z-SignFedAvg: A Unified Sign-based Stochastic Compression for Federated Learning

Anonymous Author(s)
Affiliation
Address
`email`

Abstract

Federated learning is a promising privacy-preserving distributed learning paradigm but suffers from high communication cost when training large-scale machine learning models. Sign-based methods, such as SignSGD [Bernstein et al., 2018], have been proposed as a biased gradient compression technique for reducing the communication cost. However, sign-based compression could diverge under heterogeneous data, which motivate developments of advanced techniques, such as the error-feedback method and stochastic sign-based compression, to fix this issue. Nevertheless, these methods still suffer significantly slower convergence rate than uncompressed algorithms. Besides, none of them allow local multiple SGD updates like FedAvg [McMahan et al., 2017]. In this paper, we propose a novel noisy perturbation scheme with a general symmetric noise distribution for sign-based compression, which not only allows one to flexibly control the tradeoff between gradient bias and convergence performance, but also provides a unified viewpoint to existing sign-based methods. More importantly, we propose the very first sign-based FedAvg algorithm (z-SignFedAvg). Theoretically, we show that z-SignFedAvg achieves a faster convergence rate than existing sign-based methods and, under the uniformly distribtued noise, can even enjoy the same convergence rate as its uncompressed counterpart. Extensive experiments are conducted to demonstrate that our proposed z-SignFedAvg can achieve competitive empirical performance on real datasets.

1 Introduction

In this paper, we consider the Federated Learning (FL) network with one parameter server and n clients [McMahan et al., 2017, Li et al., 2020a], with the focus on solving the following distributed learning problem

$$\min_{x\in\mathbb{R}^d} f(x) = \frac{1}{n}\sum_{i=1}^{n} f_i(x), \tag{1}$$

where $f_i(\cdot)$ is the local objective function for the i-th client, for $i = 1, \dots, n$. Throughout this work, we assume that each f_i is smooth and possibly non-convex. The local objective functions are generated from the local dataset owned by each client. When designing distributed algorithms to solve (1), a crucial aspect is the communication efficiency since each client needs to transmit their local gradients to the server frequently [Li et al., 2020a]. As one of the most popular FL algorithms, the federated averaging (FedAvg) algorithm [McMahan et al., 2017, Konečný et al., 2016] considers local multiple SGD updates with periodic communications to reduce the communication cost. Another way is to compress the local gradients before sending them to the server [Li et al., 2020a, Alistarh et al., 2017, Reisizadeh et al., 2020]. Among the existing compression methods, a simple yet elegant technqiue is to take the sign of each coordinate of the local gradients, which requires only one bit for

Submitted to 36th Conference on Neural Information Processing Systems (NeurIPS 2022). Do not distribute.

transmitting each coordinate. For any $x \in \mathbb{R}$, we define the sign operator as: $\text{Sign}(x) = 1$ if $x \geq 0$ and -1 otherwise.

Recently, optimization algorithms with sign-based compression have attracted much attention as they enjoy a great communication efficiency while still achieving comparable empirical performance as uncompressed algorithms [Bernstein et al., 2018, Karimireddy et al., 2019, Safaryan and Richtárik, 2021]. However, for distributed learning, especially the scenarios with heterogeneous data, i.e., $f_i \neq f_j$ for every $i \neq j$, a naive application of the sign-based algorithm cannot guarantee convergence [Karimireddy et al., 2019, Chen et al., 2020a, Safaryan and Richtárik, 2021]. There are mainly two approaches to fix this issue in the existing literature. The first one is the stochastic sign-based method, which intoduces stochasticity into the sign operation [Jin et al., 2020, Safaryan and Richtárik, 2021, Chen et al., 2020a], and the second one is the Error-Feedback (EF) method [Karimireddy et al., 2019, Vogels et al., 2019, Tang et al., 2019]. However, these works are still unstastifactory. Specifically, first, the theoretical convergence rates of these algorithms are still worse than uncompressed algorithms like [Ghadimi and Lan, 2013, Yu et al., 2019]. Second, none of them allows the clients to have multiple local SGD updates within one communication round like the FedAvg. This work aims at addresing these issues and closing the gaps for sign-based methods. A detailed review for related works is given in Appendix A.

Main contributions. Our contributions are summarized as follows.

(1) **A unified family of stochastic sign operators.** We show an intriguing fact: The bias brought by the sign-based compression can be flexibly controlled by injecting a proper amount of random noise before the sign operation. In particular, our analysis is based on a novel noisy perturbation scheme with a general symmetric noise distribution, and therefore provides a unified viewpoint and incorporates existing stochastic sign-based methods, including [Jin et al., 2020, Safaryan and Richtárik, 2021, Chen et al., 2020a], as special instances.

(2) **The first sign-based federated averaging algorithm.** In contrast to the existing sign-based shcemes which do not allow local multiple SGD updates within one communication round, based on the proposed stochastic sign-based compression, we design a novel family of sign-based federated averaging algorithms (z-SignFedAvg) that can achieve the best of both worlds: communication efficiency and convergence performance.

(3) **New theoretical convergence rate analyses.** By a clever use of the asymptotic unbiasedness property of the stochastic sign-based compression, we derive a series of theoretical results which exhibit better convergence rate than the existing sign-based methods. Moreover, we show that by injecting a suffciently large uniform noise, our proposed algorithm can have a matching convergence rate with the uncompressed algorithms.

Organization of this paper. In Section 2, the proposed general noisy perturbation scheme for the sign-based compression and its key propoerty about asymptotic unbiasedness are presented. Inspired by this result, the main algorithms are developed in Section 3 together with their convergence analyses under different noise distribution parameters. We evaluate our proposed algorithms on real datasets and benchmark with existing FL methods in Section 4. Finally, conclusions are drawn in Section 5.

Notations. For any $x \in \mathbb{R}^d$, we denote $x(j)$ as the j-th element of the vector x. We define the ℓ_p norm for any $p \geq 1$ as $\|x\|_p = (\sum_{j=1}^d |x(j)|^p)^{\frac{1}{p}}$. We denote that $\|\cdot\| = \|\cdot\|_2$, and $\|x\|_\infty = \max_{j\in\{1,...,d\}} |x(j)|$. For any function $f(x)$, we denote $f^{(k)}(x)$ as its k-th derivative, and for a vector $x = [x(1), ..., x(d)]^\top \in \mathbb{R}^d$, we define $\text{Sign}(x) = [\text{Sign}(x(1)), ..., \text{Sign}(x(d))]^\top$.

2 Sign operator with symmetric and zero-mean noise is asymptotically unbiased.

In this section, we introduce a general noisy perturbation scheme for the sign-based compression and analyze its asymptotic unbiasedness. The result serves as the foundation of the proposed algorithm designs in subsequent sections. Specifically, let us consider the following family of noise distribution parameterized by a postive interger $z \in \mathbb{Z}_+$.

Definition 1 (z-distribution)**.** *A random variable ξ_z follows z-distribution if its p.d.f is*

$$p_z(t) = \frac{1}{2\eta_z} e^{-\frac{t^{2z}}{2}}, \tag{2}$$

where $\eta_z = 2^{\frac{1}{2z}}\Gamma\left(1+\frac{1}{2z}\right)$ *and* $\Gamma(z) = \int_0^{+\infty} t^{z-1}e^{-t}dt$ *is the Gamma function.*

It can be verified that (2) is a valid p.d.f. When $z = 1$, it corresponds to the standard Gaussian distribution. In addition, one can also show that (2) converges to the uniform distribution when $z \to +\infty$ (Lemma 2 in Appendices). This family of distribution has a nice property that can be leveraged to bound the bias caused by the sign compression, as stated in the following lemma.

Lemma 1. *For any* $x \in \mathbb{R}^d$ *and* $\sigma > 0$*,*

$$\|\eta_z\sigma\mathbb{E}\left[\mathrm{Sign}(\mathrm{x}+\sigma\xi_z)\right] - x\|^2 \leq \frac{\|x\|_{4z+2}^{4z+2}}{4(2z+1)^2\sigma^{4z}}, \tag{3}$$

where $\xi_z(1), ..., \xi_z(d)$ *follow the i.i.d.* z*-distribution.*

Remark 1. *One can see that the RHS of* (3) *involves the term* $\left(\|x\|_{4z+2}/\sigma\right)^{4z}$*. Therefore, as long as* $\sigma > \|x\|_\infty$*, the LHS of* (3) *converges to zero when* $z \to +\infty$*. Since Lemma 2 implies that* ξ_∞ *is a vector whose elements follow i.i.d uniform distribution at* $[-1, 1]$*, we obtain* $\sigma\mathbb{E}\left[\mathrm{Sign}(\mathrm{x}+\sigma\xi_\infty)\right] = x$ *as long as* $\sigma > \|x\|_\infty$*. It is interesting to remark that the stochastic sign operators proposed in [Jin et al., 2020, Safaryan and Richtárik, 2021] is exactly the sign operator injected with a sufficient amount of uniform noise.*

3 Stochastic sign-based federated averaging with z-distribution.

In this section, based on the anaysis in Section 2, we propose the following sign-based federated averaging algorithm, termed as z-SignFedAvg. The algorithm details are stated in Algortihm 1. Note that in practice, we only implement z-SignFedAvg with $z = 1$ and $z = +\infty$ which correspond to the Gaussian distribution and uniform distribution, respectively. This is because, to the best of our knowledge, there is currently no efficient way to sampling from the distribution $p_z(t)$ with other choices of z. Nevertheless, we are interested in the convergence properteis of z-SignFedAvg for a general positive interger z since it provides better insights on how z impacts the convergence rate.

Algorithm 1 z-SignFedAvg (or z-SignSGD when $E = 1$)

Require: Total communication rounds T, Number of local steps E, Number of clients n, Clients stepsize γ, Server stepsize η, Noise coefficient σ, parameter of noise distribution z.

```
Initialize x_0 and for i = 1, ..., n.
for t = 1 to T do
    On Clients:
    for i = 1 to n in parallel do
        x^i_{t-1,0} = x_{t-1}
        for s = 1 to E do
            g^i_{t-1,s} = g_i(x^i_{t-1,s-1}), where g_i(·) is the minibatch gradient oracle of the i-th client.
            x^i_{t-1,s} = x^i_{t-1,s-1} − γ g^i_{t-1,s}.
        end for
        Δ^i_{t-1} = Sign((x_{t-1} − x^i_{t-1,E})/γ + σ ξ_z), where ξ_z(1), ..., ξ_z(d) ~ p_z(t) i.i.d.
        Send Δ^i_{t-1} to the server.
    end for
    On Server:
    x_t = x_{t-1} − ηγ (1/n) Σ_{i=1}^n Δ^i_{t-1}.
    Send x_t to the clients.
end for
return x_T.
```

We first state the following standard assumptions.

Assumption 1. *We assume that each* $f_i(x)$ *has the following properties:*

A.1 The minibatch gradient is unbiased and has bounded variance, i.e., $\mathbb{E}[g_i(x)] = \nabla f_i(x)$*,* $\mathbb{E}[\|g_i(x) - \nabla f_i(x)\|_2^2] \leq \zeta^2$.

A.2 Each f_i *is smooth, i.e., for any* $x, y \in \mathbb{R}^d$*, there exists some non-negative constans* $L_1, ..., L_d$ *such that* $f(y) - f(x) \leq \langle \nabla f(x), y - x\rangle + \frac{\sum_{j=1}^d L_j(y(j)-x(j))^2}{2}$.

A.3 There exists some constant f^* such that $f(x) \geq f^*, \forall x \in \mathbb{R}^d$

A.4 There exists a constant $G \geq 0$ such that $\|\nabla f_i(x)\| \leq G$, $\forall i = 1, \ldots, n$, and $x \in \mathbb{R}^d$.

For the convergence rate analysis, we consider two cases, namely, the case with $z < +\infty$ and the case of $z = \infty$.

3.1 Case 1: $z < +\infty$

As we can see from Lemma 1, there always exists some gradient bias when $z < +\infty$. In order to bound it, we further assume that a higher order moment of the minibatch gradient noise is bounded.

