x}_t\|^2 + \delta_t} \mathbf{x}_t (y_t - \mathbf{x}_t^T \mathbf{w}_t)$  and controls the regularization parameter  $\delta$  rather than the stepsize. (Refer to [11] for a review of this and other dynamic control approaches.) In particular, optmizing  $\delta$  to minimize the expected misalignment in each step, the authors obtain

$$\delta_t^{\text{opt}} = \frac{L\sigma_z^2}{\mathrm{E}[\|\mathbf{w}_t - \mathbf{w}^*\|^2]}.$$

For P=1, this essentially coincides with the definition of GA-OBML (except that GA-OBML is based on  $\|\mathbf{w}_t - \mathbf{w}^*\|^2$  rather than its expectation). Indeed, just as APA is a generalization

of NLMS, GA-OBML can be thought of as a generalization of regularization-optimized NLMS to setting P > 1.

The best way to extend the NLMS dynamic control literature to APA and APA-like algorithms not obvious, and a number of different approaches have been defined, typically controlling some combination of step size  $\mu$ , regularization parameter  $\delta$ , and memory order P. Variable step-size methods are common, and [20] modifies the memory order P together with the step size. Closest to our approach are two methods of dynamically adapting the regularization parameter of APA (VR-APA) [14] and [15]. These methods define different cost functions, which are then greedily optimized in each iteration. The estimate is updated after every sample, unlike GA-OBML which updates rather after blocks of P samples.

The key difference between our proposed method and existing solutions is that we fully characterize the convergence behavior of GA-OBML for i.i.d. inputs. In particular, we give an explicit upper bound on the evolution of the expected misalignment over time, and show that that the upper bound approaches the performance of offline algorithms over long time windows. Beyond LMS with variable step size [18], to our knowledge this is the first such result in this domain with convergence guarantee to the true parameters.

Additionally, in deriving the GA-OBML rule, we used maximum likelihood estimation of an analytically motivated approximate Bayesian model, rather than greedy minimization of a cost function. This approach is interesting in its own right, as it allows us to derive a related algorithm, incremental maximum likelihood, which is harder to analyze, but which empirically performs even better. This algorithm is discussed in the next section.

# IV. INCREMENTAL MAXIMUM LIKELIHOOD (IML)

In Section III, we proposed and studied GA-OBML, an adaptive MLE-based echo cancellation algorithm that updates its estimate of the echo filter coefficients every P samples. Most existing echo cancellation algorithms on the other hand, update their estimates every time sample. In this section, inspired by GA-OBML, we propose an alternative solution, which updates its online ML estimate of the parameters at every time step. We refer to the new solution as the incremental maximum likelihood (IML) algorithm. As we will show in Section V, in practice, the performance of IML in general is as good as that of OBML.

Recall that 
$$X_t = [\mathbf{x}_t, \dots, \mathbf{x}_{t-P+1}]$$
, and  $\mathbf{y}_t = [y_t, \dots, y_{t-P+1}]^T$ . Let

$$U_t = [\mathbf{x}_t, \dots, \mathbf{x}_{t-P+2}],$$

and  $\mathbf{v}_t = [y_t, \dots, y_{t-P+2}]$ . Using these definitions,

$$X_t = [\mathbf{x}_t, U_{t-1}], \quad \mathbf{y}_t = [y_t, \mathbf{v}_{t-1}^T]^T.$$

Assume that at time t, we are given  $\mathbf{w}_{t-P}$  which is an unbiased estimate of  $\mathbf{w}^*$ , such that  $\mathbf{w}_{t-P} \sim \mathcal{N}(\mathbf{w}^*, m_{t-P}I_L)$ . Then, we compute two possible updates: i)  $\mathbf{w}_1$ : ML estimate of  $\mathbf{w}^*$  given  $\mathbf{w}_{t-P}$  and P-1 new observations  $(U_{t-1}, \mathbf{v}_{t-1})$ , and

ii) ML estimate of  $\mathbf{w}^*$  given  $\mathbf{w}_{t-P}$  and P new observations  $(X_t, \mathbf{y}_t)$ . From (6), we have

$$\mathbf{w}_1 = (c^{-1}I_L + U_{t-1}U_{t-1}^T)^{-1}(\mathbf{w}_{t-P} + U_{t-1}\mathbf{v}_{t-1}), \quad (12)$$

and

$$\mathbf{w}_2 = (c^{-1}I_L + X_t X_t^T)^{-1} (\mathbf{w}_{t-P} + X_t \mathbf{y}_t), \tag{13}$$

where  $c=\frac{m_{t-P}}{\sigma_z^2}$ . But,  $X_tX_t^T=U_{t-1}U_{t-1}^T+\mathbf{x}_t\mathbf{x}_t^T$  and  $X_t\mathbf{y}_t=U_{t-1}\mathbf{v}_{t-1}+y_t\mathbf{x}_t$ . Let  $B_t=c^{-1}I_L+U_{t-1}U_{t-1}^T$ . Using the Woodbury matrix identity,

$$(c^{-1}I_L + X_t X_t^T)^{-1} = B_t^{-1} - \frac{B_t^{-1} \mathbf{x}_t \mathbf{x}_t^T B_t^{-1}}{1 + \mathbf{x}_t^T B_t^{-1} \mathbf{x}_t}.$$

Therefore,  $\mathbf{w}_1$  and  $\mathbf{w}_2$  can be connected as

$$\mathbf{w}_{2} = \left(B_{t}^{-1} - \frac{B_{t}^{-1} \mathbf{x}_{t} \mathbf{x}_{t}^{T} B_{t}^{-1}}{1 + \mathbf{x}_{t}^{T} B_{t}^{-1} \mathbf{x}_{t}}\right) (\mathbf{w}_{t} + U_{t-1} V_{t-1} + y_{t} \mathbf{x}_{t})$$

$$= \left(I_{L} - \frac{B_{t}^{-1} \mathbf{x}_{t} \mathbf{x}_{t}^{T}}{1 + \mathbf{x}_{t}^{T} B_{t}^{-1} \mathbf{x}_{t}}\right) (\mathbf{w}_{1} + y_{t} B_{t}^{-1} \mathbf{x}_{t})$$

$$= \mathbf{w}_{1} + \frac{B_{t}^{-1} \mathbf{x}_{t} (y_{t} - \mathbf{x}_{t}^{T} \mathbf{w}_{1})}{1 + \mathbf{x}_{t}^{T} B_{t}^{-1} \mathbf{x}_{t}}.$$
(14)

