Price Stability and Improved Buyer Utility with Presentation Design: A Theoretical Study of The Amazon Buy Box

Anonymous

Abstract

Platforms design the form of presentation by which sellers are shown to the buyers. This design not only shapes the buyers' experience but also leads to different market equilibria or dynamics. One component in this design is through the platform's mediation of the search frictions experienced by the buyers for different sellers. We take a model of monopolistic competition and show that, on one hand, when all sellers have the same inspection costs, the market sees no stable price since the sellers always have incentives to undercut each other, and, on the other hand, the platform may stabilize the price by giving prominence to one seller chosen by a carefully designed mechanism. This calls to mind Amazon's Buy Box design. We study natural mechanisms for choosing the prominent seller, characterize the range of equilibrium prices implementable by them, and find that in certain scenarios the buyers' surplus improves as the search friction increases.

CCS Concepts

• **Theory of computation → Market equilibria**.

Keywords

Do, Not, Us, This, Code, Put, the, Correct, Terms, for, Your, Paper

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1 Introduction

Platforms that enable transactions between buyers and providers have become major venues of the e-commerce. As a few salient examples, Amazon and eBay allow buyers to purchase from a vast body of sellers, and Airbnb connects travelers and hosts. An important activity on these websites is for one side of the market to search on the other side for a service or good. Search is sequential and time-consuming, and usually cannot be exhaustive. The way in which the platform *presents* products and services to be searched by the buyers, therefore, crucially affects the market dynamics and outcomes. In most platforms, the providers/sellers set their own prices — third-party sellers on Amazon, for example, set up their own prices, and so do hosts on Airbnb. In wielding its power to mediate the interface, the platform must take into consideration both the sellers' pricing strategies and the buyers' search policies.

Our starting point is a monopolistic competition model where symmetric but differentiated sellers post their prices visible to all; a buyer must incur an inspection cost to determine her value for any specific seller, and this cost is the same for each seller.[1] We observe that such a market does not have a pure equilibrium, i.e., no prices are stable, because the sellers are incentivized to undercut each other to be searched first by the buyer. In practice, price volatility caused by "algorithmic pricing" has been voiced by the media [8, 19] and has been a concern of consumers.

In this work, we analyze a particularly simple scheme of presentation, wherein *one* seller is made prominent, with drastically reduced inspection cost, while all the other sellers are treated equally. The platform's rule for choosing the prominent seller sets up a mechanism, in response to which the sellers strategically set their prices. We show that this presentation scheme, coupled with appropriate mechanisms, can stabilize prices at pure equilibria. We further analyze the range of prices achievable at equilibria, and derive implications on welfare and consumer surplus. An interesting discovery, among others, is that an increased inspection cost often increases the consumers' surplus, because the higher search barrier can induce sellers to lower their prices in order to gain prominence.

Before detailing our contributions, we take a moment to introduce Amazon's so-called *Buy Box*, a typical interface design that features one prominent seller. The Buy Box serves as a major motivation for our work; in the rest of the paper we refer to the prominent seller as the one *in the Buy Box.*

The Amazon Buy Box. When a buyer reaches a page on Amazon for a particular product, a *Buy Box* is shown (red box in Figure 1). There are typically multiple sellers for the product, but the Buy Box shows only one seller, with that seller's price and the buttons to press (either "Add to Cart" or "Buy now with 1-click") if the buyer were ready to purchase from the Buy Box seller.

Below the Buy Box (and not always readily visible to a typical buyer), is an additional link outside Buy Box (the green box in Figure 1) to the other sellers of the same product (bottom of Figure 1), with some information about the prices they offer.

By presenting the sellers in this way, Amazon de facto reduces the inspection cost for the Buy Box seller, and hence increasing the chance of a sale to this seller. For the other sellers of this same product, Amazon has, for all practical purposes, increased the search cost.

For the same product, the sellers' offers differ in aspects such as shipping time, location of seller, return and refund policy, product

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[1]As we explain later, this is essentially the classical model by Wolinsky [21], except that in our setting the prices are visible prior to search. We note that prices usually do not take much effort to see in most online platforms.

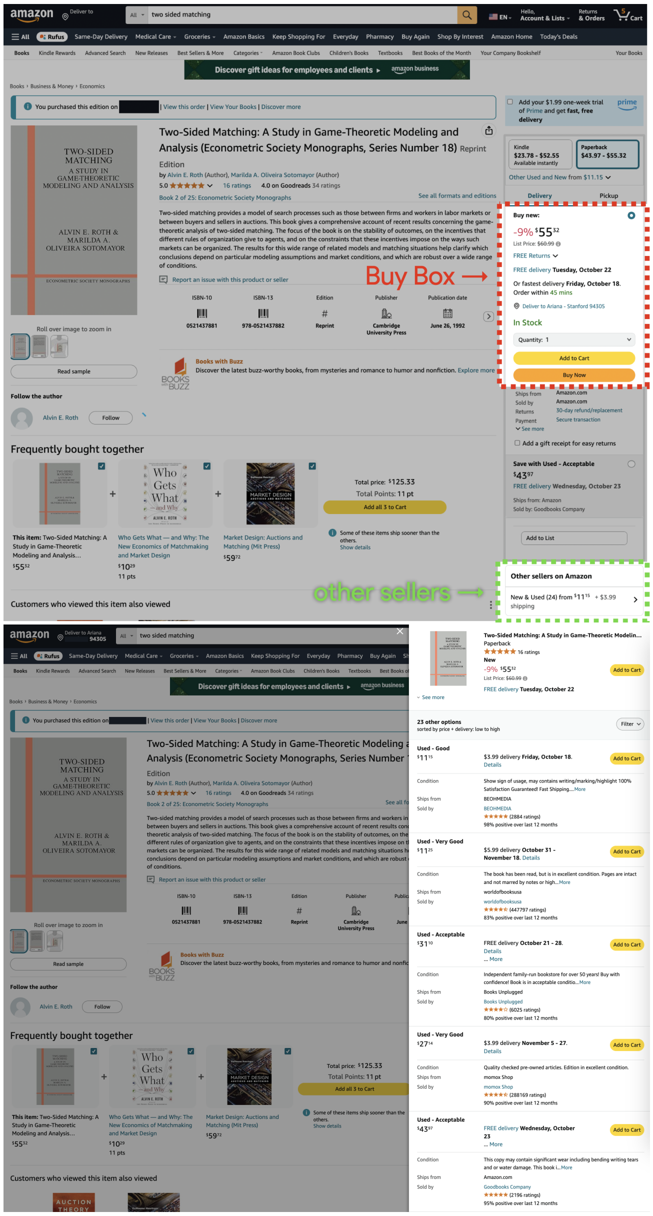

Figure 1: Illustration of Amazon's Buy Box.
Top: The Buy Box is highlighted in red. To view other sellers not in the Buy Box, a buyer needs to scroll down and click on the link we highlight in green.
Bottom: A view of sellers not in the Buy Box.

description (especially for used items), and whether they are "fulfilled by Amazon". For any particular seller, a buyer can be certain that her value should be in a certain interval, but must spend effort to determine her exact value, according to her preference.

Placement in the Buy Box has a large impact on the demand for a seller's product, and Amazon's *Buy Box mechanism* for determining the Buy Box seller has drawn attention from both the industry [13] and academic works [9]. The precise mechanism Amazon uses is proprietary information, and is generally considered to depend on many factors, including the sellers' prices, their past performances, etc.. Pricing on Amazon has long been known to involve software that helps to update prices dynamically; at least part of the drive for the incessant price changes is to compete for the Buy Box.

Wild price fluctuation on Amazon has been well documented [8, 17, 19]; Chen et al. [9] showed by simulation and by empirical observation that "sellers that use algorithmic strategies to maintain low prices relative to their competitors are likely to gain a large advantage in the struggle to win the Buy Box". With fluctuating prices, not only should the sellers be constantly ready to change their prices, but buyers have also the added burden of uncertainty and should be strategic on the time of purchase.

In this work we build a theoretical framework and study a simplified scenario where the Buy Box mechanism only reacts to the prices set by the sellers. We show that the mechanism which rewards the Buy Box to the most competitive price is intrisically unstable in that no pure equilibrium exists. We propose mechanisms that do support pure equilibria, study the range of equilibria prices, and derive welfare and surplus implications.

Our Contributions. In a monopolistic competition, m sellers offer differentiated goods/services at prices set by themselves. For each seller i, a buyer has value v_i drawn i.i.d. from a known distribution F. We normalize values so that F is supported on $[V, V+1]$ for some $V > 0$; for many results we assume $V \geq 2$, and $c < \mathbf{E}[v_i] - V$. For scenarios such as online retail platforms, these assumptions seem realistic. (See further discussion in Section 2.) The buyer knows F and can see the prices, but must incur an inspection cost c to be certain of any particular v_i. The buyer performs a sequential search, with free recall, in order to maximize her expected utility. The optimal search policy is given by Weitzman [20]'s renowned index-based policy, which we describe in detail in Section 2. One feature of sequential search is that, whenever the buyer has found a value "high enough", she should stop the search. Sellers that are inspected earlier therefore have an advantage.

We start with a model we call *plain presentation* (without a Buy Box), the buyer's optimal policy goes over the sellers in the order of decreasing prices, and this gives the sellers incentives to undercut each other. We formally show that this market admits no pure symmetric equilibrium.[2] Interestingly, for a market with two sellers, when $V \geq 2$, we are able to show that no asymmetric pure equilibrium exists either. For this result (Theorem 1), we use a technique from Armstrong and Zhou [4], which allows a nontrivial derivation of the sellers' respective probabilities of selling had they been at an equilibrium; a contradiction ensues from this calculation.

When prominence is given to a seller, e.g. in the form of a Buy Box, we assume for simplicity that this seller has no inspection cost, i.e., a buyer always knows her value for this seller before any search, whereas every other seller still incurs an inspection cost c. As many platforms claim to aim for the lowest, most competitive prices, it is tempting to use a Buy Box mechanism to induce low prices, e.g. by giving prominence to only the seller with the lowest

[2] Armstrong and Zhou [3] mention in passing that "this framework, where a consumer's match utility is independently distributed across firms, apparently does not lead to a tractable solution for how firms choose prices". Our proof may be seen as a formalization of this observation.

price. It is not difficult to see that such an approach gives even more incentive for the sellers to undercut each other, and non-existence of equilibrium from the plain presentation persists.

We therefore explore mechanisms that admit pure equilibria. The first mechanism we consider is a *dictator* — the platform stipulates a price t, and the Buy Box is awarded to only sellers that post price t; if there are multiple of them, pick one uniformly at random. The mechanism is somewhat unnatural, but we show that it has the advantage that it encompasses all equilibria, in the sense that, if any natural mechanism (Definition 3) admits a pure symmetric equilibrium, the dictator mechanism admits the same equilibrium (by stipulating that price). In Section 4.2, we completely characterize the range of equilibria realizable in the dictator mechanism (and hence all possible symmetric equilibria in natural mechanisms). Our main finding is that, the prices implementable at equilibrium always constitute an interval. The higher the inspection cost c, the more likely a pure equilibrium exists, and, when it does, the smaller the lower end of the interval, that is, the smaller the lowest price implementable at equilibrium. For the case of two sellers, we show more crisply that, as c increases, the set of implementable prices grows inclusion-wise. These findings suggest that, if a platform uses prominence solely to drive down prices, it is effective to increase the inspection cost for non-prominent sellers.

Towards more practical mechanisms, we then consider the *threshold mechanisms*, which stipulate a price t and award the Buy Box to sellers that post prices below t; again, if there are multiple of them, pick one uniformly at random. The threshold t acts like a reserve price in auctions, but unlike in optimal auctions, the mechanism does not discriminate among sellers setting prices below t. This makes it possible for the threshold mechanism to admit pure equilibria. If the platform uses a threshold mechanism and means to drive down prices, it may lower the threshold t (to the extent that equilibrium still exists), rather than directly encourage competition among the sellers. We again characterize the range of prices implementable at equilibria in threshold mechanisms (Theorem 5). It is not surprising that there are settings where the dictator mechanism implements a larger range of prices than the threshold mechanism (Proposition 7); what we do find interesting is that, whenever the threshold mechanism admits an equilibrium, the lowest price it is capable of maintaining at equilibrium is the same as that of the dictator mechanism (Corollary 2). This suggests that, if the platform aims to drive down prices, the more practical-looking threshold mechanism is just as powerful as the stringent dictator mechanism, as long as the inspection cost is calibrated to a level that allows equilibria to exist.

Finally, we discuss welfare and consumer surplus at equilibria under the Buy Box mechanisms. Since the inspection cost is a "burned" effort, and prices paid by the buyer and received by the sellers cancel out, the social welfare decreases as c increases (Theorem 6). The case of consumer surplus is more complex and interesting. As c increases, the lowest possible price implementable at equilibrium by a Buy Box mechanism decreases, but it is also more costly for the consumer to search. The two factors work in opposite directions for the consumer surplus. For many value distributions we experiment with, the consumer surplus largely increases as c increases, but it is not necessarily the case that the surplus attains its maximum when c is as large as possible. In Theorem 7, we derive a sufficient condition on F under which the consumer surplus is maximized at an inspection cost not at its maximum, if the platform implements the lowest equilibrium price. We see this result as a proof of concept, illustrating that the relationship between the consumer surplus and the inspection cost is a complicated one.

Related Works

The two works most closely related to the current work are Armstrong et al. [2] and Armstrong and Zhou [3], as both explicitly considered monopolistic competitions with search frictions and with prominent sellers, but both have key differences from our work. Armstrong et al. [2] considers a market where the a seller's price is invisible to the buyer unless the seller is inspected. Price changes therefore do not directly affect the buyer's search order, which allows pure equilibria to exist. We observe that in most online platforms the buyer sees sellers' prices quite easily, and take this as our starting point. Armstrong and Zhou [3] Section 2 consider markets with visible prices, but circumvented the trouble of nonexistence of equilibria by considering negatively correlated values, which are stylized in a Hotelling model. For online shopping, we consider such negative correlation less well motivated, and stick to the classical i.i.d. setting. Moreover, we emphasize the mechanism design aspect in assigning the Buy Box, in addition to static equilibrium analysis. For other works on markets with search frictions, we refer the reader to Armstrong [1] for a beautiful survey.

There has been a small but growing number of works on platforms' presentation of sellers or products which explicitly model buyers' search behavior. Chu et al. [10], e.g., consider the ranking of heterogeneous products for a buyer whose search order is determined by this ranking, and they consider a multiobjective optimization that includes surplus and sales revenue. Derakhshan et al. [11] propose a two-stage search policy motivated by empirical evidence, and study product ranking in response to such search in order to maximize welfare and purchase probability. Branco et al. [7] proposes a continuous inspection procedure for a buyer to evaluate a single seller, and studies the seller's optimal pricing in response. Interestingly, in this setting very different from ours, it is also observed that increased search cost can sometimes benefit the buyer in equilibrium, due to the lower price set by the seller.

The way we normalize the support of F to $[V, V+1]$ and a proof in Section 3 both follow Armstrong and Zhou [4], although the problem they study is on consumers' private signaling, quite distant from our setting.

Organization

The rest of the paper is organized as follows. In Section 2, we present the monopolistic competition with search frictions, with a description of the buyer's optimal search procedure. Section 3 formally proves that the plain presentation sees no stable prices in equilibrium. Section 4 defines Buy Box mechanisms and presents the prices implementable at equilibria by the two mechanisms we study. Section 5 analyzes welfare and consumer surplus with the Buy Box. Most proofs are relegated to the appendix.

