Association, Covariation and Causality

Weinan Qian
Yuanpei College
Peking University
ypqwn@stu.pku.edu.cn

Abstract

Association, covariation and causality are three pivotal concepts in the realm of statistics and cognitive reasoning. While in everyday discourse, these terms tend to be used interchangeably, they actually bear distinct definitions. In probability theory, it is widely recognized that covariation simply implies association, yet association may not always encompass covariation. We leverage the intuitive definition and conduct analyses of causality, contending that causality implies association but not covariation. Moreover, it is crucial to note that neither association nor covariation can directly establish causality.

1 Introduction

Imagine a person suggesting that your poor performance in the final exam might cause your skipping breakfast. You would likely find his words quite baffling and may even suspect that you misheard him or that his statement was in the wrong order. Quickly noticing your confused expression, he presents you with the statistical data demonstrating a strong positive correlation between students' scores on final exams and their frequency of having breakfast. You might chuckle and kindly correct him, explaining that it is skipping breakfast that causes the poor exam results.

This scenario is so commonplace that most people tend to overlook it and not give it much consideration. However, these everyday interactions contain a wealth of knowledge involving association, covariation, and causality. You unconsciously ponder the cause-and-effect relationship in the scenario, assuming that skipping breakfast is the cause and performing poorly in the exams is the effectly; the reverse seems implausible. Such causal reasoning can be activated almost automatically and irresistibly [2, 13]. Nevertheless, it is intriguing to demonstrate that these natural assumptions might also be incorrect. It transpires that those who do not eat breakfast are also more likely to be absent or tardy, and it is absenteeism that is playing a significant role in their poor performance [6]. Researchers have posited that breakfast primarily aids undernourished children in performing better [12].

The example above vividly illustrates the significance of distinguishing among the concepts of association, covariation and causality. We can readily discern the association and covariation between having breakfast and exam performance. Nonetheless, definitely establishing causality proves challenging due to the presence of hidden intermediate factors. In probability theory, there are also numerous instances that demonstrate two associated random variables having zero covariance.

This paper endeavors to elucidate the distinctions among association, covariation and causality, emphatically asserting that they are distinctly separate concepts that should not be confused. In Sec. 2, we provide a concise introduction to the fundamental definitions of association, covariation and causality, clearly demonstrating that while association is a necessary condition, it is insufficient for covariation. Sec. 3 expounds that causality implies association, but the reverse is not true. Finally, in Sec. 4, we establish that causality cannot provide information regarding covariation, and vice versa.

We will consistently assume that the random variables discussed in this paper have finite expectation values for the sake of convenience in our discussion.

2 Definition

To engage more precisely in the discourse about the relationship among association, covariation and causality, we will begin by providing the fundamental definitions of these three concepts. This will encompass accurate probabilistic definitions of association and covariation, alongside an intuitive definition of causality.

Definition 2.1 (Association). We say two random variables A and B are associated if they are not independent, or

$$F_{A,B}(x,y) \neq F_A(x)F_B(y),\ \exists x, y \in \mathbb{R}, \tag{1}$$

where $F_{A,B}(x,y)$ is the joint cumulative distribution function (CDF) of two random variables (A,B), and $F_A(x),\ F_B(y)$ denote the marginal CDFs of A and B, respectively. We also say that there exhibits an association between A and B. Specifically, two events A and B are associated if (Assume $\Pr(A) > 0,\ \Pr(B) > 0$)

$$\Pr(AB) \neq \Pr(A)\Pr(B) \iff \Pr(A|B) \neq \Pr(A),\ \Pr(B|A) \neq \Pr(B). \tag{2}$$

Definition 2.2 (Covariation). We say two random variables A and B covariate if their covariance is not zero:

$$\mathrm{Cov}(A,B) = \mathbb{E}[(A - \mathbb{E}(A))(B - \mathbb{E}(B))] \neq 0. \tag{3}$$

It is obvious that two variables covariate if and only if their Pearson correlation efficient is not zero:

$$\rho(A,B) = \frac{\mathrm{Cov}(A,B)}{\sqrt{D(A)D(B)}} \neq 0. \tag{4}$$

Given that the Pearson correlation efficient is a commonly employed tool for assessing the correlation between two random variables, it can be stated that the covariation of two random variables is equivalent to their correlation.