Thus if we had access to the ML estimate of  $\mathbf{w}^*$  based on the previous P-1 samples  $(\mathbf{w}_1)$ , given the new observation point  $(\mathbf{x}_t, y_t)$ , we can use (14) to obtain the ML estimate based on P samples  $(\mathbf{w}_2)$  incrementally, without reference to  $\mathbf{w}_{t-P}$ . This leads to our proposed GA-IML update rule. At time t, we assume that our current estimate of the filter coefficients  $\mathbf{w}_t$  is the ML estimate of  $\mathbf{w}^*$  based on the previous P-1 samples and  $\mathbf{w}_{t-P}$ . Then, given new data  $(\mathbf{x}_t, y_t)$  we design  $\mathbf{w}_{t+1}$  to be the ML estimate based on the past P samples and  $\mathbf{w}_{t-P}$ . That is,

$$\mathbf{w}_{t+1} = \mathbf{w}_t + \frac{B_t^{-1} \mathbf{x}_t (y_t - \mathbf{x}_t^T \mathbf{w}_t)}{1 + \mathbf{x}_t^T B_t^{-1} \mathbf{x}_t},$$
(15)

Intuitively, this update rule extracts relevant information from the latest sample, assuming that the previous P-1 samples have already been used effectively, in a maximum likelihood framework. Defining  $\tilde{\mathbf{x}}_t = c_t^{-1} B_t^{-1} \mathbf{x}_t$ , we can also write

$$\mathbf{w}_{t+1} = \mathbf{w}_t + \frac{\tilde{\mathbf{x}}_t(y_t - \mathbf{x}_t^T \mathbf{w}_t)}{c_t^{-1} + \mathbf{x}_t^T \tilde{\mathbf{x}}_t},$$
(16)

In this form, the update is seen to be similar to a VR-NLMS update, but with  $\tilde{\mathbf{x}}_t$  replacing  $\mathbf{x}_t$  in some places. Computation of  $\tilde{\mathbf{x}}_t$  only requires a matrix inverse of size P-1, since again by the Woodbury matrix identity,

$$B_t^{-1} = cI_L - cU_{t-1}(c^{-1}I_{P-1} + U_{t-1}^T U_{t-1})^{-1}U_{t-1}^T.$$

IML's update rule can also be written in an alternate form closer to OBML's. Note that

$$(c^{-1}I_L + X_t X_t^T) \frac{B_t^{-1} \mathbf{x}_t}{1 + \mathbf{x}_t^T B_t^{-1} \mathbf{x}_t} = \frac{(B_t + \mathbf{x}_t \mathbf{x}_t^T) B_t^{-1} \mathbf{x}_t}{1 + \mathbf{x}_t^T B_t^{-1} \mathbf{x}_t} = \mathbf{x}_t$$

so that (15) becomes

$$\mathbf{w}_{t+1} = \mathbf{w}_t + \left(c^{-1}I_L + X_t X_t^T\right)^{-1} \mathbf{x}_t (y_t - \mathbf{x}_t^T \mathbf{w}_t)$$

$$= \mathbf{w}_t + \left(c^{-1}I_L + X_t X_t^T\right)^{-1} X_t \begin{bmatrix} \epsilon_t \\ \mathbf{0}_{P-1} \end{bmatrix}$$

$$= \mathbf{w}_t + X_t \left(c^{-1}I_P + X_t^T X_t\right)^{-1} \begin{bmatrix} \epsilon_t \\ \mathbf{0}_{P-1} \end{bmatrix}$$
(17)

where  $\epsilon_t = y_t - \mathbf{w}_t^T \mathbf{x}_t$ . In this form, the IML update rule is very similar to that of OBML, except that in IML, the filter coefficient estimates are updated after every sample, instead of every P samples, and all but the first term in the error vector are zeroed out. Pseudocode for GA-IML in this form is provided in Algorithm 2.

# Algorithm 2 GA-IML algorithm

Require: 
$$(\mathbf{x}_1, \dots, \mathbf{x}_T)$$
, with  $\mathbf{x}_t \in \mathbb{R}^L$ ,  $(y_1, \dots, y_T)$ ,  $\mathbf{w}^* \in \mathbb{R}^L$ ,  $P, L$  and  $\sigma_z$ 
Ensure:  $\mathbf{w}$ 
Initialize  $\mathbf{w} \leftarrow \mathbf{0}_L$ 
for  $t \leftarrow 1$  to  $T$  do
$$X \leftarrow [\mathbf{x}_t, \dots, \mathbf{x}_{t-P+1}]^T$$

$$\mathbf{y} \leftarrow [y_t, \dots, y_{t-P+1}]^T$$

$$c \leftarrow \|\mathbf{w} - \mathbf{w}^*\|^2/(L\sigma_z^2)$$

$$e \leftarrow y_t - \mathbf{x}_t^T \mathbf{w}$$

$$\mathbf{e} \leftarrow \begin{bmatrix} e \\ \mathbf{0}_{P-1} \end{bmatrix}$$

$$\mathbf{w} \leftarrow \mathbf{w} + X(\frac{1}{c}I_P + X^TX)^{-1}\mathbf{e}$$
end for

# V. SIMULATION RESULTS

Our goal in this section is to compare the GA-OBML and GA-IML methods with each other, with exising adaptive filtering methods, and with the analytical results of this paper in a simulated acoustic echo cancellation setting. The first simulation uses a white noise input signal to be able to compare with analytyical results, while the subsequent simulations are based on recorded audio signals as input. We also introduce and evaluate practical versions of GA-IML and GA-OBML, referred to as IML and OBML, in which the misalignment  $\|\mathbf{w}_t - \mathbf{w}^*\|^2$  is estimated rather than provided by a genie.

