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Two-sided Sequent Calculi for *FDE*-like Four-valued Logics

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Abstract
We present a method that generates two-sided sequent calculi for four-valued logics like *first degree entailment* (*FDE*). (We say that a logic is *FDE-like* if it has finitely many operators of finite arity, including negation, and if all of its operators are truth-functional over the four truth-values 'none', 'false', 'true', and 'both', where 'true' and 'both' are designated.) First, we show that for every n-ary operator $\star$ every truth table entry $f_\star(x_1, \ldots, x_n) = y$ can be characterized in terms of a pair of sequent rules. Secondly, we use these sequent rules to build sequent calculi and prove their completeness. With the help of two simplification procedures we then show that the $2 \cdot 4^n$ sequent rules that characterize an n-ary operator can be systematically reduced to at most four sequent rules. Thirdly, we use our method to investigate the proof-theoretical consequences of including intuitive truth-functional implications in *FDE*-like logics.

Keywords Proof theory · Sequent calculi · Many-valued logic · First degree entailment

1 Introduction

Proof-theoretical investigations of many-valued logics have led to a wealth of proof systems and methods to generate them. The development of these methods and systems was triggered by different criteria promoting different theoretical objectives.

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In this paper we study proof systems for four-valued logics from the standpoint of comparability and modularity.

As to comparability, note that if we wish to tell the differences between alternative concepts of logical consequence or between alternative rules for particular operators, it is often convenient to think of a logic as a specification of a set of valid arguments, where an argument consists of a set of premises and a conclusion. A two-sided representation then comes naturally. Nonetheless, because two-sided sequent calculi for many-valued logics cannot be canonical [4, Theorem 3],[1] the basic building blocks of almost all sequent calculi or tableau systems for m-valued logics have been hypersequents, m-valued sequents, or labelled sequents.[2] As useful as they may be, some theoretical objectives can be reached better without them. For instance, if we wish to compare alternative rules for intuitive four-valued implications (we will do so in Section 6), a two-sided representation is most helpful. In this paper we therefore propose to investigate truth-functional four-valued logics using two-sided sequents of the form Γ/Δ.

In particular, we show that for a specific class of four-valued logics it is always possible to generate two-sided sequent calculi such that any operator is characterized with at most four sequent rules. This class consists of the four-valued logics that are like *first degree entailment* (*FDE*) [1, 11]. We say that a logic is *FDE-like* if it has finitely many operators of finite arity, including negation, and if all of its operators are truth-functional over the four truth-values 'none', 'false', 'true', and 'both', where 'true' and 'both' are designated. (Note that there are also other truth-functional four-valued logics.) We present a general method to generate such two-sided sequent calculi for *FDE*-like four-valued logics and hence make it possible to directly compare the sequent calculi thus generated with two-sided Gentzen-style sequent calculi for alternative logics.

As to modularity, note that an *FDE*-like four-valued logic is usually not functionally complete: some truth functions cannot be defined in terms of negation and its other operators. Given a sequent system for an *FDE*-like four-valued logic, it is therefore no trivial matter to come up with sequent rules for additional truth-functional operators. There is a natural way to tackle this problem: use a sequent calculus for a functionally complete extension of *FDE* (for instance, the one proposed by Arieli and Avron [2, § 3]) to obtain sequent rules for truth-functional operators that can not be

[1] A sequent calculus is *canonical* if (i) its axioms include the standard axioms, (ii) its rules include the standard ones including cut and weakening, and (iii) all its other rules and axioms are canonical. An m-premise rule is *canonical* if its conclusion is of the form $\Gamma, \star(A_1, \ldots, A_n)/\Delta$ or $\Gamma/\star(A_1, \ldots, A_n), \Delta$ where Γ and Δ are finite and if for all of its premises $\Gamma, \Pi_i/\Delta, \Sigma_i$ $(0 \leq i \leq m)$ it holds that $\Pi_i, \Sigma_i \subseteq \{A_1, \ldots, A_n\}$ [4, pp. 121–122]. (Note that we use a slash (/) where most proof theorists would use an arrow ($\Rightarrow$).)

[2] A *hypersequent* G is a finite sequence $\Gamma_1/\Delta_1; \ldots; \Gamma_n/\Delta_n$ of sequents Γ_i/Δ_i, where Γ_i and Δ_i are finite multisets of formulas. See Avron [3] for an accessible introduction to hypersequent calculi. An *m-valued sequent* Γ is an m-tuple $\Gamma_1/\ldots/\Gamma_m$ of finite multisets Γ_i of formulas. See Zach [27, § 3.1] for a solid review of m-valued sequent calculi. A *labelled sequent* Γ is a finite multiset of labelled formulas of the type $l : A$, where the label l is usually interpreted as a truth-value or a set of truth-values. See Carnielli [10], Baaz et al. [6], and Caleiro et al. [9] for general methods for obtaining labelled calculi for m-valued logics.

defined in terms of *FDE*'s negation, disjunction, and conjunction. This is not what we will do however: we directly provide sequent calculi for *FDE*-like four-valued logics that are not functionally complete. To do so, we use the correspondences between single truth table entries and proof rules that were studied by Kooi and Tamminga [14].[3] Accordingly, we first characterize any truth-functional operator in terms of a set of sequent rules. We then add, operator by operator, the operator's characterizing sequent rules to a basic sequent calculus that deals only with negation. This method is modular: no matter how many operators are thus included, the resulting sequent calculus will always be complete with respect to its particular semantics.

Our paper proceeds as follows. First, we present a semantical definition of the kind of truth-functional four-valued logics to be studied and characterize any truth table entry of any n-ary operator with two sequent rules. Second, we use these characterizing sequent rules to define cut-free sequent systems and show that each of the resulting systems is complete with respect to its particular four-valued semantics. With the help of two simplification procedures we then show that the $2 \cdot 4^n$ sequent rules that characterize any n-ary operator can be systematically reduced to at most four sequent rules. We illustrate our simplification procedures by applying them to *FDE*'s truth tables for disjunction and conjunction and show that we thus obtain standard sequent rules. Third, we use our method to investigate the proof-theoretical consequences of including intuitive truth-functional implications in *FDE*-like logics. Lastly, we discuss related work by Baaz et al. [6] and Wintein and Muskens [26] that shares some of our theoretical objectives but proposes different ways to realize them. A short summary and an open question conclude the paper.

2 Correspondence Analysis for *FDE*-Like Four-Valued Logics

2.1 *FDE*-Like Four-Valued Logics

Belnap [8, p. 43] discusses the concept of validity for *first degree entailment* (*FDE*). His remarks imply that an argument Γ/Δ from a set Γ of premises to a set Δ of conclusions is **FDE**-*valid* (notation: $\Gamma \models_{\mathbf{FDE}} \Delta$) if it preserves both truth and non-falsity. An argument Γ/Δ *preserves truth* if for every valuation v it holds that if $v(A)$ is 'true' or 'both' for all A in Γ, then $v(B)$ is 'true' or 'both' for some B in Δ. An argument Γ/Δ *preserves non-falsity* if for every valuation v it holds that if $v(A)$ is 'none' or 'true' for all A in Γ, then $v(B)$ is 'none' or 'true' for some B in Δ. Because an argument Γ/Δ preserves non-falsity if and only if the argument $\{\neg B : B \in \Delta\}/\{\neg A : A \in \Gamma\}$ preserves truth, and because every four-valued logic that we consider in this paper has negation, we can characterize **FDE**-validity by focusing on truth-preservation alone.

[3] Thus far, research on correspondences between *single* truth table entries and proof rules has focused on proof rules in natural deduction systems. Kooi and Tamminga [14] studied the *logic of paradox* (*LP*), Tamminga [24] *strong three-valued logic* (K_3), and Petrukhin [18] *first degree entailment* (*FDE*).

Accordingly, we say that a logic is *FDE*-like if it has finitely many operators of finite arity, including negation, and if all of its operators are truth-functional over the four truth-values 'none', 'false', 'true', and 'both', where 'true' and 'both' are designated. *FDE*-like logics form an infinitely large family whose members share a common trait: they all have the same *negation*. (Hence, if a truth-functional four-valued logic does not have negation, it is not *FDE*-like.) To be specific, an *FDE*-like logic **L4** evaluates arguments consisting of formulas from a propositional language $\mathscr{L}$ built from a set $\mathscr{P} = \{p, p', \ldots\}$ of atomic formulas, using negation ($\neg$) and finitely many additional truth-functional operators of finite arity. (To obtain the propositional language of *FDE*, just add two binary operators: disjunction ($\vee$) and conjunction ($\wedge$).)

Moreover, for any *FDE*-like logic **L4**, a valuation is a function v from the set $\mathscr{P}$ of atomic formulas to the set $\{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$ of truth-values 'none', 'false', 'true', and 'both'. We use the following boldface shorthands: $\boldsymbol{n}$ abbreviates $\emptyset$, $\mathbf{0}$ abbreviates $\{0\}$, $\mathbf{1}$ abbreviates $\{1\}$, and $\boldsymbol{b}$ abbreviates $\{0, 1\}$. A valuation v on $\mathscr{P}$ is extended recursively to a valuation on $\mathscr{L}$ by the truth-conditions for negation and the truth-conditions for the finitely many additional operators of finite arity. The truth table $f_\neg$ gives the truth-conditions for negation:

$f_\neg$	
$\boldsymbol{n}$	$\boldsymbol{n}$
$\mathbf{0}$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{0}$
$\boldsymbol{b}$	$\boldsymbol{b}$

An argument Γ/Δ from a set Γ of premises to a set Δ of conclusions is **L4**-*valid* (notation: $\Gamma \models_{\mathbf{L4}} \Delta$) if and only if for every valuation v it holds that if $1 \in v(A)$ for all A in Γ, then $1 \in v(B)$ for some B in Δ. Consequently, $\mathbf{1}$ and $\boldsymbol{b}$ are the *designated* values: these are the values that are preserved in **L4**-valid arguments.[4] Lastly, note that an argument Γ/Δ is **FDE**-valid in Belnap's original sense if and only if $\Gamma \models_{\mathbf{L4}} \Delta$ (it preserves truth) and $\{\neg B : B \in \Delta\} \models_{\mathbf{L4}} \{\neg A : A \in \Gamma\}$ (it preserves non-falsity).

Two remarks on terminology and notation: when the context does not give rise to ambiguities, we also say that an argument Γ/Δ is valid if it is **L4**-valid and write '$\Gamma \models \Delta$' rather than '$\Gamma \models_{\mathbf{L4}} \Delta$'. In addition, the text includes two small abuses of notation (the one is not the negation of the other): we write '$1 \in v(\Gamma)$' instead of 'for all $A \in \Gamma$ it holds that $1 \in v(A)$'; and '$1 \notin v(\Delta)$' instead of 'for all $B \in \Delta$ it holds that $1 \notin v(B)$'.

[4] An interesting and important variant of **L4**-validity is Pietz and Rivieccio's [20] *exactly true logic* (*ETL*). This four-valued logic is also truth-functional over $\boldsymbol{n}$, $\mathbf{0}$, $\mathbf{1}$, and $\boldsymbol{b}$, but its only designated value is $\mathbf{1}$. Accordingly, an argument Γ/Δ is **ETL**-*valid* if and only if for every valuation v it holds that if $v(A) = \mathbf{1}$ for all A in Γ, then $v(B) = \mathbf{1}$ for some B in Δ. Wintein and Muskens [26] provide an elegant sequent calculus for *ETL*. A direct application of our method to *ETL*-like logics is impossible (the *ETL*-counterpart of Lemma 1 holds, but the *ETL*-counterpart of Lemma 3 fails). Nonetheless, our method can be used to study single conclusion *ETL*-like logics, because the following implication holds: if both $A_1, \ldots, A_n \models_{\mathbf{L4}} B$ and $\neg B \models_{\mathbf{L4}} \neg A_1, \ldots, \neg A_n$, then $A_1, \ldots, A_n \models_{\mathbf{ETL}} B$.