Assumption 2. *There exists a constant $Q_z \geq 0$ such that for any $x \in \mathbb{R}^d$, we have*

$$\mathbb{E}[\|g_i(x) - \nabla f_i(x)\|_{4z+2}^{4z+2}] \leq Q_z. \tag{4}$$

Theorem 1. *Suppose that Assumption 1 and 2 hold. Denote $\bar{x}_{t,s} = \frac{1}{n}\sum_{i=1}^n x_{t,s}^i$ and $L_{\max} = \max_j L_j$. Then, for $\eta = \eta_z\sigma$ and $\gamma \leq \frac{1}{L_{\max}}$, we have*

$$\mathbb{E}\left[\frac{1}{TE}\sum_{t=1}^{T}\sum_{s=1}^{E}\|\nabla f(\bar{x}_{t-1,s-1})\|^2\right] \leq \underbrace{\frac{2\mathbb{E}[f(x_0) - f^*]}{TE\gamma} + \frac{\gamma\zeta^2 L_{\max}}{n} + \frac{(E-1)(2E-1)\gamma^2 L_{\max}^2 G^2}{3}}_{\text{(a). Standard terms in FedAvg}} \tag{5a}$$

$$+ \underbrace{\frac{2^{2z}E^{2z}\sqrt{Q_z + G^{4z+2}}G}{(2z+1)\sigma^{2z}} + \frac{\gamma 2^{4z}E^{4z+1}(Q_z + G^{4z+2})L_{\max}}{2(2z+1)^2\sigma^{4z}}}_{\text{(b). Bias terms}} \tag{5b}$$

$$+ \underbrace{\frac{4\eta_z^2\gamma\sigma^2\sum_{j=1}^d L_j}{En}}_{\text{(c). Variance term}}. \tag{5c}$$

When is the bound non-trivial? Since we assume that the ℓ_2-norm of gradient is bounded by G, all the terms in the RHS of (5) should be no larger than G^2. For example, one can check that to have the fisrt term in (5b) less than G^2, one requires σ to be greater than $2E\left(Q_z + G^{4z}\right)^{\frac{1}{4z}}/(2z+1)^{\frac{1}{2z}}$.

Bias-variance trade-off. An interesting observation from Theorem 1 is that there exists a trade-off between the bias and variance terms. One can see that the terms in (5b) is caused by the gradient bias of the sign operation (see (1)) and is an infinitesimal of σ with $\mathcal{O}\left(\sigma^{-2z}\right)$, while the term in (5c) is due to the injected noise and is in the order of $\mathcal{O}\left(\gamma\sigma^2\right)$. Specifically, the first term in (b) only depends on the noise scale σ and mostly affects the final learning performance. In the meanwhile, the variance term mainly affects the convergence speed because a smaller stepsize is required for it to diminish.

Theoretically, we can choose an iteration-dependent noise scale σ so as to making the algorithm converge. In the following results, we denote $\tau = TE$ as the total number of gradient queries to the local objective function.

Corollary 1 (Informal). *Let $\gamma = \min\{n^{\frac{z}{2z+1}}\tau^{-\frac{z+1}{2z+1}}, \frac{1}{L_{\max}}\}$ and $\sigma = (n\tau)^{\frac{1}{4z+2}}$ in Theorem 1, and let $E \leq n^{-\frac{3z}{4z+2}}\tau^{\frac{z+2}{4z+2}}$. We have*

$$\mathbb{E}\left[\frac{1}{\tau}\sum_{t=1}^{T}\sum_{s=1}^{E}\|\nabla f(\bar{x}_{t-1,s-1})\|^2\right] = \mathcal{O}\left((n\tau)^{-\frac{z}{2z+1}}\right). \tag{6}$$

Achieveing linear speedup. From Corollary 1, we can see that the z-SignFedAvg needs $(n\tau)^{\frac{3z}{4z+2}}$ communication rounds to achieve a linear speedup convergence rate. Particularly, when $z = 1$, the corresponding convergence speed is $\mathcal{O}((n\tau)^{-\frac{1}{3}})$ and the required communication rounds is $(n\tau)^{\frac{1}{2}}$.

3.2 Case 2: $z = +\infty$

In this case, the injected noise in z-SignFedAvg is uniformly distributed at $[-1, 1]$. From Remark 1 we have learned that the bias term in (5b) will either blow up when σ is lower than some threshold, or vanish on the other hand. To quantify this threshold, we need the limit form of Assumption 2:

Assumption 3. *There exists a constant $Q_\infty \geq 0$ such that, with probability 1 we have*
$$\|g_i(x) - \nabla f_i(x)\|_\infty \leq Q_\infty, \forall x \in \mathbb{R}^d. \tag{7}$$

Theorem 2 (Informal)**.** *Suppose that Assumption 1 and 3 hold. For $\gamma = \min\{n^{\frac{1}{2}}\tau^{-\frac{1}{2}}, \frac{1}{L_{\max}}\}$, $\eta = \sigma$, $E \leq n^{-\frac{3}{4}}\tau^{\frac{1}{4}}$, and $\sigma > E(G + Q_\infty)$, we have*
$$\mathbb{E}\left[\frac{1}{\tau}\sum_{t=1}^{T}\sum_{s=1}^{E}\|\nabla f(\bar{x}_{t-1,s-1})\|^2\right] = \mathcal{O}\left((n\tau)^{-\frac{1}{2}}\right). \tag{8}$$

We can see that (8) implies ∞-SignFedAvg recovers the convergence rate of uncompressed algorithms [Yu et al., 2019].

Remark 2. *It is worthwhile to point out that the condition of sufficeintly large noise scale $\sigma > E(G + Q_\infty)$ is necessary and cannot be spared. By intuition, this is because when $\sigma \leq E(G + Q_\infty)$ in Theorem 2, the injected uniform noise cannot change the sign of gradients in the worst case. For example, if ξ_∞ follows uniform distribution on $[-1, 1]$, and now $\sigma < A$ for some $A > 0$, we have $\text{Sign}(x + \sigma\xi_\infty) = \text{Sign}(x)$ for any $x \geq A$.*

Remark 3. *By comparing the required thresholds for σ in Theorem 1 and Theorem 2, we can see that when there is no minibatch gradient noise (i.e., $\zeta = 0$), Case 2 demands less noise injection and may perform better than Case 1. On the contrary, when the minibatch gradient noise has a long tail such as $Q_z \ll Q_\infty^{4z}$, Case 1 may be better. Despite of the distinction in theory, as we will see in Section 4, Case 1 and Case 2 have almost the same behavior on real datasets.*

More detailed theoretical results and comparsion with existing methods are relegated to Appendix B.

4 Experiments

In this section, we present the experiment results on real datasets. All the figures are obtained by 10 independent runs and are visualized in the form of mean±std. We also conduct an experiment on synthetic data where there is no minibatch gradient noise, and the results is relegated to Appendix D.

Noise scale as a hyperparameter. Although we explicitly characterize how the performance of Algorithm 1 depends on the noise scale σ in previous section, we treat σ as a tunable hyperparameter in practice. Because, on one hand, it usually impossible to compute the moment and support of the gradient noise in reality. One the other hand, since the theoretical results above only provide a worst-case guarantee, for some real problems, the optimal noise scales selected from grid search can be much smaller than the choice suggested by theory.

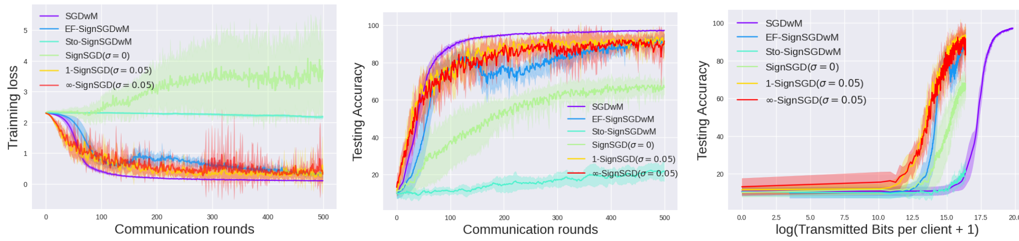
4.1 An extremely non-i.i.d setting

In this section, we consider an extremely non-i.i.d setting with the well-known dataset MNIST [Deng, 2012] which is a hand-written digits recogonition dataset. Specifically, we split the dataset into 10 parts based on the labels and each client only have the data of one digit. In such a highly heterogeneous setting, there is no benefit from local computation with periodic commnunication. Therefore, we compare with the listed algorithms: SGDwM [Ghadimi and Lan, 2013], EF-SignSGDwM [Karimireddy et al., 2019, Vogels et al., 2019], Sto-SignSGDwM [Safaryan and Richtárik, 2021]. Some baseline algorithms have an additional hyperparameter for the momentum of gradient. For all the algorithms, we select the their own optimal hyperparameters like stepsize, momentum coefficient, noise scale via grid search. For more details like hyperparameters for all the tested algorthms and the performance of 1-SignSGD and ∞-SignSGD under different noise scales, we refer the readers to Appendix E.1.

Results. From Figure 1a, 1b, we can observe that 1-SignSGD and ∞-SignSGD have roughly the same performance which outperform other sign-based algorithms and is slightly inferior to the uncompressed algorithm. Our theory is verified by comparing the performance of noiseless SignSGD and our proposed algortihms. If we compare the performance with respect to the total number of transmitted bits, our algorithms achieve the state-of-the-art performance on this task as we can see in Figure 1c.

4.2 Federated Learning on EMNIST

In this section, we verify the performance of our proposed Algorithm 1 on EMNIST[Cohen et al., 2017]. We mainly compare the performance of three algorithms: FedAvg without any compression [McMahan et al., 2017, Yu et al., 2019] and our proposed Algorithm 1 with $z = 1$ and $z = \infty$.

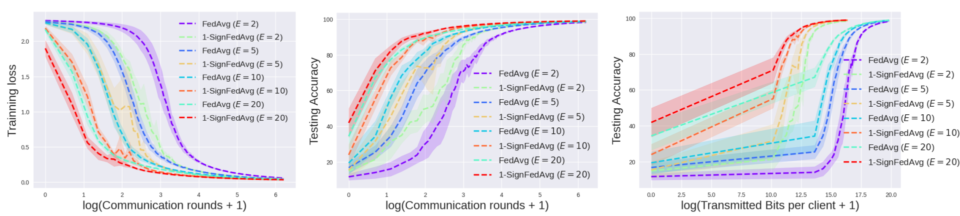


(a) Trainning Loss (b) Test Accuracy (c) Test Accuracy w.r.t bits

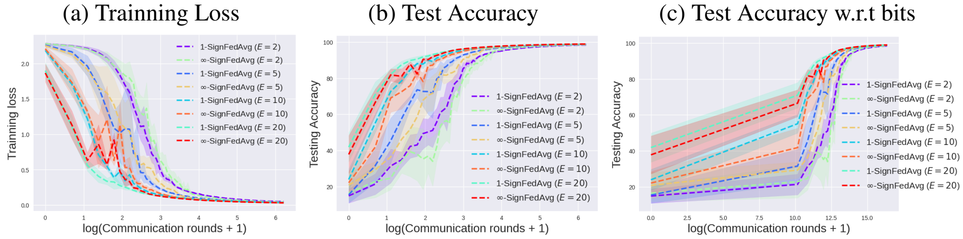
Figure 1: Performance of various algorithms on non-i.i.d MNIST

Settings. We follow a similar setting to [Reddi et al., 2020]. We also consider the scenario with partial participation. Specifically, for the EMNIST dataset, there are 3579 clients in total and we sample 100 clients uniformly to upload their local gradients at each communication round.

Results. The hyperparameters for the algorithms are tuned via grid search and details are in Appendix E.2. Specifically, we use $\sigma = 0.01$ for both 1-SignFedAvg and ∞-SignFedAvg on EMNIST dataset. We can see from Figure 2 that all the algorithms can benefit from multiple local steps, and more surprisingly, both 1-SignFedAvg and ∞-SignFedAvg can outperform the umcompressed algorithm FedAvg. This is probably because the EMNIST dataset is less non-i.i.d as the dataset we use in Section 4.1. The performance of 1-SignFedAvg and ∞-SignFedAvg under various choices of noise scale are relegated to the Figure 6 and 7 in Appendix E.2, which also matches our theoretical results.



