Nonequispaced fast Fourier transforms for bandlimited functions

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***Abstract*—In this paper we consider the problem of approximating function evaluations $f(\boldsymbol{x}_j)$ at given nonequispaced points $\boldsymbol{x}_j$, $j=1,\dots N$, of a bandlimited function from given values $\hat{f}(\boldsymbol{k})$, $\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{M}}$, of its Fourier transform. Note that if a trigonometric polynomial is given, it is already known that this problem can be solved by means of the nonequispaced fast Fourier transform (NFFT). In other words, we introduce a new NFFT-like procedure for bandlimited functions, which is based on regularized Shannon sampling formulas.**

I. Introduction

The nonequispaced fast Fourier transform (NFFT) is a fast algorithm to evaluate a trigonometric polynomial

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{M}}} \hat{f}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}}, \quad \boldsymbol{x}\in\mathbb{T}^d,$$

with given Fourier coefficients $\hat{f}_{\boldsymbol{k}}\in\mathbb{C}$, $\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{M}}$, at nonequispaced points $\boldsymbol{x}_j\in\mathbb{T}^d$, $j=1,\dots,N$, $N\in\mathbb{N}$, where for $M\in 2\mathbb{N}$ we define the index set $\mathcal{I}_{\boldsymbol{M}} := \mathbb{Z}^d \cap \big[-\frac{M}{2},\frac{M}{2}\big)^d$ with cardinality $|\mathcal{I}_{\boldsymbol{M}}| = M^d$, and $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ denotes the d-dimensional torus with $d\in\mathbb{N}$.

In this paper we focus on the analogous problem for bandlimited functions, where we aim to approximate evaluations $f(\boldsymbol{x}_j)$, $j=1,\dots,N$, of a function

$$f(\boldsymbol{x}) = \int_{\left[-\frac{M}{2},\frac{M}{2}\right)^d} \hat{f}(\boldsymbol{v})\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{v}\boldsymbol{x}}\,\mathrm{d}\boldsymbol{v}, \quad \boldsymbol{x}\in\mathbb{R}^d, \tag{1}$$

from given measurements $\hat{f}(\boldsymbol{k})\in\mathbb{C}$, $\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{M}}$, of its Fourier transform $\hat{f}$.

To do so, this paper is organized as follows. Firstly, in Section II we review the NFFT for trigonometric polynomials. Subsequently, in Section III we give an overview of the regularized Shannon sampling formulas, which play the key role in introducing the NFFT-like procedure for bandlimited functions in Section IV. Finally, in Section V we compare this new method to the classical NFFT.

II. The NFFT

For given nonequispaced nodes $\boldsymbol{x}_j\in\mathbb{T}^d$, $j=1,\dots,N$, and given coefficients $\hat{f}_{\boldsymbol{k}}\in\mathbb{C}$, $\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{M}}$, we consider the computation of the sums

$$f(\boldsymbol{x}_j) = \sum_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{M}}} \hat{f}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}_j}, \quad j=1,\dots,N, \tag{2}$$

where the inner product of two vectors shall be defined as usual as $\boldsymbol{k}\boldsymbol{x} := k_1x_1+\dots+k_dx_d$. A fast approximate algorithm, the so-called *nonequispaced fast Fourier transform (NFFT)*, can be summarized as follows, see e. g. [6], [2], [29], [10], [13] or [22, pp. 413–417].

Algorithm II.1 (NFFT)**.**

For $d,N\in\mathbb{N}$ and $M\in 2\mathbb{N}$ let $\boldsymbol{x}_j\in\mathbb{T}^d$, $j=1,\dots,N$, be given nodes as well as $\hat{f}_{\boldsymbol{k}}\in\mathbb{C}$, $\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{M}}$, given Fourier coefficients. Furthermore, we are given the oversampling factor $\sigma\geq 1$ with $M_\sigma := 2\lceil\lceil\sigma M\rceil/2\rceil\in 2\mathbb{N}$, as well as the window function φ, the truncated function φ_m with $m\ll M_\sigma$, and their 1-periodic versions $\tilde{\varphi}$ and $\tilde{\varphi}_m$.

0. *Precomputation:*
   a) *Compute the nonzero Fourier coefficients $\hat{\varphi}(\boldsymbol{k})$ for $\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{M}}$.*
   b) *Compute the nonzero values $\tilde{\varphi}_m\big(\boldsymbol{x}_j-\frac{\boldsymbol{\ell}}{M_\sigma}\big)$ for $j=1,\dots,N$, and $\boldsymbol{\ell}\in\mathcal{I}_{\boldsymbol{M}_\sigma,m}(\boldsymbol{x}_j)$, cf.* (6).
1. *Set* $\mathcal{O}(|\mathcal{I}_{\boldsymbol{M}}|)$

$$\hat{g}_{\boldsymbol{k}} := \begin{cases} \frac{\hat{f}_{\boldsymbol{k}}}{\hat{\varphi}(\boldsymbol{k})} & : \boldsymbol{k}\in\mathcal{I}_{\boldsymbol{M}}, \\ 0 & : \boldsymbol{k}\in\mathcal{I}_{\boldsymbol{M}_\sigma}\setminus\mathcal{I}_{\boldsymbol{M}}. \end{cases}$$

2. *Compute* $\mathcal{O}(|\mathcal{I}_{\boldsymbol{M}}|\log(|\mathcal{I}_{\boldsymbol{M}}|))$

$$g_{\boldsymbol{\ell}} := \frac{1}{|\mathcal{I}_{\boldsymbol{M}_\sigma}|}\sum_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{M}}} \hat{g}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{\ell}/M_\sigma}, \quad \boldsymbol{\ell}\in\mathcal{I}_{\boldsymbol{M}_\sigma},$$

   by means of a d-variate iFFT.
3. *Compute the short sums* $\mathcal{O}(N)$

$$\tilde{f}_j := \sum_{\boldsymbol{\ell}\in\mathcal{I}_{\boldsymbol{M}_\sigma,m}(\boldsymbol{x}_j)} g_{\boldsymbol{\ell}}\, \tilde{\varphi}_m\big(\boldsymbol{x}_j-\tfrac{\boldsymbol{\ell}}{M_\sigma}\big), \quad j=1,\dots,N.$$