2 Model and Preliminaries

Consider a unit-demand buyer (she) who looks to buy an item from one of m sellers (them) on a platform. The buyer has potentially different values for purchasing from different sellers; her value v_i for each seller i is a random variable drawn i.i.d. from a distribution F. The prior F is public knowledge to both the buyer and all the sellers, but neither party knows the realization v_i. We assume that $F(\cdot)$ is continuous and supported on $[V, V+1]$ for some $V > 0$, an assumption also made by Armstrong and Zhou [4]. Also following [4], for much of the paper, we assume $V \geq 2$. Each seller i sets their price p_i, visible to the buyer.

To learn her value v_i for seller i, the buyer may pay an *inspection cost* c_i. The inspection takes place in a sequential manner. Namely, at any time, the buyer may inspect the value of a seller of her choice (and incur a cost), or to buy from one of the inspected sellers and quit, or to quit without purchase. The inspection costs obviously affect a buyer's behavior, and are controlled by the platform. The three parties therefore participate in a game, whose order of actions we detail below.

Order of Actions. The platform, the sellers and the buyer take actions in the following order.

- **Stage 1**: The platform specifies a *mechanism* $\mathcal{M}$ that maps seller prices $\mathbf{p} = (p_1, \ldots, p_m)$ to their inspection costs $c_1, \ldots, c_m$.
- **Stage 2**: The sellers, aware of $\mathcal{M}$ and knowing that buyers will screen sellers and purchase in accordance with their optimal search strategy (described in detail below), decide and commit to their prices $p_1, \ldots, p_m$.
- **Stage 3**: $\mathcal{M}$ maps the prices $\mathbf{p}$ to inspection costs $c_1, \ldots, c_m$.
- **Stage 4**: The buyer performs an optimal sequential search given the value prior F, the prices $\mathbf{p}$, and the inspection costs.

Utilities and Objectives. Each seller i tries to maximize their expected *revenue*, which is their price p_i times the probability the buyer buys from them. The buyer tries to maximize her expected payoff, which is the value of the item she eventually buys minus its price, minus all the inspection costs she pays along the way. Formally, for each $i \in [m]$, let Z_i be the indicator random variable for the event that the buyer inspects seller i, and Y_i the indicator for the buyer purchasing from seller i, then at price profile $\mathbf{p} = (p_1, \ldots, p_m)$ and inspection costs $c_1, \ldots, c_m$, the buyer's expected utility is

$$\mathrm{CS} := \mathbf{E}\left[\sum_{i \in [m]} Y_i (v_i - p_i)\right] - \mathbf{E}\left[\sum_{i \in [m]} c_i Z_i\right];$$

This is sometimes referred to as the *consumer surplus*. The *social welfare* is the sum of the sellers' revenues and the consumer surplus. Formally,

$$\mathrm{SW} := \mathbf{E}\left[\sum_i (v_i Y_i - c_i Z_i)\right].$$

Optimal Sequential Search. The buyer's optimal sequence of actions is given by the optimal policy for the *Pandora's Box* problem, first discovered by Weitzman [20]. We describe below this optimal policy, and refer the reader to Kleinberg et al. [16] for a proof of its correctness.

(1) For each seller i, compute an *index* θ_i, which is the unique solution to the equation:

$$\mathbb{E}_{v_i \sim F}\left[[v_i - p_i]^+ - \theta\right]^+ = c_i,$$

where $[x]^+ := \max(x, 0)$. Tag each seller i with the index θ_i.

(2) Among all sellers whose tags are non-negative indices, inspect seller i with the highest index θ_i, see the value v_i and replace the tag by the *utility* of buying from seller i: $v_i - p_i$.

(3) If at any point the largest tag on a seller is a utility (instead of an index), purchase from that seller and quit; otherwise go back to the previous step. If all the sellers have been inspected, then
   - if any seller's price is below their value, make a purchase from the seller that yields the highest utility;
   - otherwise leave without any purchase.

A moment of thought shows that, following this policy, the buyer ends up buying from the seller i that maximizes $\kappa_i := \min(v_i - p_i, \theta_i)$, if $\max_i \kappa_i \geq 0$. This policy is also known as the *Weitzman's algorithm*,

Plain Presentation Without BuyBox. When the platform presents the sellers "plainly", only their prices $\mathbf{p} := (p_1, p_2, \ldots, p_m)$ are shown to the buyer. Each seller's inspection cost is $c_i = c$, no matter what prices they set.

Presentation with Buy Box. In a presentation with Buy Box, one seller is prominent, i.e., with inspection cost 0, and all the other sellers' inspection costs are set to some $c > 0$. The prominent seller is said to be *in the Buy Box*. We assume all the sellers' prices are visible to the buyer, but she only sees her value for the seller in the Buy Box unless she performs (costly) inspection. The identity of the seller in the Buy Box is the result of the platform's mechanism, which we define formally in Section 4.

For a seller not in the Buy Box, we often use $\theta_0(c)$ to denote their Weizman index if they post price 0, i.e., $\theta_0(c)$ is the solution to the equation $\mathbb{E}_{v_i \sim F}[v_i - \theta]^+ = c$. When the inspection cost c is clear from the context, we omit it and write θ_0.

Assumption of Non-degeneracy. Throughout the paper we assume $\theta_0 > 0$: when this is not the case, a seller not in the Buy Box is never inspected even if they post price 0, and the setting degenerates to a monopoly of the seller in the Buy Box. For most results in the paper, we further assume $\theta_0 > V$. When this is not the case, the inspection cost c is larger than $\mathbf{E}[v_i] - V$, which is the uncertainty the buyer has about her value for a seller. For one thing, search frictions in online shopping are typically not this big. For another, when an inspection costs more than the expected uncertainty a buyer has about the value, it is realistic that the buyer may make a purchase *without inspection*, an option interesting by itself [5, 6, 12, 14, 15] but not captured by Weitzman's algorithm. This work focuses on the regime with inspection costs small relative to the value uncertainty. Let $\bar{c}$ be the cost such that $\theta_0(\bar{c}) = V$. Assuming $\theta_0 > V$ is equivalent to assuming $c < \bar{c}$.

A seller with a negative index is never considered by the buyer. We therefore mostly restrict our attention to prices low enough to guarantee nonnegative indices. For such prices, the calculation of Weizman indices is simplified by the following proposition:

Proposition 1. *If $\theta_i \geq 0$ under price p_i, then $\theta_i = \theta_0 - p_i$.*

Proof. For $\theta_i \geq 0$, $\mathbb{E}[[v_i - p_i]^+ - \theta_i]^+ = [v_i - p_i - \theta_i]^+$. By definition of θ_i, $\mathbb{E}[v_i - p_i - \theta_i]^+ = c$; comparing this with $\mathbb{E}[v_i - \theta_0]^+ = c$ gives the proposition. □

Equilibria. With or without the Buy Box, the expected revenue of any seller i may be calculated from the price profile: let $D_i(\mathbf{p})$ be the probability the buyer buys from seller i if the price profile is $\mathbf{p}$, then the seller's revenue is $\mathrm{Rev}_i(\mathbf{p}) = p_i D_i(\mathbf{p})$. A price profile $\mathbf{p}$ constitutes a *pure Nash equilibrium* if, for any seller $i \in [m]$,

$$\mathrm{Rev}_i(\mathbf{p}) \geq \mathrm{Rev}_i(p_i', \mathbf{p}_{-i}), \forall p_i'.$$

3 Non-Existence of Pure Equilibria in Plain Presentations

We show that a plain presentation without a Buy Box generally does not admit pure Nash equilibria. The argument is most intuitive for the non-existence of symmetric equilibria (Proposition 2), where the stability of any symmetric positive price profile is broken by the sellers' incentives to undercut each other. It may come as a surprise that, under mild technical assumptions, even asymmetric equilibria do not exist for two sellers (Theorem 1). This follows from a nontrivial argument similar to a proof from Armstrong and Zhou [4]. Both results contrast with monopolistic competitions *without* inspection costs, where pure equilibria often exist [18].

Proposition 2. *For any $p > 0$, $\mathbf{p} = (p, \ldots, p)$ is not an equilibrium in a plain presentation. When $\theta_0 > V$, there is no symmetric equilibrium in a plain presentation.*

The proposition is intuitive: when every seller posts the same price p, a seller may slightly cut down their price and make sure they are inspected first by the buyer, which produces a jump increase in their demand. The proof lower bounds this increase, and argues that a small enough undercutting guarantees strict revenue increase.

For two sellers, we can show that even *asymmetric* equilibrium does not exist.

Theorem 1. *In a plain presentation with two sellers, whose values are supported on the interval $[V, V+1]$ for $V \geq 2$, there is no pure Nash equilibrium.*

The proof is technically interesting and involved. To prove by contradiction, one may assume an asymmetric equilibrium exists and write the equilibrium condition for the two sellers: either seller, if deviating to a different price, should see weakly less revenue. These give upper and lower bounds on the demand function of a seller as they vary their price. We derive in our setting Lemma 1, similar to one by Armstrong and Zhou [4], which allows us to express the bounds and the demand all in terms of one common variable, so that the demand is sandwiched between the two bounds. Another crucial observation by Armstrong and Zhou [4] (made in a different setting) is that, since the three functions coincide at the equilibrium point, their derivative at that point must be equal. This immediately allows one to express precisely the demand of the sellers at the equilibrium, which leads to a contradiction. (See Figure 2 for an illustration.)

Lemma 1. *In a plain presentation, for F supported on $[V, V+1]$ with $V \geq 2$, the buyer always makes a purchase if the the sellers' prices reach an equilibrium.*

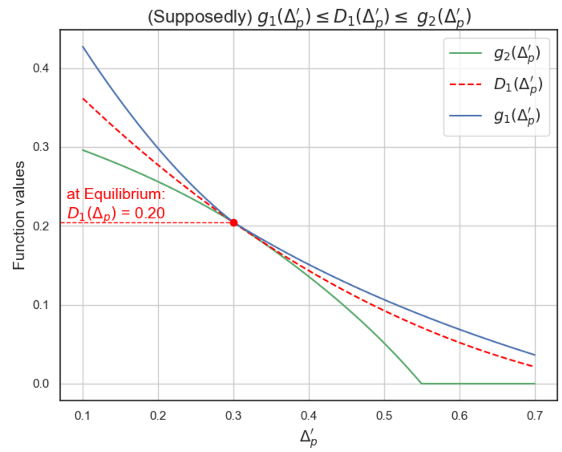

Figure 2: Illustration for a step in the proof of Theorem 1. The equilibrium conditions, together with Lemma 1, require that seller 1's demand, as the seller varies their price, is sandwiched between an upper bound g_1 and a lower bound g_2, all three functions expressed in a common variable Δ_p'. They coincide at the point $\Delta_p' = \Delta_p$, which is the hypothetical equilibrium point. The three functions must have the same derivative at $\Delta_p' = \Delta_p$.

The lemma has analogs in later settings we consider as well, but requires more conditions and modifications.

We can further show that even ϵ-approximate *symmetric* equilibrium does not exist in plain presentations with two players.

Theorem 2. *In a plain presentation, for any F supported on $[V, V+1]$ and any $c > 0$, there exists a $\Delta > 0$, such that for any positive $\epsilon < \Delta$, there is no pure symmetric ϵ-equilibrium.*

4 Buy Box Mechanisms

As set up in Section 2, the Buy Box gives prominence to one seller, who is chosen by a mechanism in response to the price profile. This mechanism, called the *Buy Box mechanism*, is at the crux of the three-party game. We formally define it below, and quickly observe that the most naïve mechanism gives no more price stability than a plain presentation. In Section 4.1, we introduce a family of powerful mechanism called dictators, and show them to encompass all price equilibria; that is, any symmetric equilibrium implementable by a *standard* mechanism is implementable under a dictator mechanism. Then in section 4.2, we give a full characterization of the range of prices implementable at equilibria. In Section 4.3, we introduce threshold mechanisms, which are less stringent and more natural looking. We characterize the equilibria implementable in threshold mechanisms, and compare the range with that of dictator mechanisms.

Buy Box Mechanisms. A *Buy Box mechanism* $\mathcal{M}$ maps a price profile $\mathbf{p} = (p_1, p_2, \ldots, p_m)$ to $\mathbf{x}^{\mathcal{M}}(\mathbf{p}) := (x_1, x_2, \ldots, x_m)$, with $\sum x_i \leq 1$, where x_i denotes the probability with which seller i wins the Buy Box.

If the platform aims to lower the equilibrium price, it is tempting to simply reward the Buy Box to the seller with the lowest price:

Definition 1. *The* Lowest Price First (LPF) Mechanism *assigns the Buy Box to the seller offering the lowest price, with ties broken uniformly at random.*

It takes only a moment of thought to realize that, compared with the plain presentation, LPF only offers more incentive for the sellers to undercut each other, and can no better sustain pure equilibria.

Proposition 3. *(a) For any $p > 0$, the symmetric price profile $\mathbf{p} = (p, \dots, p)$ is not an equilibrium under the LPF mechanism. If $\theta_0 \geq V$, LPF admits no symmetric equilibrium.*
(b) With two sellers, the LPF mechanism admits no pure equilibrium.

4.1 The Dictator Mechanisms

We now introduce a family of powerful mechanisms, where a seller has to post a price dictated by the platform to be in the Buy Box. It turns out that these mechanisms encompass all symmetric equilibria in a natural family of mechanisms (Definition 3, Theorem 3).

Definition 2 (Dictator-t mechanism)**.** *The* Dictator-t mechanism *is a mechanism parameterized with a target price $t > 0$. It assigns the Buy Box to sellers who set their prices at t, breaking ties uniformly at random. Formally, for any price profile $\mathbf{p} = (p_1, p_2, \dots, p_m)$, the Buy Box allocation vector $\mathbf{x}(\mathbf{p})$ is*

$$\mathbf{x}^{\textit{Dictator-t}}(\mathbf{p}) = \left(\frac{\mathbb{I}[p_1 = t]}{N_t}, \frac{\mathbb{I}[p_2 = t]}{N_t}, \dots, \frac{\mathbb{I}[p_m = t]}{N_t} \right),$$

where $N_t = \sum_{i \in [m]} \mathbb{I}[p_i = t]$ counts the number of sellers pricing at t, and $\mathbb{I}[\cdot]$ is the indicator function.

Definition 3. *A Buy Box mechanism is said to be*

(a) anonymous, *if for any permutation σ on $[m]$ and any price profile $\mathbf{p} := (p_1, p_2, \dots, p_m)$, the allocation satisfies $x_i(\mathbf{p}) = x_{\sigma(i)}(p_{\sigma(1)}, \dots, p_{\sigma(m)})$ for every $i \in [m]$;*
(b) always allocating, *if for any price profile $\mathbf{p}$, $\sum_{i=1}^m x_i(\mathbf{p}) = 1$.*

A mechanism is said to be standard *if it is both anonymous and always allocating.*

Theorem 3. *For any standard Buy Box mechanism $\mathcal{M}$, if $\mathbf{p} = (p, \dots, p)$ is a symmetric equilibrium, then $\mathbf{p}$ is also an equilibrium for the Dictator-p mechanism.*

The theorem is proved by observing that, in any standard mechanism, at a symmetric pure equilibrium price p, each seller must be in the Buy Box with probability $\frac{1}{m}$, the same as in the Dictator-p mechanism. It remains only to show that a deviation in the dictator mechanism cannot be more profitable than the deviation in any other standard mechanism.

4.2 Implementable Prices

In this section we characterize prices implementable at symmetric equilibria by standard mechanisms. Throughout the rest of the paper, we assume $V \geq 2$ and $\theta_0 > V$, i.e., $c < \bar{c}$. We have discussed these conditions in Section 2 (Assumption of Non-Degeneracy).