For the simulations with recorded audio input signals, we have used the publicly available audio dataset Librispeech, described in [21]. For every simulation result, we performed 10 trials using randomly selected audio segments from this dataset, and reported the average performance. The signals is sampled at  $f_s = 16$  kHz. In most simulation results the echo channel response is of length L = 500 corresponding to 31.25 (ms). In the misalignment plots, the normalized misalignment is defined as  $20 \log_{10}(\|\mathbf{w}_t - \mathbf{w}_0\|/\|\mathbf{w}_0\|)$ .

In deriving our theoretical results, we assumed that the input vectors  $\mathbf{x}_t$  are independent from each other, and also that the elements within a given vector are uncorrelated. For AEC, (and related applications), neither of these assumptions hold, as consecutive vectors have considerable overlap, and also

consecutive time samples are strongly correlated. (Recall that  $\mathbf{x}_{t+1} = [x_{t+1}, x_t, \dots, x_{t-L+2}]$  and  $\mathbf{x}_t = [x_t, \dots, x_{t-L+1}]$ .) However, in our simulation results, we show that the IML method (outperforming OBLM) achieves state-of-the-art performance, even in AEC scenarios where the input vectors are not independent and have overlap.

#### A. Performance under i.i.d. input

To illustrate directly the consequences of Theorem 1, we first simulate an echo cancellation scenario in which the farend signal is an iid Gaussian sequence.

We average the performance over 10 randomly generated audio channels of length L=500 and random iid Gaussian input sequences. The SNR, defined as  $10\log_{10}\frac{\sigma_y^2}{\sigma_z^2}$ , is set to 40 dB.

In Fig. 3, we quantiy the normalized misalignment as a function of time, for various adaptive filtering methods. In this scenario, GA-OBML and GA-IML perform very similarly; only GA-IML is shown, to simplify the plot. Differences between the two are clearer in an audio input setting, further below. In addition to the performance of GA-IML, with memory orders P=2 and P=10, we show the performance of APA with the orders P=1,2 and 10 and with fixed stepsizes  $\mu=1/P.$  Note that APA with P=1 is equivalent to NLMS. In addition to these first order methods, we plot the performance of recursive least squares with  $\eta=1-10^{-5}$ , a second order method with quadratic complexity.

As is well known, APA with fixed, large step size converges quickly initially, after which the performance plateaus at a level that depends inversely on the stepsize. RLS, with a properly tuned parameter  $\eta$ , converges even more quickly and continues to improve gradually over time as additional samples come in. GA-IML is a first order method that converges as quickly as the APA methods initiallly, and then continues to improve over time. As predicted by our analysis of GA-OBML following Theorem 1, the performance eventually approaches that of RLS.

Fig. 3 suggests that both for APA and IML, if the source is memoryless, there is no advantage in considering a memory parameter P larger than one. This is in agreement with the analysis we carried out in Sections III-A and III-B, where the initial and asymptotic convergence, as a function of the number of samples s=Pt, are independent of P.

In contrast, the memory order parameter P does make a difference when input signals are strongly correlated, as when the signals are recorded audio segments, as we show in following subsections.

# B. Performance under audio input

1) GA-IML vs. GA-OBML: In this and the remaining experiments, we average the performance over 10 randomly generated audio channels of length L=500 and randomly selected audio signals from Librispeech [21].

On, Fig. 4 (a) and Fig. 4 (b), we compare the performance of GA-OBML against the performance of GA-IML, for SNR values 30 dB and 40 dB, and for three difference memory

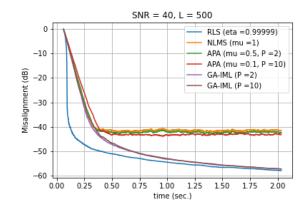


Fig. 3: Comparing misalignment performances of GA-IML with NLMS, APA and RLS. (Input is i.i.d. Gaussian samples, SNR = 40 dB and L = 500.)

orders, namely P=2, P=5, and P=10. GA-IML consistently performs a little better than GA-OBML, in all these scenarios, although the difference becomes less significant for larger values of P. Note however that since GA-OBML only updates its coefficients after every sample P samples, compared with GA-IML which updates after every sample, GA-OBML typically has a lower average computational burden and may be preferred in some applications from an engineering perspective.

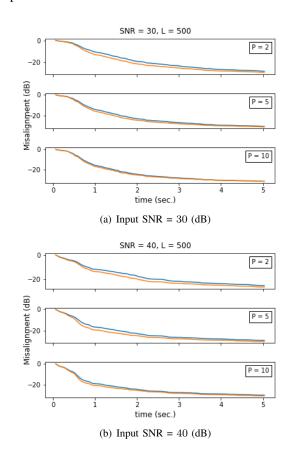


Fig. 4: Comparing misalignment performance of GA-BOML (blue) and GA-IML (orange).

2) GA-IML vs. existing methods: In Figure 5, we demonstrate the importance of using higher memory orders with adaptive control. The first two curves show the misalignment performance of NLMS with a fixed stepsize GA-OBML with P=1. As noted previously, in the case P=1, GA-OBML essentially coincides with the optimized adaptive NLMS method proposed in [11]. The benefits of the adaptive control over the fixed stepsize are clearly visible. The further advantages of GA-OBML over adaptive NLMS are clearly seen when applying larger memory orders P=2 and P=10.

Next we compare the performance of GA-IML with the other adaptive filtering methods previously discussed with respect to iid input signals in 3. We consider algorithms with linear scaling, such as NLMS and APA, and use RLS as a representative of algorithms with quadratic scaling. Figure 6 compares the performance of GA-IML with these other methods for SNR = 40 (dB). In this figure, we have set the step size  $\mu$  of NLMS equal to one, and no regularization parameter.

Figure 6 (b) shows a zoomed in version of the misalignment performance for  ${\rm SNR}=40$  (dB). It can be observed that in both cases APA and GA-IML with P=10 show much faster convergence compared to other linear-scaling methods. Moreover, their convergence behavior is comparable to that of RLS, which is computationally very intensive and impractical in many applications. However, as time proceeds, while the performance of GA-IML continues to follows that of RLS, the performance of APA with fixed step size saturates at a relatively high level of misalignment.