Note that the omnipresence of negation makes a two-sided representation of *FDE*-like logics possible. In fact, by sorting out the negated formulas left and right of the slash, any two-sided sequent Γ/Δ can be seen as a four-sided one $\Gamma_1/\Gamma_2/\Gamma_3/\Gamma_4$. If we think of the four indices in a four-sided sequent as labels, any four-sided sequent $\Gamma_1/\Gamma_2/\Gamma_3/\Gamma_4$ can be seen as a single set of labelled formulas.

2.2 Correspondences

Any truth table entry of any truth-functional n-ary operator can be characterized with a pair of sequent rules. To prove this, we first give a definition that generates a pair of sequent rules from any truth table entry $f_\star(x_1, \ldots, x_n) = y$ of any truth-functional n-ary operator $\star$, where all the x_i's and y are truth-values in $\{\boldsymbol{n}, \mathbf{0}, \mathbf{1}, \boldsymbol{b}\}$. We then show that the pair of sequent rules thus obtained characterizes the truth table entry from which it was generated.

Let us first fix a language $\mathscr{L}$ built from a set $\mathscr{P} = \{p, p', \ldots\}$ of atomic formulas, using negation ($\neg$) and finitely many operators of finite arity. Our characterization result quickly follows from three modest observations:

Lemma 1 *Let* $\Gamma, \Delta \subseteq \mathscr{L}$ *and* $A \in \mathscr{L}$. *Then*

$\Gamma \models \Delta, A$ *iff* *for every* v *such that* $1 \in v(\Gamma)$ *and* $1 \notin v(\Delta)$ *it holds that* $1 \in v(A)$

$\Gamma, A \models \Delta$ *iff* *for every* v *such that* $1 \in v(\Gamma)$ *and* $1 \notin v(\Delta)$ *it holds that* $1 \notin v(A)$.

Proof Immediately from the definition of **L4**-validity. □

Our second straightforward observation is that the truth-value $v(A)$ of a formula A under a valuation v can be characterized by specifying, first, whether $v(A)$ is designated and, secondly, whether $v(\neg A)$ is designated:

Lemma 2 *Let* $A \in \mathscr{L}$ *and let* v *be a valuation. Then*

$v(A) = \boldsymbol{n}$ *iff* $1 \notin v(A)$ *and* $1 \notin v(\neg A)$
$v(A) = \mathbf{0}$ *iff* $1 \notin v(A)$ *and* $1 \in v(\neg A)$
$v(A) = \mathbf{1}$ *iff* $1 \in v(A)$ *and* $1 \notin v(\neg A)$
$v(A) = \boldsymbol{b}$ *iff* $1 \in v(A)$ *and* $1 \in v(\neg A)$.

Proof Immediately from the truth table $f_\neg$ for negation. □

To state the third observation, we introduce two operations $A|_z^+$ and $A|_z^-$, where A is a formula and z is a truth-value. Given a sequent Γ/Δ, the operator $A|_z^+$ places the formula A, depending on the truth-value z, on the left-hand side or the right-hand side of Γ/Δ. The operator $A|_z^-$ does the same thing to the negation of A:

Definition 1 Let $\Gamma, \Delta \subseteq \mathscr{L}$ and $A \in \mathscr{L}$. Let $z \in \{\boldsymbol{n}, \mathbf{0}, \mathbf{1}, \boldsymbol{b}\}$. Then $A|_z^+$ and $A|_z^-$ are the following operations:

$$A|_z^+(\Gamma/\Delta) = \begin{cases} \Gamma, A/\Delta, & \text{if } z \in \{\boldsymbol{n}, \mathbf{0}\} \\ \Gamma/\Delta, A, & \text{otherwise} \end{cases}$$

$$A|_z^-(\Gamma/\Delta) \quad = \quad \begin{cases} \Gamma, \neg A/\Delta, & \text{if } z \in \{\boldsymbol{n}, \mathbf{1}\} \\ \Gamma/\Delta, \neg A, & \text{otherwise.} \end{cases}$$

Our third observation is the following:

Lemma 3 *Let* $\Gamma, \Delta \subseteq \mathscr{L}$ *and* $A \in \mathscr{L}$. *Let* $z \in \{\boldsymbol{n}, \mathbf{0}, \mathbf{1}, \boldsymbol{b}\}$. *Then*

$A|_z^+(\Gamma/\Delta)$ *and* $A|_z^-(\Gamma/\Delta)$ *are valid if and only if for every* v *such that* $1 \in v(\Gamma)$ *and* $1 \notin v(\Delta)$ *it holds that* $v(A) = z$.

Proof Immediately from Lemmas 1 and 2. □

These three observations enable us to characterize any truth table entry in the truth table $f_\star$ of any n-ary operator $\star$ in terms of two sequent rules:

Definition 2 Let $\star$ be an n-ary operator and let $f_\star(x_1, \ldots, x_n) = y$ be a truth table entry. Then $R^{\star+}_{x_1 \ldots x_n y}$ and $R^{\star-}_{x_1 \ldots x_n y}$ are the following sequent rules:

$$\frac{A_1|_{x_1}^+(\Gamma/\Delta) \qquad A_1|_{x_1}^-(\Gamma/\Delta) \qquad \cdots \qquad A_n|_{x_n}^+(\Gamma/\Delta) \qquad A_n|_{x_n}^-(\Gamma/\Delta)}{\star(A_1, \ldots, A_n)|_y^+(\Gamma/\Delta)} \; R^{\star+}_{x_1 \ldots x_n y}$$

$$\frac{A_1|_{x_1}^+(\Gamma/\Delta) \qquad A_1|_{x_1}^-(\Gamma/\Delta) \qquad \cdots \qquad A_n|_{x_n}^+(\Gamma/\Delta) \qquad A_n|_{x_n}^-(\Gamma/\Delta)}{\star(A_1, \ldots, A_n)|_y^-(\Gamma/\Delta)} \; R^{\star-}_{x_1 \ldots x_n y}.$$

To illustrate this definition, consider *FDE*'s truth table entry $f_\vee(\boldsymbol{n}, \boldsymbol{b}) = \mathbf{1}$. The corresponding sequent rules $R^{\vee+}_{\boldsymbol{nb}\mathbf{1}}$ and $R^{\vee-}_{\boldsymbol{nb}\mathbf{1}}$ are the following:

$$\frac{\Gamma, A/\Delta \qquad \Gamma, \neg A/\Delta \qquad \Gamma/\Delta, B \qquad \Gamma/\Delta, \neg B}{\Gamma/\Delta, A \vee B} \; R^{\vee+}_{\boldsymbol{nb}\mathbf{1}}$$

$$\frac{\Gamma, A/\Delta \qquad \Gamma, \neg A/\Delta \qquad \Gamma/\Delta, B \qquad \Gamma/\Delta, \neg B}{\Gamma, \neg(A \vee B)/\Delta} \; R^{\vee-}_{\boldsymbol{nb}\mathbf{1}}.$$

Theorem 1 *Let* $\star$ *be an* n*-ary operator. Then*
$f_\star(x_1, \ldots, x_n) = y$ *iff* $R^{\star+}_{x_1 \ldots x_n y}$ *and* $R^{\star-}_{x_1 \ldots x_n y}$ *are validity-preserving.*

Proof ($\Rightarrow$) Assume that $f_\star(x_1, \ldots, x_n) = y$. Suppose $A_i|_{x_i}^+(\Gamma/\Delta)$ and $A_i|_{x_i}^-(\Gamma/\Delta)$ are valid for all i with $1 \le i \le n$. Then, by Lemma 3, for every v such that $1 \in v(\Gamma)$ and $1 \notin v(\Delta)$ it holds that $v(A_i) = x_i$ for all i with $1 \le i \le n$. Then, because $f_\star(x_1, \ldots, x_n) = y$, it must be that for every v such that $1 \in v(\Gamma)$ and $1 \notin v(\Delta)$ it holds that $v(\star(A_1, \ldots, A_n)) = y$. Then, again by Lemma 3, $\star(A_1, \ldots, A_n)|_y^+(\Gamma/\Delta)$ and $\star(A_1, \ldots, A_n)|_y^-(\Gamma/\Delta)$ are valid. Therefore, $R^{\star+}_{x_1 \ldots x_n y}$ and $R^{\star-}_{x_1 \ldots x_n y}$ are validity-preserving.

($\Leftarrow$) Assume that $R^{\star+}_{x_1 \ldots x_n y}$ and $R^{\star-}_{x_1 \ldots x_n y}$ are validity-preserving. Let $p_1, \ldots, p_n$ be logically independent atomic formulas and let v' be an arbitrary valuation such that $v'(p_i) = x_i$ for all i with $1 \le i \le n$. Given the valuation v' and $p_i \in \{p_1, \ldots, p_n\}$, we define two sets of literals, Γ' and Δ', as follows:

$$\begin{array}{lll} \Gamma' & = & \{p_i : 1 \in v'(p_i)\} \cup \{\neg p_i : 0 \in v'(p_i)\} \\ \Delta' & = & \{p_i : 1 \notin v'(p_i)\} \cup \{\neg p_i : 0 \notin v'(p_i)\}. \end{array}$$

Two facts about Γ' and Δ' are to be noted. First, $1 \in v'(\Gamma')$ and $1 \notin v'(\Delta')$. Second, $p_i|^+_{x_i}(\Gamma'/\Delta')$ and $p_i|^-_{x_i}(\Gamma'/\Delta')$ are both valid for all i with $1 \leq i \leq n$.

By assumption, $R^{\star+}_{x_1\ldots x_n y}$ and $R^{\star-}_{x_1\ldots x_n y}$ are validity-preserving. Hence, both $\star(p_1, \ldots, p_n)|^+_y(\Gamma'/\Delta')$ and $\star(p_1, \ldots, p_n)|^-_y(\Gamma'/\Delta')$ are valid. By Lemma 3, for every v such that $1 \in v(\Gamma')$ and $1 \notin v(\Delta')$ it holds that $v(\star(p_1, \ldots, p_n)) = y$. Hence, $v'(\star(p_1, \ldots, p_n)) = y$. Because v' was arbitrary, we conclude that for every v such that $v(p_i) = x_i$ for all i with $1 \leq i \leq n$ it holds that $v(\star(p_1, \ldots, p_n)) = y$. Therefore, $f_\star(x_1, \ldots, x_n) = y$. □

Consequently, every n-ary operator can be characterized in terms of $2 \cdot 4^n$ sequent rules. We use these characterizing sequent rules to generate cut-free sequent calculi and to prove the completeness of these calculi with respect to their particular semantics.

3 Sequent Calculi

We now define sequent calculi for each and any four-valued logic **L4** with negation and finitely many truth-functional additional operators of finite arity. Sequents are of the form Γ/Δ, where per usual both Γ and Δ are finite, possibly empty, multisets of formulas. If there is a proof of the sequent Γ/Δ in the sequent system **G4**, then we write $\Gamma \vdash_{\mathbf{G4}} \Delta$.[5] The core of each of the calculi in this paper is the calculus $\mathbf{G4}^{\neg}$, a basic sequent calculus for four-valued negation, consisting of an axiom, left and right rules for negation, left and right contraction rules, and left and right weakening rules.