Definition 4. *A Buy Box price t is* implementable *if there exists a standard Buy Box mechanism under which the price profile $\mathbf{p} = (t, \dots, t)$ is an equilibrium. Let $T(c)$ denote the set of implementable prices when the inspection cost is c.*

By Theorem 3, to characterize $T(c)$, it suffices to characterize equilibria under the dictator mechanisms. $T(c)$ turns out to be an interval, whose endpoinds are best expressed as the extreme values of two functions. The following demand function for deviations is crucial in expressing these endpoints.

Lemma 2. *[Demand for a Seller Deviating from the Symmetric Equilibrium under the Dictator Mechanism] For $c \in (0, \bar{c})$, consider a symmetric equilibrium price t. If a seller deviates to a price p, then their demand is a function of the amount of deviation $x = p - t$. Specifically, the demand after deviating to p is*

$$\mathcal{D}_c(x) := \begin{cases} 1, & \text{if } x \leq -1, \\ \int_V^{\theta_0(c)-x} \Big[\int_{\max(\theta_0(c)+x, v_2+x)}^{V+1} \mathrm{d}F(v_1)\, \mathrm{d}F(v_2) \\ \quad + \int_{v_2+x}^{\max(\theta_0(c)+x, v_2+x)} F^{m-2}(v_1 - x)\, \mathrm{d}F(v_1) \Big] \mathrm{d}F(v_2), \\ \quad \text{if } -1 < x < 0, \\ \int_V^{\theta_0(c)-x} (1 - F(v+x))\, \mathrm{d}F^{m-1}(v), \\ \quad \text{if } 0 \leq x < \theta_0(c) - V, \\ 0, & \text{if } \theta_0(c) - V \leq x. \end{cases} \tag{1}$$

Theorem 4. *For $c \in (0, \bar{c})$, the set of implementable prices $T(c)$ is a closed interval determined by the inspection cost c:*

$$T(c) = [t^*(c), \bar{t}(c)],$$

if $t^*(c) \leq \bar{t}(c)$, where $t^*(c) := \sup_{x>0} \left\{ \frac{x \mathcal{D}_c(x)}{\frac{1}{m} - \mathcal{D}_c(x)} \right\}$ and $\bar{t}(c) := \inf_{x<0} \left\{ \left(\frac{(-x)\mathcal{D}_c(x)}{\mathcal{D}_c(x) - \frac{1}{m}} \right)^+ \right\} \setminus \{0\}$.

When $t^*(c) > \bar{t}(c)$, no standard mechanism admits any symmetric equilibrium.

As an example, for the uniform distribution $F = U[2, 3]$ and two sellers, we work out this interval. For this distribution, $\bar{c} = \frac{1}{2}$. For $c \in (0, \bar{c})$,

$$t^*(c) = \frac{\sqrt{-((2\sqrt{2c} - 2c) + \frac{1}{2}) + \sqrt{2(2\sqrt{2c} - 2c) + \frac{1}{4}}}}{\frac{1}{2}\sqrt{2(2\sqrt{2c} - 2c) + \frac{1}{4}} - 1} \times \frac{1}{2}\left(\frac{3}{2} - \sqrt{2(2\sqrt{2c} - 2c) + \frac{1}{4}} \right)$$

$$\bar{t}(c) = 2.$$

For any $c \in (0, \bar{c})$, $t^*(c) < \bar{t}(c)$. Therefore for all small enough c, the dictator mechanism admits a pure equilibrium.

Proposition 4. *[Properties of lowest implementable price] For $c \in (0, \bar{c})$, $t^*(c)$ is* continuous *and* monotonically decreasing *in c. Moreover,*

(i) $\lim_{c \to \bar{c}^-} t^*(c) = 0$;

(ii) $\lim_{c \to 0^+} t^*(c) = t_0$, *if a symmetric equilibrium* $(t_0, \ldots, t_0)$ *exists for the monopolistic competition without cost of inspection.*

Proposition 5. *When there are two sellers, for* $c \in (0, \bar{c})$, $\bar{t}(c)$ *is continuous and monotonically decreasing in* c.

Corollary 1. *For a market with two sellers, if* $0 < c < c' < \bar{c}$, *then* $T(c) \subseteq T(c')$.

An extension of Lemma 1 to the Buy Box setting plays an important role in these characterizations.

Lemma 3. *If* $V \geq 2$ *and* $c \leq \bar{c}$, *then under any equilibrium of any Buy-Box mechanism, the buyer always makes a purchase.*

The assumption $c < \bar{c}$ is necessary for this extension, and hence the assumption throughout Section 4.2. The proof of Proposition 6 constructs a Buy Box mechanism where prices well above V are sustained as an equilibrium for sufficiently large c.

Proposition 6. *For sufficiently large* c, *there exists a Buy Box mechanism with an equilibrium at which prices are all higher than* V.

4.3 Threshold Mechanisms

Dictator mechanisms encompass all pure symmetric equilibria, but are unnatural, and seem to abuse the platform's power. In this section we consider threshold mechanisms, which we consider more practical. A threshold mechanism only sets a price upper bound for sellers to be eligible for the Buy Box, but otherwise does not discriminate among the eligible sellers.

We again completely characterize the range of prices implementable at equilibria by the threshold mechanisms (Theorem 5), and compare it with that of the dictator mechanisms. An interesting finding is that, even though the threshold mechanisms' range of equilibria prices is generally smaller, and more prone to be empty, whenever an equilibrium does exist, the lowest price implementable at equilibrium is the same as that of the dictator mechanisms (and hence of all standard mechanisms).

Definition 5. *For a price* $t > 0$, *the* Threshold-t *mechanism selects a seller uniformly at random from those whose prices are no higher than* t *for placement in the Buy Box. If all prices are higher than* t, *no seller is in the Buy Box.*

For inspection cost c, let $\tilde{T}(c)$ denote the set of prices implementable at symmetric equilibria under a threshold mechanism.

Theorem 5. *For* $c \in (0, \bar{c})$, $\tilde{T}(c)$ *is an interval, given by*

$$\tilde{T}(c) = [t^*(c), \hat{t}(c)],$$

where $t^*(c) = \sup_{x>0} \frac{x\mathcal{D}_c(x)}{\frac{1}{m} - \mathcal{D}_c(x)}$ *is the same as in Theorem 4, and* $\hat{t}(c) = \sup_{x<0} \frac{(-x)\tilde{\mathcal{D}}_c(x)}{\tilde{\mathcal{D}}_c(x) - \frac{1}{m}}$, *where* $\tilde{\mathcal{D}}_c(x)$ *is defined for* $x < 0$:

$$\tilde{\mathcal{D}}_c(x) = \frac{1}{m}\mathcal{D}_c(x) + \left(1 - \frac{1}{m}\right) \times \left(1 - F(\theta_0(c) + x) + \int_V^{\theta_0(c)-x} (1 - F(v + x))\, \mathrm{d}F^{m-1}(v)\right).$$

If $t^*(c) > \hat{t}(c)$, $\tilde{T}(c) = \emptyset$.

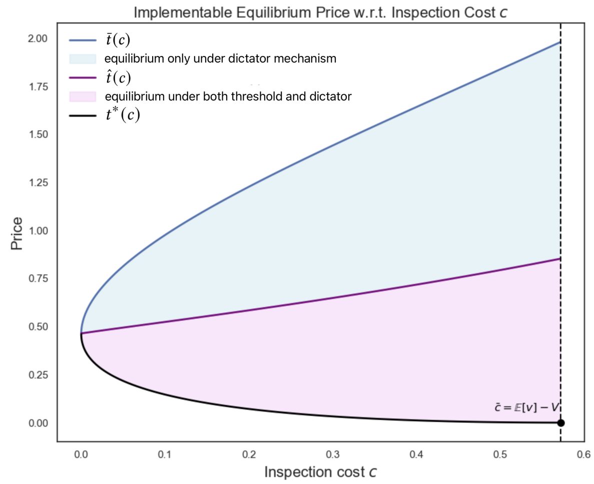

Figure 3: Illustration of implementable prices with threshold mechanisms. For two sellers, with F supported on $[2, 3]$ and pdf proportional to e^{v-2}, the magenta area illustrates the range of prices implementable with the threshold mechanism as the inspection cost c varies. The blue area is the range of prices implementable with the dictator mechanism but not with the threshold mechanism.

The following is immediate from Theorem 5 and Theorem 4.

Corollary 2. *Given* $c \in (0, \bar{c})$, *if* $\tilde{T}(c) \neq \emptyset$, *then the lowest equilibrium price implementable by any standard mechanism is implementable by a threshold mechanism.*

The upper endpoint of the range, $\hat{t}(c)$, is generally strictly smaller than $\bar{t}(c)$, the upper endpoint of $T(c)$. We give the following concrete example.

Proposition 7. *There is an instance where the Threshold-t mechanisms do not encompass all symmetric equilibria.*

The instance has two sellers, where the value distribution, supported on $[2, 3]$, has pdf $f(v) \propto e^{v-2}$. In Figure 3,we visualize $\tilde{T}(c)$, the range of prices implementable by the threshold mechanism, and compare it with $T(c)$, the range of prices implementable by the dictator mechanism. Note that the lowest price implementable by the two mechanisms is the same. In this particular example, both mechanisms admit pure equilibria for any $c \in (0, \bar{c})$.

5 Analysis of Welfare and Surplus

This section analyzes social welfare and consumer surplus at equilibrium under the Buy Box mechanisms. We pay special attention to the impact of the inspection cost changes on welfare and consumer surplus.

By our assumptions of non-degeneracy and Lemma 3, the buyer always makes a purchase. Price change therefore does not directly affect the welfare. This allows us to show that, at a given inspection cost c, the social welfare does not depend on the price at equilibrium.

Theorem 6. *For* $c < \bar{c}$, *under any standard Buy-Box mechanism and any symmetric equilibrium price* t *implemented the mechanism, the* *social welfare is determined by c only, and monotonically decreases as c increases. Consumer surplus* $\mathrm{CS} = \mathrm{SW} - t$.

Proof. Consider any symmetric pure equilibrium of some standard Buy Box mechanism, where the price is t. We show that the welfare does not depend on t. By Lemma 3, we have $t \le V$, i.e., the buyer always makes a purchase.

Without loss of generality, assume seller 1 is in the Buy Box. By Proposition 1, the index of each seller i not in the Buy Box is $\theta_i(t) = \theta_0(c) - t$, and the index of seller 1 is $\theta_1(t) = V + 1 - t$. Recall from Section 2 that the buyer's surplus is $\mathbf{E}[\max_i \kappa_i(t)]^+$, where $\kappa_i(t) = \min(\theta_i(t), v_i - t)$. Since $\theta_0 \ge V$ by assumption (as $c < \bar{c}$), and since $t \le V$, we see that $\kappa_i(t) \ge 0$ for each i. For any other equilibrium price t', we have $\theta_i(t') - \theta(t) = t - t'$ for each i, and hence $\kappa_i(t') - \kappa_i(t') = t - t'$. Therefore, $\mathbf{E}[\max_i \kappa_i(t')]^+ - \mathbf{E}[\max_i \kappa_i(t)]^+ = t - t'$. That is to say, the consumer surplus changes linearly in the opposite direction with the equilibrium price.

On the other hand, the total revenue of the sellers is exactly the equilibrium price since the buyer always makes a purchase. Therefore the welfare does not depend on the equilibrium price.

The fact that the welfare decreases monotonically as c increases is immediate from the argument above, by noticing that $\theta_0(c)$ decreases with c. □

By the theorem, we may write the social welfare at any symmetric equilibrium under a standard mechanism as a function $\mathrm{SW}(c)$.

Corollary 3. *Given $c \in (0, \bar{c})$, the highest consumer surplus achievable under any standard Buy Box mechanisms at a symmetric equilibrium is* $\mathrm{SW}(c) - t^*(c)$.

For a platform that aims to maximize its customers' surplus, how large the search friction c should it set? As c increases, on the one hand, the lowest equilibrium price $t^*(c)$ decreases, which is good for the surplus; on the other hand, search becomes most costly, which hurts the surplus. We experimented with many distributions, and saw that, for most of them, the consumer surplus largely increases as c increases for small values of c. For some distributions, the surplus increases all the way as c approaches $\bar{c}$, whereas for others, the surplus takes its maximum at an intermediate value of c, as we illustrate in Figure 4. The next theorem gives a sufficient condition for the latter to be the case.

Theorem 7. *For two sellers, a sufficient condition for the consumer surplus to be optimized at an intermediate level of inspection cost is* $f(V^+) < \frac{1}{13} f'(V^+)$.

The proof of the theorem is rather technical. One step in the proof that makes use of there being only two sellers is the following relatively succinct expression for the social welfare.

Lemma 4. *Under standard Buy Box mechanism with inspection cost $c \in (0, \bar{c})$, at any symmetric equilibrium, for $m = 2$ sellers,*

$$\mathrm{SW}(c) = \int_V^{\theta_0(c)} F(s)(1 - F(s))\,\mathrm{d}s + \mathbb{E}[v]. \qquad (2)$$

6 Conclusion

This work builds on a classical model of monopolistic competition with search friction and studies market outcomes when a prominent position is allotted to a seller by a mechanism. We find that

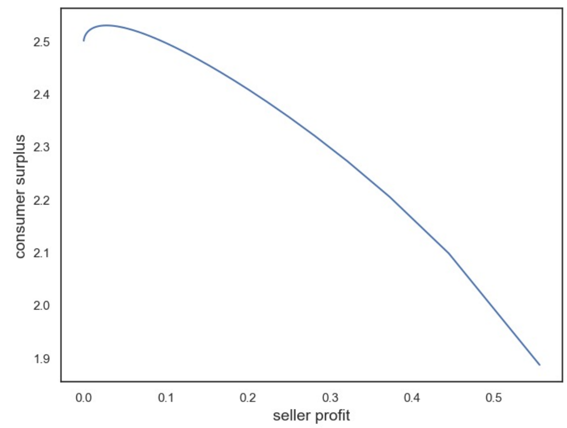

Figure 4: Illustration of trade-off between consumer surplus and seller revenue. For any c, the consumer surplus is maximized when prices are lowest at $t^*(c)$. For two sellers with $F = U[2, 3]$, we plot the Pareto frontier between the maximum consumer surplus and the corresponding seller revenue, as the inspection cost c varies on $(0, \frac{1}{2})$. In this example, the consumer surplus largely increases with the inspection cost c, but attains its maximum before c reaches $\bar{c}$.

visibility of prices, in presence of search frictions, may cause fluctuation in prices, and this can be exacerbated by a mechanism that unobstructedly encourages competition by rewarding prominence. We show that properly designed mechanisms for prominence can stabilize prices. We analyze the range of prices implementable at equilibrium, and propose the threshold mechanism which is detail free, easy to implement, and moderately encouraging competition. With the sellers competing for prominence, the consumer surplus may sometimes increase with higher search friction, the disutility of search being offset by the price decrease due to heightened competition.

Amazon's Buy Box is a salient example of platform-mediated prominence mechanisms and largely motivates this work. We recognize that the Buy Box mechanism in practice is far more complicated, and reacts to much more information than prices set by the sellers. This work takes an analytical approach towards a theoretical understanding: keeping most factors unchanged or symmetric, singling out one factor, that of prices, and studying its behavior in response to the Buy Box mechanisms.

Even though, to derive our analytic results, we specify the buyer's search policy (Weitzman's index-based policy in this case), we believe that certain insights we obtain do not rely on this particular policy. For instance, any policy that searches in the order of decreasing price and does not necessarily exhaust all sellers, is likely to incentivize sellers to undercut each other in order to be searched early.

Prominence in presentation, exemplified by the Amazon Buy Box, is a design simpler than many other forms of presentation, and arguably lends itself more readily for modeling and analysis. We see our work as taking a step towards a fuller understanding of its design principles.