As mentioned in Section I-B, it has been previiously recognized that some sort of adaptive control would be needed to achieve fast convergence and high asymptotic accuracy with APA-like methods, and various methods of adaptively controlling P or  $\mu$  for APA have been developed, which can avoid this saturation effect. Rather than providing a detailed comparison with all of these methods, the focus of this paper is to show the effectiveness of GA-IML as an adaptive filtering methods which i) is closely connected to a method (GA-OBML) which is *provably* able to reduce misalignment to arbitrarily low levels at the optimal rate under iid input and ii) empirically is competitive with quadratic complexity methods under realistic audio inputs and iii) is essentially parameter-free, as it only involves one controlled parameter ( $c_t$ ), whose optimal value is analytically expressed.

#### C. Practical IML

Finally, a key remaining issue that needs to be addressed is that the instantaneous misalignment value  $\|\mathbf{w}_t - \mathbf{w}^*\|^2$  needed to run GA-IML or GA-OBML, is not available in practice. Various methods for misalignment estimation have been proposed. Here, we test the effectiveness of one such method, the delay-and-extrapolate technique [19]. To apply this approach, we artificially delay the microphone signals by M samples, and increase the length of our adaptive filter from L to L+M. Thus, instead of learning  $\mathbf{w}^* \in \mathbb{R}^L$ , we estimate  $\tilde{\mathbf{w}}^* \in \mathbb{R}^{M+L}$ , where  $y_t = \tilde{\mathbf{x}}_t^T \tilde{\mathbf{w}}^* + z_t$ , where

$$\tilde{\mathbf{x}}_t^T = [x_{t+M}, x_{t+M-1}, x_{t-L+1}].$$

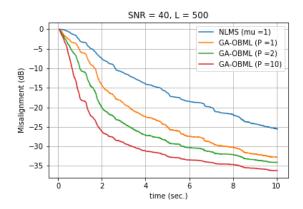
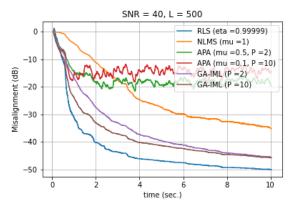
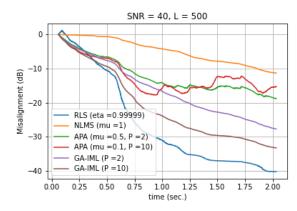


Fig. 5: Comparing the misalignment performances of GA-OBML and NLMS (SNR = 40 dB)



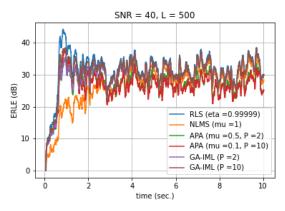
(a) Misalignment versus time



(b) Zoomed view of misalignment during initial steps

Fig. 6: Comparing misalignment performances of GA-IML with NLMS, APA and RLS. (Input is random audio segments from Librispeech dataset. SNR = 40 dB and L = 500.)

Clearly, by the causality of the channel,  $\tilde{\mathbf{w}}^* = [\mathbf{0}_M, \mathbf{w}^*]$ . Therefore, the misalignment error due to the first M coefficients can be computed exactly as  $\sum_{i=1}^M \tilde{\mathbf{w}}_{t,i}^2$ . Assuming that the errors are uniformly distributed over the filter coefficients, the full misalignment, i.e.,  $\|\tilde{\mathbf{w}}_t - \tilde{\mathbf{w}}^*\|^2$ , can then be estimated by extrapolation. The resulting estimated confidence parameter



(a) ERLE versus time

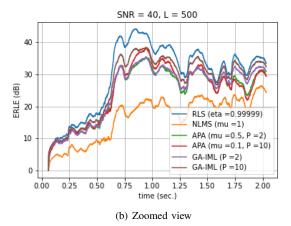


Fig. 7: Comparing ERLE performances of GA-IML with NLMS, APA and RLS. (Input is random audio segments from Librispeech dataset.  ${\rm SNR}=40~{\rm dB}$  and L=500.)

is then

$$\hat{c}_t = \frac{1}{M\sigma_z^2} \sum_{i=1}^{M} \tilde{w}_{t,i}^2.$$
 (18)

Note that, if the algorithm starts from  $\mathbf{w}_0 = \mathbf{0}_L$ , the assumption that filter error is uniformly distributed does not hold during the initial steps, and so that the confidence parameter  $\hat{c}_t$  estimated from this formula would be a significant underestimate. Infact,  $c_t$  should be large during those steps, because  $\mathbf{w}_t$  is in fact far from  $\mathbf{w}^*$ . To address this issue, in practice, we could initialize  $\hat{c}_t$  by a relatively large number for the initial iterations, and afterwards switch to using (18). In our implementations, we simply set  $\hat{c}_t = \frac{1}{\sigma_t^2}$  for the first initial M steps, and then use (18) to estimate  $c_t$  for t > M. (Algorithm 3 shows the steps of the IML algorithm. Unlike Algorithm 1 and Algorithm 2,  $\mathbf{w}^*$  is not needed as an input.)

Finally, note that to compute  $c_t$ , one needs to have access to the noise power  $\sigma_z$  as well. Like many other papers in this domain, in our simulation results, we have assumed that  $\sigma_z$  is known. There are various methods to estimate  $\sigma_z$  in practice. (Refer to [9], [22], [23] for an overview of some such methods.) However, exploring different options and their impacts on the performance of the algorithm is beyond the

# Algorithm 3 IML algorithm

Require: 
$$(\tilde{\mathbf{x}}_1,\dots,\tilde{\mathbf{x}}_T)$$
, with  $\tilde{\mathbf{x}}_t \in \mathbb{R}^{L+M}$ ,  $(y_1,\dots,y_T)$ ,  $P$ ,  $L$ ,  $M$  and  $\sigma_z$ 