From these two direct probabilistic definitions of association and covariation, we can readily demonstrate the following theorem:

Theorem 2.1. Two covariated random variables are also associated, and conversely, two associated variables do not necessarily covariate.

We provide the proof of Theorem 2.1 in Appendix A.1. This theorem illustrates that association and covariation, which are often used interchangeably in discourse, are actually distinct concepts. Covariation contains more information about the association of two random variables. This theorem also emphasizes the importance of separately discussing the relationships between causality and association, as well as between causality and covariation.

Definition 2.3 (Causality). Causality, also called causation, or cause and effect, is influence by which one event, process, state, or object (a cause) contributes to the production of another event, process, state, or object (an effect) [8].

This is an intuitive definition of causality. While there exist models like the Rubin causal model [5] and Pearl's probabilistic graphical model (*i.e.*, causal Bayesian networks (CBNs)) [3] to describe and infer causality, given the complexity of these models and the focus of this paper, we will utilize the intuitive definition provided above for our analysis.

3 Causality and Association

3.1 Causality Implies Association

Intuitively, this proposition holds true because for two independent random variables, A and B, one variable cannot have any impact on the othe. This violates the intuitive definition of causality and indicates that A and B do not exhibit causality. We aim to elucidate this point in a more mathematical manner. For the sake of simplicity in the proof, we will consider only two events, A and B. If A and B are not associated, *i.e.*, they are independent, the following proposition holds true based on Eq. (2):

$$\Pr(A|B) = \Pr(A) \land \Pr(B|A) = \Pr(B). \tag{5}$$

Eq. (5) demonstrates that when we consider adding a condition B, the probability of event A occurring will remain unaffected. Similarly, adding A as a condition will not change the probability of event B occurring. However, if A and B are causally related, the definition of causality implies that A should contribute to the production of B, or vice versa. This contradicts the implication of Eq. (5), leading to the conclusion that there is no causality between A and B. Therefore, the absence of association implies the absence of causality, and its contrapositive, which is what we aim to prove, holds true.

3.2 Association Does Not Imply Causality

This proposition is also inherently evident because the concept of association is extraordinarily broad, and we cannot definitively ascertain whether the association between two variables implies causality. More specifically, at least part of the observed association between two variables may be attributed to reverse causality or to the confounding effect of a third variable [1].

Reverse causality is an informal fallacy of questionable cause where cause and effect are reversed [10]. For instance, it is common sense that the sunrise causes the rooster to crow. If we mistakenly assume that the crowing of the rooster is the cause and the sunrise is the effect, we commit the error of reverse causality. More generally, when C is the cause of E, it is usually false that E is the cause of C. Association is always bidirectional and has symmetry, while causality is often unidirectional and does not have symmetry.

The confounding effect of a third variable is also a common occurrence in mistaken causal reasoning. When both X and Y are caused by a factor Z, despite the strong association between X and Y, we still cannot infer a direct causal link between them. For instance, consider a study conducted at the University of Pennsylvania Medical Center in 1999 [4]. This study suggested that young children who sleep with the light on are much more likely to develop myopia in later life, implying that sleeping with the light on causes myopia. However, a subsequent study at Ohio State University did not find evidence that infants sleeping with the light on directly caused the development of myopia. Instead, it discovered that myopic parents were more likely to leave a light on in their children's bedroom [7, 11]. The common underlying cause for both conditions is parental myopia, underscoring the necessity for caution when inferring causality between sleeping with the light on and myopia.

4 Causality and Covariation

4.1 Covariation Does Not Imply Causality

The analysis of this proposition is similar to Sec. 3.2. This is because two associated variables are also correlated, or covariate, in most cases, since in Sec. 2 we have already proven that correlation is equivalent to covariation. In econometrics, we often use the following linear model to perform multiple linear regression and obtain derive meaningful conclusions:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k + u. \tag{6}$$

If one of the slope coefficient given by the ordinary least squares method, say $\hat{\beta}_1$ (assuming it is positive), is economically large and statistically significant (this can be verified by a t test; here we omit the details), we can observe a positive covariance between y and x_1. This conclusion holds true only if the explanatory variables do not provide any information about the mean of the error term u, or if the zero conditional mean assumption $\mathbb{E}(u|x_1, \cdots, x_k) = 0$ is valid. If there are other unobserved factors other than $x_1, \cdots, x_k$ that have correlations with both explanatory variables and the explained variable, the causality between x_1 and y cannot be explained through this model, in spite of the positive covariance of y and x_1.