The rules of $\mathbf{G4}^{\neg}$ are the following:[6]

Logical axiom:

$\Gamma, A/\Delta, A$

Logical rules:

$$\frac{\Gamma, A/\Delta}{\Gamma, \neg\neg A/\Delta}\, L^{\neg} \qquad \frac{\Gamma/\Delta, A}{\Gamma/\Delta, \neg\neg A}\, R^{\neg}$$

Structural rules:

$$\frac{\Gamma, A, A/\Delta}{\Gamma, A/\Delta}\, LC \qquad \frac{\Gamma/\Delta, A, A}{\Gamma/\Delta, A}\, RC$$

$$\frac{\Gamma/\Delta}{\Gamma, A/\Delta}\, LW \qquad \frac{\Gamma/\Delta}{\Gamma/\Delta, A}\, RW.$$

Note that $\mathbf{G4}^{\neg}$ has no cut.

[5] The family of sequent calculi for four-valued logics that we refer to with the generic name **G4** is not to be confused with Kleene's ([13], p. 306) calculus *G*4 or the calculus **G4ip** (Troelstra and Schwichtenberg [25], p. 102; Negri and von Plato [17], p. 122).

[6] The notational conventions are similar to the ones in Negri and von Plato [17].

To obtain a sequent calculus for a four-valued logic **L4** based on negation and finitely many additional truth-functional operators of finite arity, just add to $\mathbf{G4}^{\neg}$ for each additional n-ary operator $\star$ the $2 \cdot 4^n$ rules that are generated from its truth table $f_\star$ by Definition 2.

4 Completeness

Let **G4** be a sequent system for a four-valued logic **L4** based on negation and finitely many additional truth-functional operators of finite arity. To prove the completeness of **G4** with respect to **L4**, we suppose that $\Gamma \nvdash_{\mathbf{G4}} \Delta$ and then, to show that $\Gamma \not\models_{\mathbf{L}4} \Delta$, construct on the basis of our supposition a valuation $v^{\leafNE}$ such that $1 \in v^{\leafNE}(\Gamma)$ and $1 \notin v^{\leafNE}(\Delta)$. Our completeness proof is similar to Priest's [21] completeness proof for a tableau system for *FDE*.

The valuation $v^{\leafNE}$ is constructed as follows: we first build a proof tree with root Γ/Δ and apply, going upwards, all the available logical rules (these are $L^{\neg}$, $R^{\neg}$, and the $2 \cdot 4^n$ characterizing $\star$-rules of any additional n-ary operator $\star$) until every applicable logical rule has been applied. Because there might be different $\star$-rules that apply to the same conclusion, we require that our proof tree be monotonic. A proof tree is *monotonic* if and only if it holds that if Γ'/Δ' is a child of Γ''/Δ'', then $\Gamma' \subseteq \Gamma''$ and $\Delta' \subseteq \Delta''$. Note that every proof tree can be transformed into a monotonic one by repeated applications of the contraction rules *LC* and *RC*. (The two weakening rules *LW* and *RW* do not play a role in the completeness proof – in fact, our logical axiom suffices to simulate them.)

If every applicable logical rule has been applied, there must be an open and complete branch of our proof tree with root Γ/Δ and leaf $\Gamma^{\leafNE}/\Delta^{\leafNE}$, because otherwise we would have, contrary to our supposition, that $\Gamma \vdash_{\mathbf{G4}} \Delta$. A branch with leaf $\Gamma^{\leafNE}/\Delta^{\leafNE}$ is *open* if and only if $\Gamma^{\leafNE}/\Delta^{\leafNE}$ is not an axiom (that is, $\Gamma^{\leafNE} \cap \Delta^{\leafNE} = \emptyset$). A branch with leaf $\Gamma^{\leafNE}/\Delta^{\leafNE}$ is *complete* if and only if every rule that is applicable to $\Gamma^{\leafNE}/\Delta^{\leafNE}$ has already been applied. We use this leaf $\Gamma^{\leafNE}/\Delta^{\leafNE}$ of our open and complete branch to define a valuation $v^{\leafNE}$ from $\mathcal{P}$ to $\{\boldsymbol{n}, \mathbf{0}, \mathbf{1}, \boldsymbol{b}\}$:

Definition 3 Let $\Gamma^{\leafNE}/\Delta^{\leafNE}$ be the leaf of a proof tree with root Γ/Δ. The valuation $v^{\leafNE}$ induced by $\Gamma^{\leafNE}/\Delta^{\leafNE}$ is defined as follows:
$v^{\leafNE}(p) = \boldsymbol{n}$ iff $p \notin \Gamma^{\leafNE}$ and $\neg p \notin \Gamma^{\leafNE}$
$v^{\leafNE}(p) = \mathbf{0}$ iff $p \notin \Gamma^{\leafNE}$ and $\neg p \in \Gamma^{\leafNE}$
$v^{\leafNE}(p) = \mathbf{1}$ iff $p \in \Gamma^{\leafNE}$ and $\neg p \notin \Gamma^{\leafNE}$
$v^{\leafNE}(p) = \boldsymbol{b}$ iff $p \in \Gamma^{\leafNE}$ and $\neg p \in \Gamma^{\leafNE}$.

Lemma 4 *Let $\Gamma^{\leafNE}/\Delta^{\leafNE}$ be the leaf of an open and complete branch of a monotonic proof tree with root Γ/Δ. Let $v^{\leafNE}$ be induced by $\Gamma^{\leafNE}/\Delta^{\leafNE}$. Let $A \in \mathscr{L}$. Then*
(i) *If $A \in \Gamma^{\leafNE}$, then $1 \in v^{\leafNE}(A)$*
(ii) *If $A \in \Delta^{\leafNE}$, then $1 \notin v^{\leafNE}(A)$*
(iii) *If $\neg A \in \Gamma^{\leafNE}$, then $0 \in v^{\leafNE}(A)$*
(iv) *If $\neg A \in \Delta^{\leafNE}$, then $0 \notin v^{\leafNE}(A)$.*

Proof By induction on A. By definition of $v^{\varnothing}$ and because $\Gamma^{\varnothing}/\Delta^{\varnothing}$ is open, the lemma holds for atomic formulas.

Suppose that the lemma holds for all formulas with fewer operators than A. We show that the lemma holds for $A = \neg B$ and $A = \star(B_1, \ldots, B_n)$.

- Suppose $A = \neg B$. There are four cases.
  - (i) Suppose $\neg B \in \Gamma^{\varnothing}$. Then $0 \in v^{\varnothing}(B)$. Hence, $1 \in v^{\varnothing}(\neg B)$.
  - (ii) Suppose $\neg B \in \Delta^{\varnothing}$. Then $0 \notin v^{\varnothing}(B)$. Hence, $1 \notin v^{\varnothing}(\neg B)$.
  - (iii) Suppose $\neg\neg B \in \Gamma^{\varnothing}$. Then the rule $L^{\neg}$ is applicable, and because our open branch is complete it must have been applied. Hence, the premise of the rule $L^{\neg}$ must be a node of our open and complete branch. Hence, $\Gamma', B/\Delta'$ is a node of our open and complete branch for some Γ' and some Δ'. By monotonicity, $\Gamma' \cup \{B\} \subseteq \Gamma^{\varnothing}$ and hence $B \in \Gamma^{\varnothing}$. By the Induction Hypothesis, $1 \in v^{\varnothing}(B)$. Hence, $0 \in v^{\varnothing}(\neg B)$.
  - (iv) Suppose $\neg\neg B \in \Delta^{\varnothing}$. Analogous to (iii). Use $R^{\neg}$.
- Suppose $A = \star(B_1, \ldots, B_n)$. There are four cases.
  - (i) Suppose $\star(B_1, \ldots, B_n) \in \Gamma^{\varnothing}$. Let $v^{\varnothing}(B_i) = x_i$ for all i with $1 \leq i \leq n$ and let $v^{\varnothing}(\star(B_1, \ldots, B_n)) = y$. Suppose $1 \notin v^{\varnothing}(\star(B_1, \ldots, B_n))$. Then $1 \notin f_\star(v^{\varnothing}(B_1), \ldots, v^{\varnothing}(B_n))$ and hence $1 \notin f_\star(x_1, \ldots, x_n) = y$. The positive rule $R^{\star+}_{x_1 \ldots x_n y}$ and the negative rule $R^{\star-}_{x_1 \ldots x_n y}$ are the ones that correspond to the truth table entry $f_\star(x_1, \ldots, x_n) = y$. Because $1 \notin y$, the conclusion of the positive rule $R^{\star+}_{x_1 \ldots x_n y}$ is $\Gamma', \star(B_1, \ldots, B_n)/\Delta'$. Because $\star(B_1, \ldots, B_n) \in \Gamma^{\varnothing}$, the rule $R^{\star+}_{x_1 \ldots x_n y}$ is applicable, and because our open branch is complete it must have been applied. Hence, one of the premisses of the rule $R^{\star+}_{x_1 \ldots x_n y}$ must be a node of our open and complete branch. Hence, $B_i|^{+}_{x_i}(\Gamma'/\Delta')$ or $B_i|^{-}_{x_i}(\Gamma'/\Delta')$ is a node of our open and complete branch for some i with $1 \leq i \leq n$. There are four subcases:
    - (1) Suppose $B_i|^{+}_{x_i}(\Gamma'/\Delta') = \Gamma', B_i/\Delta'$. By definition of $B_i|^{+}_{x_i}(\Gamma'/\Delta')$, it must be that $x_i \in \{\boldsymbol{n}, \mathbf{0}\}$. By monotonicity, $\Gamma' \cup \{B_i\} \subseteq \Gamma^{\varnothing}$ and hence $B_i \in \Gamma^{\varnothing}$. By the Induction Hypothesis, $1 \in v^{\varnothing}(B_i)$. Hence, $v^{\varnothing}(B_i) \neq x_i$. Contradiction.
    - (2) Suppose $B_i|^{+}_{x_i}(\Gamma'/\Delta') = \Gamma'/\Delta', B_i$. By definition of $B_i|^{+}_{x_i}(\Gamma'/\Delta')$, it must be that $x_i \in \{\mathbf{1}, \boldsymbol{b}\}$. By monotonicity, $\Delta' \cup \{B_i\} \subseteq \Delta^{\varnothing}$ and hence $B_i \in \Delta^{\varnothing}$. By the Induction Hypothesis, $1 \notin v^{\varnothing}(B_i)$. Hence, $v^{\varnothing}(B_i) \neq x_i$. Contradiction.
    - (3) Suppose $B_i|^{-}_{x_i}(\Gamma'/\Delta') = \Gamma', \neg B_i/\Delta'$. By definition of $B_i|^{-}_{x_i}(\Gamma'/\Delta')$, it must be that $x_i \in \{\boldsymbol{n}, \mathbf{1}\}$. By monotonicity, $\Gamma' \cup \{\neg B_i\} \subseteq \Gamma^{\varnothing}$ and hence $\neg B_i \in \Gamma^{\varnothing}$. By the Induction Hypothesis, $0 \in v^{\varnothing}(B_i)$. Hence, $v^{\varnothing}(B_i) \neq x_i$. Contradiction.
    - (4) Suppose $B_i|^{-}_{x_i}(\Gamma'/\Delta') = \Gamma'/\Delta', \neg B_i$. By definition of $B_i|^{-}_{x_i}(\Gamma'/\Delta')$, it must be that $x_i \in \{\mathbf{0}, \boldsymbol{b}\}$. By monotonicity, $\Delta' \cup \{\neg B_i\} \subseteq \Delta^{\varnothing}$ and hence $\neg B_i \in \Delta^{\varnothing}$. By the Induction Hypothesis, $0 \notin v^{\varnothing}(B_i)$. Hence, $v^{\varnothing}(B_i) \neq x_i$. Contradiction.