Ensure:  $\mathbf{w}$ 
Initialize  $\mathbf{w} \leftarrow \mathbf{0}_L$ 
Initialize  $\tilde{\mathbf{w}} \leftarrow \mathbf{0}_{L+M}$ 
for  $t \leftarrow 1$  to  $T$  do
$$\tilde{X} \leftarrow [\tilde{\mathbf{x}}_t,\dots,\tilde{\mathbf{x}}_{t-P+1}]$$

$$\mathbf{y} \leftarrow [y_t,\dots,y_{t-P+1}]^T$$
if  $t < M$  then
$$\hat{c} \leftarrow \frac{1}{\sigma_z^2}$$
else
$$\hat{c} \leftarrow \frac{1}{M\sigma_z^2} \sum_{i=1}^M \tilde{w}_i^2.$$
end if
$$e \leftarrow y_t - \mathbf{x}_t^T \mathbf{w}$$

$$\mathbf{e} \leftarrow \begin{bmatrix} e \\ \mathbf{0}_{P-1} \end{bmatrix}$$

$$\tilde{\mathbf{w}} \leftarrow \tilde{\mathbf{w}} + \tilde{X}(\frac{1}{\hat{c}}I_P + \tilde{X}^T \tilde{X})^{-1}\mathbf{e}$$
end for
$$\mathbf{w} \leftarrow [\tilde{w}_{M+1},\dots,\tilde{w}_{M+L}]^T$$

scope of this paper.

Fig. 8 (a) and Fig. 8 (b) show the effectiveness of this approach for  ${\rm SNR}=30$  (dB), and  ${\rm SNR}=40$  (dB), respectively. In both figures M is set to be 100, and as before L=500. The results demonstrate that the performance of GA-IML can be closely approximated by IML based on delay-and-extrapolate misalignment estimation.

## VI. PROOF OF THE MAIN RESULTS

Before presenting the proof of Theorem 1, we reproduce a lemma from [24] that we will need about the concentration of the singular values of i.i.d. random matrices.

**Lemma 1.** Let the elements of an  $m \times n$  matrix A, m < n, be drawn independently from  $\mathcal{N}(0,1)$ . Then, for any h > 0,

$$P(\sqrt{n} - \sqrt{m} - \sqrt{h} \le \sigma_{\min}(A)$$
  
 
$$\le \sigma_{\max}(A) \le \sqrt{n} + \sqrt{m} + \sqrt{h} \ge 1 - 2e^{-\frac{h}{2}}.$$

Proof of Theorem 1. Recall that  $\mathbf{w}_{t+1} = \mathbf{w}_t + X_t(\frac{1}{c_t}I_P + X_t^TX_t)^{-1}(\mathbf{y}_t - X_t^T\mathbf{w}_t)$ , where  $r_t = \|\mathbf{w}_t - \mathbf{w}^*\|^2$  and  $c_t = \frac{r_t}{L\sigma^2}$ . Since  $\mathbf{y}_t = X_t^T\mathbf{w}^* + Z_t$ , it follows that

$$\mathbf{w}_{t+1} - \mathbf{w}^* = A_t(\mathbf{w}_t - \mathbf{w}^*) + X_t(\frac{1}{c_t}I_P + X_t^T X_t)^{-1} Z_t,$$

where

$$A_t = I_L - X_t (\frac{1}{c_t} I_P + X_t^T X_t)^{-1} X_t^T.$$
 (19)

Since,  $Z_t$  is zero-mean and independent of  $X_t$ , we have

$$E[\|\mathbf{w}_{t+1} - \mathbf{w}^*\|^2] = E[\|A_t(\mathbf{w}_t - \mathbf{w}^*)\|^2] + E[\|X_t(\frac{1}{c_t}I_P + X_t^T X_t)^{-1} Z_t\|^2].$$
(20)

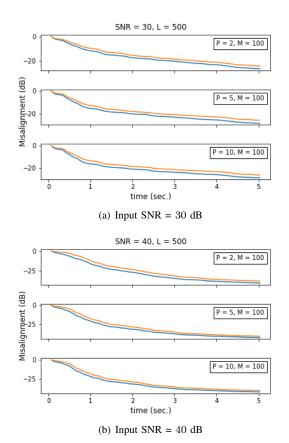


Fig. 8: Comparing misalignment performance of GA-IML (blue) and IML (orange).

For  $t=1,2,\ldots$ , define the (misalignment) error vector  $\mathbf{e}_t$  as  $\mathbf{e}_t=\mathbf{w}_t-\mathbf{w}^*.$ 

Also, recall that earlier, we defined  $r_t$  as  $r_t = \|\mathbf{e}_t\|^2$ . Then, given the independence of  $X_t$  and  $Z_t$ , from (20) it follows that

$$E[\|\mathbf{e}_{t+1}\|^{2}] = E[\mathbf{e}_{t}^{T} A_{t}^{2} \mathbf{e}_{t}] + \sigma_{z}^{2} E[\mathbf{tr}(X_{t}(\frac{1}{c_{t}} I_{P} + X_{t}^{T} X_{t})^{-2} X_{t}^{T})]. \quad (21)$$

The random matrix  $X_t$  is rotationally invariant, i.e.  $\tilde{X} = UX_t$  has the same distribution as  $X_t$  for any fixed orthonormal  $U \in \mathbb{R}^{L \times L}$ . Therefore, it is easy to see that the distribution of  $A_t^2$  is invariant to the transformation  $UA_t^2U^T$ , and thus  $\mathrm{E}[\mathbf{e}_t^TA_t^2\mathbf{e}_t]$  depends on  $\mathbf{e}_t$  only through its norm  $\sqrt{r_t}$ . Denoting by  $\tilde{\mathbf{e}}_m$  the m-th column of  $\sqrt{r_t}I$ , for each m, we have

$$E[\mathbf{e}_t^T A_t^2 \mathbf{e}_t] = E[\tilde{\mathbf{e}}_m^T A_t^2 \tilde{\mathbf{e}}_{tm}] = r_t E[(A_t^2)_{mm}].$$

Summing over m, it follows that  $E[\mathbf{e}_t^T A_t^2 \mathbf{e}_t] = \frac{r_t}{L} E[\mathbf{tr}(A_t^2)].$ 