Output: $\tilde{f}_j\approx f(\boldsymbol{x}_j)$ **Complexity:** $\mathcal{O}(|\mathcal{I}_{\boldsymbol{M}}|\log(|\mathcal{I}_{\boldsymbol{M}}|)+N)$

Remark II.2. *Note that Algorithm II.1 is part of the software packages [12] and [1], respectively.*

By defining the *nonequispaced Fourier matrix*

$$\boldsymbol{A} = \boldsymbol{A}_{|\mathcal{I}_{\boldsymbol{M}}|} := \big(\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}_j}\big)_{j=1,\,\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{M}}}^{N} \in\mathbb{C}^{N\times|\mathcal{I}_{\boldsymbol{M}}|},$$

as well as the vectors $\boldsymbol{f} := (f(\boldsymbol{x}_j))_{j=1}^N$ and $\hat{\boldsymbol{f}} := (\hat{f}_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{M}}}$, the computation of the sums (2) can be written as $\boldsymbol{f} = \boldsymbol{A}\hat{\boldsymbol{f}}$. By additionally defining the diagonal matrix

$$\boldsymbol{D} := \operatorname{diag}\left(\frac{1}{|\mathcal{I}_{\boldsymbol{M_\sigma}}| \cdot \hat{\varphi}(\boldsymbol{k})}\right)_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{M}}} \in \mathbb{C}^{|\mathcal{I}_{\boldsymbol{M}}|\times|\mathcal{I}_{\boldsymbol{M}}|}, \quad (3)$$

the truncated *Fourier matrix*

$$\boldsymbol{F} := \left(\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{\ell}/M_{\boldsymbol{\sigma}}}\right)_{\boldsymbol{\ell}\in\mathcal{I}_{\boldsymbol{M_\sigma}},\, \boldsymbol{k}\in\mathcal{I}_{\boldsymbol{M}}} \in \mathbb{C}^{|\mathcal{I}_{\boldsymbol{M_\sigma}}|\times|\mathcal{I}_{\boldsymbol{M}}|}, \quad (4)$$

and the $(2m+1)^d$-sparse matrix

$$\boldsymbol{B} := \left(\tilde{\varphi}_m\big(\boldsymbol{x}_j - \tfrac{\boldsymbol{\ell}}{M_\sigma}\big)\right)_{j=1,\, \boldsymbol{\ell}\in\mathcal{I}_{\boldsymbol{M_\sigma}}}^N \in \mathbb{R}^{N\times|\mathcal{I}_{\boldsymbol{M_\sigma}}|}, \quad (5)$$

where by definition of the index set

$$\mathcal{I}_{\boldsymbol{M_\sigma},m}(\boldsymbol{x}_j) := \big\{\boldsymbol{\ell}\in\mathcal{I}_{\boldsymbol{M_\sigma}} : \exists \boldsymbol{z}\in\mathbb{Z}^d \text{ with } -m\cdot\mathbf{1}_d \le M_\sigma\cdot(\boldsymbol{x}_j+\boldsymbol{z}) - \boldsymbol{\ell} \le m\cdot\mathbf{1}_d\big\} \quad (6)$$

each row of $\boldsymbol{B}$ contains at most $(2m+1)^d$ nonzeros, the NFFT in Algorithm II.1 can be formulated in matrix-vector notation as $\boldsymbol{A} \approx \boldsymbol{BFD}$, cf. [22, p. 419]. This is to say, using the definition of these matrices, the NFFT performs the approximation

$$\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}_j} \approx \sum_{\boldsymbol{\ell}\in\mathcal{I}_{\boldsymbol{M_\sigma},m}(\boldsymbol{x}_j)} \frac{\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{\ell}/M_\sigma}\,\tilde{\varphi}_m\big(\boldsymbol{x}_j - \frac{\boldsymbol{\ell}}{M_\sigma}\big)}{|\mathcal{I}_{\boldsymbol{M_\sigma}}|\cdot\hat{\varphi}(\boldsymbol{k})} \quad (7)$$

for $\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{M}}$ and $\boldsymbol{x}_j\in\mathbb{T}^d$, $j = 1,\ldots,N$.

III. Regularized Shannon sampling formulas

A function $f\colon\mathbb{R}^d\to\mathbb{C}$ is said to be *bandlimited* with *bandwidth* $M\in\mathbb{N}$, if the support of its *(continuous) Fourier transform*

$$\hat{f}(\boldsymbol{v}) := \int_{\mathbb{R}^d} f(\boldsymbol{x})\,\mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{v}\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x}, \quad \boldsymbol{v}\in\mathbb{R}^d, \quad (8)$$

is contained in $\big[-\frac{M}{2}, \frac{M}{2}\big]^d$. The space of all bandlimited functions with bandwidth M shall be denoted by

$$\mathcal{B}_{M/2}(\mathbb{R}^d) := \Big\{f\in L_2(\mathbb{R}^d)\colon \operatorname{supp}(\hat{f}) \subseteq \big[-\tfrac{M}{2}, \tfrac{M}{2}\big]^d\Big\},$$

which is also known as the *Paley–Wiener space*. Note that

$$\mathcal{B}_{M/2}(\mathbb{R}^d) \subseteq L_2(\mathbb{R}^d)\cap C_0(\mathbb{R}^d)\cap C^\infty(\mathbb{R}^d), \quad (9)$$

cf. [14, Lemma 4.1]. Thus, the Fourier inversion theorem, see e. g. [22, Theorem 2.23], guarantees that the *inverse Fourier transform* of f can be written as given in (1).

