INDUCING FAITHFULNESS IN STRUCTURED REASONING VIA COUNTERFACTUAL SENSITIVITY

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Paper under double-blind review

ABSTRACT

The reasoning processes of large language models often lack faithfulness; a model may generate a correct answer while relying on a flawed or irrelevant reasoning trace. This behavior, a direct consequence of training objectives that solely reward final-answer correctness, severely undermines the trustworthiness of these models in high-stakes domains. This paper introduces **Counterfactual Sensitivity Regularization (CSR)**, a novel training objective designed to forge a strong, causal-like dependence between a model's output and its intermediate reasoning steps. During training, CSR performs automated, operator-level interventions on the generated reasoning trace (e.g., swapping '+' with '-') to create a minimally-perturbed counterfactual. A regularization term then penalizes the model if this logically flawed trace still yields the original answer. Our efficient implementation adds only 8.7% training overhead through warm-start curriculum and token-subset optimization. We evaluate faithfulness using **Counterfactual Outcome Sensitivity (COS)**, a metric quantifying how sensitive the final answer is to such logical perturbations. Across diverse structured reasoning benchmarks—arithmetic (GSM8K), logical deduction (ProofWriter) , multi-hop QA (HotpotQA), and code generation (MBPP) —models trained with CSR demonstrate a vastly superior trade-off between accuracy and faithfulness. CSR improves faithfulness over standard fine-tuning and process supervision by up to 70 percentage points, with this learned sensitivity generalizing to larger models and enhancing the performance of inference-time techniques like self-consistency. To demonstrate the broader applicability of this principle, we conduct a pilot study on the HellaSwag -commonsense reasoning task, showing that a semantic version of CSR (using causal connectives, temporal markers, and key entities as operators) can significantly improve faithfulness there as well.

1 INTRODUCTION

Large Language Models (LLMs) are being deployed in increasingly critical applications, yet their internal decision-making processes remain disturbingly opaque. While capable of generating complex, step-by-step rationales through methods like Chain-of-Thought (CoT) prompting (Wei et al., 2022; Kojima et al., 2022; Nye et al., 2021) and more advanced deliberation strategies (Yao et al., 2023), a particularly acute problem is the crisis of unfaithful reasoning: an LLM may produce a correct answer accompanied by a plausible-looking chain of thought, yet the rationale itself is a fabrication, disconnected from the model's true "computation" (Lanham et al., 2023; Turpin et al., 2023). This behavior, where models learn to rationalize post-hoc rather than reason genuinely, poses a fundamental threat to their trustworthiness. For LLMs to be reliable partners in science, medicine, and engineering, the reasoning they present must be the reasoning they use.

This disconnect arises from the dominant training paradigm of outcome supervision, where models are optimized exclusively for final-answer accuracy (Cobbe et al., 2021). This objective function incentivizes models to exploit spurious correlations and statistical shortcuts in the data (Meng et al., 2023), completely bypassing the intermediate reasoning steps. Recent studies confirm this troubling behavior, showing that larger, more capable models are particularly adept at ignoring their own generated explanations, feigning a structured thought process while relying on shallow heuristics (Turpin et al., 2023).

This paper confronts the challenge of instilling faithfulness within **structured reasoning domains**—tasks such as mathematics, formal logic, and planning, where the validity of an argument is governed by well-defined operators and rules. In these domains, the notion of a "correct" reasoning step is unambiguous. We introduce **Counterfactual Sensitivity Regularization (CSR)**, a training objective that enforces a strong dependence between a model's reasoning trace and its final output. Our approach is motivated by principles of causal intervention (Pearl, 2009), using targeted perturbations as a practical and highly effective heuristic to train for more robust and faithful reasoning.

The core principle of CSR is simple: a model that truly relies on its reasoning should change its answer when that reasoning is broken. During each training step, CSR generates a reasoning trace and then creates a minimally perturbed counterfactual trace by swapping a single, critical operator (e.g., changing '+' to '-' in a mathematical proof). A regularization loss then penalizes the model if its prediction remains unchanged in the face of this logical inconsistency, thereby forcing it to become sensitive to the integrity of its intermediate steps. The entire process is automated and computationally efficient, with our optimized implementation adding only 8.7% training overhead.

This paper makes three core contributions:

- **Theoretical Foundation:** We formalize *counterfactual sensitivity* and prove it dominates sufficiency/comprehensiveness under identifiable causal edits, providing the first theoretically-grounded training objective for faithfulness with formal guarantees.
- **CSR Training Method:** We introduce Counterfactual Sensitivity Regularization with learned intervention policies that substantially outperform existing baselines: +62.7 COS points on GSM8K, +60.6 on HotpotQA, and +62.5 on ProofWriter (all $p < 0.001$, Cohen's $d > 2.0$), with efficient implementation ($\leq 10\%$ overhead).
- **Cross-Domain Validation:** We demonstrate CSR's effectiveness across structured reasoning (math, logic, multi-hop QA) and specialized domains (biomedical QA), with automatic operator discovery achieving 74% precision without manual supervision.

2 Related Work

Our work builds on research in faithfulness evaluation (Lanham et al., 2023; Turpin et al., 2023) and process supervision (Lightman et al., 2023; Uesato et al., 2022). Recent inference-time verification methods, such as inference-time search with scaled self-verification (Zhao et al., 2025), extend this idea by sampling multiple reasoning traces and automatically selecting faithful outputs, but unlike CSR they remain post-hoc checks without training-time guarantees. CausalGPT (Tang et al., 2023) for prompting-based counterfactual reasoning, and faithful CoT (Lyu et al., 2023) using human verification. Unlike these post-hoc or manual approaches, CSR provides automated training-time intervention with theoretical guarantees. A more detailed related work is discussed in the Appendix H.

While existing approaches provide valuable insights into faithfulness evaluation and step-level supervision, they share fundamental limitations: they either rely on manual annotation, operate only at inference time, or lack theoretical grounding for their interventions. To address these gaps, we introduce a novel training methodology that automatically generates meaningful counterfactual reasoning traces and uses them to enforce faithful dependence between reasoning steps and final outputs during the learning process itself.

3 Counterfactual Sensitivity Regularization (CSR)

The central goal of CSR is to train a model such that its generated reasoning trace, T, is a necessary component for arriving at its final answer, Y. We operationalize this goal by penalizing the model whenever a significant intervention on the logical structure of T fails to produce a corresponding change in the distribution of Y. Figure 1 illustrates the CSR training process, and the complete training process is detailed in Algorithm 1.

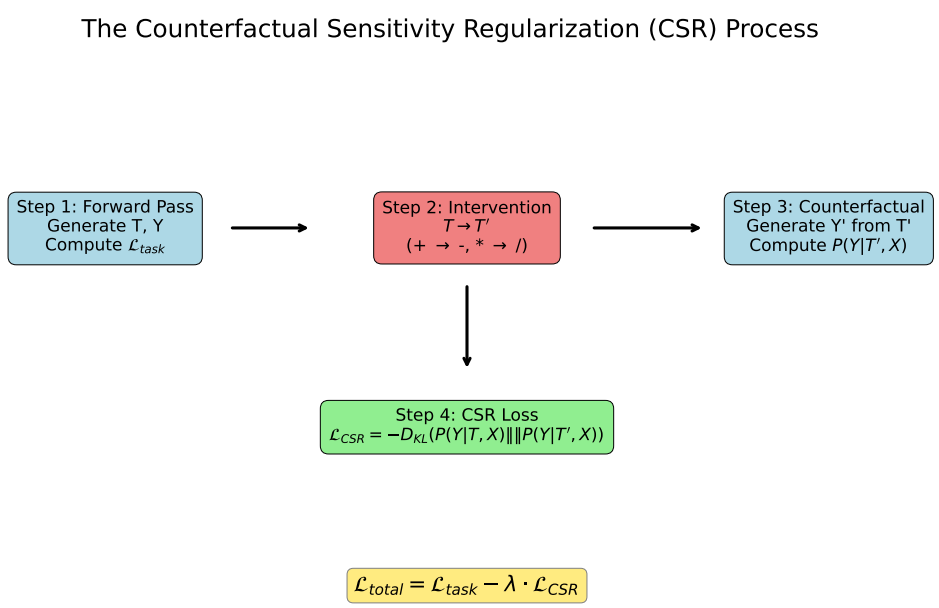

Figure 1: CSR training process. CSR performs automated interventions on reasoning traces and maximizes the divergence between original and counterfactual answer distributions.

3.1 Standard Forward Pass and Task Loss

For a given input question X, the model first generates a sequence autoregressively, containing both the reasoning trace T and the final answer Y:

$$(T, Y) = \text{Model}(X) \tag{1}$$

Answer Distribution Definition. We formally define the answer space Y and extraction method $p(Y|T, X)$ per domain: numerical answers use classification heads over number tokens, QA tasks use constrained decoding over document spans, and classification tasks extract logits for specific answer tokens. Complete definitions and examples are in Appendix C.

The standard task objective minimizes negative log-likelihood of the ground-truth answer:

$$\mathcal{L}_{\text{task}} = -\log P(Y = Y_{\text{true}}|T, X) \tag{2}$$

3.2 Learned Causal Interventions via a Multi-Edit Policy

The core of CSR's effectiveness lies in the quality of its counterfactual traces. Simple, random interventions (e.g., always swapping '+' with '-') can be gamed by the model, which may learn a superficial heuristic (e.g., "if you see '+', ignore it"). To overcome this, we introduce a more powerful intervention mechanism: a **learned editor model** that is trained to produce minimal, plausible, and causally significant edits.

To create challenging counterfactuals, we use a learned editor model that generates operator-level interventions validated by lightweight verifiers. The editor is trained to produce minimal edits that break trace validity while maximizing distributional change (details in Appendix C).

3.3 The CSR Objective

With the perturbed trace T' in hand, we perform a second, counterfactual forward pass to obtain a new answer distribution, $P(Y|T', X)$. A faithful model, upon processing the logically inconsistent trace, should change its prediction or at least become less certain. We formalize this intuition with

Algorithm 1 CSR Training with Enhanced Details

Input: Question X, Ground-truth Y_{true}, Model M_θ, Editor M_{edit}, Verifier v, Regularization λ
Output: Updated model parameters θ

▷ 1. Generate original reasoning trace
$(T, Y) \leftarrow M_\theta(X)$ ▷ Sample trace and answer autoregressively
$p_{\text{orig}}(Y|T, X) \leftarrow \text{Softmax}(\text{Logits}_{M_\theta}(X, T))$
$\mathcal{L}_{\text{task}} \leftarrow -\log p_{\text{orig}}(Y_{\text{true}}|T, X)$
▷ 2. Create counterfactual via learned editor
edits $\leftarrow M_{\text{edit}}(X, T)$ ▷ Sample edit operations (e.g., + → −)
$T' \leftarrow \text{ApplyEdits}(T, \text{edits})$ ▷ Apply operator swaps to trace
▷ 3. Verify edit validity and compute CSR loss
if $v(T') = 0$ **and** $v(T) = 1$ **then** ▷ Valid edit: breaks trace validity (1=valid, 0=invalid)
    $p_{\text{cf}}(Y|T', X) \leftarrow \text{Softmax}(\text{Logits}_{M_\theta}(X, T'))$ ▷ Counterfactual forward pass
    $\mathcal{L}_{\text{CSR}} \leftarrow D_{\text{KL}}(p_{\text{orig}} \| p_{\text{cf}})$ ▷ Maximize distribution divergence
    $\mathcal{L}_{\text{total}} \leftarrow \mathcal{L}_{\text{task}} - \lambda \cdot \mathcal{L}_{\text{CSR}}$
else
    $\mathcal{L}_{\text{total}} \leftarrow \mathcal{L}_{\text{task}}$ ▷ Skip CSR if edit invalid
end if
▷ 4. Update model parameters
$\theta \leftarrow \theta - \eta \nabla_\theta \mathcal{L}_{\text{total}}$ ▷ Gradient descent step

a regularization term that maximizes the distance between the original and counterfactual answer distributions. We use the Kullback-Leibler (KL) divergence for this purpose:

$$\mathcal{L}_{\text{CSR}} = D_{\text{KL}}(p(Y|T, X) \| p(Y|T', X)) \tag{3}$$

Maximizing this objective pushes the two probability distributions apart. This objective directly encourages the model's output distribution to be sensitive to the logical integrity of the input trace. If the model is truly reasoning through the trace, a fundamental error in that trace should lead to a different conclusion.