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A Omitted Proofs from Section 3

Proposition 2. *For any $p > 0$, $\mathbf{p} = (p, \dots, p)$ is not an equilibrium in a plain presentation. When $\theta_0 > V$, there is no symmetric equilibrium in a plain presentation.*

Proof. Consider any symmetric price profile $\mathbf{p} = (p, \dots, p)$.

If $p > 0$, we show that there exists a $p' < p$ to which some seller may deviate and improve their revenue. Let θ be the Weitzman index of a seller posting price p. Let A be the event that there are at least two sellers from whom the buyer has value at least $\theta + p$. For $c > 0$, $\Pr[A] > 0$. When A happens, the index algorithm always leads the buyer to buy from such a seller. No matter how the tie is broken, there exists a seller i such that, the probability that the buyer buys from i when A happens is at most $\mathbf{Pr}[A]/m$. On the other hand, let B be the event that $v_i \geq \theta + p$. Then $\mathbf{Pr}[B \cap A] > \mathbf{Pr}[A]/m$. If i deviates to any $p' < p$, with a new index $\theta' > \theta$, then whenever B happens, the buyer buys from i. In particular, when both A and B happens, the buyer buys from i. When $v_i < \theta + p$, the buyer buys from seller i with a probability no less than before the deviation. Therefore, for any $p' < p$, the demand from seller i jump increases by at least $\Delta := \mathbf{Pr}[B \cap A] - \frac{\mathbf{Pr}[A]}{m} > 0$. Let D be the demand from seller i before the deviation. There must exist $p' \in (0, p)$ such that $p'(D + \Delta) > pD$. Such a p' is a profitable deviation for seller i.

For $p = 0$ and $\theta_0 > V$, there is a small enough $p' > 0$ such that the index θ' corresponding to p' is still strictly positive. A seller i deviating to p' sells their product when $\kappa_i > \kappa_j$ for all $j \neq i$. This is true at least when $v_i - p' \geq \theta'$ (in which case $\kappa_i = \theta'$) and $v_j < \theta' < \theta_0$ (in which case $\kappa_j = v_j$). So seller i's demand is at least $(1 - F(\theta' + p'))F^{m-1}(\theta') = (1 - F(\theta_0))F^{m-1}(\theta')$. If $\theta_0 > V$, θ' can be taken to be greater than V, so that this demand is strictly positive, in which case the deviation to p' is profitable for seller i. □

Theorem 1. *In a plain presentation with two sellers, whose values are supported on the interval $[V, V + 1]$ for $V \geq 2$, there is no pure Nash equilibrium.*

Proof. By Proposition 2, we only need to further show that asymmetric equilibria do not exist. For the sake of contradiction, suppose (p_1, p_2), with $p_1 > p_2$, is an equilibrium. Then, $\theta_2 > \theta_1 \geq 0$. It is straightforward to see that the buyer buys from seller 1 with probability at most $\frac{1}{2}$. We show that (p_1, p_2) being an equilibrium would contradict this fact.

The buyer first inspects seller 2, after which, she only inspects seller 1 if $v_2 - p_2 < \theta_1 = \theta_0 - p_1$ (Proposition 1). Lastly, after inspecting seller 1, she buys from seller 1 if $v_2 < v_1 - (p_1 - p_2)$. (Recall from Lemma 1 that the buyer must buy from one of the sellers.) Therefore, the demands of the two sellers are determined by the difference of their prices, $\Delta_p := p_1 - p_2$:

$$D_1(\Delta_p) := \int_V^{\theta_0 - \Delta_p} \left[1 - F(v + \Delta_p)\right] \, \mathrm{d}F(v),$$
$$D_2(\Delta_p) := 1 - D_1(\Delta_p).$$

Consider small unilateral deviations by the two sellers, respectively. Let Δ'_p be close but not equal to Δ_p. Let $p'_1 = p_2 + \Delta'_p$ and $p'_2 = p_2$ (for seller 1's deviation), and $p'_2 = p_1 - \Delta'_p$ and $p'_1 = p_1$ (for seller 2's deviation). When seller 1 deviates, p_2 remains $p_2 < 2$ by Lemma 1. When seller 2 deviates to $p'_2 = p_2 - (\Delta'_p - \Delta_p)$, as long as Δ'_p is close enough to Δ_p, $p'_2 < 2$ still holds. Therefore, the buyer is still guaranteed to always make a purchase after either of these deviations. Therefore, the demands for both sellers' can still be expressed in terms of the price difference Δ'_p:

$$D_1(\Delta'_p) := \int_V^{\theta_0 - \Delta'_p} \left[1 - F(v + \Delta'_p)\right] \, \mathrm{d}F(v),$$
$$D_2(\Delta'_p) := 1 - D_1(\Delta'_p).$$

Since (p_1, p_2) is assumed to be an equilibrium, we have:

$$\text{Seller 1:} \quad p'_1 D_1(\Delta'_p) = (p_2 + \Delta'_p) D_1(\Delta'_p) \le p_1 D_1(\Delta_p);$$
$$\text{Seller 2:} \quad p'_2 D_2(\Delta'_p) = (p_1 - \Delta'_p) D_2(\Delta'_p) \le p_2 D_2(\Delta_p).$$

Since $D_1(\Delta'_p) + D_2(\Delta'_p) \equiv 1$, we can combine these inequalities:

$$1 - \frac{p_2 D_2(\Delta_p)}{p'_2} \le D_1(\Delta'_p) \le \frac{p_1 D_1(\Delta_p)}{p'_1}. \tag{3}$$

As Δ'_p approaches Δ_p, the left hand side and the right hand side both approach $\mathcal{D}_1(\Delta_p)$ (Figure 2). For these inequalities to hold in a neighborhood of Δ_p, the derivatives of the left-hand side and right-hand side with respect to Δ'_p at Δ_p must be equal, implying:

$$-\frac{p_2 D_2(\Delta_p)}{(p_2)^2} = -\frac{p_1 D_1(\Delta_p)}{(p_1)^2}.$$

Solving this along with $D_1(\Delta_p) + D_2(\Delta_p) = 1$ (Lemma 1), we find that $D_1(\Delta_p) = \dfrac{p_1}{p_1 + p_2} > \dfrac{1}{2}$ (since $p_1 > p_2$ by assumption), which contradicts the earlier observation that $D_1(\Delta_p) \le \dfrac{1}{2}$.

Therefore, (p_1, p_2) with $p_1 > p_2$ cannot be an equilibrium. Overall, under plain presentation without a Buy Box, no pure equilibrium exists in the presence of inspection costs for two i.i.d. sellers. □

Lemma 1. *In a plain presentation, for F supported on $[V, V+1]$ with $V \ge 2$, the buyer always makes a purchase if the the sellers' prices reach an equilibrium.*

Proof. Suppose $\mathbf{p} := (p_1, p_2, \dots, p_m)$ is an equilibrium. Without loss of generality, suppose $p_1 \ge p_2 \ge \dots \ge p_m$, and $0 \le \theta_1 \le \dots \le \theta_m$.

- If there is some $p_i = 0$, then $p_m = 0$. and $\kappa_m \ge 0$ by our assumption on the inspection cost. In this case, the buyer would at least inspect seller m and obtain positive utility $v_m - p_m = v_m \ge V$ if purchasing from seller m.
- If $\forall i, p_i > 0$, notice that any seller i should have strictly positive revenue $\mathrm{Rev}_i(\mathbf{p}) > 0$ in an equilibrium. This is because, fixing all others' prices, seller i can always post a positive price smaller than all others' to be inspected first and to guarantee some revenue.
  Now, any seller i must have $p_i \le p_m + 1$, otherwise a buyer always prefers seller m to i (since $v_i \le v_m + 1$ with probability 1, and seller m is inspected before i), and i's demand must be zero. On the other hand, if any seller deviates to a price lower than $p_m - 1$, he would get all the demand. In an equilibrium, since any seller cannot be better off with this deviation, we have $p_i D_i(\mathbf{p}) \ge p_m - 1, \forall i \in [m]$. Summing these inequalities over i, we have:

$$\begin{aligned} m &\ge m p_m - \sum_{i \in [m]} p_i D_i(\mathbf{p}) \\ &\ge m p_m - \sum_{i \in [m]} p_1 D_i(\mathbf{p}) \\ &\ge m p_m - p_1 \ge m p_m - 1 - p_m, \end{aligned}$$

  which gives $p_m \le \frac{m+1}{m-1} < 2$. Therefore, the buyer at least finds seller m's product desirable (i.e. $v_m - p_m > 0, \forall v_m \in [V, V+1]$ for $V \ge 2$), and must make a purchase from some seller.

□

Theorem 2. *In a plain presentation, for any F supported on $[V, V+1]$ and any $c > 0$, there exists a $\Delta > 0$, such that for any positive $\epsilon < \Delta$, there is no pure symmetric ϵ-equilibrium.*

Proof. Consider symmetric price profile $(p, p, \dots, p)$, we will show that this cannot be any ϵ-equilibrium for any $\epsilon \le \Delta$. Δ is given by

$$\Delta = \min\left(\frac{1}{m} r(x^*), (m-1) r(x^*) \left(\Pr[B_0 \cap A_0] - \frac{\Pr[A_0]}{m}\right)\right)$$

where

- $r(x)$ as will be shown later, corresponds to seller i's post-deviate demand:
  $$r(x) := x \int_V^{\theta_0 - x} (1 - F(v + x)) \, \mathrm{d}F^{m-1}(v).$$
  And, $x^* = \arg\max_{x \ge 0} r(x)$.
- A_0 is the event that there are at least two sellers from whom the buyer has value at least θ_0. B_0 is the event that $v_i \ge \theta_0$, for a given seller i.

We will now show that any price p cannot support symmetric ϵ-equilibrium $(p, \dots, p)$ for $\epsilon < \Delta$. Discuss over the value of p:

- Case: $p \le (m-1) r(x^*)$:
  When seller i deviates to $p_i > p$, she will be inspected last in plain presentation. Her demand is
  $$\begin{aligned} &D_i(p_i; p_{-i} = p) \\ &= \int_V^{\theta_0 - (p_i - p)} (1 - F(v + p_i - p)) \, \mathrm{d}F^{m-1}(v). \end{aligned}$$
  Take $x = p_i - p, x > 0$, notice that
  $$r(x) = (p_i - p) D_i(p_i; p_{-i} = p)$$
  Observe that $x^* = \arg\max_{x \ge 0} r(x) > 0$ because first, $r(x) < 1$. And at least if we take, say $\hat{x} = V + \frac{\theta_0}{2}$, we have
  $$\begin{aligned} r(\hat{x}) &= \hat{x} \int_V^{V + \frac{\theta_0}{2}} (1 - F(v + \hat{x})) \, \mathrm{d}F^{m-1}(v) \\ &\ge \hat{x} \int_V^{V + \frac{\theta_0}{2}} (1 - F(\theta_0)) \, \mathrm{d}F^{m-1}(v) \\ &= \left(V + \frac{\theta_0}{2}\right) \frac{\theta_0}{2} (1 - F(\theta_0)) F^{m-1} \left(V + \frac{\theta_0}{2}\right) \\ &> 0 = r(0). \end{aligned}$$
  So, it's safe to conclude that $x^* > 0$ and $r(x^*) > 0$.

At symmetric price p, seller i can deviates to $p_i = p + x^*$ to obtain at least Δ-more revenue: (notice that, at symmetric price p, every seller's revenue is less than $\frac{1}{m}p$)

$$\begin{aligned}\text{Rev}_i\,(p,\dots,p) + \Delta &\leq \frac{1}{m}p + \Delta \\ &\leq \frac{m-1}{m}r(x^*) + \frac{1}{m}r(x^*) \\ &= r(x^*) \quad (\text{let } p_i = x^* + p) \\ &= (p_i - p)D_i(p_i; p_{-i} = p) \\ &\leq p_i D_i(p_i; p_{-i} = p).\end{aligned}$$

In other words, this breaks the ϵ-equilibrium for $\epsilon < \Delta$.

- Case: $p > (m-1)r(x^*)$ and $\theta \leq 0$: for price profile $(p,\dots,p)$, all seller's revenue would be zero because the buyer does not even care to inspect any seller. For seller i, by posting price $p_i = V$, by assumption $\theta_0 > V$, her index becomes positive:
$$\mathbb{E}[(v_i - V) - 0]^+ \geq \mathbb{E}[v_i - \theta_0]^+ = c.$$
Then, every buyer would inspect and buy from seller i, so that seller i's revenue rises to V. So this deviate breaks the ϵ-equilibrium.
- Case: $p > (m-1)r(x^*)$ and $\theta > 0$:
Similar as in proof A of Proposition 2, let A be the event that there are at least two sellers from whom the buyer has value at least $\theta + p$. Let B be the event that $v_i \geq \theta + p$. By Proposition 1, $\theta = \theta_0 - p$. So $A_0 = A$ and $B_0 = B$.
As demonstrated earlier in in proof A of Proposition 2, the demand for seller i when he deviates to some $p_i < p$, fix all other $p_{-i} = (p,\dots,p)$, increases at least by $\Pr[B \cap A] - \frac{\Pr[A]}{m}$. Assume seller i deviates to $p - \delta$:
$$\text{Rev}_i\,(p,\dots,p) + \Delta \tag{4}$$
$$\leq \text{Rev}_i\,(p,\dots,p) + (p-\delta)\left(\Pr[B\cap A] - \frac{\Pr[A]}{m}\right) \tag{5}$$
$$\leq \text{Rev}_i\,(p_i = p-\delta, p_{-i} = (p,\dots,p)) \tag{6}$$
Inequality (4) holds by the assumption we made for Δ. Inequality (5) holds as $\delta \to 0^+$. (6) holds from proof A of Proposition 2.

Therefore, in any cases for symmetric price profile p, it cannot be an ϵ-equilibrium for $\epsilon < \Delta$. □

B Omitted Proofs from Section 4

Proposition 3. *(a) For any $p > 0$, the symmetric price profile $\mathbf{p} = (p,\dots,p)$ is not an equilibrium under the LPF mechanism. If $\theta_0 \geq V$, LPF admits no symmetric equilibrium.*
(b) With two sellers, the LPF mechanism admits no pure equilibrium.

Proof. (a) Consider any symmetric price profile $\mathbf{p} = (p,\dots,p)$. Compared with presentation without Buy Box, deviating to a lower price is even (weakly) more profitable, and so, for $p > 0$, an argument identical to that of Proposition 2 shows that there exists a profitable deviation $p' < p$.
For $p = 0$ and $\theta_0 > V$, we show there is a small enough $p' > 0$ such that the index θ' corresponding to p' is still strictly positive. A seller i deviating to p' sells their product when $\kappa_i > \kappa_j$ for all $j \neq i$. The only difference from the proof of Proposition 2 is that here seller j maybe in the Buy Box, in which case their index is $V + 1$. We see that essentially the same argument goes through: we have $\kappa_i > \kappa_j$ at least when $v_i - p' \geq \theta'$ (in which case $\kappa_i = \theta'$) and $v_j < \theta' < \max(\theta_0, V+1)$ (in which case $\kappa_j = v_j$). So seller i's demand is at least $(1 - F(\theta' + p'))F^{n-1}(\theta') = (1 - F(\theta_0))F^{n-1}(\theta')$. If $\theta_0 > V$, θ' can be taken to be greater than V, so that this demand is strictly positive, in which case the deviation to p' is profitable for seller i.
(b) This directly follows from Theorem 1. For i.i.d. sellers, the LPF mechanism prioritizes the lower-priced seller, who is also inspected first in scenarios without a Buy Box. The behavior under plain presentation without the Buy Box is analogous to that under the LPF mechanism (Lemma 1). Consequently, if no equilibrium exists in the plain presentation scenario, it follows that no equilibrium can be sustained under the LPF mechanism either.