It is convenient to define the normalized misalignment  $a_t = r_t \sigma_x^2/\sigma_z^2$ . In an echo cancellation setting, this is also known as the normalized misadjustment: the ratio of additional output power due to imperfect filter coefficients, divided by the output power obtained with perfect coefficients. In these terms, we

have

$$E[a_{t+1}] = \frac{a_t}{L} E[\mathbf{tr}(A_t^2)] + \sigma_x^2 E[\mathbf{tr}(X_t(\frac{1}{c_t}I_P + X_t^T X_t)^{-2} X_t^T)].$$
(22)

Consider the singular value decomposition (SVD) of  $X_t$  defined as  $X_t = USV^T$ , where  $U \in \mathbb{R}^{L \times P}$ ,  $S \in \mathbb{R}^{P \times P}$  and  $V \in \mathbb{R}^{P \times P}$ . Denoting the P singular values as  $S_1, \ldots, S_P$ , we can compute the eigenvalues of the matrices in (22). The matrix  $A_t$  has L-P eigenvalues equal to 1, and the remaining P eigenvalues equal to  $1 - \frac{S_t^2}{c_t^{-1} + S_t^2} = \frac{1}{1 + c_t S_t^2}$ , while the matrix  $X_t(\frac{1}{c_t}I_P + X_t^TX_t)^{-2}X_t^T$  has L-P eigenvalues equal to zero and the remaining P eigenvalues equal to  $\frac{c_t^2 S_t^2}{(1 + c_t S_t^2)^2}$ . Thus

$$\begin{split} \mathbf{E}[a_{t+1}] = & \frac{a_t}{L} \left( L - P + \sum_{i=1}^{P} \mathbf{E} \left[ \frac{1}{(1 + c_t S_i^2)^2} \right] \right) \\ & + \sigma_x^2 \sum_{i=1}^{P} \mathbf{E} \left[ \frac{c_t^2 S_i^2}{(1 + c_t S_i^2)^2} \right]. \\ = & a_t (1 - \frac{P}{L}) + \frac{a_t}{L} \sum_{i=1}^{P} \mathbf{E} \left[ \frac{1}{(1 + a_t \tilde{S}_i^2)^2} + \frac{a_t \tilde{S}_i^2}{(1 + a_t \tilde{S}_i^2)^2} \right] \\ = & a_t \left( 1 - \frac{P}{L} + \frac{1}{L} \sum_{i=1}^{P} \mathbf{E} \left[ \frac{1}{1 + a_t \tilde{S}_i^2} \right] \right) \\ = & a_t \left( 1 - \frac{1}{L} \sum_{i=1}^{P} \mathbf{E} \left[ \frac{a_t \tilde{S}_i^2}{1 + a_t \tilde{S}_i^2} \right] \right), \end{split}$$

where we have defined normalized singular values  $\tilde{S}_i = S_i/\sqrt{L\sigma_x^2}$ , and used  $c_t = r_t/(L\sigma_z^2) = a_t/(L\sigma_x^2)$ .

To complete the proof, we will show that

$$\operatorname{E}\left[\frac{a_t \tilde{S}_j^2}{1 + a_t \tilde{S}_i^2}\right] \ge (1 - \gamma_L) \frac{a_t}{1 + a_t}$$

for a constant  $\gamma_L$  that goes to zero, as  $L \to \infty$ .

This can be established because the normalized singular values concentrate around unity for large L. For convenience, define

$$\psi(S, a) = \frac{aS^2}{1 + aS^2}.$$

Since for any  $a_t > 0$ ,  $\psi(S, a_t)$  is non-negative and increasing in S > 0, for any  $\tau \ge 0$ , we have

$$E[\psi(\tilde{S}_i, a_t)] = E[\psi(\tilde{S}_i, a_t)|S_i < \tau]P[\tilde{S}_i < \tau]$$
(23)

$$+ E[\psi(\tilde{S}_i, a_t)|\tilde{S}_i > \tau]P[\tilde{S}_i > \tau] \tag{24}$$

$$>\psi(\tau, a_t)P[\tilde{S}_i > \tau].$$
 (25)

Moreover, if  $\tau \leq 1$ ,  $\psi(\tau, a_t) \geq \psi(1, a_t)\tau^2$ , so for any  $\tau \in (0, 1)$ , we have

$$E[\psi(\tilde{S}_i, a_t)] \ge \psi(1, a_t) \tau^2 P[\tilde{S}_i \ge \tau].$$

On the other hand, from Lemma 1, we know that

$$P\left[\tilde{S}_i \ge 1 - \sqrt{\frac{P}{L}} - \sqrt{\frac{h}{L}}\right] \ge 1 - 2e^{-h/2}$$

for any  $0 \le h \le \sqrt{L} - \sqrt{P}$ . Thus taking  $\tau =$ 

$$\left(1-\sqrt{\frac{P}{L}}-\sqrt{\frac{h}{L}}\right)$$
, we have

$$\mathbb{E}[\psi(\tilde{S}_i, a_t)] \ge \psi(1, a_t) \left(1 - \sqrt{\frac{P}{L}} - \sqrt{\frac{h}{L}}\right)^2 \left(1 - 2e^{-h/2}\right)$$

Choosing  $h = \log L$ , assuming  $\log L \le \sqrt{L} - \sqrt{P}$ , we obtain the desired bound with

$$\gamma_L = 1 - \left(1 - \sqrt{\frac{P}{L}} - \sqrt{\frac{\log L}{L}}\right)^2 \left(1 - \frac{2}{\sqrt{L}}\right).$$

For large L,

$$\gamma_L \approx 2 \frac{\sqrt{P} + 1 + \sqrt{\log L}}{\sqrt{L}}$$

establishing that  $\gamma_L = O(\sqrt{\log L/L})$ .