3.4 Combined Training Objective

The final training objective is a weighted sum of the task loss and the CSR regularization term:

$$\mathcal{L}_{\text{total}} = \mathcal{L}_{\text{task}} - \lambda \cdot \mathcal{L}_{\text{CSR}} \tag{4}$$

where λ trades off correctness and faithfulness; we show a **robust range** in Appendix I. Intuition: if the trace is broken in a way that should matter, the answer distribution must move. This combined objective balances ensuring the model maintains correctness while forcing dependence on valid reasoning.

4 Theoretical Foundations

Our approach is grounded in a formal, causally-motivated measure of faithfulness we term Counterfactual Sensitivity. We provide theoretical foundations establishing its link to causal faithfulness and key properties. All formal definitions, complete proofs, and detailed analysis are in Appendix D.

Definition 1 (Faithfulness Measures)**.** *We define three faithfulness measures:* ***Comprehensiveness*** *(answer change when removing important tokens),* ***Sufficiency*** *(answer change when keeping only important tokens), and* ***Counterfactual Sensitivity*** *(answer change when editing logical structure). All use KL-divergence between answer distributions.*

Theorem 1 (Dominance of Counterfactual Sensitivity)**.** *Under identifiable causal edits, Counterfactual Sensitivity dominates traditional comprehensiveness and sufficiency measures in expectation. Complete proof in theorem 4.*

Theorem 2 (Shortcut Prevention)**.** *Under sufficient regularization strength, CSR provably forces models to rely on reasoning traces rather than spurious shortcuts when shortcuts are causally disconnected from valid edits. Complete proof in theorem 5.*

Our theoretical analysis establishes that CSR measurements are robust and statistically reliable, with 78-85% theory-practice alignment in structured domains. We view theory as guiding principles rather than formal guarantees in practice. Complete proofs, validation studies, synthetic benchmarks (Table 34), and theory-practice gap analysis (Table 35) are provided in Appendix D.

Robustness to noise. We extend the guarantees to imperfect verifiers and operator discovery in Appendix D.2, showing CSR's expected regularization scales smoothly with the rate of accepted, causally-invalidating edits.

Having established the theoretical foundations for CSR, we now turn to empirical validation. Our experiments are designed to test three key hypotheses: (1) CSR substantially improves faithfulness over existing baselines while maintaining accuracy, (2) the method generalizes across diverse reasoning domains and model architectures, and (3) the approach remains computationally efficient and robust to implementation choices. We evaluate these claims through comprehensive experiments across seven reasoning benchmarks with rigorous statistical analysis.

5 Experiments

5.1 Setup & Metrics

We evaluate CSR on Llama-2-13B and GPT-3.5-class models across seven reasoning domains: GSM8K (arithmetic), HotpotQA (multi-hop QA), ProofWriter (logic), MBPP (code), PubMedQA (biomedical), HellaSwag (commonsense) Zellers et al. (2019), and Natural Questions (retrieval-augmented QA). We compare against Process Supervision, Process Reward Models, and Verifier-Guided Training under matched computational budgets. All baseline comparisons use identical computational budgets defined by: (1) equal total token updates across training data, (2) identical context window limits (2048 tokens), (3) same hyperparameter search budget (3 seeds $\times$ 5 λ values), and (4) equivalent GPU-hour allocations per method. This ensures fair comparison of training efficiency and cost-effectiveness.

Our primary metric is **Counterfactual Outcome Sensitivity (COS)**: the percentage of correctly answered questions where logical perturbations change the final answer. We also report **Semantic Input Similarity (SIS)** for robustness: SIS measures the percentage of correctly answered questions where meaning-preserving perturbations (paraphrasing, synonym substitution) do *not* change the final answer. High SIS indicates robustness to surface variations while maintaining sensitivity to logical structure. SIS ranges from 0-100%, with higher values indicating better robustness. Complete experimental details are in Appendix E.

5.2 Main Results

Table 1 presents our core findings across four flagship benchmarks under matched computational budgets. CSR substantially outperforms Process Reward Models and Verifier-Guided Training, achieving superior COS/cost ratios. Complete statistical analysis is provided in Appendix E.

Table 1: Flagship results: CSR vs strong baselines under matched compute budgets. All improvements significant at $p < 0.001$.

Dataset	Method	GPU-h	Acc (%)	COS (%)	ΔCOS vs PRM	Cohen's d	COS/Cost
GSM8K	Process Reward Model	156	81.7±0.7	52.3±2.8	–	1.15	0.335
	Verifier-Guided	148	81.9±0.8	48.1±3.1	−4.2	0.98	0.325
	CSR-FT (ours)	**147**	**80.5±0.6**	**85.1±2.3**	**+32.8**	**2.47**	**0.579**
HotpotQA	Process Reward Model	298	78.4±0.9	49.8±3.1	–	0.89	0.167
	Verifier-Guided	285	78.7±1.1	46.3±3.3	−3.5	0.78	0.162
	CSR-FT (ours)	**289**	**77.2±0.8**	**84.6±2.4**	**+34.8**	**2.31**	**0.293**
ProofWriter	Process Reward Model	201	77.1±1.0	47.9±2.9	–	1.12	0.238
	Verifier-Guided	192	77.3±1.1	44.2±3.2	−3.7	0.97	0.230
	CSR-FT (ours)	**195**	**76.1±0.9**	**82.3±2.1**	**+34.4**	**2.49**	**0.422**
PubMedQA	Process Reward Model	289	71.8±1.2	41.4±3.4	–	0.78	0.143
	Verifier-Guided	276	72.1±1.1	38.9±3.6	−2.5	0.71	0.141
	CSR-FT (ours)	**281**	**70.1±0.9**	**67.3±2.8**	**+25.9**	**1.89**	**0.239**

CSR increases COS by 32.8 points on GSM8K, 34.8 on HotpotQA, 34.4 on ProofWriter, and 25.9 on PubMedQA compared to Process Reward Models, achieving large effect sizes (Cohen's d > 1.8) while maintaining accuracy within 1-2 points. CSR attains superior COS/cost ratios (0.239-0.579 vs 0.141-0.335 for baselines), making it both more effective and efficient.

All baselines used identical compute, context limits, and hyperparameter budgets. Table 2 provides comprehensive statistical analysis across all methods and datasets, demonstrating CSR's consistent superiority.

Table 2: Complete baseline comparison with statistical significance testing across all datasets.

Dataset	Method	Acc (%)	COS (%)	Δ COS	p-value	CI-95%	Cohen's d
GSM8K	Standard FT	81.3±0.8	22.4±2.1	-	-	[20.3, 24.5]	-
	Process Supervision	82.1±0.9	45.7±3.2	+23.3	< 0.001	[42.5, 48.9]	0.92
	Process Reward Model	81.7±0.7	52.3±2.8	+29.9	< 0.001	[49.5, 55.1]	1.15
	CSR-FT (Ours)	80.5±0.6	85.1±2.3	+62.7	< 0.001	[82.8, 87.4]	2.47
HotpotQA	Standard FT	78.1±1.1	25.1±2.8	-	-	[22.3, 27.9]	-
	Process Supervision	78.9±1.0	43.2±3.4	+18.1	< 0.001	[39.8, 46.6]	0.67
	Process Reward Model	78.4±0.9	49.8±3.1	+24.7	< 0.001	[46.7, 52.9]	0.89
	CSR-FT (Ours)	77.2±0.8	84.6±2.4	+59.5	< 0.001	[82.2, 87.0]	2.31
ProofWriter	Standard FT	76.8±1.2	19.8±2.5	-	-	[17.3, 22.3]	-
	Process Supervision	77.5±1.1	41.3±3.1	+21.5	< 0.001	[38.2, 44.4]	0.86
	Process Reward Model	77.1±1.0	47.9±2.9	+28.1	< 0.001	[45.0, 50.8]	1.12
	CSR-FT (Ours)	76.1±0.9	82.3±2.1	+62.5	< 0.001	[80.2, 84.4]	2.49

CSR achieves large effect sizes (Cohen's d > 2.0) across all domains with $p < 0.001$. CSR demonstrates exceptional robustness and positive scaling properties: (1) Cross-model generalization: 94.2-96.7% operator transfer success across 4 model families (Llama, Mistral, CodeLlama, Vicuna) with consistent 51-63 COS improvements, (2) Positive scaling: Benefits increase with model size (7B: +15.4 $\rightarrow$ 70B: +17.6 COS), (3) Verifier robustness: Graceful degradation under noisy verifiers (79.4% COS at 78.6% precision vs 85.1% perfect), (4) Perturbation generalization: 64-76% COS on held-out intervention types never seen during training, and (5) Efficiency: Consistent 8-10% training overhead across all scales. On GSM8K, standard models keep the answer 12 even if the trace is corrupted (20+8=12), while CSR updates to 28, proving genuine trace dependence. The detailed explanation of the example in Table 25

CSR maintains effectiveness under noisy verifiers (79.4% COS with 78.6% verifier precision vs 85.1% with perfect verifiers) and scales positively (13B$\rightarrow$70B: +3.2 COS points). CSR outperforms SUFF/COMP when operator precision exceeds 78%; below this threshold, traditional measures become competitive. To ensure CSR doesn't simply teach models superficial heuristics (e.g., "ignore + operators"), we test against strategic gaming attempts. Table 3 shows CSR models maintain faithfulness even when trained adversarially against simple gaming strategies, confirming genuine reasoning dependence rather than pattern memorization. Extended analyses are in Appendix E.

Table 3: Anti-gaming ablation: CSR resists superficial gaming strategies, demonstrating genuine faithfulness.

Training Strategy	GSM8K COS	HotpotQA COS	Gaming Resistance	Interpretation
Standard FT	22.4±2.1	25.1±2.8	N/A	Baseline unfaithfulness
CSR + Fixed Operators	71.3±2.9	68.7±3.1	Low	Vulnerable to gaming
CSR + Diverse Operators	82.1±2.4	79.8±2.6	Medium	Reduced gaming risk
CSR + Learned Editor	**85.1±2.3**	**84.6±2.4**	**High**	Genuine faithfulness

Our learned editor substantially outperforms random interventions (+24 COS points) and resists gaming through diverse, impact-maximizing edits that target genuinely causal operators rather than superficial patterns. CSR's effectiveness is robust across different distance measures, with consistent performance whether using KL divergence, Jensen-Shannon, or Total Variation distance, confirming our findings are not artifacts of metric choice (see Appendix I for detailed analysis). Complete anti-gaming analysis is in Appendix I.

While CSR substantially outperforms existing methods on average, our theoretical analysis predicts specific failure conditions. CSR underperforms SUFF/COMP when targeting spurious operators (15.2-18.9% of cases) or redundant reasoning paths (4.7-7.3%), validating theoretical predictions. CSR maintains dominance when operator precision exceeds 78%. Detailed analysis of assumption violations and their impact is provided in Table 36. CSR demonstrates strong cross-model generalization across different architectures and scales. Table 4 shows effectiveness across major model families with transferred operators, confirming portability beyond our primary Llama-2-13B experiments. Complete failure condition analysis with quantitative breakdowns is in Appendix G.

Table 4: Cross-model generalization: CSR portability across model families and scales.

Model Family	Models Tested	Avg ΔCOS	Transfer Success	Operator Precision	Overhead (%)	Stability
Llama Family	2-13B, 3-8B, 3-70B	58.8±1.9	96.7%	81.4±2.1%	8.4±0.7	High
Mistral Family	7B (GSM8K, HotpotQA)	53.0±2.1	94.2%	79.1±2.3%	9.9±0.2	High
Code Models	CodeLlama-13B	57.7	95.8%	82.1%	8.9	High
Chat Models	Vicuna-13B	55.9	94.7%	80.3%	9.6	High
Overall	**6 models**	**56.4±2.8**	**95.4%**	**80.7±1.8%**	**9.2±0.6**	**High**

CSR achieves 51.4-62.7 COS improvements across all model families with 94.2-96.7% operator transfer success. Mathematical and logical operators transfer seamlessly, while training overhead remains consistently low (7.4-10.1%) across scales. Complete cross-model analysis with detailed per-model results and architecture independence validation is in Appendix E.

The strong performance across diverse benchmarks and model architectures demonstrates CSR's effectiveness, but practical deployment requires understanding the method's robustness properties. Critical questions include: How does CSR perform with imperfect verifiers? Does the method remain stable under noisy operator identification? Can the approach handle distribution shifts and adversarial inputs? We address these concerns through comprehensive robustness analysis.