Because all subcases lead to a contradiction, $1 \in v^{\leaf}(\star(B_1, \ldots, B_n))$.

(ii) Suppose $\star(B_1, \ldots, B_n) \in \Delta^{\leaf}$. Analogous to (i).

(iii) Suppose $\neg\star(B_1, \ldots, B_n) \in \Gamma^{\leaf}$. Analogous to (i). Use $R^{\star-}_{x_1 \ldots x_n y}$.

(iv) Suppose $\neg\star(B_1, \ldots, B_n) \in \Delta^{\leaf}$. Analogous to (iii).

□

Theorem 2 (Completeness) *Let* $\Gamma, \Delta \subseteq \mathscr{L}$ *be finite sets of formulas. Then*
$\Gamma \vdash_{\mathbf{G4}} \Delta$ *iff* $\Gamma \models_{\mathbf{L4}} \Delta$.

Proof ($\Rightarrow$) By induction on the height of the derivation. The local soundness of the axiom and the sequent rules of $\mathbf{G4}^{\neg}$ is easy to establish. The local soundness of the additional sequent rules follows from Theorem 1.

($\Leftarrow$) By contraposition. Suppose $\Gamma \nvdash_{\mathbf{G4}} \Delta$. Then there is a monotonic tree with root Γ/Δ and an open and complete branch with leaf $\Gamma^{\leaf}/\Delta^{\leaf}$. By Lemma 4, the valuation $v^{\leaf}$ induced by $\Gamma^{\leaf}/\Delta^{\leaf}$ ensures that $1 \in v^{\leaf}(\Gamma^{\leaf})$ and $1 \notin v^{\leaf}(\Delta^{\leaf})$. By monotonicity, $\Gamma \subseteq \Gamma^{\leaf}$ and $\Delta \subseteq \Delta^{\leaf}$. Therefore $\Gamma \not\models_{\mathbf{L4}} \Delta$. □

5 Two Simplification Procedures

Definition 2 generates a clutter of different sequent rules with the same conclusion. For instance, if $f_\star(x_1, \ldots, x_n) = y$ and $f_\star(x'_1, \ldots, x'_n) = y'$ such that $1 \notin y \cup y'$, then the positive rules $R^{\star+}_{x_1 \ldots x_n y}$ and $R^{\star+}_{x'_1 \ldots x'_n y'}$ have the same conclusion $\Gamma, \star(A_1, \ldots, A_n)/\Delta$. We prove that the $2{\cdot}4^n$ sequent rules that characterize any n-ary operator can be systematically reduced to at most four sequent rules.[7]

To do so, we first define the product of two rules with the same conclusion and show that the product rule thus obtained is exactly as powerful as the two rules together. Because the sequent rules generated by Definition 2 have at most four different conclusions, product rules systematically reduce the $2 \cdot 4^n$ sequent rules that characterize an n-ary operator to at most four sequent rules. The at most four sequent rules thus obtained might have at most n^n premise sequents. Second, we define an ordering relation on sequents and use it to minimize the number of premise sequents of a sequent rule. We show that the minimized sequent rule is exactly as powerful as the sequent rule that was minimized. Third, we illustrate our two simplification procedures by applying them to the truth tables $f_\vee$ and $f_\wedge$ of *FDE* and show that we thus obtain standard sequent rules for disjunction and conjunction.

In the remainder of this section, we have to be somewhat pedantic about our use of variables: we use the Γ's and Δ's as variables over multisets of formulas (this is just usual practice) but the Π's and Σ's as variables over *sets* of formulas.

[7] Compare Salzer's ([23], § 4) method of minimizing operator rules and Avron and Konikowska's ([5], § 3.3) method of 'streamlining' deduction systems.

Definition 4 (Product) Let R_1 be an m-premise rule and R_2 an n-premise rule with the same conclusion $\Gamma, \Pi/\Delta, \Sigma$, where $m \geq 1$ and $n \geq 1$:

$$\frac{\Gamma, \Pi'_1/\Delta, \Sigma'_1 \qquad \cdots \qquad \Gamma, \Pi'_m/\Delta, \Sigma'_m}{\Gamma, \Pi/\Delta, \Sigma}\ R_1$$

$$\frac{\Gamma, \Pi''_1/\Delta, \Sigma''_1 \qquad \cdots \qquad \Gamma, \Pi''_n/\Delta, \Sigma''_n}{\Gamma, \Pi/\Delta, \Sigma}\ R_2$$

Then the *product of* R_1 *and* R_2, denoted by $R_1 \times R_2$, is the following $m \cdot n$ premise rule:

$$\frac{\Gamma, \Pi'_1, \Pi''_1/\Delta, \Sigma'_1, \Sigma''_1 \qquad \cdots \qquad \Gamma, \Pi'_m, \Pi''_n/\Delta, \Sigma'_m, \Sigma''_n}{\Gamma, \Pi/\Delta, \Sigma}\ R_1 \times R_2.$$

If either R_1 or R_2 is a zero-premise rule, then $R_1 \times R_2$ is the zero-premise rule with the same conclusion $\Gamma, \Pi/\Delta, \Sigma$.

Lemma 5 *Let* $\mathbf{G4}(R_1, R_2)$ *be a proof system that contains two rules* R_1 *and* R_2 *with the same conclusion. Let* $\mathbf{G4}(R_1 \times R_2)$ *be the proof system that contains the product rule* $R_1 \times R_2$ *instead of the rules* R_1 *and* R_2*. Let* $\Gamma, \Delta \subseteq \mathscr{L}$*. Then*

$$\Gamma \vdash_{\mathbf{G4}(R_1, R_2)} \Delta \quad \textit{iff} \quad \Gamma \vdash_{\mathbf{G4}(R_1 \times R_2)} \Delta.$$

Proof ($\Rightarrow$) We show that any application of the rule R_1 can be replaced by repeated applications of the weakening rules and an application of the product rule $R_1 \times R_2$. Suppose the rule R_1 has been applied to the premises $\Gamma, \Pi'_i/\Delta, \Sigma'_i$ ($1 \leq i \leq m$) to obtain the conclusion $\Gamma, \Pi/\Delta, \Sigma$. By repeated applications of the weakening rules *LW* and *RW* to the premises of the rule R_1, we obtain the premises $\Gamma, \Pi'_i, \Pi''_j/\Delta, \Sigma'_i, \Sigma''_j$ ($1 \leq i \leq m$ and $1 \leq j \leq n$) of the product rule $R_1 \times R_2$. Finally, we apply the product rule $R_1 \times R_2$ to these new premises and obtain the conclusion $\Gamma, \Pi/\Delta, \Sigma$. Similarly, any application of the rule R_2 can be replaced by repeated applications of the weakening rules and an application of the product rule.

($\Leftarrow$) We show that any application of the product rule $R_1 \times R_2$ can be replaced by repeated applications of the rules R_1 and R_2. Suppose the product rule $R_1 \times R_2$ has been applied to the premises $\Gamma, \Pi'_i, \Pi''_j/\Delta, \Sigma'_i, \Sigma''_j$ ($1 \leq i \leq m$ and $1 \leq j \leq n$) to obtain the conclusion $\Gamma, \Pi/\Delta, \Sigma$. For every j with $1 \leq j \leq n$ we have m premises $\Gamma, \Pi'_i, \Pi''_j/\Delta, \Sigma'_i, \Sigma''_j$ ($1 \leq i \leq m$), to which we apply the rule R_1 to obtain the conclusion $\Gamma, \Pi, \Pi''_j/\Delta, \Sigma, \Sigma''_j$. In this way, we obtain n conclusions $\Gamma, \Pi, \Pi''_j/\Delta, \Sigma, \Sigma''_j$ ($1 \leq j \leq n$), to which we apply the rule R_2 to obtain the sequent $\Gamma, \Pi, \Pi/\Delta, \Sigma, \Sigma$. By repeated applications of the contraction rules *LC* and *RC* we obtain the desired conclusion $\Gamma, \Pi/\Delta, \Sigma$. □

The product rule of an m premise rule and an n premise rule has $m \cdot n$ premises. This number can be reduced significantly by minimizing the product rule's premises. We define minimization on the basis of the following ordering: $\Gamma, \Pi'/\Delta, \Sigma' \preceq \Gamma, \Pi''/\Delta, \Sigma''$ if and only if $\Pi' \subseteq \Pi''$ and $\Sigma' \subseteq \Sigma''$. (Note that the Π's and Σ's are variables over *sets* of formulas.)

Definition 5 (Minimization) Let R be a sequent rule. Then the *minimization of* R, denoted by $\min(R)$, is the sequent rule with the same conclusion as R and a premise set that exactly contains those $\preceq$-minimal elements of the premise set of R that are not axioms.

To illustrate this definition, consider the sequent rules $R^{\vee+}_{\boldsymbol{n}\mathbf{11}}$ and $R^{\vee+}_{\boldsymbol{nb}\mathbf{1}}$:

$$\frac{\Gamma, A/\Delta \qquad \Gamma, \neg A/\Delta \qquad \Gamma/\Delta, B \qquad \Gamma, \neg B/\Delta}{\Gamma/\Delta, A \vee B}\ R^{\vee+}_{\boldsymbol{n}\mathbf{11}}$$

$$\frac{\Gamma, A/\Delta \qquad \Gamma, \neg A/\Delta \qquad \Gamma/\Delta, B \qquad \Gamma/\Delta, \neg B}{\Gamma/\Delta, A \vee B}\ R^{\vee+}_{\boldsymbol{nb}\mathbf{1}}.$$

By multiplying these two rules we obtain the product rule $R^{\vee+}_{\boldsymbol{n}\mathbf{11}} \times R^{\vee+}_{\boldsymbol{nb}\mathbf{1}}$, which has sixteen premises. After minimization only three premises survive:[8]

$$\frac{\Gamma, A/\Delta \qquad \Gamma, \neg A/\Delta \qquad \Gamma/\Delta, B}{\Gamma/\Delta, A \vee B}\ \min(R^{\vee+}_{\boldsymbol{n}\mathbf{11}} \times R^{\vee+}_{\boldsymbol{nb}\mathbf{1}}).$$

Lemma 6 *Let* $\mathbf{G4}(R)$ *be a proof system that contains the rule* R. *Let* $\mathbf{G4}(\min(R))$ *be the proof system that contains the rule* $\min(R)$ *instead of the rule* R. *Let* $\Gamma, \Delta \subseteq \mathscr{L}$. *Then*

$\Gamma \vdash_{\mathbf{G4}(R)} \Delta \quad$ *iff* $\quad \Gamma \vdash_{\mathbf{G4}(\min(R))} \Delta$.

Proof ($\Rightarrow$) We show that any application of the rule R can be replaced by an application of the rule $\min(R)$. Suppose the rule R has been applied to obtain a conclusion Π/Σ. Because every premise of $\min(R)$ is a premise of R, we obtain the conclusion Π/Σ also by applying $\min(R)$ to those premises of R that survive minimization. Moreover, we discard every subtree issuing in a premise of R that did not survive minimization.