□

Theorem 3. *For any standard Buy Box mechanism $\mathcal{M}$, if $\mathbf{p} = (p,\dots,p)$ is a symmetric equilibrium, then $\mathbf{p}$ is also an equilibrium for the Dictator-p mechanism.*

Proof. Consider the probability that the buyer purchases from a seller priced at q, when all other sellers are priced at p: if the seller with price q is in the Buy Box, let this probability be denoted as $D_{\text{BB}}(p,q)$; otherwise it is denoted by $D_{\text{NBB}}(p,q)$. (Note that in the latter case, one of the sellers who post price p *is* in the Buy Box.) It is evident that $D_{\text{BB}}(p,q) \geq D_{\text{NBB}}(p,q)$ for all p and q.

Since $\mathcal{M}$ is anonymous and always allocates, if all sellers post price p, they are in the Buy Box each with probability $1/m$; the same is true for the Dictator-p mechanism. So at price profile $\mathbf{p}$, each seller's revenue in $\mathcal{M}$ is the same as in the Dictator-p mechanism. If any seller unilaterally deviates to price q in $\mathcal{M}$, with certain probability the seller is in the Buy Box, in which case her demand is $D_{\text{BB}}(q,p)$, and with the rest of the probability the seller is not in the BuyBox, while another seller with price p is shown in the BuyBox, in which case the seller's demand is $D_{\text{NBB}}(q,p)$. The same deviation to q in the Dictator mechanism would result in demand $D_{\text{NBB}}(q,p)$ with probability 1. Since $D_{\text{NBB}}(q,p) \leq D_{\text{BB}}(q,p)$, if q is not a profitable deviation in $\mathcal{M}$, it is not profitable in the Dictator mechanism either.

□

Lemma 3. *If $V \geq 2$ and $c \leq \bar{c}$, then under any equilibrium of any Buy-Box mechanism, the buyer always makes a purchase.*

Proof. Suppose $\mathbf{p} = (p_1, p_2, \dots, p_m)$ is an equilibrium price profile, ordered such that $p_1 \geq p_2 \geq \cdots \geq p_m$. We consider two cases based on the value of the lowest price p_m:

- **Case 1:** $p_m \leq V$**.**
In the Buy-Box mechanism, the buyer is first presented with the Buy-Box seller's value $\hat{v}$ and price $\hat{p}$ without incurring the inspection cost. The buyer's utility from the Buy-Box seller is $u_{\text{BB}} = \hat{v} - \hat{p}$.
  - If $u_{\text{BB}} \geq 0$, the buyer will at least purchase from the Buy-Box seller.

– If $u_{\text{BB}} < 0$, the buyer considers inspecting seller m. The expected utility gain from inspecting seller m is:

$$\begin{aligned} &\mathbb{E}\left[v_m - p_m\right]^+ - c \\ =& \mathbb{E}[v_m] - p_m - c \\ \geq& \mathbb{E}[v] - V - c \geq 0. \end{aligned}$$

The above inequalities come from the assumptions $p_m \leq V$ and $c \leq \bar{c} = \mathbb{E}[v] - V$.
In this case, when the lowest-price seller is not presented in the Buy-Box, if the buyer finds the Buy-Box seller's offer to be of negative utility, the buyer finds herself better off to further inspect seller m. After inspection, the buyer's utility from purchasing from seller m is at least $v_m - p_m \geq 0$, so the buyer will make a purchase from seller m.

- **Case 2:** $p_m > V$**.** We will show that this case cannot occur in equilibrium.
Consider a seller deviating to $p_m - 1$. If this seller is in the Buy-Box, the buyer would simply purchase from this seller. But, even if the seller is not in the Buy-Box, the buyer would **always** opt to inspect and purchase from this deviating seller. This is because as shown below, the expect utility gain from inspecting this deviating seller is positive, regardless of what the Buy Box offers:

$$\begin{aligned} &\mathbb{E}\left[v - (p_m - 1) - u_{\text{BB}}\right]^+ - c \\ \geq& \mathbb{E}\left[v - (p_m - 1) - (V + 1 - p_m)\right]^+ - c \\ =& \mathbb{E}\left[v - V\right]^+ - c \geq 0. \end{aligned}$$

After inspecting this deviating seller, the buyer will prefer the deviating seller's product since the price is lower:

$$\begin{aligned} &v - (p_m - 1) \\ \geq& V - (p_m - 1) \\ =& (V + 1) - p_m \\ \geq& \max_i (v_i - p_i). \end{aligned}$$

Therefore, similar to Lemma 1, assuming $p_m > V$, when any seller deviates and sets their price to $p_m - 1$, their demand increases to 1. For the equilibrium condition to hold, it is required that each seller does not find this deviation profitable; that is, for any seller $i \in [m]$,

$$\text{Rev}_i(\mathbf{p}) = p_i D_i(\mathbf{p}) \geq p_m - 1$$

Notice the seller's equilibrium revenue is supposed to be strictly positive, given we assumed that $p_m - 1 \geq V - 1 \geq 0$. Summing these inequalities over i, we have

$$\sum_{i=1}^{m} p_i D_i(\mathbf{p}) \geq m(p_m - 1).$$

On the other hand, any seller must have $p_i \leq p_m + 1$. Because otherwise, $p_i > p_m + 1 > V + 1$ and this seller's demand and revenue would be zero. So

$$\sum_{i=1}^{m} p_i D_i(\mathbf{p}) \leq \sum_{i=1}^{m} (p_m + 1) D_i(\mathbf{p}) \leq p_m + 1,$$

Combining the inequalities B and B, we get

$$p_m \leq \frac{m-1}{m+1} \leq 2.$$

This contradicts the assumption that $p_m > V \geq 2$.

To conclude, the second case $p_m > V$ cannot occur in equilibrium. Thus, in equilibrium, $p_m \leq V$, and from the above analysis of Case 1, the buyer always makes a purchase at those prices. □

Proposition 6. *For sufficiently large c, there exists a Buy Box mechanism with an equilibrium at which prices are all higher than V.*

Proof. Consider the Dictator's Mechanism that sets the target price at the monopoly price:

$$p^\star := \arg\max_p \, p\,(1 - F(p)).$$

As long as the inspection cost is high enough ($c > \mathbb{E}[v] - \frac{p^\star}{m}$), it is optimal for every seller to maintain the monopoly price $p^\star$. This is because any seller deviating to a price $p' \neq p^\star$ would lose the Buy-Box status and could only be inspected if all other sellers' values satisfy $(v - p^\star)_+ \leq \theta_0 - p'$. This requires that p' is low enough: $p' \leq \theta_0$. Therefore, the deviating seller's revenue is at most θ_0.

Furthermore, as long as the monopoly profit satisfies:

$$\frac{p^\star}{m} \geq \theta_0,$$

and under our assumption of a high enough inspection cost c:

$$\mathbb{E}\left[(v - \theta_0)^+\right] = c > \mathbb{E}\left[\left(v - \frac{p^\star}{m}\right)^+\right] \implies \frac{p^\star}{m} \geq \theta_0,$$

the monopoly price vector $\vec{p}^\star = (p^\star, \ldots, p^\star)$ will be an equilibrium. □

Lemma 2. *[Demand for a Seller Deviating from the Symmetric Equilibrium under the Dictator Mechanism] For $c \in (0, \bar{c})$, consider a symmetric equilibrium price t. If a seller deviates to a price p, then their demand is a function of the amount of deviation $x = p - t$. Specifically, the demand after deviating to p is*

$$\mathcal{D}_c(x) := \begin{cases} 1, \quad \text{if}\, x \leq -1, \\ \int_V^{\theta_0(c)-x} \Big[\int_{\max(\theta_0(c)+x, v_2+x)}^{V+1} \mathrm{d}F(v_1)\,\mathrm{d}F(v_2) \\ \quad + \int_{v_2+x}^{\max(\theta_0(c)+x, v_2+x)} F^{m-2}(v_1 - x)\,\mathrm{d}F(v_1) \Big] \,\mathrm{d}F(v_2), \\ \qquad \text{if} -1 < x < 0, \\ \int_V^{\theta_0(c)-x} (1 - F(v + x))\,\mathrm{d}F^{m-1}(v), \\ \qquad \text{if}\, 0 \leq x < \theta_0(c) - V, \\ 0, \quad \text{if}\, \theta_0(c) - V \leq x. \end{cases} \tag{1}$$

Proof. Since the equilibrium is symmetric, without loss of generality, assume that the deviating seller is seller 1. We may occasionally omit the seller subscript when there is no ambiguity. Let $x = p' - t$, where p' is the deviated price. We will analyze the demand case by cas with respect to x's value.

- **Case** $x \leq -1$ **(i.e.,** $p' < t + 1$**).** This scenario is covered in the proof of Lemma 3. In this case, the seller's demand is equal to 1.

- **Case** $-1 < x < 0$ **(i.e.,** $t-1 < p' < t$**).** Note that, in general, p' could be less than 0. However, this does not affect our analysis of the demand, although setting such a price would result in negative revenue for the seller.
  Under the Dictator Mechanism, the deviating seller does not receive any Buy Box allocation. Since all other sellers are identical, to analyze the deviating seller's demand, we assume that seller 2 receives the Buy Box. There are two mutually exclusive scenarios in the buyer's search behavior that result in a purchase from seller 1:
  - **Scenario 1:** The buyer sees seller 1 in the Buy Box, inspects seller 2, buys from seller 1, and then leaves. This requires:
    (i) The buyer chooses to inspect seller 1 after observing seller 2's value in the Buy Box:
    $$v_2 - t < \theta_0(c) - p' \iff: v_2 < \theta_0(c) - x.$$
    (ii) After inspecting seller 1, the buyer prefers seller 1 over seller 2:
    $$v_1 - p' \geq v_2 - t \iff v_1 \geq v_2 + x.$$
    (iii) After inspecting seller 1, the buyer does not inspect any additional sellers:
    $$\begin{aligned} &\max(v_1 - p', v_2 - t) \geq \theta_0(c) - t \\ \iff &\underbrace{\max(v_1 - x, v_2) = v_1 - x}_{\text{by condition ii}} \geq \theta_0(c). \end{aligned}$$
    The demand contribution from this scenario is
    $$\int_V^{\max(\theta_0(c)-x,V)} \int_{\max(\theta_0(c)+x,v_2+x)}^{V+1} \mathrm{d}F(v_1)\,\mathrm{d}F(v_2).$$
  - **Scenario 2:** The buyer sees seller 1 in the Buy Box, first inspects seller 2, then inspects all sellers, compares all products, and finally buys from seller 1 before leaving. This requires:
    (i) The buyer chooses to inspect seller 1 after observing seller 2's value in the Buy Box:
    $$v_2 - t < \theta_0(c) - p' \iff v_2 < \theta_0(c) - x.$$
    (ii) After inspecting all sellers, the buyer prefers seller 1's product:
    $$\begin{aligned} &v_1 - p' \geq \max_{i=2,\ldots,m}\{v_i - t\} \\ \iff &v_1 - x \geq \max_{i=2,\ldots,m}\{v_i\}. \end{aligned}$$
    (iii) The buyer chooses to inspect all remaining sellers ($i = 3, \ldots, m$):
    $$\begin{aligned} &\max\left(v_1 - p', \max_{i=2,\ldots,m}\{v_i - t\}\right) \geq \theta_0(c) - t \\ \iff &\underbrace{\max\left(v_1 - x, \max_{i=2,\ldots,m}\{v_i\}\right) = v_1 - x}_{\text{by condition ii}} \geq \theta_0(c). \end{aligned}$$
    The demand contribution from this scenario is
    $$\int_V^{\max(\theta_0(c)-x,V)} \int_{v_2+x}^{\max(\theta_0(c)+x,v_2+x)} F^{m-2}(v_1 - x)\,\mathrm{d}F(v_1)\,\mathrm{d}F(v_2).$$

  These two scenarios are mutually exclusive. Therefore, the total demand of the deviating seller when setting the price $p' = t + x$ for $x \in (-1, 0)$ is
  $$\begin{aligned} &\mathcal{D}_c(x) := \\ &\int_V^{\max(\theta_0(c)-x,V)} \int_{\max(\theta_0(c)+x,v_2+x)}^{V+1} \mathrm{d}F(v_1)\,\mathrm{d}F(v_2) \\ &+ \int_V^{\max(\theta_0(c)-x,V)} \int_{v_2+x}^{\max(\theta_0(c)+x,v_2+x)} \\ &F^{m-2}(v_1 - x)\,\mathrm{d}F(v_1)\,\mathrm{d}F(v_2). \end{aligned}$$
- **Case** $0 \leq x < \theta_0(c) - V$ **(i.e.,** $t \leq p' < \theta_0(c)$**).** In this case, particularly for $p' = t$ and define $\mathcal{D}_c(0) := \lim_{x\to 0^+} \mathcal{D}_c(x)$. For $x \in (0, \theta_0(c) - V)$, since the deviating seller's price $p' = t + x > t$, they are inspected last only if the values of all other sellers $i = 2, \ldots, m$ satisfy $v_i \leq \theta_0(c) - x$. Additionally, the deviating seller 1 is purchased from only if
  $$v_1 - p' \geq \max_{i=2,\ldots,m}\{v_i - t\}.$$
  These conditions translate to
  $$\mathcal{D}_c(x) = \int_V^{\theta_0(c)-x} (1 - F(v + x))\ \mathrm{d}F^{m-1}(v).$$
- **Case** $x \geq \theta_0(c) - V$ **(i.e.,** $p' > \theta_0(c)$**).** In this scenario, the deviating seller 1 will never be inspected. Suppose that seller 2 in the Buy Box yields a value v_2:
  $$\begin{aligned} v_2 - t &\geq V - t \\ &= V - (p' - x) = V - x - p' \geq \theta_0(c) - p'. \end{aligned}$$
  Since buyers will never choose to inspect seller 1, the seller's demand is zero.

This concludes the proof. The deviating seller's demand is thus expressed as a function of the deviation $x = p' - t$, *independent from* t, where p' is the deviated price and t is the symmetric equilibrium price of the remaining sellers. □

Theorem 4. *For $c \in (0, \bar{c})$, the set of implementable prices $T(c)$ is a closed interval determined by the inspection cost c:*
$$T(c) = [t^*(c), \bar{t}(c)],$$
if $t^*(c) \leq \bar{t}(c)$, where $t^*(c) := \sup_{x>0}\left\{\frac{x\mathcal{D}_c(x)}{\frac{1}{m} - \mathcal{D}_c(x)}\right\}$ and $\bar{t}(c) := \inf_{x<0}\left\{\left(\frac{(-x)\mathcal{D}_c(x)}{\mathcal{D}_c(x) - \frac{1}{m}}\right)^+\right\} \setminus \{0\}$.

When $t^*(c) > \bar{t}(c)$, no standard mechanism admits any symmetric equilibrium.

Proof. Theorem 3 states that it is sufficient to consider dictator's mechanism for any implementable equilibrium price. Consider Dictator mechanism setting target price at t, inducing symmetric equilibrium $(t, t, \ldots, t)$.

If any seller deviates to $p' = t+x$, Lemma 2 gives the seller's post-deviate demand expressed in x: $\mathcal{D}_c(\cdot)$. The equilibrium condition can then be expressed as:

$$\underbrace{(x+t)}_{p'=x+t} \cdot \underbrace{\mathcal{D}_c(x)}_{\text{seller 1's demand}} \le \underbrace{\frac{1}{m}t}_{\text{Eq. revenue}}, \quad \forall x \in \mathbb{R} \tag{7}$$

By transforming this condition algebraically, we obtain

$$\left(\frac{1}{m} - \mathcal{D}_c(x)\right) t \ge x\mathcal{D}_c(x). \tag{8}$$

Discuss over potential value of x (i.e. $p' = t + x$).