(a) Trainning Loss (b) Test Accuracy (c) Test Accuracy w.r.t bits



(d) Trainning Loss (e) Test Accuracy (f) Test Accuracy w.r.t bits

Figure 2: Performance of FedAvg, 1-SignFedAvg and ∞-SignFedAvg on EMNIST dataset.

5 Conclusion

In this work, we have proposed the z-SignFedAvg: a FedAvg-type algorithm with a novel family of sign-based stochastic compression. Throughout extensive theoretical analysis and empirical evaluation, we have shown that z-SignFedAvg can perform comparably, sometimes even better, as the uncompressed FedAvg algorithm with a significantly reduced number of bits transmitted from the clients to the server. However, a vital issue in z-SignFedAvg is that it involves a new hyperparameter, i.e., the noise scale σ, which needs to be carefully chosen for achieving a good convergence performance. An interesting observation from the experiments is that the less heterogeneous the local data are, the smaller the optimal noise scale is, which is consisten with the theoretical insights. In the future, we will futher study the relationship between the client's heterogeneity and the optimal noise scale. As a final remark, we note that the stochastic sign-based compression proposed in this work is of independent interest and can be directly combined with other adaptive FL algorithms like those in Karimireddy et al. [2020], Reddi et al. [2020].

References

Jeremy Bernstein, Yu-Xiang Wang, Kamyar Azizzadenesheli, and Animashree Anandkumar. signsgd: Compressed optimisation for non-convex problems. In *International Conference on Machine Learning*, pages 560–569. PMLR, 2018.

Brendan McMahan, Eider Moore, Daniel Ramage, Seth Hampson, and Blaise Aguera y Arcas. Communication-efficient learning of deep networks from decentralized data. In *Artificial intelligence and statistics*, pages 1273–1282. PMLR, 2017.

Tian Li, Anit Kumar Sahu, Ameet Talwalkar, and Virginia Smith. Federated learning: Challenges, methods, and future directions. *IEEE Signal Processing Magazine*, 37(3):50–60, 2020a.

Jakub Konečnỳ, H Brendan McMahan, Felix X Yu, Peter Richtárik, Ananda Theertha Suresh, and Dave Bacon. Federated learning: Strategies for improving communication efficiency. *arXiv preprint arXiv:1610.05492*, 2016.

Dan Alistarh, Demjan Grubic, Jerry Li, Ryota Tomioka, and Milan Vojnovic. Qsgd: Communication-efficient sgd via gradient quantization and encoding. *Advances in neural information processing systems*, 30, 2017.

Amirhossein Reisizadeh, Aryan Mokhtari, Hamed Hassani, Ali Jadbabaie, and Ramtin Pedarsani. Fedpaq: A communication-efficient federated learning method with periodic averaging and quantization. In *International Conference on Artificial Intelligence and Statistics*, pages 2021–2031. PMLR, 2020.

Sai Praneeth Karimireddy, Quentin Rebjock, Sebastian Stich, and Martin Jaggi. Error feedback fixes signsgd and other gradient compression schemes. In *International Conference on Machine Learning*, pages 3252–3261. PMLR, 2019.

Mher Safaryan and Peter Richtárik. Stochastic sign descent methods: New algorithms and better theory. In *International Conference on Machine Learning*, pages 9224–9234. PMLR, 2021.

Xiangyi Chen, Tiancong Chen, Haoran Sun, Steven Z Wu, and Mingyi Hong. Distributed training with heterogeneous data: Bridging median-and mean-based algorithms. *Advances in Neural Information Processing Systems*, 33:21616–21626, 2020a.

Richeng Jin, Yufan Huang, Xiaofan He, Huaiyu Dai, and Tianfu Wu. Stochastic-sign sgd for federated learning with theoretical guarantees. *arXiv preprint arXiv:2002.10940*, 2020.

Thijs Vogels, Sai Praneeth Karimireddy, and Martin Jaggi. Powersgd: Practical low-rank gradient compression for distributed optimization. *Advances in Neural Information Processing Systems*, 32, 2019.

Hanlin Tang, Chen Yu, Xiangru Lian, Tong Zhang, and Ji Liu. Doublesqueeze: Parallel stochastic gradient descent with double-pass error-compensated compression. In *International Conference on Machine Learning*, pages 6155–6165. PMLR, 2019.

Saeed Ghadimi and Guanghui Lan. Stochastic first- and zeroth-order methods for nonconvex stochastic programming. *SIAM Journal on Optimization*, 23(4):2341–2368, 2013. doi:10.1137/120880811. URL `https://doi.org/10.1137/120880811`.

Hao Yu, Sen Yang, and Shenghuo Zhu. Parallel restarted sgd with faster convergence and less communication: Demystifying why model averaging works for deep learning. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 33, pages 5693–5700, 2019.

Li Deng. The mnist database of handwritten digit images for machine learning research [best of the web]. *IEEE signal processing magazine*, 29(6):141–142, 2012.

Gregory Cohen, Saeed Afshar, Jonathan Tapson, and Andre Van Schaik. Emnist: Extending mnist to handwritten letters. In *2017 international joint conference on neural networks (IJCNN)*, pages 2921–2926. IEEE, 2017.

Sashank Reddi, Zachary Charles, Manzil Zaheer, Zachary Garrett, Keith Rush, Jakub Konečnỳ, Sanjiv Kumar, and H Brendan McMahan. Adaptive federated optimization. *arXiv preprint arXiv:2003.00295*, 2020.

Sai Praneeth Karimireddy, Satyen Kale, Mehryar Mohri, Sashank Reddi, Sebastian Stich, and Ananda Theertha Suresh. Scaffold: Stochastic controlled averaging for federated learning. In *International Conference on Machine Learning*, pages 5132–5143. PMLR, 2020.

Jinshuo Dong, Aaron Roth, and Weijie Su. Gaussian differential privacy. *Journal of the Royal Statistical Society*, 2021.

Yanmeng Wang, Yanqing Xu, Qingjiang Shi, and Tsung-Hui Chang. Quantized federated learning under transmission delay and outage constraints. *IEEE Journal on Selected Areas in Communications*, 40(1):323–341, 2021.

Cynthia Dwork, Aaron Roth, et al. The algorithmic foundations of differential privacy. *Found. Trends Theor. Comput. Sci.*, 9(3-4):211–407, 2014.

Robin C Geyer, Tassilo Klein, and Moin Nabi. Differentially private federated learning: A client level perspective. *arXiv preprint arXiv:1712.07557*, 2017.

Martin Abadi, Andy Chu, Ian Goodfellow, H Brendan McMahan, Ilya Mironov, Kunal Talwar, and Li Zhang. Deep learning with differential privacy. In *Proceedings of the 2016 ACM SIGSAC conference on computer and communications security*, pages 308–318, 2016a.

Yiwei Li, Tsung-Hui Chang, and Chong-Yung Chi. Secure federated averaging algorithm with differential privacy. In *2020 IEEE 30th International Workshop on Machine Learning for Signal Processing (MLSP)*, pages 1–6. IEEE, 2020b.

Naman Agarwal, Ananda Theertha Suresh, Felix Xinnan X Yu, Sanjiv Kumar, and Brendan McMahan. cpsgd: Communication-efficient and differentially-private distributed sgd. *Advances in Neural Information Processing Systems*, 31, 2018.

Zhiqi Bu, Jinshuo Dong, Qi Long, and Weijie J Su. Deep learning with gaussian differential privacy. *Harvard data science review*, 2020(23), 2020.

Qinqing Zheng, Shuxiao Chen, Qi Long, and Weijie Su. Federated f-differential privacy. In *International Conference on Artificial Intelligence and Statistics*, pages 2251–2259. PMLR, 2021.

Saba Amiri, Adam Belloum, Sander Klous, and Leon Gommans. Compressive differentially-private federated learning through universal vector quantization. 2021.

Lun Wang, Ruoxi Jia, and Dawn Song. D2p-fed: Differentially private federated learning with efficient communication. *arXiv preprint arXiv:2006.13039*, 2020.

Xiangru Lian, Ce Zhang, Huan Zhang, Cho-Jui Hsieh, Wei Zhang, and Ji Liu. Can decentralized algorithms outperform centralized algorithms? a case study for decentralized parallel stochastic gradient descent. *Advances in Neural Information Processing Systems*, 30, 2017.

Ilya Mironov. Rényi differential privacy. In *2017 IEEE 30th computer security foundations symposium (CSF)*, pages 263–275. IEEE, 2017.

Shahab Asoodeh, Jiachun Liao, Flavio P Calmon, Oliver Kosut, and Lalitha Sankar. Three variants of differential privacy: Lossless conversion and applications. *IEEE Journal on Selected Areas in Information Theory*, 2(1):208–222, 2021.

Igal Sason and Sergio Verdú. f-divergence inequalities. *IEEE Transactions on Information Theory*, 62(11):5973–6006, 2016.

Laurent Condat, Kai Yi, and Peter Richtárik. Ef-bv: A unified theory of error feedback and variance reduction mechanisms for biased and unbiased compression in distributed optimization. *arXiv preprint arXiv:2205.04180*, 2022.

Shuai Zheng, Ziyue Huang, and James Kwok. Communication-efficient distributed blockwise momentum sgd with error-feedback. *Advances in Neural Information Processing Systems*, 32, 2019.

Xiangyi Chen, Steven Z Wu, and Mingyi Hong. Understanding gradient clipping in private sgd: A geometric perspective. *Advances in Neural Information Processing Systems*, 33:13773–13782, 2020b.

Xinwei Zhang, Xiangyi Chen, Mingyi Hong, Zhiwei Steven Wu, and Jinfeng Yi. Understanding clipping for federated learning: Convergence and client-level differential privacy. *arXiv preprint arXiv:2106.13673*, 2021.

Zhiwei Tang, Tsung-Hui Chang, Xiaojing Ye, and Hongyuan Zha. Low-rank matrix recovery with unknown correspondence. *arXiv preprint arXiv:2110.07959*, 2021a.

Peter Kairouz, Ziyu Liu, and Thomas Steinke. The distributed discrete gaussian mechanism for federated learning with secure aggregation. In *International Conference on Machine Learning*, pages 5201–5212. PMLR, 2021.

John T Chu. On bounds for the normal integral. *Biometrika*, 42(1/2):263–265, 1955.

Hanlin Tang, Shaoduo Gan, Ammar Ahmad Awan, Samyam Rajbhandari, Conglong Li, Xiangru Lian, Ji Liu, Ce Zhang, and Yuxiong He. 1-bit adam: Communication efficient large-scale training with adam's convergence speed. In *International Conference on Machine Learning*, pages 10118–10129. PMLR, 2021b.

Jianyu Wang and Gauri Joshi. Cooperative sgd: A unified framework for the design and analysis of local-update sgd algorithms. *Journal of Machine Learning Research*, 22, 2021.

Frank Seide, Hao Fu, Jasha Droppo, Gang Li, and Dong Yu. 1-bit stochastic gradient descent and its application to data-parallel distributed training of speech dnns. In *Fifteenth annual conference of the international speech communication association*. Citeseer, 2014.

Martín Abadi, Paul Barham, Jianmin Chen, Zhifeng Chen, Andy Davis, Jeffrey Dean, Matthieu Devin, Sanjay Ghemawat, Geoffrey Irving, Michael Isard, et al. {TensorFlow}: a system for {Large-Scale} machine learning. In *12th USENIX symposium on operating systems design and implementation (OSDI 16)*, pages 265–283, 2016b.

Adam Paszke, Sam Gross, Soumith Chintala, Gregory Chanan, Edward Yang, Zachary DeVito, Zeming Lin, Alban Desmaison, Luca Antiga, and Adam Lerer. Automatic differentiation in pytorch. 2017.