By the famous Whittaker–Kotelnikov–Shannon sampling theorem ([31], [17], [28]) any bandlimited function $f\in\mathcal{B}_{M/2}(\mathbb{R}^d)$ can be recovered from its samples $f\big(\frac{\boldsymbol{\ell}}{L}\big)$, $\boldsymbol{\ell}\in\mathbb{Z}^d$, with $L\ge M$, $L\in\mathbb{N}$, in the form

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} f\big(\tfrac{\boldsymbol{\ell}}{L}\big)\operatorname{sinc}\big(L\pi\big(\boldsymbol{x} - \tfrac{\boldsymbol{\ell}}{L}\big)\big), \quad \boldsymbol{x}\in\mathbb{R}^d, \quad (10)$$

where the sinc function is given by $\operatorname{sinc}(\boldsymbol{x}) := \prod_{t=1}^d \operatorname{sinc}(x_t)$ with

$$\operatorname{sinc}(x) := \begin{cases} \frac{\sin x}{x} & : x\in\mathbb{R}\setminus\{0\},\\ 1 & : x = 0.\end{cases}$$

It is well known that the series in (10) converges absolutely and uniformly on whole $\mathbb{R}^d$.

Unfortunately, the numerical use of this classical Whittaker–Kotelnikov–Shannon sampling series (10) is limited, since it requires infinitely many samples, which is impossible in practice, and its truncated version is not a good approximation due to the slow decay of the sinc function, see [11]. In addition to this rather poor convergence, it is known, see [8], [9], [5], that in the presence of noise in the samples $f\big(\frac{\boldsymbol{\ell}}{L}\big)$, $\boldsymbol{\ell}\in\mathbb{Z}^d$, the convergence of the Shannon sampling series (10) may even break down completely.

Based on this observation, numerous approaches for numerical realizations have been developed, where the Shannon sampling series was regularized with a suitable window function. Note that many authors such as [4], [20], [26], [21], [30] used window functions in the frequency domain, but the recent study [16] has shown that it is much more beneficial to employ a window function in the spatial domain, cf. [24], [25], [30], [19], [18], [3], [15].

Therefore, for a given $m\in\mathbb{N}$ with $2m \ll L$ we introduce the set $\Phi_{m,L}$ of all window functions $\varphi\colon\mathbb{R}\to[0,1]$ with the following properties:

- φ is compactly supported on $\big[-\frac{m}{L}, \frac{m}{L}\big]$, belongs to $L_1(\mathbb{R})\cap C_0(\mathbb{R})$ and is even.
- φ restricted to $[0,\infty)$ is monotonously non-increasing with $\varphi(0) = 1$.

Remark III.1. *As examples of such window functions we consider the* sinh-type window function

$$\varphi_{\sinh}(x) := \frac{1}{\sinh\beta}\,\sinh\Big(\beta\sqrt{1-\big(\tfrac{Lx}{m}\big)^2}\Big)\,\chi_{[-\frac{m}{L},\frac{m}{L}]}(x) \quad (11)$$

with certain $\beta > 0$, and the continuous Kaiser–Bessel window function

$$\varphi_{\mathrm{cKB}}(x) := \frac{\Big(I_0\Big(\beta\sqrt{1-\big(\frac{Lx}{m}\big)^2}\Big) - 1\Big)}{I_0(\beta) - 1}\,\chi_{[-\frac{m}{L},\frac{m}{L}]}(x)$$

with certain $\beta > 0$, where I_0 denotes the modified Bessel function of first kind. *Note that these window functions are well-studied in the context of the NFFT, see e. g. [23].*

Then, for a fixed window function $\varphi\in\Phi_{m,L}$ we study the *regularized Shannon sampling formula with localized sampling*

$$(R_{\varphi,m}f)(\boldsymbol{x}) := \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} f\big(\tfrac{\boldsymbol{\ell}}{L}\big)\operatorname{sinc}\big(L\pi\big(\boldsymbol{x}-\tfrac{\boldsymbol{\ell}}{L}\big)\big)\,\varphi\big(\boldsymbol{x}-\tfrac{\boldsymbol{\ell}}{L}\big), \quad \boldsymbol{x}\in\mathbb{R}^d. \quad (12)$$

Note that this is an *interpolating approximation* of f on $\frac{1}{L}\,\mathbb{Z}^d$, i. e., we have

$$f\big(\tfrac{\boldsymbol{k}}{L}\big) = (R_{\varphi,m}f)\big(\tfrac{\boldsymbol{k}}{L}\big), \quad \boldsymbol{k}\in\mathbb{Z}^d,$$

since by assumption $\varphi(0)=1$ and $\operatorname{sinc}(\pi(k-\ell))=\delta_{k,\ell}$ for all $k,\ell\in\mathbb{Z}$ with the Kronecker symbol $\delta_{k,\ell}$. Then it is known that the regularized Shannon sampling formula $R_{\varphi,m}f$ in (12) with suitable window function $\varphi\in\Phi_{m,L}$ yields a good approximation of f, cf. [15], [16], [14].