Proof of Theorem 2. The arguments in the proof of Theorem 1 up to equation (22) apply here too, except that we need to substitute a generic regularization parameter  $\delta$  and consider that  $X_s$  now includes the past s samples instead of the past s samples. Also, in the case that  $\delta = 0$  matrix inverses are replaced by pseudo-inverses. Instead of (22), we have

$$E[a_s^{(\delta)}] = \frac{a_0}{L} E[\mathbf{tr}(A_t^2)] + \sigma_x^2 E[\mathbf{tr}(X_t(\delta I_s + X_t^T X_t)^{-2} X_t^T)],$$
(26)

where  $A_t$  is defined in (19). When  $s \leq L$ ,  $A_t$  has L-s eigenvalues equal to 1, and s eigenvalues equal to  $\frac{\delta}{\delta + S_t^2}$ , while the matrix  $X_t(\delta I_s + X_t^T X_t)^{-2} X_t^T$  has L-s eigenvalues equal to zero and the remaining s eigenvalues equal to  $\frac{S_t^2}{(\delta + S_s^2)^2}$ . Thus

$$\begin{split} \mathbf{E}[a_s^{(\delta)}] = & \frac{a_0}{L} \left( L - s + \sum_{i=1}^s \mathbf{E} \left[ \frac{\delta^2}{(\delta + S_i^2)^2} \right] \right) \\ & + \sigma_x^2 \sum_{i=1}^s \mathbf{E}[\frac{S_i^2}{(\delta + S_i^2)^2}] \\ = & a_0 (1 - \frac{s}{L}) + \sigma_x^2 \sum_{i=1}^s \mathbf{E} \left[ \frac{c_0 \delta^2 + S_i^2}{(\delta + S_i^2)^2} \right], \end{split}$$

where, as before, we have defined  $c_0 = a_0/(L\sigma_x^2)$ . By examing the first derivative, it may be easily verified that for  $S^2 > 0$  and c > 0, the function  $(c\delta^2 + S^2)/(\delta + S^2)^2$  has a unique minimum on  $\delta \geq 0$ , achieved at  $\delta = 1/c$ , and equal to  $1/(c^{-1} + S^2)$ . If  $S^2 = 0$ , the function is constant and equal to c; so the same bound holds in that case. Since  $\delta = c_0^{-1}$  is the minimizer regardess of the value of the singular value  $S_i$ ,

we have  $\mathrm{E}[a_s^{(\delta)}] \geq \mathrm{E}[a_s^{(c_0^{-1})}]$ . Further,

$$E[a_s^{(c_0^{-1})}] \ge a_0(1 - \frac{s}{L}) + \sigma_x^2 \sum_{i=1}^s E\left[\frac{1}{c_0^{-1} + S_i^2}\right]$$

$$\ge a_0(1 - \frac{s}{L}) + \sigma_x^2 s E\left[\frac{1}{c_0^{-1} + S_I^2}\right]$$

$$\ge a_0(1 - \frac{s}{L}) + \sigma_x^2 s \frac{1}{c_0^{-1} + L\sigma_x^2}$$

$$\ge a_0(1 - \frac{s}{L}) + \frac{s}{L} \frac{a_0}{a_0 + 1}$$

$$\ge a_0\left(1 - \frac{s}{L} + \frac{a_0}{1 + a_0}\right). \tag{28}$$

Here,  $S_I$  denotes a singular value chosen uniformly at random from among the s singular values, with  $\mathrm{E}[S_I^2] = L\sigma_x^2$ , and (27) is obtained by the Jensen's inequality.

For the case  $s \geq L$ ,  $A_s$  has L eigenvalues equal to  $\frac{\delta}{\delta + S_i^2}$  and  $X_t(\delta I_s + X_t^T X_t)^{-2} X_t^T$  has s - L eigenvalues equal to zero and the remaining L eigenvalues equal to  $\frac{S_i^2}{(\delta + S_s^2)^2}$ . Thus

$$\begin{split} \mathbf{E}[a_s^{(\delta)}] = & \frac{a_0}{L} \sum_{i=1}^L \mathbf{E} \left[ \frac{\delta^2}{(\delta + S_i^2)^2} \right] \\ &+ \sigma_x^2 \sum_{i=1}^L \mathbf{E} \left[ \frac{S_i^2}{(\delta + S_i^2)^2} \right] \\ = & \sigma_x^2 \sum_{i=1}^L \mathbf{E} \left[ \frac{c_0 \delta^2 + S_i^2}{(\delta + S_i^2)^2} \right]. \end{split}$$

Minimizing each term over  $\delta$  as before, we obtain

$$E[a_s^{(c_0^{-1})}] \ge \sigma_x^2 \sum_{i=1}^{L} E\left[\frac{1}{c_0^{-1} + S_i^2}\right]$$

$$\ge L\sigma_x^2 E\left[\frac{1}{c_0^{-1} + S_I^2}\right]$$

$$\ge L\sigma_x^2 \frac{1}{c_0^{-1} + s\sigma_x^2}$$

$$\ge \frac{La_0}{L + sa_0},$$
(29)

where we again use the Jensen's inequality, and in this case have  $\mathrm{E}[S_I^2] = s\sigma_x^2$ .

# VII. CONCLUSIONS

In this paper we have proposed GA-OBML and GA-IML, two novel algorithms for adaptive filtering. We have fully characterized the expected performance of GA-OBML and have shown that it is order-wise optimal. In our simulations, we explored the application of the proposed methods in AEC and showed that GA-IML consistently outperforms GA-OBML. Our theoretical result is developed for GA-OBML. Developing a similar theoretical characterization for GA-IML is also very interesting, and can shed light on the observed superiority of GA-IML compared to GA-OBML. Both GA-OBML and GA-IML in principle require to have access to the instantaneous estimation error  $\|\mathbf{w}_t - \mathbf{w}^*\|^2$ , which is not known in practice. However, in our simulation results we have

shown that known techniques such as delay and extrapolate can be employed to efficiently estimate the estimation error while minimally affecting the performance compared to the case where  $\|\mathbf{w}_t - \mathbf{w}^*\|^2$  is known accurately.

# APPENDIX A MEAN BEHAVIOR OF GA-OBML

**Lemma 2.** Consider the sequence  $a_{t+1} = f(a_t)$  with initial condition  $a_1 > 0$  and

$$f(a) = \left(1 - b\frac{a}{1+a}\right)a$$

for some 0 < b < 1. Then for any  $\alpha > 0$ , there exists T such that

$$(1-\alpha)(bt)^{-1} \le a_t \le (1+\alpha)(bt)^{-1}$$

for  $t \geq T$ .