- **Case** $x > 0$ **(i.e.** $p = x + t > t$). It's evident that post-deviate demand $\mathcal{D}_c(x) < \frac{1}{m}$. Hence the above condition translates to
$$t \ge \frac{x\mathcal{D}_c(x)}{\frac{1}{m} - \mathcal{D}_c(x)}, \forall x > 0.$$
This gives a lower bound for implementable price t.
- **Case** $x < 0$ **(i.e.** $p = x + t < t$). Notice that the left-hand-side of the inequality 8, $x\mathcal{D}_c(x) < 0$ — so the condition automatically holds for $\frac{1}{m} - \mathcal{D}_c(x) \ge 0$. Now considers x such that $\mathcal{D}_c(x) > \frac{1}{m}$, inequality 8 translates to
$$t \le \frac{(-x)\mathcal{D}_c(x)}{\mathcal{D}_c(x) - \frac{1}{m}}, \quad \forall x < 0 \text{ and } \frac{1}{m} - \mathcal{D}_c(x) < 0$$
.

And this sums up to be the condition for the theorem. □

As an example, for the uniform distribution $F = U[2, 3]$ and two sellers, we work out this interval. For this distribution, $\bar{c} = \frac{1}{2}$. For $c \in (0, \bar{c})$,

$$t^*(c) = \frac{\sqrt{-((2\sqrt{2c} - 2c) + \frac{1}{2}) + \sqrt{2(2\sqrt{2c} - 2c) + \frac{1}{4}}}}{\frac{1}{2}\sqrt{2(2\sqrt{2c} - 2c) + \frac{1}{4}} - 1} \times \frac{1}{2}\left(\frac{3}{2} - \sqrt{2(2\sqrt{2c} - 2c) + \frac{1}{4}}\right)$$
$$\bar{t}(c) = 2.$$

For any $c \in (0, \bar{c})$, $t^*(c) < \bar{t}(c)$. Therefore for all small enough c, the dictator mechanism admits a pure equilibrium.

Find $t^*(c), \bar{t}(c)$ for two uniform seller. For convenience, use θ to denote $\theta_0(c)$. First calculate $\mathcal{D}_c(x)$:

$$\mathcal{D}_c(x) = \begin{cases} 1 & x < -1 \\ \frac{1}{2}x^2 + \frac{1}{2} & x \in [-1, \theta - 3) \\ -x + \frac{1}{2}(\theta - 2)^2 & x \in [\theta - 3, 0) \\ -\frac{1}{2}x^2 + \frac{1}{2}(\theta - 2)^2 & x \in [0, \theta - 2) \\ 0 & x \ge \theta - 2 \end{cases}$$

Find $t^*(c)$ and $\bar{t}(c)$: let

$$\chi^c(x) := \frac{x\mathcal{D}_c(x)}{\frac{1}{2} - \mathcal{D}_c(x)}$$

then

$$\chi^{c\prime}(x) = \frac{1}{(\frac{1}{2} - \mathcal{D}_c(x))^2}\left(\mathcal{D}_c(x)(\frac{1}{2} - \mathcal{D}_c(x)) + \frac{1}{2}xD_c'(x)\right).$$

Let $\tilde{\chi}(x) := \mathcal{D}_c(x)(\frac{1}{2} - \mathcal{D}_c(x)) + \frac{1}{2}xD_c'(x)$,

$$\tilde{\chi}^{c\prime}(x) = \mathcal{D}_c'(x)(1 - 2\mathcal{D}_c(x)) + \frac{1}{2}x\mathcal{D}_c''(x).$$

To get $t^*(c) = \sup_{x>0} \chi^c(x)$, notice that only needs to consider $x \in (0, \theta - 2)$ where $\mathcal{D}_c(x) > 0$. Within this region, $\mathcal{D}_c(x) < \frac{1}{2}$, $\mathcal{D}_c(x) = -x < 0$ and $\mathcal{D}_c''(x) = -1$, therefore

$$\tilde{\chi}^{c\prime}(c) \le 0.$$

And

$$\tilde{\chi}^c(x) = -\frac{1}{4}x^4 + (\frac{1}{2}(\theta - 2)^2 - \frac{1}{2})x^2 + \frac{1}{2}(\theta - 2)^2\left(\frac{1}{2} - \frac{1}{2}(\theta - 2)^2\right).$$

So we can solve for the unique interior supremum point x^* that maximizes $\chi^c(x)$:

$$(x_c^*)^2 = -(\hat{\theta} + \frac{1}{2}) + \sqrt{2\hat{\theta} + \frac{1}{4}},$$

where $\hat{\theta} = 1 - (\theta - 2)^2$. Plug in x_c^* into $\chi^c(x)$ (because $t^*(c) = \sup_{x>0} \chi^c(x)$) we obtain

$$\begin{aligned} t^*(c) &= \frac{x_c^*\mathcal{D}_c(x_c^*)}{\frac{1}{2} - \mathcal{D}_c(x_c^*)} \\ &= \frac{\sqrt{-(\hat{\theta} + \frac{1}{2}) + \sqrt{2\hat{\theta} + \frac{1}{4}}} \times \frac{1}{2}\left(\frac{3}{2} - \sqrt{2\hat{\theta} + \frac{1}{4}}\right)}{\frac{1}{2}\sqrt{2\hat{\theta} + \frac{1}{4}} - 1} \\ &= \frac{\sqrt{-((2\sqrt{2c} - 2c) + \frac{1}{2}) + \sqrt{2(2\sqrt{2c} - 2c) + \frac{1}{4}}}}{\frac{1}{2}\sqrt{2(2\sqrt{2c} - 2c) + \frac{1}{4}} - 1} \\ &\quad \times \frac{1}{2}\left(\frac{3}{2} - \sqrt{2(2\sqrt{2c} - 2c) + \frac{1}{4}}\right) \end{aligned}$$

Vice versa, we can directly calculate $\bar{t}(c)$.

For $x \in [\theta - 3, 0)$, let $\tilde{\theta} = (\theta - 2)^2$, so $\mathcal{D}_c(x) = -x + \frac{1}{2}\tilde{\theta}$, and

$$\begin{aligned} \chi^c(x) &= \frac{x(-x + \frac{1}{2}\tilde{\theta})}{\frac{1}{2} - (-x + \frac{1}{2}\tilde{\theta})} \\ &= (\frac{1}{2}\tilde{\theta} - x - \frac{1}{2}) + \frac{\frac{1}{4}(1 - \tilde{\theta})}{(\frac{1}{2}\tilde{\theta} - x - \frac{1}{2})} + 1 - \frac{1}{2}\tilde{\theta} \\ &\ge 2. \end{aligned}$$

As for $x \in [-1, \theta - 3)$,

$$\begin{aligned} \chi^c(x) &= \frac{x(-x^2 + \frac{1}{2})}{-\frac{1}{2}x^2} \\ &= -(x + \frac{1}{x}) \ge 2. \end{aligned}$$

Therefore,

$$\bar{t}(c) \equiv 2.$$

□

Proposition 4. *[Properties of lowest implementable price] For* $c \in (0, \bar{c})$, $t^*(c)$ *is* continuous *and* monotonically decreasing *in* c. *Moreover,*

(i) $\lim_{c \to \bar{c}^-} t^*(c) = 0$;
(ii) $\lim_{c \to 0^+} t^*(c) = t_0$, *if a symmetric equilibrium* $(t_0, \ldots, t_0)$ *exists for the monopolistic competition without cost of inspection.*

Proof of Proposition 4. We will first prove that $t^*(\cdot)$ is monotonically decreasing and continuous. Then prove its endpoint property.

Monotonicity of $t^*(c)$. Theorem 4 gives

$$t^*(c) := \sup_{x>0}\left\{\frac{x\mathcal{D}_c(x)}{\frac{1}{m}-\mathcal{D}_c(x)}\right\}.$$

Fix x, consider inspection costs $c_1 \le c_2$ with $\theta_0(c_1) \ge \theta_0(c_2)$, we have, for $x \ge 0$:

$$\begin{aligned}\mathcal{D}_{c_1}(x) - \mathcal{D}_{c_2}(x) &= \int_V^{\theta_0(c_1)-x} (1-F(v+x))\,\mathrm{d}F^{m-1}(v)\\ &\quad - \int_V^{\theta_0(c_2)-x} (1-F(v+x))\,\mathrm{d}F^{m-1}(v)\\ &= \int_{\theta_0(c_2)-x}^{\theta_0(c_1)-x} (1-F(v+x))\,\mathrm{d}F^{m-1}(v) \ge 0.\end{aligned}$$

So, fix x, $\mathcal{D}_c(x)$ decreases as c increases. Therefore, for any $c_1 \ge c_2$, $\mathcal{D}_{c_1}(x) \le \mathcal{D}_{c_2}(x)$. For any $x > 0$:

$$\begin{aligned}\frac{x\mathcal{D}_{c_1}(x)}{\frac{1}{m}-\mathcal{D}_{c_1}(x)} &= (-x) + \frac{\frac{1}{m}}{\frac{1}{m}-\mathcal{D}_{c_1}(x)}\\ &\le (-x) + \frac{\frac{1}{m}}{\frac{1}{m}-\mathcal{D}_{c_2}(x)} = \frac{x\mathcal{D}_{c_2}(x)}{\frac{1}{m}-\mathcal{D}_{c_2}(x)}.\end{aligned}$$

This implies $t^*(c_2) \ge t^*(c_1)$, hence proving $t^*(\cdot)$ monotonically decreases.

Continuity of $t^*(c)$. The expression $t^*(c) := \sup_{x>0}\left\{\frac{x\mathcal{D}_c(x)}{\frac{1}{m}-\mathcal{D}_c(x)}\right\}$ takes supremum over $x > 0$. Because $\mathcal{D}_c(x) = 0$ for $x \ge \theta_0(c) - V$, it suffices to consider the open interval $x \in (0, \theta_0(c) - V)$.

Define $\phi(c,x) := \frac{x\mathcal{D}_c(x)}{\frac{1}{m}-\mathcal{D}_c(x)}$. The function ϕ is jointly continuous in (c,x) for $x > 0$. Given that the supremum point x^* that maximizes $\phi(c,x)$ within the interval $(0, \theta_0(c)-V)$ is obtained in an interior point, by pointwise convergence, $t^*(c) = \inf_{x\in(0,\theta_0(c)-V)} \phi(c,x)$ will also exhibit continuity.

The limiting convergence as $c \to \bar{c}^-$. Discuss over the region of x that we're taking supremum over for $t^*(c) := \sup_{x>0}\left\{\frac{x\mathcal{D}_c(x)}{\frac{1}{m}-\mathcal{D}_c(x)}\right\}$:

- $x \in [\theta_0(c)-V, \infty)$, $\mathcal{D}_c(x) = 0$. So this part doesn't contribute to $t^*(c)$.
- $x \in (0, \theta_0(c) - V)$: Look at $\mathcal{D}_c(x)$ as $x \to 0^+$:

$$\begin{aligned}\lim_{x\to 0^+} \mathcal{D}_c(x) &= \lim_{x\to 0^+} \int_V^{\theta_0(c)-x} (1-F(v+x))\ \mathrm{d}F^{m-1}(v)\\ &= \int_V^{\theta_0(c)} (1-F(v))\ \mathrm{d}F^{m-1}(v)\\ &= \int_V^{\theta_0(c)} \mathrm{d}F^{m-1}(v) - \frac{m-1}{m}\int_V^{\theta_0(c)} \mathrm{d}F^m(v)\\ &= F^{m-1}(\theta_0(c)) - \frac{m-1}{m}F^m(\theta_0(c)),\end{aligned}$$

we have, as $x \to 0^+$

$$\begin{aligned}&\lim_{x\to 0^+}\left(\frac{1}{m} - \mathcal{D}_c(x)\right)\\ &= \frac{1}{m} - \left(F^{m-1}(\theta_0(c)) - \frac{m-1}{m}F^m(\theta_0(c))\right).\end{aligned}$$

Let $g(\alpha) = \frac{1}{m} - \left(\alpha^{m-1} - \frac{m-1}{m}\alpha^m\right)$, for $\alpha \in [0,1]$. $g(1) = 0$, $g'(\alpha) = -(m-1)\alpha^{m-2}(1-\alpha) < 0, \forall\alpha \in (0,1)$. So $g(\alpha) > 0, \forall\alpha \in (0,1)$. As $c \to \bar{c}^+$, $\theta_0(c) \to V$ so for sure $F(\theta_0(c)) \in (0,1)$ (actually $F(\theta_0(c))$ is bounded far away from 1). Plug in $\alpha = F(\theta_0(c))$, we have

$$\lim_{x\to 0^+}\left(\frac{1}{m} - \mathcal{D}_c(x)\right) = g(F(\theta_0(c))) > 0.$$

Notice that as $c \to c^-$, $\theta_0(c) \to V^+$:

$$\begin{aligned}&\lim_{c\to\bar{c}^-}\ \sup_{x\in(0,\theta_0(c)-V)}\left\{\frac{x\mathcal{D}_c(x)}{\frac{1}{m}-\mathcal{D}_c(x)}\right\}\\ &\le \lim_{c\to\bar{c}^-}\ (\theta_0(c)-V)\sup_{x\in(0,\theta_0(c)-V)}\left\{\frac{\mathcal{D}_c(x)}{\frac{1}{m}-\mathcal{D}_c(x)}\right\}\\ &\le \lim_{c\to\bar{c}^-}\ (\theta_0(c)-V)\sup_{x\in(0,\theta_0(c)-V)}\left\{\frac{1}{\frac{1}{m}-\mathcal{D}_c(x)}\right\}\\ &= 0.\end{aligned}$$

Taking together the two regions of x discussed above, $\lim_{c\to\bar{c}^-} t^*(c) = 0$.

The limiting convergence as $c \to 0^+$. First, consider the equilibrium **without** inspection cost. In this case, the demand function for the deviated seller $\mathcal{D}_c(x)$ naturally extends to $c = 0$, when $\theta_0(0) = V + 1$:

$$\mathcal{D}_0(x) = \int_V^{V+1-x} (1-F(v+x))\,\mathrm{d}F^{m-1}(v).$$

Denote the equilibrium price as $(t_0, t_0, \ldots, t_0)$. The equilibrium condition for any deviate $x \in [\max(-1,-t_0), 1]$:

$$(t_0 + x)\mathcal{D}_0(x) \le \frac{1}{m}t \tag{9}$$

$$\Leftrightarrow \mathcal{D}_0(x) \le \frac{1}{m}\frac{t_0}{t_0+x}, \forall x \in [\max(-1,-t_0), 1) \tag{10}$$

Notice that, the inequality is strict at $x = 0$—when $\mathcal{D}_0(x) = \frac{1}{m} = \frac{1}{m}\frac{t_0}{t_0+x}\Big|_{x=0}$. And since $\mathcal{D}_0(x)$ is continuous, this situation implies that the slope of $\mathcal{D}_0$ and the right-hand side's equal-revenue curve coincide, yielding:

$$t_0 = \frac{1}{m(m-1)\int_V^{V+1} F^{m-2}(v)f^2(v)\,\mathrm{d}v}.$$

If (10) holds for t_0, then $(t_0, \ldots, t_0)$ constitutes the unique symmetric equilibrium when the market is free of inspection costs.