IV. NFFT-like procedure for bandlimited functions

Now assume we are given the values $\hat f(\boldsymbol{k})$, $\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{M}}$, of the Fourier transform (1) of a bandlimited function $f\in\mathcal{B}_{M/2}(\mathbb{R}^d)$, and we are looking for function evaluations $f(\boldsymbol{x}_j)$ at given nonequispaced points $\boldsymbol{x}_j$, $j=1,\dots N$. Further we assume that the function f fulfills the condition

$$\sum_{\boldsymbol{\ell}\in\mathbb{Z}^d}\left|f\left(\tfrac{\boldsymbol{\ell}}{L}\right)\right|<\infty, \tag{13}$$

such that the Fourier series

$$\hat f(\boldsymbol{v})=\frac{1}{L^d}\sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} f\left(\tfrac{\boldsymbol{\ell}}{L}\right)\mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{\ell}\boldsymbol{v}/L},\quad \boldsymbol{v}\in\left[-\tfrac{L}{2},\,\tfrac{L}{2}\right]^d,$$

converges absolutely and uniformly, see [16, p. 12], and thus the considered problem is well-defined. Note that by [27, Lemma 2] it is known that $f\in L_1(\mathbb{R}^d)$ fulfill the condition (13). Moreover, $f\in L_1(\mathbb{R}^d)$ directly implies $\hat f\in C_0(\mathbb{R}^d)$.

Remark IV.1. *Note that the recent work [7] derived error estimates for a familiar problem, where however for equispaced points $x_j=\frac{2j-N}{2N}$, $j=1,\dots,N$, and functions $f\in C(\mathbb{R})$ satisfying certain decay and smoothness conditions simply the FFT can be used.*

In order to compute the values $f(\boldsymbol{x}_j)$, $j=1,\dots N$, we aim to make use of the regularized Shannon sampling formulas, see Section III. Inserting the approximation (12) into the Fourier transform (8) and using the definition of the regularized sinc function

$$\psi(\boldsymbol{x}):=\operatorname{sinc}(L\pi\boldsymbol{x})\,\varphi(\boldsymbol{x}), \tag{14}$$

we have

$$\begin{aligned}\hat f(\boldsymbol{v})&=\int_{\mathbb{R}^d} f(\boldsymbol{x})\,\mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{v}\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x}\approx\int_{\mathbb{R}^d}(R_{\varphi,m}f)(\boldsymbol{x})\,\mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{v}\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x}\\&=\int_{\mathbb{R}^d}\sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} f\left(\tfrac{\boldsymbol{\ell}}{L}\right)\psi\left(\boldsymbol{x}-\tfrac{\boldsymbol{\ell}}{L}\right)\mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{v}\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x}\\&=\sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} f\left(\tfrac{\boldsymbol{\ell}}{L}\right)\mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{v}\boldsymbol{\ell}/L}\int_{\mathbb{R}^d}\psi(\boldsymbol{y})\,\mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{v}\boldsymbol{y}}\,\mathrm{d}\boldsymbol{y}\\&=\left(\sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} f\left(\tfrac{\boldsymbol{\ell}}{L}\right)\mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{v}\boldsymbol{\ell}/L}\right)\cdot\hat\psi(\boldsymbol{v}),\end{aligned} \tag{15}$$

where summation and integration may be interchanged by the theorem of Fubini–Tonelli. By defining

$$\hat\nu(\boldsymbol{v}):=\sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} f\left(\tfrac{\boldsymbol{\ell}}{L}\right)\mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{v}\boldsymbol{\ell}/L},\quad\boldsymbol{v}\in\mathbb{R}^d, \tag{16}$$

we recognize that this function $\hat\nu$ is L-periodic. Thus, due to the fact that the Fourier transform of the bandlimited function $f\in\mathcal{B}_{M/2}(\mathbb{R}^d)$ is non-periodic, the approximation (15) can only be reasonable for $\boldsymbol{v}\in\left[-\tfrac{L}{2},\tfrac{L}{2}\right]^d$.

As the goal is to recover the nonequispaced samples $f(\boldsymbol{x}_j)$, $j=1,\dots,N$, by means of a regularized Shannon sampling formula (12), we need access to as many equispaced samples $f\left(\tfrac{\boldsymbol{\ell}}{L}\right)$ as possible, i. e., we are looking for an inversion formula for (16). To this end, note that (16) can be written as

$$\begin{aligned}\hat\nu(\boldsymbol{v})&=\sum_{\boldsymbol{\ell}\in\mathcal{I}_{\boldsymbol{\Theta}}} f\left(\tfrac{\boldsymbol{\ell}}{L}\right)\mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{v}\boldsymbol{\ell}/L}\\&\quad+\sum_{\boldsymbol{r}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}}\sum_{\boldsymbol{\ell}\in\mathcal{I}_{\boldsymbol{\Theta}}} f\left(\tfrac{\boldsymbol{\ell}+\boldsymbol{r}\Theta}{L}\right)\mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{v}(\boldsymbol{\ell}+\boldsymbol{r}\Theta)/L},\quad\boldsymbol{v}\in\mathbb{R}^d,\end{aligned}$$

with the index set $\mathcal{I}_{\boldsymbol{\Theta}}$ with $\boldsymbol{\Theta}=\Theta\cdot\boldsymbol{1}_d$, $\Theta\in 2\mathbb{N}$. Since $f\in\mathcal{B}_{M/2}(\mathbb{R}^d)\subseteq C_0(\mathbb{R}^d)$, see (9), the equispaced samples $f\left(\tfrac{\boldsymbol{\ell}}{L}\right)$ are negligible for all $\|\boldsymbol{\ell}\|_\infty\geq\frac{\Theta}{2}$ with suitably chosen Θ. In order to avoid aliasing in the computation we assume that $\Theta=L$ is sufficient. Hence, we consider

$$\hat\nu(\boldsymbol{v})\approx\hat\vartheta(\boldsymbol{v}):=\sum_{\boldsymbol{\ell}\in\mathcal{I}_{\boldsymbol{L}}} f\left(\tfrac{\boldsymbol{\ell}}{L}\right)\mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{v}\boldsymbol{\ell}/L},\quad\boldsymbol{v}\in\mathbb{R}^d, \tag{17}$$

and thus by (15) the approximation

$$\hat f(\boldsymbol{v})\approx\hat\vartheta(\boldsymbol{v})\cdot\hat\psi(\boldsymbol{v}),\quad\boldsymbol{v}\in\left[-\tfrac{L}{2},\tfrac{L}{2}\right]^d. \tag{18}$$