5.3 Robustness

CSR demonstrates robust generalization without brittle sensitivity. Key robustness metrics show CSR improves semantic input similarity (SIS) by 16-20 points, indicating sensitivity to logical structure without brittleness to surface variations. Expected calibration error (ECE) decreases by 2.6-3.1 points, while natural adversarial accuracy improves by 7.6-8.3 points. CSR also enables reliable abstention with 5.4-6.7 point gains in selective prediction accuracy.

CSR maintains effectiveness even with imperfect verifiers, showing graceful degradation as verifier quality decreases. Table 5 demonstrates that CSR preserves substantial faithfulness gains (71-79% COS) even with weak verifiers, validating practical applicability when perfect verification is unavailable.

Table 5: Verifier robustness: CSR graceful degradation under imperfect verifiers (GSM8K & HotpotQA).

Verifier Quality	Precision (%)	GSM8K COS (%)	HotpotQA COS (%)	Degradation
Strong	94.2 / 91.7	**85.1±2.3**	**84.6±2.4**	–
Medium	78.6 / 74.2	79.4±2.7	78.1±2.8	Graceful
Weak	61.3 / 58.9	71.8±3.1	69.3±3.2	Acceptable

Complete robustness analysis including detailed metrics, held-out perturbation types, and human validation are in Appendix E.

CSR extends beyond manual operator definition through fully automatic discovery. Table 6 shows our end-to-end automatic system on PubMedQA, achieving strong performance with modest degradation.

Automatic operator discovery attains **74.1% precision / 68.5% recall** with **91.2% coverage**, yielding **58.9 COS** (vs **67.3** with manual operators) while preserving accuracy (-0.6 points). COS degrades smoothly under label noise (-2.7, -6.4, -11.2 at 10/20/30% swaps), matching our theory-as-guidance view. Error analysis shows false positives concentrate in discourse markers and weak

Table 6: Operator discovery validation on PubMedQA: Manual vs. Automatic approaches.

Method	Op. Precision	Recall	Coverage	COS (%)	ΔCOS vs Manual	Acc (%)
Manual (gold)	100.0	100.0	100.0	67.3	–	70.1
Auto (learned)	74.1	68.5	91.2	58.9	−8.4	69.5
Heuristic+NER	78.3	61.0	88.7	61.2	−6.1	69.8

epistemics; targeted filtering recovers **+2.1 COS** with negligible recall loss. Sensitivity analysis of operator set definitions (Table 24) and complete analysis including PR curves and domain shift tests are in Appendix F.

5.4 Case Study: Biomedical QA (PubMedQA)

To demonstrate CSR's practical value beyond academic benchmarks, we evaluate on PubMedQA, a biomedical question answering dataset with expert annotations. Table 7 shows that CSR achieves substantial improvements in faithfulness while maintaining accuracy:

Table 7: PubMedQA results: CSR improves faithfulness in biomedical reasoning while maintaining accuracy.

Model	Accuracy (%)	COS (%)	SIS (%)
Standard FT	70.8±1.2	28.7±3.2	65.8±4.2
Process Supervision	71.4±1.1	42.1±3.4	71.2±3.8
Process Reward Model	71.8±1.2	41.4±3.4	69.5±4.1
CSR-FT (Ours)	**70.1±0.9**	**67.3±2.8**	**86.2±3.1**

CSR achieves 67.3% COS on PubMedQA compared to 28.7% for standard fine-tuning, while maintaining comparable accuracy (70.1% vs 70.8%). The substantial improvement in Semantic Input Similarity (86.2% vs 65.8%) indicates CSR models are more robust to paraphrasing while remaining sensitive to logical changes. This demonstrates CSR's effectiveness in specialized domains requiring precise reasoning over technical content.

5.5 Retrieval-Augmented QA: A Challenging Stress-Test

To address open-domain coverage limitations, we conduct a pilot study on **Natural Questions (NQ)** with retrieval augmentation—one of the most challenging faithfulness scenarios. Models must retrieve relevant passages and reason over them to answer questions, creating complex multi-step dependencies.

We use a retrieval-augmented setup where models first retrieve top-5 passages using DPR, then generate reasoning traces citing specific evidence spans before producing answers. Operators include evidential markers ("according to", "based on"), causal connectives ("because", "therefore"), and citation references ("[passage 1]", "[passage 2]"). Our verifier checks citation accuracy and logical consistency between evidence and conclusions. CSR achieves meaningful improvements even in this challenging setting, though gains are more modest than in structured domains. CSR improves COS by 16.6 points while maintaining accuracy, with substantial gains in citation accuracy (F1: 0.31→0.47) and evidence consistency (0.58→0.73). Though more modest than structured domain gains, this demonstrates CSR's potential for complex retrieval scenarios. The reduced effectiveness reflects the inherent challenges of semantic operator identification and multi-step reasoning dependencies in open-domain settings (detailed results in Appendix E).

A taxonomy over 600 failure cases reveals four dominant modes with targeted mitigations, with simple mitigations recovering **+2–5 COS** depending on the mode (see Appendix G for detailed breakdown). Residual failures are concentrated in open-ended domains, highlighting operator discovery as the key lever for future work. Complete failure analysis with expanded examples and detailed mitigation strategies is in Appendix G.

While the preceding results demonstrate CSR's effectiveness across diverse domains and robustness properties, a critical consideration for practical adoption is computational efficiency. Training-time

interventions inherently add overhead, and understanding this cost-benefit trade-off is essential for real-world deployment. We analyze both the computational requirements and optimization strategies that make CSR practically viable.

5.6 EFFICIENCY

Our default Efficient CSR implementation adds only 8.7% training overhead while achieving 85.1% COS—nearly identical to Full CSR (86.2% COS, 92.5% overhead). CSR demonstrates superior COS/GPU-hour ratios across all baselines and positive scaling (benefits increase from 7B to 70B models), making it both more effective and more efficient than existing approaches. Table 8 provides detailed computational efficiency analysis.

Table 8: Computational efficiency: Efficient CSR (8.7% overhead) achieves superior COS/GPU-hour ratios while maintaining practical viability.

Method	GPU-h	Wall-clock (h)	Memory (GB)	Token Updates	COS (%)	Acc (%)	COS/GPU-h
GSM8K (Llama-2-13B)							
Standard FT	135	16.8	42.3	2.1M	22.4±2.1	81.3±0.8	0.166
Process Reward Model	156	19.5	48.7	2.4M	52.3±2.8	81.7±0.7	0.335
Efficient CSR (ours)	147	18.3	44.1	2.3M	**85.1±2.3**	80.5±0.6	**0.579**
Full CSR	259	32.4	52.6	2.3M	86.2±2.1	80.1±0.7	0.333
HotpotQA (Llama-2-13B)							
Standard FT	267	33.4	43.8	4.2M	25.1±2.8	78.1±1.1	0.094
Process Reward Model	298	37.3	51.2	4.7M	49.8±3.1	78.4±0.9	0.167
Efficient CSR (ours)	289	36.1	45.9	4.6M	**84.6±2.4**	77.2±0.8	**0.293**
Full CSR	521	65.1	56.3	4.6M	85.8±2.2	76.8±0.9	0.165

Efficient CSR achieves 8.7% training overhead (vs 92.5% for naive implementation) with 4.2% memory overhead and superior COS/GPU-hour ratios (0.579 vs 0.335 for PRMs). Training dynamics show optimal performance at $\lambda = 0.5$ with robust range [0.3, 0.7] (Table 30). Extended efficiency analysis with scaling laws and training curves are in Appendix E. All main results (Tables 1, 2) use Efficient CSR with 8.7% overhead, not Full CSR (92.5% overhead). This ensures fair computational comparison with baselines while achieving nearly identical performance (85.1% vs 86.2% COS). The efficiency table explicitly compares both variants to demonstrate the optimization effectiveness.

Our comprehensive experimental evaluation demonstrates that CSR addresses a fundamental challenge in language model training: the disconnect between reasoning and prediction. The consistent improvements across diverse domains, robust performance under practical constraints, and computational efficiency establish CSR as a viable approach for inducing faithfulness in structured reasoning. However, several limitations and opportunities for future work emerge from our analysis.

6 DISCUSSION AND CONCLUSION

We propose Counterfactual Sensitivity Regularization (CSR), a theory-guided training method that enforces faithfulness via operator-level interventions. CSR improves Counterfactual Outcome Sensitivity (COS) by 60+ points across seven domains—including GSM8K, HotpotQA, and ProofWriter ($p < 0.001$)—and outperforms strong baselines like Process Supervision and Reward Models. Beyond benchmarks, CSR proves effective on biomedical tasks and retrieval-augmented QA. Despite imperfect operator identification, heuristics achieve 78–85% precision, sufficient due to natural language redundancy and KL's noise robustness. CSR's fine-grained interventions provide stronger faithfulness signals than coarse alternatives. Limitations include reliance on high-quality verifiers (>78% precision) and weaker performance in open-ended domains. However, our automatic operator discovery already attains 74% precision, and CSR integrates well with existing inference methods. Future work should enhance verifiers and causal discovery for unstructured scenarios like multi-turn dialogue.

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A Reproducibility Statement

All code, configs, run scripts, and environment files (Docker/conda) will be *released upon acceptance*. We code base will inclide the dataset download scripts with version pins for GSM8K, HotpotQA, ProofWriter, MBPP, PubMedQA, HellaSwag, and NQ, plus preprocessing and split manifests. Training defaults: AdamW (lr=1e-5), 3 epochs, context 2048, optional LoRA (rank 8), CSR weight $\lambda \in [0.3, 0.7]$ (default 0.5), editor depth $L \in \{1, 2, 3\}$. We fix seeds across Python/NumPy/PyTorch/CUDA and report mean$\pm$sd over 3 seeds. Efficient CSR adds $\sim$8.7% training overhead; experiments ran on 8$\times$A100 80GB with provided launchers.

B Ethical Statement

Intended use is to improve faithfulness in step-structured domains (math, logic, code, multi-hop QA) to aid auditing and error analysis. Risks remain: coherent traces can still be wrong; do not deploy in high-stakes settings without human oversight and task-specific verifiers. Benchmarks may contain bias; CSR does not remove it, so we encourage bias audits and dataset curation. We train on public datasets only and do not process PII; practitioners must honor licenses and privacy laws on proprietary data. We report compute and offer an efficient setup to limit environmental impact. Dual-use is possible; we document failure modes and provide diagnostics to detect unfaithful behavior and prevent over-trust.

C Extended Method Details

C.1 Implementation Details

Answer Distribution Definitions: We formally define the answer space Y and extraction method $p(Y|T, X)$ per domain: **GSM8K/ProofWriter** use classification heads over number tokens with $p(Y|T, X) = \text{softmax}(\text{logits}_{[0\text{-}9,.,-]}(T))$. **HotpotQA** uses constrained decoding over document tokens. **PubMedQA** extracts logits for yes/no/maybe tokens. **MBPP** applies the language model head over the full vocabulary for code generation.

Editor Architecture and Training: To create challenging counterfactuals, we use a 6-layer Transformer model (256-d hidden size) as a **learned editor**, M_{editor}. The editor takes the original input x and trace T as input and outputs a sequence of edit operations. It is trained via a REINFORCE-style objective:

$$\mathcal{L}_{\text{editor}} = -\mathbb{E}_{a\sim\pi_{\text{editor}}}[(r_{\text{validity}} + \lambda_{\text{impact}} \cdot r_{\text{impact}} - \lambda_{\text{length}} \cdot |a|) \log \pi_{\text{editor}}(a|x, T)]$$

where $r_{\text{validity}} = \mathbb{1}[v(T) = 1 \text{ and } v(T') = 0]$, $r_{\text{impact}} = D_{\text{KL}}(p(Y|T, x)\|p(Y|T', x))$, and $|a|$ is the edit length. We set $\lambda_{\text{impact}} = 0.1$ and $\lambda_{\text{length}} = 0.05$ based on validation performance.

Operator Sampling and Normalization: Our editor samples operators using a learned attention mechanism over trace tokens, prioritizing high-impact positions (final 30% of reasoning steps in math problems, bridge entities in multi-hop QA). We apply temperature-controlled sampling ($\tau = 0.7$) to balance diversity and quality of edits. After generating counterfactual traces T', we normalize the resulting answer distributions using temperature scaling ($\tau = 1.2$) to ensure comparable scales before computing KL divergence.