Alex Krizhevsky and Geoff Hinton. Convolutional deep belief networks on cifar-10. *Unpublished manuscript*, 40(7):1–9, 2010.

Tim Van Erven and Peter Harremos. Rényi divergence and kullback-leibler divergence. *IEEE Transactions on Information Theory*, 60(7):3797–3820, 2014.

Dimitri P Bertsekas. Nonlinear programming. *Journal of the Operational Research Society*, 48(3): 334–334, 1997.

Naman Agarwal, Peter Kairouz, and Ziyu Liu. The skellam mechanism for differentially private federated learning. *Advances in Neural Information Processing Systems*, 34:5052–5064, 2021.

Yangsibo Huang, Samyak Gupta, Zhao Song, Kai Li, and Sanjeev Arora. Evaluating gradient inversion attacks and defenses in federated learning. *Advances in Neural Information Processing Systems*, 34:7232–7241, 2021.

Jonas Geiping, Hartmut Bauermeister, Hannah Dröge, and Michael Moeller. Inverting gradients-how easy is it to break privacy in federated learning? *Advances in Neural Information Processing Systems*, 33:16937–16947, 2020.

Ilya Mironov, Kunal Talwar, and Li Zhang. R\'enyi differential privacy of the sampled gaussian mechanism. *arXiv preprint arXiv:1908.10530*, 2019.

Checklist

The checklist follows the references. Please read the checklist guidelines carefully for information on how to answer these questions. For each question, change the default **[TODO]** to [Yes] , [No] , or [N/A] . You are strongly encouraged to include a **justification to your answer**, either by referencing the appropriate section of your paper or providing a brief inline description. For example:

- Did you include the license to the code and datasets? [Yes] See Section **??**.
- Did you include the license to the code and datasets? [No] The code and the data are proprietary.
- Did you include the license to the code and datasets? [N/A]

Please do not modify the questions and only use the provided macros for your answers. Note that the Checklist section does not count towards the page limit. In your paper, please delete this instructions block and only keep the Checklist section heading above along with the questions/answers below.

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? **[TODO]**
   (b) Did you describe the limitations of your work? **[TODO]**
   (c) Did you discuss any potential negative societal impacts of your work? **[TODO]**
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? **[TODO]**
2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? **[TODO]**
   (b) Did you include complete proofs of all theoretical results? **[TODO]**
3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? **[TODO]**
   (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? **[TODO]**
   (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? **[TODO]**
   (d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? **[TODO]**
4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...
   (a) If your work uses existing assets, did you cite the creators? **[TODO]**
   (b) Did you mention the license of the assets? **[TODO]**
   (c) Did you include any new assets either in the supplemental material or as a URL? **[TODO]**
   (d) Did you discuss whether and how consent was obtained from people whose data you're using/curating? **[TODO]**
   (e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? **[TODO]**
5. If you used crowdsourcing or conducted research with human subjects...
   (a) Did you include the full text of instructions given to participants and screenshots, if applicable? **[TODO]**
   (b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? **[TODO]**
   (c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? **[TODO]**

Appendices

A Dicussion on related works

Sign-based optimization algorithms such as the SignSGD in [Bernstein et al., 2018] have gained much popularity recently because of their simple compression rule and comparable performance to uncompressed algorithms. In this work, we focus on the scenario with heterogeneous data, and as we have discussed in Section 1, a naive application of sign-based compression in this scenario is problematic. Besides, we consider using sign-based compression only for the uplink communication in this work, while it is worth mentioning that [Tang et al., 2019, Jin et al., 2020, Chen et al., 2020a] also compress for the downlink communication. In the following paragraphs, we review a few related works on similar topics.

Stochastic sign-based method. The setting considered by [Safaryan and Richtárik, 2021] is the closest to ours because [Jin et al., 2020, Chen et al., 2020a] also compresses the server-to-client communication with majority vote. Aside from the difference in setting, the algorithms in them achieve the same convergence rate $O(\tau^{-\frac{1}{4}})$ w.r.t different convergence metrics, where τ is the number of gradient queries to the objective function. As will be discussed in Appendix A, these rates are usually inferior to that of uncompressed algorithms. Our proposed algorithm also belongs to this category. Compared to existing works, we require a slightly stronger assumption on the gradient noise, and the convergence speed of our algorithm is either $O(\tau^{-\frac{1}{3}})$ or $O(\tau^{-\frac{1}{2}})$ with the commonly used squared ℓ_2 norm of gradients as the convergence metric. Moreover, we also show that our proposed sign-based algorithm can achieve a linear speedup when the number of clients increases, and such a result is not known in previous works.

Error Feedback method. The error feedback (EF) method is first proposed by [Seide et al., 2014] and then theoretically justified by [Karimireddy et al., 2019]. Then [Vogels et al., 2019, Tang et al., 2019, 2021b] further extend this EF framework into distributed non-i.i.d setting and adaptive gradient method. The key idea is to show that the sign operator multiplying with one norm is a contractive compressor, and the error induced by the contractive compressor can be fixed by the error-compensated gradient method. However, unlike the pure sign-based gradient method, it must transmit one extra real number for the one norm. Besides, the convergence rate for the EF algorithms is $\mathcal{O}(\tau^{-\frac{1}{2}} + \tau^{-1}/\delta^2)$, where δ is the parameter of contractive compressor. In the worst case, the sign operator multiplying with one norm is a contractive compressor with $\delta = 1/d$, where d is the dimension of the gradient. Therefore, the convergence rate of it becomes $\mathcal{O}(\tau^{-\frac{1}{2}} + d^2\tau^{-1})$, which could become very bad especially for high-dimension optimization problem. Besides, to our knowledge, no one has extended the error feedback method to the scenario with periodic aggregation.

It is often tricky to compare the convergence results of sign-based methods because some works like [Chen et al., 2020a, Safaryan and Richtárik, 2021] do not use the standard convergence metric. To better compare existing results and ours, in Appendix A, we provide a detailed discussion on the existing convergence metrics and summarize the representative algorithms and their theoretical results in Table 1.

Table 1 gives a brief summary for a few representative works related to this work. In this table, we review the algorithms in these works on the convergence rate along with the used convergence metrics, communication complexity, assumptions required, and also whether they can deal with periodic aggregation. Particularly, [Chen et al., 2020a, Safaryan and Richtárik, 2021] adopt convergence metrics other than the commonly used average squared ℓ_2 norm of gradients. Due to the additional step of server-to-client compression, [Chen et al., 2020a] use a convergence metric mixed with ℓ_2 norm and ℓ_1 norm, while [Safaryan and Richtárik, 2021] use ℓ_2 norm instead of squared ℓ_2 norm. For communication complexity, we only compare the unlink communication, and to compute the used bits per communication, we assume that all the algorithms need 32 bits to represent a float number as this is the most common setting in Tensorflow [Abadi et al., 2016b] and Pytorch [Paszke et al., 2017].

Among the works in Table 1, the setting considered by [Safaryan and Richtárik, 2021] is the closest to ours. [Safaryan and Richtárik, 2021] propose an algorithm that can achieve convergence rate $\mathcal{O}(\tau^{-\frac{1}{4}})$ with average ℓ_2 norm of gradients as the metric. We remark that this is inferior to the convergence rate $\mathcal{O}(\tau^{-\frac{1}{2}})$ with squared ℓ_2 norm as the metric. To illustrate this point, we denote a serie of vector

Algorithm	Convergence metric / rate	Used bits per communication	Extra Assumptions?	Can achieve linear speedup?	Can allow multiple local steps?
[Ghadimi and Lan, 2013]	$\mathcal{O}(\tau^{-\frac{1}{2}})$ squared ℓ_2	$32d$	No	✓	✗
[Yu et al., 2019]	$\mathcal{O}(\tau^{-\frac{1}{2}})$ squared ℓ_2	$32d$	• Bounded gradient	✓	✓
[Karimireddy et al., 2019]	$\mathcal{O}(\tau^{-\frac{1}{2}} + d^2\tau^{-1})$ squared ℓ_2	$d + 32$	• Bounded gradient	✗	✗
[Safaryan and Richtárik, 2021]	$\mathcal{O}(\tau^{-\frac{1}{4}})$ ℓ_2	d	No	✗	✗
[Jin et al., 2020]	$\mathcal{O}(\tau^{-\frac{1}{4}})$ squared ℓ_2	d	• Bounded gradient • n is an odd number	✗	✗
[Chen et al., 2020a]	$\mathcal{O}(\tau^{-\frac{1}{4}})$ mixed	d	• Bounded gradient • n is an odd number	✗	✗
1-SignFedAvg (ALG. 1) **This work**	$\mathcal{O}(\tau^{-\frac{1}{3}})$ squared ℓ_2	d	• Bounded gradient • Bounded 6th moment of gradient noise	✓	✓
∞-SignFedAvg (ALG. 1) **This work**	$\mathcal{O}(\tau^{-\frac{1}{2}})$ squared ℓ_2	d	• Bounded gradient • Bounded support of gradient noise	✓	✓

Table 1: Summary for related works.

$\{\alpha_1, ..., \alpha_\tau, ...\}$ with $\alpha_i \in \mathbb{R}^d$. If now

$$\frac{1}{\tau}\sum_{i=1}^{\tau}\|\alpha_i\| = \mathcal{O}(\tau^{-\frac{1}{4}}), \tag{9}$$

in the worst case, we can only guarantee that

$$\frac{1}{\tau}\sum_{i=1}^{\tau}\|\alpha_i\|^2 \le \tau\left(\frac{1}{\tau}\sum_{i=1}^{\tau}\|\alpha_i\|\right)^2 = \mathcal{O}(\tau^{\frac{1}{2}}) \tag{10}$$

for squared ℓ_2 norm. As a simple example, the equality in (10) holds when there is only one non-zero term in $\{\alpha_1, ..., \alpha_\tau\}$.

On the contrary, if

$$\frac{1}{\tau}\sum_{i=1}^{\tau}\|\alpha_i\|^2 = \mathcal{O}(\tau^{-\frac{1}{2}}), \tag{11}$$

we have

$$\frac{1}{\tau}\sum_{i=1}^{\tau}\|\alpha_i\| \le \sqrt{\frac{1}{\tau}\sum_{i=1}^{\tau}\|\alpha_i\|^2} = \mathcal{O}(\tau^{-\frac{1}{4}}). \tag{12}$$

Consider the scenario $E = 1$, the algorithm in [Safaryan and Richtárik, 2021] is equivalent to our Algorithm 1 with σ chosen to be $\|g^i_{t-1,s}\|$. On one hand, this choice of noise scale σ make it unable to be extended to the federated averaging algorithm, because each client use a different noise scale. On the other hand, this choice is linearly increaseing w.r.t problem dimension and hence is too conservative. From Figure 3 and 1 we can see that this input-dependent noise scale result in an extremely slow convergence for high-dimension problems.

B Theoretical results

In this section, we state the formal version of Corollary 1 and Theorem 2.

Corollary 2 (Formal version of Corollary 1)**.** *If we choose* $\gamma = \min\{n^{\frac{z}{2z+1}}\tau^{-\frac{z+1}{2z+1}}, \frac{1}{L_{\max}}\}$ *and* $\sigma = (n\tau)^{\frac{1}{4z+2}}$ *in Theorem 1, we have*

$$\mathbb{E}[\frac{1}{\tau}\sum_{t=1}^{T}\sum_{s=1}^{E}]\|\nabla f(\bar{x}_{t-1,s-1})\|^2 \le \frac{2\mathbb{E}[f(x_0)-f^*]}{(n\tau)^{\frac{z}{2z+1}}} + \frac{\zeta^2 L_{\max}}{(n\tau)^{\frac{z+1}{2z+1}}} + \frac{(E-1)(2E-1)n^{\frac{2z}{2z+1}}L_{\max}^2 G^2}{3\tau^{\frac{2z+2}{2z+1}}} \tag{13a}$$

$$+ \frac{2^{2z}E^{2z}\sqrt{Q_z + G^{4z+2}}G}{(2z+1)(n\tau)^{\frac{z}{2z+1}}} + \frac{2^{4z}E^{4z+1}(Q_z+G^{4z+2})L_{\max}}{2(2z+1)^2 n^{\frac{z}{2z+1}}\tau^{\frac{3z+1}{2z+1}}} \tag{13b}$$

$$+ \frac{4\eta_z^2\sum_{j=1}^{d}L_j}{E(n\tau)^{\frac{z}{2z+1}}}. \tag{13c}$$

Furthermore, if $E \le n^{-\frac{3z}{4z+2}}\tau^{\frac{z+2}{4z+2}}$*, the upper bound above will converge as* $\mathcal{O}\left((n\tau)^{-\frac{z}{2z+1}}\right)$.