As for the limiting equilibrium price when inspection cost approaches zero, $t^*(0)$, because $t^*(c) = \inf_x \phi(c,x)$ is continuous in c, all that remains is to verify $\sup_x \phi(0,x) = t_0$. This is confirmed at the limit as $x \to 0^+$:

$$\lim_{x\to 0^+} \phi(0,x) = \lim_{x\to 0^+} \frac{x\mathcal{D}_0(x)}{\frac{1}{m} - \mathcal{D}_0(x)} = \frac{\frac{\partial}{\partial x} x\mathcal{D}_0(x)}{\frac{\partial}{\partial x}[\frac{1}{m} - \mathcal{D}_0(x)]}\Big|_{x=0} = \frac{1}{m(m-1)\int_V^{V+1} F^{m-2}(v)f(v)^2 \, \mathrm{d}v}.$$

By the monotonicity and continuity of $t^*(c)$, it would be contradictory for some $x > 0$ if $\phi(0,x) > t_0$, as this would imply, for some negligible cost of inspection $\epsilon \to 0$, that $\phi(\epsilon, x) > t_0$, leading to a contradiction. Therefore, presuming t_0 exists as a symmetric equilibrium absent inspection costs, $\lim_{c\to 0^+} t^*(c) = t_0$. □

Proposition 5. *When there are two sellers, for $c \in (0, \bar{c})$, $\bar{t}(c)$ is continuous and monotonically decreasing in c.*

Proof. Theorem 4 states that

$$\bar{t}(c) := \inf_{x<0} \left\{ \left(\frac{(-x)\mathcal{D}_c(x)}{\mathcal{D}_c(x) - \frac{1}{m}} \right)^+ \right\} \setminus \{0\},$$

so we focus on $x < 0$ and study the behavior of $\mathcal{D}_c(x)$ with respect to c. For convenience, denote

$$\begin{aligned}\tilde{h}_x(c) &= \mathcal{D}_c(x) \\ &= \int_V^{\theta_0(c)-x} \int_{\max(\theta_0(c)+x, v_2+x)}^{V+1} \mathrm{d}F(v_1)\,\mathrm{d}F(v_2) \\ &+ \int_V^{\theta_0(c)-x} \int_{v_2+x}^{\max(\theta_0(c)+x, v_2+x)} F^{m-2}(v_1 - x)\,\mathrm{d}F(v_1)\,\mathrm{d}F(v_2).\end{aligned}$$

Note that this is still inconvenient. Let

$$\begin{aligned}h_x(\theta) &:= \tilde{h}_x(c) \circ \theta_0(c) \\ &= \int_V^{\theta-x} \int_{\max(\theta+x, v_2+x)}^{V+1} \mathrm{d}F(v_1)\,\mathrm{d}F(v_2) \\ &+ \int_V^{\theta-x} \int_{v_2+x}^{\max(\theta+x, v_2+x)} F^{m-2}(v_1 - x)\,\mathrm{d}F(v_1)\,\mathrm{d}F(v_2) \\ &= \int_V^{\theta} (1 - F(\theta + x))\,\mathrm{d}F(v_2) \\ &+ \int_V^{\theta} \int_{v_2+x}^{\theta+x} F^{m-2}(v_1 - x)\,\mathrm{d}F(v_1)\,\mathrm{d}F(v_2) \\ &+ \int_\theta^{\theta-x} \int_{v_2+x}^{V+1} \mathrm{d}F(v_1)\,\mathrm{d}F(v_2) \\ &= (1 - F(\theta + x))\,F(\theta) \\ &+ \int_V^{\theta} \int_{v_2+x}^{\theta+x} F^{m-2}(v_1 - x)\,\mathrm{d}F(v_1)\,\mathrm{d}F(v_2) \\ &+ \int_\theta^{\theta-x} (1 - F(v_2 + x))\,dF(v_2)\end{aligned}$$

So

$$\begin{aligned}h'_x(\theta) &= -f(\theta + x)F(\theta) + \int_V^{\theta} F^{m-2}(\theta)f(\theta + x)\,\mathrm{d}F(v_2) \\ &+ (1 - F(\theta))\,f(\theta - x) - (1 - F(\theta + x))\,f(\theta) \\ &= \left(F^{m-2}(\theta) - 1\right) f(\theta + x)F(\theta) + (1 - F(\theta))\,f(\theta - x).\end{aligned}$$

Since $\theta_0(c)$ is the solution for $\mathbb{E}[v - \theta]^+ = c$, $\theta_0'(c) \le 0$, and this implies

$$\tilde{h}'_x(c) = \theta_0'(c)h'_x(\theta).$$

When $m = 2$

$$h'_x(\theta) = (1 - F(\theta))\,f(\theta - x) \ge 0.$$

This implies that $\tilde{h}'_x(c) \le 0$, and that the demand for deviated seller with $p' > t$ would decrease as inspection cost c increases. Thus, $c_1 \le c_2 \Rightarrow \mathcal{D}_{c_1}(x) \ge \mathcal{D}_{c_2}(x), \forall x < 0$ when $m = 2$. So $\forall x < 0$,

$$\frac{(-x)\mathcal{D}_{c_1}(x)}{\mathcal{D}_{c_1}(x) - \frac{1}{m}} = (-x) + \frac{(-x)\frac{1}{m}}{\mathcal{D}_{c_1}(x) - \frac{1}{m}} \tag{11}$$

$$\le (-x) + \frac{(-x)\frac{1}{m}}{\mathcal{D}_{c_2}(x) - \frac{1}{m}} = \frac{(-x)\mathcal{D}_{c_2}(x)}{\mathcal{D}_{c_2}(x) - \frac{1}{m}}. \tag{12}$$

And this implies $\bar{t}(c_1) \le \bar{t}(c_2), \forall 0 < c_1 \le c_2 \le \bar{c}$, thereby confirming the monotone property of $\bar{t}(c)$. □

Theorem 5. *For $c \in (0, \bar{c})$, $\tilde{T}(c)$ is an interval, given by*

$$\tilde{T}(c) = [t^*(c), \hat{t}(c)],$$

where $t^*(c) = \sup_{x>0} \frac{x\mathcal{D}_c(x)}{\frac{1}{m} - \mathcal{D}_c(x)}$ is the same as in Theorem 4, and $\hat{t}(c) = \sup_{x<0} \frac{(-x)\tilde{\mathcal{D}}_c(x)}{\tilde{\mathcal{D}}_c(x) - \frac{1}{m}}$, where $\tilde{\mathcal{D}}_c(x)$ is defined for $x < 0$:

$$\tilde{\mathcal{D}}_c(x) = \frac{1}{m}\mathcal{D}_c(x) + \left(1 - \frac{1}{m}\right) \times \left(1 - F(\theta_0(c) + x) + \int_V^{\theta_0(c)-x} (1 - F(v + x))\,\mathrm{d}F^{m-1}(v)\right).$$

If $t^*(c) > \hat{t}(c)$, $\tilde{T}(c) = \emptyset$.

Proof. We investigate the robustness of the Threshold-t mechanism in sustaining an equilibrium. Assume that symmetric equilibrium $(t, t, \ldots, t)$ holds, where t is the threshold below which a seller would be allocated the Buy Box uniformly at random. Consider seller 1 who deviates from equilibrium price t to $p_1' = t + x$. When $x > 0$, the buyer will find seller 1's offer least attractive and only inspected him at last if all other sellers are not satisfying enough (exactly similar to the case in Dictator's mechanism)

$$D_1^{\text{Threshold}}(p_1', t) = \mathcal{D}_c(p_1' - t).$$

When seller 1 deviates to a lower price, i.e. $x < 0$, he gets the Buy Box with probability $1/m$. For the rest $1 - 1/m$, he is not in the Buy Box, conditional on which his demand corresponds to the case as if he is in Dictator mechanism. We can derive seller 1's demand as follows:

$$D_1^{\text{Threshold}}(p_1', t) = \frac{1}{m}D_{BB}(p_1', t) + (1 - \frac{1}{m})\mathcal{D}_c(x)$$

where, $D_{BB}(p_1', t)$ is seller 1's demand when he is placed in the Buy Box:

$$\begin{aligned} D_{BB}(p_1', t) &= P[\text{buyer buys seller 1 without inspecting others}] \\ &\quad + P[\text{buyer inspect all sellers and buys 1}] \\ &= P[v_1 - p_1' + t \geq \theta] \\ &\quad + P[v_1 - p_1' + t \geq \theta \text{ and } v_1 - p_1' \geq \max_{i=2,\dots,m} v_i - t] \\ &= P[v_1 - x \geq \theta] + P[v_1 - x \geq \theta \text{ and } v_1 - x \geq \max_{i=2,\dots,m} v_i] \\ &= 1 - F(\theta + x) + \int_V^{\theta+x} (1 - F(v + x))\, \mathrm{d}F^{m-1}(v). \end{aligned}$$

Notice that, the demand of the deviated seller can again be delineated w.r.t. the price differential x. Denote $\tilde{\mathcal{D}}_c(x) := D_1^{\text{Threshold}}(p, t)$ as the demand of seller 1. Specifically, for $x = 0$, let $\tilde{\mathcal{D}}_c(x) = \frac{1}{m}t$ so that it's left-continuous. And $\tilde{\mathcal{D}}_c(x)$ can be summarized as:

$$\tilde{\mathcal{D}}_c(x) = \begin{cases} \frac{1}{m}\mathcal{D}_c(x) + (1 - \frac{1}{m})\times \\ \quad \left(1 - F(\theta_0(c) + x) + \int_V^{\theta_0(c)-x} (1 - F(v + x))\, \mathrm{d}F^{m-1}(v)\right) \\ \qquad\qquad \text{for } x < 0 \\ \mathcal{D}_c(x) = \int_V^{\theta - V} (1 - F(v + x))\, \mathrm{d}F^{m-1}(v) \\ \qquad\qquad \text{for } x > 0. \end{cases}$$

Threshold-t mechanism can sustain a symmetric equilibrium $(t, t, \dots, t)$ if and only if $t \in \tilde{T}(c)$, for $\tilde{T}(c)$ being articulated as:

$$\tilde{T}(c) := \left\{ t : (x + t)\tilde{\mathcal{D}}_c(x) \leq \frac{1}{m}t, \forall x \in [\max(-1, -t), 1] \right\}.$$

Following the methodological approach in the proof of Theorem 4, $\tilde{T}(c)$, for $t \in \tilde{(}c)$ should satisfies the following criteria simultaneously:

$$t \leq \inf_{x<0} \frac{(-x)\tilde{\mathcal{D}}_c(x)}{\tilde{\mathcal{D}}_c(x) - \frac{1}{m}},$$
$$t \geq \sup_{x>0} \frac{x\tilde{\mathcal{D}}_c(x)}{\frac{1}{m} - \tilde{\mathcal{D}}_c(x)}.$$

The set $\tilde{T}(c)$ is nonempty, indicating the existence of a symmetric equilibrium $(t, \dots, t)$, if and only if:

$$\sup_{x>0} \frac{x\tilde{\mathcal{D}}_c(x)}{\frac{1}{m} - \tilde{\mathcal{D}}_c(x)} \leq \inf_{x<0} \frac{(-x)\tilde{\mathcal{D}}_c(x)}{\tilde{\mathcal{D}}_c(x) - \frac{1}{m}}.$$

Let $\hat{t}(c) := \inf_{x<0} \frac{(-x)\tilde{\mathcal{D}}_c(x)}{\tilde{\mathcal{D}}_c(x) - \frac{1}{m}}$, $t^*(c) := \sup_{x>0} \frac{x\tilde{\mathcal{D}}_c(x)}{\frac{1}{m} - \tilde{\mathcal{D}}_c(x)}$. For the Threshold-$t$ mechanism with inspection cost c, t can sustain an equilibrium if and only if $t^*(c) \leq \hat{t}(c)$. If so, the implementable region of t for Threshold mechanism is given by the following interval

$$[t^*(c), \hat{t}(c)].$$

□

Proposition 7. *There is an instance where the Threshold-t mechanisms do not encompass all symmetric equilibria.*

Proof. We examin a specific instance of two i.i.d. sellers, where their valuations are distributed according to the cumulative distribution function $F(v) = \frac{1}{e-1}(e^{v-2} - 1)$, supported on $[2, 3]$. By Theorem 3, it suffices to look at Dictator's mechanism for all symmetric equilibrium supported by the class of *standard* mechanism. Therefore, for Dictator's mechanism and Threshold mechanism, the implementable price regions under symmetric equilibrium is given by Theorem 5 and Theorem 4:

$$\text{Dictator } T(c) = [t^*(c), \bar{t}(c)]$$
$$\text{Threshold } \tilde{T}(c) = [t^*(c), \hat{t}(c)]$$

We vary inspection cost c to investigate the change of implementable regions, as shown in Figure 3. It can be observed that, the upper bound $\hat{t}(c)$ of Threshold mechanism is lower than that of Dictator's mechanism. This implies, Threshold mechanism cannot sustain any symmetric equilibrium price above that line, yet Dictator's mechanism goes far beyond that bound and can sustain even higher prices in equilibrium. □

C Omitted Proofs from Section 5

Lemma 4. *Under standard Buy Box mechanism with inspection cost $c \in (0, \bar{c})$, at any symmetric equilibrium, for $m = 2$ sellers,*

$$\mathrm{SW}(c) = \int_V^{\theta_0(c)} F(s)(1 - F(s))\, \mathrm{d}s + \mathbb{E}[v]. \tag{2}$$

Proof. We first solve in closed form for social welfare, and demonstrate how it might be interpreted as the simplified representation stated in the theorem.

Before we start we simplify some notations. Because every $c \in [0, \bar{c}]$ corresponds to a unique index value $\theta_0 \in [V, V+1]$, we replace c and $\theta_0(c)$ with shortened expression θ when there is no confusion. Also, in some expressions, replace inspection cost c, $\theta_0(c)$ with corresponding index θ. For example, social welfare at a specific inspection cost level ($\mathrm{SW}(c)$) as $\mathrm{SW}(\theta)$. For two i.i.d. sellers, the match value v_c (or, denoted as v_θ, equals

$$v_\theta = P[v_2 \geq \theta]E[v_2 | v_2 \geq \theta] + P[v_2 < \theta]E[\max(v_1, v_2) | v_2 < \theta].$$

Social welfare can be expressed as (WLOG assumes seller 2 is inspected first),

$$\begin{aligned} SW(\theta) &= -P[v_2 \leq \theta]c + u_\theta \\ &= -P[v_2 \leq \theta]c \\ &\quad + P[v_2 \geq \theta]E[v_2 | v_2 \geq \theta] \\ &\quad + P[v_2 < \theta]E[\max(v_1, v_2) | v_2 < \theta] \\ &= -cF(\theta) \\ &\quad + \int_\theta^{V+1} v_2 \mathrm{d}F v_2 \\ &\quad + \int_V^\theta \left(\int_V^\theta \max(v_1, v_2)\, \mathrm{d}F v_1 + \int_\theta^{V+1} v_1 \mathrm{d}F v_1\right) \mathrm{d}F v_2 \\ &= -cF(\theta) \\ &\quad + \left(\int_\theta^{V+1} v\, \mathrm{d}F v\right)(1 + F(\theta)) \\ &\quad + \int_V^\theta \left(\int_V^{v_2} v_2\, \mathrm{d}F(v_1) + \int_{v_2}^\theta v_1\, \mathrm{d}F(v_1)\right) \mathrm{d}F(v_2) \\ &= -cF(\theta) \\ &\quad + \left(\int_\theta^{V+1} v\, \mathrm{d}F v\right)(1 + F(\theta)) \\ &\quad + \int_V^\theta v_2 F(v_2)\, \mathrm{d}F(v_2) + \int_V^\theta \int_{v_2}^\theta v_1\, \mathrm{d}F(v_1) \mathrm{d}F(v_2) \end{aligned}$$

Whereas the last component above $\int_V^\theta \int_{v_2}^\theta v_1 \, \mathrm{d}F(v_1) \, \mathrm{d}F(v_2) = \int_V^\theta \int_V^{v_1} v_1 \, \mathrm{d}F(v_2) \mathrm{d}F(v_1) = \int_V^\theta v_1 F(v_1) \, \mathrm{d}F(v_1)$. And also notice that $c = \mathbb{E}[v-\theta]^+ = \int_\theta^{V+1} (v-\theta) \, \mathrm{d}F(v) = \int_\theta^{V+1} v \, \mathrm{d}F(v) - \theta(1-F(\theta))$. Putting together

$$\begin{aligned}
\mathrm{SW}(\theta) = & -cF(\theta) \\
& + (\int_\theta^{V+1} v \, \mathrm{d}Fv)(1+F(\theta)) \\
& + \int_V^\theta v_2 F(v_2) \, \mathrm{d}F(v_2) + \int_V^\theta v_1 F(v_1) \, \mathrm{d}F(v_1) \\
= & -cF(\theta) + (\int_\theta^{V+1} v \, \mathrm{d}Fv)(1+F(\theta)) + 2\int_V^\theta vF(v) \, \mathrm{d}F(v) \\
= & \int_\theta^{V+1} v \, \mathrm{d}F(v) + \theta F(\theta)(1-F(\theta)) + 2\int_V^\theta vF(v) \, \mathrm{d}F(v)
\end{aligned}$$

And

$$\begin{aligned}
\partial_\theta \, \mathrm{SW}(\theta) &= -\theta f(\theta) + F(\theta)(1-F(\theta)) \\
&\quad + \theta f(\theta)(1-F(\theta)) - \theta F(\theta) f(\theta) + 2\theta F(\theta) f(\theta) \\
&= F(\theta)(1-F(\theta)) \geq 0
\end{aligned}$$

When $\theta = V$, $\mathrm{SW}(\theta) = \mathbb{E}[v]$. So that social welfare can be written in the following simplified form

$$\mathrm{SW}(\theta) = \mathbb{E}[v] + \int_V^\theta F(s)(1-F(s)). \, ds$$

□

Theorem 7. *For two sellers, a sufficient condition for the consumer surplus to be optimized at an intermediate level of inspection cost is* $f(V^+) < \frac{1}{13} f'(V^+)$.