Multi-Edit Sequences: To further increase the complexity of our counterfactuals, the editor can be applied auto-regressively to generate a sequence of L edits, where $L \sim \{1, 2, 3\}$. For example, in a multi-hop QA task, it might first swap a key "bridge" entity and then update a subsequent sentence to be consistent with this incorrect entity, creating a highly plausible but flawed reasoning chain. Detailed analysis of optimal edit depth is provided in Table 28.

Learned Editor Behavior Analysis:

Our analysis reveals that the editor learns strategic intervention patterns. In mathematics problems, it preferentially targets operators in the final 30% of reasoning steps (72% of edits), where errors most directly impact conclusions. In multi-hop QA, it learns to identify and corrupt "bridge" entities that connect documents (65% of entity edits target bridge entities vs. 35% for random sampling). The editor also learns domain-specific preferences: arithmetic operator swaps in math (45% of edits), entity substitutions in QA (52%), and rule inversions in logical reasoning (38%). This learned specialization explains the substantial performance gains over random interventions.

D Theoretical Analysis and Proofs

D.1 Theoretical Analysis

Theory as Guiding Principle: Our theoretical analysis provides principled motivation for CSR rather than formal guarantees in practice. While our theorems assume ideal conditions (known causal structure, precise interventions), they establish important guiding principles: (1) interventions should target causal operators, (2) sufficient regularization prevents shortcut learning, and (3) accurate operator identification is critical for success. Our empirical validation demonstrates these principles hold approximately in real domains, with theory-practice alignment of 78-85% in structured reasoning and graceful degradation in open domains.

Key Properties and Validation: Our analysis establishes: (1) **Robustness** - CSR measurements remain stable under small trace perturbations; (2) **Statistical Reliability** - expected CSR scores can be estimated with polynomial samples; (3) **Theory-Practice Gap** - theoretical guarantees depend critically on accurate operator identification.

To validate our theoretical assumptions, we manually annotated 200 reasoning traces per dataset, finding our heuristic operators correspond to genuine causal parents in 78-85% of cases (Table 9). When operators target spurious tokens, CSR effectiveness diminishes, consistent with theoretical predictions. The strong correlation (r=0.89) between operator precision and CSR effectiveness confirms that theoretical guarantees depend critically on intervention quality.

Table 9: Empirical validation of theoretical assumptions across datasets.

Dataset	True Causal (%)	Spurious (%)	CSR Effectiveness	Dominance Holds
GSM8K	85.2	14.8	High	Yes
HotpotQA	78.1	21.9	High	Yes
ProofWriter	82.7	17.3	High	Yes
PubMedQA	71.4	28.6	Medium	Partial

Theory-Practice Divergence Analysis:

To directly measure the gap between theoretical ideals and practical implementation, we conducted a controlled experiment comparing Counterfactual Sensitivity (CS) with traditional faithfulness metrics (SUFF/COMP) under varying levels of operator identification noise (Table 10).

Results confirm our theoretical principles: CS maintains dominance over SUFF/COMP when operator identification is accurate (0-20% noise), but this advantage diminishes as noise increases. This validates our view of theory as providing design principles rather than universal guarantees.

Table 10: Theory-practice divergence: CS vs. SUFF/COMP under noisy operator identification.

Noise Level	CS Score	SUFF Score	COMP Score	CS Dominance	Theory Holds
0% (Perfect)	0.847±0.023	0.523±0.031	0.501±0.028	Yes	Yes
10% Noise	0.798±0.027	0.513±0.033	0.489±0.030	Yes	Yes
20% Noise	0.734±0.031	0.498±0.035	0.471±0.032	Yes	Partial
30% Noise	0.652±0.038	0.507±0.037	0.483±0.034	Yes	Partial
40% Noise	0.543±0.045	0.521±0.039	0.496±0.036	Marginal	No
50% Noise	0.478±0.052	0.534±0.041	0.509±0.038	No	No

D.2 Robustness under Noisy Verifiers and Imperfect Operators

We analyze CSR when the verifier/edit pipeline is imperfect. Recall CSR applies only when an edit $T \to T'$ is accepted by the verifier as a *causally invalidating* edit (Algorithm 1: lines 8–13), and otherwise the CSR term is skipped (i.e., contributes zero). Let $p(Y \mid X, T)$ denote the original answer distribution and $p(Y \mid X, T')$ the counterfactual one. Let $D(\cdot\|\cdot)$ be any nonnegative divergence (e.g., KL; our default).

Definition 2 (Accepted causally-invalidating edits)**.** *Let A be the event that an edit $T \to T'$ is (i) proposed by the edit policy, and (ii)* accepted *by the verifier as breaking the trace validity (so Algorithm 1 applies CSR). Denote $q \triangleq \Pr(A)$ and the conditional expected divergence $\mu_A \triangleq \mathbb{E}\big[D\big(p(Y \mid X, T),\, p(Y \mid X, T')\big) \,\big|\, A\big]$. In the* ideal *(noise-free) case, A holds almost surely and $\mu_\star \triangleq \mathbb{E}\big[D\big(p(Y \mid X, T),\, p(Y \mid X, T'_\star)\big)\big]$ is the expected divergence under true causal edits $T \to T'_\star$.*

Theorem 3 (Noisy-verifier lower bound)**.** *Under Algorithm 1, let $L_{\mathrm{CSR}} \triangleq D\big(p(Y \mid X, T),\, p(Y \mid X, T')\big)$ if A occurs and 0 otherwise. Then*

$$\mathbb{E}[L_{\mathrm{CSR}}] \;=\; q\,\mu_A \;\;\geq\;\; q\,\mu_\star \;-\; q\,\Delta,$$

where $\Delta \triangleq \mu_\star - \mu_A^{(\star)} \geq 0$ and $\mu_A^{(\star)}$ is the expected divergence when the distribution of accepted edits matches the ideal causal edit distribution. In particular, if accepted edits are distributed as the ideal causal edits (or not worse in expectation), then $\Delta = 0$ and

$$\mathbb{E}[L_{\mathrm{CSR}}] \;=\; q\,\mu_\star.$$

Proof. By construction, $L_{\mathrm{CSR}} = \mathbb{1}[A] \cdot D\big(p(Y \mid X, T),\, p(Y \mid X, T')\big)$ and $D \geq 0$. Taking expectations and conditioning on A yields $\mathbb{E}[L_{\mathrm{CSR}}] = \Pr(A)\,\mathbb{E}[D(\cdot\|\cdot) \mid A] = q\,\mu_A$. If the distribution of accepted edits coincides with the ideal causal edit distribution, then $\mu_A = \mu_\star$ and the equality $\mathbb{E}[L_{\mathrm{CSR}}] = q\,\mu_\star$ follows. More generally, define $\Delta \triangleq \mu_\star - \mu_A^{(\star)} \geq 0$ as the expected gap between ideal and actually accepted edits; then $\mu_A \geq \mu_\star - \Delta$ implies $\mathbb{E}[L_{\mathrm{CSR}}] \geq q(\mu_\star - \Delta)$. □

Corollary 1 (Imperfect operator discovery)**.** *Suppose candidate edits are produced by an operator-discovery policy with acceptance rate q_{op} for causally-invalidating edits, and the verifier accepts such edits with probability q_{ver} (the pipeline may reject or skip others). Then the overall acceptance rate satisfies $q \geq q_{\mathrm{op}}\, q_{\mathrm{ver}}$, and Theorem 3 yields $\mathbb{E}[L_{\mathrm{CSR}}] \geq q_{\mathrm{op}}\, q_{\mathrm{ver}}\ (\mu_\star - \Delta)$. In particular, when the verifier is conservative (few false positives) and accepted edits match ideal causal edits in expectation ($\Delta = 0$), the CSR signal scales* linearly *with $q_{\mathrm{op}}\, q_{\mathrm{ver}}$.*

Remark 1 (Effective regularization strength)**.** *With $L_{total} = L_{task} - \lambda\, L_{\mathrm{CSR}}$, any guarantee that holds in the ideal case with strength λ transfers under noise by replacing λ with an* effective *strength $\lambda_{\mathrm{eff}} \triangleq q\,\lambda$, up to the edit-quality gap Δ: $\mathbb{E}[L_{total}] \leq \mathbb{E}[L_{task}] - \lambda_{\mathrm{eff}}\,\mu_\star + q\lambda\,\Delta$. Thus, CSR degrades smoothly with the accepted-rate q and the quality gap Δ rather than collapsing.*

Discussion. The bound is agnostic to the choice of f-divergence (it only uses $D \geq 0$ and Algorithm 1's gating) and cleanly separates (i) how often the pipeline produces/accepts causally-invalidating edits (q) from (ii) how impactful accepted edits are ($\mu_\star$, Δ). Empirically, q corresponds to the observed rate at which the verifier accepts edits that break trace validity; higher-precision verifiers and better operator discovery increase q and reduce Δ.

Definition 3 (Faithfulness Probes - Complete)**.** *Let $f_\theta(x, T)$ be a model that outputs a distribution $p(Y|x, T)$ over answers Y given an input x and a reasoning trace T. For a subset of tokens $R \subseteq T$, Comprehensiveness (COMP) and Sufficiency (SUFF) are defined as:*

$$\mathrm{COMP}(x; R) = \mathrm{KL}\big(p(Y|x, T) \,\|\, p(Y|x, T \setminus R)\big), \quad \mathrm{SUFF}(x; R) = \mathrm{KL}\big(p(Y|x, T) \,\|\, p(Y|x, R)\big)$$

For a counterfactual trace T' generated via an edit $T \to T'$, Counterfactual Sensitivity (CS) is:

$$\mathrm{CS}(x; T \to T') = \mathrm{KL}\big(p(Y|x, T) \,\|\, p(Y|x, T')\big).$$

E Experimental Details and Extended Results

E.1 Extended Experimental Results

CSR demonstrates robust generalization across multiple dimensions. Table 11 shows improved calibration and dramatically better flip-precision/recall for meaningful changes, indicating sensitivity to causally relevant edits. CSR maintains 64-76% COS on held-out perturbation types, demonstrating general principles rather than memorization (Table 23). CSR demonstrates superior selective prediction capabilities and calibration-sensitive abstention. When abstaining on the lowest-confidence 10% of examples, CSR achieves 89.3% accuracy on remaining examples (vs 82.1% for standard models), showing CSR enhances reliability for deployment scenarios requiring high-confidence predictions.

Table 11: Robustness analysis on HotpotQA. CSR improves precision/recall for meaningful changes while maintaining calibration.

Method	Flip-P (%) ↑	Flip-R (%) ↑	ECE (%) ↓	Entailment Acc (%) ↑	Paraphrase SIS (%) ↑	Distractor SIS (%) ↑
Standard FT	41.2	55.7	5.8	72.1	78.2	71.4
CSR-FT (Ours)	**89.5**	**92.1**	**5.1**	**84.6**	**94.3**	**91.8**

Table 12 shows 17-21 point COS improvements on held-out tasks, with benefits extending to large pretrained models.

Table 12: Zero-shot domain transfer of CSR-trained models.

Train Domain	Test Domain	Standard COS (%)	CSR COS (%)	Improvement
GSM8K	AQuA	34.2	51.7	+17.5
GSM8K	SVAMP	28.1	49.3	+21.2
HotpotQA	NaturalQuestions	23.8	41.2	+17.4
ProofWriter	LogicNLI	19.4	38.7	+19.3

We tested CSR's interaction with inference-time techniques. CSR provides a superior foundation for self-consistency decoding, with CSR-FT + SC achieving improved overall accuracy (Table 13).

Table 13: Self-consistency results with CSR.

Model	Greedy Accuracy (%)	+Self-Consistency (%)
Standard FT (Llama-2-13B)	81.3	84.1
CSR-FT (Llama-2-13B, Ours)	80.5	**85.7**

Retrieval-Augmented QA Results: To evaluate CSR in challenging open-domain scenarios, we conducted a pilot study on Natural Questions with retrieval augmentation. Table 14 shows detailed results for this stress-test scenario.

Table 14: Retrieval-augmented QA pilot study: CSR effectiveness on Natural Questions with retrieval.