Relationship to [Chen et al., 2020a]. [Chen et al., 2020a] also studies the sign-based optimization algorithm with symmetric and zero-mean noise and prove that the convergence rate is $\mathcal{O}(\tau^{-\frac{1}{4}})$ using a similar iteration-dependent noise scale like us. However, there are two difference between their result and our Theorem 1. First, since they also consider the downlink compression, the convergence metric they used is no longer ℓ_2 norm and hard to interpret. Second, unlike [Chen et al., 2020a] whose result is rooted in median-based algorithm, our analysis directly exploits the property of sign operation and hence can provide a better and more interpretable result.

Theorem 3 (Formal version of Theorem 2)**.** *Given that Assumption 1 and 3 hold, and we choose* $\eta = \sigma$*, if* $\gamma \le \frac{1}{L_{\max}}$*, if* $\sigma > E(G + Q_\infty)$*, we have*

$$\mathbb{E}[\frac{1}{TE}\sum_{t=1}^{T}\sum_{s=1}^{E}]\|\nabla f(\bar{x}_{t-1,s-1})\|^2 \le \underbrace{\frac{2\mathbb{E}[f(x_0)-f^*]}{TE\gamma} + \frac{\gamma\zeta^2 L_{\max}}{n} + \frac{(E-1)(2E-1)\gamma^2 L_{\max}^2 G^2}{3}}_{\text{standard terms in federated averaging}} \tag{14a}$$

$$+ \underbrace{\frac{4\gamma\sigma^2\sum_{j=1}^{d}L_j}{En}}_{\text{variance term}}. \tag{14b}$$

otherwise, if $\sigma \le E(G + Q_\infty)$*, there exists a problem where the algorithm cannot converge.*

If we further choose $\gamma = \min\{n^{\frac{1}{2}}\tau^{-\frac{1}{2}}, \frac{1}{L_{\max}}\}$*, we have*

$$\mathbb{E}[\frac{1}{\tau}\sum_{t=1}^{T}\sum_{s=1}^{E}]\|\nabla f(\bar{x}_{t-1,s-1})\|^2 \le \frac{2\mathbb{E}[f(x_0)-f^*]}{(n\tau)^{\frac{1}{2}}} + \frac{\zeta^2 L_{\max}}{(n\tau)^{\frac{1}{2}}} + \frac{(E-1)(2E-1)nL_{\max}^2 G^2}{3\tau} \tag{15}$$

$$+ \frac{4\sigma^2\sum_{j=1}^{d}L_j}{E(n\tau)^{\frac{1}{2}}}. \tag{16}$$

Furthermore, if $E \le n^{-\frac{3}{4}}\tau^{\frac{1}{4}}$*, the upper bound above will converge as* $\mathcal{O}\left((n\tau)^{-\frac{1}{2}}\right)$*, which recovers the convergence result of uncompressed algorithm [Yu et al., 2019].*

Remark 4. *When* $\sigma \le E(G + Q_\infty)$ *in Theorem 3, the injected uniform noise cannot change the sign of gradients in the worst case. For example, if* ξ_∞ *follows uniform distribution on* $[-1, 1]$*, and now* $\sigma < A$ *for some* $A > 0$*, we have* $\mathrm{Sign}(x + \sigma\xi_\infty) = \mathrm{Sign}(x)$ *for any* $x \ge A$*.*

Relationship to [Jin et al., 2020, Safaryan and Richtárik, 2021]. We remark that both the stochastic sign operators proposed in [Jin et al., 2020, Safaryan and Richtárik, 2021] are equivalent to the sign operator with uniform noise considered in Case 2. In particular, [Jin et al., 2020] also consider downlink compression and hence its convergence results are not directly comparable to the Case 2. [Safaryan and Richtárik, 2021] adopts an input-dependent noise scale and proves $\mathcal{O}(\tau^{-\frac{1}{4}})$ convergence rate with ℓ_2 norm of gradient as the metric. We remark that this rate is usually worse than the rate $\mathcal{O}(\tau^{-\frac{1}{2}})$ with squared ℓ_2 norm as the metric. Such input-dependent noise scale make it possible to prove convergence without the bounded support of gradient noise assumed in this work. But there are two disadvantages for their choice of noise scale. First, it can not be extended to federated averaging algorithm. Second, it often leads to slow convergence in practice when the problem dimension is very high. More discussions on Safaryan and Richtárik [2021] are provided in Appendix A.

C Missing proofs

Lemma 2. *z-distribution weakly converges to uniform distribution at $[-1,1]$ when $z \to +\infty$.*

Proof of Lemma 2. Now we denote the p.d.f of uniform distribution as

$$p_\infty(x) = \begin{cases} \frac{1}{2} & |x| \le 1, \\ 0 & |x| > 1. \end{cases} \tag{17}$$

Without loss of generality, for any $x > 1$ and $z \in \mathbb{Z}_+$, we have

$$\left| \int_{-\infty}^{x} \frac{1}{2\eta_z} e^{-\frac{t^{2z}}{2}} dt - \int_{-\infty}^{x} p_\infty(t) dt \right| = \left| \int_0^x \left(\frac{1}{2\eta_z} e^{-\frac{t^{2z}}{2}} - p_\infty(t) \right) dt \right| \tag{18a}$$

$$\le \int_0^1 \left| \frac{1}{2\eta_z} e^{-\frac{t^{2z}}{2}} - \frac{1}{2} \right| dt + \int_1^x \frac{1}{2\eta_z} e^{-\frac{t^{2z}}{2}} dt. \tag{18b}$$

For any $0 < \epsilon < \min\{1, x-1\}$, we have

$$\int_0^1 \left| \frac{1}{2\eta_z} e^{-\frac{t^{2z}}{2}} - \frac{1}{2} \right| dt = \int_0^{1-\epsilon} \left| \frac{1}{2\eta_z} e^{-\frac{t^{2z}}{2}} - \frac{1}{2} \right| dt + \int_{1-\epsilon}^1 \left| \frac{1}{2\eta_z} e^{-\frac{t^{2z}}{2}} - \frac{1}{2} \right| dt \tag{19a}$$

$$\le \left| \frac{1}{2\eta_z} e^{-\frac{(1-\epsilon)^{2z}}{2}} - \frac{1}{2} \right| + \epsilon. \tag{19b}$$

Since $\lim_{z\to\infty} \frac{1}{2\eta_z} = \lim_{z\to\infty} \frac{z}{2^{\frac{1}{2z}}\Gamma(\frac{1}{2z})} = \frac{1}{2}$ and $\lim_{z\to\infty} e^{-\frac{(1-\epsilon)^{2z}}{2}} = 1$, there exists an interger $Z_1 > 0$ such that if $z > Z_1$, we have

$$\left| \frac{1}{2\eta_z} e^{-\frac{(1-\epsilon)^{2z}}{2}} - \frac{1}{2} \right| \le \epsilon.$$

Similarly, we have

$$\int_1^x \frac{1}{2\eta_z} e^{-\frac{t^{2z}}{2}} dt = \int_1^{1+\epsilon} \frac{1}{2\eta_z} e^{-\frac{t^{2z}}{2}} dt + \int_{1+\epsilon}^x \frac{1}{2\eta_z} e^{-\frac{t^{2z}}{2}} dt \tag{20a}$$

$$\le \epsilon + \frac{1}{2\eta_z} e^{-\frac{(1+\epsilon)^{2z}}{2}} (x - 1 - \epsilon). \tag{20b}$$

Since $\lim_{z\to\infty} e^{-\frac{(1+\epsilon)^{2z}}{2}} = 0$, there exists an interger $Z_2 > 0$ such that if $z > Z_2$, we have

$$\int_1^x \frac{1}{2\eta_z} e^{-\frac{t^{2z}}{2}} dt \le \epsilon. \tag{21}$$

In all, for any $0 < \epsilon < 1$, if z is sufficiently large, we have

$$\left| \int_{-\infty}^{x} \frac{1}{2\eta_z} e^{-\frac{t^{2z}}{2}} dt - \int_{-\infty}^{x} p_\infty(t) dt \right| \le 4\epsilon. \tag{22}$$

Take $\epsilon \to 0$ and $z \to \infty$, we have

$$\lim_{z\to\infty}\left|\int_{-\infty}^{x}\frac{1}{2\eta_z}e^{-\frac{t^{2z}}{2}}dt - \int_{-\infty}^{x}p_\infty(t)dt\right| = 0. \tag{23}$$

$\square$

Proof of Lemma 1. We first state a useful inequality on the c.d.f of z distribution:

Lemma 3. *For any $x \in \mathbb{R}$*

$$|x| - \frac{|x|^{2z+1}}{2(2z+1)} \le |\Psi_z(x)| \le |x|, \text{ where } \Psi_z(x) \overset{\text{def.}}{=} \int_0^x e^{-\frac{t^{2z}}{2}}dt. \tag{24}$$

Then, we have

$$\|\eta_z\sigma\mathbb{E}\left[\mathrm{Sign}(\mathrm{x}+\sigma\xi_\mathrm{z})\right] - x\|^2 = \left\|x - \sigma\Psi_z(\frac{x}{\sigma})\right\|^2 = \sum_{j=1}^{d}\left(x(j) - \sigma\Psi_z(\frac{x(j)}{\sigma})\right)^2 \tag{25a}$$

$$\le \sum_{j=1}^{d}\frac{x(j)^{4z+2}}{4(2z+1)^2\sigma^{4z}} = \frac{\|x\|_{4z+2}^{4z+2}}{4(2z+1)^2\sigma^{4z}}. \tag{25b}$$

$\square$

Proof of Lemma 3. Without loss of generality, we prove it for $x \ge 0$.

First,

$$\int_0^x e^{-\frac{t^{2z}}{2}}dt \le \int_0^x 1dt \le x. \tag{26}$$

Now we define $F(x) \overset{\text{def.}}{=} \int_0^x e^{-\frac{t^{2z}}{2}}dt - x + \frac{x^{2z+1}}{2(2z+1)}$. Note that $F(0) = 0$.