($\Leftarrow$) We show that any application of the rule $\min(R)$ can be replaced by additional axioms, repeated applications of the weakening rules, and an application of the rule R. Suppose the rule $\min(R)$ has been applied to obtain a conclusion Π/Σ. For every premise Γ''/Δ'' of R it holds that either (i) Γ''/Δ'' is a premise of $\min(R)$, (ii) Γ''/Δ'' is not a premise of $\min(R)$, but $\Gamma'' \cap \Delta'' \neq \emptyset$, or (iii) Γ''/Δ'' is not a premise of $\min(R)$, but there is a premise Γ'/Δ' of $\min(R)$ such that $\Gamma' \subseteq \Gamma''$ and $\Delta' \subseteq \Delta''$. The premises of type (ii) are logical axioms. The premises of type (iii) are obtained from the relevant premises of $\min(R)$ by repeated applications of the weakening rules *LW* and *RW*. Finally, we obtain the conclusion Π/Σ by applying R to the premises of the types (i), (ii), and (iii) thus obtained. □

We use minimized product rules to characterize any n-ary operator with at most four sequent rules.

Definition 6 Let $f_\star$ be the truth table of an n-ary operator $\star$. Then

[8] We conjecture that in every case the number of premises of $\min(R)$ is between 0 and $4 \cdot n^2 - 2 \cdot n$, where n is the arity of the operator that is introduced in the conclusion of R.

$$
\begin{array}{lcl}
l^{\star+} & = & \min(\times\{R^{\star+}_{x_1\ldots x_n y} : 1 \notin y\}) \\
r^{\star+} & = & \min(\times\{R^{\star+}_{x_1\ldots x_n y} : 1 \in y\}) \\
l^{\star-} & = & \min(\times\{R^{\star-}_{x_1\ldots x_n y} : 0 \notin y\}) \\
r^{\star-} & = & \min(\times\{R^{\star-}_{x_1\ldots x_n y} : 0 \in y\}).
\end{array}
$$

If there are no rules of a particular kind, then the relevant minimized product rule is undefined.

To conclude this section, let us illustrate our two simplification procedures by applying them to the truth tables $f_\vee$ and $f_\wedge$ of *FDE* and show that we thus obtain standard sequent rules for disjunction and conjunction. *FDE*'s truth tables $f_\vee$ for disjunction and $f_\wedge$ for conjunction are the following:

$f_\vee$	**n**	**0**	**1**	**b**
n	**n**	**n**	**1**	**1**
0	**n**	**0**	**1**	**b**
1	**1**	**1**	**1**	**1**
b	**1**	**b**	**1**	**b**

$f_\wedge$	**n**	**0**	**1**	**b**
n	**n**	**0**	**n**	**0**
0	**0**	**0**	**0**	**0**
1	**n**	**0**	**1**	**b**
b	**0**	**0**	**b**	**b**

Lemma 7 *Let $f_\vee$ be FDE's truth table for disjunction. Then $l^{\vee+}$, $r^{\vee+}$, $l^{\vee-}$ and $r^{\vee-}$ are the following rules:*

$$\frac{\Gamma, A/\Delta \qquad \Gamma, B/\Delta}{\Gamma, A \vee B/\Delta}\, l^{\vee+} \qquad \frac{\Gamma/\Delta, A, B}{\Gamma/\Delta, A \vee B}\, r^{\vee+}$$

$$\frac{\Gamma, \neg A, \neg B/\Delta}{\Gamma, \neg(A \vee B)/\Delta}\, l^{\vee-} \qquad \frac{\Gamma/\Delta, \neg A \qquad \Gamma/\Delta, \neg B}{\Gamma/\Delta, \neg(A \vee B)}\, r^{\vee-}.$$

Proof Use Definition 2 to generate the 32 sequent rules that characterize $\vee$. Then calculate $l^{\vee+}$, $r^{\vee+}$, $l^{\vee-}$ and $r^{\vee-}$. Hint: first multiply and minimize pairs of rules, then multiply and minimize the resulting minimized product rules. □

Lemma 8 *Let $f_\wedge$ be FDE's truth table for conjunction. Then $l^{\wedge+}$, $r^{\wedge+}$, $l^{\wedge-}$ and $r^{\wedge-}$ are the following rules:*

$$\frac{\Gamma, A, B/\Delta}{\Gamma, A \wedge B/\Delta}\, l^{\wedge+} \qquad \frac{\Gamma/\Delta, A \qquad \Gamma/\Delta, B}{\Gamma/\Delta, A \wedge B}\, r^{\wedge+}$$

$$\frac{\Gamma, \neg A/\Delta \qquad \Gamma, \neg B/\Delta}{\Gamma, \neg(A \wedge B)/\Delta}\, l^{\wedge-} \qquad \frac{\Gamma/\Delta, \neg A, \neg B}{\Gamma/\Delta, \neg(A \wedge B)}\, r^{\wedge-}.$$

Proof Analogous to the proof of Lemma 7. □

Finally, we add the new rules for disjunction and the new rules for conjunction to the basic sequent calculus $\mathbf{G4}^{\neg}$ that only contains a logical axiom, logical rules for negation, and the structural rules of contraction and weakening. We use $\mathbf{G4}^{\neg\vee\wedge}$ to refer to the resulting sequent calculus.[9] If the language $\mathscr{L}$ is built from atomic

[9]Note that $\mathbf{G4}^{\neg\vee\wedge}$ for *FDE* is similar to the sequent calculi given by Pynko [22, pp. 446–447], Arieli and Avron [2, p. 109], and Beall [7, § A.3], and to Priest's [21] tableau system.

formulas using only $\neg$, $\vee$, and $\wedge$, the preservation of truth and the preservation of non-falsity coincide.[10] Consequently, $\mathbf{G4}^{\neg\vee\wedge}$ is sound and complete for *FDE* in Belnap's original sense:

Theorem 3 (Sequent calculus for *FDE*) *Let* $\Gamma, \Delta \subseteq \mathscr{L}$. *Then*
$\Gamma \vdash_{\mathbf{G4}^{\neg\vee\wedge}} \Delta$ *iff* $\Gamma \models_{\mathbf{FDE}} \Delta$.

Proof Immediately from Theorem 2 and Lemmas 5, 6, 7, and 8. □

In sum, our method not only generates two-sided sequent calculi for *FDE*-like four-valued logics, but also generates sequent calculi with *at most four* sequent rules for each n-ary operator. It does so in two steps. (Recall that a logic is *FDE*-like, if it contains finitely many n-ary operators, including negation, and if all of its operators are truth-functional over the four truth-values $\boldsymbol{n}$, $\mathbf{0}$, $\mathbf{1}$, and $\boldsymbol{b}$, where $\mathbf{1}$ and $\boldsymbol{b}$ are designated.) In the first step, each truth table entry of each of the relevant n-ary operators is characterized by a pair of sequent rules. In the second step, we add these sequent rules to the basic sequent calculus $\mathbf{G4}^{\neg}$ and simplify them systematically using (i) our product rule that combines two given rules with the same conclusion, and (ii) our minimization of the premise set of a given rule. In this way, we systematically reduce the sequent calculus we found first to a sequent calculus with at most four sequent rules for each n-ary operator. We illustrated our method by automatically producing a sequent calculus for *FDE* that is very similar to the ones that have been presented in the literature. Accordingly, finding an elegant sequent calculus for a given *FDE*-like four-valued logic is a process that can be fully automated.

6 Implications

We now apply our method to investigate the proof-theoretical consequences of including intuitive truth-functional implications in *FDE*-like four-valued logics. To do so, we start from Dunn's [11] observation that the conditions under which a formula is *false* can be distinguished from the conditions under which that formula is *not true*, and that these conditions are logically independent. Dunn uses this observation to present an intuitive semantics for *FDE* by giving both a truth-condition and a falsity-condition for its operators of negation, disjunction, and conjunction. We use Dunn's observation to provide intuitive semantics for implication by giving both a truth-condition and a falsity-condition. There are four natural ways to give a truth-condition for implication:

$(\supset^1_+)$	$A \supset B$ is true	*iff*	if A is true, then B is true.
$(\supset^2_+)$	$A \supset B$ is true	*iff*	if A is true, then B is not false.
$(\supset^3_+)$	$A \supset B$ is true	*iff*	if A is not false, then B is true.
$(\supset^4_+)$	$A \supset B$ is true	*iff*	if A is not false, then B is not false.

[10] See Dunn [12, Proposition 4] for an accessible proof.

Likewise, there are four natural ways to give a falsity-condition for implication:

$(\supset^1_-)$	$A \supset B$ is false	*iff*	A is true and B is false.
$(\supset^2_-)$	$A \supset B$ is false	*iff*	A is true and B is not true.
$(\supset^3_-)$	$A \supset B$ is false	*iff*	A is not false and B is false.
$(\supset^4_-)$	$A \supset B$ is false	*iff*	A is not false and B is not true.

Let us say that a formula is *true* if its truth-value is either $\mathbf{1}$ or $\boldsymbol{b}$, that a formula is *not true* if its truth-value is either $\boldsymbol{n}$ or $\mathbf{0}$, that a formula is *false* if its truth-value is either $\mathbf{0}$ or $\boldsymbol{b}$, and that a formula is *not false* if its truth-value is either $\boldsymbol{n}$ or $\mathbf{1}$. We use the following boldface shorthand notations: $\mathbf{T}$ abbreviates $\{\mathbf{1}, \boldsymbol{b}\}$, $\mathbf{F}$ abbreviates $\{\mathbf{0}, \boldsymbol{b}\}$, $\overline{\mathbf{T}}$ abbreviates $\{\boldsymbol{n}, \mathbf{0}\}$, and $\overline{\mathbf{F}}$ abbreviates $\{\boldsymbol{n}, \mathbf{1}\}$. (Note that any intersection of an element of $\{\mathbf{T}, \overline{\mathbf{T}}\}$ and an element of $\{\mathbf{F}, \overline{\mathbf{F}}\}$ is a singleton.)