*Proof.* For any a > 0, we have 0 < f(a) < a, so  $a_t$  is a decreasing sequence with limit point f(0) = 0. We will show that for any initial condition, eventually  $a_t \sim (bt)^{-1}$ .

The function can be expressed  $f(a)=(1-b)a+b\frac{a}{1+a}$ , hence  $f'(a)=1-b+\frac{b}{(1+a)^2}$  and we see that f is monotonically increasing for  $a\geq 0$ .

Lower bound. We will show that the sequence  $\gamma_t := a_t^{-1} - bt$  is decreasing. Then for the lower bound, given  $\alpha > 0$ , we set  $T \ge \gamma_1 (1 - \alpha)/(\alpha b)$ . For  $t \ge T$ , we have

$$bta_t = \frac{bt}{bt + \gamma_t} \geq \frac{bt}{bt + \gamma_1} \geq \frac{bT}{bT + \gamma_1} \geq 1 - \alpha.$$

To show that  $\gamma_t$  is decreasing, we use f(a) = (1 - b)a + ba/(1 + a), such that

$$a_{t+1} = f\left(\frac{1}{bt + \gamma_t}\right)$$

$$= \frac{1 - b}{bt + \gamma_t} + \frac{b}{bt + \gamma_t + 1}$$

$$= \frac{bt + \gamma_t + (1 - b)}{(bt + \gamma_t)(bt + \gamma_t + 1)}$$

Further algebraic manipulation then shows that

$$\gamma_{t+1} = a_{t+1}^{-1} - b(t+1) = \gamma_t - \frac{b(1-b)}{bt + \gamma_t + (1-b)} < \gamma_t.$$

Upper bound. Given  $a_t$ , we define the related sequence  $\alpha_t = bta_t - 1$ , so that  $a_t = (1 + \alpha_t)(bt)^{-1}$ . For the upper bound, for any initial condition  $a_1$  we construct a sequence  $\beta_t$  with  $\beta_t \geq \alpha_t$  and with  $\beta_t$  converging to zero.

First we establish a bound on  $\alpha_{t+1}$  given a bound  $a_t \leq$ 

 $(1+\beta_t)(bt)^{-1}$ . We have

$$\alpha_{t+1} = b(t+1)a_{t+1} - 1 = b(t+1)f(a_t) - 1$$

$$\leq b(t+1)f\left((1+\beta_t)(bt)^{-1}\right) - 1$$

$$\leq (1+\beta_t)\left[1 - b\frac{1+\beta_t}{bt+1+\beta_t}\right]\frac{t+1}{t} - 1$$

$$\leq (1+\beta_t)\left[\frac{bt+(1-b)(1+\beta_t)}{bt+1+\beta_t}\right]\frac{t+1}{t} - 1$$

$$\leq (1+\beta_t)\left[1 - \frac{bt\beta_t - (1-b)(1+\beta_t)}{t(bt+1+\beta_t)}\right] - 1$$

$$\leq \beta_t - \frac{(1+\beta_t)(bt\beta_t - (1-b)(1+\beta_t))}{t(bt+1+\beta_t)}$$

$$\leq \beta_t \left[1 - b\frac{1+\beta_t}{(bt+1+\beta_t)}\left(1 - \frac{(1-b)(1+\beta_t)}{bt\beta_t}\right)\right]$$

One consequence of this expression is that if we want our bound on  $\alpha_{t+1}$  to be less than our bound  $\alpha_t \leq \beta_t$ , we need  $(1-b)(1+\beta_t) < bt\beta_t$ , or equivalently  $\beta_t > (1-b)(bt-(1-b))^{-1}$ . Thus, we will design  $\beta_t$  for  $t > b^{-1}$  to meet the sufficient condition that  $\beta_t \geq (bt-1)^{-1}$ .

Suppose that for some  $t > b^{-1}$ ,  $\beta_t \ge \alpha_t$  and  $\beta_t \ge (bt - 1)^{-1}$ . Then  $\beta_t/(1+\beta_t) \ge (bt)^{-1}$  and  $(1+\beta_t)/(bt+1+\beta_t) \ge (bt)^{-1}$ . Our bound on  $\alpha_{t+1}$  then becomes

$$\alpha_{t+1} \leq \beta_t \left[ 1 - b \frac{1}{bt} \left( 1 - (1 - b) \right) \right]$$

$$\leq \beta_t \left( 1 - \frac{b}{t} \right) \tag{30}$$

To complete our definition of the sequence  $\beta_t$ , first suppose that  $\alpha_t \leq 1/(bt-1)$  for all  $t > b^{-1}$ . In this case, we can take  $\beta_t = bt(bt-1)^{-1} - 1 = (bt-1)^{-1}$  for all  $t > b^{-1}$ . Then  $\beta_t \geq \alpha_t$  and  $\beta_t \to 0$  as desired.

Otherwise, let T be the first  $T>b^{-1}$  with  $a_T>1/(bT-1)$ . We define  $\beta_T=\alpha_T$ , and for t>T, recursively define  $\beta_{t+1}=\max\left\{\beta_t\left(1-\frac{b}{t}\right)\,,\,\frac{1}{bt-1}\right\}$ . By virtue of (30), we have  $\beta_t\geq\alpha_t$  for all t>T.

To see that  $\beta_t \to 0$ , consider the related sequence defined by  $\beta_{t+1}^* = \beta_t^*(1-b/t)$ . Taking logs,  $\log(\beta_{t+1}^*) = \log(\beta_t^*) + \log(1-b/t) \approx \log(\beta_t^*) - b/t$  for sufficiently large t. Since the sum of 1/t is infinite,  $\beta_t^*$  decreases without bound.

For any  $\alpha > 0$ , recursively define

$$\beta_{t+1}^{(\alpha)} = \max \left\{ \beta_t^{(\alpha)} \left( 1 - \frac{b}{t} \right), \, \frac{1}{bt - 1}, \, \alpha \right\}.$$

Clearly  $\beta_t^{(\alpha)} \geq \beta_t$ , and  $\beta_t^{(\alpha)} \to \alpha$  as  $t \to \infty$ . Since this is true for any  $\alpha > 0$ , we have  $\beta_t \to 0$ .

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