Then, we can prove that $F(x) \ge 0$ by

$$F'(x) = e^{-\frac{t^{2z}}{2}} - x + \frac{t^{2z}}{2} \ge 0. \tag{27}$$

(27) is due to the inequality $e^{-x} - 1 + x \ge 0$ for any $x \ge 0$. $\square$

Proof of Theorem 1. Here we define the virtual aggregated update:

$$\bar{x}_{t,s} = \frac{1}{n}\sum_{i=1}^{n}x_{t,s}^{i}, \tag{28}$$

$$\bar{x}_t = \bar{x}_{t-1,E}. \tag{29}$$

We now state the two useful lemmas:

Lemma 4.

$$\mathbb{E}[f(x_t) - f(\bar{x}_t)] \le \frac{\gamma 2^{2z}E^{2z+1}\sqrt{Q_z + G^{4z+2}}G}{2(2z+1)\sigma^{2z}} + \frac{\gamma^2 2^{4z}E^{4z+2}(Q_z + G^{4z+2})L_{\max}}{4(2z+1)^2\sigma^{4z}} \tag{30a}$$

$$+\frac{2\eta_z^2\gamma^2\sigma^2\sum_{j=1}^{d}L_j}{n}. \tag{30b}$$

Lemma 5.

$$\mathbb{E}[f(\bar{x}_t) - f(x_{t-1})] \le -\frac{\gamma}{2}\sum_{s=1}^{E}\|\nabla f(\bar{x}_{t-1,s-1})\|^2 + \frac{E\gamma^2\zeta^2L_{\max}}{2n} + \frac{E(E-1)(2E-1)\gamma^3L_{\max}^2G^2}{6}. \tag{31}$$

With this two lemma, we have

$$\begin{aligned}
\mathbb{E}[f(x_t) - f(x_{t-1})] &= \mathbb{E}[f(x_t) - f(\bar{x}_t)] + E[f(\bar{x}_t) - f(x_{t-1})] && \text{(32a)}\\
&\leq -\frac{\gamma}{2}\sum_{s=1}^{E}\|\nabla f(\bar{x}_{t-1,s-1})\|^2 + \frac{E\gamma^2\zeta^2 L_{\max}}{2n} + \frac{E(E-1)(2E-1)\gamma^3 L_{\max}^2 G^2}{6} && \text{(32b)}\\
&+ \frac{\gamma 2^{2z} E^{2z+1}\sqrt{Q_z + G^{4z+2}}G}{2(2z+1)\sigma^{2z}} + \frac{\gamma^2 2^{4z} E^{4z+2}(Q_z + G^{4z+2})L_{\max}}{4(2z+1)^2\sigma^{4z}} && \text{(32c)}\\
&+ \frac{2\eta_z^2\gamma^2\sigma^2\sum_{j=1}^{d} L_j}{n}. && \text{(32d)}
\end{aligned}$$

Rearranging the terms, we have

$$\begin{aligned}
\frac{1}{E}\sum_{s=1}^{E}\|\nabla f(\bar{x}_{t-1,s-1})\|^2 &\leq \frac{2\mathbb{E}[f(x_{t-1}) - f(x_t)]}{E\gamma} + \frac{\gamma\zeta^2 L_{\max}}{n} + \frac{(E-1)(2E-1)\gamma^2 L_{\max}^2 G^2}{3} && \text{(33a)}\\
&+ \frac{2^{2z} E^{2z}\sqrt{Q_z + G^{4z+2}}G}{(2z+1)\sigma^{2z}} + \frac{\gamma 2^{4z} E^{4z+1}(Q_z + G^{4z+2})L_{\max}}{2(2z+1)^2\sigma^{4z}} && \text{(33b)}\\
&+ \frac{4\eta_z^2\gamma\sigma^2\sum_{j=1}^{d} L_j}{En}. && \text{(33c)}
\end{aligned}$$

Form the telescopic sum

$$\begin{aligned}
\mathbb{E}[\frac{1}{TE}\sum_{t=1}^{T}\sum_{s=1}^{E}]\|\nabla f(\bar{x}_{t-1,s-1})\|^2 &\leq \frac{2\mathbb{E}[f(x_0) - f^*]}{TE\gamma} + \frac{\gamma\zeta^2 L_{\max}}{n} + \frac{(E-1)(2E-1)\gamma^2 L_{\max}^2 G^2}{3} && \text{(34a)}\\
&+ \frac{2^{2z} E^{2z}\sqrt{Q_z + G^{4z+2}}G}{(2z+1)\sigma^{2z}} + \frac{\gamma 2^{4z} E^{4z+1}(Q_z + G^{4z+2})L_{\max}}{2(2z+1)^2\sigma^{4z}} && \text{(34b)}\\
&+ \frac{4\eta_z^2\gamma\sigma^2\sum_{j=1}^{d} L_j}{En}. && \text{(34c)}
\end{aligned}$$

$\square$

Proof of Lemma 4. Therefore, from smoothness we have,

$$f(x_t) - f(\bar{x}_t) \leq \langle \nabla f(\bar{x}_t), x_t - \bar{x}_t\rangle + \frac{\sum_{j=1}^{d} L_j\left(x_t(j) - \bar{x}_t(j)\right)^2}{2}. \tag{35}$$

The following equation and inequality can be checked, where the expectation is taken over the noise vector ξ_z,

$$x_t - \bar{x}_t = \frac{\gamma}{n}\sum_{i=1}^{n}\left(\eta_z\sigma\mathrm{Sign}\left(\sum_{s=1}^{E} g_{t,s}^i + \sigma\xi_z\right) - \sum_{s=1}^{E} g_{t,s}^i\right), \tag{36}$$

$$\mathbb{E}[x_t - \bar{x}_t] = \frac{\gamma}{n}\sum_{i=1}^{n}\left(\sigma\Psi_z\left(\frac{1}{\sigma}\sum_{s=1}^{E} g_{t,s}^i\right) - \sum_{s=1}^{E} g_{t,s}^i\right). \tag{37}$$

526 For any $j = 1, ..., d$, we have

$$\mathbb{E}[(x_t(j) - \bar{x}_t(j))^2] \leq \frac{\gamma^2}{n^2}\left(\sum_{i=1}^{n}\left(\sigma\Psi_z\left(\frac{1}{\sigma}\sum_{s=1}^{E} g_{t,s}^i(j)\right) - \sum_{s=1}^{E} g_{t,s}^i(j)\right)\right)^2 \tag{38a}$$

$$+ \frac{\gamma^2}{n^2}\mathbb{E}\left[\left(\sum_{i=1}^{n}\left(\eta_z\sigma\text{Sign}\left(\sum_{s=1}^{E} g_{t,s}^i(j) + \sigma\xi_z\right) - \sigma\Psi_z\left(\frac{1}{\sigma}\sum_{s=1}^{E} g_{t,s}^i(j)\right)\right)\right)^2\right] \tag{38b}$$

$$\leq \frac{\gamma^2}{n}\sum_{i=1}^{n}\left(\sigma\Psi_z\left(\frac{1}{\sigma}\sum_{s=1}^{E} g_{t,s}^i(j)\right) - \sum_{s=1}^{E} g_{t,s}^i(j)\right)^2 \tag{38c}$$

$$+ \frac{\gamma^2}{n^2}\sum_{i=1}^{n}\mathbb{E}\left[\left(\eta_z\sigma\text{Sign}\left(\sum_{s=1}^{E} g_{t,s}^i(j) + \sigma\xi_z\right) - \sigma\Psi_z\left(\frac{1}{\sigma}\sum_{s=1}^{E} g_{t,s}^i(j)\right)\right)^2\right] \tag{38d}$$

$$\leq \frac{\gamma^2}{n}\sum_{i=1}^{n}\left(\sigma\Psi_z\left(\frac{1}{\sigma}\sum_{s=1}^{E} g_{t,s}^i(j)\right) - \sum_{s=1}^{E} g_{t,s}^i(j)\right)^2 \tag{38e}$$

$$+ \frac{2\gamma^2}{n^2}\sum_{i=1}^{n}\mathbb{E}\left[\left(\eta_z\sigma\text{Sign}\left(\sum_{s=1}^{E} g_{t,s}^i(j) + \sigma\xi_z\right)\right)^2\right] \tag{38f}$$

$$+ \frac{2\gamma^2}{n^2}\sum_{i=1}^{n}\left(\sigma\Psi_z\left(\frac{1}{\sigma}\sum_{s=1}^{E} g_{t,s}^i(j)\right)\right)^2 \tag{38g}$$

$$\leq \frac{\gamma^2}{n}\sum_{i=1}^{n}\left(\sigma\Psi_z\left(\frac{1}{\sigma}\sum_{s=1}^{E} g_{t,s}^i(j)\right) - \sum_{s=1}^{E} g_{t,s}^i(j)\right)^2 + \frac{4\eta_z^2\gamma^2\sigma^2}{n}. \tag{38h}$$

$$\tag{38i}$$

527 Therefore, from Lemma 1, we have

$$\mathbb{E}[\sum_{j=1}^{d} L_j\left(x_t(j) - \bar{x}_t(j)\right)^2] \leq \frac{\gamma^2}{n}\sum_{i=1}^{n}\sum_{j=1}^{d} L_j\left(\sigma\Psi_z\left(\frac{1}{\sigma}\sum_{s=1}^{E} g_{t,s}^i(j)\right) - \sum_{s=1}^{E} g_{t,s}^i(j)\right)^2 \tag{39a}$$

$$+ \frac{4\eta_z^2\gamma^2\sigma^2\sum_{j=1}^{d} L_j}{n} \tag{39b}$$

$$\leq \frac{\gamma^2 L_{\max}}{n}\sum_{i=1}^{n}\left\|\sigma\Psi_z\left(\frac{1}{\sigma}\sum_{s=1}^{E} g_{t,s}^i\right) - \sum_{s=1}^{E} g_{t,s}^i\right\|^2 + \frac{4\eta_z^2\gamma^2\sigma^2\sum_{j=1}^{d} L_j}{n} \tag{39c}$$

$$\leq \frac{\gamma^2 L_{\max}}{4(2z+1)^2\sigma^{4z} n}\sum_{i=1}^{n}\left\|\sum_{s=1}^{E} g_{t,s}^i\right\|_{4z+2}^{4z+2} + \frac{4\eta_z^2\gamma^2\sigma^2\sum_{j=1}^{d} L_j}{n}. \tag{39d}$$

528 Now we need to bound

$$\mathbb{E}\left[\left\|\sum_{s=1}^{E} g_{t,s}^i\right\|_{4z+2}^{4z+2}\right], \tag{40}$$

where the expectation is taken over gradient noise.

$$\mathbb{E}\left[\left\|\sum_{s=1}^{E} g_{t,s}^{i}\right\|_{4z+2}^{4z+2}\right] \leq \mathbb{E}\left[E^{4z+1}\sum_{s=1}^{E}\left\|g_{t,s}^{i}\right\|_{4z+2}^{4z+2}\right] \tag{41a}$$

$$= \mathbb{E}\left[E^{4z+1}\sum_{s=1}^{E}\left\|g_{t,s}^{i} - \nabla f_i(x_{t,s-1}^{i}) + \nabla f_i(x_{t,s-1}^{i})\right\|_{4z+2}^{4z+2}\right] \tag{41b}$$

$$\leq \mathbb{E}\left[(2E)^{4z+1}\sum_{s=1}^{E}\left\|g_{t,s}^{i} - \nabla f_i(x_{t,s-1}^{i})\right\|_{4z+2}^{4z+2} + (2E)^{4z+1}\sum_{s=1}^{E}\left\|\nabla f_i(x_{t,s-1}^{i})\right\|_{4z+2}^{4z+2}\right] \tag{41c}$$

$$\leq (2E)^{4z+1}EQ_z + (2E)^{4z+1}\sum_{s=1}^{E}\left\|\nabla f_i(x_{t,s-1}^{i})\right\|_{2}^{4z+2} \leq 2^{4z+1}E^{4z+2}(Q_z + G^{4z+2}). \tag{41d}$$

In the derivation above, we use a classical result on the monotonicity of ℓ_p norm: For any $x \in \mathbb{R}^d$ and $1 < r < p$, we have

$$\|x\|_p \leq \|x\|_r \leq d^{\frac{1}{r}-\frac{1}{p}}\|x\|_p. \tag{42}$$

Therefore, by taking expectation over both ξ_z and Gradient noise, we have

$$\mathbb{E}[\sum_{j=1}^{d} L_j\left(x_t(j) - \bar{x}_t(j)\right)^2] \leq \frac{\gamma^2 2^{4z+1}E^{4z+2}(Q_z + G^{4z+2})L_{\max}}{4(2z+1)^2\sigma^{4z}} + \frac{4\eta_z^2\gamma^2\sigma^2\sum_{j=1}^{d} L_j}{n}. \tag{43}$$