Since it is additionally known that $\hat f(\boldsymbol{v})=0$ for all $\boldsymbol{v}\notin\left[-\tfrac{M}{2},\tfrac{M}{2}\right]^d$ and $\hat\psi(\boldsymbol{v})\neq 0$ for all $\boldsymbol{v}\in\left[-\tfrac{L}{2},\tfrac{L}{2}\right]^d$, we might use (18) and (17) for given $\hat f(\boldsymbol{k})$, $\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{M}}$, to approximate the equispaced samples $f\left(\tfrac{\boldsymbol{\ell}}{L}\right)$, $\boldsymbol{\ell}\in\mathcal{I}_{\boldsymbol{L}}$, by setting

$$\hat\vartheta(\boldsymbol{k})=\begin{cases}\frac{\hat f(\boldsymbol{k})}{\hat\psi(\boldsymbol{k})} & :\ \boldsymbol{k}\in\mathcal{I}_{\boldsymbol{M}},\\ 0 & :\ \boldsymbol{k}\in\mathcal{I}_{\boldsymbol{L}}\setminus\mathcal{I}_{\boldsymbol{M}},\end{cases}$$

and subsequently computing

$$f\left(\tfrac{\boldsymbol{\ell}}{L}\right)\approx\vartheta_{\boldsymbol{\ell}}:=\frac{1}{|\mathcal{I}_{\boldsymbol{L}}|}\sum_{\boldsymbol{k}\in\mathcal{I}_{\boldsymbol{L}}}\hat\vartheta(\boldsymbol{k})\,\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{\ell}/L},\quad\boldsymbol{\ell}\in\mathcal{I}_{\boldsymbol{L}}, \tag{19}$$

by means of an iFFT.

To finally approximate the samples $f(\boldsymbol{x}_j)$, $j=1,\dots,N$, we make use of the regularized Shannon sampling formula (12). Note that since we assumed that the window function $\varphi\in\Phi_{m,L}$ is compactly supported, the computation of $(R_{\varphi,m}f)(\boldsymbol{x})$ for fixed $\boldsymbol{x}\in\mathbb{R}^d\setminus\frac{1}{L}\mathbb{Z}^d$ requires only $(2m+1)^d$ samples $f\left(\tfrac{\boldsymbol{\ell}}{L}\right)$. However, we have already encountered that (19) can only be used to approximate $f\left(\tfrac{\boldsymbol{\ell}}{L}\right)$ for $\boldsymbol{\ell}\in\mathcal{I}_{\boldsymbol{L}}$ in order to avoid aliasing in the computation of the inverse Fourier transform in (19). Thereby, we are confronted with a limitation of the feasible points to $\boldsymbol{x}_j\in[-\frac12+\frac{m}{L},\frac12-\frac{m}{L})^d$, $j=1,\dots,N$, since only in this case exclusively the evaluations $f\left(\tfrac{\boldsymbol{\ell}}{L}\right)$, $\boldsymbol{\ell}\in\mathcal{I}_{\boldsymbol{L}}$, are needed for the computation. Hence, the final approximation is computed by

$$\begin{aligned}(R_{\varphi,m}f)(\boldsymbol{x}_j)\approx f_j&:=\sum_{\boldsymbol{\ell}\in\mathcal{I}_{\boldsymbol{L}}}\vartheta_{\boldsymbol{\ell}}\,\psi\left(\boldsymbol{x}_j-\tfrac{\boldsymbol{\ell}}{L}\right)\\&=\sum_{\boldsymbol{\ell}\in\mathcal{J}_{\boldsymbol{L},m}(\boldsymbol{x}_j)}\vartheta_{\boldsymbol{\ell}}\,\psi\left(\boldsymbol{x}_j-\tfrac{\boldsymbol{\ell}}{L}\right),\end{aligned}$$

where the index set of the nonzero entries

$$\mathcal{J}_{\boldsymbol{L},m}(\boldsymbol{x}_j):=\left\{\boldsymbol{\ell}\in\mathbb{Z}^d\colon\ -m+L\boldsymbol{x}_j\leq\boldsymbol{\ell}\leq m+L\boldsymbol{x}_j\right\} \tag{20}$$

contains at most $(2m+1)^d$ entries for each fixed $\boldsymbol{x}_j$, cf. (6). Thus, the obtained algorithm can be summarized as follows, cf. [14, Algorithm 5.16].

Algorithm IV.2 (NFFT-like procedure for bandlim. functions)**.**

For $d, m, N \in \mathbb{N}$, $M \in 2\mathbb{N}$, and $L = M(1+\lambda) \in \mathbb{N}$ with the oversampling parameter $\lambda \geq 0$ let $\boldsymbol{x}_j \in [-\frac{1}{2}+\frac{m}{L}, \frac{1}{2}-\frac{m}{L})^d$, $j = 1, \dots, N$, be given nodes as well as $\hat{f}(\boldsymbol{k}) \in \mathbb{C}$, $\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}$, given evaluations of the Fourier transform of the bandlimited function $f \in \mathcal{B}_{M/2}(\mathbb{R}^d)$. Furthermore, we are given the window function $\varphi \in \Phi_{m,L}$, the corresponding regularized sinc *function ψ in* (14)*, and its Fourier transform $\hat{\psi}$.*