4.2 Causality Does Not Imply Covariation

It might seem counterintuitive that causality does not imply covariation, indicating that covariation and causality are two completely distinct concepts. To elucidate this proposition, we should employ the probabilistic definition of covariation. Let us assume a standard normal random variable $X \sim \mathcal{N}(0, 1)$, and consider another random variable $Y = X^2$. It is evident that X and Y exhibit a strong causality, because X entirely determines Y. However, according to the definition of covariation and the fact that the probability density function (PDF) of a normal distribution is an odd function, we have:

$$\mathrm{Cov}(X, Y) = \mathrm{Cov}(X, X^2) = \mathbb{E}(X^3) - \mathbb{E}(X)\mathbb{E}(X^2) = 0 - 0 = 0, \tag{7}$$

which demonstrates that X and Y exhibit no covariation. Additional examples can be found in Fig. 1. In the absence of other influencing factors, one can assert that every pair of random variables in the examples displays causality as well as strong association. However, the fact that all of them exhibit zero covariance indicates that they are not correlated, or equivalently, have no covariation. That is due to the fact that covariance primarily reflects linear associations between two variables, and there are numerous instances where two variables with non-linear associations lack a linear association.

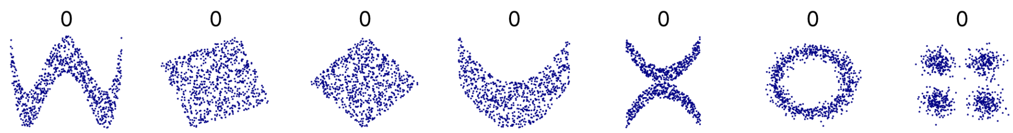

Figure 1: Several examples that two associated variables have a zero covariance. Image borrowed from [9].

5 Conclusion

To clearly distinguish the concepts of association, covariation and causality is highly significant in the fields of statistics, econometrics, cognitive reasoning, and also in our daily life. A plethora of reasoning errors have resulted from the confusion of association and covariation with causality, leading to incorrect conclusions about causality relationships. We endeavor to elucidate the explicit relationships among these three concepts through a more rigorous approach, demonstrating that (1) covariation and causality implies association, (2) association and causality do not imply covariation, and (3) association and covariation do not imply causality. Association is the most general concept, covariation is a special form of linear association between two variables, and causality is an abstract concept involving cause and effect. By understanding these concepts correctly, one can make more rational judgments in his life and draw more convincing conclusions from a wide range of social phenomena or experimental results.

References

[1] Alberto Abadie. Causal inference. *Encyclopedia of Social Measurement*, pages 259–266, 2005. 3

[2] Roberta Corrigan and Peggy Denton. Causal understanding as a developmental primitive. *Developmental review*, 16(2):162–202, 1996. 1

[3] Judea Pearl. *Causality: Models, reasoning and inference*. Cambridge University Press, 2000. 2

[4] Graham E. Quinn, Chai H. Shin, Maureen G. Maguire, and Richard A. Stone. Myopia and ambient lighting at night. *Nature*, 399:113–114, 1999. 3

[5] Donald B. Rubin. Estimating causal effects of treatments in randomized and nonrandomized studies. *Journal of Educational Psychology*, 66(5):688–701, 1974. 2

[6] George Mason University. Causation vs. correlation. URL https://flexiblelearning.auckland.ac.nz/data-to-insight/4/1/2/files/4_5_stats_causation_vs_correlation.pdf. 1

[7] Ohio State University. Night lights don't lead to nearsightedness, study suggests, 2000. URL https://www.sciencedaily.com/releases/2000/03/000309074442.htm. 3

[8] Wikipedia. Causality, . URL https://en.wikipedia.org/wiki/Causality. 2

[9] Wikipedia. Correlation, . URL https://en.wikipedia.org/wiki/Correlation. 4

[10] Wikipedia. Correlation does not imply causation, . URL https://en.wikipedia.org/wiki/Correlation_does_not_imply_causation#B_causes_A_(reverse_causation_or_reverse_causality). 3