Hence, we have

$$\mathbb{E}[f(x_t) - f(\bar{x}_t)] \leq \left\langle \nabla f(\bar{x}_t), \frac{\gamma}{n}\sum_{i=1}^{n}\left(\sigma\Psi\left(\frac{1}{\sigma}\sum_{s=1}^{E} g_{t,s}^{i}\right) - \sum_{s=1}^{E} g_{t,s}^{i}\right)\right\rangle + \frac{\sum_{j=1}^{d} L_j\left(x_t(j) - \bar{x}_t(j)\right)^2}{2} \tag{44a}$$

$$\leq \|\nabla f(\bar{x}_t)\|\left\|\frac{\gamma}{n}\sum_{i=1}^{n}\left(\sigma\Psi\left(\frac{1}{\sigma}\sum_{s=1}^{E} g_{t,s}^{i}\right) - \sum_{s=1}^{E} g_{t,s}^{i}\right)\right\| + \frac{\sum_{j=1}^{d} L_j\left(x_t(j) - \bar{x}_t(j)\right)^2}{2} \tag{44b}$$

$$\leq \frac{\gamma 2^{2z}E^{2z+1}\sqrt{Q_z + G^{4z+2}}G}{2(2z+1)\sigma^{2z}} + \frac{\gamma^2 2^{4z}E^{4z+2}(Q_z + G^{4z+2})L_{\max}}{4(2z+1)^2\sigma^{4z}} + \frac{2\eta_z^2\gamma^2\sigma^2\sum_{j=1}^{d} L_j}{n}. \tag{44c}$$

$\square$

Proof of Lemma 5.

$$f(\bar{x}_t) - f(x_{t-1}) = f(\bar{x}_{t-1,E}) - f(\bar{x}_{t-1,0}) = \sum_{s=1}^{E} f(\bar{x}_{t-1,s}) - f(\bar{x}_{t-1,s-1}) \tag{45a}$$

$$\leq \sum_{s=1}^{E}\left(-\langle\nabla f(\bar{x}_{t-1,s-1}), \bar{x}_{t-1,s-1} - \bar{x}_{t-1,s}\rangle + \frac{L_{\max}}{2}\|\bar{x}_{t-1,s} - \bar{x}_{t-1,s-1}\|^2\right) \tag{45b}$$

$$= \sum_{s=1}^{E}\left(-\gamma\langle\nabla f(\bar{x}_{t-1,s-1}), \frac{1}{n}\sum_{i=1}^{n} g_{t-1,s}^{i}\rangle + \frac{\gamma^2 L_{\max}}{2}\|\frac{1}{n}\sum_{i=1}^{n} g_{t-1,s}^{i}\|^2\right). \tag{45c}$$

Taking expectation over gradient noise, we have

$$\mathbb{E}[\|\frac{1}{n}\sum_{i=1}^{n} g_{t-1,s}^{i}\|^2] \leq \|\frac{1}{n}\sum_{i=1}^{n}\nabla f_i(x_{t-1,s-1}^{i})\|^2 + \frac{\zeta^2}{n}, \tag{46a}$$

$$\mathbb{E}[-\langle\nabla f(\bar{x}_{t-1,s-1}), \frac{1}{n}\sum_{i=1}^{n} g_{t-1,s}^{i}\rangle] = -\langle\nabla f(\bar{x}_{t-1,s-1}), \frac{1}{n}\sum_{i=1}^{n}\nabla f_i(x_{t-1,s-1}^{i})\rangle \tag{46b}$$

$$= -\frac{1}{2}\|\nabla f(\bar{x}_{t-1,s-1})\|^2 - \frac{1}{2}\|\frac{1}{n}\sum_{i=1}^{n}\nabla f_i(x_{t-1,s-1}^{i})\|^2 \tag{46c}$$

$$+ \frac{1}{2}\|\nabla f(\bar{x}_{t-1,s-1}) - \frac{1}{n}\sum_{i=1}^{n}\nabla f_i(x_{t-1,s-1}^{i})\|^2. \tag{46d}$$

Notice that from smoothness, we have for arbitrary $x, y \in \mathbb{R}^d$,

$$f(y) \leq \langle\nabla f(x), y - x\rangle + \frac{L_{\max}}{2}\|y - x\|^2, \tag{47}$$

which is equivalent to

$$\|\nabla f(x) - \nabla f(y)\| \leq L_{\max}\|y - x\|. \tag{48}$$

Now for every s, we have

$$\|\nabla f(\bar{x}_{t-1,s-1}) - \frac{1}{n}\sum_{i=1}^{n}\nabla f_i(x_{t-1,s-1}^{i})\|^2 \tag{49a}$$

$$=\|\frac{1}{n}\sum_{i=1}^{n}\nabla f_i(\bar{x}_{t-1,s-1}) - \frac{1}{n}\sum_{i=1}^{n}\nabla f_i(x_{t-1,s-1}^{i})\|^2 \tag{49b}$$

$$\leq\frac{L^2}{n}\sum_{i=1}^{n}\|\bar{x}_{t-1,s-1} - x_{t-1,s-1}^{i}\|^2 \tag{49c}$$

$$=\frac{\gamma^2 L_{\max}^2}{n}\sum_{i=1}^{n}\left\|\sum_{q=1}^{s-1}\left(\frac{1}{n}\sum_{j=1}^{n} g_{t-1,q}^{j} - g_{t-1,q}^{i}\right)\right\|^2 \tag{49d}$$

$$\leq\frac{(s-1)\gamma^2 L_{\max}^2}{n}\sum_{i=1}^{n}\sum_{q=1}^{s-1}\left\|\frac{1}{n}\sum_{j=1}^{n} g_{t-1,q}^{j} - g_{t-1,q}^{i}\right\|^2 \tag{49e}$$

$$\leq 2(s-1)^2\gamma^2 L_{\max}^2 G^2. \tag{49f}$$

In all, we have

$$\mathbb{E}[f(\bar{x}_t) - f(x_{t-1})] \leq \sum_{s=1}^{E}\Bigg(-\frac{\gamma}{2}\|\nabla f(\bar{x}_{t-1,s-1})\|^2 - \frac{\gamma}{2}\|\frac{1}{n}\sum_{i=1}^{n}\nabla f_i(x_{t-1,s-1}^{i})\|^2 + \frac{\gamma^2\zeta^2 L_{\max}}{2n} \tag{50a}$$

$$+ \frac{\gamma}{2}\|\nabla f(\bar{x}_{t-1,s-1}) - \frac{1}{n}\sum_{i=1}^{n}\nabla f_i(x_{t-1,s-1}^{i})\|^2 + \frac{\gamma^2 L_{\max}}{2}\|\frac{1}{n}\sum_{i=1}^{n}\nabla f_i(x_{t-1,s-1}^{i})\|^2\Bigg) \tag{50b}$$

$$\leq \sum_{s=1}^{E}\left(-\frac{\gamma}{2}\|\nabla f(\bar{x}_{t-1,s-1})\|^2 + \frac{\gamma^2\zeta^2 L_{\max}}{2n} + (s-1)^2\gamma^3 L_{\max}^2 G^2\right) \tag{50c}$$

$$= -\frac{\gamma}{2}\sum_{s=1}^{E}\|\nabla f(\bar{x}_{t-1,s-1})\|^2 + \frac{E\gamma^2\zeta^2 L_{\max}}{2n} + \frac{E(E-1)(2E-1)\gamma^3 L_{\max}^2 G^2}{6}. \tag{50d}$$

□

Proof of Theorem 3. We need a similar lemma like Lemma 4.

Lemma 6. *If $\sigma > E(G + Q_\infty)$, then*

$$\mathbb{E}[f(x_t) - f(\bar{x}_t)] \leq \frac{2\gamma^2\sigma^2 \sum_{j=1}^d L_j}{n}. \tag{51}$$

Following similar idea in the proof of Theorem 1, we have

$$\mathbb{E}[f(x_t) - f(x_{t-1})] = \mathbb{E}[f(x_t) - f(\bar{x}_t)] + E[f(\bar{x}_t) - f(x_{t-1})] \tag{52a}$$

$$\leq -\frac{\gamma}{2}\sum_{s=1}^{E} \|\nabla f(\bar{x}_{t-1,s-1})\|^2 + \frac{E\gamma^2\zeta^2 L_{\max}}{2n} + \frac{E(E-1)(2E-1)\gamma^3 L_{\max}^2 G^2}{6} + \frac{2\gamma^2\sigma^2\sum_{j=1}^d L_j}{n}. \tag{52b}$$

Rearranging the terms, we have

$$\frac{1}{E}\sum_{s=1}^{E} \|\nabla f(\bar{x}_{t-1,s-1})\|^2 \leq \frac{2\mathbb{E}[f(x_{t-1}) - f(x_t)]}{E\gamma} + \frac{\gamma\zeta^2 L_{\max}}{n} + \frac{(E-1)(2E-1)\gamma^2 L_{\max}^2 G^2}{3} \tag{53a}$$

$$+ \frac{4\gamma\sigma^2 \sum_{j=1}^d L_j}{En}. \tag{53b}$$

Form the telescopic sum

$$\mathbb{E}[\frac{1}{TE}\sum_{t=1}^{T}\sum_{s=1}^{E}]\|\nabla f(\bar{x}_{t-1,s-1})\|^2 \leq \frac{2\mathbb{E}[f(x_0) - f^*]}{TE\gamma} + \frac{\gamma\zeta^2 L_{\max}}{n} + \frac{(E-1)(2E-1)\gamma^2 L_{\max}^2 G^2}{3} \tag{54a}$$

$$+ \frac{4\gamma\sigma^2 \sum_{j=1}^d L_j}{En}. \tag{54b}$$

Here we provide a simple example where $\sigma < E(G + Q_\infty)$ and the algorithm cannot converge. Consider $E = 1$, $Q_\infty = 0$ and the problem

$$\min_{x\in\mathbb{R}} (x - A)^2 + (x + A)^2,$$

where $A > 0$ is some postive number. If we choose the initial to be $x_0 = \frac{A}{2}$. As we can, the gradient at x_0 for the two parts of the objective function are $-A$ and $3A$ respectively. We denote that ξ_∞ as the random noise following uniform distribution at $[-1, 1]$. If now $\sigma < A$, we have

$$\text{Sign}(-A + \sigma\xi_\infty) + \text{Sign}(3A + \sigma\xi_\infty) = 0, \tag{55}$$

i.e., this algorithm never update the variable. □

Proof of Lemma 6. We first note that, when $z = +\infty$, we have

$$\Psi_\infty(x) = \begin{cases} x & x \in [-1, 1], \\ 1 & x < -1, \\ 1 & x > 1. \end{cases} \tag{56}$$

Now, from L-smoothness we have,

$$f(x_t) - f(\bar{x}_t) \leq \langle \nabla f(\bar{x}_t), x_t - \bar{x}_t \rangle + \frac{\sum_{j=1}^d L_j \left(x_t(j) - \bar{x}_t(j)\right)^2}{2}. \tag{57a}$$

The following equation and inequality can be checked, where the expectation is taken over ξ_∞,

$$\mathbb{E}[x_t - \bar{x}_t] = \mathbb{E}\left[\frac{\gamma}{n}\sum_{i=1}^{n}\left(\sigma\text{Sign}\left(\sum_{s=1}^{E} g_{t,s}^i + \sigma\xi_\infty\right) - \sum_{s=1}^{E} g_{t,s}^i\right)\right] = 0, \tag{58}$$

because from the condition of σ we can see that $\sigma > \|\sum_{s=1}^E g_{t,s}^i\|_\infty$ almost surely. □

For any $j = 1, ..., d$, we have

$$\mathbb{E}[(x_t(j) - \bar{x}_t(j))^2] \leq \frac{\gamma^2}{n^2}\mathbb{E}\left[\left(\sum_{i=1}^{n}\left(\sigma \text{Sign}\left(\sum_{s=1}^{E} g_{t,s}^i(j) + \sigma\xi_\infty(j)\right) - \sigma\Psi_\infty\left(\frac{1}{\sigma}\sum_{s=1}^{E} g_{t,s}^i(j)\right)\right)\right)^2\right] \tag{59}$$

$$\leq \frac{4\gamma^2\sigma^2}{n}. \tag{60}$$

Hence, we have

$$\mathbb{E}[f(x_t) - f(\bar{x}_t)] \leq + \frac{\sum_{j=1}^{d} L_j \left(x_t(j) - \bar{x}_t(j)\right)^2}{2} \tag{61}$$

$$\leq \frac{2\gamma^2\sigma^2 \sum_{j=1}^{d} L_j}{n}. \tag{62}$$

D A simple simulated experiment.