0. *Precomputation:*
   a) *Compute the nonzero values $\hat{\psi}(\boldsymbol{k})$ for $\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}$.*
   b) *Compute the evaluations $\psi\big(\boldsymbol{x}_j - \frac{\boldsymbol{\ell}}{L}\big)$ for $j = 1, \dots, N$, and $\boldsymbol{\ell} \in \mathcal{J}_{\boldsymbol{L},m}(\boldsymbol{x}_j)$, cf.* (20)*.*
1) *Set* $\mathcal{O}(|\mathcal{I}_{\boldsymbol{M}}|)$

$$\hat{\vartheta}(\boldsymbol{k}) := \begin{cases} \frac{\hat{f}(\boldsymbol{k})}{\hat{\psi}(\boldsymbol{k})} & : \quad \boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}, \\ 0 & : \quad \boldsymbol{k} \in \mathcal{I}_{\boldsymbol{L}} \setminus \mathcal{I}_{\boldsymbol{M}}. \end{cases}$$

2) *Compute* $\mathcal{O}(|\mathcal{I}_{\boldsymbol{M}}| \log(|\mathcal{I}_{\boldsymbol{M}}|))$

$$\vartheta_{\boldsymbol{\ell}} := \frac{1}{|\mathcal{I}_{\boldsymbol{L}}|} \sum_{\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{L}}} \hat{\vartheta}(\boldsymbol{k}) \, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}\boldsymbol{\ell}/L}, \quad \boldsymbol{\ell} \in \mathcal{I}_{\boldsymbol{L}},$$

by means of a d-variate iFFT.

3) *Compute the short sums* $\mathcal{O}(N)$

$$f_j := \sum_{\boldsymbol{\ell} \in \mathcal{J}_{\boldsymbol{L},m}(\boldsymbol{x}_j)} \vartheta_{\boldsymbol{\ell}} \, \psi\big(\boldsymbol{x}_j - \tfrac{\boldsymbol{\ell}}{L}\big), \quad j = 1, \dots, N.$$

Output: $f_j \approx f(\boldsymbol{x}_j)$ **Complexity:** $\mathcal{O}(|\mathcal{I}_{\boldsymbol{M}}| \log(|\mathcal{I}_{\boldsymbol{M}}|) + N)$

Note that by defining the vector $\hat{\boldsymbol{f}} := (\hat{f}(\boldsymbol{k}))_{\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}}$ as well as the diagonal matrix

$$\boldsymbol{D}_{\hat{\psi}} := \mathrm{diag}\left(\frac{1}{|\mathcal{I}_{\boldsymbol{L}}| \cdot \hat{\psi}(\boldsymbol{k})} \right)_{\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}} \in \mathbb{C}^{|\mathcal{I}_{\boldsymbol{M}}| \times |\mathcal{I}_{\boldsymbol{M}}|} \tag{21}$$

and the $(2m+1)^d$-sparse matrix

$$\boldsymbol{\Psi} := \left(\psi\big(\boldsymbol{x}_j - \tfrac{\boldsymbol{\ell}}{L}\big) \right)_{j=1,\, \boldsymbol{\ell} \in \mathcal{I}_{\boldsymbol{L}}}^{N} \in \mathbb{R}^{N \times |\mathcal{I}_{\boldsymbol{L}}|}, \tag{22}$$

the approximation of Algorithm IV.2 is given by

$$\boldsymbol{f} = \boldsymbol{\Psi} \boldsymbol{F} \boldsymbol{D}_{\hat{\psi}} \hat{\boldsymbol{f}}, \tag{23}$$

where $\boldsymbol{F} \in \mathbb{C}^{|\mathcal{I}_{\boldsymbol{L}}| \times |\mathcal{I}_{\boldsymbol{M}}|}$ denotes the Fourier matrix (4) with $L = M_\sigma$.

V. Comparison to the classical NFFT

Note that one might also directly apply an equispaced quadrature rule to the inverse Fourier transform (1), i. e., consider the approximation

$$f(\boldsymbol{x}) = \int_{[-\frac{M}{2}, \frac{M}{2}]^d} \hat{f}(\boldsymbol{v}) \, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{v}\boldsymbol{x}} \, \mathrm{d}\boldsymbol{v} \approx \sum_{\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}} \hat{f}(\boldsymbol{k}) \, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}\boldsymbol{x}},$$

such that the function evaluations $f(\boldsymbol{x}_j)$, $j = 1, \dots, N$, could also be approximated efficiently by means of an NFFT. Since this raises the question of which of the two methods, Algorithm II.1 or Algorithm IV.2, is more advantageous, this section deals with the comparison of the two approaches.

Considering the matrix notations $\boldsymbol{BFD}$ and $\boldsymbol{\Psi F D}_{\hat{\psi}}$, cf. (7) and (23), the first thing to realize is that for $\boldsymbol{B} \in \mathbb{R}^{N \times |\mathcal{I}_{\boldsymbol{L}}|}$ in (5) the window function $\varphi_m(\boldsymbol{x})$ is used, while for $\boldsymbol{\Psi} \in \mathbb{R}^{N \times |\mathcal{I}_{\boldsymbol{L}}|}$ in (22) we consider the regularized sinc function $\psi(\boldsymbol{x})$ in (14). A similar remark can also be made about the diagonal matrices $\boldsymbol{D} \in \mathbb{C}^{|\mathcal{I}_{\boldsymbol{M}}| \times |\mathcal{I}_{\boldsymbol{M}}|}$ in (3) and $\boldsymbol{D}_{\hat{\psi}} \in \mathbb{C}^{|\mathcal{I}_{\boldsymbol{M}}| \times |\mathcal{I}_{\boldsymbol{M}}|}$ in (21).

It is also important to note that the two methods can only be compared for $\boldsymbol{x} \in [-\frac{1}{2}+\frac{m}{L}, \frac{1}{2}-\frac{m}{L})^d$, as the approximation by Algorithm IV.2 is only reasonable in this case. This implies that the matrix $\boldsymbol{B}$ in (5) is, unlike usual, non-periodic, whereas the matrix $\boldsymbol{\Psi}$ in (22) is inherently non-periodic by definition.