[11] Karla Zadnik, Lisa A. Jones, Brett C. Irvin, Robert N. Kleinstein, Ruth E. Manny, Julie A. Shin, and Donald O. Mutti. Myopia and ambient lighting at night. *Nature*, 404:143–144, 2000. 3

[12] Katie ZAdolphus, Clare L. Lawton, and Louise Dye. The effects of breakfast on behavior and academic performance in children and adolescents. *Frontiers in human neuroscience*, 7:425, 2013. 1

[13] Yixin Zhu, Tao Gao, Lifeng Fan, Siyuan Huang, Mark Edmonds, Hangxin Liu, Feng Gao, Chi Zhang, Siyuan Qi, Ying Nian Wu, et al. Dark, beyond deep: A paradigm shift to cognitive ai with humanlike common sense. *Engineering*, 6(3):310–345, 2020. 1

A Appendix

A.1 Proof of Theorem 2.1

On one hand, we prove that two covariated random variables A and B are also associated. Suppose A and B are not associated or, equivalently, independent, we have

$$F_{A,B}(x,y) = F_A(x)F_B(y),\ \forall x, y \in \mathbb{R}.$$

Due to the fact that A and B are independent, we also have the following equation, which will practically be used in our proof:

$$\Pr(A > x,\ B > y) = \Pr(A > x)\Pr(B > y),\ \forall x, y \in \mathbb{R}.$$

We firstly assume that A and B are non-negative. Therefore, we have

$$\mathbb{E}(A) = \int_0^{+\infty} \Pr(A > x)\,\mathrm{d}x,\ \mathbb{E}(B) = \int_0^{+\infty} \Pr(B > y)\,\mathrm{d}y.$$

It is worth noting that this definition for expectation value is always true as long as $\mathbb{E}(|A|) < +\infty$ and $\mathbb{E}(|B|) < +\infty$, even when A or B is neither discrete nor continuous. This is because the function $G_A(x) = \Pr(A > x) = 1 - F_A(x)$ is monotonically decreasing, hence is always integrable (this proposition can be proved through the definition of Riemann integral; here we omit its detailed proof).

Multiplying these two equations gives us

$$\begin{aligned}\mathbb{E}(A)\mathbb{E}(B) &= \int_0^{+\infty} \Pr(A > x)\,\mathrm{d}x \int_0^{+\infty} \Pr(B > y)\,\mathrm{d}y \\ &= \int_0^{+\infty}\int_0^{+\infty} \Pr(A > x)\Pr(B > y)\,\mathrm{d}x\mathrm{d}y \\ &= \int_0^{+\infty}\int_0^{+\infty} \Pr(A > x, B > y)\,\mathrm{d}x\mathrm{d}y = \mathbb{E}(AB),\end{aligned}$$

where the last equality holds true only for non-negative random variables AB.

Now we prove the most general case. Let $X^+ := \max\{X, 0\}$, $X^- := \max\{-X, 0\}$ for any random variable X. According to the definition of expectation, we have $\mathbb{E}(X) = \mathbb{E}(X^+) - \mathbb{E}(X^-)$. Now it can be easily proved that

$$(AB)^+ = A^+B^+ + A^-B^-,\ (AB)^- = A^+B^- + A^-B^+.$$

Combined with the conclusion we obtain in the non-negative case, we finally have

$$\begin{aligned}\mathbb{E}(A)\mathbb{E}(B) &= [\mathbb{E}(A^+) - \mathbb{E}(A^-)][\mathbb{E}(B^+) - \mathbb{E}(B^-)] \\ &= \mathbb{E}(A^+B^+) - \mathbb{E}(A^-B^+) - \mathbb{E}(A^+B^-) + \mathbb{E}(A^-B^-) \\ &= \mathbb{E}[(AB)^+] - \mathbb{E}[(AB)^-] = \mathbb{E}(AB),\end{aligned}$$

which implies that $\mathrm{Cov}(A, B) = \mathbb{E}(AB) - \mathbb{E}(A)\mathbb{E}(B) = 0$ and concludes the proof.

On the other hand, we give a counterexample to prove that two associated random variables A and B may have no covariation. In fact, this counterexample has been already given in Sec. 4.2, because the condition $Y = X^2$ also suggests that X and Y have a strong association, but the zero covariance between X and Y shows that they have no covariation.