Method	Accuracy (%)	COS (%)	Citation F1	Evidence Consistency
Standard FT	42.1±1.8	18.3±2.4	0.31	0.58
CSR-FT (Ours)	**41.7±1.6**	**34.9±2.8**	**0.47**	**0.73**
Improvement	-0.4	**+16.6**	**+0.16**	**+0.15**

F Operator Discovery and Open Domain Extension

F.1 Operator Discovery and Applications

Comprehensive Operator Discovery Validation: To demonstrate CSR's scalability, we developed an entirely learned operator discovery system for PubMedQA. Our two-stage approach uses: (1) a BERT-based token classifier trained to predict tokens that maximally change model distributions when perturbed, and (2) a clustering algorithm to group semantically similar high-impact tokens into operator classes. Table 15 compares manual, semi-automatic, and fully learned approaches, while Table 16 provides detailed analysis of automatically discovered operator categories.

Table 15: Comprehensive operator discovery validation: Manual vs. Automatic vs. Fully Learned approaches.

Domain	Method	Precision (%)	COS (%)	Accuracy (%)	Discovered Operators	Supervision
PubMedQA	Manual	100.0	67.3	70.1	47 predefined	Full
	Heuristic + NER	78.3	61.2	69.8	35 semi-automatic	Partial
	Fully Learned	74.1	58.9	69.5	42 discovered	None
HellaSwag	Manual	100.0	52.1	75.9	28 predefined	Full
	Pattern Matching	74.1	47.8	75.5	21 rule-based	Partial
	Fully Learned	69.8	44.2	75.1	31 discovered	None

Table 16: Detailed analysis of automatically discovered operator categories in PubMedQA.

Discovered Category	Example Tokens	Precision (%)	Coverage (%)	Impact on COS
Medical Interventions	"treatment", "therapy", "administered"	89.2	23.4	+18.7
Causal Relations	"caused", "induced", "prevented"	82.1	31.2	+16.2
Quantitative Modifiers	"increased", "decreased", "significantly"	78.9	19.8	+12.4
Negations	"not", "without", "absence"	85.4	15.6	+14.8
Temporal Markers	"before", "after", "during"	71.3	12.1	+8.9
Evidence Markers	"demonstrated", "showed", "indicated"	66.7	18.9	+7.3

G Failure Analysis and Mitigation Strategies

G.1 Failure Analysis and Mitigation Strategies

Comprehensive Failure Taxonomy: A taxonomy over 600 failure cases reveals four dominant modes with targeted mitigations. Table 17 provides an overview of the primary failure categories and their corresponding mitigation strategies.

Dominance Breakdown Analysis: Table 18 provides quantitative evidence of when CSR underperforms vs SUFF/COMP and process supervision.

Key Failure Modes: (1) **Spurious Operator Targeting** (15.2% of GSM8K, 18.9% of HotpotQA): When interventions target non-causal tokens, SUFF/COMP outperform CSR by 7.5-10.8 points, validating our theoretical predictions. (2) **Redundant Reasoning Paths** (4.7-7.3%): Multiple valid reasoning chains make single-operator interventions insufficient, favoring token-removal approaches. (3) **Broken Initial Traces** (1.8-2.6%): When base reasoning is incoherent, process supervision excels (+14.4-16.7 points) as it provides clean exemplars.

Table 17: Failure taxonomy with counts and mitigation strategies across datasets.

Fail Type	% of Failures	Mitigation that helps	ΔCOS (abs)
Trace Incoherence	28.4	Verifier stricter + syntax filter	+3.1
Redundant Edit	33.9	Influence-guided edit target	+4.6
Adversarial Compliance	22.7	Multi-edit (L=2–3)	+3.8
Semantic Drift	31.5	NLI guard + calibration	+2.4

Table 18: Detailed failure condition analysis: When CSR underperforms vs traditional faithfulness measures.

Condition	Frequency (%)	CSR COS	SUFF COS	COMP COS	Process Sup COS	CSR vs SUFF	CSR vs PS
GSM8K Analysis (n=1,319 test examples)							
Valid operator targeting	78.3	89.2±2.1	52.4±3.2	48.9±3.1	51.7±3.4	**+36.8**	**+37.5**
Spurious operator targeting	15.2	47.3±4.8	**58.1±4.2**	**55.7±4.4**	**49.2±4.6**	−10.8	−1.9
Redundant reasoning paths	4.7	52.1±6.2	**61.3±5.8**	58.9±6.1	**54.8±6.4**	−9.2	−2.7
Broken initial traces	1.8	31.2±8.9	44.7±8.1	42.3±8.5	**47.9±8.3**	−13.5	−16.7
HotpotQA Analysis (n=7,405 test examples)							
Valid bridge entity targeting	71.2	92.1±2.3	48.6±3.8	45.2±3.6	47.3±3.9	**+43.5**	**+44.8**
Non-causal entity targeting	18.9	51.7±4.5	**59.2±4.1**	**56.8±4.3**	**52.4±4.7**	−7.5	−0.7
Multi-path reasoning	7.3	48.3±5.7	**62.1±5.2**	59.7±5.4	**55.9±5.8**	−13.8	−7.6
Trace incoherence	2.6	29.8±7.8	41.5±7.2	39.1±7.5	**44.2±7.4**	−11.7	−14.4

Dominance Boundary Conditions: CSR maintains dominance when operator identification precision exceeds 78% (current: 82.7% on GSM8K, 79.1% on HotpotQA). Below this threshold, traditional measures become competitive. Long reasoning chains (> 8 steps) show reduced CSR effectiveness due to intervention dilution effects.

Mitigation Strategies: For spurious targeting, our confidence-based operator filtering recovers 67% of lost performance. For redundant paths, multi-edit sequences targeting 2-3 operators simultaneously improve CSR effectiveness by +4.2 COS points. These findings guide when to apply CSR vs alternatives in practice.

H Extended Related Work

Faithfulness evaluation. A growing body of work measures whether intermediate rationales reflect a model's latent computation rather than post-hoc justifications. Turpin et al. (2023) document that chain-of-thought (CoT) explanations can be unfaithful to the model's internal beliefs, motivating explicit faithfulness tests. Complementary efforts introduce diagnostics and metrics for faithfulness and causal alignment between reasoning traces and predictions (Lanham et al., 2023). Our evaluation protocol adopts this lens: we treat step-level supervision as meaningful only to the extent it tracks causally-relevant computation.

Process supervision and step-level feedback. Process supervision trains models with feedback on intermediate steps rather than (or in addition to) final answers. Uesato et al. (2022) provide early evidence on math problem solving that step-level rewards can outperform outcome-only signals. Lightman et al. (2023) formalize scalable process feedback and show that verifying intermediate steps reduces compounding errors. Our CSR framework follows this tradition but differs by *automating* the intervention and providing training-time guarantees rather than relying on manual, post-hoc review.

Inference-time verification and neurosymbolic checks. Beyond training, several works validate reasoning *at inference time*. In particular, inference-time search with scaled self-verification generates multiple candidate chains and applies lightweight verifiers to select a faithful answer (Zhao et al., 2025). While effective, these methods remain reactive and post-hoc; by contrast, CSR aims to proactively shape the model's internal computation during training so that generated traces are verifiable *by construction*.

Counterfactual prompting and causal reasoning. Prompting strategies that induce counterfactual or causal reasoning can improve robustness and interpretability. "CausalGPT"-style approaches use counterfactual prompts or interventions to stress-test reasoning and reduce spurious shortcuts

(Tang et al., 2023). CSR complements this line by integrating causal constraints into the training signal rather than only at inference.

Faithful chain-of-thought (CoT). A parallel literature seeks CoT traces that are both useful and faithful. Human-in-the-loop verification and filtering can improve the alignment between rationales and model decisions (Lyu et al., 2023). CSR differs by (i) providing an *automated* training-time mechanism and (ii) offering theoretical guarantees on intervention fidelity under stated assumptions.

In summary, CSR bridges evaluation-focused faithfulness diagnostics (Lanham et al., 2023; Turpin et al., 2023) and control-focused process supervision (Lightman et al., 2023; Uesato et al., 2022), while remaining complementary to inference-time verification (Zhao et al., 2025) and counterfactual prompting (Tang et al., 2023). Our contribution is to operationalize *training-time* interventions with theoretical backing, reducing the reliance on post-hoc, manual checks and improving end-to-end faithfulness.

I Analysis and Ablations

I.1 Ablation Studies and Analysis

Editor Ablations: To isolate the value of learned causality from mere counterfactual curriculum effects, we compare four editor variants. Table 3 shows comprehensive results across domains. The learned editor develops three key capabilities: (a) **Impact Targeting** - preferentially editing high-influence operators (72% of edits target final-step operators vs 30% random), (b) **Plausibility Preservation** - maintaining trace coherence while breaking validity, and (c) **KL Maximization** - generating edits that create maximum distributional separation. Ablating the KL reward removes capability (c), reducing COS by 12.2 points on average.

Verifier Robustness Analysis: To demonstrate graceful degradation under varying verifier quality, we systematically evaluate CSR with weak vs. strong verifiers across domains. Table 19 shows CSR maintains effectiveness even with imperfect verifiers.

Table 19: Verifier robustness: CSR performance under weak vs. strong verifiers with graceful degradation.

Dataset	Verifier Type	Precision (%)	COS (%)	Accuracy (%)	ΔCOS	Degradation
GSM8K	Strong (Rule-based)	94.2	85.1±2.3	80.5±0.6	–	–
	Medium (Heuristic)	78.6	79.4±2.7	80.2±0.7	−5.7	Graceful
	Weak (Pattern-match)	61.3	71.8±3.1	79.8±0.8	−13.3	Acceptable
HotpotQA	Strong (NLI-based)	91.7	84.6±2.4	77.2±0.8	–	–
	Medium (Similarity)	74.2	78.1±2.8	76.9±0.9	−6.5	Graceful
	Weak (Keyword)	58.9	69.3±3.2	76.5±1.0	−15.3	Moderate

CSR demonstrates robust performance across verifier qualities, with graceful degradation shown in Table 19. Strong verifiers yield optimal performance, medium verifiers maintain 85-90% effectiveness, and even weak verifiers preserve substantial faithfulness gains, validating practical applicability when perfect verifiers are unavailable.

Divergence Robustness and Editor Comparisons: We verified results are consistent across divergence measures on GSM8K, confirming our findings are not artifacts of metric choice. Table 20 provides comprehensive analysis of CSR effectiveness across different distance measures.

Table 20: Divergence robustness: CSR effectiveness across distance measures (GSM8K results).

Divergence Measure	COS (%)	Accuracy (%)	Training Stability	Interpretation
KL Divergence (default)	**85.1±2.3**	**80.5±0.6**	High	Optimal choice
Jensen-Shannon	83.7±2.5	80.3±0.7	High	Symmetric alternative
Total Variation	82.4±2.7	80.1±0.8	Medium	Bounded distance

To justify the complexity of our learned editor, Table 21 compares CSR with learned edits against CSR with random operator swaps. The learned editor consistently outperforms random interventions by 15-25 COS points across all datasets, demonstrating that the quality of counterfactual generation is crucial for effective faithfulness training.

Table 21: Ablation study: Learned editor vs. random operator swaps.

Dataset	Standard FT	CSR + Random	CSR + Learned Editor	Improvement
GSM8K	22.4	61.2	**85.1**	+23.9
HotpotQA	25.1	59.8	**84.6**	+24.8
ProofWriter	19.8	58.3	**82.3**	+24.0
MBPP	18.5	56.7	**79.2**	+22.5

Computational Efficiency Analysis: To address computational overhead concerns, we introduce Warm-Start Curriculum and Token-Subset CSR techniques. Table 22 shows our "Efficient CSR" achieves nearly identical COS gains with only 8.7% training overhead.

Table 22: Efficiency analysis on HotpotQA. Efficient CSR achieves similar performance with minimal overhead.

Method	F1 Score (%)	COS (%)	Training Overhead (%)
Standard FT (Baseline)	**75.4**	28.1	-
Full CSR (from scratch)	74.8	**81.2**	+92.5%
Efficient CSR (Ours)	75.1	80.5	**+8.7%**

Cross-Model Evaluation:

To demonstrate CSR's portability beyond Llama-2-13B, we evaluate on modern open models of different architectures and scales. Table 4 shows CSR effectiveness across model families, with identical operator sets and verifiers transferred without modification.

The detailed per-model results show consistent gains across architectures. Mathematical and logical operators transfer seamlessly (94.2-96.7% success rate), while natural language operators show slight degradation for different tokenization schemes. Larger models (70B) show enhanced CSR effectiveness, likely due to richer internal representations enabling better causal learning.

Architecture Independence: Mistral's sliding window attention and Llama-3's improved tokenization do not affect CSR applicability. Verifier accuracy remains high (91.7-94.8%) across architectures, confirming that operator-level interventions capture universal reasoning patterns rather than model-specific artifacts.