To study the quality of both approaches, note that by the NFFT we are given the approximation

$$\mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}\boldsymbol{x}} \approx \frac{1}{|\mathcal{I}_{\boldsymbol{L}}| \cdot \hat{\varphi}(\boldsymbol{k})} \sum_{\boldsymbol{\ell} \in \mathcal{I}_{\boldsymbol{L}}} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}\boldsymbol{\ell}/L} \, \tilde{\varphi}_m\big(\boldsymbol{x} - \tfrac{\boldsymbol{\ell}}{L}\big), \tag{24}$$

for $\boldsymbol{x} \in \mathbb{T}^d$ and $\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}$ fixed, cf. (7) with $L = M_\sigma$, where $\tilde{\varphi}_m(\boldsymbol{x}) = \sum_{\boldsymbol{r} \in \mathbb{Z}^d} \varphi_m(\boldsymbol{x} + \boldsymbol{r})$ denotes the 1-periodic version of the compactly supported window function φ_m. Thus, we look for a comparable approximation of the exponential function using our newly proposed method in Algorithm IV.2. For this purpose, note that $g(\boldsymbol{x}) := \hat{\psi}(\boldsymbol{x}) \, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}\boldsymbol{x}}$ with $\boldsymbol{k} \in \mathbb{R}^d$ fixed possesses the Fourier transform $\hat{g}(\boldsymbol{v}) = \psi(\boldsymbol{k} - \boldsymbol{v})$. Therefore, we have $g \in \mathcal{B}_{M/2}(\mathbb{R}^d)$ for all $\boldsymbol{k} \in \big[-\frac{M}{2}+\frac{m}{L}, \frac{M}{2}-\frac{m}{L}\big]^d$, i. e., considering (12) for this function g yields

$$\hat{\psi}(\boldsymbol{x}) \, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}\boldsymbol{x}} \approx \sum_{\boldsymbol{\ell} \in \mathbb{Z}^d} \hat{\psi}\big(\tfrac{\boldsymbol{\ell}}{L}\big) \, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}\boldsymbol{\ell}/L} \, \psi\big(\boldsymbol{x} - \tfrac{\boldsymbol{\ell}}{L}\big), \quad \boldsymbol{x} \in \mathbb{R}^d,$$

or rather

$$\mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}\boldsymbol{x}} \approx \sum_{\boldsymbol{\ell} \in \mathcal{I}_{\boldsymbol{L}}} \frac{\hat{\psi}\big(\frac{\boldsymbol{\ell}}{L}\big)}{\hat{\psi}(\boldsymbol{x})} \, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}\boldsymbol{\ell}/L} \, \psi\big(\boldsymbol{x} - \tfrac{\boldsymbol{\ell}}{L}\big)$$

for $\boldsymbol{x} \in \big[-\frac{1}{2}+\frac{m}{L}, \frac{1}{2}-\frac{m}{L}\big)^d$. Since numerical experiments have shown that $\hat{\psi}(\boldsymbol{y}) \approx \frac{1}{|\mathcal{I}_{\boldsymbol{L}}|}$, $\boldsymbol{y} \in \big[-\frac{M}{2}, \frac{M}{2}\big)^d$, for the window functions mentioned in Remark III.1, the above approximation simplifies to

$$\mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}\boldsymbol{x}} \approx \sum_{\boldsymbol{\ell} \in \mathcal{I}_{\boldsymbol{L}}} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}\boldsymbol{\ell}/L} \, \psi\big(\boldsymbol{x} - \tfrac{\boldsymbol{\ell}}{L}\big), \tag{25}$$

which equals the approximation $\boldsymbol{\Psi F D}_{\hat{\psi}}$ of Algorithm IV.2, since $|\mathcal{I}_{\boldsymbol{L}}| \, \hat{\psi}(\boldsymbol{k}) \approx 1$, $\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}$. Therefore, we can compare the quality of the two methods by considering the approximations (24) and (25) of the exponential function.

For simplicity we restrict ourselves to the one-dimensional setting $d = 1$ for the visualization. To estimate the quality of the approaches, we consider the approximation error

$$e(v) := \max_{x_p, p=1,\dots,P} \big| \mathrm{e}^{2\pi \mathrm{i} v x_p} - h(x_p) \big|, \tag{26}$$

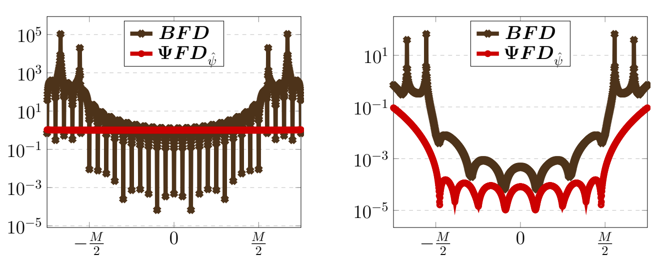
where the term $h(x_p)$ is a placeholder for the right-hand sides of (24) and (25), respectively, evaluated at a fine grid of $P = 10^5$ equispaced points x_p, $p = 1, \dots, P$. This approximation error (26) shall now be computed for several values

$$v_s = -\tfrac{M}{2} - m + \tfrac{s}{S} \in \left[-\tfrac{M}{2} - m, \tfrac{M}{2} + m\right], \qquad s = 0, \dots, S(M + 2m), \tag{27}$$

where $S = 1$ corresponds to integer evaluation, whereas we use $S = 32$ to examine the approximation at non-integer points as well. Note that (24) is expected to provide a good approximation only for $v \in \left[-\frac{M}{2}, \frac{M}{2}\right]$, while (25) is expected to do so only for $v \in \left[-\frac{M}{2} + \frac{m}{L}, \frac{M}{2} - \frac{m}{L}\right]$. Nevertheless, we test for v from a larger interval to confirm these assumptions.