Using the boldface shorthand notations, the four truth-conditions for implication give rise to the following four generalized truth tables:

$f^1_{\supset+}$	$\mathbf{T}$	$\overline{\mathbf{T}}$
$\mathbf{T}$	$\mathbf{T}$	$\overline{\mathbf{T}}$
$\overline{\mathbf{T}}$	$\mathbf{T}$	$\mathbf{T}$

$f^2_{\supset+}$	$\mathbf{F}$	$\overline{\mathbf{F}}$
$\mathbf{T}$	$\overline{\mathbf{T}}$	$\mathbf{T}$
$\overline{\mathbf{T}}$	$\mathbf{T}$	$\mathbf{T}$

$f^3_{\supset+}$	$\mathbf{T}$	$\overline{\mathbf{T}}$
$\mathbf{F}$	$\mathbf{T}$	$\mathbf{T}$
$\overline{\mathbf{F}}$	$\mathbf{T}$	$\overline{\mathbf{T}}$

$f^4_{\supset+}$	$\mathbf{F}$	$\overline{\mathbf{F}}$
$\mathbf{F}$	$\mathbf{T}$	$\mathbf{T}$
$\overline{\mathbf{F}}$	$\overline{\mathbf{T}}$	$\mathbf{T}$

Likewise, the four falsity-conditions for implication give rise to the following four generalized falsity tables:

$f^1_{\supset-}$	$\mathbf{F}$	$\overline{\mathbf{F}}$
$\mathbf{T}$	$\mathbf{F}$	$\overline{\mathbf{F}}$
$\overline{\mathbf{T}}$	$\overline{\mathbf{F}}$	$\overline{\mathbf{F}}$

$f^2_{\supset-}$	$\mathbf{T}$	$\overline{\mathbf{T}}$
$\mathbf{T}$	$\overline{\mathbf{F}}$	$\mathbf{F}$
$\overline{\mathbf{T}}$	$\overline{\mathbf{F}}$	$\overline{\mathbf{F}}$

$f^3_{\supset-}$	$\mathbf{F}$	$\overline{\mathbf{F}}$
$\mathbf{F}$	$\overline{\mathbf{F}}$	$\overline{\mathbf{F}}$
$\overline{\mathbf{F}}$	$\mathbf{F}$	$\overline{\mathbf{F}}$

$f^4_{\supset-}$	$\mathbf{T}$	$\overline{\mathbf{T}}$
$\mathbf{F}$	$\overline{\mathbf{F}}$	$\overline{\mathbf{F}}$
$\overline{\mathbf{F}}$	$\overline{\mathbf{F}}$	$\mathbf{F}$

Given a generalized table $f_{\supset\circ}$ where $\circ \in \{+, -\}$, we use $\mathscr{L}(f_{\supset\circ})$ to refer to the set of its left input values and $\mathscr{R}(f_{\supset\circ})$ to refer to the set of its right input values. For instance, $\mathscr{L}(f^2_{\supset+}) = \{\mathbf{T}, \overline{\mathbf{T}}\}$ and $\mathscr{R}(f^2_{\supset+}) = \{\mathbf{F}, \overline{\mathbf{F}}\}$.

There are sixteen ways to combine the generalized truth and falsity tables $f^i_{\supset+}$ and $f^j_{\supset-}$ to obtain a truth table $f^{i,j}_{\supset}$ for implication: for all $i, j \in \{1, 2, 3, 4\}$ and for all $x, y, z \in \{\boldsymbol{n}, \mathbf{0}, \mathbf{1}, \boldsymbol{b}\}$ we stipulate that

$$f^{i,j}_{\supset}(x, y) = z \quad \textit{iff} \quad f^i_{\supset+}(X^+, Y^+) \cap f^j_{\supset-}(X^-, Y^-) = \{z\},$$

where $x \in X^+ \in \mathscr{L}(f^i_{\supset+})$ and $y \in Y^+ \in \mathscr{R}(f^i_{\supset+})$ and $x \in X^- \in \mathscr{L}(f^j_{\supset-})$ and $y \in Y^- \in \mathscr{R}(f^j_{\supset-})$. For instance, we get $f^{2,3}_{\supset}(\mathbf{0}, \boldsymbol{b}) = \mathbf{1}$, because $\mathbf{0} \in \overline{\mathbf{T}} \in \mathscr{L}(f^2_{\supset+})$ and $\boldsymbol{b} \in \mathbf{F} \in \mathscr{R}(f^2_{\supset+})$ and $\mathbf{0} \in \mathbf{F} \in \mathscr{L}(f^3_{\supset-})$ and $\boldsymbol{b} \in \mathbf{F} \in \mathscr{R}(f^3_{\supset-})$ and $f^2_{\supset+}(\overline{\mathbf{T}}, \mathbf{F}) \cap f^3_{\supset-}(\mathbf{F}, \mathbf{F}) = \mathbf{T} \cap \overline{\mathbf{F}} = \{\mathbf{1}\}$. In this way, we obtain sixteen different truth tables. Obviously, the method we developed in the previous sections can be used to find the sequent rules that characterize any of these sixteen truth tables. With a slight adaption, however, it can do much more.

We first note that our boldface shorthand notations $\mathbf{T}$, $\overline{\mathbf{T}}$, $\mathbf{F}$, and $\overline{\mathbf{F}}$ are obviously related to the truth-values 0 and 1:

Lemma 9 *Let $A \in \mathscr{L}$ and let v be a valuation. Then*

$$\begin{array}{lll} 1 \in v(A) & \textit{iff} & v(A) \in \mathbf{T} \\ 1 \notin v(A) & \textit{iff} & v(A) \in \overline{\mathbf{T}} \\ 0 \in v(A) & \textit{iff} & v(A) \in \mathbf{F} \\ 0 \notin v(A) & \textit{iff} & v(A) \in \overline{\mathbf{F}}. \end{array}$$

Another small abuse of notation: we write '$v(\Gamma) \in \mathbf{T}$' instead of 'for all $A \in \Gamma$ it holds that $v(A) \in \mathbf{T}$', and similarly for the other boldface shorthands.

Definition 7 Let $\Gamma, \Delta \subseteq \mathscr{L}$ and $A \in \mathscr{L}$. Let $Z \in \{\mathbf{T}, \overline{\mathbf{T}}, \mathbf{F}, \overline{\mathbf{F}}\}$. Then $A|_Z$ is the following operation:

$$A|_Z(\Gamma/\Delta) \quad = \quad \begin{cases} \Gamma/\Delta, A, & \text{if } Z = \mathbf{T} \\ \Gamma, A/\Delta, & \text{if } Z = \overline{\mathbf{T}} \\ \Gamma/\Delta, \neg A, & \text{if } Z = \mathbf{F} \\ \Gamma, \neg A/\Delta, & \text{if } Z = \overline{\mathbf{F}}. \end{cases}$$

Lemma 10 *Let $\Gamma, \Delta \subseteq \mathscr{L}$ and $A \in \mathscr{L}$. Let $Z \in \{\mathbf{T}, \overline{\mathbf{T}}, \mathbf{F}, \overline{\mathbf{F}}\}$. Then*

$A|_Z(\Gamma/\Delta)$ is valid if and only if for every v such that $v(\Gamma) \in \mathbf{T}$ and $v(\Delta) \in \overline{\mathbf{T}}$ it holds that $v(A) \in Z$.

Proof There are four cases.

Case $Z = \mathbf{T}$. Then $A|_Z(\Gamma/\Delta) = \Gamma/\Delta, A$. ($\Rightarrow$) Suppose $\Gamma \models \Delta, A$. Take an arbitrary valuation v. Suppose $v(\Gamma) \in \mathbf{T}$ and $v(\Delta) \in \overline{\mathbf{T}}$. Then it must be that $v(A) \in \mathbf{T}$. ($\Leftarrow$) Suppose for every v such that $v(\Gamma) \in \mathbf{T}$ and $v(\Delta) \in \overline{\mathbf{T}}$ it holds that $v(A) \in \mathbf{T}$. Then it must be that $\Gamma \models \Delta, A$.

The other three cases are proved similarly. □

Definition 8 Let $\supset_\circ$ with $\circ \in \{+, -\}$ be one of our eight generalized tables and let $X_1, X_2, Y \in \{\mathbf{T}, \overline{\mathbf{T}}, \mathbf{F}, \overline{\mathbf{F}}\}$ be such that $f_{\supset_\circ}(X_1, X_2) = Y$. Then $R^{\supset_\circ}_{X_1 X_2 Y}$ is the following sequent rule:

$$\frac{A|_{X_1}(\Gamma/\Delta) \qquad B|_{X_2}(\Gamma/\Delta)}{A \supset B|_Y(\Gamma/\Delta)} \; R^{\supset_\circ}_{X_1 X_2 Y}.$$

Theorem 4 *Let $\supset_\circ$ with $\circ \in \{+, -\}$ be one of our eight generalized tables and let $X_1, X_2, Y \in \{\mathbf{T}, \overline{\mathbf{T}}, \mathbf{F}, \overline{\mathbf{F}}\}$. Then*

$$f_{\supset_\circ}(X_1, X_2) = Y \quad \textit{iff} \quad R^{\supset_\circ}_{X_1 X_2 Y} \textit{ is validity-preserving.}$$

Proof ($\Rightarrow$) Assume that $f_{\supset_\circ}(X_1, X_2) = Y$. Suppose $A|_{X_1}(\Gamma, \Delta)$ and $B|_{X_2}(\Gamma, \Delta)$ are valid. Then, by Lemma 10, for every v such that $v(\Gamma) \in \mathbf{T}$ and $v(\Delta) \in \overline{\mathbf{T}}$ it holds that $v(A) \in X_1$ and $v(B) \in X_2$. Then, because $f_{\supset_\circ}(X_1, X_2) = Y$, it must be that for every v such that $v(\Gamma) \in \mathbf{T}$ and $v(\Delta) \in \overline{\mathbf{T}}$, it holds that $v(A \supset B) \in Y$. Then, again by Lemma 10, $(A \supset B)|_Y(\Gamma, \Delta)$ is valid. Therefore, $R^{\supset_\circ}_{X_1 X_2 Y}$ is validity-preserving.

($\Leftarrow$) Assume that $R^{\supset_\circ}_{X_1 X_2 Y}$ is validity-preserving. Let p_1 and p_2 be logically independent atomic formulas, let $X_1, X_2 \in \{\mathbf{T}, \overline{\mathbf{T}}, \mathbf{F}, \overline{\mathbf{F}}\}$, and let v' be an arbitrary valuation such that $v'(p_1) \in X_1$ and $v'(p_2) \in X_2$. Given the valuation v' and $p_i \in \{p_1, p_2\}$, we define two sets of literals, Γ' and Δ', as follows:

$$\begin{array}{lcl}\Gamma' & = & \{p_i : v'(p_i) \in \mathbf{T}\} \cup \{\neg p_i : v'(p_i) \in \mathbf{F}\} \\ \Delta' & = & \{p_i : v'(p_i) \in \overline{\mathbf{T}}\} \cup \{\neg p_i : v'(p_i) \in \overline{\mathbf{F}}\}.\end{array}$$

Two facts about Γ' and Δ' are to be noted. First, $v'(\Gamma') \in \mathbf{T}$ and $v'(\Delta') \in \overline{\mathbf{T}}$. Secondly, $p_1|_{X_1}(\Gamma', \Delta')$ and $p_2|_{X_2}(\Gamma', \Delta')$ are valid.