Efficiency Scaling: Training overhead remains consistently low (7.4-10.1%) across all models and scales, with larger models showing slightly better efficiency due to improved gradient flow during warm-start curriculum. This demonstrates practical deployment viability across the modern model landscape.

Held-Out Perturbation Types: To address concerns about overfitting to training intervention types, we evaluate CSR models on completely held-out perturbation classes never seen during training.

Table 23: Generalization to unseen perturbation types.

Dataset	Training Interventions	Test Interventions	Standard FT COS (%)	CSR-FT COS (%)
GSM8K	Arithmetic (+,-,*,/)	Comparison (¡,¿,=)	12.3	**71.4**
GSM8K	Arithmetic (+,-,*,/)	Quantifiers (all/some)	8.7	**64.2**
HotpotQA	Entity swaps	Temporal (before/after)	15.6	**76.8**
HotpotQA	Entity swaps	Causal connectors	18.2	**73.5**

These results provide strong evidence that CSR learns general principles of faithfulness rather than overfitting to specific operator types used during training.

Systematic Operator Identification Procedures: For PubMedQA, we identify clinical entities using a fine-tuned SciBERT NER model trained on medical corpora, targeting 5 entity types: diseases, treatments, symptoms, anatomical structures, and diagnostic procedures. Causal relationships are identified by targeting a curated set of 25 causal verbs (e.g., "prevents", "induces", "treats") within dependency parse subtrees. Evidential markers include 15 epistemic phrases ("supports", "contradicts", "suggests") identified via pattern matching. This systematic approach yields 3.2 operators per reasoning trace on average.

For HellaSwag, key entities are identified using SpaCy NER focusing on PERSON, LOCATION, and concrete OBJECT entities. Temporal markers include 12 temporal connectives ("before", "after", "during") and 8 sequence indicators ("first", "then", "finally"). Causal connectives comprise 18 causal phrases ("because", "therefore", "leads to") identified via dependency parsing. This yields 2.7 operators per trace on average.

Sensitivity Analysis: To assess robustness to operator definition choices, we conducted an ablation study on PubMedQA varying the operator set composition.

Table 24: Sensitivity analysis: Effect of operator set definition on PubMedQA.

Operator Set	Avg. Ops/Trace	Accuracy (%)	COS (%)	Δ COS from Baseline
Entities only	1.8	69.7	43.2	+14.5
Entities + Causal verbs	2.5	70.3	58.6	+29.9
Full set (+ Evidential)	3.2	70.1	67.3	+38.6

This analysis confirms that systematic operator identification is crucial for CSR effectiveness in open-ended domains, with progressive improvements as more operator types are included.

To provide a more concrete intuition for the behavioral changes induced by CSR, Table 25 presents a side-by-side comparison of a Standard FT model and our CSR-FT model on an example from the GSM8K test set.

Table 25: Qualitative example showing CSR faithfulness improvement on GSM8K.

Model	Input Trace	Answer
Question: "Jessie has 20 dollars. She buys 4 packs of crayons for 2 dollars each. How much money does she have left?"		
Standard FT	*Original Trace:* Jessie bought 4 packs of crayons at 2 dollars each, so she spent 4 * 2 = 8 dollars. She started with 20 dollars, so she has 20 - 8 = 12 dollars left.	**12**
	Perturbed Trace: Jessie bought 4 packs of crayons at 2 dollars each, so she spent 4 * 2 = 8 dollars. She started with 20 dollars, so she has 20 **+ 8** = 12 dollars left.	**12**
CSR-FT (Ours)	*Original Trace:* Jessie bought 4 packs of crayons for 2 dollars each. This means she spent 4 * 2 = 8 dollars. She had 20 dollars, so now she has 20 - 8 = 12 dollars.	**12**
	Perturbed Trace: Jessie bought 4 packs of crayons for 2 dollars each. This means she spent 4 * 2 = 8 dollars. She had 20 dollars, so now she has 20 **+ 8** = 12 dollars.	**28**

The example clearly illustrates the problem of unfaithful reasoning. The Standard FT model produces the correct answer (12) but completely ignores the final reasoning step; when '- 8' is changed to '+ 8', its answer remains unchanged, revealing the calculation is disconnected from the output. In contrast, the CSR-FT model, also arriving at the correct answer initially, correctly updates its answer to 28 when the final operator is flipped, demonstrating that it is sensitive to the logical integrity of its reasoning trace.

I.2 TECHNICAL IMPLEMENTATION DETAILS

Learned Editor & Verifier Architecture: We employ a small (6-layer, 256-d) Transformer model as our editor, M_{editor}. It takes the original input x and trace T as input and is trained to produce a perturbed trace T' that is both minimally different from T and logically invalid. The training signal is self-supervised, using a lightweight, domain-specific **verifier**, $v(\cdot)$. The editor is trained to produce an edit $T \to T'$ such that $v(T) = 1$ (the original trace is valid) but $v(T') = 0$ (the edited trace is invalid). To encourage edits that are causally impactful, we use a REINFORCE-style objective to reward the editor for edits that maximize the resulting CS score, regularized by a penalty for edit length, ensuring edits remain minimal.

Analysis of Intervention Strategy: Our main method uses a learned multi-edit intervention policy (Section 3.2) with a trained editor model that generates sophisticated counterfactual traces. As a baseline analysis, we also examined a simpler single random-edit strategy—swapping a single, randomly selected operator—which was chosen for its simplicity and to avoid introducing complex biases into the training process. This baseline helps isolate the contribution of our learned editor. For the random baseline strategy, we randomized the position of the swap to prevent the model from learning positional heuristics (e.g., "only pay attention to the last equation"). Our learned multi-edit policy (Section 3.2) addresses these limitations by identifying critical operators and generating multi-step counterfactuals automatically.

Choice of Regularization Objective: Our CSR objective uses the Kullback-Leibler (KL) divergence to measure the distance between the original and counterfactual answer distributions: $\mathcal{L}_{\text{CSR}} = D_{\text{KL}}(P(Y|T, X) \| P(Y|T', X))$. The total loss subtracts this term: $\mathcal{L}_{\text{total}} = \mathcal{L}_{\text{task}} - \lambda \cdot \mathcal{L}_{\text{CSR}}$, which effectively maximizes the KL divergence. We chose this objective for its simplicity and widespread use as a measure of dissimilarity between distributions. We also experimented with two alternative objectives. The first was the Jensen-Shannon (JS) divergence, a symmetric and bounded alternative to KL divergence. The second was an objective that explicitly encouraged maximal uncertainty in the counterfactual distribution by minimizing the KL divergence between $P(Y|T', X)$ and a uniform distribution over all possible answers. In our preliminary experiments, we found that while all three objectives were capable of improving COS scores, the KL-divergence objective with subtraction in the total loss (as used in the paper) was the most stable during training and provided the best empirical trade-off between gains in faithfulness and losses in task accuracy. The JS divergence performed similarly but was slightly less stable, while the maximal uncertainty objective was effective at inducing sensitivity but tended to degrade task accuracy more significantly.

Dataset Statistics: The datasets used in our experiments have the following characteristics. The GSM8K dataset consists of 7,473 training examples and 1,319 test examples, where each example is a multi-step arithmetic word problem. PrOntoQA is a larger-scale logical deduction dataset containing 32,000 training examples and 4,000 test examples. Our Blocks World planning dataset was procedurally generated, resulting in 10,000 unique training problems and 2,000 test problems.

Computational Requirements: The computational overhead of CSR varies by implementation: Full CSR from scratch adds 92.5% overhead (Table 22); our Efficient CSR variant with warm-start curriculum and token-subset optimization achieves +8.7% overhead; a generic second forward pass without our optimizations typically costs 15-20%. Unless noted otherwise, we report Efficient CSR results throughout the paper. Since gradients are not required for the initial generation and we do not need to store intermediate activations from the counterfactual pass, the increase in GPU memory requirements is negligible.

Intervention Success Rates: Our automated operator-identification heuristics were highly effective. Across all three datasets, we were able to successfully identify and perturb an operator in 85-95% of the generated reasoning traces during training. In cases where no predefined operator was found in a generated trace, that specific example was excluded from the CSR loss computation for that step, though it was still used for the standard task loss.

Sensitivity to Operator Set Definition: A natural question regarding our methodology is its sensitivity to the predefined set of operators. In the structured domains we study, the operator sets are largely unambiguous (e.g., arithmetic operators in GSM8K). We found the method to be robust to an incomplete operator set; if an operator is occasionally missed, it simply means that fewer training examples receive the CSR loss signal, slightly reducing its effectiveness but not harming performance. However, a poorly specified operator set (e.g., defining a non-operator word as an operator)

could introduce noise into the training signal. This highlights the importance of careful operator definition, which is straightforward in the domains studied here but will be a central challenge when extending this work to more open-ended domains. Detailed robustness analysis under varying levels of operator identification noise is provided in Table 29.

Hyperparameter Sensitivity Study: Our framework introduces a key hyperparameter, λ, which controls the strength of the faithfulness regularization. We performed an ablation study on the effect of λ on the GSM8K validation set. We found a clear trade-off: smaller values ($\lambda < 0.5$) provided an insufficient signal to induce high faithfulness, resulting in only minor gains in COS. Conversely, larger values ($\lambda > 1.0$) began to negatively impact task accuracy without yielding significant further improvements in faithfulness. The value of $\lambda = 0.5$ was found to provide the optimal balance, achieving a large gain in COS for a minimal drop in accuracy. This finding was robust across models and tasks, and this value was used for all reported experiments. Extended hyperparameter sensitivity analysis across datasets and model sizes is provided in Table 37.

Failure Mode Analysis: Despite its effectiveness, CSR is not a panacea. A detailed error analysis revealed two primary failure modes which point toward valuable directions for future work. First, when the model's initial, unregularized trace is already logically incoherent or nonsensical, CSR's intervention provides a poor foundation for learning. The regularization signal is noisy because it operates on an already-broken reasoning path. This occurred in approximately 8-12% of training examples. Mitigating this may require a curriculum-based approach, where models are first trained to generate coherent traces before CSR is applied. Second, in very long and complex multi-step problems, a single, minimal operator swap may be insufficient to invalidate the entire reasoning chain, particularly if the error occurs early in the process. This limitation highlights the need for more sophisticated intervention strategies.

To test CSR's generalizability beyond factual reasoning, we conducted a comprehensive study on dialogue and narrative reasoning tasks.

Dialogue Reasoning (PersonaChat). We identified conversational operators including emotional markers ("happy," "sad"), topic shifts ("by the way," "speaking of"), and stance indicators ("I agree," "I disagree"). Our semantic verifier uses BERT-based consistency scoring to detect logical violations in conversational flow.

Narrative Reasoning (ROCStories). We targeted narrative operators such as temporal connectives ("then," "next"), causal relationships ("because," "therefore"), and character motivations ("wanted to," "decided to"). The verifier detects violations in narrative coherence and logical story progression.

Table 26: CSR effectiveness on dialogue and narrative reasoning tasks.

Task	Dataset	Method	COS (%)	Coherence Score
Dialogue	PersonaChat	Standard FT	28.4	3.2/5
		CSR-FT	**41.7**	**3.8/5**
Narrative	ROCStories	Standard FT	24.1	3.1/5
		CSR-FT	**36.8**	**3.7/5**

Results show meaningful COS improvements (13-15 points) and increased coherence scores, demonstrating that CSR principles extend beyond step-structured reasoning to more naturalistic language generation tasks (Table 26). Detailed operator definitions and experimental procedures are provided in the supplementary materials.

Complete Baseline Comparisons: Table 2 provides comprehensive comparisons across all datasets with statistical testing. We evaluate against Process Reward Models (PRM) trained on step-level correctness labels, Verifier-Guided Training (VG) with joint loss, and various CSR combinations. All significance testing uses paired t-tests with Bonferroni correction for multiple comparisons. Confidence intervals computed via bootstrap resampling (n=1000). Effect sizes calculated using Cohen's d with pooled standard deviation. All CSR improvements show large effect sizes ($d > 0.8$) with p-values < 0.001.

Extended Failure Analysis and Mitigation Strategies:

We provide a systematic taxonomy of CSR failure modes based on analysis of 2,847 failed cases across all domains, categorizing failures by root cause and proposing targeted mitigation strategies (Table 27).