The corresponding outcomes when computing the approximations (24) and (25) using the sinh-type window function (11) as well as the parameters $M = 20$, $\lambda = 1$, $L = (1 + \lambda)M$, and $m = 5$, are displayed in Figure 1. For $x \in \left[-\frac{1}{2}, \frac{1}{2}\right)$ it is easy to see that our newly proposed method (25) indeed does not provide reasonable results, while the approximation (24) by means of the NFFT is only useful at integer points v. For the truncated interval $x \in \left[-\frac{1}{2} + \frac{m}{L}, \frac{1}{2} - \frac{m}{L}\right)$, however, both approximations (24) and (25) are clearly beneficial for non-integer points v as well, but as expected these methods only succeed when $|v| \le \frac{M}{2}$. Nevertheless, although also the approximation (24) by means of the NFFT yields better results in this setting, the approximation (25) by means of our newly proposed method easily outperforms the classical NFFT in terms of the approximation error (26).

That is to say, Figure 1 demonstrates that the novel NFFT-like approach in Algorithm IV.2 is better suited for bandlimited functions, while this superiority is not limited to $k \in \mathcal{I}_M$ but extends to the entire domain $v \in \left[-\frac{M}{2}, \frac{M}{2}\right]$. Moreover, the error of Algorithm IV.2 is bounded by the error estimates of the regularized Shannon sampling formulas in Section III, whereas the quadrature error of the NFFT remains unclear.



(a) $x \in \left[-\frac{1}{2}, \frac{1}{2}\right)$ (b) $x \in \left[-\frac{1}{2} + \frac{m}{L}, \frac{1}{2} - \frac{m}{L}\right)$

Fig. 1: Maximum approximation error (26) for $P = 10^5$ computed for (27) with $S = 32$ using the sinh-type window function (11) as well as $M = 20$, $\lambda = 1$, $L = (1 + \lambda)M$, and $m = 5$ in the one-dimensional setting $d = 1$.

Example V.1. *To finally examine the approximation quality of the NFFT-like procedure in Algorithm IV.2 for bandlimited functions we provide a function f with its corresponding Fourier transform $\hat f$ in* (8)*, such that we have access to the exact values $\hat f(k)$, $k \in \mathcal{I}_M$, as input for Algorithm IV.2, as well as the exact function evaluations $f(x_j)$, $j = 1, \dots, N$. In doing so, we can compare the result f_j, $j = 1, \dots, N$, of Algorithm IV.2 to the exact function evaluations $f(x_j)$ by computing the maximum approximation error*

$$\max_{j=1,\dots,N} |f_j - f(x_j)|. \tag{28}$$

For comparison we also compute the approximation error (28) *when f_j is the result of the classical NFFT in Algorithm II.1.*

We consider the one-dimensional setting $d = 1$ and for several bandwidth parameters $M \in \{20, 40, \dots, 1000\}$ we study the function $f(x) = \operatorname{sinc}^2\left(\frac{M}{2}\pi x\right)$ with the Fourier transform

$$\hat f(v) = \frac{2}{M} \cdot \begin{cases} 1 - \left|\frac{2v}{M}\right| & : |v| \le \frac{M}{2}, \\ 0 & : \textit{otherwise}. \end{cases}$$

Note that the function f is scaled such that $\max_{x \in \mathbb{R}} f(x) = 1$ independent of the bandwidth M and thereby the approximation errors (28) *are comparable for all considered M. As evaluation points $x_j \in \left[-\frac{1}{2} + \frac{m}{L}, \frac{1}{2} - \frac{m}{L}\right]$, $j = 1, \dots, N$, we choose the scaled Chebyshev nodes*

$$x_j = \cos\left(\tfrac{(j-1)\pi}{N}\right) \cdot \left(\tfrac{1}{2} - \tfrac{m}{L}\right), \quad j = 1, \dots, N, \tag{29}$$

with $N = \frac{M}{2}$, $m = 5$, as well as $M_\sigma = L = M(1 + \lambda)$ with $\lambda = 1$, and we use the sinh*-type window function* (11).

The corresponding results are depicted in Figure 2. As expected by Figure 1, the new NFFT-like procedure in Algorithm IV.2 performs much better than the classical NFFT in Algorithm II.1. While for $M \le 80$ both approaches exhibit the same maximum approximation error (28)*, for larger bandwidth M the approximation error* (28) *gets smaller only for the NFFT-like procedure in Algorithm IV.2. That is to say, when approximating the evaluations $f(x_j)$, $j = 1, \dots, N$, of the bandlimited function $f \in \mathcal{B}_{M/2}(\mathbb{R})$ by given samples $\hat f(k)$, $k \in \mathcal{I}_M$, of the corresponding Fourier transform* (8)*, reasonable results can be obtained by the NFFT in Algorithm II.1, yet evidence indicates that our newly proposed NFFT-like procedure for bandlimited functions in Algorithm IV.2 yields results that are at least as good, if not superior. Accordingly, we conclude that the NFFT-like procedure in Algorithm IV.2 is the preferred approach in this context.*

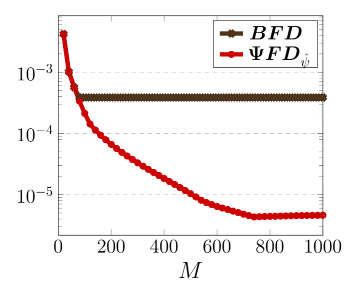


Fig. 2: Maximum approximation error (28) of Algorithms II.1 and IV.2 using the sinh-type window function (11) computed for the function $f(x) = \operatorname{sinc}^2\left(\frac{M}{2}\pi x\right)$, $M \in \{20, 40, \dots, 1000\}$, and the scaled Chebyshev nodes (29) with $N = \frac{M}{2}$, $m = 5$, $M_\sigma = L = (1 + \lambda)M$, as well as $\lambda = 1$ and $d = 1$.

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