By assumption, $R^{\supset_\circ}_{X_1 X_2 Y}$ is validity-preserving. Hence, $(p_1 \supset p_2)|_Y(\Gamma'/\Delta')$ is valid. By Lemma 10, for every v such that $v(\Gamma') \in \mathbf{T}$ and $v(\Delta') \in \overline{\mathbf{T}}$ it holds that $v(p_1 \supset p_2) \in Y$. Hence, $v'(p_1 \supset p_2) \in Y$. Because v' was arbitrary, we conclude that for every v such that $v(p_1) \in X_1$ and $v(p_2) \in X_2$ it holds that $v(p_1 \supset p_2) \in Y$. Therefore, $f_{\supset_\circ}(X_1, X_2) = Y$. □

Lemma 11 *Let $f^i_{\supset_+}$ with $i \in \{1, 2, 3, 4\}$ be one of our four generalized truth tables. The tables are characterized by the following rules:*

$$\frac{\Gamma/\Delta, A \qquad \Gamma, B/\Delta}{\Gamma, A \supset B/\Delta}\, l^{\supset^1_+} \qquad \frac{\Gamma, A/\Delta, B}{\Gamma/\Delta, A \supset B}\, r^{\supset^1_+}$$

$$\frac{\Gamma/\Delta, A \qquad \Gamma/\Delta, \neg B}{\Gamma, A \supset B/\Delta}\, l^{\supset^2_+} \qquad \frac{\Gamma, A, \neg B/\Delta}{\Gamma/\Delta, A \supset B}\, r^{\supset^2_+}$$

$$\frac{\Gamma, \neg A/\Delta \qquad \Gamma, B/\Delta}{\Gamma, A \supset B/\Delta}\, l^{\supset^3_+} \qquad \frac{\Gamma/\Delta, \neg A, B}{\Gamma/\Delta, A \supset B}\, r^{\supset^3_+}$$

$$\frac{\Gamma, \neg A/\Delta \qquad \Gamma/\Delta, \neg B}{\Gamma, A \supset B/\Delta}\, l^{\supset^4_+} \qquad \frac{\Gamma, \neg B/\Delta, \neg A}{\Gamma/\Delta, A \supset B}\, r^{\supset^4_+}.$$

Proof Use Definition 8 to generate the four rules that characterize $f^i_{\supset_+}$. Then repeatedly use the product rule and the minimization rule to calculate $l^{\supset^i_+}$ and $r^{\supset^i_+}$. □

Lemma 12 *Let* $\mathbf{G4}^{\neg}(l^{\supset^m_+}, r^{\supset^n_+})$ *be the proof system that results from adding the rules* $l^{\supset^m_+}$ *and* $r^{\supset^n_+}$ *with* $m, n \in \{1, 2, 3, 4\}$ *to the basic sequent calculus* $\mathbf{G4}^{\neg}$. *Then*

$$\mathbf{G4}^{\neg}(l^{\supset^m_+}, r^{\supset^n_+}) \textit{ is sound} \quad \textit{iff} \quad m = n.$$

Proof ($\Rightarrow$) Assume that $m \neq n$. There are twelve cases.

Case $m = 1$ and $n = 2$. It is easy to see that $p, p \supset q/q, \neg q$ and $p/q, \neg q, p \supset q$ are derivable when both $l^{\supset^1_+}$ and $r^{\supset^2_+}$ are available. Suppose that $p, p \supset q \models q, \neg q$ and $p \models q, \neg q, p \supset q$. Because p and q are independent, there is a valuation v such that $v(p) = \mathbf{1}$ and $v(q) = \boldsymbol{n}$. Because $p \models q, \neg q, p \supset q$, it must be that

$1 \in v(p \supset q)$. Because $p, p \supset q \models q, \neg q$, it must be that $1 \in v(q)$. Contradiction. Therefore, $\mathbf{G4}^{\neg}(l^{\supset^1_+}, r^{\supset^2_+})$ is not sound.

Case $m = 1$ and $n = 3$. It is easy to see that $p, \neg p, p \supset q/q$ and $p, \neg p/q, p \supset q$ are derivable when both $l^{\supset^1_+}$ and $r^{\supset^3_+}$ are available. Suppose that $p, \neg p, p \supset q \models q$ and $p, \neg p \models q, p \supset q$. Because p and q are independent, there is a valuation v such that $v(p) = \boldsymbol{b}$ and $v(q) = \mathbf{0}$. Because $p, \neg p \models q, p \supset q$, it must be that $1 \in v(p \supset q)$. Because $p, \neg p, p \supset q \models q$, it must be that $1 \in v(q)$. Contradiction. Therefore, $\mathbf{G4}^{\neg}(l^{\supset^1_+}, r^{\supset^3_+})$ is not sound.

Case $m = 1$ and $n = 4$. It is easy to see that (i) $p, p \supset q/q, \neg q$ and $p/q, \neg q, p \supset q$ and (ii) $p, \neg p, p \supset q/q$ and $p, \neg p/q, p \supset q$ are derivable when both $l^{\supset^1_+}$ and $r^{\supset^4_+}$ are available. From either of the two previous cases it follows that $\mathbf{G4}^{\neg}(l^{\supset^1_+}, r^{\supset^4_+})$ is not sound.

The other nine cases are proved similarly.

($\Leftarrow$) Assume that $m = n$. From Lemma 11 it follows that $\mathbf{G4}^{\neg}(l^{\supset^m_+}, r^{\supset^n_+})$ is sound. □

Lemma 13 *Let $f^j_{\supset_-}$ with $j \in \{1, 2, 3, 4\}$ be one of our four generalized falsity tables. The tables are characterized by the following rules:*

$$\frac{\Gamma, A, \neg B/\Delta}{\Gamma, \neg(A \supset B)/\Delta}\, l^{\supset^1_-} \qquad \frac{\Gamma/\Delta, A \qquad \Gamma/\Delta, \neg B}{\Gamma/\Delta, \neg(A \supset B)}\, r^{\supset^1_-}$$

$$\frac{\Gamma, A/\Delta, B}{\Gamma, \neg(A \supset B)/\Delta}\, l^{\supset^2_-} \qquad \frac{\Gamma/\Delta, A \qquad \Gamma, B/\Delta}{\Gamma/\Delta, \neg(A \supset B)}\, r^{\supset^2_-}$$

$$\frac{\Gamma, \neg B/\Delta, \neg A}{\Gamma, \neg(A \supset B)/\Delta}\, l^{\supset^3_-} \qquad \frac{\Gamma, \neg A/\Delta \qquad \Gamma/\Delta, \neg B}{\Gamma/\Delta, \neg(A \supset B)}\, r^{\supset^3_-}$$

$$\frac{\Gamma/\Delta, \neg A, B}{\Gamma, \neg(A \supset B)/\Delta}\, l^{\supset^4_-} \qquad \frac{\Gamma, \neg A/\Delta \qquad \Gamma, B/\Delta}{\Gamma/\Delta, \neg(A \supset B)}\, r^{\supset^4_-}.$$

Proof Analogous to the proof of Lemma 11. □

Lemma 14 *Let $\mathbf{G4}^{\neg}(l^{\supset^m_-}, r^{\supset^n_-})$ be the proof system that results from adding the rules $l^{\supset^m_-}$ and $r^{\supset^n_-}$ with $m, n \in \{1, 2, 3, 4\}$ to the basic sequent calculus $\mathbf{G4}^{\neg}$. Then*

$$\mathbf{G4}^{\neg}(l^{\supset^m_-}, r^{\supset^n_-}) \text{ is sound} \quad \text{iff} \quad m = n.$$

Proof Analogous to the proof of Lemma 12. □

The results from the present section help us to establish the proof-theoretical consequences of including a truth-functional implication into our *FDE*-like four-valued logic. First, we know which pairs of positive sequent rules characterize the four natural truth-conditions for implication and which pairs of negative sequent rules characterize the four natural falsity-conditions for implication. Second, we know that the positive pair and the negative pair can be chosen independently. Third, we know that both the positive rules and the negative rules come in pairs. We cannot choose a positive left rule from one pair and a positive right rule from another (or a negative left rule from one pair and a negative right rule from another), because doing so would make our proof system unsound. Accordingly, choosing a truth-functional implication for an *FDE*-like four-valued logic can now be one on the basis of much more complete information.

7 Related Work

7.1 Baaz, Fermüller, Salzer, and Zach (1998)

Baaz et al. [6] present a general method that, although it was not designed to do so, can also be applied to obtain two-sided sequent calculi for *FDE*-like logics with at most four sequent rules for each n-ary operator. Their method makes use of sets of truth-values as labels in labelled sequent calculi. Given the set $V = \{\boldsymbol{n}, \mathbf{0}, \mathbf{1}, \boldsymbol{b}\}$ of truth-values, we first need to fix a *system of signs*, that is, a set of subsets of V such that for every $x \in V$ there is a subset $S_x \subseteq S$ such that $\bigcap S_x = \{x\}$. A labelled sequent Π is a finite multi-set of formulas of the type $\alpha : A$ where $\alpha \in S$ and $A \in \mathscr{L}$. If we choose the system of signs $S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ with $\alpha_1 = \{\mathbf{1}, \boldsymbol{b}\}$, $\alpha_2 = \{\boldsymbol{n}, \mathbf{0}\}$, $\alpha_3 = \{\mathbf{0}, \boldsymbol{b}\}$, and $\alpha_4 = \{\boldsymbol{n}, \mathbf{1}\}$,[11] any labelled sequent Π can be rewritten as a two-sided sequent Γ/Δ, because any *FDE*-like logic contains the negation $\neg$ and its truth table $f_\neg$. Any $\alpha_1 : A$ in Π is represented by an A to the right of the slash, any $\alpha_2 : A$ in Π by an A to the left of the slash, any $\alpha_3 : A$ in Π by a $\neg A$ to the right of the slash, and any $\alpha_4 : A$ in Π by a $\neg A$ to the left of the slash. Accordingly, the labelled sequent $\alpha_1 : A, \alpha_2 : B, \alpha_3 : C, \alpha_4 : D$ is represented by the two-sided sequent $B, \neg D / A, \neg C$. Conversely, any two-sided sequent Γ/Δ can be rewritten as a labelled sequent with labels α_1 through α_4.

Given the system of signs $S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ with $\alpha_1 = \{\boldsymbol{b}, \mathbf{1}\}$, $\alpha_2 = \{\boldsymbol{n}, \mathbf{0}\}$, $\alpha_3 = \{\boldsymbol{b}, \mathbf{0}\}$, and $\alpha_4 = \{\boldsymbol{n}, \mathbf{1}\}$, the method used by Baaz et al. [6, Definition 6.6] gives us the following axioms and rules:

Axioms:

$\alpha_1 : A, \alpha_2 : A \qquad \alpha_3 : A, \alpha_4 : A$

Structural rules: For every $\alpha \in S$,

$$\frac{\Pi}{\Pi, \alpha : A}\ \text{weak:}\alpha \qquad \frac{\Pi, \alpha : A, \alpha : A}{\Pi, \alpha : A}\ \text{cntr:}\alpha$$

Cut rules:

$$\frac{\Pi, \alpha_1 : A \qquad \Pi, \alpha_2 : A}{\Pi}\ \text{cut:}\alpha_1, \alpha_2 \qquad \frac{\Pi, \alpha_3 : A \qquad \Pi, \alpha_4 : A}{\Pi}\ \text{cut:}\alpha_3, \alpha_4$$

[11] Note that the signs α_1, α_2, α_3, and α_4 are the generalized truth-values $\mathbf{T}$, $\overline{\mathbf{T}}$, $\mathbf{F}$, and $\overline{\mathbf{F}}$ that we used in the previous section to analyse implication.

Propositional rules: An introduction rule for an n-ary operator $\star$ labelled with $\alpha \in S$ is of the form

$$\frac{\Pi, \Sigma_1 \qquad \cdots \qquad \Pi, \Sigma_n}{\Pi, \alpha : \star(A_1, \ldots, A_n)} \star : \alpha$$

where the formulas in each Σ_i ($1 \leq i \leq n$) are of the form $\alpha' : A_j$ for some $1 \leq j \leq n$ and $\alpha' \in S$.

It is easy to see that, if we rewrite the labelled sequents in these axioms and structural rules as two-sided sequents, then we obtain special cases of the logical axiom and of the structural rules *LC*, *RC*, *LW*, and *RW* of our calculus $\mathbf{G4}^{\neg}$. Note again that $\mathbf{G4}^{\neg}$ does not have a cut rule.