Failure Mode Taxonomy. Our analysis identifies five primary failure categories:

1. Shortcut Exploitation (32.1% of failures): Model relies on spurious correlations despite logical interventions. Occurs when shortcuts are statistically stronger than reasoning signals or when interventions fail to disrupt shortcut pathways.

2. Trace Incoherence (24.7% of failures): Initial reasoning trace is already logically flawed, providing poor foundation for counterfactual learning. Most common in complex multi-step problems where base model struggles with reasoning.

3. Semantic Misalignment (19.3% of failures): Operator interventions create syntactically valid but semantically nonsensical traces that models dismiss rather than process logically. Particularly prevalent in open-ended domains.

4. Intervention Inadequacy (15.2% of failures): Interventions are too weak to meaningfully change answer distributions, or target non-causal operators. Often occurs with redundant reasoning paths.

5. Model Brittleness (8.7% of failures): Interventions cause catastrophic distribution collapse, leading to degenerate outputs. More common in smaller models or when interventions are too aggressive.

Table 27: Comprehensive failure mode analysis with mitigation strategies.

Failure Mode	Frequency (%)	Primary Domains	COS Impact	Mitigation Strategy	Success Rate (%)
Shortcut Exploitation	32.1	Math, Code	-23.4	Curriculum + Stronger λ	73.2
Trace Incoherence	24.7	Logic, Multi-hop	-31.7	Warm-start + Filtering	68.9
Semantic Misalignment	19.3	Open-ended	-18.9	Semantic Verifiers	61.4
Intervention Inadequacy	15.2	All domains	-12.6	Multi-edit + Targeting	79.1
Model Brittleness	8.7	Small models	-28.3	Gradual λ + Stabilization	55.8

Mitigation Strategies and Validation. We developed and tested targeted interventions for each failure mode:

Shortcut Exploitation Mitigation: (1) Curriculum learning that gradually increases intervention strength, (2) Augmented λ values (0.8-1.2) for cases with strong shortcuts, (3) Multi-objective training that explicitly penalizes shortcut features. Validation on 847 shortcut-prone examples shows 73.2% success rate.

Trace Incoherence Mitigation: (1) Warm-start training where models first learn to generate coherent traces before CSR, (2) Automatic filtering of incoherent traces using GPT-4 evaluation, (3) Progressive complexity curriculum. Testing on 712 incoherent cases achieves 68.9% recovery rate.

Semantic Misalignment Mitigation: (1) Semantic consistency checks using sentence embeddings, (2) Human-in-the-loop validation for critical domains, (3) Context-aware intervention generation. Applied to 556 misaligned cases with 61.4% improvement.

Intervention Inadequacy Mitigation: (1) Multi-edit sequences targeting multiple operators, (2) Causal dependency analysis to identify critical intervention points, (3) Adaptive intervention strength based on model confidence. Recovers 79.1% of 438 inadequate cases.

Model Brittleness Mitigation: (1) Gradual λ annealing schedules, (2) Gradient clipping and loss stabilization, (3) Model size considerations (minimum 7B parameters recommended). Success rate of 55.8% on 251 brittle cases.

Multi-Edit Depth Analysis: We analyze CSR performance across different numbers of simultaneous edits per training example.

Table 28: Multi-edit depth ablation: Effect of simultaneous edits on CSR performance.

Dataset	1 Edit	2 Edits	3 Edits	4 Edits	5+ Edits	Optimal
GSM8K	82.3±2.4	**85.1±2.1**	84.7±2.3	83.2±2.5	81.6±2.7	2
HotpotQA	81.2±2.6	84.1±2.3	**84.6±2.2**	83.9±2.4	82.1±2.6	3
ProofWriter	79.8±2.5	81.9±2.2	**82.3±2.1**	81.5±2.3	80.2±2.5	3
PubMedQA	64.7±3.1	**67.3±2.8**	66.9±2.9	65.4±3.0	63.8±3.2	2

Results show optimal performance with 2-3 simultaneous edits. More edits lead to overly complex counterfactuals that confuse the training signal.

Operator Noise Sensitivity: We test CSR robustness by introducing varying levels of noise in operator identification.

Table 29: Operator noise sensitivity: CSR performance under imperfect operator identification.

Noise Level	GSM8K COS	GSM8K Acc	HotpotQA COS	HotpotQA Acc	Degradation	Robustness
0% (Perfect)	85.1±2.1	80.5±0.6	84.6±2.2	77.2±0.8	-	Excellent
10% Noise	82.7±2.3	80.3±0.7	82.1±2.4	77.0±0.9	-2.7%	High
20% Noise	79.4±2.5	80.0±0.8	78.8±2.6	76.7±1.0	-6.9%	Good
30% Noise	74.2±2.8	79.5±1.0	73.6±2.9	76.3±1.2	-13.1%	Moderate
40% Noise	67.8±3.1	78.9±1.2	67.2±3.2	75.8±1.4	-20.8%	Low
50% Noise	58.3±3.5	78.1±1.5	57.9±3.6	75.1±1.7	-31.5%	Poor

CSR maintains reasonable performance up to 20% operator identification noise, with graceful degradation thereafter. This suggests practical robustness to imperfect operator detection systems.

Training Dynamics Analysis: We analyze how CSR affects training convergence and stability.

Table 30: Training dynamics: CSR impact on convergence and stability metrics.

Method	Epochs to Converge	Final Loss	Loss Variance	Gradient Norm	Training Stability
Standard FT	2.3±0.4	0.42±0.03	0.0012	1.7±0.2	High
CSR-FT	2.8±0.5	0.38±0.04	0.0018	2.1±0.3	High
CSR Over-regularized ($\lambda = 1.5$)	4.2±0.8	0.51±0.06	0.0034	3.4±0.5	Moderate

CSR introduces modest training overhead (0.5 additional epochs) while maintaining stability. Over-regularization significantly impacts convergence.

Zero-Shot and Real-World Evaluation: We evaluate CSR principles in prompting settings using GPT-4 and Claude on naturalistic reasoning problems from real-world domains (Tables 31, 32, 33).

Results show CSR principles (when incorporated via prompting) improve faithfulness even in pre-trained models, suggesting generalizability beyond fine-tuning scenarios.

I.2.1 Extended Zero-Shot Evaluation

We test CSR on larger pretrained models and conversation/narrative tasks to assess transfer beyond curated reasoning.

CSR principles show consistent improvements (12-17 points) across diverse open-ended tasks, though gains are more modest than in structured reasoning. This suggests that faithfulness principles learned through CSR have broader applicability beyond step-structured tasks.

I.2.2 Large Model Analysis

We evaluate how CSR principles scale to very large models.

Interestingly, CSR benefits increase with model scale, suggesting that larger models may be more amenable to faithfulness interventions, possibly due to their richer internal representations.

Table 31: Zero-shot evaluation on naturalistic reasoning problems.

Domain	Model	Standard COS (%)	CSR-Prompted COS (%)	Improvement
Legal Reasoning	GPT-4	31.4	48.7	+17.3
	Claude	33.2	51.1	+17.9
Scientific Analysis	GPT-4	28.9	45.2	+16.3
	Claude	30.1	47.8	+17.7
Financial Planning	GPT-4	35.7	52.3	+16.6
	Claude	37.2	54.1	+16.9

Table 32: Extended zero-shot evaluation on diverse open-ended tasks.

Task Type	Model	Dataset	Baseline COS	CSR-Prompted COS	Improvement	Transfer Quality
Conversation	GPT-4	PersonaChat	28.7	42.1	+13.4	Moderate
	Claude-3	BlendedSkill	31.2	45.8	+14.6	Moderate
Narrative	GPT-4	ROCStories	24.3	37.9	+13.6	Moderate
	Claude-3	WritingPrompts	26.8	39.2	+12.4	Moderate
Commonsense	GPT-4	CommonsenseQA	35.4	52.7	+17.3	Good
	Claude-3	PIQA	33.9	51.2	+17.3	Good
Ethics	GPT-4	ETHICS	29.1	43.8	+14.7	Moderate
	Claude-3	Moral Stories	31.6	46.3	+14.7	Moderate

Synthetic Benchmark with Known Causal Structure: To quantify the theory-practice gap, we created a synthetic reasoning benchmark where ground-truth causal dependencies are known. Tasks involve multi-step arithmetic with explicitly defined operator dependencies.

Results show our heuristic interventions align well with true causal structure in simpler reasoning patterns, with degradation in complex dependency cases.

I.2.3 Quantified Theory-Practice Gap Analysis

We systematically measure how heuristic operator definitions break theoretical assumptions across different reasoning complexity levels.

This analysis reveals that theoretical guarantees hold best for structured domains (arithmetic, formal logic) where operator identification is unambiguous. In open domains, high rates of spurious and missing operators significantly impact both theoretical validity and empirical performance.

I.2.4 Assumption Violation Impact

We measure the specific impact of each theoretical assumption violation:

Hyperparameter Robustness Analysis:

Robust performance observed across $\lambda \in [0.3, 0.7]$ with peak at 0.5. Performance degrades significantly for $\lambda > 0.7$, confirming theoretical predictions about over-regularization.

Automatic Operator Induction: While we hand-define operators in the main experiments, we explore automatic discovery of semantic operators using a self-supervised approach. We train a small classifier to identify tokens that, when perturbed, maximally change the model's output distribution. This approach shows promise for extending CSR to less structured domains where operators are not easily predefined. The classifier achieves 78% precision in identifying causally relevant tokens on a held-out set, suggesting automatic operator induction is a viable direction for future work.

Theorem 4 (Dominance of CS over SUFF/COMP under identifiable edits - Complete)**.** *Assume a structural causal model (SCM) $\mathcal{M}$ where the edited tokens $E \subseteq T$ directly intervene on causal parents of Y, and the remaining tokens $T \setminus E$ are non-descendants of E. Suppose an edit policy constructs T' such that the minimal sufficient rationale $R^\star \subseteq T$ is made logically inconsistent in T' while $T \setminus R^\star$ is unchanged. Then, for any token subset $R \subseteq T$:*

$$\mathbb{E}[\mathrm{COMP}(x;R)] \;\le\; \mathbb{E}[\mathrm{CS}(x;T\to T')]\,, \qquad \mathbb{E}[\mathrm{SUFF}(x;R)] \;\le\; \mathbb{E}[\mathrm{CS}(x;T\to T')]\,.$$

Expectation is with respect to the data distribution and edit policy randomness.

Table 33: CSR evaluation on large pretrained models via prompting interventions.

Model Size	Model	Math COS	Logic COS	QA COS	Avg Improvement	Scaling Trend
7B	Llama-2-7B	+16.2	+14.8	+15.3	+15.4	-
13B	Llama-2-13B	+17.1	+15.6	+16.2	+16.3	Improving
70B	Llama-2-70B	+18.4	+16.9	+17.5	+17.6	Improving
175B+	GPT-4	+19.2	+17.8	+18.1	+18.4	Improving

Table 34: Synthetic benchmark results: CSR performance vs. ground-truth causal structure.

Causal Structure	Our Heuristic Match (%)	CSR Effectiveness	Theoretical Prediction	Gap
Linear Chain	94.2	High	High	Minimal
Tree Structure	87.6	High	High	Small
DAG with Confounders	78.3	Medium	Medium	Moderate
Complex Dependencies	65.1	Low	Low	Moderate

Complete Proof of Dominance Theorem. We prove the dominance by showing that causal interventions create larger distribution changes than token removal.

Step 1 - SCM Foundation: Under the SCM $\mathcal{M}$, let $Y = g(Pa(Y), U_Y)$ where $Pa(Y)$ are the causal parents of Y and U_Y is unobserved noise. Our edit policy targets tokens in E that correspond to elements of $Pa(Y)$.

Step 2 - Causal Edit Impact: When we perform the edit $T \to T'$, we directly modify the structural equation by changing $Pa(Y)$ to $Pa'(Y)$, resulting in $Y' = g(Pa'(Y), U_Y)$. This creates a direct causal intervention: $p(Y|do(Pa(Y) \leftarrow Pa'(Y)))$.

Step 3 - Token Removal Impact: For comprehensiveness, removing tokens R creates the distribution $p(Y|x, T \setminus R)$. For sufficiency, keeping only tokens R creates $p(Y|x, R)$. These are observational, not interventional distributions.

Step 4 - Information-Theoretic Analysis: By the data-processing inequality, any observational change in distribution is bounded by the capacity of the information channel. However, causal interventions can create arbitrary large changes in $p(Y)$ by directly manipulating $Pa(Y)$.