Proof. When the inspection cost takes its upper bound $\bar{c} = \mathbb{E}[v] - V$, at which buyers simply ignore all sellers outside the Buy Box at any symmetric equilibrium, the equilibrium price can be arbitrarily close to 0 (Proposition 4). In this case, social welfare coincide with the consumer surplus (Proposition 4, Lemma 4):

$$\mathrm{SW}(\bar{c}) = \mathrm{CS}(\bar{c}) = \mathbb{E}[v].$$

By Lemma 4, when the inspection cost c drops in the range $(0, \bar{c})$, social welfare rises to

$$\mathrm{SW}(c) = \int_V^{\theta_0(c)} F(v)(1-F(v)) \, \mathrm{d}v + \mathrm{SW}(\bar{c})$$

But the lowest equilibrium price $t^*(c)$ rises as well (Proposition 4). Starting at $\bar{c}$, we wish to decrease inspection cost to increase consumer surplus using the Buy Box mechanism. Lowering inspection cost increases social welfare, meanwhile we would wish the rise in (implementable) equilibrium price lowerbound does not offset the rise in social welfare. Put together the two forces, consumer surplus would be better off if, for inspection cost c that is slightly around the left-neighbourhood of $\bar{c}$:

$$\mathrm{SW}(c) - \mathrm{SW}(\bar{c}) > t^*(c) - 0 \tag{13}$$

$$\Leftrightarrow \tag{14}$$

$$\int_V^\theta F(v)(1-F(v)) \, \mathrm{d}v > \sup_{x \in (0, \theta - V)} \frac{x \mathcal{D}_c(x)}{\frac{1}{2} - \mathcal{D}_c(x)} \tag{15}$$

With a slight abuse of notation, define $\mathcal{D} : [V, V+1] \times \mathbb{R} \to \mathbb{R}$ as:

$$\mathcal{D}(\theta, x) := \int_V^{\theta_0(c) - x} (1 - F(v+x)) \, \mathrm{d}F(v) \equiv \mathcal{D}_c(x).$$

Then, the condition for consumer surplus being better off (15) can be expressed in the following equivalent form:

$$\exists \theta \in (V, V+1), \exists x \in (0, \theta - V) \text{ such that:} \tag{16}$$

$$x\mathcal{D}(\theta, x) + \varphi(\theta)\mathcal{D}(\theta, x) - \frac{1}{2}\varphi(\theta) < 0, \tag{17}$$

where, $\varphi(\theta) := \int_V^\theta F(v)(1-F(v)) \, \mathrm{d}v$. To find θ, x such that (17) holds, define $\mu : \mathbb{R}^2 \to \mathbb{R}$ as

$$\mu(\theta, x) := \begin{cases} x\mathcal{D}(\theta, x) + \varphi(\theta)\mathcal{D}(\theta, x) - \frac{1}{2}\varphi(\theta), \\ \qquad \text{for } \theta \in (V, V+1), x \in (0, \theta - V); \\ 0, \\ \qquad \text{otherwise.} \end{cases} \tag{18}$$

Notice that $\lim_{x \to 0} \mu(\theta, x) = \lim_{x \to \theta - V} \mu(\theta, x) = 0$. Let

$$x^*(\theta) := \begin{cases} \arg\min_x \mu(\theta, x) & \theta \in (V, V+1) \\ 0 & \text{otherwise.} \end{cases}$$

$x^*(\theta)$ is thus a well defined continuous function with $\lim_{\theta \to V} x^*(\theta) = 0$. Let $g(\theta) := \mu(\theta, x^*(\theta))$. Then, $g(V) = 0$. Study the derivative of $g(\cdot)$ when $\theta \to 0^+$. By envelope's theorem

$$\begin{aligned}
g'(\theta) &= \frac{\partial \mu}{\partial \theta}(\theta, x^*(\theta)) \\
&= (x^*(\theta) + \varphi(\theta)) \frac{\partial \mathcal{D}}{\partial \theta}(\theta, x^*(\theta)) + \varphi'(\theta)(\mathcal{D}(\theta, x^*(\theta)) - \frac{1}{2}),
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \mathcal{D}}{\partial \theta}(\theta, x) &= \frac{\partial}{\partial \theta} \int_V^{\theta - x} (1 - F(v+x)) \, \mathrm{d}F(v), \\
&= (1 - F(\theta)) f(\theta - x) \\
\varphi'(\theta) &= F(\theta)(1 - F(\theta)),
\end{aligned}$$

which jointly implies

$$\begin{aligned}
g'(\theta) &= (1 - F(\theta)) f(\theta - x^*(\theta))(x^*(\theta) + \varphi(\theta)) \\
&\quad + F(\theta)(1 - F(\theta))(\mathcal{D}(\theta, x^*(\theta)) - \frac{1}{2}) \\
&= (1 - F(\theta)) \left(f(\theta - x^*(\theta))(x^*(\theta) + \varphi(\theta)) \right) \\
&\quad + (1 - F(\theta)) \left(F(\theta)(\mathcal{D}(\theta, x^*(\theta)) - \frac{1}{2}) \right).
\end{aligned}$$

Since $\varphi(V) = 0, x^*(V) = 0$:

$$\lim_{\theta \to V^+} g'(\theta) = 0.$$

Unfortunately, this implies that we need to study higher order derivatives of g and how they behave around $\theta \to V^+$. For convenience, in the subsequent analysis, for any general function $f(\cdot)$, denote $\lim_{x \to V^+} f(x)$ as $f(V^+)$. We first solve $x^{*\prime}(V^+)$ and $x^{*\prime\prime}(V^+)$:

Taking a step back: for general $\rho : [V, V+1] \to \mathbb{R}$, define $\mathbb{D}_\rho(\theta) := \mathcal{D}(\theta, \rho(\theta))$, so that:

$$\mathbb{D}_\rho(\theta) = \int_V^{\theta - \rho(\theta)} (1 - F(v + \rho(\theta))) \, \mathrm{d}F(v)$$

and its first-order derivative:

$$\mathbb{D}'_{\rho}(\theta) = (1-\rho'(\theta))(1-F(\theta))f(\theta-\rho(\theta)) - \rho'(\theta)\left(\int_V^{\theta-\rho(\theta)} f(v)f(v+\rho(\theta))\,\mathrm{d}v\right),$$

and, second-order derivative:

$$\begin{aligned}\mathbb{D}''_{\rho}(\theta) = &(1-F(\theta))f'(\theta-\rho(\theta))(1-\rho'(\theta))^2 \\ &+ (1-F(\theta))f(\theta-\rho(\theta))(-\rho''(\theta)) \\ &- \rho''(\theta)\int_V^{\theta-\rho(\theta)} f(v)f(v+\rho(\theta))\,\mathrm{d}v \\ &- \rho'(\theta)\Bigg[(1-\rho'(\theta))f(\theta-\rho(\theta))f(\theta) \\ &\quad + \int_V^{\theta-\rho(\theta)} f(v)f'(v+\rho(\theta))\rho'(\theta)\,\mathrm{d}v\Bigg].\end{aligned}$$

Plug in $\rho = x^*, \theta \to V^+$:

$$\mathbb{D}'_{x^*}(V^+) = (1-x^{*\prime}(V^+))f(V^+) \tag{19}$$

$$\mathbb{D}''_{x^*}(V^+) = f'(V^+)(1-x^{*\prime}(V^+))^2 - f(V^+)(x^{*\prime\prime}(V^+)) \tag{20}$$

$$- x^{*\prime}(V^+)(1-x^{*\prime}(V^+))f^2(V^+) \tag{21}$$

Recall, $x^*(\cdot)$ is defined as the x^* that achives $\inf_x \mu(\theta, x)$. Since $\mu(\theta, x) = 0$ for $x \notin (0, \theta - V)$, fix θ, $\mu(\theta, x)$'s minimum is attained at some interior $x \in (0, \theta - V)$, thus for $x^*(\theta) = \arg\min_x \mu(\theta, x)$:

$$\frac{\partial\mu}{\partial x}(\theta, x^*(\theta)) = \frac{\partial\mathcal{D}}{\partial x}(\theta, x^*(\theta))(x^*(\theta)+\varphi(\theta)) + \mathcal{D}(\theta, x^*(\theta)) \equiv 0 \tag{22}$$

Define $h(\theta) := \frac{\partial\mathcal{D}}{\partial x}(\theta, x^*(\theta))$, there is

$$\begin{aligned}h(\theta) &= -(1-F(\theta))f(\theta-x^*(\theta)) \\ &\quad - \int_V^{\theta-x^*(\theta)} f(v)f(v+x^*(\theta))\,\mathrm{d}v \\ h'(\theta) &= f(\theta)f(\theta-x^*(\theta)) \\ &\quad - (1-F(\theta))f'(\theta-x^*(\theta))(1-x^{*\prime}(\theta)) \\ &\quad - \Bigg[(1-x^{*\prime}(\theta))f(\theta-x^*(\theta))f(\theta) \\ &\quad + \int_V^{\theta-x^*(\theta)} f(v)f'(\theta-x^*(\theta))x^{*\prime}(\theta)\,\mathrm{d}v\Bigg].\end{aligned}$$

Notice that $h(V^+) = -f(V^+)$, $h'(V^+) = \frac{1}{2}f^2(V^+) - \frac{1}{2}f'(V^+)$. On the basis of notation $h(\cdot)$ the preceding condition (22) of $\frac{\partial\mu}{\partial x}(\theta, x^*(\theta)) \equiv 0$ is equivalent to

$$h(\theta)(x^*(\theta)+\varphi(\theta)) + \mathcal{D}(\theta, x^*(\theta)) \equiv 0, \forall\theta \in (V, V+1). \tag{23}$$

Essentially, taking double derivative of (23) w.r.t. θ on both side yields

$$\text{(once) } h'(\theta)(x^*(\theta)+\varphi(\theta)) \tag{24}$$

$$+ h(\theta)(x^{*\prime}(\theta)+\varphi'(\theta)) + \mathbb{D}'_{x^*}(\theta) = 0 \tag{25}$$

$$\text{(twice) } 2h'(\theta)(x^{*\prime}(\theta)+\varphi'(\theta)) \tag{26}$$

$$+ h''(\theta)(x^*(\theta)+\varphi(\theta)) \tag{27}$$

$$+ h(\theta)(x^{*\prime\prime}(\theta)+\varphi''(\theta)) + \mathbb{D}''_{x^*}(\theta) = 0. \tag{28}$$

And $\varphi'(\theta) = F(\theta)(1-F(\theta))$, $\varphi''(\theta) = f(\theta) - 2F(\theta)f(\theta)$. So, (24), adding (19) simplifies to

$$-f(V^+)x^{*\prime}(V^+) + (1-x^{*\prime}(V^+))f(V^+) = 0 \Rightarrow x^{*\prime}(V^+) = \frac{1}{2}$$

Notice that $x^*(V^+) + \varphi(V^+) = 0$, so that we need not further calculate $h''(V^+)$. Integrating $x^{*\prime}(V^+) = \frac{1}{2}$ into 20 we obtain

$$\mathbb{D}''_{x^*}(V^+) = \frac{1}{4}f'(V^+) - f(V^+)x^{*\prime\prime}(V^+) - \frac{1}{4}f^2(V^+)$$

plug in 24 solves

$$x^{*\prime\prime}(V^+) = -\frac{f'(V^+) + 3f^2(V^+)}{8f(V^+)}$$

Back to the function g. $g(V^+) = 0 = g'(V^+) = 0$. To study its higher-order derivatives, take $g'(\theta) = (1-F(\theta))\gamma(\theta)$, we'd have

$$\begin{aligned}\gamma(\theta) &:= f(\theta-x^*(\theta))(x^*(\theta)+\varphi(\theta)) + F(\theta)(\mathcal{D}(\theta, x^*(\theta)) - \frac{1}{2}) \\ \gamma'(\theta) &= (1-x^{*\prime}(\theta))f'(\theta-x^*(\theta))(x^*(\theta)+\varphi(\theta)) \\ &\quad + f(\theta-x^*(\theta))(x^{*\prime}(\theta)+\varphi'(\theta)) \\ &\quad + f(\theta)(\mathbb{D}_{x^*}(\theta) - \frac{1}{2}) + F(\theta)\mathbb{D}'_{x^*}(\theta) \\ \gamma''(\theta) &= (x^*(\theta)+\varphi(\theta))f''(\theta-x^*(\theta)) \\ &\quad + 2f'(\theta-x^*(\theta))(1-x^{*\prime}(\theta))(x^{*\prime}(\theta)+\varphi'(\theta)) \\ &\quad + f(\theta-x^*(\theta))(x^{*\prime\prime}(\theta)+\varphi''(\theta)) + f'(\theta)(\mathbb{D}_{x^*}(\theta) - \frac{1}{2}) \\ &\quad + 2f(\theta)\mathbb{D}'_{x^*}(\theta) + F(\theta)\mathbb{D}''_{x^*}(\theta)\end{aligned}$$

Interestingly, $\gamma'(V^+) = 0$. But

$$\begin{aligned}\gamma''(V^+) &= \frac{1}{2}f'(V^+) + f(V^+)(x^{*\prime\prime}(V^+) + f(V^+)) - \frac{1}{2}f'(V^+) \\ &= f(V^+)\left(x^{*\prime\prime}(V^+) + 2f(V^+)\right) \\ &= \frac{1}{8}\left(13f^2(V^+) - f'(V^+)\right)\end{aligned}$$

Therefore, when $f^2(V^+) < f'(V^+)$, $\exists\theta^\star \in N^+(V)$ such that $g(\theta^\star) < 0$, corresponding to $\mu(\theta^\star, x^*(\theta^\star) < 0$ — that starting from its upper bound $\bar{c}$, decrease in inspection cost (at least, with a small amount) will benefit consumer surplus. □

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