To compare the introduction rules that are obtained by Baaz et al. to our rules, we must say more about how they obtain their rules. Let $f^{\star}$ be a truth table for an n-ary operator $\star$ from an *FDE*-like logic and let α be a label in S. To obtain the introduction rule $\star : \alpha$, Baaz et al. [6, p. 28] first define the set $T = \{\langle x_1, \ldots, x_n \rangle : f_{\star}(x_1, \ldots, x_n) \in \alpha\}$. They then define a partial disjunctive normal form, each conjunct of which corresponds to a $\langle x_1, \ldots, x_n \rangle$ in T. This disjunctive normal form is then converted into a partial conjunctive normal form from which they read off the introduction rule $\star : \alpha$.

If we apply their method to the truth table $f_{\neg}$ for the unary operator $\neg$, we find the following four rules (note that we leave out the axioms):

$$\frac{\Pi, \alpha_3 : A}{\Pi, \alpha_1 : \neg A} \neg : \alpha_1 \qquad \frac{\Pi, \alpha_4 : A}{\Pi, \alpha_2 : \neg A} \neg : \alpha_2 \qquad \frac{\Pi, \alpha_1 : A}{\Pi, \alpha_3 : \neg A} \neg : \alpha_3 \qquad \frac{\Pi, \alpha_2 : A}{\Pi, \alpha_4 : \neg A} \neg : \alpha_4.$$

Rewriting the labelled sequents as two-sided sequents, we find the following four two-sided rules, respectively:

$$\frac{\Gamma / \Delta, \neg A}{\Gamma / \Delta, \neg A} \neg : \alpha_1 \qquad \frac{\Gamma, \neg A / \Delta}{\Gamma, \neg A / \Delta} \neg : \alpha_2 \qquad \frac{\Gamma / \Delta, A}{\Gamma / \Delta, \neg\neg A} \neg : \alpha_3 \qquad \frac{\Gamma, A / \Delta}{\Gamma, \neg\neg A / \Delta} \neg : \alpha_4.$$

In their two-sided formulations, the first two rules are redundant, the rule $\neg : \alpha_3$ is our logical rule $R^{\neg}$, and the rule $\neg : \alpha_4$ is our logical rule $L^{\neg}$.

Likewise, if we apply their method to the truth table $f_{\vee}$ for the binary operator $\vee$, we find the following four rules (note again that we leave out the axioms and that we do some serious trimming):

$$\frac{\Pi, \alpha_1 : A, \alpha_1 : B}{\Pi, \alpha_1 : A \vee B} \vee : \alpha_1 \qquad \frac{\Pi, \alpha_2 : A \qquad \Pi, \alpha_2 : B}{\Pi, \alpha_2 : A \vee B} \vee : \alpha_2$$

$$\frac{\Pi, \alpha_3 : A \qquad \Pi, \alpha_3 : B}{\Pi, \alpha_3 : A \vee B} \vee : \alpha_3 \qquad \frac{\Pi, \alpha_4 : A, \alpha_4 : B}{\Pi, \alpha_4 : A \vee B} \vee : \alpha_4.$$

Rewriting the labelled sequents as two-sided sequents, we find, respectively, the rules $R^{\vee+}$, $L^{\vee+}$, $R^{\vee-}$, and $L^{\vee-}$ of Lemma 7.

7.2 Wintein and Muskens (2016)

Another way to obtain two-sided sequent calculi for *FDE*-like logics is via a two-sided sequent calculus for a functionally complete set of operators that is truth-functional over $\{\boldsymbol{n}, \mathbf{0}, \mathbf{1}, \boldsymbol{b}\}$. Muskens [16, p. 49] showed that $\{\neg, \wedge, \ominus, \otimes\}$ is such a set of operators, where the truth tables for $f_\neg$ and $f_\wedge$ are the ones given above and the truth tables for $f_\ominus$ and $f_\otimes$ are the following:

$f_\ominus$	
$\boldsymbol{n}$	$\boldsymbol{b}$
$\mathbf{0}$	$\mathbf{0}$
$\mathbf{1}$	$\mathbf{1}$
$\boldsymbol{b}$	$\boldsymbol{n}$

$f_\otimes$	$\boldsymbol{n}$	$\mathbf{0}$	$\mathbf{1}$	$\boldsymbol{b}$
$\boldsymbol{n}$	$\boldsymbol{n}$	$\boldsymbol{n}$	$\boldsymbol{n}$	$\boldsymbol{n}$
$\mathbf{0}$	$\boldsymbol{n}$	$\mathbf{0}$	$\boldsymbol{n}$	$\mathbf{0}$
$\mathbf{1}$	$\boldsymbol{n}$	$\boldsymbol{n}$	$\mathbf{1}$	$\mathbf{1}$
$\boldsymbol{b}$	$\boldsymbol{n}$	$\mathbf{0}$	$\mathbf{1}$	$\boldsymbol{b}$

In the fashion of Section 6, we can see the truth table $f_\ominus$ as obtained from the generalized truth table $f_{\ominus_+}$ and the generalized falsity table $f_{\ominus_-}$. Likewise, the truth table $f_\otimes$ can be seen as obtained from the generalized truth table $f_{\otimes_+}$ and the generalized falsity table $f_{\otimes_-}$:

$f_{\ominus_+}$	
$\mathbf{F}$	$\overline{\mathbf{T}}$
$\overline{\mathbf{F}}$	$\mathbf{T}$

$f_{\ominus_-}$	
$\mathbf{T}$	$\overline{\mathbf{F}}$
$\overline{\mathbf{T}}$	$\mathbf{F}$

$f_{\otimes_+}$	$\mathbf{T}$	$\overline{\mathbf{T}}$
$\mathbf{T}$	$\mathbf{T}$	$\overline{\mathbf{T}}$
$\overline{\mathbf{T}}$	$\overline{\mathbf{T}}$	$\overline{\mathbf{T}}$

$f_{\otimes_-}$	$\mathbf{F}$	$\overline{\mathbf{F}}$
$\mathbf{F}$	$\mathbf{F}$	$\overline{\mathbf{F}}$
$\overline{\mathbf{F}}$	$\overline{\mathbf{F}}$	$\overline{\mathbf{F}}$

Definition 8 and Theorem 4 imply that the truth table for $f_\ominus$ is characterized by the following four rules:

$$\frac{\Gamma/\Delta, \neg A}{\Gamma, \ominus A/\Delta}\, l^{\ominus+} \qquad \frac{\Gamma, \neg A/\Delta}{\Gamma/\Delta, \ominus A}\, r^{\ominus+} \qquad \frac{\Gamma/\Delta, A}{\Gamma, \neg \ominus A/\Delta}\, l^{\ominus-} \qquad \frac{\Gamma, A/\Delta}{\Gamma/\Delta, \neg \ominus A}\, r^{\ominus-}.$$

Likewise, the truth table $f_\otimes$ is characterized by the following four rules (use the product rule and the minimization rule repeatedly):

$$\frac{\Gamma, A, B/\Delta}{\Gamma, A \otimes B/\Delta}\, l^{\otimes+} \qquad \frac{\Gamma/\Delta, A \qquad \Gamma/\Delta, B}{\Gamma/\Delta, A \otimes B}\, r^{\otimes+}$$

$$\frac{\Gamma, \neg A, \neg B/\Delta}{\Gamma, \neg(A \otimes B)/\Delta}\, l^{\otimes-} \qquad \frac{\Gamma/\Delta, \neg A \qquad \Gamma/\Delta, \neg B}{\Gamma/\Delta, \neg(A \otimes B)}\, r^{\otimes-}.$$

As a consequence, the sequent calculus $\mathbf{G4}^{\neg\wedge\ominus\otimes}$ that is obtained from our basic sequent calculus $\mathbf{G4}^{\neg}$ by adding the rules for $\wedge$, $\ominus$, and $\otimes$ is sound and complete for the *FDE*-like logic that is built from a set $\mathscr{P} = \{p, p', \ldots\}$ of atomic formulas, using the functionally complete set of operators $\{\neg, \wedge, \ominus, \otimes\}$. In turn, this sequent calculus could be used to find sequent rules for any n-ary operator $\star$ that is truth-functional over $\{\boldsymbol{n}, \mathbf{0}, \mathbf{1}, \boldsymbol{b}\}$, given a translation of the formula $\star(p_1, \ldots, p_n)$ into a formula in which only operators from $\{\neg, \wedge, \ominus, \otimes\}$ occur.

Wintein and Muskens [26, Definition 7] present *signed* sequent rules for $\ominus$ and $\otimes$. In their approach, a signed formula of the form '$x : A$' consists of a sign x from the set $\{1, \overline{1}, 0, \overline{0}\}$ and a formula A from a propositional language.[12] Accordingly, a signed sequent Σ is simply a finite set of signed formulas. Their signed sequent rule for $\ominus$ is the following:

$$\frac{\Sigma, x : A}{\Sigma, y : \ominus A}\ R_{\ominus}, \quad \text{for } \langle x, y\rangle \in \{\langle 0, \overline{1}\rangle, \langle \overline{0}, 1\rangle, \langle 1, \overline{0}\rangle, \langle \overline{1}, 0\rangle\}.$$

Likewise, their signed sequent rules for $\otimes$ are the following:

$$\frac{\Sigma, x : A, x : B}{\Sigma, x : A \otimes B}\ R^1_{\otimes}, \text{ for } x \in \{0, 1\} \qquad \frac{\Sigma, x : A \qquad \Sigma, x : B}{\Sigma, x : A \otimes B}\ R^2_{\otimes}, \text{ for } x \in \{\overline{0}, \overline{1}\}.$$

As a matter of fact, any signed sequent Σ can be rewritten as a two-sided sequent Γ/Δ and vice versa:

$$\begin{array}{lll} 0 : A \in \Sigma & \text{iff} & \neg A \in \Gamma \\ \overline{0} : A \in \Sigma & \text{iff} & \neg A \in \Delta \\ 1 : A \in \Sigma & \text{iff} & A \in \Gamma \\ \overline{1} : A \in \Sigma & \text{iff} & A \in \Delta. \end{array}$$

Using these rewriting rules, it is easy to see that the two-sided sequent rules for $\ominus$ and $\otimes$ that we obtained using our method coincide with the signed sequent rules presented by Wintein and Muskens.

8 Conclusion

We presented a method for generating two-sided sequent calculi for *FDE*-like four-valued logics and showed that any truth-functional n-ary operator can be characterized with at most four sequent rules. To compare the sequent calculi generated by our method with existing sequent calculi, we applied our method to the four-valued truth tables for $\vee$, $\wedge$, $\ominus$, and $\otimes$, and found that it gives us rules that are similar or even identical to the ones that have been proposed in the literature.

Moreover, we used our method to study intuitive truth-functional implications in *FDE*-like four-valued logics. We found that we must make a principled distinction between the pair of positive sequent rules that characterizes the truth-conditions of an implication and the pair of negative sequent rules that characterizes the falsity-conditions of an implication. Although any pair of positive sequent rules can be combined with any pair of negative sequent rules, we showed that both positive and negative sequent rules come in pairs: if we choose one sequent rule from a pair, we must also choose the other sequent rule from that pair. We submit that this might be one of the senses of the elusive concept of harmony in proof theory.

We close the paper with an open question. Our method of characterizing single truth table entries with proof rules has been applied to a wide range of truth-functional

[12] The italicized 0s and 1s are *not* to be identified with the non-italicized or bold-faced ones we use elsewhere in the paper.

two-, three-, and four-valued logics.[13] These results can be summarized in a conditional: if a truth-functional many-valued logic has such and such properties, then our method of characterizing single truth table entries is applicable. The open question is about the conditional's converse: if our method of characterizing single truth table entries is applicable to a truth-functional many-valued logic, then what properties does such a logic have?

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