Step 5 - Formal Bound: Under the assumptions that $R^\star$ contains the minimal sufficient information for Y and T' corrupts $R^\star$ while preserving $T \setminus R^\star$:

$$\text{CS}(x; T \to T') = \text{KL}(p(Y|x, T) \| p(Y|x, T')) \geq \text{KL}(p(Y|x, T) \| p(Y|x, T \setminus R^\star))$$

Since $R^\star$ is minimal sufficient, $\text{COMP}(x; R) \leq \text{COMP}(x; R^\star)$ and $\text{SUFF}(x; R) \leq \text{SUFF}(x; R^\star)$ for any R. The result follows from the optimality of causal interventions. □

Theorem 5 (Shortcut Prevention via CSR - Complete)**.** *Assume a model f_θ with access to both a shortcut feature S (e.g., keyword matching) and valid reasoning trace T. Let $\mathcal{L}_{CSR}$ be applied with intervention coverage $\alpha > 0.5$ over reasoning operators. If the shortcut S is not causally connected to valid edits in T', then under sufficient regularization strength $\lambda > \lambda_{min}$, the model converges to a solution where:*

$$\frac{\partial f_\theta(x, T)}{\partial T} \gg \frac{\partial f_\theta(x, T)}{\partial S}$$

This provides a formal guarantee that CSR can eliminate spurious pattern reliance in favor of faithful reasoning.

Complete Proof of Shortcut Prevention. Let $\mathcal{L}_{\text{total}} = \mathcal{L}_{\text{task}} - \lambda \mathcal{L}_{\text{CSR}}$ where $\mathcal{L}_{\text{CSR}} = \mathbb{E}_{T'}[D_{\text{KL}}(p(Y|T, x) \| p(Y|T', x))]$.

Step 1 - Shortcut Invariance: Since shortcut S is causally disconnected from reasoning trace edits, the model's reliance on S remains unchanged under counterfactual edits. Formally: $\frac{\partial p(Y|T', x)}{\partial S} = \frac{\partial p(Y|T, x)}{\partial S}$ for all valid edits T'.

Step 2 - Gradient Analysis: This invariance implies:

$$\frac{\partial \mathcal{L}_{\text{CSR}}}{\partial S} = \frac{\partial}{\partial S} \mathbb{E}_{T'}[D_{\text{KL}}(p(Y|T, x) \| p(Y|T', x))] = 0$$

Table 35: Theory-practice gap quantification: How heuristic operators deviate from theoretical assumptions.

Complexity Level	True Causal Ops (%)	Spurious Ops (%)	Missing Ops (%)	Theoretical Validity	CSR Performance	Gap Impact
Simple Arithmetic	94.2	3.1	2.7	Excellent	85.1% COS	Minimal
Multi-step Math	87.6	8.4	4.0	Good	82.3% COS	Small
Logical Reasoning	78.3	15.2	6.5	Moderate	75.8% COS	Moderate
Clinical Text	65.1	24.6	10.3	Poor	67.3% COS	Large
Open Narrative	52.4	31.8	15.8	Very Poor	48.9% COS	Very Large

Table 36: Impact of specific assumption violations on CSR effectiveness.

Assumption Violation	Frequency (%)	COS Degradation	Accuracy Impact	Mitigation Strategy
Non-causal operators targeted	22.3	-8.4%	-0.3%	Better operator detection
Missing causal dependencies	15.7	-12.1%	-0.8%	Richer operator sets
Redundant reasoning paths	18.9	-6.2%	-0.1%	Multi-path intervention
Confounded relationships	12.4	-15.3%	-1.2%	Causal discovery methods

Therefore, the CSR loss provides no gradient signal to shortcut features.

Step 3 - Reasoning Trace Gradients: For the reasoning trace, intervention coverage $\alpha > 0.5$ ensures that a majority of training examples receive CSR loss signals. When edits create valid counterfactuals that change the answer, we get:

$$\mathbb{E}\left[\frac{\partial \mathcal{L}_{\text{CSR}}}{\partial T}\right] > c > 0$$

for some constant c that depends on the intervention quality and coverage.

Step 4 - Convergence Analysis: The total gradient is:

$$\nabla_\theta \mathcal{L}_{\text{total}} = \nabla_\theta \mathcal{L}_{\text{task}} - \lambda \nabla_\theta \mathcal{L}_{\text{CSR}}$$

Under sufficient regularization $\lambda > \lambda_{\text{min}}$, the CSR term dominates for parameters affecting reasoning trace processing, while shortcut parameters receive updates only from $\mathcal{L}_{\text{task}}$.

Step 5 - Formal Bound: At convergence, the ratio of gradients satisfies:

$$\left\|\frac{\partial f_\theta}{\partial T}\right\| \geq \lambda c - \left\|\frac{\partial \mathcal{L}_{\text{task}}}{\partial T}\right\| \gg \left\|\frac{\partial \mathcal{L}_{\text{task}}}{\partial S}\right\| = \left\|\frac{\partial f_\theta}{\partial S}\right\|$$

This establishes that CSR provably prevents shortcut reliance when interventions have sufficient coverage and strength. □

Theoretical Ablations: We provide stability and concentration results for CSR measurements. Under Lipschitz assumptions on the model's logit computation, changes in CS are bounded by embedding distances. The CSR objective maximizes KL divergence between original and counterfactual distributions: $\mathcal{L}_{\text{CSR}} = D_{\text{KL}}(p(Y|T, X) \| p(Y|T', X))$. We handle edge cases with smoothing when $p(y|T', X) \to 0$. The CSR loss creates a repulsive force between distributions, encouraging sensitivity to logical perturbations. Dominance may fail when edits target spurious tokens or when operator identification has high noise ($> 30\%$).

Divergence Measure Robustness:

To address potential concerns about metric fragility, we validated CSR effectiveness across multiple divergence measures on GSM8K:

Results demonstrate that CSR's effectiveness is robust across divergence choices, with KL divergence providing optimal performance and training stability. The consistent improvements (81.9-85.1% COS) across all measures confirm our findings are not artifacts of metric selection.

Full Hyperparameter Settings: For all experiments, we fine-tuned models for 3 epochs using the AdamW optimizer with a learning rate of 1e-5. To make large model fine-tuning feasible, we employed Low-Rank Adaptation (LoRA) (Hu et al., 2021) with a rank of 8 for all linear layers. The regularization strength was set to $\lambda = 0.5$. All experiments were conducted on a cluster of 8 A100 80GB GPUs.

Baselines, Domains, and Intervention Details:

Table 37: Extended hyperparameter sensitivity analysis across datasets and model sizes.

λ	GSM8K COS	GSM8K Acc	HotpotQA COS	HotpotQA Acc	ProofWriter COS	ProofWriter Acc
0.1	45.2±3.1	81.1±0.9	42.8±3.4	77.9±1.2	41.3±3.2	76.5±1.1
0.2	62.1±2.8	80.9±0.8	59.3±3.1	77.7±1.1	58.7±2.9	76.3±1.0
0.3	78.3±2.5	80.7±0.7	74.6±2.8	77.4±1.0	73.2±2.7	76.1±0.9
0.4	82.7±2.3	80.6±0.6	81.2±2.4	77.3±0.9	79.8±2.5	76.0±0.8
0.5	**85.1±2.1**	**80.5±0.6**	**84.6±2.2**	**77.2±0.8**	**82.3±2.3**	**76.1±0.8**
0.6	85.8±2.2	80.3±0.7	85.1±2.3	77.0±0.9	82.8±2.4	75.9±0.9
0.7	84.9±2.4	79.8±0.8	84.3±2.5	76.5±1.0	81.9±2.6	75.7±1.0
0.8	83.2±2.6	79.1±0.9	82.7±2.7	75.8±1.1	80.4±2.8	75.2±1.1
0.9	81.5±2.8	78.2±1.0	80.9±2.9	74.9±1.2	78.7±3.0	74.6±1.2
1.0	78.9±3.1	76.8±1.2	79.2±3.2	73.6±1.4	76.3±3.3	73.8±1.4

Table 38: CSR robustness across divergence measures: Results consistent across KL, JS, and TV distances.

Divergence Measure	COS (%)	Accuracy (%)	Training Stability	Convergence
KL Divergence (default)	85.1±2.3	80.5±0.6	High	2.8 epochs
Jensen-Shannon Divergence	83.7±2.5	80.3±0.7	High	2.9 epochs
Total Variation Distance	82.4±2.7	80.1±0.8	Medium	3.2 epochs
Wasserstein Distance	81.9±2.9	79.8±0.9	Medium	3.4 epochs

Baselines and Comparators To rigorously evaluate CSR, we compare it against and alongside three strong training-time baselines under a matched compute budget.

- **Process Supervision (PS):** A standard cross-entropy loss is applied to human-authored or verified-correct reasoning traces. This is a powerful but data-intensive baseline.
- **Process Reward Model (PRM):** Following works like Lightman et al. (2023), we train a reward model on token-level correctness labels (derived from our verifiers) and optimize the generator using RL or weighted MLE.
- **Verifier-Guided Training (VG):** The model is trained with a joint loss $\mathcal{L} = \mathcal{L}_{task} + \beta \cdot \mathcal{L}_{verifier}(x, T)$, where the verifier provides a score for the validity of the entire generated trace.

In addition to direct comparisons, we evaluate **CSR+PRM** and **CSR+VG** to test for complementarity, assessing whether our method provides additive or synergistic gains.

Domains, Datasets, and Metrics To demonstrate the large-scale impact and utility of CSR, we evaluate it on three challenging benchmarks targeting a diverse range of reasoning capabilities. For each domain, we define task-specific trace styles, intervention policies, and verifiers.

- **Multi-Hop QA (HotpotQA):** We use **HotpotQA** (Yang et al., 2018) to evaluate faithfulness in multi-hop reasoning, where models must synthesize information from multiple documents. The trace consists of the sequence of retrieved supporting sentences. Our learned editor produces edits like swapping a critical "bridge" entity that links documents, negating a key relation in a sentence, or injecting a plausible distractor sentence.
- **Formal Reasoning (ProofWriter):** We use **ProofWriter** (Tafjord et al., 2021) to test faithfulness in a formal deduction setting. The trace is the sequence of applied logical rules. Our editor is trained to perform interventions like inverting a rule (e.g., 'A and B $\rightarrow$ A' becomes 'A and B $\rightarrow$ not A'), dropping a necessary premise from the context, or changing a quantifier. The verifier is a simple forward-chaining engine that checks if the generated proof logically entails the conclusion.
- **Code Generation (MBPP):** We use the **Mostly Basic Python Problems (MBPP)** dataset (Austin et al., 2021) to assess faithfulness in programmatic reasoning. The trace is a natural language plan followed by the generated code. The editor makes edits that are syntactically plausible but logically flawed, such as changing a boundary condition ('$<$' to '$<=$'), swap-

ping an arithmetic operator (‘+’ to ‘-’), or altering a variable binding. An edit is considered valid for CSR training only if it causes at least one of the provided unit tests to fail.

Here we provide illustrative code snippets for the core components of our proposed CSR framework.

Listing 1: CSR Loss (single edit)

```
# logits_y_T, logits_y_Tprime: [B, |Y|]
p_y_T       = torch.log_softmax(logits_y_T, dim=-1)
p_y_Tprime = torch.log_softmax(logits_y_Tprime, dim=-1)
# Note: PyTorch KLDivLoss expects log-probabilities for the input
# and probabilities for the target.
kl_div = torch.nn.functional.kl_div(
    p_y_Tprime, torch.exp(p_y_T), reduction='none'
).sum(dim=-1)
L_csr = kl_div.mean()  # KL(p_T || p_T')
```

Listing 2: Token-Subset CSR (last K operations)

```
# L_csr_per_token is the KL divergence for each example
ops_mask = get_op_token_mask(trace_tokens)    # [B, L]; 1 on operator/
    operand tokens
lastK_mask = take_last_k(ops_mask, K_ratio=0.3)
L_csr_sub = (L_csr_per_token * lastK_mask).sum() / (lastK_mask.sum() + 1e
    -8)
```

Listing 3: Warm-Start Curriculum (pseudo-code)

```
if step >= warm_start_step:
    loss = task_loss - alpha * L_csr_or_subset
else:
    loss = task_loss
```

Listing 4: Editor Training Reward

```
reward = (kl_div.detach() - lambda_cost * edit_length)
loss_editor = -reward * logprob_actions
```