In this section, we verify our theorical results in Section 3 on a simple simulated experiment without any gradient noise. Specifically, we consider the following distributed optimization problem with 10 clients,

$$\min_{x \in \mathbb{R}^d} \frac{1}{2}\sum_{i=1}^{10} \|x - y_i\|^2. \tag{63}$$

Here we generate $y_1, ..., y_{10} \in \mathbb{R}^d$ using i.i.d standard Gaussian distribution, where d is the problem dimension. We compare the performance of the following algorithms. For all the algorithms, we use the same stepsize 0.01 and all-zero initialization. We denote the tested algorithms as:

- GD: Distributed gradient descent without any compression.
- Sto-SignSGD: The algorithm proposed by [Safaryan and Richtárik, 2021].
- SignSGD: (Algorithm 1 with $z = 1$, $E = 1$ and $\sigma = 0$.).
- 1-SignSGD (Algorithm 1 with $z = 1$ and $E = 1$.)
- ∞-SignSGD (Algorithm 1 with $z = +\infty$ and $E = 1$.)

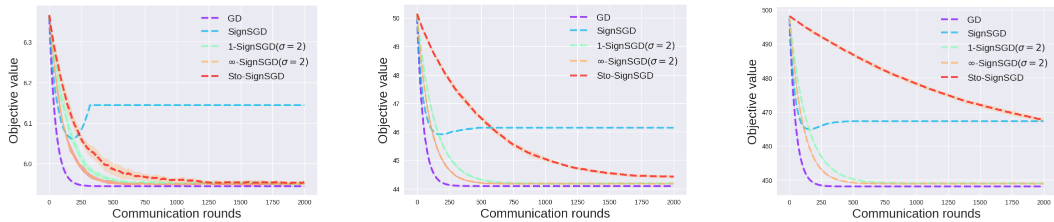
Results. As we can see from Figure 3, all the stochastic sign-based algorithms can converge to the optimal solution, while the SignSGD without any noise fail to converge to the optimal solution. Besides, 1-SignSGD and ∞-SignSGD have roughly the same convergence speed which is slightly slower than the uncompressed gradient descent. It is also verified that the input-dependent noise scale adopted by [Safaryan and Richtárik, 2021] could lead to slow convergence when the problem dimension is high, as we have discussed in Section 3.2. The optimal noise scales of 1-SignSGD and ∞-SignSGD are selected based on Figure 4. We can see that there is a clear bias-variance trade-off in 4 which corroborates our prediction in Section 3. Moreover, it worth to mention that in this experiment, the optimal σ for ∞-SignSGD is much smaller than the conservative choice suggested by theory.

E Experiment details

E.1 Details for the experiment in Section 4.1

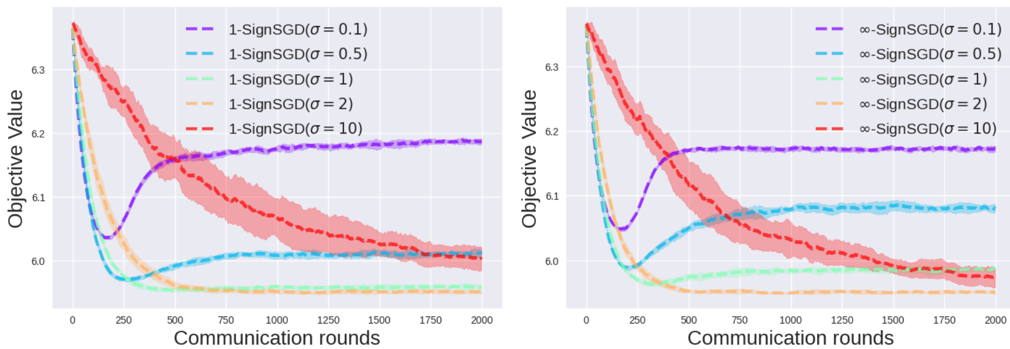
In Table 2, we provide the tuned hyperparameters for all the tested algorithms on non-i.i.d MNIST. Generally, we tune the hyperparameters via grid search: $[0.1, 0.05, 0.01, 0.005]$ for stepsize, $[0, 0.3, 0.5, 0.7, 0.9]$ for momentum coefficient, $[0, 0.02, 0.05, 0.01, 0.03, 0.05, 0.1, 0.3, 0.5]$ for noise scale.

In Figure 5, we visualize the performance of 1-SignSGD and ∞-SignSGD under different noise scales. As we can see, the results for 1-SignSGD and ∞-SignSGD are almost the same, except that the ∞-SignSGD is slighly better than 1-SignSGD when the noise scale is large.



(a) $d = 10$ (b) $d = 100$ (c) $d = 1000$

Figure 3: Performance of algorithms under different problem dimension.



(a) 1-SignSGD (b) ∞-SignSGD

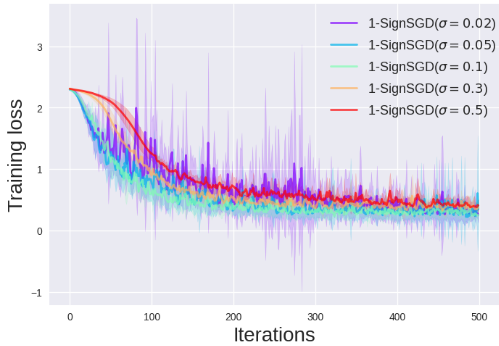
Figure 4: Algorithm 1 under different noise scales

E.2 Details for the experiment in Section 4.2

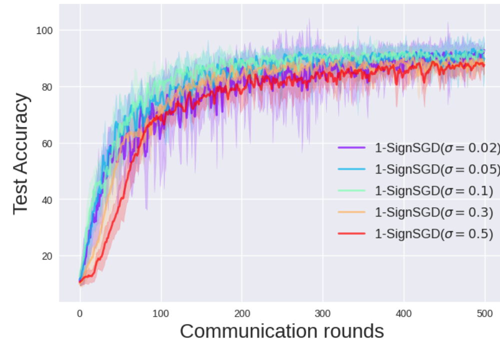
For the experiment on EMNIST, we fix the client stepsize as 0.05. Then we tune the server stepsize, noise scales via grid search: $[1, 0.5, 0.1, 0.05, 0.01, 0.005]$ for stepsize, $[0, 0.005, 0.02, 0.05, 0.01, 0.03, 0.05, 0.1, 0.2]$ for noise scale. The used hyperparameter in the Figure 2 are summarized in Table 3. We also visualize the performance of 1-SignFedAvg and ∞-SignFedAvg under various noise scales and local steps in Figure 6, 7, where we use SignFedAvg to represent Algorithm 1 with $\sigma = 0$.

Algorithm	**Stepsize**	**Momentum coefficient**	**Noise scale**
SGDwM	0.05	0.9	
EF-SignSGDwM	0.05	0.9	
Sto-SignSGDwM	0.01	0.9	
SignSGD	0.01	0	0
1-SignSGD	0.01	0	0.05
∞-SignSGD	0.01	0	0.05

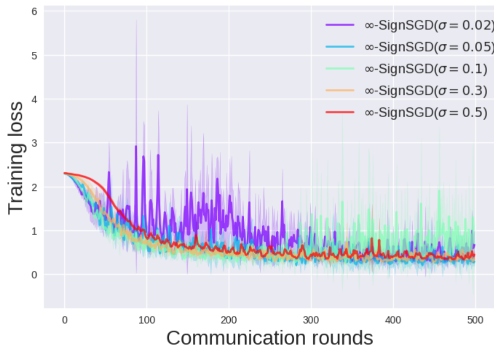
Table 2: Hyperparameters for tested Algorithms on non-i.i.d MNIST.



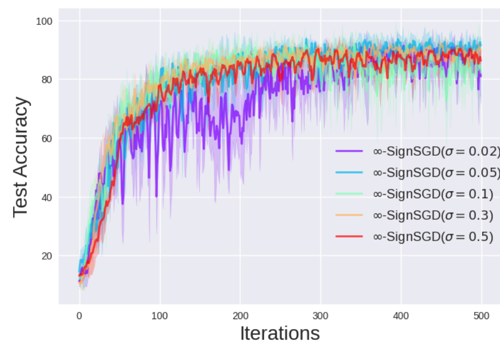
(a) Trainning Loss:1-SignSGD



(b) Test Accuracy:1-SignSGD



(c) Trainning Loss:∞-SignSGD

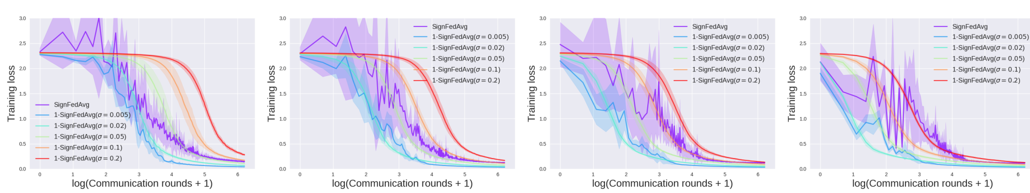


(d) Test Accuracy:∞-SignSGD

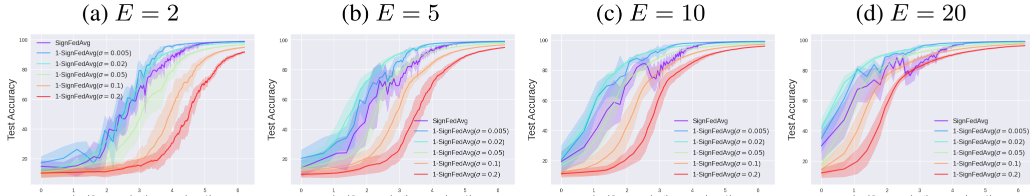
Figure 5: ALG 1 under different noise scales on non-i.i.d MNIST

Algorithm	Server stepsize	Noise scale
1-SignFedAvg	0.03	0.01
∞-SignFedAvg	0.03	0.01

Table 3: Hyperparameters for tested Algorithms on EMNIST.

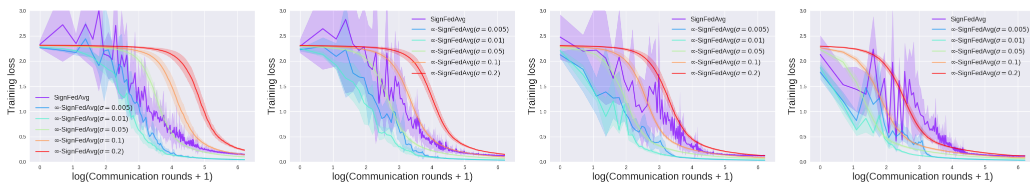


(a) $E = 2$ (b) $E = 5$ (c) $E = 10$ (d) $E = 20$

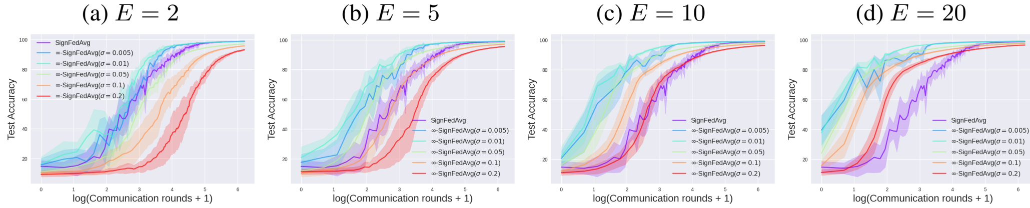


(e) $E = 2$ (f) $E = 5$ (g) $E = 10$ (h) $E = 20$

Figure 6: 1-SignFedAvg under different noise scales and local steps



(a) $E = 2$ (b) $E = 5$ (c) $E = 10$ (d) $E = 20$



(e) $E = 2$ (f) $E = 5$ (g) $E = 10$ (h) $E = 20$

Figure 7: ∞-SignFedAvg under different noise scales and local steps